EDUCATOR'S PRACTICE GUIDE

A set of recommendations to address challenges in classrooms and schools

WHAT WORKS CLEARINGHOUSE™

Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students

NCEE 2015-4010
U.S. DEPARTMENT OF EDUCATION

NATIONAL CENTER FOR EDUCATION EVALUATION AND REGIONAL ASSISTANCE

Institute of Education Sciences

About this practice guide

The Institute of Education Sciences (IES) publishes practice guides in education to provide educators with the best available evidence and expertise on current challenges in education. The What Works Clearinghouse (WWC) develops practice guides in conjunction with an expert panel, combining the panel's expertise with the findings of existing rigorous research to produce specific recommendations for addressing these challenges. The WWC and the panel rate the strength of the research evidence supporting each of their recommendations. See Appendix A for a full description of practice guides.

The goal of this practice guide is to offer educators specific, evidence-based recommendations that address the challenges of teaching algebra to students in grades 6 through 12. This guide synthesizes the best available research and shares practices that are supported by evidence. It is intended to be practical and easy for teachers to use. The guide includes many examples in each recommendation to demonstrate the concepts discussed.

Practice guides published by IES are available on the What Works Clearinghouse website at http://whatworks.ed.gov.

How to use this guide

This guide provides educators with instructional recommendations that can be implemented in conjunction with existing standards or curricula and does not recommend a particular curriculum. Teachers can use the guide when planning instruction to prepare students for future mathematics and post-secondary success. The recommendations are appropriate for use with all students learning algebra in grades 6–12 and in diverse contexts, including for use during both formative and summative assessment.

Administrators, professional development providers, program developers, and researchers can also use this guide. Administrators and professional development providers can use the guide to implement evidence-based instruction and align instruction with state standards or to prompt teacher discussion in professional learning communities. Program developers can use the guide to create more effective algebra curricula and interventions. Finally, researchers may find opportunities to test the effectiveness of various approaches to algebra and explore gaps or variations in the algebra instruction literature.

IES Practice Guide

Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students

April 2015

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Introduction

Introduction to the *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students* Practice Guide

Algebra is often the first mathematics subject that requires extensive abstract thinking, a challenging new skill for many students. Algebra moves students beyond an emphasis on arithmetic operations to focus on the use of symbols to represent numbers and express mathematical relationships. Understanding algebra is a key for success in future mathematics courses, including geometry and calculus. Many mathematics experts also consider algebra knowledge and skills important for post-secondary success as well as for producing a skilled workforce for scientific and technical careers.[1] Algebra requires proficiency with multiple representations, including symbols, equations, and graphs, as well as the ability to reason logically, both of which play crucial roles in advanced mathematics courses.

Most states have standards for student knowledge in high school algebra. However, these standards do not typically provide **evidence-based**[2] instructional guidance for implementing the standards. This practice guide provides educators with recommendations and resources to help students achieve success in algebra.

This practice guide presents evidenced-based suggestions for how to improve algebra skills and knowledge for students in grades 6–12. The guide offers three recommendations that provide teachers with specific, actionable guidance for implementing these practices in their classrooms. It also provides a level of supporting evidence for each recommendation, examples to use in class, and solutions to potential implementation challenges.

Overarching themes

This practice guide highlights three general and interrelated themes for improving the teaching and learning of algebra.

- **Developing a deeper understanding of algebra.** Although proficiency in arithmetic operations is important to becoming proficient in algebra, the recommendations advocate algebra instruction that moves students beyond superficial mathematics knowledge and toward a deeper understanding of algebra. This includes encouraging students to make connections between algebraic concepts and the procedures present in problems, and helping students recognize how the placement of the quantities relative to the operations in problems impacts the solution **strategy**. Teachers can prompt students to consider: What am I being asked to do in this problem? What do I know about the form of this expression or equation? What are the relationships between the quantities in this expression or equation? How can I check that my **solution** is correct?

- **Promoting process-oriented thinking.** The guide emphasizes moving beyond a primary focus on the correct final answer to algebra problems to also promoting the understanding of the processes by which one arrives at an answer. For example, the guide encourages students to consider questions such as the following: What decisions did you make to solve the problem? What steps did you take to solve the problem? Was this a good strategy? Why or why not? Are there other ways to solve the problem? Can you show (through manipulatives, pictures, or number-lines) how you solved the problem?

- **Encouraging precise communication.** The guide prompts teachers to provide frequent opportunities for students to reason with and talk about mathematical concepts, procedures, and strategies using **precise mathematical language**. This communication plays a key role in helping students develop mathematical understanding. For example, the guide encourages teachers to ask students: How would you describe this problem using precise mathematical language? How would you describe your strategy for solving this problem using precise mathematical language?

Overview of the recommendations

Recommendation 1. Use solved problems to engage students in analyzing algebraic reasoning and strategies.

1. Have students discuss solved problem structures and solutions to make connections among strategies and reasoning.
2. Select solved problems that reflect the lesson's instructional aim, including problems that illustrate common errors.
3. Use whole-class discussions, small-group work, and independent practice activities to introduce, elaborate on, and practice working with solved problems.

Recommendation 2. Teach students to utilize the structure of algebraic representations.

1. Promote the use of language that reflects mathematical structure.
2. Encourage students to use reflective questioning to notice structure as they solve problems.
3. Teach students that different algebraic representations can convey different information about an algebra problem.

Recommendation 3. Teach students to intentionally choose from alternative algebraic strategies when solving problems.

1. Teach students to recognize and generate strategies for solving problems.
2. Encourage students to articulate the reasoning behind their choice of strategy and the mathematical validity of their strategy when solving problems.
3. Have students evaluate and compare different strategies for solving problems.

Summary of supporting research

Practice guide staff conducted a thorough literature search, identified studies that met protocol requirements, and then reviewed those studies using the What Works Clearinghouse (WWC) design standards. This literature search focused on studies published within 20 years from the start of the review, as is standard on WWC literature searches. The time frame was established to help ensure that the guide characterizes effectiveness from the research base relative to conditions similar to those in schools today and to define a realistic scope of work for the review.

A search for literature related to algebra instruction published between 1993 and 2013 yielded more than 2,800 citations. These studies were all screened for relevance according to eligibility criteria described in the practice guide protocol.[3] Studies that did not include populations of interest, measure relevant outcomes, or assess the effectiveness of replicable practices used to teach algebra were excluded. Consistent with the protocol, the literature search and screening excluded general policies, such as an extra period of algebra, that are typically determined by administrators and not teachers. Of the eligible studies, 30 studies used experimental and quasi-experimental designs to examine the effectiveness of the practices found in this guide's recommendations. From this subset, 15 studies met the WWC's rigorous evidence group design standards and were related to the panel's recommendations. Studies were classified as having a positive or negative effect when the result was either statistically significant (unlikely to occur by chance) or substantively important (producing considerable differences in outcomes).

In this guide, the different types of knowledge and skills that algebra students are expected to master are classified into three domains: conceptual knowledge, procedural knowledge, and procedural **flexibility**. Conceptual knowledge includes understanding algebraic ideas, operations, procedures, and notation. Procedural knowledge includes choosing operations and procedures to solve algebra problems, as well as applying operations and procedures to arrive at the correct solution to problems. Procedural flexibility includes identifying and implementing multiple methods to solve algebra problems, as well as choosing the most appropriate method. (For more information about the domains and how outcomes were classified into the domains, see Appendix D.) Some of the recommendations in this guide

are more effective in improving achievement in certain domains than others. For example, most of the supporting evidence for Recommendation 3 suggested that the practices in this recommendation are most effective in improving procedural flexibility. The evidence linking Recommendation 3 to improvement in conceptual knowledge and procedural knowledge was not as strong.

The recommendations can be implemented individually in any order or together at the same time with one exception: as the evidence suggests and as the panel also believes, the practices proposed in Recommendation 3 are most effective if they are implemented once students have some fluency with algebra procedures and strategies. One recommended practice is no more or less important than another. A practice from one recommendation can also be used to implement another recommendation.

For example, **solved problems** (Recommendation 1) can be used to teach algebraic **structure** (Recommendation 2) and multiple solution strategies (Recommendation 3). The panel believes that each recommendation can be used to develop conceptual knowledge, procedural knowledge, and procedural flexibility. For example, solved problems can help students learn concepts, strategies, reasoning, and algebraic structure.

The evidence level for each recommendation is based on an assessment of the relevant evidence supporting each recommendation. (Appendix A describes the criteria for each level of evidence.) Table 1 shows the level of evidence rating for each recommendation as determined by WWC guidelines outlined in Table A.1 in Appendix A. (Appendix D presents more information on the body of evidence supporting each recommendation.)

Table 1. Recommendations and corresponding levels of evidence

	Levels of Evidence		
Recommendation	**Strong Evidence**	**Moderate Evidence**	**Minimal Evidence**
1. Use solved problems to engage students in analyzing algebraic reasoning and strategies.			◆
2. Teach students to utilize the structure of algebraic representations.			◆
3. Teach students to intentionally choose from alternative algebraic strategies when solving problems.		◆	

How to use this guide

This guide provides educators with instructional recommendations that can be implemented in conjunction with existing standards or curriculum and does not prescribe the use of a particular curricula. Teachers can use the guide when planning instruction to prepare students for future mathematics and post-secondary success. The recommendations are appropriate for use with all students learning algebra in grades 6–12 and in diverse contexts, including for use during both formative and summative assessment.

Administrators, professional development providers, program developers, and researchers can also use this guide. Administrators and professional development providers can use the guide to implement evidence-based instruction and align instruction with state standards or to prompt teacher discussion in professional learning communities. Developers can use the guide to develop more effective algebra curricula and interventions. Finally, researchers may find opportunities to test the effectiveness of various approaches to algebra and explore gaps or variations in the algebra instruction literature.

Use solved problems to engage students in analyzing algebraic reasoning and strategies.

Compared to elementary mathematics work like arithmetic, solving algebra problems often requires students to think more abstractly. Algebraic reasoning requires students to process multiple pieces of complex information simultaneously, which can limit students' capacity to develop new knowledge. (Such reasoning is sometimes described as imposing high cognitive load or challenging working memory, which can interfere with students' ability to learn.[4]) Solved problems can minimize the burden of ***abstract reasoning*** *by allowing students to see the problem and many* ***solution steps*** *at once—without executing each step—helping students learn more efficiently.[5]*

Analyzing and discussing solved problems can also help students develop a deeper understanding of the logical processes used to solve algebra problems.[6] Discussion and the use of incomplete or ***incorrect solved problems*** *can encourage students to think critically.*

Solved problem: An example that shows both the problem and the steps used to reach a solution to the problem. A solved problem can be pulled from student work or curricular materials, or it can be generated by the teacher. A solved problem is also referred to as a "worked example."

Sample solved problem:

Solve for x in this equation: $3^{4x+3} = 81$	$3^{4x+3} = 81$ $3^{4x+3} = 3^4$ $4x + 3 = 4$ $4x = 1$ $x = \frac{1}{4}$

Summary of evidence: Minimal Evidence

Four studies examined the effects of using solved problems in algebra instruction and met WWC group design standards without reservations (see Appendix D).[7] Of the four, three studies showed positive effects[8] on conceptual knowledge.[9] One of these three studies showed a positive effect of providing correct and incorrect solved problems and prompting students to explain the solution, compared to students who were prompted similarly as they solved practice problems.[10] Two additional studies demonstrated that providing students with solved problems alongside practice problems had positive effects on student achievement, compared to students who received additional practice problems.[11] These three studies examined solved problems with students in remedial, regular, and honors algebra classes. The remaining study found that solved problems had negative effects on conceptual and procedural knowledge.[12] This study compared students who studied solved problems to students who used **reflective questioning** (a practice suggested in Recommendation 2).[13] This body of evidence suggests that, compared to asking students to solve practice problems alone, studying solved problems can improve achievement.

How to carry out the recommendation

1. Have students discuss solved problem structures and solutions to make connections among strategies and reasoning.

Create opportunities for students to discuss and analyze solved problems by asking students to describe the steps taken in the solved problem and to explain the reasoning used. Ask students specific questions about the solution strategy, and whether that strategy is logical and mathematically correct. Asking these questions encourages active student engagement. Vary the questions based on the needs of students and the types of problems being discussed. The questions can be asked verbally or written down for students to reference. Example 1.1 presents general questions that could be applicable to many types of algebra problems. (See later examples in this recommendation for ideas about how to incorporate these and other specifically tailored questions to sample solved problems.)

Example 1.1. Questions to facilitate discussion of solved problems

- What were the steps involved in solving the problem? Why do they work in this order? Would they work in a different order?
- Could the problem have been solved with fewer steps?
- Can anyone think of a different way to solve this problem?
- Will this strategy always work? Why?
- What are other problems for which this strategy will work?
- How can you change the given problem so that this strategy does not work?
- How can you modify the solution to make it clearer to others?
- What other mathematical ideas connect to this solution?

Structure: The underlying mathematical features and relationships of an expression, representation, or equation. Structure includes quantities, variables, operations, and relationships (including equality and inequality). More complex structures are built out of simple ones, and recognizing structure involves the ability to move between different levels of complexity.

Foster extended analysis of solved problems by asking students to notice and explain different aspects of a problem's structure (Example 1.2). Carefully reviewing and discussing the structure and each solution step of a solved problem helps students recognize the sequential nature of solutions and anticipate the next step in solving a problem. This can improve students' ability to understand the reasoning behind different problem-solving strategies.

Example 1.2. Questions to facilitate discussion of the structure of problems

- What quantities—including numbers and variables—are present in this problem?
- Are these quantities discrete or continuous?
- What operations and relationships among quantities does the problem involve? Are there multiplicative or additive relationships? Does the problem include equality or inequality?
- How are parentheses used in the problem to indicate the problem's structure?

2. Select solved problems that reflect the lesson's instructional aim, including problems that illustrate common errors.

Use solved problems to accomplish diverse learning objectives (Example 1.3). Choose solved problems that are linked to the learning objectives from student examples (including from current or past students and other teachers' students) or curricular materials, or make up examples. Specific types of solved problems—different problems solved with similar solution steps and incorrect solved problems—can be used for specific learning objectives. During the lesson, allow students to consult solved problems independently or in groups to understand different ways to solve a problem when they are unsure how to proceed.

Presenting several solved problems that use similar solution steps can help students see how to approach different problems that have similar structures (Example 1.4). To incorporate multiple solved problems into a lesson, consider the following approaches:

- Select problems with varying levels of difficulty and arrange them from simplest to most complex applications of the same concept.
- Display the multiple examples simultaneously to encourage students to recognize patterns in the solution steps across problems.
- Alternatively, show the problems individually, one after the other, to facilitate more detailed discussion on each problem.

Example 1.3. Examples of solved problems for different learning objectives[14]

Problem	Jada's response	Questions to guide student discussion
Objective 1: Solve a system of equations with two unknowns.		
Solve for x and y. $2x - 4y = 10$ $5x - 2y = 9$	$2x - 4y = 10$ $2x = 10 + 4y$ $x = 5 + 2y$	What did Jada do first? Why? Does this make sense as a first step? Can anyone think of a different first step to solve this problem?
	$5x - 2y = 9$ $5(5 + 2y) - 2y = 9$ $25 + 10y - 2y = 9$ $25 + 8y = 9$ $8y = -16$ $y = -2$	What did Jada do next? Why? Do all of her steps make sense? Can anyone think of a different way to solve for y?
	$x = 5 + 2(-2)$ $x = 5 + (-4)$ $x = 1$	How did Jada solve for x? Will this strategy always work? Why? Can anyone think of a different way to solve for x?
	Verify my solution: $2(1) - 4(-2) = 10$ $2 - (-8) = 10$ ✓ $5(1) - 2(-2) = 9$ $5 - (-4) = 9$ ✓	Are Jada's solutions correct? How do we know Jada did not make any errors? How can we tell whether she made any mistakes in reasoning? Will Jada's strategy always work? Would the steps she took work in another order? Could the equations have been solved in a simpler way or with fewer steps?
Objective 2: Factor quadratic expressions.		
$2x^2 - 16x + 32$	$2x^2 - 16x + 32$ $2(x^2 - 8x + 16)$ $2(x - 4)(x - 4)$ $2(x - 4)^2$	What did Jada do first? Can you explain why she did this first? Why did Jada factor out a 2? Can anyone think of a different way to factor this expression? How might Jada check her answer?
Objective 3: Solve quadratic equations by completing the square.		
$x^2 + 8x = 9$	$x^2 + 8x = 9$ $x^2 + 8x + 16 = 9 + 16$ $(x + 4)^2 = 25$ $x + 4 = \pm 5$	Why did Jada add 16 to both sides?
	$x + 4 = 5$ $\quad$ $x + 4 = -5$ $x = 5 - 4$ $\quad$ $x = -5 - 4$ $x = 1$ $\quad$ $x = -9$	Why are there two solutions to this equation?
	Verify my solution: $(1)^2 + 8(1) = 9$ $1 + 8 = 9$ $9 = 9$ ✓ $(-9)^2 + 8(-9) = 9$ $81 + (-72) = 9$ $9 = 9$ ✓	Are Jada's solutions correct? How do we know that Jada did not make any errors? How can we tell whether Jada made any mistakes in reasoning? Can anyone think of a different way of solving this equation?

Example 1.4. Presenting multiple solved problems with varying degrees of difficulty

Each row below has multiple solved problems for a specific topic. Within each row, the examples range from simplest on the left to most difficult on the right. Display the examples in each row all at once or one at a time, depending on students' understanding of the topic. Encourage students to notice how the problems' structures are similar and how they are solved in similar ways.

Factoring[15]		
1. $x^2 - 81$ $(x+9)(x-9)$	**2.** $2x^3 - 32x$ $2x(x^2 - 16)$ $2x(x+4)(x-4)$	**3.** $x^4 - 9y^4$ $(x^2)^2 - (3y^2)^2$ $(x^2 + 3y^2)(x^2 - 3y^2)$
Solving quadratic equations by completing the square		
1. $x^2 + 8x - 9 = 0$ $x^2 + 8x = 9$ $x^2 + 8x + 16 = 9 + 16$ $(x+4)^2 = 25$ $x + 4 = \pm 5$ $x + 4 = 5$ $\quad$ $x + 4 = -5$ $x = 5 - 4$ $\quad$ $x = -5 - 4$ $x = 1$ $\quad$ $x = -9$	**2.** $4x^2 + 16x - 120 = 8$ $4x^2 + 16x = 128$ $x^2 + 4x = 32$ $x^2 + 4x + 4 = 32 + 4$ $(x+2)^2 = 36$ $x + 2 = \pm 6$ $x + 2 = 6$ $\quad$ $x + 2 = -6$ $x = 6 - 2$ $\quad$ $x = -6 - 2$ $x = 4$ $\quad$ $x = -8$	**3.** $4x^2 = 10x - 3$ $4x^2 - 10x = -3$ $x^2 - \frac{10}{4}x = \frac{-3}{4}$ $x^2 - \frac{5}{2}x = \frac{-3}{4}$ $x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{-3}{4} + \frac{25}{16}$ $(x - \frac{5}{4})^2 = \frac{-12}{16} + \frac{25}{16}$ $(x - \frac{5}{4})^2 = \frac{13}{16}$ $x - \frac{5}{4} = \pm\sqrt{\frac{13}{16}}$ $x - \frac{5}{4} = \pm\frac{\sqrt{13}}{\sqrt{16}}$ $x - \frac{5}{4} = \pm\frac{\sqrt{13}}{4}$ $x = \frac{5}{4} \pm \frac{\sqrt{13}}{4}$ $x = \frac{5 + \sqrt{13}}{4}$ $\quad$ $x = \frac{5 - \sqrt{13}}{4}$
Operating with functions, given $f(x) = 2x^2 + 10x$ and $g(x) = 4 - 3x^3$		
1. Find $f(x^2) + f(x)$. $[2(x^2)^2 + 10(x^2)] + [2x^2 + 10x]$ $2x^4 + 10x^2 + 2x^2 + 10x$ $2x^4 + 12x^2 + 10x$	**2.** Find $(f - g)(x)$. $(2x^2 + 10x) - (4 - 3x^3)$ $2x^2 + 10x - 4 + 3x^3$ $3x^3 + 2x^2 + 10x - 4$	**3.** Find $(f \circ g)(x)$. $2(4 - 3x^3)^2 + 10(4 - 3x^3)$ $2(16 - 24x^3 + 9x^6) + 40 - 30x^3$ $32 - 48x^3 + 18x^6 + 40 - 30x^3$ $18x^6 - 78x^3 + 72$

After reviewing correct solved problems, use incorrect solved problems to help students deepen their understanding of concepts and correct solution strategies by analyzing strategic, reasoning, and procedural errors.

Students can discuss problems and strategies that they know are incorrect to develop a better understanding of the process used to obtain a correct solution (Example 1.5).

Example 1.5. One way to introduce incorrect solved problems

- Give students correct solved problems to study and discuss.
- Once students have an understanding of correct strategies and problems, present an incorrect solved problem to students.
- Display the incorrect solved problem by itself or side-by-side with a correct version of the same problem.
- Clearly label that the problem is solved incorrectly.
- Engage in discussion of the error and what steps led to the incorrect answer. For example, teachers could use the dialogue below when presenting the following pair of solved problems, one incorrect and one correct:

Solve the system of linear equations $3x - 2y = 12$ and $-x - 2y = -20$.

Adriana's Response: Correct	Jimmy's Response: Incorrect
$3x - 2y = 12$ $-x - 2y = -20$	$3x - 2y = 12$ $-x - 2y = -20$
$3x - 2y - (-x - 2y) = 12 - (-20)$ $3x - 2y + x + 2y = 12 + 20$ $4x = 32$ $x = 8$	$3x - 2y + (-x - 2y) = 12 + (-20)$ $2x = -8$ $x = -4$
$3(8) - 2y = 12$ $24 - 2y = 12$ $-2y = -12$ $y = 6$	$3(-4) - 2y = 12$ $-12 - 2y = 12$ $-2y = 24$ $y = -12$
Solution: (8, 6)	**Solution: (–4, –12)**

Teacher: What did Jimmy do *correctly*?

Student 1: It looks like the substitution was done correctly after x was found.

Teacher: What lets you know he substituted correctly?

Student 2: He substituted –4 for x in the first equation, and then he solved for y.

Teacher: That's right. So where is the error in Jimmy's work? What does he not understand?

Student 3: He got the wrong solution for x.

Teacher (further prompting): So the error occurred earlier in the solution. In the original equations, are either of the x or y terms the same?

Student 2: The y terms are the same.

Teacher: If either of the terms are the same, what is an efficient first step to solve a system of equations?

Student 3: Subtract the equations.

Teacher: So looking at the incorrect solution, and thinking about what we've just discussed, what was the error in the incorrect solution?

Student 1: Jimmy added the two equations to each other, but he mistakenly subtracted the y terms instead of adding them.

Teacher: That's right. What did Adriana do differently?

Student 3: She subtracted the two equations correctly.

Teacher: How can we verify Adriana's solution?

Student 2: We can substitute the solution into the two equations.

When analyzing an incorrect solved problem, students should explain why identified errors led to an incorrect answer so they can better understand the correct processes and strategies.

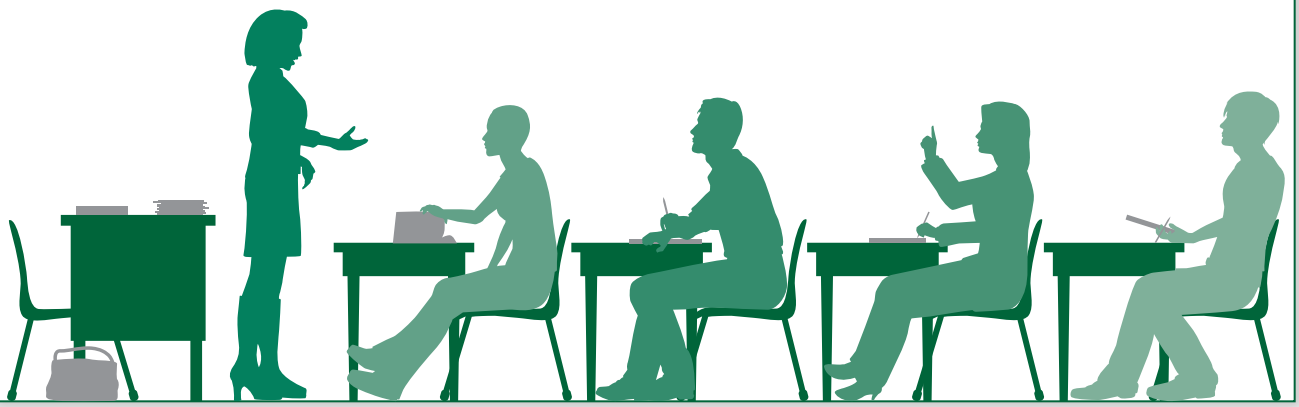
Present an incorrect solved problem by itself, or display it beside the same problem solved correctly and ask students to compare the two strategies used. Clearly label correct and incorrect examples, so students do not confuse correct and incorrect strategies. One option, as shown in Example 1.6, is to show a correct solved problem alongside multiple incorrect solved problems and ask probing questions to draw students' attention to the errors and help them understand what was done incorrectly.

Example 1.6. Parallel correct and incorrect solved problems, factoring

Factor: $x^2 - 10x - 24$		
Correct solved problem	**Incorrect solved problem: Sum of the integers does not equal the middle term**	**Incorrect solved problem: Sum of the integers does not equal the middle term**
Student A factored this expression correctly: $x^2 - 10x - 24$ $(x - 12)(x + 2)$	Student B did *not* factor this expression correctly: $x^2 - 10x - 24$ $(x - 4)(x + 6)$	Student C did *not* factor this expression correctly: $x^2 - 10x - 24$ $(x + 12)(x - 2)$

Sample questions to guide discussion of the errors:

1. How can you show that the answers from students B and C are incorrect?
2. What advice would you give to students B and C to help them avoid factoring this type of problem incorrectly in the future?
3. How can you check that student A factored this expression correctly?
4. What strategy would you use to factor this expression and why did you choose that strategy?



Include different types of errors—for example, **strategic errors** and **procedural (or computational) errors**—next to a correct solved problem (Example 1.7). Errors can arise from using an incorrect strategy (a strategic or reasoning error) or from making a mathematical operations mistake (a procedural or computational error). Encourage students to think critically about how different steps and choices can lead to errors.

Example 1.7. Parallel correct and incorrect solved problems, completing the square

Show students the correct and incorrect solved problems together. Ask students to describe the error (shown in bold text below), and guide students' discussion of why the error occurred.

	Correct solved problem	Incorrect solved problem: Strategic and reasoning error	Incorrect solved problem: Procedural error
Equation $x^2 + 6x = 27$	$x^2 + 6x = 27$ $x^2 + 6x + 9 = 27 + 9$ $(x + 3)^2 = 36$ $x + 3 = \pm 6$ $x + 3 = 6$ $\quad x + 3 = -6$ $x = 6 - 3$ $\quad x = -6 - 3$ $x = 3$ $\quad x = -9$	$x^2 + 6x = 27$ $x^2 + 6x + 9 = 27 + 9$ $(x + 3)^2 = 36$ $\mathbf{x + 3 = 6}$ $x = 6 - 3$ $x = 3$	$x^2 + 6x = 27$ $\mathbf{x^2 + 6x + 9 = 27}$ $(x + 3)^2 = 27$ $x + 3 = \pm 3\sqrt{3}$ $x = -3 + 3\sqrt{3}$ $\quad x = -3 - 3\sqrt{3}$
Description of error	N/A	The student did not include the negative square root as a solution.	The student did not add 9 to both sides when completing the square. This means the new equation is not equivalent to the previous equation.
Questions to guide discussion of the error	N/A	If a number squared is 36, what could the number be equal to? What properties of numbers and operations can we use to justify each step in the example?	If you add something to one side of the equation, what else do you need to do? Why? What property is this? The original equation tells us how $x^2 + 6x$ and 27 are related. What is that relationship? If 27 and $x^2 + 6x$ equal each other, then what should be the relationship between 27 and $x^2 + 6x + 9$?

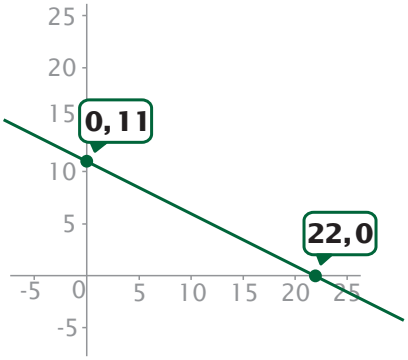
3. Use whole-class discussions, small-group work, and independent practice activities to introduce, elaborate on, and practice working with solved problems.

Although the studies that meet WWC design standards did not examine the effect of solved problems across different situations, the panel believes that solved problems are useful in a variety of contexts. Introduce solved problems during whole-class instruction to provide an overview of a solution strategy. For example, after beginning to explain slope-intercept form, use solved problems to help introduce the concept of graphing linear equations (Example 1.8).

Example 1.8. Using solved problems in whole-class discussion

Display the left column, which has the steps of the solved problem, on the board for students. Use the questions in the right column to guide student analysis and discussion of the solved problem.

The slope of a line is $-\frac{1}{2}$. The point (6, 8) is on the line. What is the equation for the line in slope-intercept form? What is the *y*-intercept? Graph this line.

Asha's response	Questions to guide whole-class discussion of the solved problem
$y = mx + b$	In slope-intercept form, what does *b* represent? What does *m* represent?
$y = -\frac{1}{2}x + b$	What did Asha do first? Why? What does $-\frac{1}{2}$ represent? Can anyone think of a different first step to solve this equation?
$8 = -\frac{1}{2}(6) + b$ $8 = -3 + b$ $8 + 3 = b$ $11 = b$	What did Asha do in this series of steps? Why is it important to solve for *b*? What can we substitute into the slope-intercept equation?
$y = -\frac{1}{2}x + 11$	How did Asha get this equation? How can we know this equation is correct? Does the point (6, 8) lie on this line?
The *y*-intercept is 11, and the slope is $-\frac{1}{2}$.	Does the line go through the point (6, 8)? Explain how you know.
	What conditions are given in the problem? How does the graph meet these conditions? Is any other line possible? Why or why not?

Create activities for pairs or small groups of students to critically analyze solved problems. For example, present solved problems, including a combination of incorrect and correct solved problems, with accompanying questions for students to answer in small groups of two or three (Example 1.9). The questions can ask about the steps, structures, and strategies in the solved problems. After students work in small groups, bring the whole class together to discuss the problems further using mathematically appropriate language, and engage all students to address relevant components of the problem.

Example 1.9. Solved problems in small-group work[16]

Students can work with a partner or in small groups to study these solved problems and answer the questions.

1. $3 = \frac{4x}{8x+10}$

Jack solved this equation correctly. Here are the steps that he took to solve the equation for x:

Assume $8x + 10 \neq 0$, because if it is, then the right side of the equation is not defined.

$3 \times (8x + 10) = \frac{4x}{8x+10} \times (8x + 10)$

$3(8x + 10) = 4x$

$24x + 30 = 4x$

$24x + 30 - 30 = 4x - 30$

$24x = 4x - 30$

$24x - 4x = 4x - 30 - 4x$

$20x = -30$

$x = \frac{-30}{20}$

$x = \frac{-3}{2}$

a. What did Jack do first to solve the equation?

b. Was this an appropriate first step to solve this equation? Why or why not?

c. How does the placement of the quantities and the operations impact what Jack did first?

d. How did Jack reason through this problem?

e. How can you show that Jack got the correct solution to the equation?

2. $7x = 12x - 8 + 3$

Luis did *not* solve this equation correctly. To the right are the steps that he took to solve the equation for x:

a. What did Luis do first to solve the equation?

b. Was this an appropriate first step to solve this equation? Why or why not?

c. How did Luis reason through this problem?

d. What error did Luis make?

e. What strategy could you use to solve this equation correctly?

Luis

$7x = 12x - 8 + 3$

$\quad\quad\quad -3 \quad -3$

$7x = 12x - 11$

$11 = 5x$

$\frac{11}{5} = x$

Solved problems can replace some exercises in students' independent practice. After a lesson, give students **incomplete solved problems** and ask students to complete the solutions (Example 1.10).

Example 1.10. Incomplete solved problems

Include incomplete solved problems in students' independent practice, asking students to fill in the blank steps of the solved problems.

$-x + 7 \geq 9$ $-x \geq 2$ _____	$3(x + 2) + 12 \leq 4(1 - x)$ _____ $3x + 18 \leq 4 - 4x$ $7x \leq -14$ $x < -2$	$2(x + 7) - 5(3 - 2x) \geq 7x - 4$ $2x + 14 - 15 + 10x \geq 7x - 4$ _____ $5x \geq -3$ $x \geq -\frac{3}{5}$

In the incomplete solved problems, determine which step of the problem should remain blank, depending on the students' familiarity with the topic and the goal of the lesson.

Another strategy for incorporating solved problems into independent practice activities is alternating solved problems with unsolved problems that are similar to the solved problems in terms of problem structure or solution strategy (Example 1.11). Create exercises that show a solved problem followed by an unsolved problem for students to complete. Students can use the solved problem as an example to guide their independent work, while the unsolved problem provides a way for students to actively engage with the solved problem.

Example 1.11. Independent practice that incorporates solved problems

Give these problems to students to complete for independent practice. Ask students to first study the solved problem in the left column and to then expand the expressions in the right column in each row. Encourage students to notice the steps used in the solved problem, and explain that these steps are similar to the steps they will need to use to solve the expressions in the right column. Ask students to show how they reached their solutions to the problems in the right column and to think about how they could show another student that their solutions are correct.

1. $(3x + 4)(x - 6)$ $3x^2 - 18x + 4x - 24$ $3x^2 - 14x - 24$	**2.** $(2x - 8)(5x + 1)$
3. $(2x - 7)(9x - 1)$ $18x^2 - 2x - 63x + 7$ $18x^2 - 65x + 7$	**4.** $(4x - 5)(5x - 4)$

Potential roadblocks and suggested approaches

Roadblock 1.1. *I already use solved problems during whole-class instruction, but I'm not sure students are fully engaged with them.*

Suggested Approach. Whole-class discussion and analysis of solved problems can help guide students to notice aspects of the solved problems that are important. Asking questions and fostering discussion encourages students to think critically about solved problems. During whole-class instruction, model engaging with solved problems by using **think-aloud questions**. See Example 1.1 for examples of the types of questions that can foster discussion and analysis, and see Example 1.2 for important aspects of problems to highlight.

Look for opportunities to incorporate solved problems into many aspects of the lesson beyond whole-class instruction. During small group or independent work, help students engage with solved problems in a more meaningful way (as opposed to simply studying them) by providing students with probing questions (as in Example 1.9), requiring students to fill in missing steps of incomplete solved problems (as in Example 1.10), or asking students to solve problems themselves after studying a similar solved problem (as in Example 1.11). Class assessments can also ask students to analyze solved problems in the same manner in which they were incorporated in the lessons.

Roadblock 1.2. *I do not know where to find solved problems to use in my classroom and do not have time to make new examples for my lessons.*

Suggested Approach. Solved problems can come from many sources. Often, curriculum materials or textbooks may include sample or solved problems. In addition, student work can provide solved problems. Consider current students' responses on homework, projects, and assessments, including unique strategies to solve a problem, to find suitable material for solved problems. Past student work can also provide examples for solved problems, particularly related to common strategic or procedural errors. Finally, ask other teachers in your school if they have any solved problems that they can share, or if they have any sample student work that could be adapted into solved problems.

Roadblock 1.3. *I'm worried that showing students incorrect solved problems will confuse them.*

Suggested Approach. Exposing students to incorrect solved problems can help students understand the difference between correct and incorrect solution strategies and can highlight the reasoning behind common errors. If students are not accustomed to analyzing work and interpreting the math, they may initially resist analyzing incorrect solved problems. Setting the expectation that understanding the process is as important as a final correct answer helps students adjust and become more comfortable with using and discussing incorrect solved problems. Clearly identify and label incorrect solved problems to ensure that students recognize that there is an error within the solution steps. Presenting parallel correct solved problems alongside incorrect solved problems can help students recognize and analyze the error. Engage students in a discussion about each step of the incorrect solved problem, thoroughly explaining and discussing the nature of the error to reduce confusion about which strategies were applied correctly and incorrectly. Students who can identify and discuss the reasons for where and why an error exists may be better able to recognize when and how to use correct strategies as they later solve problems independently. In addition, discussion of incorrect solved problems allows students to confront their own potential errors in a nonthreatening way. When evaluating hypothetical student work, students can be more open and honest in their analysis of errors than they might be if critiquing their own work. See Examples 1.5, 1.6, and 1.7 for examples of how to implement incorrect solved problems to minimize student confusion.

Teach students to utilize the structure of algebraic representations.

*Structure refers to an **algebraic representation's** underlying mathematical features and relationships, such as*

- *the number, type, and position of quantities, including variables*
- *the number, type, and position of operations*
- *the presence of an equality or inequality*
- *the relationships between quantities, operations, and equalities or inequalities*
- *the range of complexity among expressions, with simpler expressions nested inside more complex ones*

Example 2.1. Seeing structure in algebraic representations

Consider these three equations:

$$\mathbf{2}x + 8 = 14$$

$$\mathbf{2}(x + 1) + 8 = 14$$

$$\mathbf{2}(3x + 4) + 8 = 14$$

Though the equations appear to differ, they have similar structures: in all three equations,

2
multiplied by a quantity,
plus 8,
equals 14.

*Paying attention to structure helps students make connections among problems, solution strategies, and representations that may initially appear different but are actually mathematically similar (Example 2.1). With an understanding of structure, students can focus on the mathematical similarities of problems that may appear to be different, which can simplify solving algebra problems. In particular, recognizing structure helps students understand the characteristics of **algebraic expressions** and problems regardless of whether the problems are presented in symbolic, numeric, verbal, or graphic forms.*

Summary of evidence: Minimal Evidence

Six studies contributed to the level of evidence for this recommendation. Four studies met WWC group design standards without reservations, and two studies did not meet WWC group design standards but adequately controlled for selection bias (see Appendix D).[17] Four of the six studies found positive effects[18] on procedural knowledge,[19] and three studies found positive effects on conceptual knowledge.[20] However, none of the studies examined an important component of the recommendation: the use of language that reflects mathematical structure. Two of the six studies taught students to use reflective questioning, one method of focusing on structure.[21] One study showed positive effects on procedural and conceptual knowledge for reflective questioning compared to having students study solved problems in small groups, and the other study showed positive effects on procedural knowledge, for reflective questioning compared to regular instruction without reflective questioning.[22] The remaining four studies compared using graphical representations—another recommended method of focusing on structure—to similar instruction that did not involve graphical representations.[23] Two of the four studies had positive effects on procedural knowledge.[24] The other two studies found that using graphical representations had positive effects on conceptual knowledge.[25] The populations included in the six studies varied; two of the studies were conducted outside of the United States,[26] and two study samples included students with specific learning challenges.[27] Overall, the body of evidence indicated that teaching reflective questioning or using graphical representations improves procedural and conceptual knowledge across diverse populations.

How to carry out this recommendation

1. Promote the use of language that reflects mathematical structure.

Although no study that meets WWC group design standards examines the effect of using precise mathematical language, the panel believes that using such language is a key component to understanding structure and sets the foundation for the use of reflective questioning, multiple representations, and diagrams (discussed in more detail below). When talking to students, phrase algebra solution steps in precise mathematical language to communicate the logical meaning of a problem's structure, operations, solution steps, and strategies (Example 2.2). Use precise mathematical language to help students analyze and verbally describe the specific features that make up the structure of algebraic representations. When introducing a new topic or concept, use and model precise mathematical language to encourage students to describe the structure of algebra problems with accurate and appropriate terms.

During whole-class instruction, teachers can rephrase student solutions and responses to questions using appropriate mathematical language instead of vague, non-mathematical language (Example 2.3). The panel believes that precise mathematical language more accurately and completely describes the **mathematical validity** of a problem, helping students better understand quantities and operations, along with the relationships between them. In addition, as suggested in Example 2.3, a vague expression might have more than one meaning.

Example 2.2. Modeling precise mathematical language, the distributive property

A teacher describes and illustrates the process for multiplying two binomials by using the distributive property. As a result of the teacher's modeling, students can understand the mathematical meaning behind using the distributive property as a solution strategy.

Teacher: Let's first notice the structure of this expression.

$$(2x + 5)(4x - 3)$$

We have two binomials, and each binomial consists of the sum or difference of two quantities. We can use extensions of the distributive property of multiplication over addition to rewrite the expression. The first binomial, $2x + 5$, is the sum of $2x$ and 5. We can distribute—from the right—the second binomial, $4x - 3$, over the first binomial:

$$(2x + 5)(4x - 3)$$
$$(2x)(4x - 3) + (5)(4x - 3)$$

We can then distribute each monomial, 2x and 5, over the binomial:

$$(2x)(4x - 3) + (5)(4x - 3)$$
$$(2x)(4x) - (2x)(3) + (5)(4x) - (5)(3)$$

Carrying out multiplication, we have

$$8x^2 - 6x + 20x - 15$$
or
$$8x^2 + 14x - 15.$$

Example 2.3. Imprecise vs. precise mathematical language

Imprecise language	Precise mathematical language
Take out the *x*.	Factor *x* from the expression. Divide both sides of the equation by *x*, with a caution about the possibility of dividing by 0.
Move the 5 over.	Subtract 5 from both sides of the equation.
Use the rainbow method. Use FOIL.	Use the distributive property.
Solve an expression.	Solve an equation. Rewrite an expression.
A is apples.	Let *a* represent the number of apples. Let *a* represent the cost of the apples in dollars. Let *a* represent the weight of the apples in pounds.
Plug in the 2.	Substitute 2 for *x*.
To simplify, flip it and multiply. To divide a fraction, invert and multiply.	To simplify, multiply both sides by the reciprocal. To divide fractions, multiply by the reciprocal.
Do the opposite to each side.	Use inverse operations. Add the opposite to each side.
The numbers cancel out.	The numbers add to zero. The numbers divide to one.
Plug it into the expression.	Evaluate the expression.

2. Encourage students to use reflective questioning to notice structure as they solve problems.

By asking themselves questions about a problem they are solving, students can think about the structure of the problem and the potential strategies they could use to solve the problem. First, model reflective questioning to students by thinking aloud while solving a problem. Teachers can write down the questions they ask themselves to clearly demonstrate the steps of their thinking processes. Then present a problem during whole-class instruction, and ask students to write down what questions they might ask themselves to solve the problem. Students can practice the think-aloud process while working in pairs or share their written ideas with a partner. This process will help students use reflective questioning on their own during independent practice to explore algebraic structure (Example 2.4).

Example 2.4. Student using reflective questioning

In the example below, a student completes the following task using reflective questioning (shown in the left column) to articulate his or her thoughts and reasoning (shown in the right column).

Rewrite the following expression: $\frac{2x}{x-1}+\frac{3x}{x+1}$

What can I say about the form of the expression?	It is a sum of rational expressions. I can think about rewriting this expression in terms of adding fractions, beginning with a common denominator of $x-1$ and $x+1$.	$\frac{2x}{x-1}+\frac{3x}{x+1}$ $\frac{2x}{x-1}\times\frac{x+1}{x+1}+\frac{3x}{x+1}\times\frac{x-1}{x-1}$ $\frac{(2x)(x+1)}{(x-1)(x+1)}+\frac{(3x)(x-1)}{(x+1)(x-1)}$
What do I notice about the denominator of each expression?	Both are binomials. The terms, x and 1, are the same, but one expression is the sum of these terms and the other is the difference. Binomials like these are factors for a difference of perfect squares.	
What has happened in problems that I solved before?	Sometimes I was able to see common factors in numerators and denominators after adding two rational expressions. I won't rewrite the denominators yet.	$\frac{(2x)(x+1)+(3x)(x-1)}{(x-1)(x+1)}$ $\frac{2x^2+2x+3x^2-3x}{(x-1)(x+1)}$ $\frac{5x^2-x}{(x-1)(x+1)}$
Do I see any common factor of the numerator and denominator?	Neither factor of the denominator is a factor of the numerator, so I'll rewrite the numerator and the denominator.	$\frac{5x^2-x}{x^2-1}$

To help students with reflective questioning, hang a poster or distribute a list of common questions students can ask themselves while solving a problem (Example 2.5). Initial question lists can be updated as new questions are used in class, helping students connect the questions to new learning experiences.

Example 2.5. Reflective questions for noticing structure

- What am I being asked to do in this problem?
- How would I describe this problem using precise mathematical language?
- Is this problem structured similarly to another problem I've seen before?
- How many variables are there?
- What am I trying to solve for?
- What are the relationships between the quantities in this expression or equation?
- How will the placement of the quantities and the operations impact what I do first?

3. Teach students that different algebraic representations can convey different information about an algebra problem.

Recognizing and explaining corresponding features of the structure of two representations can help students understand the relationships among several algebraic representations, such as equations, graphs, and word problems. Examples 2.6 and 2.7 demonstrate how equations represented in different forms provide different information. Teachers can present students with equations in different forms and ask students to identify the similarities and differences. Working in pairs, students can then discuss the similarities and differences they identified.

Example 2.6. Equations of the same line in different forms

Compare different forms of equations for the same line.

	Similarities	Differences
Slope-intercept form $y = mx + b$ $y = 2x - 3$	Both are equations of straight lines It is easy to see that the slope is 2. It is hard to see what the x-intercept is.	**Slope-intercept form** makes it easy to see what the y-intercept is.
Point-slope form $y - y_1 = m(x - x_1)$ $y - 5 = 2(x - 4)$		**Point-slope form** makes it easy to see that the point (4, 5) is on the line.

Example 2.7. Equations of the same parabola in different forms[28]

The "Throwing Horseshoes" exercise provides an opportunity for students to compare and interpret different forms of the same parabola.

Exercise: The height (in feet) of a thrown horseshoe t seconds into flight can be described by the following expression:

$$h = 1\frac{3}{16} + 18t - 16t^2$$

The expressions (a) through (d) below are equivalent. Which of them most clearly reveals the maximum height of the horseshoe's path? Explain your reasoning.

a. $h = 1\frac{3}{16} + 18t - 16t^2$

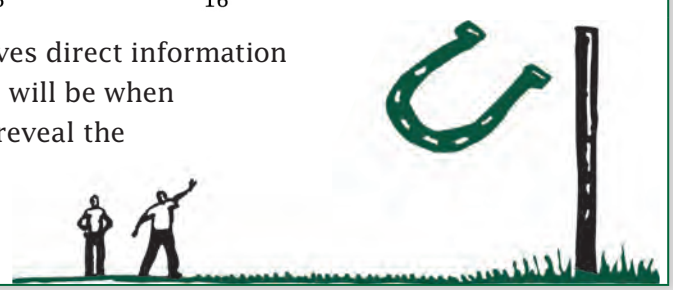
b. $h = -16(t - \frac{19}{16})(t + \frac{1}{16})$

c. $h = \frac{1}{16}(\frac{19}{16}t)(16t + 1)$

d. $h = -16(t - \frac{9}{16})^2 + \frac{100}{16}$

Solution: The maximum height corresponds to the greatest value of the expression. Choice (d) expresses the height as the sum of a negative number, –16, multiplied by a squared expression, $(t - \frac{9}{16})^2$, added to a positive number, $\frac{100}{16}$. The value of the squared expression is always either positive or zero, and so the value of the product of it and –16 is negative or zero. Since the maximum value of that product is 0 and it is added to a constant, the maximum value of the entire expression for the height is the value of the constant, $\frac{100}{16}$. This maximum value occurs only if the value of the squared expression is equal to zero, namely when $t = \frac{9}{16}$. The horseshoe will reach its maximum height of $\frac{100}{16}$ feet when $t = \frac{9}{16}$ seconds.

Among the listed representations, choice (b) gives direct information about the time when the height is 0 feet, which will be when the horseshoe is on the ground, so it does not reveal the maximum height.

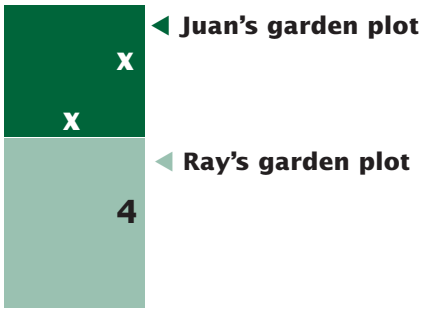


Help students see that different representations based on the same information can display the information differently, as in Example 2.8. Specific representations can exhibit some information about the problem's structure more readily than other representations. For example, students might find it easier to find the x-intercepts of the graph of a quadratic function if it is expressed in factored form rather than in standard form.

As needed, incorporate diagrams into instruction to demonstrate the similarities and differences between representations of algebra problems to students (Example 2.9). Encourage students to use a diagram to visualize the structure of a problem, organize and document the solution steps of the problem, and translate the problem into another representation, as illustrated in several studies reviewed for this guide.[29]

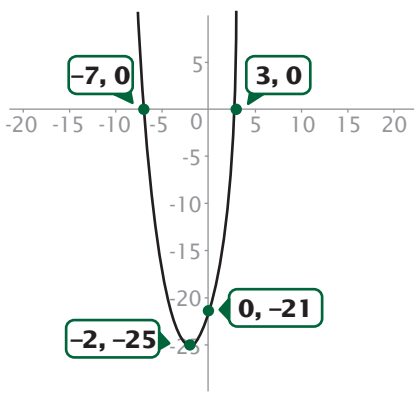
Example 2.8. Multiple algebraic representations

This table presents a problem and several representations of that problem. Students do not need to move in a linear fashion from one representation to the next, but should instead recognize that different representations based on the same problem can display the information differently.

Representation	Example	Notes
Word problem	Ray and Juan both have community garden plots. Ray has a rectangular plot. Juan has a square plot, and each side of his plot is x yards wide. Ray and Juan's plots share one full border; the length of Ray's plot on an unshared side is 4 yards. If Juan and Ray put in a fence around both of their plots, the area of the fenced space would be 21 square yards. How wide is the shared border?	The statement of the problem is one representation of a relationship among three quantities, which are the total area of 21 square yards, the area of Ray's plot, and the area of Juan's plot. Students typically move to other representations to solve the problem. They might draw a diagram and produce an equation, and then solve the equation algebraically or graphically.
Diagram	x = the length in yards of one of the sides of Juan's plot 	The diagram represents the two garden plots with a common border and a 4-yard unshared side of Ray's plot. The diagram also represents one large rectangle composed of two rectangles to illustrate that the total area is equal to the area of Ray's plot plus the area of Juan's plot. Using the rectangles, the given lengths, and the total area of 21 square yards, students can produce and solve an equation. Students can use the diagram to see the structure of the problem as the equivalence of a total area to the sum of two parts and to express it as an equation. After solving the equation for x, students can explain why there are two possible solutions for the quadratic equation, and why –7 does not yield an answer to the question in the word problem.

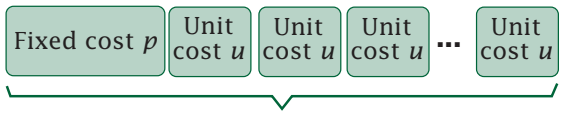
(continued)

Example 2.8. Multiple algebraic representations *(continued)*

Equations	Equation representing the equivalent areas in square yards: $21 = x(4 + x)$ Equation in standard form: $0 = x^2 + 4x - 21$ Equation in factored form: $0 = (x + 7)(x - 3)$ Total area = 21 yd^2 Area = length × width Area = $x(4 + x)$ $21 = x(4 + x)$ $21 = 4x + x^2$ $0 = x^2 + 4x - 21$ $0 = (x + 7)(x - 3)$ $x = -7 \quad x = 3$	Students will likely come to the standard form first when solving this problem, then will need to factor to reach the possible solutions for x. Students should recognize that the quadratic expression can be factored. The values of x that make the factored expression on the right side of the equation equal to zero can be read from the structure of the expression as a product. For a product to be zero, one of the factors has to be zero, so x is –7 or 3.
Graph		Students can find where an expression equals zero by thinking of the expression as a function and graphing it, seeing where the graph crosses the horizontal axis. The x-intercepts of the parabola can be read from the factored form. The y-intercept can be read from the standard form, and that form is helpful in determining the vertex of the parabola. The graph is a parabola because it is a quadratic equation, and the direction in which the parabola opens depends on the sign of the coefficient of x^2.

Example 2.9. Using diagrams to compare algebraic representations

Compare a diagram and an equation to represent a company's total utility costs per month if the company has a fixed/starting cost (p) of \$100 plus a unit cost (u) of \$6.50 for every unit manufactured. The company manufactured 15 units last month. What was its total utility cost?

	Similarities	Differences
Diagram showing the total cost (T) 	The quantities of T, p, n, and u are present in the diagram and the equation. Both show the unit cost and the fixed/starting cost. Both show the structure of multiplying the unit cost by the number of units and then adding that sum to the fixed/starting cost to find the total cost.	The exact value of the unit cost, the starting cost, and the number of units manufactured appear in the equation but not in the diagram. The diagram needs to be interpreted to see the costs as the lengths of rectangles.
Equation representing total cost (T), where n is the number of units manufactured: $T = p + nu$ $T = 100 + 15(6.50)$ $T = \$197.50$		

Potential roadblocks and suggested approaches

Roadblock 2.1. *I like to simplify mathematical language, and my students seem to respond positively to my use of simplified and informal language as well. Doesn't this approach make it easier for students than using complicated mathematical language?*

Suggested Approach. Informal language often relies on superficial features such as the position of symbols on the page, rather than on the underlying mathematical operations. Informal language can introduce misconceptions and confusion during standardized assessments where precise language is used, adding unnecessary complexity by giving students another idea to understand. While it is not harmful to use informal language in the classroom, it is necessary for students to know precise mathematical language and understand the logical meaning behind it.

Precise mathematical language is not necessarily more complicated, but it is more mathematically accurate. For example, instructing students to "Add the opposite to each side" is a precise but still simple way to tell them to use inverse operations, instead of the more imprecise phrase "Do the opposite to each side." Precise language facilitates mathematical communication beyond the classroom by promoting the use of common language across classrooms, grade levels, and assessments. Precision in language serves to highlight mathematical structure, helping students notice problem components such as quantities, operations, and their interrelationships. Referring to these components frequently and with precise language helps make them more accessible to students.

Roadblock 2.2. *My students race through problems. How do I get students to slow down, pause to ask themselves questions, and think about the problem?*

Suggested Approach. If students are moving through problems quickly without pausing to think or ask themselves questions, this might suggest one of two possibilities. First, the problems may in fact be too familiar and easy for students if they can successfully solve the problems without a lot of reflective thought. If this is the case, consider offering students variations of familiar problems that have similar mathematical structure to what they are familiar with but that may look very different from what they are used to. Challenge students to solve problems in more than one way and to explain the differences between the solution strategies. As a second possibility, students may have developed well-practiced strategies that they prefer to use, so they may not consider alternative strategies, even when their preferred strategies are not optimal. Consider assigning tasks that require students to respond to reflective questions, perhaps even instead of actually solving the problem. In either case, the expert panel believes it might be fruitful to use cooperative learning strategies, such as those presented in Example 2.10, to encourage students to use reflective questioning and carefully think through algebra problems.

Roadblock 2.3. *Diagrams don't seem to be very useful to some of my students.*

Suggested Approach. Some students will find the correct answer to a problem without the need for a diagram. However, sometimes a student can work toward the right answer without noticing or attending to the problem's structure or even without understanding what the problem is about. Diagrams can illuminate the mathematical structure of a problem and facilitate understanding of the mathematics behind the problem. Thus, even when a diagram is not perceived to be needed for getting the right answer, teachers can encourage students to recognize that diagrams continue to be essential tools in highlighting structure and fostering deeper understanding.

Example 2.10. Examples of cooperative learning strategies

- ☑ **Think, write, pair, share.** Give students time to think independently about the problem and write their ideas before they share the ideas with their partners and then with the entire group. As a whole group, identify the reflective questions that students naturally used to help their own thinking and to help their partners.
- ☑ **Confident, shaky, relearn.** Students can ask themselves what aspects of a task they feel confident about, what aspects they feel shaky about, and what aspects they need to relearn. When a student feels confident about a task, he or she can explain the task and mathematical validity of the solution strategy to himself or herself and to others.
- ☑ **Partner coaching/Quiz, quiz, trade.** Students quiz each other on assigned problems or tasks. While partnered, one student solves a problem, and the other student provides feedback on the solution and solution strategy. Then students can switch roles.
- ☑ **Directions for a friend.** Rather than asking students to solve a problem, ask them to write down the directions for how to solve it. For example, provide the following instructions to students: "Say your friend emails you for help with problem 7 on the assignment. How would you describe how to solve the problem to him or her? What would you write?" Then, have students trade directions with a partner, and have each student try to solve the problem according to the partner's directions.
- ☑ **Jigsaw.** Arrange students in groups and give each group member a different problem. For example, in each "jigsaw" group, student 1 is given information on problem A, student 2 is given information on problem B, and student 3 is given information on problem C. Then group members collaborate with members from other groups who were assigned the same problem to discuss their ideas and strategies. Finally, students come back to their jigsaw groups to discuss the ideas and strategies they heard from students in the other groups.
- ☑ **Numbered heads together.** Assign students to groups and give each student a number. Ask the class a question and encourage students to "put their heads together" to answer the question. Call out a number and have the student who was assigned that number act as the spokesperson for the group and explain the group's answer. Because students do not know what number will be called, group members must work together to come up with an answer and all group members must be prepared to answer the question.

Teach students to intentionally choose from alternative algebraic strategies when solving problems.

A strategy involves a general approach for accomplishing a task or solving a problem. Unlike an ***algorithm****, which contains a sequence of steps that are intended to be executed in a particular order, a strategy may require students to make choices based on the specifics of the problem as well as their problem-solving goals. A strategy might also include alternative approaches that consider variations of a problem or unexpected results a student might encounter while implementing the steps of the solution. Strategies are general and broadly applicable, making them useful in solving a variety of problems.*

Strategy: A general approach for accomplishing a task or solving a problem that may include sequences of steps to be executed, as well as the rationale behind the use and effectiveness of these steps.

By learning from and having access to multiple algebraic strategies, students learn to approach algebra problems with flexibility, recognizing when to apply specific strategies, how to execute different solution strategies correctly, and which strategies are more appropriate for particular tasks. This can help students develop beyond the memorization of one approach, allowing them to extend their knowledge and think more abstractly.

Comparing correct solution strategies can help deepen students' conceptual understanding and allow students to notice similarities and differences between problem structures and solution strategies. Having students compare strategies may be more conducive to learning than having students study individual strategies, because comparison activities enable students to reference their prior knowledge of one strategy to learn new strategies.

While this recommendation promotes the understanding and use of multiple solution strategies, the recommendation does not advocate that students be fluent in all possible strategies for solving a given problem type. By learning alternative strategies, students can select from different options when they encounter a familiar or unfamiliar problem.

Summary of evidence: Moderate Evidence

Six studies examined the effects of teaching alternative algebraic strategies and met WWC group design standards without reservations (see Appendix D).[30] Four studies showed positive effects[31] of teaching alternative algebraic strategies in at least one of the three outcome domains (procedural knowledge, conceptual knowledge, and procedural flexibility),[32] and two studies found negative or mixed effects.[33] Three of the studies with positive effects examined the effects of studying different solutions—presented side by side—to the same solved problem, compared to students that studied solved problems with methods presented on different pages, or to students that studied two different solved problems solved using the same solution method.[34] The fourth study found positive effects of asking students to solve a problem one way and then solve it a second time using a different method or ordering of steps (compared to students asked to just solve a problem one way).[35] The two studies with mixed or negative results involved students with no prior knowledge of algebra, and they compared the use of multiple strategies to the use of just one strategy to solve a problem.[36] One study found negative effects on all three outcome domains for students with no knowledge of algebra, but found no significant effects in any domain for students who had attempted some algebra on the pretest.[37] The other study found a negative effect on procedural knowledge but a positive effect on procedural flexibility.[38] These findings indicate that teaching alternative algebraic strategies can improve achievement, especially procedural flexibility, once students have developed some procedural knowledge of algebra.

How to carry out this recommendation

1. Teach students to recognize and generate strategies for solving problems.

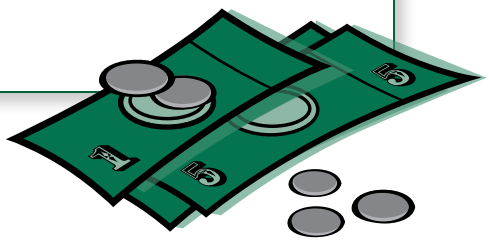
Provide students with examples that illustrate the use of multiple algebraic strategies. Include standard strategies that students commonly use, as well as alternative strategies that may be less obvious. Students can observe that strategies vary in their effectiveness and **efficiency** for solving a problem. Example 3.1 illustrates conventional and alternative solution strategies for several different solved problems. In the linear equations in this example, conventions might be to distribute for multiplication over addition or subtraction before applying properties of equality or to work with linear terms before working with constants.

Solved problems can demonstrate how the same problem could be solved with different strategies (Example 3.2) and how different problems could be solved with the same strategy (Example 3.3). These kinds of examples emphasize that strategies can be used flexibly; that is, students are not limited to using one strategy for a particular problem type, and similarly, problems that appear different on the surface may in some cases be solved using the same strategy. Label, compare, and provide mathematical justification for each solution step in these solved problems to illustrate how the strategies differ. Have students articulate the rationale behind a strategy to identify misconceptions and to ensure they understand that a strategy is more than a series of steps to memorize. Students also can discuss their ideas for alternative solution strategies with a partner.

Through facilitated whole-class discussion, students can articulate how different strategies can be used for the same problem, and whether certain strategies are appropriate or effective in solving a problem.

Example 3.1. Alternative and conventional solution strategies[39]

Solution via conventional method	Solution via alternative method
Evaluate $2a + 4b - 7a + 2b - 8a$ if $a = 1$ and $b = 7$	
$2a + 4b - 7a + 2b - 8a$ $2(1) + 4(7) - 7(1) + 2(7) - 8(1)$ $2 + 28 - 7 + 14 - 8$ 29	$2a + 4b - 7a + 2b - 8a$ $-13a + 6b$ $-13(1) + 6(7)$ 29
Our restaurant bill, including tax but before tip, was $16.00. If we wanted to leave exactly 15% tip, how much money should we leave in total?	
$16.00 * 1.15 = x$ $x = \$18.40$	10% of $16.00 is $1.60, and half of $1.60 is $0.80, which totals $2.40, so the total bill with tip would be $16.00 + $2.40 or $18.40.
Solve for x: $3(x + 1) = 15$	
$3(x + 1) = 15$ $3x + 3 = 15$ $3x = 12$ $x = 4$	$3(x + 1) = 15$ $x + 1 = 5$ $x = 4$
Solve for x: $7(x - 2) = 3(x - 2) + 16$	
$7(x - 2) = 3(x - 2) + 16$ $7x - 14 = 3x - 6 + 16$ $7x - 14 = 3x + 10$ $4x - 14 = 10$ $4x = 24$ $x = 6$	$7(x - 2) = 3(x - 2) + 16$ $4(x - 2) = 16$ $x - 2 = 4$ $x = 6$
Solve for x: $4(x - 2) + 2x + 10 = 2(3x + 1) + 4x + 8$	
$4(x - 2) + 2x + 10 = 2(3x + 1) + 4x + 8$ $4x - 8 + 2x + 10 = 6x + 2 + 4x + 8$ $6x + 2 = 10x + 10$ $2 = 4x + 10$ $-8 = 4x$ $-2 = x$	$4(x - 2) + 2x + 10 = 2(3x + 1) + 4x + 8$ $4(x - 2) + 2x + 2 = 2(3x + 1) + 4x$ $4x - 8 + 2x + 2 = 6x + 2 + 4x$ $6x - 6 = 10x + 2$ $-6 = 4x + 2$ $-8 = 4x$ $-2 = x$



Example 3.2. The same problem solved using two different solution strategies[40]

Strategy 1: Devon's solution—apply distributive property first	
Solution steps	**Labeled steps**
$10(y + 2) = 6(y + 2) + 16$	Distribute
$10y + 20 = 6y + 12 + 16$	Combine like terms
$10y + 20 = 6y + 28$	Subtract $6y$ from both sides
$4y + 20 = 28$	Subtract 20 from both sides
$4y = 8$	Divide by 4 on both sides
$y = 2$	

Strategy 2: Elena's solution—collect like terms first	
Solution steps	**Labeled steps**
$10(y + 2) = 6(y + 2) + 16$	Subtract $6(y + 2)$ on both sides
$4(y + 2) = 16$	Divide by 4 on both sides
$y + 2 = 4$	Subtract 2 from both sides
$y = 2$	

Prompts to accompany the comparison of problems, strategies, and solutions

- What similarities do you notice? What differences do you notice?
- To solve this problem, what did each person do first? Is that valid mathematically? Was that useful in this problem?
- What connections do you see between the two examples?
- How was Devon reasoning through the problem? How was Elena reasoning through the problem?
- What were they doing differently? How was their reasoning similar?
- Did they both get the correct solution?
- Will Devon's strategy always work? What about Elena's? Is there another reasonable strategy?
- Which strategy do you prefer? Why?

Example 3.3. Two different problems, each solved with the same strategy

Strategy: Apply distributive property first			
	Solution steps	**Labeled steps**	**Verify my solution**
Equation A	$2(x + 5) + 10 = 3x - 6$	Distribute	$2(26 + 5) + 10 = 3(26) - 6$
	$2x + 10 + 10 = 3x - 6$	Combine like terms	$2(31) + 10 = 78 - 6$
	$2x + 20 = 3x - 6$	Add 6 to both sides	$62 + 10 = 72$
	$2x + 26 = 3x$	Subtract $2x$ from both sides	$72 = 72$ ✓
	$26 = x$		
	Solution steps	**Labeled steps**	**Verify my solution**
Equation B	$3(2x - 5) + 3 = \frac{1}{2}(x + 8) + 6$	Distribute	$3(2x - 5) + 3 = \frac{1}{2}(x + 8) + 6$
	$6x - 15 + 3 = \frac{1}{2}x + 4 + 6$	Combine like terms	$3[2(4) - 5] + 3 = \frac{1}{2}(4 + 8) + 6$
	$6x - 12 = \frac{1}{2}x + 10$	Subtract 10 from both sides	$3(8 - 5) + 3 = (12) + 6$
	$6x - 22 = \frac{1}{2}x$	Multiply both sides by 2	$3(3) + 3 = 6 + 6$
	$12x - 44 = x$	Subtract $12x$ from both sides	$9 + 3 = 12$
	$-44 = -11x$	Divide both sides by -11	$12 = 12$ ✓
	$4 = x$		

To avoid overwhelming students, introduce one or two different solution strategies at a time. Show the class work from past or current students to demonstrate examples of students using multiple strategies. Students who are struggling can be overwhelmed by the rapid introduction of multiple strategies. Before introducing a new strategy, provide sufficient time to practice the strategies they already know and ensure students understand when to use each strategy.

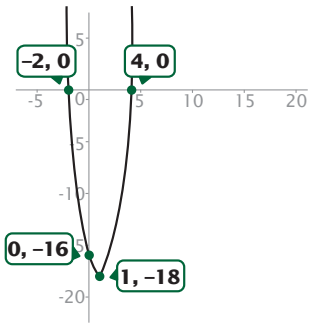
After students are introduced to different strategies, help them develop skills for selecting which strategy to use. Present a problem during whole-class instruction, and ask students to write down questions they might ask themselves to determine appropriate solution strategies. Alternatively, provide a list of questions that students may ask themselves (Example 3.4). Students can then draw on this list of reflective questions as they work independently or in small groups. Teachers can also provide students with model answers to the questions in Example 3.4. First, teachers can present students with a problem and corresponding answers to each of the questions as they relate to the problem. Once students have examined these answers, teachers can present a new problem and ask students to go through the same exercise to answer the questions from Example 3.4 on their own.

After students find a solution to a problem, challenge them to solve the problem in another way and to generate additional strategies during group work and independent practice. Use tables similar to those in Examples 3.5 and 3.6 to demonstrate a few strategies that may be useful for solving quadratic equations and linear equations, respectively. Present alternative strategies to students after they become comfortable with a few standard strategies. Emphasize that not all strategies are appropriate for all problems; different strategies have advantages and disadvantages, depending on the problem. If students run into roadblocks or challenges when solving a problem using one strategy, encourage students to try a different strategy.

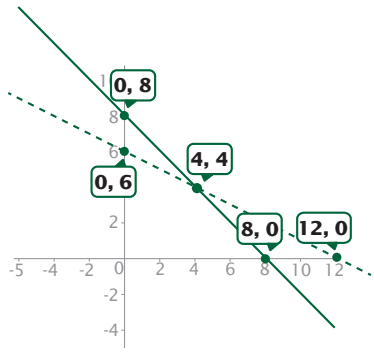
Example 3.4. Reflective questions for selecting and considering solution strategies

- What strategies could I use to solve this problem? How many possible strategies are there?
- Of the strategies I know, which seem to best fit this particular problem? Why?
- Is there anything special about this problem that suggests that a particular strategy is or is not applicable or a good idea?
- Why did I choose this strategy to solve this problem?
- Could I use another strategy to check my answer? Is that strategy sufficiently different from the one I originally used?

Example 3.5. Some possible strategies for solving quadratic equations

Equation *(Solution strategy)*	**Solution steps**	**Notes about strategies**
$x^2 - x - 12 = 0$ *(Factoring)*	$x^2 - x - 12 = 0$ $(x + 3)(x - 4) = 0$ $x = -3 \qquad x = 4$	Factoring is often done by guessing and checking, which can be time consuming, depending on the problem.
$2x^2 - 4x - 16 = 0$ *(Graph and find intercepts)*	$2x^2 - 4x - 16 = 0$ $x = -2 \qquad x = 4$  	Graphing can be done by hand or by using technology, and the choice might depend on such things as whether the intercepts are known to be integers or irrational numbers.
$x^2 - 6x + 5 = 0$ *(Make a table)*	$x^2 - 6x + 5 = 0$ $x \quad x^2 - 6x + 5$ 0 5 1 0 2 –3 3 –4 4 –3 5 0	Tables may be useful in illustrating how changes in one variable are related to changes in the other variable. Starting by substituting 0 and then 1 for x allows for simple calculations. Knowing that the value of $x^2 - 6x + 5$ decreases then increases as the value of x increases suggests that we should continue by substituting 2 for x. Symmetry in the values becomes apparent after substituting 4 for x, leading to the discovery of the second zero.
$0 = -x^2 + 4x + 5$ *(Complete the square)*	$0 = -x^2 + 4x + 5$ $0 = -x^2 + 4x - 4 + 4 + 5$ $0 = -(x^2 - 4x + 4) + 9$ $0 = -(x - 2)^2 + 9$ $(x - 2)^2 = 9$ $x - 2 = \pm 3$ $x = 5 \qquad x = -1$	This strategy always works, though it might be cumbersome. Although completing the square may be a relatively complex algebraic process, this method is very useful in helping students notice the mathematical structure that unites quadratic equations with squared quantities.
$(x + 7)^2 = 25$ *(Take the square root)*	$(x + 7)^2 = 25$ $x + 7 = \pm 5$ $x = -12 \qquad x = -2$	As with completing the square, awareness of this strategy can reinforce the structure of quadratics. The steps taken in this strategy parallel a subset of the steps used to complete the square.
$2x^2 + 7x + 6 = 0$ *(Use the quadratic formula)*	$2x^2 + 7x + 6 = 0$ $ax^2 + bx + c = 0$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \dfrac{-7 \pm \sqrt{7^2 - 4(2)(6)}}{2(2)}$ $x = \dfrac{-7 \pm 1}{4}$ $x = \dfrac{-8}{4} = -2 \qquad x = \dfrac{-6}{4} = \dfrac{-3}{2}$	Structurally, awareness of the connection between the quadratic formula and the completing the square method is key. The quadratic formula applies to an equation of the form $ax^2 + bx + c = 0$. When equations are presented in a different form, it is necessary to decide whether to rewrite the equation in this form or use a different strategy.

Example 3.6. Some possible strategies for solving linear systems[41]

Problem statement	Solution strategy	Solution steps	Notes about strategies
$5x + 10y = 60$ $x + y = 8$	Graph using x- and y-intercepts	$5x + 10y = 60$ (12, 0) (0, 6) $x + y = 8$ (8, 0) (0, 8)  	The x- and y-intercepts are integers and easy to find in these two equations, so graphing by hand to find the point of intersection might be a good strategy to use.
$-2x + y = 7$ $x = 6y + 2$	Substitution	$-2x + y = 7$ $x = 6y + 2$ $-2(6y + 2) + y = 7$ $-12y - 4 + y = 7$ $-11y = 11$ $y = -1$	Because one of the equations in this system is already written in the form of "$x =$", it makes sense to use the substitution strategy.
$2x + y = 6$ $x - y = 9$	Elimination	$2x + y = 6$ $\underline{x - y = 9}$ $3x = 15$ $x = 5$ $2(5) + y = 6$ $y = -4$	Because the coefficients of the y terms are equal in absolute value but have opposite signs, the strategy of elimination may be a natural fit for this system.
$3x - 2y = 7$ $4x + 3y = 5$	Matrices (Cramer's rule)	$\begin{vmatrix} 3 & -2 \\ 4 & 3 \end{vmatrix} \times \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 7 \\ 5 \end{vmatrix}$	Matrices may be useful for more complicated linear systems, such as with three equations and three unknowns.
$y = 100 + 4x$ $y = 25 + 7x$	Properties of equality	$y = 100 + 4x$ $y = 25 + 7x$ $100 + 4x = 25 + 7x$ $75 = 3x$ $25 = x$	Since both equations are in the form of "$y =$", it would be logical to set the two expressions in x equal to each other and solve for x.

2. Encourage students to articulate the reasoning behind their choice of strategy and the mathematical validity of their strategy when solving problems.

Have students describe their reasoning while analyzing the problem structure, determining their solution strategy, solving a problem, and analyzing another student's solution.

Describing their reasoning helps students understand the choices they make and goals they set when selecting a strategy. Students should communicate their reasoning verbally

and through written work. Ask students working independently to write out their strategic reasoning in addition to solving the problem. Ask students working in pairs or small groups to discuss their solution strategies and explain their mathematical reasoning for why a strategy works (i.e., its mathematical validity). Provide a list of prompts, such as those in Example 3.7, to accompany practice problems or initiate small-group discussion to encourage students to articulate their reasoning.

When initially introducing group activities, model how to work with a partner to discuss potential strategies, how to label the steps of each strategy, and how to explain the similarities and differences observed between strategies. (Example 3.8 presents a sample dialogue discussing Devon and Elena's solution strategies from Example 3.2.) After students become more accustomed to these tasks, they can work to complete them independently.

Example 3.7. Prompts to encourage students to articulate their reasoning

- What did you notice first about the problem structure? How did that influence your solution strategy? What strategy is appropriate for solving this problem, and why?
- What choices did you have to make in solving this problem?
- What goal were you trying to achieve?
- How did you get your answer? How do you know it is correct?
- Describe to another student how to solve this problem.
- What was most difficult about this problem? Did you run into any challenges? If so, what did you do to overcome them?

Example 3.8. Sample student dialogue discussing two different solution strategies[42]

Divide students into small groups to discuss two solved problems. Instruct students to discuss each solution, check each answer, and explain which solution strategy they would select and why. In the following sample dialogue, two students, Ben and Krista, discuss Devon and Elena's solution strategies from Example 3.2.

Krista: What did Elena do?

Ben: She subtracted $6(y + 2)$ as one whole term, so she was left with $4(y + 2)$ on one side of the equation. She subtracted $6(y + 2)$ from $10(y + 2)$ to get $4(y + 2)$.

Krista: Yeah, Elena didn't distribute.

Ben: Then Elena divided by 4 on both sides and then subtracted 2 on both sides.

Krista: It looks like Devon distributed first and then combined like terms before subtracting on both sides.

Ben: They both got the same answer. Devon just did a few extra steps, I believe.

Krista: They both did the problem correctly, but they just did it in different ways.

Ben: Devon distributed…

Krista: …and combined, while Elena subtracted like terms. Both Elena and Devon basically did the same steps after that, but just in a different order.

Ben: Elena's strategy is quicker and more efficient.

Teacher: What relationship between the steps of Devon's strategy and those of Elena's strategy helps explain why Elena's strategy is more efficient?

Krista: Devon's first step of distributing the 10 and 6 made the expressions look less complicated, but it didn't eliminate the need to subtract twice and divide once.

3. Have students evaluate and compare different strategies for solving problems.

Encourage students to compare problem structures and solution strategies to discover the relationships among similar and different problems, strategies, and solutions. Begin comparison activities after students understand one strategy, so that students are able to identify similarities and differences between the familiar strategy and the newly learned strategy. Teachers can incorporate prompts such those presented in Examples 3.1, 3.3, and 3.7 into whole-class discussion or independent practice activities that encourage students to explain why they might choose one solution strategy over another. Comparison activities can be carried into small-group discussions, as illustrated in Example 3.9.

Use solved problems showing two strategies side by side to enable students to see the number, type, and sequence of solution steps. This will allow them to compare solution strategies and consider the accuracy, efficiency, and applicability of various combinations of solution steps. When presenting pairs of solved problems to communicate a particular instructional goal to students, use solved problems that are moderately similar to each other (as opposed to highly different from each other) to help students look past the surface features of the problems and instead focus on the underlying solution structure. By visually aligning the steps of different solution strategies, students can make connections among strategies and analyze their similarities and differences. Teachers can initially provide reasoning for the steps of solution strategies and later ask students to label the steps of the strategies they use.

Example 3.9. Small-group comparison and discussion activity

Objectives:

- ☑ Share and compare multiple solution strategies
- ☑ Use precise mathematical language to describe solution steps
- ☑ Explain reasoning and mathematical validity

Directions: Pair students off to work on algebra problems so that students with different strategies have the opportunity to talk with each other. For example, if two strategies are prevalent and approximately half of the students use each, students may be put into groups A and B based on like strategies and then each paired with a student from the other group. Partners can discuss the strategies they used to solve the first problem (e.g., What strategy did each person use? How did the strategies differ from one another? What was the partner's rationale for using a different strategy? Did both strategies produce the same answer?). Challenge students to use their partner's strategy when solving the next problem. Conclude the activity by asking students to reflect on what they discussed with their partners, explaining the most important ways in which the two strategies differ. Have students record the strategies discussed by the class.

Potential roadblocks and suggested approaches

Roadblock 3.1. *I'm worried about confusing my students by teaching them multiple strategies for solving a problem. They have a hard enough time learning one strategy! Isn't it easier for them to master one strategy for solving algebra problems?*

Suggested Approach. The goal in teaching alternative strategies is not to have students master all strategies, but rather to have students compare them in order to develop a deeper understanding of the strategy they choose to use. With this understanding, they can use their selected strategy flexibly to solve problems they have not seen before. Multiple strategies also do not need to be taught all at once. Teachers may introduce one strategy and give students time to become comfortable with it before introducing alternative strategies in a structured way to emphasize the comparison of strategies to develop student understanding.

Although one student may grasp a particular strategy very well, another student may find that strategy confusing or unclear. Seeing alternative strategies provides students with more than one way to think about solutions to algebra problems, moving beyond rote memorization and toward a deeper conceptual understanding of the problem, the thought process required to solve the problem, and the reasons why particular solution strategies work. Note that this recommendation is not advocating that students be fluent in all possible strategies for solving a given problem type. However, seeing alternative strategies provides students with different options that they can choose from when they encounter a familiar or unfamiliar problem, and the flexibility in choice can lead to greater success, confidence, and knowledge in algebra.

Students naturally encounter new strategies as their classmates offer ideas about how to solve problems. Including alternative strategies in the classroom conversation allows students to see ways to evaluate and use the alternatives they encounter. Seeing several solution strategies can help students who are struggling in algebra choose a strategy they feel comfortable with, understand, and may be able to better remember, as opposed to relying on a single strategy that has been memorized by rote and may be easily forgotten.

Roadblock 3.2. *My special-education students need a very structured process for solving algebra problems. Introducing multiple strategies and asking students to choose among strategies might be hard on them.*

Suggested Approach. Special-education students may require explicit instruction in how to solve problems, but it is important to distinguish between providing explicit instruction and teaching only a single solution strategy and asking students to memorize the steps of that strategy. Students are better served if they come to view mathematics *not* as a game like "Memory" in which they associate a problem with a specific method. This view of mathematics places even greater demands on students' memory—given the number of problem types and methods to be memorized—and thus presents particular challenges for students in special education. Teachers can help special-education students understand alternative strategies by being explicit not only about the steps of a strategy but also about its underlying rationale, including how, when, and why it is applicable or useful for particular problems.

Roadblock 3.3. *I can't seem to teach my students multiple strategies without them becoming confused. I presented and compared five algebra strategies for solving quadratic equations during one class period, and my students didn't understand. What should I do?*

Suggested Approach. Comparing several strategies at once can overwhelm students who are trying to learn many new concepts at one time. Teachers can begin simply—by introducing one strategy to students. After students are comfortable with this strategy, teachers may compare that strategy with one or two other new strategies for the purpose of deepening student understanding of the underlying mathematics. Additional strategies may be introduced if they are useful as students are ready. Often just one alternative strategy will help students acquire a strategic approach to problem solving. Reviewing multiple strategies at once may be useful prior to assessments to remind students of the many solution strategies available to them.

Roadblock 3.4. *Teaching students to use and compare multiple strategies requires knowledge of many strategies, and our textbook presents only one strategy.*

Suggested Approach. Examples 3.5 and 3.6 list some of the multiple solution strategies available for solving quadratic equations and linear equations. Devoting time to discussing multiple strategies in professional learning communities enables and supports teachers in deepening their own understanding of strategies. Teachers may also learn new strategies from students, who frequently develop their own strategies. Teachers may choose to post different strategies in the classroom or create handouts for students to reference at their desks.

Roadblock 3.5. *How can I stay on schedule teaching everything required to meet state standards and still have time to teach students to use multiple strategies?*

Suggested Approach. Teaching multiple strategies is not about adding additional content to teachers' lesson plans or making time for students to work on each strategy repeatedly. Instead, the focus is on teaching students how to think about an algebra problem and recognize when an alternative strategy may be appropriate. Encouraging these types of skills can happen within the existing framework of a teacher's lesson plan, instead of placing any additional burden on teachers.

Teaching students alternative strategies creates a strong foundation of reasoning skills as students learn to select appropriate solution methods based on the problems they encounter. Teaching students multiple strategies equips them with an array of tools to use to master state standards.

Glossary

A

Abstract reasoning is processing and analyzing complex, non-concrete concepts.

An **algebraic expression** is a symbol or combination of symbols for variables, numbers, and arithmetic operations used to represent a quantity. Examples of algebraic expressions are $9 - a^2$ and $3x - 4y + 7$.

An **algebraic representation** is a way to display an algebraic relationship or quantity. Algebraic representations can have symbolic, numeric, verbal, or graphic forms.

An **algorithm** is a series of steps that, when executed in the prescribed order, lead to the desired or expected outcome.

E

Strategies can be evaluated on the basis of their **efficiency**, when a more efficient strategy is one that can be executed relatively easily, quickly, and/or without error, as compared to another strategy that may be more difficult or more tedious to implement and/or that may be more likely to lead to error. Determining which strategy is the most efficient for solving a problem may depend on the problem's structure, the features of the strategy, and the knowledge and goals of the person implementing the strategy.

Evidence-based practices, policies, or recommendations are those that are supported by studies that meet WWC design standards with or without reservations.

F

Flexibility includes the knowledge of a range of strategies for solving a given problem and the ability to select the most appropriate strategy for that problem. An appropriate strategy may be the most efficient for the problem and/or the one that is the best match for the problem's features and structure.

I

An **incomplete solved problem** for instructional purposes is a solved problem or worked example that shows both the problem and some of the steps taken to solve the problem. One or more steps are left blank for students to complete.

An **incorrect solved problem** for instructional purposes is a solved problem or worked example that shows both the problem and the steps taken to try to solve the problem, with at least one error in the steps, so that a correct outcome is not reached. An error could arise from using the wrong strategy to solve the problem, using improper mathematical reasoning, or making a computational or strategic error.

M

Mathematical validity is a characteristic of a solution or the mathematical reasoning that logically flows from valid assumptions; essentially, it is *why* a strategy works.

P

Precise mathematical language can convey the underlying logical, accurate meaning of algebraic structure, operations, solution steps, and strategies.

A **procedural** or **computational error** is an error in applying a procedure and can occur when a step of a procedure is not executed correctly, such as combining like terms incorrectly or incorrectly applying the order of operations.

R

Reflective questioning is a process in which students ask themselves relevant questions as they work to solve a problem.

S

The **solution** set to an equation is the set of values for which the equation is true. The **solution steps** illustrate the process used to reach the solution.

A **solved problem** for instructional purposes is an example that shows both the problem and the steps used to reach the problem's solution. A solved problem is also referred to as a "worked example."

A **strategic error** is a mistake that occurs because of incorrect reasoning, a misunderstanding of mathematical structure, or an incorrect choice of strategy.

A **strategy** comprises the overarching choices made and steps taken in attempting to solve a problem. A strategy may include sequences of steps to be executed, as well as the rationale behind the use and effectiveness of these steps.

Structure refers to the underlying mathematical features and relationships of an expression, representation, or equation. Structure includes quantities, variables, arithmetic operations, and relationships among these things (including equality and inequality). For example, "the difference of two squares" is one way to describe the underlying structure of $a^2 - 9$.

T

Think-aloud questions are questions that guide students to verbalize their internal thinking. They can be used to help students analyze solved problems.

Appendix A

Postscript from the Institute of Education Sciences

What is a practice guide?

The Institute of Education Sciences (IES) publishes practice guides to share evidence and expert guidance on addressing education-related challenges not readily solved with a single program, policy, or practice. Each practice guide's panel of experts develops recommendations for a coherent approach to a multifaceted problem. Each recommendation is explicitly connected to supporting evidence. Using What Works Clearinghouse (WWC) design standards, the supporting evidence is rated to reflect how well the research demonstrates the effectiveness of the recommended practices. Strong evidence means positive findings are demonstrated in multiple well-designed, well-executed studies, leaving little or no doubt that the positive effects are caused by the recommended practice. Moderate evidence means well-designed studies show positive impacts, but there are questions about whether the findings can be generalized beyond the study samples or whether the studies definitively show evidence that the practice is effective. Minimal evidence means that there is not definitive evidence that the recommended practice is effective in improving the outcome of interest, although there may be data to suggest a correlation between the practice and the outcome of interest. (See Table A.1 for more details on levels of evidence.)

How are practice guides developed?

To produce a practice guide, IES first selects a topic. Topic selection is informed by inquiries and requests to the WWC Help Desk, a limited literature search, and evaluation of the topic's evidence base. Next, IES recruits a panel chair who has a national reputation and expertise in the topic. The chair, working with IES and WWC staff, then selects panelists to help develop the guide. Panelists are selected based on their expertise in the topic area and the belief that they can work together to develop relevant, evidence-based recommendations. Panels include two practitioners with expertise in the topic.

Relevant studies are identified through panel recommendations and a systematic literature search. These studies are then reviewed using the WWC design standards by certified reviewers who rate each effectiveness study. The panel synthesizes the evidence into recommendations. WWC staff summarize the research and help draft the practice guide.

IES practice guides are then subjected to external peer review. This review is done independently of the IES staff that supported the development of the guide. A critical task of the peer reviewers of a practice guide is to determine whether the evidence cited in support of particular recommendations is up-to-date and that studies of similar or better quality that point in a different direction have not been overlooked. Peer reviewers also evaluate whether the level of evidence category assigned to each recommendation is appropriate. After the review, a practice guide is revised to meet any concerns of the reviewers and to gain the approval of the standards and review staff at IES.

Institute of Education Sciences levels of evidence for What Works Clearinghouse practice guides

This section provides information about the role of evidence in IES's WWC practice guides. It describes how practice guide panels determine the level of evidence for each recommendation and explains the criteria for each of the three levels of evidence (strong evidence, moderate evidence, and minimal evidence).

The level of evidence assigned to each recommendation in this practice guide represents the panel's judgment of the quality of the existing research to support a claim that, when these practices were implemented in past research, positive effects were observed on student outcomes. After careful review of

the studies supporting each recommendation, panelists determine the level of evidence for each recommendation using the criteria in Table A.1. The panel first considers the relevance of individual studies to the recommendation and then discusses the entire evidence base, taking the following into consideration:

- the number of studies
- the study designs
- the internal validity of the studies
- whether the studies represent the range of participants and settings on which the recommendation is focused
- whether findings from the studies can be attributed to the recommended practice
- whether findings in the studies are consistently positive

A rating of *strong evidence* refers to consistent evidence that the recommended strategies, programs, or practices improve student outcomes for a diverse population of students.[43] In other words, there is strong causal and generalizable evidence.

A rating of *moderate evidence* refers either to evidence from studies that allow strong causal conclusions but cannot be generalized with assurance to the population on which a recommendation is focused (perhaps because the findings have not been widely replicated) or to evidence from studies that are generalizable but have some causal ambiguity. It also might be that the studies that exist do not specifically examine the outcomes of interest in the practice guide, although the studies may be related to the recommendation.

A rating of *minimal evidence* suggests that the panel cannot point to a body of evidence that demonstrates the practice's positive effect on student achievement. In some cases, this simply means that the recommended practices would be difficult to study in a rigorous, experimental fashion;[44] in other cases, it means that researchers have not yet studied this practice, or that there is weak or conflicting evidence of effectiveness. A minimal evidence rating does not indicate that the recommendation is any less important than other recommendations with a strong or moderate evidence rating.

In developing the levels of evidence, the panel considers each of the criteria in Table A.1. The level of evidence rating is determined by the lowest rating achieved for any individual criterion. Thus, for a recommendation to get a strong rating, the research must be rated as strong on each criterion. If at least one criterion receives a rating of moderate and none receives a rating of minimal, then the level of evidence is determined to be moderate. If one or more criteria receive a rating of minimal, then the level of evidence is determined to be minimal.

The panel relied on WWC design standards to assess the quality of evidence supporting education programs and practices. The WWC evaluates evidence for the causal validity of instructional programs and practices according to WWC design standards. Information about these standards is available at http://whatworks.ed.gov. Eligible studies that meet WWC designs standards without reservations or meet WWC design standards with reservations are indicated by **bold text** in the endnotes and references pages.

A final note about IES practice guides

In policy and other arenas, expert panels typically try to build a consensus, forging statements that all its members endorse. Practice guides do more than find common ground; they create a list of actionable recommendations. Where research clearly shows which practices are effective, the panelists use this evidence to guide their recommendations. However, in some cases research does not provide a clear indication of what works. In these cases, the panelists' interpretation of the existing (but incomplete) research plays an important role in guiding the recommendations. As a result, it is possible that two teams of recognized experts working

independently to produce a practice guide on the same topic would come to very different conclusions. Those who use the guides should recognize that the recommendations represent, in effect, the advice of consultants. However, the advice might be better than what a school or district could obtain on its own. Practice guide authors are nationally recognized experts who collectively endorse the recommendations, justify their choices with supporting evidence, and face rigorous independent peer review of their conclusions. Schools and districts would likely not find such a comprehensive approach when seeking the advice of individual consultants.

Institute of Education Sciences

Table A.1. Institute of Education Sciences levels of evidence for What Works Clearinghouse practice guides

Criteria	STRONG Evidence Base	MODERATE Evidence Base	MINIMAL Evidence Base
Validity	High internal validity (high-quality causal designs). Studies must meet WWC design standards with or without reservations.[45] AND High external validity (requires multiple studies with high-quality causal designs that represent the population on which the recommendation is focused). Studies must meet WWC design standards with or without reservations.	High internal validity but moderate external validity (i.e., studies that support strong causal conclusions but generalization is uncertain). OR High external validity but moderate internal validity (i.e., studies that support the generality of a relation but the causality is uncertain).[46]	The research may include evidence from studies that do not meet the criteria for moderate or strong evidence (e.g., case studies, qualitative research).
Effects on relevant outcomes	Consistent positive effects without contradictory evidence (i.e., no statistically significant negative effects) in studies with high internal validity.	A preponderance of evidence of positive effects. Contradictory evidence (i.e., statistically significant negative effects) must be discussed by the panel and considered with regard to relevance to the scope of the guide and intensity of the recommendation as a component of the intervention evaluated.	There may be weak or contradictory evidence of effects.
Relevance to scope	Direct relevance to scope (i.e., ecological validity)—relevant context (e.g., classroom vs. laboratory), sample (e.g., age and characteristics), and outcomes evaluated.	Relevance to scope (ecological validity) may vary, including relevant context (e.g., classroom vs. laboratory), sample (e.g., age and characteristics), and outcomes evaluated. At least some research is directly relevant to scope (but the research that is relevant to scope does not qualify as strong with respect to validity).	The research may be out of the scope of the practice guide.
Relationship between research and recommendations	Direct test of the recommendation in the studies or the recommendation is a major component of the intervention tested in the studies.	Intensity of the recommendation as a component of the interventions evaluated in the studies may vary.	Studies for which the intensity of the recommendation as a component of the interventions evaluated in the studies is low; and/or the recommendation reflects expert opinion based on reasonable extrapolations from research.
Panel confidence	The panel has a high degree of confidence that this practice is effective.	The panel determines that the research does not rise to the level of strong but is more compelling than a minimal level of evidence. The panel may not be confident about whether the research has effectively controlled for other explanations or whether the practice would be effective in most or all contexts.	In the panel's opinion, the recommendation must be addressed as part of the practice guide; however, the panel cannot point to a body of research that rises to the level of moderate or strong.
Role of expert opinion	Not applicable	Not applicable	Expert opinion based on defensible interpretations of theory (theories). (In some cases, this simply means that the recommended practices would be difficult to study in a rigorous, experimental fashion; in other cases, it means that researchers have not yet studied this practice.)
When assessment is the focus of the recommendation	For assessments, meets the standards of *The Standards for Educational and Psychological Testing*.[47]	For assessments, evidence of reliability that meets *The Standards for Educational and Psychological Testing* but with evidence of validity from samples not adequately representative of the population on which the recommendation is focused.	Not applicable

Appendix B

About the Authors

Panelists

Jon R. Star, Ph.D., (chair) is the Nancy Pforzheimer Aronson Associate Professor in Human Development and Education in the Graduate School of Education at Harvard University. A former middle school and high school mathematics teacher, his current research focuses on the learning and teaching of mathematics, particularly algebra. Dr. Star's work has been supported by several grants from the Institute of Education Sciences (IES) and from the National Science Foundation. He is the author of more than 50 peer-reviewed articles, many of which have been published in leading journals in mathematics education and educational psychology. In addition, Dr. Star has served as an author on two prior IES practice guides (*Encouraging Girls in Math and Science* and *Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools*). He currently serves as an editor at the *Elementary School Journal* and *Cognition and Instruction*. Dr. Star is also an experienced teacher of in-service and pre-service mathematics teachers.

Anne Foegen, Ph.D., is a professor of special education and the Director of Graduate Education in the School of Education at Iowa State University. Dr. Foegen's research has explored the development and implementation of progress-monitoring assessments in mathematics for students with, or at risk for, disabilities. Her most recent work has focused on measures to reflect student progress during initial algebra instruction at the middle school and high school levels. Dr. Foegen has also conducted research on the development and use of mathematics progress-monitoring measures for the primary grades and for middle school mathematics. She has been principal investigator on three U.S. Department of Education grants and sub-award principal investigator on a fourth, all of which have supported her work in mathematics progress monitoring.

Matthew R. Larson, Ph.D., is the K–12 mathematics curriculum specialist for the Lincoln Public Schools in Lincoln, NE. He has served on the board of directors for the National Council of Teachers of Mathematics (NCTM), as past chair of NCTM's research committee, and as a member of NCTM's task force on linking research and practice. He is the author or coauthor of numerous journal articles, professional books, and K–12 mathematics programs. He has taught mathematics at the secondary and college levels.

William G. McCallum, Ph.D., is a University Distinguished Professor of Mathematics at the University of Arizona. He received his Ph.D. in mathematics from Harvard University in 1984, under the supervision of Barry Mazur. After spending two years at the University of California, Berkeley, and one at the Mathematical Sciences Research Institute in Berkeley, he joined the faculty at the University of Arizona in 1987. In 1989, he joined the Harvard calculus consortium, and he is the lead author of the consortium's multivariable calculus and college algebra texts. In 1993/94, he spent a year at the Institut des Hautes Études Scientifiques, and in 1995/96 he spent a year at the Institute for Advanced Study on a Centennial Fellowship from the American Mathematical Society. He received the Director's Award for Distinguished Teaching Scholars from the National Science Foundation in 2005. In 2006, he founded the Institute for Mathematics and Education at the University of Arizona, and he is currently its director. He was one of the lead writers for the Common Core State Standards in Mathematics. His professional interests include arithmetical algebraic geometry and mathematics education. He has received grants and written articles, essays, and books in both areas.

Jane Porath (Peterson) has been teaching secondary mathematics for 18 years. She is currently an 8th-grade mathematics teacher at Traverse City East Middle School in Traverse City, Michigan. In 2013, Jane was selected as a state finalist for the Presidential Award of Excellence of Mathematics and Science Teaching. In 2012, Jane was named the Community Champion of Traverse City for bringing the Annual Conference of the Michigan Council of Teachers of Mathematics (MCTM) to the region. She served on the executive board for the MCTM and currently serves as its publications director. She has been involved professionally with the Connected Mathematics Project 3 (CMP3) for most of her career. She has piloted the curriculum,

worked on the development of the assessments for CMP3, and provided professional development around the country to teachers implementing the curriculum. Through this work, she has focused extensively on pedagogy and helping students make sense of mathematics. Jane was also an adjunct faculty member for Central Michigan University, teaching physical science to pre-service elementary teachers in Traverse City. Jane is involved professionally with the National Council of Teachers of Mathematics (NCTM). She is currently serving on the board of directors and the program committee for the 2013 NCTM Louisville Kentucky Regional Conference. Jane also served on the editorial panel for *Mathematics Teaching in the Middle School* as a member and chair from 2008 to 2010. Jane authored a book chapter in *A Decade of Middle School Mathematics Curriculum Implementation: Lessons Learned from the Show-Me Project* in 2007.

Rose Mary Zbiek, Ph.D., is a professor of mathematics education at the Pennsylvania State University and is the Director of Graduate Studies in Curriculum and Instruction. Dr. Zbiek's research has focused on the use of technology in mathematics learning and teaching, relationships between teacher knowledge and classroom practice, and the roles of mathematical modeling and technology in secondary school curricula in general and algebra curricula in particular. Her work includes the development and evaluation of mathematics curriculum materials that capitalize on the use of technology and engage students in mathematical modeling for the learning of curricular mathematics. Her current research investigates relationships among mathematical processes, technological expertise, and classroom practice of beginning and experienced secondary school mathematics teachers. Dr. Zbiek has served as principal investigator or co–principal investigator for grants from the National Science Foundation and corporate foundations and has received recognition for classroom-based research and the integration of research, teaching, and service. In addition to having been a member of editorial teams for both research and practice publications, she is the series editor of the National Council of Teachers of Mathematics 16-volume *Essential Understanding* series and is a past associate editor of the *Journal for Research in Mathematics Education*.

Staff

Pia Caronongan, Ed.D., is a researcher at Mathematica Policy Research. She oversaw efforts to assess the rigor and relevance of studies related to the panel's recommendations. Outside of the What Works Clearinghouse, Dr. Caronongan focuses on early childhood programs and policies and has played key roles on studies of child-care quality rating systems, home-visiting programs, universal preschool, and Early Head Start, among others.

Joshua Furgeson, Ph.D., is a senior researcher at Mathematica Policy Research and a former high school teacher. He is the lead methodologist for the special education topic areas in the What Works Clearinghouse, and he leads the practice guides under Mathematica's contract. He oversaw efforts to assess the rigor and relevance of studies related to the panel's recommendations for the *Improving Mathematical Problem Solving in Grades 4 Through 8* practice guide.

Betsy Keating, M.P.P., is a research analyst at Mathematica Policy Research. She helped coordinate the review of studies related to the panel's recommendations and helped draft the text of the practice guide. Ms. Keating is certified as a What Works Clearinghouse reviewer. She has worked on various other projects focused on education, including an implementation study of district-charter collaboration grants and a pilot study of teacher-effectiveness ratings in Pennsylvania.

Julia Lyskawa, M.P.P., is a research analyst at Mathematica Policy Research. She led the panel's efforts to translate research findings into practitioner-friendly text. Ms. Lyskawa also contributed to the *Teaching Math to Young Children* practice guide and summary. She is certified as a What Works Clearinghouse reviewer. She has worked on many other education-focused projects, including an evaluation of child outcomes and coaching methods in the Los Angeles Universal Preschool Program, a feasibility study of extending learning time in elementary schools, and a redesign of the Head Start Family and Child Experiences Survey.

Appendix C

Disclosure of Potential Conflicts of Interest

Practice guide panels are composed of individuals who are nationally recognized experts on the topics about which they are making recommendations. IES expects the experts to be involved professionally in a variety of matters that relate to their work as a panel. Panel members are asked to disclose these professional activities and institute deliberative processes that encourage critical examination of their views as they relate to the content of the practice guide. The potential influence of the panel members' professional activities is further muted by the requirement that they ground their recommendations in evidence that is documented in the practice guide. In addition, before all practice guides are published, they undergo an independent external peer review focusing on whether the evidence related to the recommendations in the guide has been presented appropriately.

The professional activities reported by each panel member that appear to be most closely associated with the panel recommendations are noted below.

Anne Foegen is the principal investigator on two current IES-funded research projects that involve the development and implementation of measures to monitor student progress in algebra. She provides in-person training for teachers to learn how to administer, score, and interpret the progress monitoring measures and data that are produced.

Matthew R. Larson is a K–12 textbook author with the Houghton Mifflin Harcourt Publishing Company based in Boston, MA. This includes coauthorship of a first- and second-year algebra textbook. He receives royalties and consulting compensation from this relationship.

Jane Porath provides professional development consulting services for Michigan State University related to the Connected Mathematics Project.

Jon R. Star is currently a paid consultant for both Scholastic and Pearson, two publishers who are developing algebra curricula that are intended to be aligned with the Common Core State Standards.

Appendix D

Rationale for Evidence Ratings[48]

The level of evidence rating is based on the findings of studies that examined the effectiveness of recommended practices and meet What Works Clearinghouse (WWC) design standards.[49] The studies were primarily identified through a keyword search of several databases. The search was limited to studies published between 1993 and 2013 that examined practices for teaching algebra to students in grades 6–12. The search was supplemented with studies recommended by the panel.

The search identified more than 2,800 studies, including 30 eligible group-design studies reviewed using WWC group design standards and four single-case design studies reviewed according to WWC pilot single-case design standards.[50] No single-case design studies met WWC single-case design standards. Fifteen group-design studies met evidence standards with or without reservations and tested interventions related to one or more recommendations. Study effects were calculated and classified as having a positive or negative effect when the result was either

- statistically significant ($p \leq 0.05$)[51] or
- substantively important as defined by the WWC[52]

Some studies met WWC group design standards but did not adjust statistical significance when there were multiple significance tests or when the unit of assignment was different from the unit of analysis ("clustering," for example, when classrooms are assigned to conditions but individual children's test scores are analyzed). In these cases, the WWC adjusted for clustering and multiple tests within a domain.[53]

The text and tables in this appendix focus on total or full-scale scores on the outcome closest to the end of the intervention; these are labeled *posttests*. All outcome measures administered after the posttest are described in table notes.

The review team for each study classified each outcome into one of three domains, consulting with the panel chair as needed to determine the proper domain. Table D.1 provides definitions and representative sample items from the reviewed studies for each outcome domain.

Table D.1. Outcome domain definitions and sample items

Domain	Definition	Sample Items
Conceptual knowledge	Understanding algebraic ideas, operations, procedures, and notation	▪ Identifying that $-5x + 6$ is equal to $6 - 5x$ and to $6 + (-5x)$ but not equal to $-6 + 5x$ [*examines student understanding of operations and notation; student does not solve the problem*][54] ▪ Translating text ("five less than a number") into an algebraic expression ($y - 5$) [*examines student understanding of notation; student does not solve the problem*][55] ▪ Understanding that the equations $98 = 21x$ and $98 + 2(x + 1) = 21x + 2(x + 1)$ are equivalent [*examines student understanding of the additive property of the equality principle; student does not solve the problem*]
Procedural knowledge	Choosing operations and procedures to solve algebra problems, as well as applying operations and procedures to arrive at the correct solutions to problems	▪ Solving $x^2 + 10x - 20 =$ for x [*examines student ability to solve the equation*] ▪ Solving $\frac{16}{x-7} = 2x$ for x [*examines student ability to solve the equation*][56]
Procedural flexibility	Identifying and implementing multiple methods to solve algebra problems, as well as choosing the most appropriate method	▪ Solving $98 + 2(x + 1) = 21x + 2(x + 1)$ using two different methods [*examines student ability to solve problems using multiple methods*]: — Distribute first on both sides and then solve for x — Eliminate like terms on both sides of the equation and then solve for x[57] ▪ Solving $2(x + 1) = 18$ using two different methods [*examines student ability to solve problems using multiple methods*]: — Distribute first on the left side of the equation and then solve for x — Divide both sides of the equation by 2 first and then solve for x[58]

Conceptual knowledge, procedural knowledge, and procedural flexibility are distinct competencies.[59] Mathematical proficiency results when children develop these and other competencies and form links between them. This guide groups outcomes into these three domains because, according to the National Mathematics Advisory Panel definition of *proficiency*, each domain represents an important ability or understanding.[60]

Though theoretically distinct, these competencies are often difficult to measure empirically. Even when outcomes in a study are classified into one domain, the outcomes might also measure broader knowledge or competencies beyond that domain. Some measures of conceptual and procedural knowledge adequately assess the specific competency, but studies have raised concerns about the validity and independence of several measures. For example, studies have found that conceptual and procedural knowledge measures may overlap, assess additional competencies, and can fail to adequately distinguish between the different types of knowledge.[61]

When studies have multiple posttest outcome measures administered within the same domain, effect sizes for each measure are averaged, and the overall average is reported in the tables. Findings for individual outcomes within each domain are described in table notes.

Recommendation 1. Use solved problems to engage students in analyzing algebraic reasoning and strategies.

Level of evidence: Minimal Evidence

WWC staff and the panel assigned a minimal level of evidence based on four studies that meet WWC group design standards without reservations and examined this practice (see Table D.2).[62] Although the studies collectively demonstrated strong internal validity, the level of evidence was rated as minimal because of limited generalizability. All studies finding a positive impact compare solved problems to additional practice problems; in the only study comparing solved problems to another intervention (meta-cognition, a practice listed in Recommendation 2), the authors find negative impacts.

Three of the four studies found positive effects of using solved problems, providing a preponderance of evidence of positive effects on conceptual knowledge.[63] To measure conceptual knowledge, these studies asked students to translate word problems into algebra equations or assessed whether participants could understand how the terms and operations in an expression can be written differently and still be equivalent (the latter measures understanding of the meaning of the expression features). One of these studies also examined effects on procedural knowledge (measured by whether students solved equations or algebra word problems correctly), and found neither positive nor negative effects.[64] No studies examined outcomes in the procedural flexibility domain. Each of the three studies was a randomized controlled trial with low sample attrition, resulting in high internal validity. The three studies finding positive effects examined the effect of solved problems without other intervention components, providing a direct test of the recommendation.

However, each of the three studies compared providing solved problems to having students solve practice problems, limiting generalization of the findings. Moreover, none of the interventions involved regular or sustained instruction with solved problems: two of the studies involved interventions that lasted for only one day or used one worksheet,[65] and the remaining study intervention involved eight solved problems.[66] Two studies compared students who independently studied correct solved problems alongside practice problems to students who were not provided with any solved problems and were instead provided with more practice problems.[67]

One study found that providing students with correct and incorrect solved problems and prompting them to describe the solution steps and explain why they were carried out had positive effects compared to students who were prompted similarly as they solved practice problems.[68] The interventions ranged in duration from one day to a month. In one of the studies, students completed the unit at their own pace, so the duration of the intervention varied by the individual (for example, most students completed the intervention within eight class sessions over a four-week period).[69]

The three studies included students of varying ability and were conducted in high schools across the United States (including the Midwest, West Coast and Mid-Atlantic regions). One of the studies included students in regular Algebra I classes;[70] one study included students from basic, regular, and honors algebra classes;[71] and one study included high school students enrolled in a remedial mathematics class with participants aged 14 to 17.[72] One study took place in a computer lab setting, with the intervention delivered via computer software.[73] The other two studies took place in regular classrooms, and involved worksheets.[74] For all three studies, the intervention occurred during scheduled algebra classes within the regular school day. The studies as a whole provide moderate external and ecological validity.

One randomized controlled trial that meets WWC group design standards without reservations found that solved problems had negative effects on conceptual and procedural knowledge.[75] The study compared students who studied and discussed solved problems in groups to students who used reflective questioning (a practice suggested in Recommendation 2) to solve practice problems in groups.[76] This negative finding indicates that the effectiveness of this practice can depend on what students do instead of using solved problems.

Table D.2. Studies providing evidence for Recommendation 1

Study and design	Participants	Setting	Intervention condition as implemented in the study	Comparison condition as implemented in the study	Outcome domain and effect size
Booth et al. (2013) *Randomized controlled trial*[a]	116 high school students in regular (not remedial or honors) algebra classes	3 high schools: one on the West Coast and two in the Mid-Atlantic region of the United States	Students completed the two-step equation unit of the Algebra I Cognitive Tutor software program, with 8 solved problems replacing typical guided-practice problems. Each of the solved problems illustrated either a correct or an incorrect example of solving a linear equation. Students were prompted to assess what was done in the solved problem and why the problem was incorrect. Students completed the unit at their own pace; the majority of students completed the unit within 8 class sessions (4 weeks).	Students completed the two-step equation unit of the Algebra I Cognitive Tutor software program, with only guided practice problems and no solved problems.	conceptual knowledge = 0.40
					procedural knowledge = 0.17[b]
Carroll (1994) *Randomized controlled trial*[c]	40 students, ages 15 to 17, enrolled in basic, regular, and honors algebra classes[d]	1 high school in a large school district in the Midwest region of the United States	Students received a 24-item worksheet as classwork. Half of the worksheet items were solved problems, and the worksheet presented a solved problem followed by a similar practice problem. The worksheet focused on translating mathematical phrases into linear equations or algebraic expressions. The solved problems linked mathematical phrases (e.g. "five less than a number") to algebraic expressions (e.g., "$x - 5$"). The intervention was delivered by a researcher and completed in 1 day.	Students received the same in-class and homework worksheets, but none of the problems were solved problems.	conceptual knowledge = 0.75[e]

(continued)

Table D.2. Studies providing evidence for Recommendation 1 *(continued)*

Study and design	Participants	Setting	Intervention condition as implemented in the study	Comparison condition as implemented in the study	Outcome domain and effect size
Carroll (1994) *Randomized controlled trial*[f]	24 students, ages 14 to 16, enrolled in a remedial mathematics class	1 high school in a large school district in the Midwest region of the United States	Students received up to 3 in-class worksheets, each consisting of 12 problems, half of which were solved problems. The worksheet focused on translating mathematical phrases into linear equations or algebraic expressions. The solved problems linked mathematical phrases (e.g., "five less than a number") to algebraic expressions (e.g., "$x - 5$"). Students were instructed to study each solved problem before completing the similar practice problem that followed. The intervention was completed in 1 day.	Students received up to 3 in-class worksheets, each consisting of 12 problems, none of which were solved problems.	conceptual knowledge = 0.56*[g]
Mevarech & Kramarski (2003) *Clustered randomized controlled trial*[h]	122 8th-grade students	5 heterogeneous, non-tracked math classrooms within one junior high school in Israel; classes were divided into small groups, each with 1 low-, 2 middle-, and 1 high-achieving student.	Students studied and discussed solved problems (word problems and graphs related to time, distance, and speed) in groups and then solved the same problems as the comparison group. The solved problems described each step in the solution process and provided written explanations for the problem's solution. The intervention was a 50-minute daily session delivered by teachers for about 4 weeks.	Students were taught to ask themselves reflective questions about (1) similarities and differences between each problem and previous problems they had solved, and (2) strategies for organizing the information provided. The problems were word problems and graphs related to time, distance, and speed.	conceptual knowledge = −0.60*[i] procedural knowledge = −0.47[j]

All studies in this table meet WWC group design standards without reservations. Studies are listed alphabetically by last name of the first author.

Each row in this table represents a study, defined by the WWC as an examination of the effect of an intervention on a distinct sample. In some cases, multiple studies were described in a single article.

For studies that included multiple outcomes in a domain, reported effect sizes and statistical significance are for the domain and calculated as described in the *WWC Procedures and Standards Handbook 3.0* (pp. 28–29).

* = statistically significant at 0.05 level

[a] This row summarizes Booth et al. (2013) Experiment 1. Results from Experiment 2 are not reported because all of the participants in that study were exposed to solved problems.

[b] The study examined two outcomes in this domain: isomorphic procedural knowledge and transfer procedural knowledge. For isomorphic procedural knowledge, the WWC-calculated effect size is 0.03 and the effect is not statistically significant. For transfer procedural knowledge, the WWC-calculated effect size is 0.31 and the effect is not statistically significant.

[c] This row summarizes Carroll (1994) Experiment 1.

[d] This study grouped participants into higher and lower achievers based on scores on a placement test created by the researcher. The effects of the intervention did not vary by student achievement level.

[e] The study also reports finding for a delayed posttest, administered one day after the completion of the intervention. However, there was an unknown amount of attrition and the authors did not demonstrate baseline equivalence of the analytic samples.

[f] This row summarizes Carroll (1994) Experiment 2.

[g] The outcome measure in this study was based on the number of errors. The intervention group committed fewer errors and thus had lower scores. Because fewer errors indicate a more positive impact of the intervention, we have reversed the sign on the effect size.

[h] The intervention of interest in this study was centered on reflective questions. However, because this table summarizes evidence about the effects of solved problems, we present that as the intervention condition.

[i] A delayed posttest ($n = 122$) was also administered one year after the intervention. The WWC-calculated effect size is −0.15 and the effect is not statistically significant.

[j] The study examined two posttest outcomes in this domain: an overall test score and knowledge of algebraic solution. For the overall test score, the WWC-calculated effect size is −0.34 and the effect is not statistically significant. For knowledge of algebraic solution, the WWC-calculated effect size is −0.59 and the effect is not statistically significant. A delayed posttest ($n = 122$) was also administered to assess the two outcomes one year after the intervention. For the overall test score, the WWC-calculated effect size is −0.32 and the effect is not statistically significant. For knowledge of algebraic solution, the WWC-calculated effect size is −0.15 and the effect is not statistically significant.

Recommendation 2. Teach students to utilize the structure of algebraic representations.

Level of evidence: Minimal Evidence

WWC staff and the panel assigned a minimal level of evidence based on four studies that meet WWC group design standards without reservations and examined this practice (see Table D.3).[77] In addition, two studies do not meet WWC group design standards but contributed to the level of evidence because the analyses adequately controlled for selection bias.[78] The six studies examined different methods of instruction on the structure of algebraic representations, but none of the studies examined the use of language that reflects mathematical structure (a component of Recommendation 2), resulting in a minimal level of evidence rating.

Four studies showed positive effects of instruction on the structure of algebraic representations for outcomes in the procedural knowledge domain.[79] In these studies, procedural knowledge was measured by asking students to solve equations or word problems. Three studies found a positive effect on conceptual knowledge.[80] To measure conceptual knowledge, one study asked a series of open-ended questions asking students to justify their procedures for solving systems of equations.[81] The other two studies asked students to formulate equations based on information provided in problem text.[82]

Four of the studies had high internal validity because they were randomized controlled trials with low sample attrition.[83] The fifth study was a randomized controlled trial with high sample attrition, and the sixth study used a quasi-experimental design; although the analytic samples for these studies were not shown to be equivalent at baseline, the analyses included a pretest covariate as a statistical control for selection bias.[84]

Each study conducted a direct test of a component of the recommendation. Although none of the studies evaluated the effectiveness of instruction that intentionally taught students to utilize structure, the panel determined that the interventions studied are effective because they encourage students to notice, discuss, and analyze structure.

Two studies compared students who used reflective questioning to solve practice problems in small groups to students who studied solved problems in small groups or students who received regular instruction without any explicit instruction on reflective questioning.[85] Students were taught to ask themselves questions about their comprehension of the problem, the similarities and differences between each problem and previous problems they had solved, and strategies for organizing the information provided and solving the problem.

The other four studies compared students who used graphical representations to solve problems to students who did not use graphical representations or were not encouraged to use any type of representation other than standard algebraic equations. One study used computer-based animation to transform elements of word problems into concrete objects and then mathematical symbols.[86] For example, a problem asked about the rates of damage to two types of trees, and the animation

- displayed a set of trees with the damaged trees disappearing from view,
- morphed into an abstract representation of the rates of damage, with the initial set of trees represented by a large square and the damaged trees represented by small squares within the large square, and
- showed the problem as an equation and included all the steps needed to solve the problem.

Two studies taught students to use graphical representations to depict the elements of a word problem and break down solution steps.[87] For example, in one study, students used a

table with three columns and two rows to solve a system of equations with three variables.[88] Students were taught to use the top row of the table to eliminate variables as they moved from one column to the next and to then use the bottom row of the table to substitute variables until they had solved the entire system. The final study taught students to utilize verbal, tabular, graphical, and symbolic representations to describe the same problem.[89] For example, students were asked to find errors in a sales invoice; analyze the pattern of errors using a graph, chart, table, or equation; and write a description of the pattern.

The study samples and settings were diverse. Two of the studies were conducted in schools outside of the United States: one included 8th-grade students from Israel,[90] and another included 9th-grade students from Germany.[91] The other four studies were conducted in the United States: one study included 7th- and 8th-grade students and another examined high school students in Algebra I.[92] Participants in the other two studies conducted in the United States were 6th- to 12th-grade students who had learning disabilities or emotional disorders or were at risk for failing mathematics based on teacher reports and test scores.[93] The delivery of the intervention also varied. Four of the interventions were delivered by teachers, one was delivered by a combination of doctoral students and experienced teachers,[94] and one was computer administered.[95] The studies as a whole provide moderate to high external and ecological validity.

Table D.3. Studies providing evidence for Recommendation 2

Study and design	Participants	Setting	Intervention condition as implemented in the study	Comparison condition as implemented in the study	Outcome domain and effect size
Meets WWC group design standards without reservations					
Ives (2007) *Randomized controlled trial*[a]	30 students, ages 13 to 19; 22 of the students had language-related disabilities.	4 classes (each with less than 10 students) in a Georgia private school that serves students in grades 6–12 with learning disabilities or attention disorders	Students participated in 4 lessons on systems of linear equations and were taught by a researcher to use "graphic organizers" (tables) to identify the intermediate steps required to solve the linear equations. The graphic organizer, a table with two rows and three columns, guided students to work from cell to cell in a clockwise direction starting with the top left cell. The top row was used to combine equations until an equation with one variable was produced. The bottom row guided students to solve equations for each of the variables in the system.	Students participated in the same series of 4 lessons on systems of linear equations, but were not taught to use graphic organizers.	conceptual knowledge = 1.00[b] procedural knowledge = 0.16[b]
Mevarech & Kramarski (2003) *Clustered randomized controlled trial*	122 8th-grade students[c]	5 classrooms comprised of students with mixed achievement levels within one junior high school in Israel; classes were divided into small groups, each with 1 low-, 2 middle-, and 1 high-achieving student.	Students were taught to ask themselves reflective questions about (1) similarities and differences between each problem and previous problems they had solved, and (2) strategies for organizing the information provided. The problems were word problems and graphs related to time, distance, and speed. The intervention was a 50-minute daily session delivered by teachers for about 4 weeks.	Students studied solved problems (word problems and graphs related to time, distance, and speed) in groups and then as a group discussed and solved the same problems as the intervention group. The solved problems described each step in the solution process and provided written explanations for the solution.	conceptual knowledge = 0.60*[d] procedural knowledge = 0.47*[e]

(continued)

Table D.3. Studies providing evidence for Recommendation 2 *(continued)*

Study and design	Participants	Setting	Intervention condition as implemented in the study	Comparison condition as implemented in the study	Outcome domain and effect size
Meets WWC group design standards without reservations					
Scheiter, Gerjets, & Schuh (2010) *Randomized controlled trial*	32 9th-grade students	1 high school in Germany	Students read 3 textbook chapters on biology, chemistry, and physics on a computer screen. Each chapter contained 3 algebraic solved problems accompanied by animations that depicted the problems in concrete objects (e.g., an illustration of deforestation) and then transformed these objects into mathematical symbols (e.g., the rate of deforestation by type of tree). For each of the 9 solved problems the steps of the solution were presented. The intervention, pretest, and posttest took place over 2 hours on 1 day.	Solved problems were presented as text only, with no accompanying animation. For each of the nine solved problems the steps of the solution were presented.	procedural knowledge = 0.87*[f]
Xin, Jitendra, & Deatline-Buchman (2005) *Randomized controlled trial*	22 students in grades 6–8 with learning challenges (18 with learning disabilities, 1 with an emotional disorder, and 3 at risk of failing mathematics)	1 middle school in the Northeast region of the United States	Students were taught to use a schematic diagram to represent a multiplicative compare word problem or proportion word problem, transform the diagram to a math sentence, and then solve the problem. The intervention consisted of 12 sessions of 1 hour each, 3–4 times per week. Instruction was provided by doctoral students and experienced special educators.	Students received the same amount of instruction from the same teachers but were taught general strategies to represent word problems (such as drawing a picture).	procedural knowledge = 1.87*
Does not meet WWC group design standards but contributed to the level of evidence					
Akkus, Seymour, & Hand (2007) *Quasi-experimental design*	202 students in high school Algebra I courses	10 classrooms within one high school during first semester of the year	Students were provided a template that listed questions for students to ask themselves as they attempted to solve problems. Questions included describing how they would solve the problem, the steps to solution, and their reasoning for choosing a specific strategy. The questions encouraged students to compare their solutions with their classmates and to reflect on their solution after a classroom discussion. Teachers were also given a template to organize information on their own knowledge of mathematics and their students' prior knowledge to help students learn key concepts.	Teachers taught their regular lessons without any explicit instruction on reflective questioning.	procedural knowledge = 0.36[g]
Brenner et al. (1997) *Randomized controlled trial*	128 students in grades 7–8 in their first year of pre-algebra	3 junior high schools in Southern California	Students participated in a unit on single-variable functional relationships that emphasized translation and representation of mathematical relationships from word problems and mathematical phrases into verbal, tabular, graphical, and/or symbolic modes. The intervention was delivered by teachers and consisted of 20 daily sessions over a 4-week period.	Students were taught to use equations to solve word problems. Other modes of representation (graphs, tables, symbolic) were not encouraged.	conceptual knowledge = 0.78*[h] procedural knowledge = 0.08[i]

Each row in this table represents a study, defined by the WWC as an examination of the effect of an intervention on a distinct sample. In some cases, multiple studies were described in a single article.

For studies that included multiple outcomes in a domain, the reported effect sizes and statistical significance are for the domain and calculated as described in the *WWC Procedures and Standards Handbook 3.0* (pp. 28–29).

* = statistically significant at 0.05 level

[a] This row summarizes Ives (2007) Experiment 1. Experiment 2 does not meet WWC group design standards because only one classroom was assigned to each condition.

[b] A delayed posttest ($n = 30$) was also administered two to three weeks after the intervention. The WWC-calculated effect sizes are 0.00 for procedural knowledge and 0.88 for conceptual knowledge.

[c] This study grouped participants into higher and lower achievers based on pretest scores. The effects of the intervention did not vary by student achievement level.

[d] A delayed posttest ($n = 122$) was also administered one year after the intervention. The WWC-calculated effect size is 0.15 and the effect is not statistically significant.

[e] The study examined two posttest outcomes in this domain: an overall test score and knowledge of algebraic solution. For the overall test score, the WWC-calculated effect size is 0.34 and the effect is not statistically significant. For knowledge of algebraic solutions, the WWC-calculated effect size is 0.59 and the effect is statistically significant. A delayed posttest ($n = 122$) was also administered to assess the two outcomes one year after the intervention. For the overall test score, the WWC-calculated effect size is 0.32 and the effect is not statistically significant. For knowledge of algebraic solutions, the WWC-calculated effect size is 0.15 and the effect is not statistically significant.

[f] The study examined three outcomes in this domain: equivalent problems, similar problems, and unrelated problems. For equivalent problems, the WWC-calculated effect size is 0.37 and the effect is not statistically significant. For similar problems, the WWC-calculated effect size is 1.14 and the effect is statistically significant. For unrelated problems, the WWC-calculated effect size is 1.11 and the effect is statistically significant.

[g] The authors reported an eta-squared effect size statistic, and did not respond to requests for more information. To make this effect size more similar to the standard WWC effect size (Hedge's *g*), we: (1) took the square root of eta-squared to obtain the correlation (*r*) between the treatment and the outcome; (2) used the following equation to estimate Cohen's d ($d = \frac{2r}{\sqrt{1-r^2}}$)—with the sample sizes in this study, *g* and *d* are similar; and (3) multiplied the resulting *d* by $1 - \frac{3}{(4N-9)}$, a small sample correction, to obtain an estimate for the WWC effect size.

[h] The study examined two outcomes in this domain: functional word problem representation and word problem representation. For functional word problem representation, the WWC-calculated effect size is 0.91 and the effect is statistically significant. For word problem representation, the WWC-calculated effect size is 0.64 and the effect is statistically significant.

[i] The study examined three outcomes in this domain: functional word problem correct solution, a word problem-solving test, and an equation-solving test. For functional word problem correct solution, the WWC-calculated effect size is 0.41 and the effect is not statistically significant. For the word problem-solving test, the WWC-calculated effect size is 0.11 and the effect is not statistically significant. For the equation solving test, the WWC-calculated effect size is −0.29 and the effect is not statistically significant.

Recommendation 3. Teach students to intentionally choose from alternative algebraic strategies when solving problems.

Level of evidence: Moderate Evidence

WWC staff and the panel assigned a moderate level of evidence based on six studies that meet WWC group design standards without reservations (see Table D.4).[96]

Four studies showed positive effects of teaching alternative algebraic strategies, providing a preponderance of evidence of positive effects.[97] Consistently positive effects were found in the procedural flexibility outcome domain. To measure procedural flexibility, these studies asked students to solve a problem in a different way, to identify appropriate first steps in solving a problem, and to describe whether a first step taken was appropriate and efficient. Findings pertaining to the other outcome domains were mixed.

Only one of the four studies found a positive effect on conceptual knowledge,[98] with the remaining studies finding neither significant nor substantively important effects on conceptual knowledge. These studies measured conceptual knowledge by asking students to identify equivalent equations—which measured understanding of the meaning of equation features—and to recognize equivalent terms that could be combined. A positive effect on procedural knowledge was also only found in one of the four studies,[99] with the remaining studies finding neither significant nor substantively important effects. To measure procedural knowledge, these studies asked students to solve linear equations. The studies were randomized controlled trials with low sample attrition, resulting in high internal validity.[100]

Each of these four studies examined the effects of reviewing and practicing different solution methods for the same problem—there were no other intervention components—providing

a direct test of the recommendation.[101] Three of the studies examined the effects of having students study different solutions to the same solved problem, with the solutions presented side by side on the same page to facilitate comparison.[102] In two of the three studies, students in the comparison condition also studied solved problems, but the solution methods to the problems were presented one at a time on different pages.[103] In the third study, students in the comparison condition studied two different problems solved with the same solution method.[104] The fourth study compared students who solved a problem one way and then were asked to solve it a second time using a different method or ordering of steps to students who were asked to use only one solution method to solve problems.[105] The interventions in these three studies ranged from two to five classroom periods.

The four studies included students aged 11 to 15 in 6th, 7th, and 8th grade. The studies took place in rural, suburban, and urban settings within the United States. Three interventions were implemented in classrooms during regular class time,[106] and one study was conducted during the summer.[107] The interventions were implemented by researchers, or by researchers working alongside classroom teachers. The studies as a whole provide moderate external and ecological validity.

Two studies involving students with no knowledge of algebra—as measured by a pretest—found negative or mixed effects.[108] Both studies were randomized trials with low attrition, and the interventions asked students to review and practice multiple strategies. The first study occurred at a low-performing middle school and compared the use of multiple strategies to asking students to use only one solution method to solve problems. The study reported results separately for students who did not use algebra to solve problems on a pretest and students who did, and the findings were significantly different for these two groups. For the students who had no knowledge of algebra at baseline, the study found negative effects across procedural knowledge, procedural flexibility, and conceptual knowledge; for the students who had attempted some algebra on the pretest, the study found effects that were neither significant nor substantively important in those domains.[109] The other study was conducted in multiple schools in a single district in the Midwest and compared the use of multiple strategies to asking students to solve a new problem that was similar in underlying structure to a problem they had previously solved. This study found a negative effect on procedural knowledge but a positive effect on procedural flexibility.[110] These findings indicate that teaching alternative algebraic strategies can improve achievement once students have developed some procedural knowledge of algebra.

Table D.4. Studies providing evidence for Recommendation 3

Study and design	Participants	Setting	Intervention condition as implemented in the study	Comparison condition as implemented in the study	Outcome domain and effect size
Meets WWC group design standards without reservations					
Rittle-Johnson & Star (2009) *Randomized controlled trial*[a]	98–105 students, ages 11 to 15, in grades 7 and 8[b]	1 rural public school, 1 suburban public school, and 1 urban private school	Students worked with partners to study 12 solved problems presented in pairs on the same page. The solved problems were linear equations with one unknown. Each page displayed the same problems solved in two different ways (a conventional method and a shortcut method). The intervention lasted 3 days. At the end of each day, students received 2 problems and were asked to solve each problem with both solution methods.	Students worked with partners to study 12 solved problems presented in pairs on the same page. The solved problems were linear equations with one unknown. Each page displayed 2 problems of similar structure solved with the same method. Half of the problems illustrated the conventional solution method, and half illustrated a shortcut method. At the end of each day, students received 4 problems and were asked to solve each problem using a single method of their choice.	conceptual knowledge = 0.36[c]
					procedural knowledge = −0.14[c]
					procedural flexibility = 0.36[d]
	101–106 students, ages 11 to 15, in grades 7 and 8[e]	1 rural public school, 1 suburban public school, and 1 urban private school	Students worked with partners to study 12 solved problems presented in pairs on the same page. The solved problems were linear equations with one unknown. Each page displayed 2 different types of equations, each solved with the same method. Half the problems illustrated the conventional solution method and half illustrated a shortcut method. The intervention lasted 3 days. At the end of each day, students received 4 problems and were asked to solve each problem using a single method of their choice.	Students worked with partners to study 12 solved problems presented in pairs on the same page. The solved problems were linear equations with one unknown. Each page displayed 2 problems of similar structure solved with the same method. Half of the problems illustrated the conventional solution method, and half illustrated a shortcut method. At the end of each day, students received 4 problems and were asked to solve each problem using a single method of their choice.	conceptual knowledge = 0.14[f]
					procedural knowledge = −0.17[f]
					procedural flexibility = 0.35[g]
Rittle-Johnson & Star (2007) *Randomized controlled trial*	69 students, ages 11 to 13, in grade 7	1 urban private school in the United States	Students worked with partners to study 24 solved problems presented in pairs on the same page. The solved problems were linear equations with one unknown. Each page displayed the same problems solved in two different ways. Students studied problems over a 2-day period. A separate set of problems was used on each day.	Students worked with partners to study 12 solved problems presented in pairs on the same page. The solved problems were linear equations with one unknown. Each page displayed 2 problems of similar structure solved with the same method. Half of the problems illustrated the conventional solution method, and half illustrated a shortcut method. At the end of each day, students received 4 problems and were asked to solve each problem using a single method of their choice.	conceptual knowledge = −0.19
					procedural knowledge = 0.33
					procedural flexibility = 0.40[h]

(continued)

Table D.4. Studies providing evidence for Recommendation 3 *(continued)*

Study and design	Participants	Setting	Intervention condition as implemented in the study	Comparison condition as implemented in the study	Outcome domain and effect size
Meets WWC group design standards without reservations					
Rittle-Johnson, Star, & Durkin (2012) *Randomized controlled trial*[i]	124 students, ages 13 to 15, in 8th-grade classrooms	2 public middle schools in the United States	Students worked with partners to study pairs of solved problems that used the same linear equation but provided different solution methods. Problems were presented side by side. The intervention took place in 2 classroom periods of 80–90 minutes over 2 days.	Students worked with partners to study examples, but the examples were presented illustrating one solution method at a time. The examples were linear equations with one unknown.	conceptual knowledge = –0.12[j]
					procedural knowledge = 0.13[j]
					procedural flexibility = 0.37*[k]
	115 students, ages 13 to 15, in 8th-grade classrooms	2 public middle schools in the United States	Students worked with partners to study pairs of solved problems presented side by side. On day 1, the examples in a pair used the same solution method to solve a different linear equation. On day 2, the examples in a pair solved the same equation using different solution methods. The intervention took place in 2 classroom periods of 80–90 minutes over 2 days.	Students worked with partners to study solved problems, but the examples were presented illustrating one solution method at a time. The examples were linear equations with one unknown.	procedural flexibility = –0.15[l]
Rittle-Johnson, Star, & Durkin (2009) *Randomized controlled trial*[m]	101 students in grades 7 and 8 with prior knowledge of algebra; students were categorized as having or not having prior knowledge of algebra based on whether they used algebra to solve problems at pretest.	1 low-performing urban middle school in Massachusetts	Students worked with partners to study 24 solved problems. The solved problems were linear equations with one unknown. Two different ways of solving each problem (a conventional method and a shortcut method) were presented side by side. The intervention lasted 3 days. At the end of each day, students received 2 problems and were asked to solve each problem using both solution methods.	Students worked with partners to study 24 solved problems presented one at a time. The solved problems were linear equations with one unknown. Half of the problems illustrated the conventional solution method, and half illustrated a shortcut method. At the end of each day, students received 4 problems and were asked to solve each problem using a single method of their choice.	conceptual knowledge = –0.13
					procedural knowledge = 0.19
					procedural flexibility = 0.12[n]
	55 students in grades 7 and 8 with no prior knowledge of algebra; students were categorized as having or not having prior knowledge of algebra based on whether they used algebra to solve problems at pretest.	1 low-performing urban middle school in Massachusetts	Students worked with partners to study 24 solved problems. The solved problems were linear equations with one unknown. Two different ways of solving each problem (a conventional method and a shortcut method) were presented side by side. The intervention lasted 3 days. At the end of each day, students received 2 problems and were asked to solve each problem using both solution methods.	Students worked with partners to study 24 solved problems presented one at a time. The solved problems were linear equations with one unknown. Half of the problems illustrated the conventional solution method, and half illustrated a shortcut method. At the end of each day, students received 4 problems and were asked to solve each problem using a single method of their choice.	conceptual knowledge = –0.42
					procedural knowledge = –0.44
					procedural flexibility = –0.35[o]

(continued)

Table D.4. Studies providing evidence for Recommendation 3 *(continued)*

Study and design	Participants	Setting	Intervention condition as implemented in the study	Comparison condition as implemented in the study	Outcome domain and effect size
Meets WWC group design standards without reservations					
Star & Rittle-Johnson (2008) *Randomized controlled trial*[p]	63 students who just completed 6th grade	2 large, suburban middle-class school districts	Students were encouraged to discover multiple strategies for solving linear equations with one unknown. For example, they were sometimes asked to solve a problem again using a different ordering of steps. The intervention took place over 5 consecutive days during the summer.	Instead of solving a problem again, students worked on a new problem that was similar in underlying structure. The problems were all linear equations with one unknown.	procedural knowledge = −0.16
					procedural flexibility = 0.48[q]
	66 students who just completed 6th grade	2 large, suburban middle-class school districts	Students received an 8-minute period of strategy instruction, during which a researcher demonstrated the most efficient solution for each of 3 problems involving linear equations with one unknown. The intervention took place over 5 consecutive days during the summer.	Students solved a new problem that was similar in form to a problem they had previously solved during the first day of instruction. The problems were linear equations with one unknown.	procedural knowledge = 0.06
					procedural flexibility = 0.49[r]
Star & Seifert (2006) *Randomized controlled trial*	32 6th-grade students with no prior algebra knowledge	Multiple schools within a district in a medium-sized city in the Midwest region of the United States	The intervention took place over 3 days at a local university. After the first day of instruction, students were given problems they had previously solved during the instructional lesson and were asked to solve them again using a different ordering of steps. The problems were linear equations with one unknown.	Students solved a new problem that was similar in form to a problem they had previously solved during the first day of instruction. The problems were linear equations with one unknown.	procedural knowledge = −0.35[s]
					procedural flexibility = 0.64[t]

Each row in this table represents a study, defined by the WWC as an examination of the effect of an intervention on a distinct sample. In some cases, multiple studies were described in a single article.

For studies that included multiple outcomes in a domain, the reported effect sizes and statistical significance are for the domain and calculated as described in the *WWC Procedures and Standards Handbook 3.0* (pp. 28–29).

* = statistically significant at 0.05 level

[a] This study included two intervention groups and one comparison group. The effect sizes presented for this study were calculated by comparing the outcomes for each intervention group to the single comparison group.

[b] The sample size for the conceptual knowledge and procedural flexibility outcomes was 105 students, while the sample size for the procedural knowledge outcome was 98 students.

[c] A delayed posttest was also administered two weeks after the intervention. The WWC-calculated effect sizes are 0.01 ($n = 94$) for procedural knowledge and 0.29 ($n = 101$) for conceptual knowledge.

[d] The study examined two outcomes in this domain: flexibility knowledge and flexibility use. For flexibility knowledge, the WWC-calculated effect size is 0.45 and the effect is statistically significant. For flexibility use, the WWC-calculated effect size is 0.25 and the effect is not statistically significant. A delayed posttest ($n = 101$) was also administered to assess the two outcomes two weeks after the intervention. For flexibility knowledge, the WWC-calculated effect size is 0.47 and the effect is statistically significant. For flexibility use, the WWC-calculated effect size is 0.53 and the effect is statistically significant. The WWC-calculated domain effect size for procedural flexibility on the delayed posttest is 0.50 and the effect is statistically significant.

[e] The sample size for the conceptual knowledge and procedural flexibilty outcomes was 106 students, while the sample size for the procedural knowledge outcome was 101 students.

[f] A delayed posttest was also administered two weeks after the intervention. The WWC-calculated effect sizes are −0.20 ($n = 99$) for procedural knowledge and 0.01 for conceptual knowledge ($n = 104$).

[g] The study examined two outcomes in this domain: flexibility knowledge and flexibility use. For flexibility knowledge, the WWC-calculated effect size is 0.30 and the effect is not statistically significant. For flexibility use, the WWC-calculated effect size is 0.41 and the effect is not statistically significant. A delayed posttest ($n = 104$) was also administered two weeks after the intervention. For flexibility knowledge, the WWC-calculated effect size is 0.32 and the effect is statistically significant. For flexibility use, the WWC-calculated effect size is 0.38 and the effect is not statistically significant. The WWC-calculated domain effect size for procedural flexibility on the delayed posttest is 0.35.

[h] The study examined two outcomes in this domain: flexibility measured via solution strategy use and flexibility via an independent measure. For flexibility measured via solution strategy use, the WWC-calculated effect size is 0.33 and the effect is statistically significant. For flexibility via an independent measure, the WWC-calculated effect size is 0.47 and the effect is statistically significant.

[i] The study included two intervention groups and one comparison group. The effect sizes presented for this study were calculated by comparing the outcomes for each intervention group to the single comparison group.

[j] A delayed posttest ($n = 118$) was also administered one month after the intervention. The WWC-calculated effect sizes are 0.15 for procedural knowledge and 0.14 for conceptual knowledge.

[k] The study examined two outcomes in this domain: flexibility knowledge and flexibility use. For flexibility knowledge, the WWC-calculated effect size is 0.30 and the effect is not statistically significant. For flexibility use, the WWC-calculated effect size is 0.44 and the effect is statistically significant. A delayed posttest ($n = 118$) was also administered one month after the intervention. For flexibility knowledge, the WWC-calculated effect size is 0.31 and the effect is not statistically significant. For flexibility use, the WWC-calculated effect size is 0.52 and the effect is statistically significant. The WWC-calculated domain effect size for procedural flexibility on the delayed posttest is 0.42 and the effect is statistically significant.

[l] This comparison meets WWC group design standards with reservations due to high sample attrition. Two outcomes were examined in the procedural flexibility domain: flexibility knowledge and flexibility use. For flexibility knowledge, the WWC-calculated effect size is −0.06 and the effect is not statistically significant. For flexibility use, the WWC-calculated effect size is −0.24 and the effect is not statistically significant.

[m] Effect sizes for this study are calculated using the standard deviations provided in the paper (calculated with imputed data) because standard deviations based on non-imputed data were not available.

[n] The study examined two outcomes in this domain: flexibility knowledge and flexibility use. For flexibility knowledge, the WWC-calculated effect size is 0.03 and the effect is not statistically significant. For flexibility use, the WWC-calculated effect size is 0.21 and the effect is not statistically significant.

[o] The study examined two outcomes in this domain: flexibility knowledge and flexibility use. For flexibility knowledge, the WWC-calculated effect size is −0.37 and the effect is not statistically significant. For flexibility use, the WWC-calculated effect size is −0.33 and the effect is not statistically significant.

[p] This study included two intervention groups and one comparison group. The effect sizes presented for this study were calculated by comparing the outcomes for each intervention group to the single comparison group.

[q] The study examined three outcomes in this domain: knowledge of multiple strategies, use of multiple strategies, and use of efficient strategies. For knowledge of multiple strategies, the WWC-calculated effect size is 0.68 and the effect is statistically significant. For use of multiple strategies, the WWC-calculated effect size is 0.53 and the effect is not statistically significant. For use of efficient strategies, the WWC-calculated effect size is 0.23 and the effect is not statistically significant.

[r] The study examined three outcomes in this domain: knowledge of multiple strategies, use of multiple strategies, and use of efficient strategies. For knowledge of multiple strategies, the WWC-calculated effect size is 0.52 and the effect is not statistically significant. For use of multiple strategies, the WWC-calculated effect size is 0.03 and the effect is not statistically significant. For use of efficient strategies, the WWC-calculated effect size is 0.93 and the effect is statistically significant.

[s] The study examined two outcomes in this domain: percent correct on the test, and knowledge of standard solutions. For percent correct on the test, the WWC-calculated effect size is −0.35 and the effect is not statistically significant. For knowledge of standard solutions, the WWC-calculated effect size is −0.34 and the effect is not statistically significant.

[t] The study examined two outcomes in this domain: use of multiple strategies, and use of inventions. For use of multiple strategies, the WWC-calculated effect size is 0.58 and the effect is not statistically significant. For use of inventions, the WWC-calculated effect size is 0.70 and the effect is not statistically significant.

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[a] Eligible studies that meet WWC evidence standards or meet evidence standards with reservations are indicated by **bold text** in the endnotes and references pages. For more information about these studies, please see Appendix D.

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Endnotes[a]

1. Katz (2007).
2. Bolded terms in this practice guide are defined in the Glossary.
3. The review protocol is available via the WWC website at http://www.whatworks.ed.gov.
4. Atkinson et al. (2000); Kalyuga et al. (2001); Sweller (1988).
5. Sweller, Ayres, and Kalyuga (2011); Tarmizi and Sweller (1988); Ward and Sweller (1990).
6. Sweller and Cooper (1985); Sweller, van Merrienboer, and Paas (1998); Zhu and Simon (1987).
7. **Booth et al. (2013) Experiment 1**; **Carroll (1994) Experiment 1**; **Carroll (1994) Experiment 2**; **Mevarech and Kramarski (2003)**.
8. Studies were classified as having a positive or negative effect when the result was either statistically significant (unlikely to occur by chance) or substantively important (producing considerable differences in outcomes).
9. **Booth et al. (2013) Experiment 1**; **Carroll (1994) Experiment 1**; **Carroll (1994) Experiment 2**.
10. **Booth et al. (2013) Experiment 1**.
11. **Carroll (1994) Experiment 1**; **Carroll (1994) Experiment 2**.
12. **Mevarech and Kramarski (2003)**.
13. One randomized controlled trial that meets WWC group design standards without reservations found that solved problems had negative effects on procedural knowledge. Mevarech and Kramarski (2003) compared students who studied and discussed solved problems in groups to students who used reflective questioning (a practice suggested in Recommendation 2) to solve practice problems in groups. These negative findings indicate that the effects of this practice can depend on what students do instead of using solved problems.
14. In these and other similar examples, teachers can include a "verify my solution" section in the solved problem to model how students can verify their solutions by substituting their answers into the original equation.
15. The third factoring problem is adapted from an example in *CCSS.MATH.CONTENT.HSA-SSE.A.2*.
16. Adapted from Booth, Koedinger, and Pare-Blagoev (2010) Experiment 2.
17. **Ives (2007) Experiment 1**; **Mevarech and Kramarski (2003)**; **Scheiter, Gerjets, and Schuh (2010)**; **Xin, Jitendra, and Deatline-Buchman (2005)**; Akkus, Seymour, and Hand (2007); Brenner et al. (1997). Brenner et al. (1997) was a randomized controlled trial with high sample attrition, and Akkus, Seymour, and Hand (2007) used a quasi-experimental design; although the analytic samples for these studies were not shown to be equivalent at baseline, the analyses included a pretest covariate as a statistical control for selection bias.
18. Studies were classified as having a positive or negative effect when the result was either statistically significant (unlikely to occur by chance) or substantively important (producing considerable differences in outcomes).
19. **Mevarech and Kramarski (2003)**; **Scheiter, Gerjets, and Schuh (2010)**; **Xin, Jitendra, and Deatline-Buchman (2005)**; Akkus, Seymour, and Hand (2007).
20. Brenner et al. (1997); **Ives (2007) Experiment 1**; **Scheiter, Gerjets, and Schuh (2010)**.
21. Akkus, Seymour, and Hand (2007); **Mevarech and Kramarski (2003)**.
22. **Mevarech and Kramarski (2003)**; Akkus, Seymour, and Hand (2007).
23. Brenner et al. (1997); **Ives (2007) Experiment 1**; **Scheiter, Gerjets, and Schuh (2010)**; **Xin, Jitendra, and Deatline- Buchman (2005)**.
24. **Scheiter, Gerjets, and Schuh (2010)**; **Xin, Jitendra, and Deatline-Buchman (2005)**.
25. Brenner et al. (1997); **Ives (2007) Experiment 1**.
26. **Mevarech and Kramarski (2003)**; **Scheiter, Gerjets, and Schuh (2010)**.
27. **Ives (2007) Experiment 1**; **Xin, Jitendra, and Deatline-Buchman (2005)**.
28. Adapted from the "Throwing Horseshoes" exercise from the National Council of Teachers of Mathematics (2010).
29. **Ives (2007) Experiment 1**; **Scheiter, Gerjets, and Schuh (2010)**; **Xin, Jitendra, and Deatline-Buchman (2005)**.

[a] Eligible studies that meet WWC evidence standards or meet evidence standards with reservations are indicated by **bold text** in the endnotes and references pages. For more information about these studies, please see Appendix D.

30. **Rittle-Johnson and Star (2009)**; **Rittle-Johnson and Star (2007)**; **Rittle-Johnson, Star, and Durkin (2012)**; **Rittle-Johnson, Star, and Durkin (2009)**; **Star and Rittle-Johnson (2008)**; **Star and Seifert (2006)**.
31. Studies were classified as having a positive or negative effect when the result was either statistically significant (unlikely to occur by chance) or substantively important (producing considerable differences in outcomes).
32. **Rittle-Johnson and Star (2009)**; **Rittle-Johnson and Star (2007)**; **Rittle-Johnson, Star, and Durkin (2012)**; **Star and Rittle-Johnson (2008)**.
33. **Rittle-Johnson, Star, and Durkin (2009)**; **Star and Seifert (2006)**.
34. **Rittle Johnson and Star (2009)**; **Rittle Johnson and Star (2007)**; **Rittle Johnson, Star, and Durkin (2012)**.
35. **Star and Rittle-Johnson (2008)**.
36. **Rittle-Johnson, Star, and Durkin (2009)**; **Star and Seifert (2006)**.
37. **Rittle-Johnson, Star, and Durkin (2009)**.
38. **Star and Seifert (2006)**.
39. Adapted from **Rittle-Johnson and Star (2007)**.
40. Ibid.
41. These are a subset of possible strategies for solving linear systems. Not all strategies will be applicable or useful for all types of linear systems.
42. Adapted from **Rittle-Johnson and Star (2007)**.
43. Following WWC guidelines, improved outcomes are indicated by either a positive statistically significant effect or a positive, substantively important effect size. The WWC defines substantively important, or large, effects on outcomes to be those with effect sizes greater than or equal to 0.25 standard deviations. See the WWC guidelines at http://whatworks.ed.gov.
44. For more information, see the WWC Frequently Asked Questions page for practice guides at http://whatworks.ed.gov.
45. This includes randomized control trials (RCTs) and quasi-experimental design studies (QEDs). Studies not contributing to levels of evidence include single-case designs (SCDs) evaluated with WWC pilot SCD standards and regression discontinuity designs (RDDs) evaluated with pilot RDD standards.
46. The research may include studies generally meeting WWC design standards and supporting the effectiveness of a program, practice, or approach with small sample sizes and/or other conditions of implementation or analysis that limit generalizability. The research may include studies that support the generality of a relation but do not meet WWC design standards; however, they have no major flaws related to internal validity other than lack of demonstrated equivalence at pretest for QEDs. QEDs without equivalence must include a pretest covariate as a statistical control for selection bias. These studies must be accompanied by at least one relevant study meeting WWC design standards.
47. American Educational Research Association, American Psychological Association, and National Council on Measurement in Education (1999).
48. Eligible studies that meet WWC evidence standards without reservations or meet evidence standards with reservations are indicated by **bold text** in the endnotes and references pages.
49. As indicated in Table A.1, studies that do not meet WWC design standards but have no major flaws related to internal validity other than a lack of demonstrated equivalence can contribute to a moderate level of evidence if the study includes a pretest covariate as a statistical control for selection bias.
50. Hutchinson (1993) was reviewed as both a group design study and a single-case design study. It is included in both counts.
51. A statistically significant finding is unlikely to occur by chance.
52. Recognizing that some studies lack the statistical power to classify practically important effects as statistically significant, the panel also accepts substantively important effects as evidence of effectiveness. Substantively important effects are defined as an effect size greater than or equal to 0.25 or less than or equal to −0.25, as measured by Hedge's *g*.
53. For multiple comparison adjustments and cluster corrections, see the *WWC Procedures and Standards Handbook, Version 3.0,* at http://www.whatworks.ed.gov.

54. Adapted from a problem in **Booth et al. (2013)**.
55. Taken from **Carroll (1994)**.
56. Adapted from a problem in **Booth et al. (2013)**.
57. Taken from **Rittle-Johnson, Star, and Durkin (2012)**.
58. Adapted from a problem in **Rittle-Johnson and Star (2009)**.
59. National Research Council (2001); U.S. Department of Education (2008).
60. "By the term proficiency, the panel means that students should understand key concepts; achieve automaticity as appropriate (e.g., with addition and related subtraction facts); develop flexible, accurate, and automatic execution of the standard algorithms; and use these competencies to solve problems." (U.S. Department of Education 2008; p. 22).
61. Schneider and Stern (2010); Schneider, Rittle-Johnson, and Star (2011).
62. **Booth et al. (2013) Experiment 1**; **Carroll (1994) Experiment 1**; **Carroll (1994) Experiment 2**; **Mevarech and Kramarski (2003)**.
63. **Booth et al. (2013) Experiment 1**; **Carroll (1994) Experiment 1**; **Carroll (1994) Experiment 2**.
64. **Booth et al. (2013) Experiment 1**.
65. **Carroll (1994) Experiment 1**; **Carroll (1994) Experiment 2**.
66. **Booth et al. (2013) Experiment 1**.
67. **Carroll (1994) Experiment 1**; **Carroll (1994) Experiment 2**.
68. **Booth et al. (2013) Experiment 1**.
69. Ibid.
70. Ibid.
71. **Carroll (1994) Experiment 1** grouped participants into higher and lower achievers based on scores on a placement test created by the researcher. The effects of the intervention did not vary by student achievement level.
72. **Carroll (1994) Experiment 2**.
73. **Booth et al. (2013) Experiment 1**.
74. **Carroll (1994) Experiment 1**; **Carroll (1994) Experiment 2**.
75. **Mevarech and Kramarski (2003)**.
76. Ibid.
77. **Ives (2007) Experiment 1**; **Mevarech and Kramarski (2003)**; **Scheiter, Gerjets, and Schuh (2010)**; **Xin, Jitendra, and Deatline-Buchman (2005)**.
78. Akkus, Seymour, and Hand (2007); Brenner et al. (1997).
79. Akkus, Seymour, and Hand (2007); **Mevarech and Kramarski (2003)**; **Scheiter, Gerjets, and Schuh (2010)**; **Xin, Jitendra, and Deatline-Buchman (2005)**.
80. Brenner et al. (1997); **Ives (2007) Experiment 1**; **Mevarech and Kramarski (2003)**.
81. **Ives (2007) Experiment 1**.
82. Brenner et al. (1997); **Mevarech and Kramarski (2003)**.
83. **Ives (2007) Experiment 1**; **Mevarech and Kramarski (2003)**; **Scheiter, Gerjets, and Schuh (2010)**; **Xin, Jitendra, and Deatline-Buchman (2005)**.
84. Akkus, Seymour, and Hand (2007); Brenner et al. (1997).
85. Akkus, Seymour, and Hand (2007); **Mevarech and Kramarski (2003)**.
86. **Scheiter, Gerjets, and Schuh (2010)**.
87. **Ives (2007) Experiment 1**; **Xin, Jitendra, and Deatline-Buchman (2005)**.
88. **Ives (2007) Experiment 1**.
89. Brenner et al. (1997).
90. **Mevarech and Kramarski (2003)**.
91. **Scheiter, Gerjets, and Schuh (2010)**.
92. Akkus, Seymour, and Hand (2007); Brenner et al. (1997).
93. **Ives (2007) Experiment 1**; **Xin, Jitendra, and Deatline-Buchman (2005)**.
94. **Xin, Jitendra, and Deatline-Buchman (2005)**.
95. **Scheiter, Gerjets, and Schuh (2010)**.
96. **Rittle-Johnson and Star (2009)**; **Rittle-Johnson and Star (2007)**; **Rittle-Johnson, Star, and Durkin (2012)**; **Rittle-Johnson, Star, and Durkin (2009)**; **Star and Rittle-Johnson (2008)**; **Star and Seifert (2006)**.
97. **Rittle-Johnson and Star (2009)**; **Rittle-Johnson and Star (2007)**; **Rittle-Johnson, Star, and Durkin (2012)**; **Star and Rittle-Johnson (2008)**.
98. **Rittle-Johnson and Star (2009)**.
99. **Rittle-Johnson and Star (2007)**.

100. One of the studies conducted a comparison for which there was high sample attrition. The analytic samples for this study were shown to be equivalent at baseline, resulting in moderate internal validity. The comparison found effects that were neither statistically significant nor substantively important.
101. **Rittle-Johnson and Star (2009)**; **Rittle-Johnson and Star (2007)**; **Rittle-Johnson, Star, and Durkin (2012)**; **Star and Rittle-Johnson (2008)**.
102. **Rittle-Johnson and Star (2009)**; **Rittle-Johnson and Star (2007)**; **Rittle-Johnson, Star, and Durkin (2012)**.
103. **Rittle-Johnson and Star (2007)**; **Rittle-Johnson, Star, and Durkin (2012)**.
104. **Rittle-Johnson and Star (2009)**.
105. **Star and Rittle-Johnson (2008)**.
106. **Rittle-Johnson and Star (2009)**; **Rittle-Johnson and Star (2007)**; **Rittle-Johnson, Star, and Durkin (2012)**.
107. **Star and Rittle-Johnson (2008)**.
108. **Rittle-Johnson, Star, and Durkin (2009)**; **Star and Seifert (2006)**.
109. **Rittle-Johnson, Star, and Durkin (2009)**.
110. **Star and Seifert (2006)**.

A **practice guide** presents recommendations for educators to address challenges in their classrooms and schools. They are based on reviews of research, the experiences of practitioners, and the expert opinions of a panel of nationally recognized experts.