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## Relations of Different Types of Numerical Magnitude Representations to Each Other and to Mathematics Achievement

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### **Abstract**

We examined relations between symbolic and non-symbolic numerical magnitude representations, between whole number and fraction representations, and between these representations and overall mathematics achievement in fifth graders. Fraction and whole number symbolic and non-symbolic numerical magnitude understandings were measured using both magnitude comparison and number line estimation tasks. After controlling for non-mathematical cognitive proficiency, both symbolic and non-symbolic numerical magnitude understandings were uniquely related to mathematics achievement, but the relation was much stronger for symbolic numbers. A meta-analysis of 19 published studies indicated that relations between non-symbolic numerical magnitude knowledge and mathematics achievement are present, but tend to be weak, especially beyond age six.

Keywords: fractions, magnitude representations, Approximate Number System, mathematics achievement

Relations of Symbolic and Non-symbolic Fraction and Whole Number Magnitude  
Representations to Each Other and to Mathematics Achievement

**Introduction**

Precise representations of numerical magnitudes are foundational for learning mathematics. Both correlational and causal evidence link the precision of individual children's numerical magnitude representations to their whole number and fraction arithmetic skill, memory for numbers, and other aspects of mathematical knowledge (Halberda, Mazocco, & Feigenson, 2008; Holloway & Ansari, 2009; Siegler, Thompson, & Schneider, 2011; Thompson & Siegler, 2010). Such relations have been demonstrated with both symbolically expressed numbers (e.g., choosing the larger Arabic numeral) and non-symbolic numbers (e.g., choosing the more numerous dot array).

It remains unclear, however, how the symbolic and non-symbolic numerical magnitude representation systems are related to each other and whether each is uniquely related to mathematics achievement. It has been hypothesized that non-symbolic numerical magnitude representations give rise to symbolic ones, and that both are related to mathematics achievement (e.g., Dehaene, 2011; Verguts & Fias, 2004), but experimental tests of these hypotheses have not yielded a consistent pattern. Almost all of these experiments have used a single task (magnitude comparison) and all have used a single type of number (whole numbers); none have compared alternative models of the relations between the types of magnitudes and the relation of each to overall mathematics achievement. In addition, many studies have used narrow measures of mathematics achievement such as arithmetic performance rather than broad measures such as standardized mathematics achievement test scores.

In the present study, we seek to provide a broader and more general understanding of relations between symbolic and non-symbolic numerical magnitude representations, and their unique relation to overall mathematics achievement. We pursue this goal by examining the relations of symbolic and non-symbolic numerical magnitude understanding on different tasks (magnitude comparison and number line estimation) and with different types of numbers (whole numbers and fractions), and then use these data to evaluate three models of relations among non-symbolic numerical magnitude representations, symbolic numerical magnitude representations, and mathematics achievement. We also try to explain the inconsistent relations between non-symbolic numerical magnitude knowledge and mathematics achievement by performing a meta-analysis that examines variables that might influence the relation between the two abilities.

### **Understanding of Symbolic Numerical Magnitudes**

Numerical magnitude understanding refers to the ability to comprehend, estimate, and compare the sizes of numbers (both symbolic and non-symbolic whole numbers and fractions). Numerical magnitude understanding is separate from other numerical abilities such as counting, cardinality or arithmetic and deals solely with understanding numbers as magnitudes that can be compared and ordered. Such understanding is typically assessed using comparison or estimation tasks. Symbolic magnitude comparison tasks ask which of two Arabic numerals is larger (e.g., 3 or 6,  $1/2$  or  $1/3$ ). On such tasks, speed and accuracy increase with age, experience, and the distance between the numbers being compared (e. g., Moyer & Landauer, 1967; Sekuler & Mierkiewicz, 1977). Comparisons are faster and distance effects smaller among students at selective universities than among community college students, suggesting a link between symbolic numerical magnitude understanding

and mathematics proficiency even among adults (Schneider & Siegler, 2010). Moreover, symbolic magnitude comparison performance correlates positively with arithmetic skill and mathematics achievement test scores for comparisons of whole numbers (Castronovo & Gobel, 2012; De Smedt, Verschaffel, & Ghesquiere, 2009; Holloway & Ansari, 2009; Vanbinst, Ghesquiere, & De Smedt, 2012) and fractions (Hecht & Vagi, 2010; Siegler & Pyke, 2013; Siegler et al., 2011).

Another task that is often used to measure symbolic numerical magnitude understanding is number line estimation. Participants are shown a horizontal line with a number at each end and are asked to estimate other numbers' positions on the line. For example, if the line had 0 and 1000 at the two ends, 500 would go at the midpoint. As children gain experience with increasing ranges of numbers, their number line estimates become more accurate and more closely approximate a linear function (Siegler & Opfer, 2003). As with magnitude comparison, number line estimation accuracy for symbolically expressed whole numbers and fractions is closely related to both arithmetic proficiency and mathematics achievement (Ashcraft & Moore, 2012; Geary, 2011; Siegler & Booth, 2004; Siegler & Pyke, 2013; Siegler et al., 2011).

These two tasks, though superficially different, both tap children's understanding of numerical magnitudes. In both participants are asked to compare magnitudes: comparing the two presented numbers for magnitude comparison or comparing the to-be-estimated number to the two endpoints in number line estimation. Supporting the hypothesis that the two tasks tap the same underlying construct, performance on the two types of tasks is highly correlated among preschoolers (Ramani & Siegler, 2008), elementary school students (Laski & Siegler, 2007), and middle school students (Siegler et al., 2011).

Causal evidence also links understanding of symbolic numerical magnitudes to other mathematics knowledge. Presenting randomly chosen children accurate number line representations of the magnitudes within arithmetic problems improves the children's learning of the answers to the problems (Booth & Siegler, 2008). Similarly, playing a game with randomly chosen children that emphasizes numerical magnitude understanding improves their ability to learn answers to arithmetic problems (Siegler & Ramani, 2009).

### **Understanding of Non-symbolic Numerical Magnitudes**

Humans and many other animals also have a non-symbolic Approximate Number System (ANS) that represents numerical quantity without using symbolic numerals. Discriminability of the number of items in two sets is a function of the ratio between the quantities, as described by Weber's Law. For example, discriminating between 8 and 10 objects and between 16 and 20 objects is equally difficult (e. g., Brannon, Jordan, & Jones, 2010; Libertus & Brannon, 2009). Like the precision of symbolic numerical magnitude representations, non-symbolic numerical magnitude understanding (ANS precision) increases with age. Six-month-olds do not discriminate 12 from 8 sounds, but 9-month-olds do (Lipton & Spelke, 2003); 3-year-olds discriminate dot displays that differ by 3:4 ratios, 6-year-olds 5:6 ratios, and some adults 10:11 ratios (Halberda & Feigenson, 2008; Piazza et al., 2010). Additionally, infants' ability to discriminate fractions matches their whole number discrimination (McCrink & Wynn, 2007).

As with symbolic numerical magnitude representations, individual differences in non-symbolic numerical magnitude understanding have been related to mathematics achievement. Preschoolers' non-symbolic magnitude comparison accuracy has been related to their concurrent mathematics achievement (e. g., Bonny & Lourenco, 2013;

Libertus, Feigenson, & Halberda, 2011; Mussolin, Nys, Leybaert, & Content, 2012) and to their mathematics skills two years later (Mazzocco, Feigenson, & Halberda, 2011b). Preschoolers' non-symbolic numerical magnitude understanding also predicts their growth in mathematics achievement over six months (Libertus, Feigenson, & Halberda, 2013). Moreover, non-symbolic numerical magnitude understanding in 9<sup>th</sup> grade has been related to mathematics achievement test scores during elementary school (Halberda et al., 2008) and has been found to be greater in typically achieving children than in ones with mathematics difficulties (Mazzocco, Feigenson, & Halberda, 2011a; Piazza et al., 2010). In addition, college students' non-symbolic numerical magnitude understanding has been related to their quantitative SAT scores, mental arithmetic ability, and geometry skill (Dewind & Brannon, 2012; Libertus, Odic, & Halberda, 2012; Lourenco, Bonny, Fernandez, & Rao, 2012; Lyons & Beilock, 2011).

On the other hand, many other studies have not found relations between non-symbolic numerical magnitude understanding and mathematics achievement in adults (Castronovo & Gobel, 2012; Inglis, Attridge, Batchelor, & Gilmore, 2011; Price, Palmer, Battista, & Ansari, 2012) and in elementary school children (Holloway & Ansari, 2009; Sasanguie, De Smedt, Defever, & Reynvoet, 2012; Vanbinst et al., 2012). Similarly, several researchers have found that children with and without mathematics difficulties have similar non-symbolic numerical magnitude understanding (Iuculano, Tang, Hall, & Butterworth, 2008; Rousselle & Noel, 2007). The general goal of the present study was to systematically determine how and when understandings of non-symbolic and symbolic numerical magnitudes are related to each other and to mathematics achievement.

## **The Present Study**

In pursuit of this general goal, the current research addressed four specific theoretical questions: 1) How are children's magnitude representations of symbolic fractions and whole numbers related? 2) Are children's understandings of symbolic and non-symbolic numerical magnitude representations related to each other, and are they uniquely related to overall mathematics achievement? 3) What are the direct and indirect causal pathways among non-symbolic numerical magnitude representations, symbolic numerical magnitude representations, and overall mathematics achievement? 4) What are some possible explanations for previous discrepant findings about the relation between non-symbolic numerical magnitude representations and overall mathematics achievement?

With regard to the first question, children's understanding of fractions is typically depicted as quite separate from their whole number knowledge. When the two are discussed together, the purpose is usually to contrast children's early and "natural" learning about whole numbers with their later, painstaking and often unsuccessful acquisition of fraction knowledge (Geary, 2006; Gelman & Williams, 1998; Wynn, 2002). As these theories of numerical development note, the two types of numbers differ in many ways that make fractions more difficult to learn: whole numbers are represented by a single number, whereas fractions require understanding the relation between two numbers; whole numbers have unique successors, whereas fractions do not; multiplication of whole numbers always produces a result at least as great as either multiplicand, whereas multiplication of fractions may not; and so on.

However, as noted in the integrated theory of numerical development (Siegler, Fazio, Bailey, & Zhou, 2013; Siegler et al., 2011), whole numbers and fractions share one essential



property: They both express magnitudes. Neuroscience data are consistent with the view that this property connects whole number and fraction representations. Fraction and whole number magnitudes are processed in similar areas of the brain, and both types of numbers show neuronal distance effects (see Jacob, Vallentin, & Nieder, 2012 for a review). This perspective led to the hypothesis that, as with whole numbers, the precision of fraction magnitude representations should be related to arithmetic proficiency and to mathematics achievement. Consistent with this inference, both 6<sup>th</sup> and 8<sup>th</sup> graders' symbolic fraction magnitude understanding, measured by both magnitude comparison and number line estimation, are highly correlated with their fraction arithmetic accuracy and overall mathematics achievement (Siegler & Pyke, 2013; Siegler et al., 2011). In addition, children's fraction magnitude understanding has been related to their mathematics achievement one year later (Bailey, Hoard, Nugent, & Geary, 2012) and to their concurrent whole number arithmetic abilities (Hecht, 1998; Hecht, Close, & Santisi, 2003). However, we know of no studies that have directly compared concurrent understanding of fraction and whole number magnitudes.

The current study expands upon previous research to directly compare children's understanding of symbolic whole number and fraction magnitudes. If understanding of magnitudes involves the same process regardless of the type of number, then 1) Individual differences in whole number and fraction symbolic magnitude representations should be related, and 2) This relation should be present regardless of whether symbolic magnitude knowledge is assessed through number line estimation or magnitude comparison. Whereas other theories of numerical development are agnostic about the relation between symbolic whole number magnitude understanding and the corresponding understanding of fractions,

the integrated theory of numerical development explicitly predicts strong relations between them (Siegler et al., 2011).

The second question that motivated the present study was how symbolic and non-symbolic numerical magnitude representations are related to each other and to overall mathematics achievement. As discussed above, both types of magnitude knowledge have been shown to correlate with mathematics achievement (e. g., Halberda et al., 2008; Siegler et al., 2011). However, it is unclear whether they are related to each other and whether each of their relations to mathematics achievement are present after controlling for the effect of the other type of magnitude knowledge and non-mathematical cognitive ability. Examining the contribution of each type of magnitude knowledge after statistically controlling for the other also promised to shed light on the more general relation between non-symbolic and symbolic numerical magnitude representations. If statistically controlling the contribution of one type of representation eliminated the contribution of the other, and the opposite was not the case, this would imply that only one type of knowledge was crucial to mathematics achievement. If both types of knowledge remained predictive after controlling for the effect of the other, this would support either a bidirectional relation between them or independent causal linkages between them and mathematics achievement (see Lourenco et al., 2012 for similar logic).

A third question involves causal relations among non-symbolic numerical magnitude understanding, symbolic numerical magnitude understanding, and mathematics achievement. Three simple alternatives are shown in Figure 1. One possibility (panel A) is that children with more precise understanding of non-symbolic numerical magnitudes more easily learn number words and their associated magnitudes, which in turn improves

their overall mathematics achievement. A second possibility (panel B) is that non-symbolic numerical magnitude understanding has both direct and indirect effects on mathematics achievement. A third possibility (panel C) is that non-symbolic and symbolic numerical magnitude understandings may or may not be related, but that they independently affect overall mathematics achievement. Whereas the first two alternatives predict that non-symbolic numerical magnitude understanding bootstraps symbolic understanding, this hypothesis proposes that the two abilities have separate, independent effects on mathematics achievement.

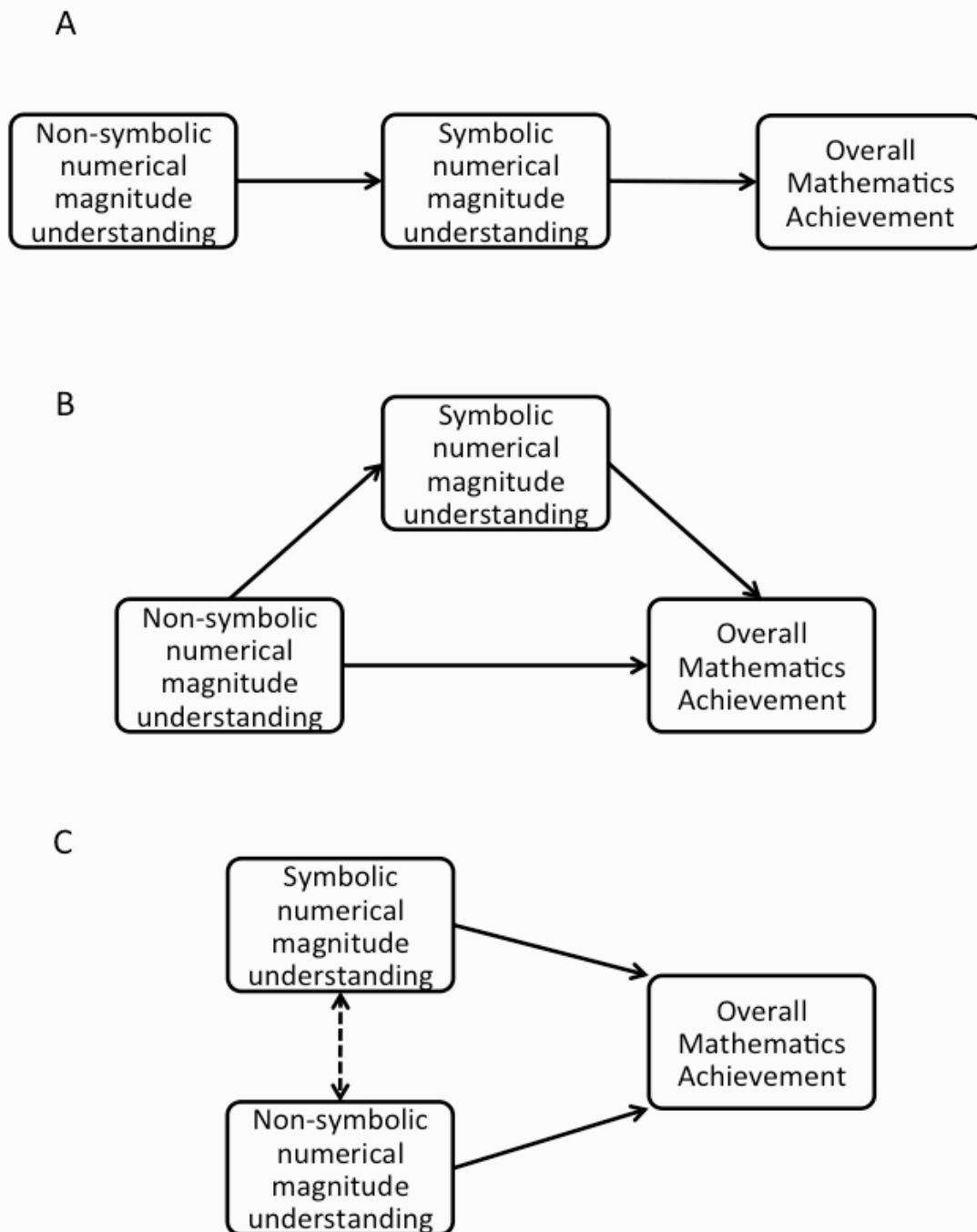


Figure 1. Possible causal links between non-symbolic numerical magnitude understanding, symbolic numerical magnitude understanding and overall mathematics achievement.

It is intuitively plausible that a more precise understanding of non-symbolic numerical magnitudes leads to a better understanding of symbolic numerical magnitudes. Indeed, several researchers have suggested such a developmental path (e.g., Dehaene, 2011; Verguts & Fias, 2004). However, evidence for a relation between symbolic and non-symbolic numerical magnitude knowledge is mixed for both children and adults. Kolkman and colleagues (2013) found a relation between symbolic and non-symbolic number line estimation accuracy in kindergarten, and Gilmore, McCarthy, and Spelke (2010) found that kindergarteners' accuracy at non-symbolic approximate addition was related to their accuracy at symbolic magnitude comparison. However, many studies have found no relation between symbolic and non-symbolic magnitude comparison in kindergarten (Desoete, Ceulemans, De Weerd, & Pieters, 2012; Kolkman, Kroesbergen, & Leseman, 2013; Sasanguie, Defever, Maertens, & Reynvoet, 2013) and elementary school (Holloway & Ansari, 2009; Iuculano et al., 2008; Landerl & Kolle, 2009; Mussolin et al., 2012; Vanbinst et al., 2012). Studies with adult participants include one that documented a relation between non-symbolic and symbolic magnitude comparison with the numbers 1-9 (Gilmore, Attridge, & Inglis, 2011) and another that found no relation in the same numerical range (Maloney, Risko, Preston, Ansari, & Fugelsang, 2010).

Our strategy for clarifying the relation between non-symbolic and symbolic numerical magnitude understanding was to broaden the range of tasks and types of numbers examined. Earlier studies have relied on magnitude comparison tasks with whole numbers. In contrast, we used both number line estimation and magnitude comparison tasks with both whole numbers and fractions to examine relations between non-symbolic and symbolic numerical magnitude representations. This broader base of tasks yields

additional data points indicating when relations between non-symbolic and symbolic numerical magnitudes are and are not present. We also conducted mediation analyses to examine whether symbolic numerical magnitude understanding partially or fully mediated the relation between non-symbolic numerical magnitude understanding and mathematics achievement.

To address these questions, we assessed fifth graders' symbolic and non-symbolic numerical magnitude representations of fractions and whole numbers on magnitude comparison and number line tasks and obtained their mathematics and reading achievement scores. The reading scores were used to rule out the possibility that relations between different types of numerical magnitude understanding and mathematics achievement were due to a common influence of non-mathematical cognition. Fifth-graders were studied because they have some knowledge of symbolic fraction magnitudes, but their understanding of whole number magnitudes is below adult levels (see Siegler & Opfer, 2003). Observing performance on these tasks allowed a systematic examination of relations between symbolic and non-symbolic numerical magnitude understanding, whole number and fraction understanding, and each of these and mathematics achievement, as well as evaluation of three models of the relation.

In order to more systematically examine the currently mixed evidence on the relation between non-symbolic numerical magnitude understanding and mathematics achievement, we also performed a meta-analysis on the previously published research. The meta-analysis provided an estimate of the true size of the relation and examined how participant age and the measure of non-symbolic numerical magnitude understanding are related to the correlation between non-symbolic numerical magnitude understanding and overall

mathematics achievement.

## **Method**

### **Participants**

The participants were 53 fifth graders at four small charter schools in or near Pittsburgh, PA ( $M_{age} = 10.72$ ,  $SD = 0.36$ ; 59% female; 50% White, 44% Black, 6% Biracial). Of the children attending these schools, 74% are eligible for free or reduced-price lunches; Pennsylvania's average is 39%. Despite the low-income population, the schools score at or above the state average on standardized achievement tests.

### **Tasks**

Children completed the eight tasks shown in Figure 2; they varied in the type of task (magnitude comparison or number line estimation), type of number (whole numbers or fractions), and form of number (symbolic or non-symbolic). Numbers of items on the tasks varied considerably based on the number of trials required for a stable estimate of performance. Time/task was roughly equivalent; each took less than five minutes. The full instructions for each task can be found in the online supplement.

MAGNITUDE UNDERSTANDING

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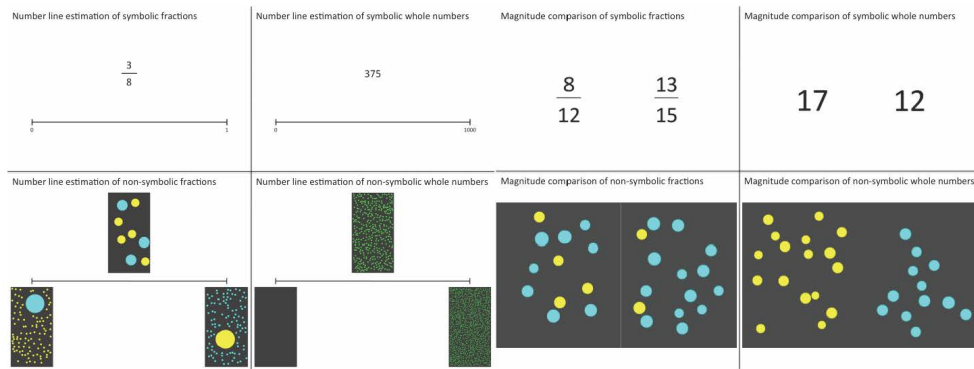


Figure 2. Sample stimuli from each of the eight tasks. For the number line estimation tasks on the left side, the child is asked to place a fraction or whole number onto the number line. On the non-symbolic versions, three-eighths is represented by three blue dots and five yellow dots (three-eighths of the dots are blue) and 375 is represented by 375 dots. For the magnitude comparison tasks on the right side, the child is asked which whole number or fraction is larger. On the non-symbolic versions, the whole numbers are represented by dots and the fractions are represented in the same way as for the number line estimation (i.e., eight-twelfths is 8 blue dots and 4 yellow dots).



**Number line estimation of symbolic whole numbers.** Twenty-six number lines were presented sequentially on a computer screen, with “0” at the left end and “1000” at the right. Above each line was the to-be-estimated number. Children made their estimates by clicking the mouse when the cursor was at the desired location. All numbers remained on the screen until the child responded. As in previous studies, there were four numbers from each of the first three tenths of the number line and two numbers from each remaining tenth: 53, 67, 83, 100, 125, 143, 167, 200, 222, 250, 286, 300, 333, 375, 417, 444, 556, 600, 625, 667, 750, 800, 833, 875, 909, and 929. Previous findings suggested that the 0 – 1000 numerical range would produce large individual differences in fifth-graders’ performance (Booth & Siegler, 2006; Siegler & Opfer, 2003).

**Number line estimation of symbolic fractions.** The fraction estimation task was almost identical to the whole number task. Children again saw 26 number lines but with the left endpoint labeled “0” and the right endpoint labeled “1”. The numbers estimated on the two tasks were at the same positions on the number line (e.g., 250 and  $\frac{1}{4}$ ;  $\frac{7}{8}$  and 875). The fractions were  $\frac{1}{19}$ ,  $\frac{1}{15}$ ,  $\frac{1}{12}$ ,  $\frac{1}{10}$ ,  $\frac{1}{8}$ ,  $\frac{1}{7}$ ,  $\frac{1}{6}$ ,  $\frac{1}{5}$ ,  $\frac{2}{9}$ ,  $\frac{1}{4}$ ,  $\frac{2}{7}$ ,  $\frac{3}{10}$ ,  $\frac{1}{3}$ ,  $\frac{3}{8}$ ,  $\frac{5}{12}$ ,  $\frac{4}{9}$ ,  $\frac{5}{9}$ ,  $\frac{3}{5}$ ,  $\frac{5}{8}$ ,  $\frac{2}{3}$ ,  $\frac{3}{4}$ ,  $\frac{4}{5}$ ,  $\frac{5}{6}$ ,  $\frac{7}{8}$ ,  $\frac{10}{11}$ , and  $\frac{13}{14}$ . Fractions can obviously be larger than one, but the 0 – 1 range allowed us to easily match positions with the whole number task and is a numerical range emphasized in most fifth-grade curricula.

**Number line estimation of non-symbolic whole numbers.** This task paralleled the number line task with symbolic whole numbers. Children were presented 26 number lines; the left and right endpoints were labeled with pictures of 0 and 1000 dots (Figure 2). The to-be-estimated number was shown as a collection of dots. To avoid activating the symbolic number system, the endpoints were labeled as “no dots” and “a lot of dots.” To

prevent counting, the to-be-estimated dots disappeared from view after 5 s. Total area covered by the dots was fixed, so that as the number of dots increased, dot size decreased. This control prevented children from responding based on area covered rather than number. Children were told to focus on the number of dots and to ignore their size. Number of dots on each trial matched the trials with symbolic whole numbers.

**Number line estimation of non-symbolic fractions.** On this task, the number line had 1 blue dot and 99 yellow dots ( $1/100$  blue dots) on the left and 99 blue dots and one yellow dot ( $99/100$  blue dots) on the right (Figure 2). Children were asked to estimate where mixed sets of blue and yellow dots belonged on the line. The endpoints were verbally labeled as “lots of yellow dots” and “lots of blue dots.” The 26 fractions matched those from number line estimation of symbolic fractions. For example, for the symbolic fraction  $5/9$ , the corresponding non-symbolic problem involved 5 blue and 4 yellow dots. On each trial, the area covered by the blue and yellow dots was equal (for the to-be-estimated fraction and each of the endpoints). This again ensured that children could not accurately respond based on area. The to-be-estimated dots disappeared from the screen after 2 s; the time was shorter than for whole numbers to prevent counting of the smaller number of dots on fraction trials.

**Magnitude comparison of symbolic whole numbers.** Children were asked which of two Arabic numerals was larger. The numbers ranged from 5 to 21 (these values matched the default values of our non-symbolic magnitude comparison task, downloaded from panamath.org). The numbers remained on the screen until the child responded. Among the 40 trials, 10 came from each of 4 ratio bins: 1.15-1.28, 1.28-1.43, 1.48-1.65, and 2.46-2.71. The ratio for each trial was determined by dividing the larger number by the

smaller number. For example, a trial with 18 blue and 15 yellow dots would have a ratio of 1.2:1, whereas a trial with 18 and 7 would have a ratio of 2.57:1 (ratios hereafter abbreviated as 1.2, 2.57, etc.). Children were asked to respond as quickly and accurately as possible.

**Magnitude comparison of symbolic fractions.** Children were asked which of two symbolically expressed fractions was larger. The denominators ranged from 5 to 21, to match the numbers used on the other comparison tasks. There were 40 trials, 10 from each of the ratio bins. To ensure that participants could not be consistently correct by attending to numerators or denominators alone, each bin included five types of problems in which, relative to the smaller fraction, the larger fraction had either 1) a larger numerator and an equal denominator; 2) an equal numerator and a smaller denominator; 3) a larger numerator and a larger denominator; 4) a larger numerator and a smaller denominator; or 5) a smaller numerator and a smaller denominator.

**Magnitude comparison of non-symbolic whole numbers.** The non-symbolic magnitude comparison task was downloaded from [panamath.org](http://panamath.org), designed by Justin Halberda; it closely resembled the task used by Halberda, Mazocco, and Feigenson (2008). On each problem, children saw blue dots on the left side of the screen, yellow dots on the right, and judged which side had more dots. There were 186 trials: 2 practice trials and 46 trials in each of the four ratio bins. After the dots were on the screen for 1382 ms, the display was replaced by a backward mask of yellow and blue dots for 500 ms. Number of dots on each side of the screen varied from 5 to 21. On half of the trials, individual dot size was equated on the two sides; on the other half, the overall area covered by the dots on the two sides was equal. These controls insured that the children could not consistently

respond correctly on the basis of the size of the dots or the total area covered. As on all magnitude comparison tasks, children were asked to respond as quickly and accurately as possible.

**Magnitude comparison of non-symbolic fractions.** Children saw blue and yellow dots on both sides of the screen. Participants were told that they should pretend that the dots were candies, that the blue candies taste best, and that they should pick the side of the screen that would give them the best chance of getting a blue candy. They were also told that the size of the candies did not matter. After 2 s, a mask of small yellow and blue dots covered the screen (the mask appeared later than in the whole number task since the task was more demanding). The children could respond either before or after the mask appeared.

There were 182 trials: two practice trials and 45 trials in each of the usual four ratio bins. In each bin, each of the five types of problems described for the symbolic fraction comparison task was presented nine times. Total number of dots on a side equaled the denominator, and the number of blue dots equaled the numerator. For example,  $5/8$  was represented by five blue and three yellow dots. The relevant ratio was that between the two fractions; if the fractions on the two sides were  $6/12$  and  $2/8$ , the ratio was 2.00. The total number of dots on each side ranged from 5 to 21, to match the number on the whole number tasks. On half of the trials, size of each dot was equivalent on the two sides; on the other half, the total area on each side covered by dots of each color was equivalent.

**Pennsylvania system of school assessment (PSSA) mathematics and reading tests.** Fifth grade PSSA scores were obtained from the schools as standardized measures of mathematics and reading achievement. The mathematics test included three open-ended

items and 60 four-alternative multiple-choice questions. The test covers numbers and operations (41-45% of questions), measurement (12-15%), geometry (12-15%), algebraic concepts (13-17%) and data analysis and probability (12-15%). Magnitude knowledge received little attention; of 65 multiple-choice items released from the fifth grade PSSA Mathematics Test, only three addressed numerical magnitudes.

The reading test included 40 multiple-choice questions and four open-ended questions. Most of the test deals with comprehension and reading skills (60-80%), the rest with interpretation and analysis of fiction and nonfiction text (20-40%).

### **Procedure**

Children performed the experimental tasks in a quiet space in their school. The eight tasks were presented over two sessions separated by an average of two days; four tasks were presented per day. Tasks and items within tasks were presented in a separate random order for each child. Children were tested individually.

### **Results**

We first present descriptive statistics for each task, examine relations among performance on the experimental tasks, and analyze relations between the experimental tasks and mathematics achievement test scores. Next, we examine the fit of the three models illustrated in Figure 1 to the data and present a meta-analysis aimed at clarifying when non-symbolic numerical magnitude understanding is related to math achievement.

Trials with RTs greater than or equal to 2.5 standard deviations from the child's mean RT (2% - 3% of trials) were removed as outliers. Children who scored more than 4 SDs from the mean were excluded from analyses of that task. This led to one child being removed from each of the following tasks: magnitude comparison of symbolic whole

numbers, magnitude comparison of non-symbolic whole numbers, and magnitude comparison of non-symbolic fractions (three separate children). All eight experimental tasks had high reliabilities ( $\alpha = .82 - .93$ ), including the three novel tasks: magnitude comparison of non-symbolic fractions (.90), number line estimation of non-symbolic whole numbers (.92) and number line estimation of non-symbolic fractions (.82). Unless otherwise noted, all results were significant at the .05 alpha level.

### **Symbolic and Non-symbolic Number Line Estimation**

Accuracy on all number line estimation tasks was measured by percent absolute error (PAE):  $(|\text{actual location of number} - \text{child's estimate}|) / \text{numerical range}$ . For example, if a child was asked to place 200 on the 0-1000 number line and placed it at 300, his PAE would be 10%  $((|200 - 300|) / 1000)$ . Thus, smaller PAEs indicate more accurate estimates.

On the symbolic number line tasks, estimation with whole numbers was more accurate than estimation with fractions: PAEs = 12% vs. 20%,  $t(52) = 5.42$ ,  $SEM = 1.54$ ,  $d = 0.86$ ; more linear,  $R^2_{lin} = .78$  vs.  $.47$ ,  $t(52) = 6.18$ ,  $SEM = 0.05$ ,  $d = 1.00$ ; and with the slope of the best fitting linear function closer to 1.00,  $.73$  vs.  $.52$ ,  $t(52) = 3.44$ ,  $SEM = 0.06$ ,  $d = 0.56$ .

On the corresponding non-symbolic number line tasks, the best fitting linear function fit estimates for whole numbers better than fractions,  $R^2_{lin} = .69$  vs.  $.48$ ,  $t(52) = 5.13$ ,  $SEM = 0.04$ ,  $d = 0.96$ . Unlike with symbolic numbers, however, estimates with non-symbolic whole numbers and fractions were equally accurate, PAEs = 22% and 19%,  $t(52) = 1.35$ , *ns.*, and had similar slopes,  $.69$  and  $.63$ ,  $t(52) = 1.03$ , *ns.*

### **Symbolic and Non-symbolic Magnitude Comparison**

Magnitude comparisons of symbolic and non-symbolic numbers were more accurate and faster with whole numbers than fractions (Table 1). On both, accuracy increased and

reaction time decreased with the larger ratios.

For the non-symbolic magnitude comparison tasks, we used Halberda and Feigenson's (2008) psychophysical model to estimate each child's Weber fraction. The Weber fraction represents the noise in the non-symbolic numerical magnitude representation; thus, lower  $w$  values indicate more precise representations. The average Weber fraction was much lower for whole numbers than fractions,  $w = .21$  versus  $.63$ ,  $t(48) = 9.62$ ,  $SEM = 0.04$ ,  $d = 1.76$ . Three participants were eliminated from analyses of the fraction task because the models of their judgments could not converge on a value of  $w$ .

We did not compute Weber fractions for symbolic magnitude comparisons. The reason for whole numbers was ceiling effects. The reason for fractions was that the model could not converge on a value for  $w$  for 45% of children. These children's accuracy did not increase with larger ratios, which resulted in the model not fitting their performance.

Table 1. Descriptive Statistics for the Magnitude Comparison Tasks

Task	Accuracy (% Correct)					Reaction Time (ms)				
	Overall	Small	Medium	Large	Largest	Overall	Small	Medium	Large	Largest
Symbolic Whole Number	96	92	94	96	100	899	975	940	890	793
Symbolic Fraction	66	64	64	66	67	2302	2438	2369	2351	2261
Non-symbolic Whole Number	87	74	86	91	98	961	1062	986	941	842
Non-symbolic Fraction	70	65	68	71	76	1527	1542	1548	1533	1496

Note. Small, Medium, Large and Largest refer to the four ratio bins 1.15-1.28, 1.28-1.43, 1.48-1.65, and 2.46-2.71.



**Consistency of Individual Differences across Experimental Tasks**

For the number line tasks, analyses were based on accuracy (PAE). For magnitude comparison, the measure varied with the task. On magnitude comparison of symbolic whole numbers, accuracy was at ceiling, so we only analyzed reaction time. On magnitude comparison of symbolic fractions, variability of strategy use (reliance on numerator, denominator, and/or fraction magnitudes) led to reaction times being uninformative, so we only analyzed percent correct.

On non-symbolic magnitude comparisons,  $w$  and mean RT were standardized ( $M = 0$ ,  $SD = 1$ ) and summed to create a measure of non-symbolic numerical magnitude understanding. Past research has varied in the measures used to examine non-symbolic magnitude comparison performance, with researchers using the slope of the linear function relating reaction time to ratio (e.g., Holloway & Ansari, 2009; Price et al., 2012), percent correct (e.g., Lourenco et al., 2012; Mazocco et al., 2011b),  $w$  (e.g., Halberda et al., 2008; Inglis et al., 2011) or both  $w$  and mean RT (e.g., Halberda, Ly, Wilmer, Naiman, & Germine, 2012; Libertus et al., 2011). We used  $w$  and RT together, because doing so yielded stronger relations to other tasks and helped to control for speed/accuracy tradeoffs. The correlations with  $w$  and RT as separate variables and with RT slope and percent correct can be found in Table S1 of the online supplement. Due to skewed distributions, RT, PAE, and  $w$  measures were log transformed prior to analysis.

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Table 2. Correlations Among Experimental Tasks and Achievement Scores.

	PSSA Reading	Symbolic Whole Number Comparison RT	Symbolic Fraction Comparison % Correct	Symbolic Whole Number NL Estimation PAE	Symbolic Fraction NL Estimation PAE	Non-symbolic Whole Number Comparison Precision ( $w + RT$ )	Non-symbolic Fraction Comparison Precision ( $w + RT$ )	Non-symbolic Whole Number NL Estimation PAE	Non-symbolic Fraction NL Estimation PAE
PSSA Mathematics	.58**	-.58**	.69**	-.56**	-.68**	-.60*	-.25	-.32*	-.46**
PSSA Reading		-.19	.43**	-.17	-.43**	-.50**	-.20	-.14	-.35*
Symbolic Whole Number Comparison RT			-.33*	.50**	.43**	.54**	-.14	.27	.20
Symbolic Fraction Comparison % Correct				-.53**	-.71**	-.39**	-.11	-.25	-.21
Symbolic Whole Number NL Estimation PAE					.46**	.42**	.21	.24	-.06
Symbolic Fraction NL Estimation PAE						.37**	.03	.29*	.28*
Non-symbolic Whole Number Comparison Precision ( $w + RT$ )							.61**	.26	.32*
Non-symbolic Fraction Comparison Precision ( $w + RT$ )								.17	.14
Non-symbolic Whole Number NL Estimation PAE									-.03

Note.  $N = 53$  for all correlations except for those with achievement scores ( $N = 43$ ), symbolic whole number comparison ( $N = 52$ ), non-symbolic whole number comparison ( $N = 52$ ) and non-symbolic fractions comparison ( $N = 49$ ). The lower  $N$  for achievement scores reflected a number of parents not giving permission for their children's test scores to be released.

NL = number line. PAE = percent absolute error. PSSA = Pennsylvania System of School Assessment.  $w$  = Weber fraction.

\*  $p < .05$ . \*\*  $p < .01$ .

As shown in Table 2, all four symbolic tasks were significantly correlated, in most cases quite strongly (Rows and Columns 3-5). The correlations among the non-symbolic tasks and between the symbolic and non-symbolic tasks were generally weaker and non-significant.

We conducted a factor analysis to further explore these relations. Based on Horn's Parallel Analysis (Horn, 1965), we retained two factors. Items with loadings greater than 0.40 were standardized ( $M = 0, SD = 1$ ) and summed to create factor scores. The first factor, symbolic knowledge, consisted of three of the four symbolic tasks: magnitude comparison of fractions, number line estimation of whole numbers, and number line estimation of fractions (loadings =  $-.72, .54,$  and  $.96,$  respectively). The second factor, non-symbolic knowledge, consisted of magnitude comparison of non-symbolic whole numbers and fractions (loadings =  $.95$  and  $.59,$  respectively). Magnitude comparison of symbolic whole numbers loaded similarly on symbolic knowledge ( $.37$ ) and non-symbolic knowledge ( $.44$ ) and was excluded from subsequent analyses that used the results of the factor analysis. The non-symbolic number line estimation tasks did not load on either factor. Details about the factor analysis are in the online supplement.

### **Relations of Experimental Tasks to Mathematics Achievement**

As shown in the first row of Table 2, children's performance on all four symbolic tasks was strongly related to their mathematics achievement,  $r_s > .55$ . Magnitude comparison of non-symbolic whole numbers was also strongly related to mathematics achievement,  $r(40) = -.60$ . The two non-symbolic number line tasks were more weakly related to mathematics achievement,  $r(41) = -.32$  and  $r(41) = -.46$ .

Finally, magnitude comparison of non-symbolic fractions was not related to mathematics achievement. However, this was likely due to the elimination of three lower achieving participants for whom the model was unable to converge on a value of  $w$ . Accuracy on the non-symbolic fraction number line task, a measure that could be computed for all participants, was related to mathematics achievement,  $r(40) = .35$  (Table S1). Overall, performance on all eight tasks was related to mathematics achievement, although the relations were stronger for the symbolic tasks. Scatterplots of the relation between each of the tasks and overall mathematics achievement are shown in Figure S1 online.

### **Testing the Three Models**

To determine which of the three models pictured in Figure 1 best fit the data, we examined whether symbolic numerical magnitude knowledge mediated the relation between non-symbolic numerical magnitude understanding and mathematics achievement. In Panel A, symbolic numerical magnitude understanding fully mediates the relation between non-symbolic numerical magnitude understanding and mathematics achievement. Panel B depicts partial mediation. Panel C depicts no mediation, although both abilities should be independently related to mathematics achievement.

To evaluate the models, we used the symbolic and non-symbolic knowledge factors identified in the above factor analysis. Both they and PSSA mathematics scores were standardized prior to running the bootstrap mediation analysis (Preacher & Hayes, 2008).

As shown in Figure 3, mathematics achievement was strongly related to knowledge of symbolic numerical magnitudes and somewhat related to knowledge of non-symbolic magnitudes, both before and after controlling for symbolic magnitude knowledge. Controlling for symbolic knowledge reduced the direct effect of non-symbolic knowledge on mathematics achievement, but not significantly ( $\beta = .17, p=.07$ ). Importantly, the relation between non-symbolic and symbolic knowledge was not significant ( $\beta = .25, p=.10$ ), suggesting that the different types of numerical magnitude knowledge independently influence math achievement.

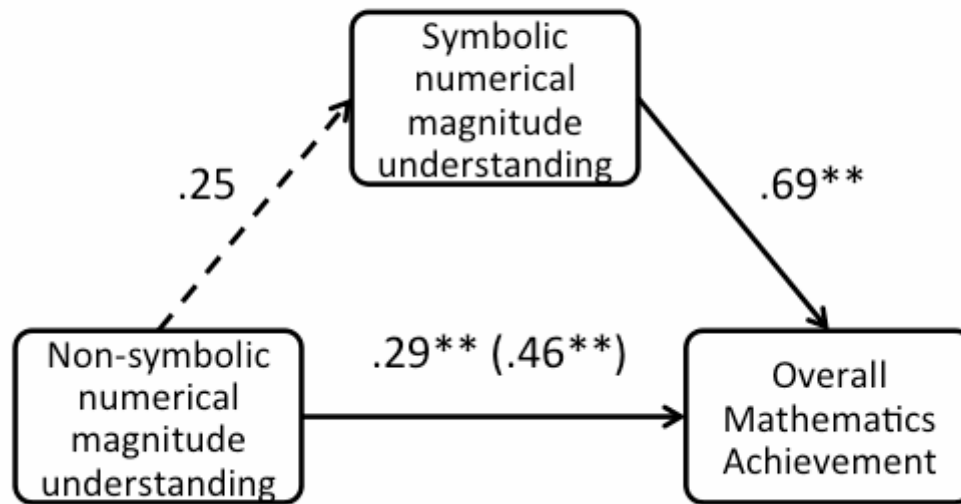


Figure 3. Mediation analysis examining whether symbolic numerical magnitude understanding mediates the relation between non-symbolic numerical magnitude understanding and overall mathematics achievement.

If symbolic and non-symbolic numerical magnitude understandings have

independent effects on mathematics achievement, then each should uniquely predict variance in mathematics achievement after controlling for the other type of knowledge and non-mathematical cognitive ability. To examine this hypothesis, we conducted two hierarchical regressions. In both models, we first entered reading achievement test scores as a measure of non-mathematical cognition, followed by the symbolic and non-symbolic factors identified in the factor analysis. In Model 1, after entering reading achievement, we entered the measure of non-symbolic numerical magnitude knowledge, followed by the measure of symbolic numerical magnitude knowledge. Model 2 included the same variables, but with symbolic magnitude knowledge entered before non-symbolic magnitude knowledge.

Reading achievement accounted for 33% of the variance in mathematics achievement,  $F(1, 36) = 17.56$ . In Model 1, non-symbolic knowledge added 7% to the explained variance,  $F(1, 35) = 4.32$ , and symbolic knowledge explained an additional 27%,  $F(1, 34) = 28.59$ . Ordered in this way, both non-symbolic and symbolic numerical magnitude representations were predictive of mathematics achievement, although symbolic knowledge explained almost four times as much variance.

In Model 2, symbolic knowledge added 31% to the 33% of variance in mathematics achievement explained by reading achievement,  $F(1, 35) = 29.44$ . Adding the measure of non-symbolic knowledge increased the explained variance by 4%,  $F(1, 34) = 4.20$ . Again, both measures explained unique variance in mathematics achievement, but the contribution of symbolic knowledge was much larger. Our results best support the model in Panel C, with non-symbolic and

symbolic numerical magnitude knowledge having independent effects on mathematics achievement.

### **Meta-analysis of the Relation between Non-symbolic Numerical Magnitude Knowledge and Overall Mathematics Achievement**

Although non-symbolic numerical magnitude understanding and mathematics achievement were related in the present data set, a number of previous studies have failed to find any relation. Several reasons for the inconsistent relations seemed plausible.

1) *Variation in sample sizes.* In most studies, the relation has been positive but weak. For example, a study of more than 10,000 participants found a small but reliable correlation ( $r = .21$ ) between non-symbolic numerical magnitude understanding and mathematics achievement (Halberda et al., 2012). A correlation of this magnitude would not have been significant with many smaller sample sizes.

2) *Achievement level of sample.* The relation has tended to be stronger for children with poor mathematics achievement (Bonny & Lourenco, 2013; Mazzocco et al., 2011a), and the percentage of such children in the sample probably varies across studies.

3) *Ages of participants.* Participant ages have varied from 3-years to adulthood. Some studies that have included participants of multiple ages have found relations at some ages but not others (Inglis et al., 2011; Mussolin et al., 2012).

4) *Measures.* As noted by Price and colleagues (2012), studies have varied in whether they measure non-symbolic numerical magnitude understanding in terms of percent correct,  $w$ , RT slope, or a combination of  $w$  and mean RT. The particular measure can influence the strength of the relation found. In the present study, using

a combination of  $w$  and RT to measure non-symbolic magnitude knowledge yielded a considerably stronger relation than using RT slope ( $r = .60$  versus  $.27$ ).

5) *Content of mathematics achievement test.* Tests that include advanced arithmetic and geometry (such as the PSSA used in the present study) have sometimes yielded stronger relations (Lourenco et al., 2012).

To systematically examine the size of the relation between non-symbolic numerical magnitude understanding and overall mathematics achievement, we conducted a meta-analysis of all published research as of February 28, 2013. We focused on how participant age and the measure of non-symbolic numerical magnitude understanding influenced the size of the correlation. The other three variables listed above might also matter, but we did not include them in the meta-analysis because the effects of sample size on significance level is well known, because the number of low achieving participants in each sample requires data about individual participants that frequently was not available, and because whether material on achievement tests taps advanced concepts is an inherently subjective judgment. To avoid ambiguities in participant age, we included only studies that examined non-symbolic numerical magnitude understanding and mathematics achievement at the same time point.

We found 19 articles that reported the bivariate correlation between performance on a non-symbolic numerical magnitude comparison task and mathematics achievement, with both constructs measured at the same age. Due to some articles containing multiple age groups or multiple experiments, the meta-analysis included 34 samples of participants (see Table S2 in the online supplement).



Many studies reported multiple measures of non-symbolic numerical magnitude understanding. For the overall meta-analysis, we included only the measure that produced the highest correlation with mathematics achievement. If a study included more than one measure of mathematics achievement, we again included only the measure that led to the largest correlation.

Using a random-effects model, the overall weighted average correlation was .22 (95% CI = .20-.25). Given that we examined only published studies, which often have larger effect sizes, the true correlation might be lower. However, Halberda et al.'s (2012) study of 10,000 participants yielded an almost identical correlation,  $r = .21$ , and it would require 16 additional samples that found no correlation between non-symbolic numerical magnitude understanding and overall mathematics achievement to reduce the estimated correlation to .15, 41 to reduce it to .10, and 116 to reduce it to .05 (Orwin, 1983). Thus, it is safe to assume that there is a small but true relation between non-symbolic numerical magnitude understanding and overall mathematics achievement.

We next conducted separate random-effects meta-analyses for each of the dependent measures; studies that reported multiple measures were included in multiple analyses. The three measures of non-symbolic numerical magnitude understanding that were based on accuracy all yielded relations: percent correct  $r = .29$  (95% CI = .18-.40,  $N = 11$ ),  $w r = .19$  (95% CI = .17-.21,  $N = 26$ ), and  $w$  & mean RT  $r = .21$  (95% CI = .19-.23,  $N = 10$ ). In contrast, when performance was measured using reaction time alone, the relations were very weak: mean RT  $r = .09$  (95% CI .06-.12,  $N = 14$ ), RT slope  $r = .03$  (95% CI = -.10-.16,  $N = 5$ ).

Because a test of homogeneity indicated that the correlations in the overall model were heterogeneous,  $Q(33) = 58$ , we examined whether participants' age could explain variability among effect sizes. We separated the samples into three groups: younger children (mean age below 6 years), older children (6- to 18-years) and adults (over 18 years). The logic for this division was that children under age 6 have usually had relatively little formal instruction on symbolically expressed numbers, those between 6 and 18 have had much more experience with symbolically expressed numbers, and those over 18 are primarily adults from the more select population receiving a college education.

Using a random-effects model, the weighted effect size for younger children was .40. This was significantly larger than the correlations for older children ( $r = .17$ ) and adults ( $r = .21$ ), which did not differ (Table 3 and Figure 4). The relation between non-symbolic numerical magnitude understanding and mathematics achievement thus appears to be stronger before children begin formal instruction in mathematics at around age six. Additional analyses are included in the online supplement.

Table 3. Effect of Participant Age on the Correlation between Non-symbolic Numerical Magnitude Understanding and Mathematics Achievement

Participant Age	Between-Group Effect ( $Q_b$ )	N	Weighted $r$	95% CI	Homogeneity within each group ( $Q_w$ )
	28.29**				
< 6		9	.40	[.33, .47]	10.01
6-18		7	.17	[.12, .21]	6.24
18+		18	.21	[.19, .23]	13.27

•  $p < .05$ . \*\*  $p < .01$ .

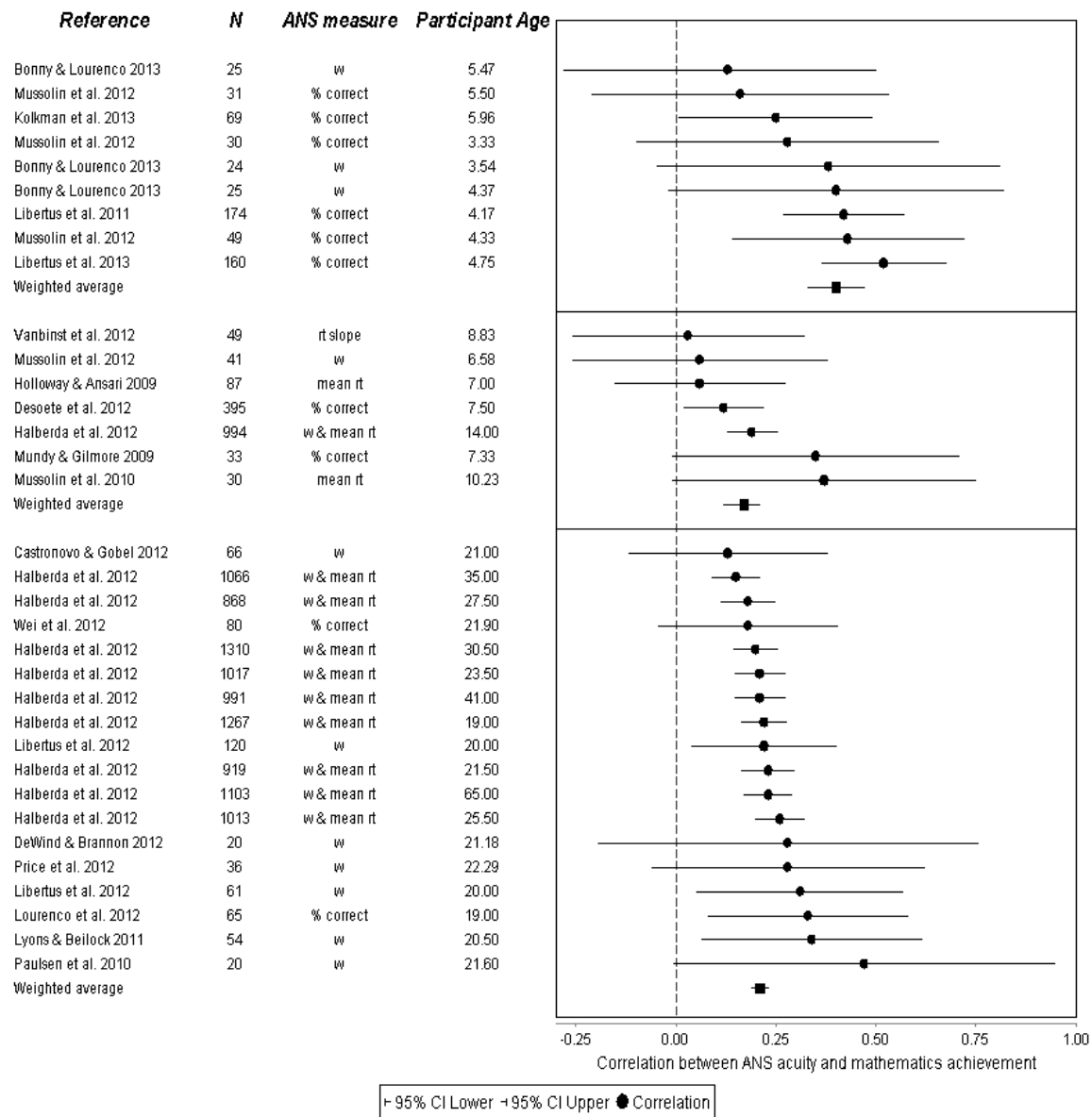


Figure 4. Forest plot of the correlations for each sample and the weighted average of each age group. The top panel includes samples with an average age less than 6 years, the middle panel shows samples with an average age between 6 and 18 years, the bottom panel shows samples whose participants are over 18 years.

## **Discussion**

The present findings help answer the four main questions that motivated the study. In this concluding section, we discuss the implications of the findings for these main questions and discuss the contribution of the novel tasks in the study.

### **Relations between Symbolic Fraction and Whole Number Representations**

Although numerous theories of numerical development emphasize the interfering effects of whole number knowledge on learning fractions, a basic prediction of the integrated theory of numerical development (Siegler et al., 2011) is that magnitude knowledge of whole numbers and fractions should be closely related. The present findings supported this prediction. Children who were more accurate at placing symbolic whole numbers on number lines and faster at comparing symbolic whole number magnitudes were also superior at the corresponding tasks with fractions. Thus, despite the many differences between symbolically expressed fractions and whole numbers, the common element that they both represent magnitudes seems to connect them psychologically as well as mathematically.

### **Relations of Magnitude Understanding to Mathematics Achievement**

The present findings indicated that knowledge of magnitudes of symbolic whole numbers and fractions is quite strongly related to mathematics achievement. Each of the four symbolic tasks accounted for 30%-45% of variance in mathematics achievement test scores. These results match those of previous studies showing large relations between symbolic numerical magnitude understanding and mathematics achievement (Ashcraft & Moore, 2012; Siegler & Booth, 2004; Siegler & Pyke, 2013; Siegler et al., 2011). Symbolic numerical magnitude representations

also explained variance in mathematics achievement test scores beyond that explained by non-mathematical cognitive proficiency and non-symbolic numerical magnitude representations. The consistency of the relation between symbolic numerical magnitude representations and math achievement test scores across different tasks and different types of numbers, even after these controls, is striking.

Non-symbolic magnitude understanding was also related to mathematics achievement, but the relations were much weaker. For example, whereas symbolic magnitude knowledge accounted for 27% of the variability in math achievement after controlling for reading scores and non-symbolic knowledge, non-symbolic knowledge accounted for only 4% of the variance after controlling for reading scores and symbolic knowledge. Our results suggest that efforts to train non-symbolic numerical magnitude representations in order to improve mathematics achievement may be misguided. Interventions that target understanding of symbolic numerical magnitudes are likely to be much more effective.

### **Direct and Indirect Relations between Numerical Magnitude Understanding and Overall Mathematics Achievement**

The present findings also help to clarify how non-symbolic and symbolic numerical magnitude knowledge affect mathematics achievement. Figure 1 depicts three ways that non-symbolic numerical magnitude understanding could affect mathematics achievement. In the first model, symbolic numerical magnitude understanding fully mediates the relation. We found no evidence to support this account. Non-symbolic numerical magnitude understanding explained significant variance in mathematics achievement, even after controlling for symbolic numerical magnitude understanding and non-mathematical cognition.

The second model suggests that non-symbolic numerical magnitude knowledge has both direct and indirect effects on mathematics achievement. We again found little support for this hypothesis. The path in the mediation analysis between non-symbolic and symbolic numerical magnitude knowledge was not significant. Moreover, controlling for symbolic numerical magnitude knowledge did not significantly reduce the direct path between non-symbolic numerical magnitude knowledge and mathematics achievement.

The results were most consistent with the third model: symbolic and non-symbolic numerical magnitude knowledge have independent effects on mathematics achievement. While both types of knowledge were related to mathematics achievement, the symbolic and non-symbolic tasks loaded on different factors and there was no correlation between symbolic and non-symbolic magnitude knowledge. These results fit with other recent findings suggesting that symbolic and non-symbolic numerical magnitude understanding may be unrelated in both adults and children (e.g., Lyons, Ansari, & Beilock, 2012; Sasanguie et al., 2013) and contrast with theories that predict strong relations between symbolic and non-symbolic numerical magnitude understanding (e.g., Dehaene, 2011; Verguts & Fias, 2004). It remains an open question how non-symbolic numerical magnitude understanding affects mathematics achievement if it is not by improving understanding of symbolic numerical magnitudes.

There might also be bidirectional relations between symbolic and non-symbolic magnitude understanding and mathematics achievement. Non-symbolic numerical magnitude understanding increases throughout childhood and into

adulthood, which raises the possibility that it could be affected by school instruction (Halberda & Feigenson, 2008; Halberda et al., 2012). Consistent with this idea, educated European adults have been found to have more precise non-symbolic numerical magnitude representations than Amazonian indigene adults without formal education (Pica, Lemer, Izard, & Dehaene, 2004).

We were unable to examine bidirectional relations with our current cross-sectional data; longitudinal data would allow tests of whether such bidirectional relations are present. We also cannot rule out the possibility that symbolic and non-symbolic numerical magnitude understandings are more closely linked earlier in development. Non-symbolic magnitude understanding may bootstrap initial learning of symbolic magnitudes, but with additional instruction on symbolic numerals and magnitudes, the two constructs might become independent by fifth-grade. However, some researchers have found no relation between non-symbolic and symbolic numerical magnitude understanding even with kindergarteners (Desoete et al., 2012; Sasanguie et al., 2013).

### **Differences in the Size of the Correlation between Non-symbolic Numerical Magnitude Understanding and Overall Mathematics Achievement**

The meta-analysis of previously published studies showed a small, but reliable, correlation between non-symbolic numerical magnitude understanding and overall mathematics achievement. Importantly, two aspects of the studies affected the size of the correlation. First, studies that measured non-symbolic knowledge using a measure of accuracy consistently showed a relation to mathematics achievement, but studies that used only reaction time measures did not show such relations. Thus, future studies should use percent correct,  $w$ , or a combination of accuracy and

RT when examining how non-symbolic numerical magnitude understanding relates to skill in formal mathematics. Second, we found that the relation between non-symbolic numerical magnitude understanding and mathematics achievement was larger for younger children who have not yet started formal schooling. Mathematics achievement tests used with younger children might be more sensitive to non-symbolic numerical magnitude knowledge; alternatively, younger children might be more reliant on non-symbolic representations than older children who have more experience with symbolic numbers. Future work should examine in greater detail how the link between non-symbolic numerical magnitude understanding and mathematics achievement changes over development.

### **Validity of the Novel Tasks**

We used three novel tasks in this experiment: magnitude comparison of non-symbolic fractions, number line estimation of non-symbolic whole numbers, and number line estimation of non-symbolic fractions. This raised the question of whether these tasks yield reliable and valid measures of non-symbolic magnitudes. A variety of evidence suggests affirmative answers. All three novel tasks had satisfactory reliability ( $\alpha = .82 - .92$ ), and all appeared to have construct validity as well. For magnitude comparison of non-symbolic fractions, one indicator that the task was assessing the magnitude representations that it was designed to measure was that accuracy increased with the ratio of the numbers being compared, an essential prediction of the ANS construct (Halberda et al., 2008). Another indicator of this task's construct validity was the large correlation between it and the widely used non-symbolic whole number magnitude comparison task. This relation



suggests that non-symbolic fraction magnitude comparison, like non-symbolic whole number magnitude comparison, reflects ANS functioning. Finally, accuracy on this task and on both non-symbolic number line estimation tasks was related to overall mathematics achievement. These data are definitely not the final word, but the present results provide at least preliminary evidence for the validity of the new tasks for measuring non-symbolic numerical magnitude understanding.

### **Conclusions**

By examining children's performance on numerical magnitude tasks that varied in the type of task (magnitude comparison or number line estimation), number (whole number or fraction), and form (symbolic or non-symbolic), we were able to answer a number of questions about children's understanding of numerical magnitudes. First, we found strong evidence supporting the integrated theory of numerical development. There were large correlations between children's understanding of symbolic whole number and fraction magnitudes. Second, we found that both symbolic and non-symbolic numerical magnitude understanding uniquely predicted overall mathematics achievement, above and beyond the influence of non-mathematical cognition and the other type of numerical magnitude understanding. Third, we found that symbolic and non-symbolic magnitude understandings have separate and independent effects on overall mathematics achievement.

A clear pattern that emerged in the present study is that, at least by fifth grade, mathematics achievement is more closely related to symbolic than non-symbolic magnitude knowledge. Performance on all four tasks that assessed symbolic

magnitude knowledge was highly correlated with overall mathematics achievement test scores. The relations were equally strong for number line estimation and magnitude comparison and were even stronger with fractions than whole numbers. Together with the causal evidence showing that manipulations that improve young children's whole number magnitude representations also improve their ability to learn novel arithmetic problems (Booth & Siegler, 2008; Siegler & Ramani, 2009), the findings suggest that interventions designed to improve symbolic magnitude representations might be useful for children considerably older than the preschoolers and first graders with whom previous interventions have been conducted.

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