CANADIAN MATHEMATICS EDUCATION
STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE
DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2013 ANNUAL MEETING / RENCONTRE ANNUELLE 2013

CMESG
GCEDM

Brock University
May 24 – May 28, 2013

EDITED BY:
Susan Oesterle, *Douglas College*
Darien Allan, *Simon Fraser University*

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PROCEEDINGS OF THE 2013 ANNUAL MEETING OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE ANNUELLE 2013 DU GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

37th Annual Meeting
Brock University
May 24 – May 28, 2013

CONTENTS / TABLES DES MATIÈRES

TOPIC SESSIONS / SÉANCES THÉMATIQUES

NEW PHD REPORTS / PRÉSENTATIONS DE THÈSES DE DOCTORAT

AD HOC SESSIONS / SÉANCES AD HOC

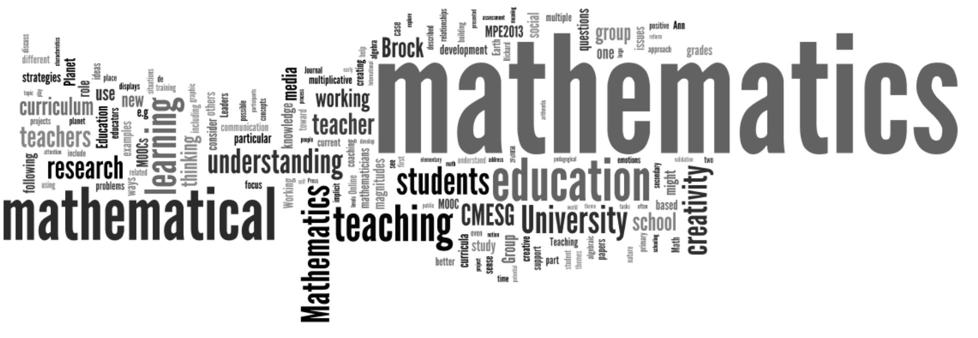
MATHEMATICS GALLERY / GALLÉRIE MATHÉMATIQUE

APPENDICES / ANNEXES

INTRODUCTION

Elaine Simmt – President, CMESG/GCEDM
University of Alberta

The 2013 Brock meeting provided CMESG members with the opportunity to consider mathematics in terms of our relationship with it as individual learners and teachers, as well as a domain embedded in rapidly transforming contexts. This *Wordle*™ demonstrates well the heart of our work.



Our deliberations were enhanced by Rosa Leikin's presentation on the relationships between mathematical knowledge, creativity and talent and Bill Ralph's presentation in which he asked, "Are we teaching Roman numerals in a digital age?" These two plenaries set the stage for three days of working group discussions in which members explored issues of teaching and learning in online environments, creativity in mathematics, multiplicative thinking, the mathematics of planet earth, and curriculum re-conception. In these proceedings the working groups share their discussions. The topic study groups too demonstrated innovations in our field through the use of role-play in mathematics teacher education and the possibilities afforded by social media for mathematics and mathematics education. With the work of 8 new PhDs shared in the meeting and the reflections on mathematics education by Dr. Eric Muller in his Elder Talk we can be reassured that the Canadian Mathematics Education Study Group continues to live its legacy of bringing together mathematicians, mathematics teacher educators and mathematics educational researchers to consider the profound impact mathematics has in Canada.

I would like to take this opportunity to thank our colleagues, Drs. Chantal Buteau and Joyce Mgombelo, for their work hosting a wonderful conference at Brock last May. They and their team ensured that we were well fed, entertained and educated. Particular thanks to the Planning Committee—Laura Broley, Jeff Irvine, Assuntina Del Gobbo, Amanjot Toor; and the volunteers—Dianne Kenton, Kristina Wamboldt, Matt Klompmaker, Josh Farkas, Ryan Racine, Jessica Varga, Matt Chang-Kit, Tyler Plyley, David Nguyen, and Mike Dubé.

Horaire

Vendredi 24 mai	**Samedi 25 mai**	**Dimanche 26 mai**	**Lundi 27 mai**	**Mardi 28 mai**
9:30 – 16:30 **Pré-Conférence** *Recherches en didactique des mathématiques et enseignement des mathématiques : illusions, réalité et opportunités*	8:00 – 9:00 **Inscription**			
	8:45 – 10:15 **Groupes de travail**	8:45 – 10:15 **Groupes de travail**	8:45 – 10:15 **Groupes de travail**	8:45 – 9:15 **Nlles thèses (3)** 9:20 – 9:50 **Nlles thèses (4)**
	10:15 – 10:45 Pause café	10:15 – 10:45 Pause café	10:15 – 10:45 Pause café	10:00 – 10:30 **Séances *ad hoc***
	10:45 – 12:15 **Groupes de travail**	10:45 – 12:15 **Groupes de travail**	10:45 – 12:15 **Groupes de travail**	10:45 – 11:30 **La parole aux anciens** 11:30 – 11:45 Pause café 11:45 – 12:30 **Séance de clôture**
12:15 – 13:30 Dîner	12:30 – 13:30 Dîner	12:30 – 13:30 Dîner	12:30 – 13:30 Dîner	
14:30 – 18:45 **Inscription** 15:30 – 16:20 **Amis de FLM**	13:30 – 14:45 **Dessert et Gallérie mathématique** 15:00 – 15:30 **Petits groupes** 15:40 – 16:30 **Plénière 1 : Séance questions** 16:30 – 17:00 Pause café	13:30 – 14:30 **Plénière 2** 14:35 – 15:50 **Assemblée générale annuelle** 15:50 – 16:20 Pause café	13:30 – 14:00 **Petits groupes** 14:00 – 15:00 **Plénière 2 : Séance questions**	
17:00 – 18:45 **Souper**	17:00 – 17:30 **Nlles thèses (1)** 17:30 – 18:00 **Nlles thèses (2)**	16:30 – 17:30 **Séances thématiques** 17:30 – 18:00 **Séances *ad hoc*** 18:30 – ? **Souper** *Pond Inlet*	16:00 – 22:00 **Excursion** ** Niagara Falls* **Niagara-on-the-Lake Winery et Downtown* ** Excursion à pied* **Souper**	
18:45 – 19:30 **Séance d'ouverture** 19:30 – 20:30 **Plénière 1** 20:30 – ? **Réception**	**Souper libre** *Dans le Niagara*	21:30 – ? **Feu de camps**		

Schedule

Friday May 25	Saturday May 26	Sunday May 27	Monday May 28	Tuesday May 29
9:30 – 16:30 **Pre-Conference** *Mathematics Education Research and Mathematics Teaching: Illusions, Reality and Opportunities*	8:00 – 9:00 **Registration**			
	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 9:15 **New PhDs (3)** 9:20 – 9:50 **New PhDs (4)**
	10:15 – 10:45 Coffee Break	10:15 – 10:45 Coffee Break	10:15 – 10:45 Coffee Break	10:00 – 10:30 ***Ad hoc***
	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	10:45 – 11:30 **Elder Talk** 11:30 – 11:45 Coffee Break 11:45 – 12:30 **Closing Session**
12:15 – 13:30 Lunch	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	
14:30 – 18:45 **Registration** 15:30 – 16:20 **Friends of FLM**	13:30 – 14:45 **Dessert and Gallery Walk** 15:00 – 15:30 **Small Groups** 15:40 – 16:30 **Plenary 1: Q&A** 16:30 – 17:00 Coffee Break	13:30 – 14:30 **Plenary 2** 14:35 – 15:50 **AGM** 15:50 – 16:20 Coffee Break	13:30 – 14:00 **Small Groups** 14:00 – 15:00 **Plenary 2: Q&A**	
17:00 – 18:45 **Dinner**	17:00 – 17:30 **New PhDs (1)** 17:30 – 18:00 **New PhDs (2)**	16:30 – 17:30 **Topic Sessions** 17:30 – 18:00 ***Ad hoc*** 18:30 – ? **Dinner** *Pond Inlet*	16:00 – 22:00 **Excursion** * *Niagara Falls* **Niagara-on-the-Lake Winery et Downtown* * *Nature Hike* **Dinner**	
18:45 – 19:30 **Opening Session** 19:30 – 20:30 **Plenary 1** 20:30 – ? **Reception**	**Dinner on your own** *in Niagara*	21:30 – ? **Bonfire**		

Plenary Lectures

Conférences plénières

ON THE RELATIONSHIPS BETWEEN MATHEMATICAL CREATIVITY, EXCELLENCE AND GIFTEDNESS

Roza Leikin
(with Mark Leikin, Miri Lev, Nurit Paz, Ilana Waisman)

University of Haifa, Israel

This paper reports on a study that searches for deep insights into the nature of mathematical giftedness with special attention to the role of mathematical creativity. It introduces and explores distinctions between general giftedness *and* excellence in mathematics*, which are widely accepted as being identical constructs. The study is of a multidimensional nature, as it attempts to understand special qualities of mathematically gifted individuals in three dimensions:* cognition, neuro-cognition *and* mathematical creativity. *The study findings lead to the following hypotheses: (i) general giftedness and excellence in mathematics are interrelated but different in nature phenomena; (ii) general giftedness is not a necessary condition for excellence in mathematics; (iii) in the fluency-flexibility-originality triad, fluency and flexibility are of a developmental nature, related to excellence in school mathematics, whereas originality is a 'gift' mainly related to general giftedness; (iv) general giftedness is strongly associated with differences related to solving insight-based tasks both in creativity and neuro-cognitive diminutions.*

BACKGROUND

GIFTEDNESS AND EXCELLENCE IN MATHEMATICS

Mathematical giftedness is an extremely complex construct which implies high mathematical abilities. In the field of gifted education, mathematical giftedness is usually considered as a distinct type of specific giftedness which is opposed to general giftedness (Piirto, 1999). The psychometric definition of giftedness determined by Terman is an IQ (Intelligence Quotient), two SDs (standard deviations) above the population mean (usually 130) (Feldman, 2003; Silverman, 2009; Winner, 2000). Over the years, the popular notion of 'unitary' intelligence has been consistently challenged, as theorists started to broaden the notion of intelligence (Callahan, 2000). A number of theorists have developed broad, multi-dimensional formulations of giftedness and talent that are now widely accepted (Gardner, 1983/2003; Marland, 1972; Sternberg, 2000). In the adult population the criteria for giftedness are more restrictive; Nevo (2008), for example, mentions Nobel laureates. In children and adolescents, gifted students "are those identified by professionally qualified persons who by virtue of outstanding abilities are capable of high performance" (Davis & Rimm, 2004, p. 18). We use this definition in our study for both general giftedness and exceptional mathematical abilities.

In mathematics education literature, such terms as *mathematical giftedness*, *mathematical excellence*, *mathematical talent* and *high ability in mathematics* are often used as equivalent terms. The precise acquisition of mathematical abilities involves a broad range of general cognitive skills, including spatial perception, visuo-spatial ability, visual perception, visuo-motor perception, attention, and memory, including working memory (e.g., Hoard, Geary, Byrd-Craven, & Nugent, 2008; Meyer, Salimpoor, Geary, & Menon, 2010).

Krutetskii's (1976) seminal study introduces components of high mathematical ability in school children, including the abilities to grasp formal structures, think logically in spatial, numeric, and symbolic relationships, generalize rapidly and broadly and be flexible with mental processes. According to Krutetskii, students with high mathematical abilities appreciate clarity, simplicity, and rationality and can be characterized by the general synthetic component called *mathematical cast of mind*. In our study we combine cognitive and neuro-cognitive exploration of mathematical abilities with relatively advanced mathematical problems. For over 20 years, Lubinski Benbow, Webb and Bleske-Rechek (2006) tracked participants, identified before age 13, who scored in the top 0.01% on cognitive-ability measures. They demonstrated that high SAT predicts occupational success comparable to that of individuals attending world-class mathematics, science, and engineering graduate training programs. Mathematical invention, which is an integral part of the activities of research mathematicians, consists of four stages: initiation, incubation, illumination, and verification (Hadamard, 1945). Special attention here is given to illumination, which largely involves intuitive thinking that leads to mathematical insight. Insight exists when a person acts adequately in a new situation (van Hiele, 1986). Thus, success in insight-based problem solving can serve as an indication of mathematical giftedness among school students. Thus mathematical creativity was chosen as one of the dimensions in the present study. In addition, insight-based problems are integrated in our research tools in creativity and neuro-cognitive dimensions.

In our research, we connect the theories in mathematics education with the theories in general giftedness as they are reflected in a sampling procedure performed according to G- and EM-factors (G-*for general giftedness*, EM-*for mathematical excellence*).

RESEARCH IN THREE STUDY DIMENSIONS

Neuro-cognitive studies

There are several electrophysiological (neuro-imaging) techniques, such as fMRI, PET, and EEG, which are used in the research of intelligence (including giftedness) and mathematical ability (including excellence). Our study makes use of the EEG (namely, ERP) approach. Event Related Potentials (ERPs) are electrophysiological measures reflecting changes in electrical activity of the central nervous system related to external stimuli or cognitive processes occurring in the brain. These measures provide information about the process in real time before the appearance of the external response (Neville, Coffey, Holcomb, & Tallal, 1993). The ERP technique has useful applications in language-related research (e.g., Kaan, 2007) and recently was adapted for the study of creativity, solving of insight-based problems and mathematical processing (e.g., Dietrich & Kanso, 2010; O'Boyle, 2005).

Neuro-imaging research focuses on the underlying brain structures of intelligence. Intelligence is associated with the reciprocity of several brain regions within a widespread brain network (Colom, Karama, Jung, & Haier, 2010; Jung & Haier, 2007). Another branch of neuro-cognitive research focuses on the establishment of the relationship between intelligence and the extent of induced brain activity during cognitive task performance (Jausovec & Jausovec, 2000). This led to the formulation of the *neural efficiency hypothesis*, stating that brighter individuals display lower, and therefore more efficient, brain activation while performing cognitive tasks (e.g., Neubauer & Fink, 2009).

Neuro-imaging research identified several brain regions in the parietal and frontal lobes connected to the processing of number sense and arithmetic (e.g., Arsalidou & Taylor, 2011; Nieder & Dehaene, 2009). There are several studies on algebraic equation-solving in adult students (e.g., Anderson, Betts, Ferris, & Fincham, 2011), different representations of functions (Thomas, Wilson, Corballis, Lim, & Yoon, 2010), calculus (Krueger et al., 2008) and geometry proof generation (Kao, Douglass, Fincham, & Anderson, 2008). Even though some recent studies have focused on brain activity when solving simple mathematical word problems, these studies have been performed without taking into account high abilities in mathematics (e.g. Lee et al., 2007).

Our study is aimed at characterising brain activity in senior high school students with extremely high abilities in mathematics. It applies theories of mathematics education to the design of mathematical items for ERP exploration (see the research procedure, described below).

<u>Memory</u>

Memory refers to the organism's ability to store, retain, and recall information. Memory that is differentially processed in circumstances that require storing material for a matter of seconds is referred to as *working memory* (WM), and in those that entail storing material for longer intervals, from minutes to years, is referred to as *long-term memory* (LTM) (Hilgard & Atkinson, 2000). *Short-term memory* (STM) entails recalling, for a period of several seconds to a minute, without rehearsal. *Working memory*, as described by Baddeley (1986), is composed of one central executive and three slave systems or buffers: the *phonological loop*, the *visuo-spatial sketchpad* and, more recently, the *episodic buffer*. Both the phonological loop and the visuo-spatial sketchpad are in direct contact with the central executive system (Baddeley & Logie, 1999; Engle, Laughlin, Tuholski, & Conway, 1999; Hoard, et al., 2008; Swanson & Jerman, 2006).

Gifted children are shown to display higher memory capacity compared to non-gifted children (Harnishfeger & Bjorklund, 1994; Gaultney, Bjorklund, & Goldstein, 1996). Beier and Ackerman (2004) found that memory tasks correlated with intelligence measures (r's > 0.70). Ackerman, Beier and Boyle (2005) conducted a meta-analysis (a literature search that ranged from 1872 through 2002) to ascertain the relationship between WM and intelligence and immediate memory and intelligence. They found a correlation between four measures of STM in which ability measures pointed to a relation between simple span memory and intelligence. Researchers examined the relations between STM, WM, and fluid intelligence (gf). Unsworth (2010) examined participants' performance on various WM tasks, including recall, recognition, fluid intelligence (gf) and crystallized intelligence (gc) measures. The results suggested that all three memory constructs were substantially related to gf, and less related to gc.

Memory characteristics are shown to be related to mathematical performance. Working memory (WM) is vital to many aspects of mathematical learning (Meyer et al., 2010). Working memory storage is fundamental to the solution of complex (multi-step) arithmetical problems (Hitch, 1978; Hoard et al., 2008). Mathematical precociousness has been associated with higher central executive performance (Swanson & Jerman, 2006; Hoard et al., 2008; Smedt et al., 2009). Phonological loop plays an important role in arithmetic, presumably in counting or in keeping track of the operands while calculating (Fürst & Hitch, 2000; Imbo, Vandierendonck, & DeRammelaere, 2007; Smedt et al., 2009). Children with high ability in mathematics use direct memory retrieval to solve addition problems significantly more frequently than children in the low-ability mathematics group. Solving mathematical problems requires reducing accessibility of less relevant information that could overload working memory and interfere during processing (Agostino, Johnson, & Pascual-Leone, 2010). Individuals with higher math ability have less difficulty in preventing unnecessary information from entering WM.

The literature review demonstrated the importance of investigation of cognitive characteristics of students possessing exceptional mathematical talents, with special attention to their general ability level and excellence in mathematics.

Mathematical creativity

One of the research questions that requires special attention of the mathematics education community is the relationship between *mathematical creativity* and *mathematical giftedness*. There is no single, authoritative perspective or definition of creativity (Mann, 2006; Sriraman, 2005). There are a variety of views on creativity and they keep changing over time. Based on research literature, Mann (2006) argues that there are more than 100 contemporary definitions of creativity.

Mathematical creativity in school mathematics is usually connected with problem solving or problem posing (e.g., Silver, 1997). Following Torrance (1974), Silver (1997) suggested developing creativity through problem solving through the following three parameters: *fluency* is developed by generating multiple ideas, multiple answers to a problem (when such exist), exploring situations, and raising multiple ideas; *flexibility* is advanced by generating new solutions when at least one has already been produced; *novelty* is advanced by exploring many solutions to a problem and generating a new one. We use a model for evaluating mathematical creativity using Multiple Solution Tasks (Leikin, 2009, 2013). By doing so, we connect advances in mathematics education research with research on creativity and giftedness. It places mathematical insight at the center of the investigation of mathematical creativity to gain insights about the relationship between mathematical genius and mathematical creativity.

THE STUDY

The study links advances in gifted education research with advances in mathematics education by carefully choosing the research sample and distinguishing between mathematical excellence and general giftedness. Additionally, we chose students with exceptional achievements in out-of-school mathematics in order to examine exceptional giftedness in mathematics. We are the first to approach this population systematically.

GOALS

The study seeks to identify specific characteristics of students with high mathematical abilities in each one of the dimensions. It is of a multidimensional nature, as it attempts to understand special qualities of mathematically gifted individuals in *three dimensions*: *cognition*, *neuro-cognition* and *mathematical creativity*. The goal of the study is to describe mental characteristics of mathematical giftedness, which are quantitative traits at the end of a continuum of abilities, as well as those characteristics that are qualitatively different, including a precise explanation of the phenomena.

RESEARCH SAMPLE

A sample of about 200[1] students was chosen out of a population of 1200 10^{th}-11^{th} grade students (16-17 years old) with similar socio-economic backgrounds. The sampling procedure was directed at examination of G- (general giftedness) and EM- (excellence in mathematics) characteristics.

G-factor: The entire research population was examined using Raven's Advanced Progressive Matrix Test (RPMT) (Raven, Raven, & Court, 2000) which was used for two purposes: first for the validation of giftedness in students from the classes for gifted students; second in order to exclude gifted students from the group of excelling students who were not identified as gifted at earlier stages. Note that the majority of students in the G group of participants studied in classes for gifted students. The students in the G group are representative of 1% of the population in their age group, at most.

EM-factor: The EM characteristic was identified according to the level at which the students studied mathematics in school, their score in school mathematics at a particular level, and the SAT-M (Scholastic Assessment Test in Mathematics, adopted from Zohar (1990)) score that students received during the sampling procedure. Students from the EM group are representative of about 5% of the general population.

After the sampling procedure, in which 1200 students took part, 200 students were subdivided into four experimental groups, determining the research population according to varying combinations of the EM and G factors:

- G-EM group: students who are identified as generally gifted and excelling in mathematics;
- G-NEM group: students who are identified as generally gifted but do not excel in mathematics;
- NG-EM group: students excelling in mathematics who are not identified as generally gifted;
- NG-NEM group: students who are identified as being neither generally gifted nor excelling in mathematics.

RESEARCH TOOLS

We used several research batteries in each one of the study dimensions. Here we briefly present tools which are relevant to results presented in this paper.

Neuro-cognitive dimension

To examine brain activity associated with solving mathematical problems we designed nine tests of which six proved to be reliable (Alpha Chronbach > 0.68). Computerized tests were designed and administered using E-Prime software (Schneider, Eschmann, & Zuccolotto, 2002). Each test included 60 tasks (trials). All tasks were presented in the center of the computer screen and displayed in black characters on a white background.

Here we report on the Insight-based Test. Each task in the test was presented in three windows with different stimuli (S1 – presenting a situation stage, S2 – question presentation stage and S3 – answer verification stage) that appeared consecutively (see example in Figure 1).

[1] The numbers varied slightly in different study dimensions.

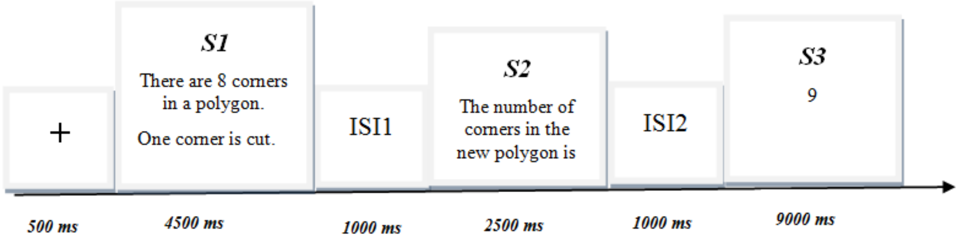


Figure 1. Examples of the item design for tests T6.

This item design was invented based on our understanding of mental processes involved in solving mathematical problems.

Accuracy for each participant is determined by calculating the percentage of correct responses. RT (reaction time) is determined as the mean RT for the answers in all trials of the test. RT for correct responses and accuracy are two interdependent measures (Pachella, 1974; Jensen, 2006) and, therefore, are examined as two measures in the same MANOVA.

<u>Electrophysiological data analysis</u>

Event Related Potentials (ERPs) were analyzed offline using the *Brain Vision Analyzer* software (*Brain-Products*). ERPs were Zero Phase Shift filtered offline (bandpass: 0.53 Hz–30 Hz) and referenced to the common average of all electrodes. Epochs with amplitude changes exceeding ±80 µV on any channel were rejected. Ocular artifacts were corrected using the Gratton, Coles and Donchin (1983) method. The ERP waveforms were time-locked to the onset of S1, to the onset of S2 and to the onset of S3. The averaged epoch for ERP, including a 200 ms pre-trigger baseline, was 2200 ms for S1, 2500 for S2 and 3200 ms for S3 (for which only the correct answers were averaged). The resulting data were baseline-corrected, and grand wave was calculated for each segment. Table 1 summarizes the statistical analysis performed for T6.

	Statistical test	Stage	Time frame	Between subjects differences	Within subjects differences	Measures
P1		S1 S2 S3	S1: 100-175 ms S2: 90-200 ms S3: 80-200 ms		Laterality: (Left, Middle, Right)	Latency and Amplitude at P, PO,O electrodes
P2	Repeated measures MANOVA	S1	175-250 ms	G factor		Latency and Amplitude at AF, F, FC electrodes
Late potentials	Follow up ANOVA Pairwise comparison	S1 S2 S3	For each of the 3 stages: 250-500 ms 500-700 ms 700-900 ms	E factor Interaction G × E	Laterality (Left, Middle, Right) Caudality (AF, F, FC, P, PO, O) Laterality × Caudality	Mean amplitude at each of 18 electrodes AF3, AFz, AF4, F3, Fz, F4, FC3, FCz, FC4, P3, Pz, P4, PO3, POz, PO4, O1, Oz, O2

Table 1. Summary of statistical analysis procedure for electrophysiological data. (For more details see, for example, Leikin, Waisman, Shaul, & Leikin (2012).)

<u>Cognitive dimension</u>

The memory functions were examined using three standardized tests:

(a) Short Term Memory - Digit Span test (WISC III, Wechsler, 1997),
(b) Working Memory for Digits and Letters test (WISC III, Wechsler, 1997), and
(c) Visio-Spatial Working Memory test (Corsi, 1972).

All three tests are known to be good indicators of memory ability (including WM). Data collection for all the tests was performed according to the standard procedures of data collection recommended by the tests' manuals (see details in Leikin, Paz-Baruch, & Leikin, 2013).

A multivariate analysis of variance test (MANOVA) was used to compare the accuracy and time of task performance (when applicable) in each test as a whole and on internal parts of the tests. *Between-subject differences* were examined for each one of the tests for G-factor, E-factor, interactions between G × E factors, and gender. *Within-subject differences* were examined for performance on the different tests.

<u>Creativity dimension</u>

The test consisted of five problems. The students were asked explicitly to solve each problem in as many ways as possible (see examples in Figure 2). Duration of time was an hour and a half. Data were collected for 665 students, of which 191 belonged to the research sample. This was done in order to allow examination of relative creativity (see details in Leikin & Lev, 2013).

	Solve in as many ways as possible:
Calculation task	Find the value of the expression: $2\frac{1}{4} \times 1.75$
Word problem 	Mali produces a strawberry jam for several food shops. She uses big jars to deliver the jam to the shops. One time she distributed 80 liters of jam equally among the jars. She decided to save 4 jars and to distribute jam from these jars equally among the other jars. She realized that she had added exactly 1/4 of the previous amount to each of the jars. How many jars did she prepare at the start?

Figure 2. Examples of tasks in the creativity test.

Correctness of the solution for a problem was evaluated according to the most complete solution produced by the student to the problem. For a complete solution, a student received 25 points. We evaluated fluency, flexibility, and originality according to the model described in the supplementary *creativity tool* file.

Multivariate analysis of variance test (MANOVA) was used for examination of main effects of G- and E-factors as well as of interactions between them with respect to the correctness of mathematical performance and each one of the creativity components on each task in the tests, with consequent ANOVAs and pair-wise comparisons (G vs. NG in EM and NEM groups and EM vs. NEM in G and NG groups, separately).

FINDINGS

In this paper we only exemplify some of the findings related to the tests described in the previous section in order to outline the main insights of our study.

NEURO-COGNITIVE DIMENSION

Behavioural findings

MANOVA demonstrated that general giftedness and excellence in school mathematics had significant effects on solving insight-based tasks (Table 2). We found that G participants were significantly more accurate than their NG counterparts. Pair-wise comparisons found that the significant difference in accuracy between G and NG was both in EM and NEM students. Additionally, EM adolescents were significantly more accurate and faster for correct responses than their NEM counterparts. However, there were no significant differences in Acc (accuracy) and in RT (reaction time) between EM and NEM participants among NG students, whereas among G students these differences were significant. Surprisingly G-NEM students had the longest RT among all four groups. Furthermore, G-NEM students were more accurate than NG-EM students when performing the tasks (though this difference was not significant) (Table 2). Additionally, we found a marginally significant interaction of G and EM factors on RTc.

Measure		Mean (SD) G	Mean (SD) NG	**Overall**	EM factor F (1,71)
Acc	EM	61.4 (7.3)	53.6 (9.9)	57.6 (9.4)	8.199**
	NEM	55.6 (7.0)	48.3 (8.2)	52.0 (8.4)	
	Overall	58.3 (7.6)	50.8 (9.3)		
	G factor F (1, 71)	15.066***			
RTc	EM	1744.2 (361.4)	2021.0 (616.7)	1878.4 (513.2)	9.035**
	NEM	2393.0 (608.2)	2153.5 (553.7)	2273.2 (586.4)	
	Overall	2086.6 (598.2)	2092.9 (578.4)		
	G factor F (1, 71)	.021			

$^{*}p \le 0.05$, $^{**}p \le 0.01$, $^{***}p \le 0.001$

Acc – Accuracy, RTc – Reaction time for correct responses

Table 2. Impact of G- and EM-factors on RT and Acc.

These findings suggest that the G-factor had a strong impact on insight-related problem-solving performance, especially in the NEM participants. In what follows, electrophysiological data provide additional information on the main effects of G- and EM-factors.

Electrophysiological findings (examples)

We observed significant interaction of Laterality and G-factor (Laterality × G-factor) in the 500-700 and 700-900 ms time frames at S2 (Figure 3).

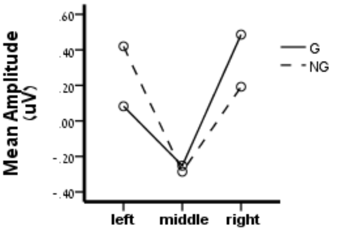


Figure 3. Mean amplitude in G vs. NG at S2 in the 500-700 ms at three laterality sites.

Among other findings, the study revealed a significant interaction between hemispheric laterality and the G-factor only at the question presentation stage (i.e., S2) which was more prominent in later parts of the time course (after 500 ms and especially in the time interval between 700 and 900 ms). That is, G participants demonstrated higher mean amplitudes in the right hemisphere, whereas NG participants activated the left hemisphere more prominently. We propose that these differences in activation patterns between these two groups may be explained by differences between the processing strategies used by G and NG individuals.

The results demonstrate that in a time period starting from 500 ms at the answer verification stage (i.e., S3), G students have a lower overall mean amplitude for correct responses; that is, they seem to exhibit a more efficient brain activation pattern during this cognitive task (Neubauer & Fink, 2009). Additionally, a significant interaction between G- and EM- factors was found in the time interval from 700 to 900 ms. In this case, the mean amplitude of G-NEM participants was not significantly different from NG-EM and NG-NEM students. However, the mean amplitude in G-EM individuals was found to be the lowest and significantly different from that in G-NEM participants.

COGNITIVE DIMENSION

MANOVA revealed significant main effects of the G-factor and gender on student performance in memory tests (Figure 4). Univariate ANOVA tests showed that the Short Term Memory test constitutes the main source for the effects of both G-factor and gender. Students' scores on Short Term Memory were higher among G students than among NG students, and among male participants more than among female participants. In contrast, a main effect of the EM-factor was found to be marginally significant on the Visio-Spatial Memory score (see Figure 4). In this case, EM students outperformed NEM students. An interaction of G- and EM-factors was found with regard to the Working Memory total score. The effect of the EM-factor was opposite for G and NG students. The Working Memory scores of EM students were similar in G and NG groups of participants. However, the Working Memory scores in NEM students were significantly higher for G than for NG students (see Figure 4).

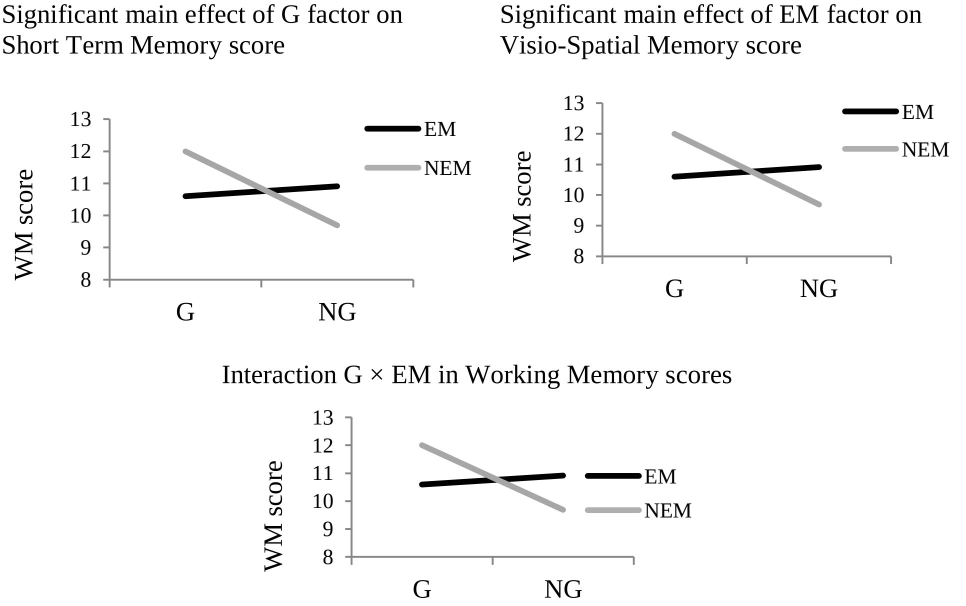
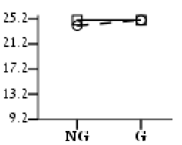
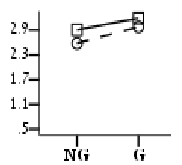
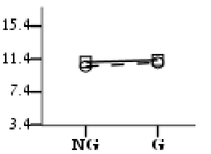
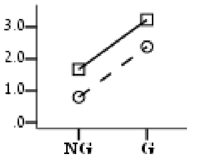


Figure 4. Significant effects and interactions of G- and EM-factors in three memory tests.

CREATIVITY DIMENSION

We report here on the results related to the calculation problem only. Figure 5 demonstrates the estimated marginal means for each criterion for the calculation problem (Figure 2).

We found that all students from all research groups (not surprisingly) except the NG-NEM group solved this arithmetic calculation problem correctly and thus received the highest score of 25 points (see Figure 5). G-EM students received the highest scores on all of the parameters examined: they all solved the problem correctly; most of them produced more than 3 solutions that belong to different groups of solutions.

	Cor	Flu	Flx	Or
$2\frac{1}{4} \times 1.75$				
		G** factor: F (1,180)= 11.618** EM** factor F (1,180)= 7.612**		G** factor: F (1,180)= 8.666**

*p < 0.05 **p < 0.01 ***p < 0.001
Cor - Correctness, Flu - Fluency, Flx - Flexibility, Or - Originality, Cr - Creativity

Figure 5. Estimated marginal means and significant between group differences.

MANOVA demonstrates that there are significant differences between the research groups that can be seen by the significant main effect of G-factor (Wilk's $\Lambda = 0.906$, $F_{(5,176)} = 3.646$, $p < 0.01$). Follow-up ANOVA for the analysis of the effect of G- and EM-factors on fluency revealed a significant main effect of both G-factor ($F_{(1,180)} = 11.618$, $p < 0.01$, $\eta^2 = 0.061$) and EM-factor ($F_{(1,180)} = 7.612$, $p < 0.01$). ANOVA also revealed a main effect of G-factor on originality and creativity (Or: $F_{(1,180)} = 8.666$, $p < 0.01$, Cr: $F_{(1,180)} = 8.312$, $p < 0.01$).

Pair-wise differences showed that for the originality criteria, the G-EM students' scores are different from the NG-NEM students' scores. Analysis of the fluency exhibited by the students when solving this arithmetic calculation problem demonstrated that the fluency of NG-NEM students differed significantly from that of gifted students in both the G-EM and G-NEM groups. This finding demonstrates that on problems that are strongly related to the mathematics curriculum and are relatively simple, the differences on the fluency and originality criteria reflect the differences between two examined characteristics – G and EM. Interestingly, no between-group differences were obtained on the flexibility and correctness criteria. While all students were able to produce at least one correct solution for the problem, they succeeded in producing different solutions. The main between-group difference appeared in the quality of solutions reflected in the originality score with significant impact of G-factor.

MAIN INSIGHTS FROM THE STUDY

The three dimensions of the study are complementary. Their combination allows us to take a step towards deeper understanding of the phenomena of mathematical giftedness and mathematical creativity. Based on the findings exemplified herein, and on the additional findings that were not presented in this paper due to space constraints, we arrive at the following hypothesis on which our current research is focusing.

G- and EM-factors reflect different personal traits: naturally G and EM appear to be interrelated personal traits, however G- and EM-factors have different main effects on performance on different types of tasks in each of the three dimensions, as well as G- and EM-factors interact on many characteristics (see for example, findings in the cognitive dimension). We argue that while G is not a necessary condition for EM, EM is not necessarily an indicator of G. Thus the G-factor allows G-NEM students to outperform NG-EM students when solving insight-based tasks.

The combination of G- and EM-factors leads to the manifestation of cognitive, neuro-cognitive and creative properties at an upper edge of a continuum (discovered on a number of tasks). Several tasks led to the observation that mathematical giftedness, expressed in G-EM students, can be characterized by qualitatively different phenomena in students. An especially important finding is related to the brain efficiency effect that consistently was found as specific for G-EM students. We also found a right shift of insight-related brain activation in gifted participants.

We hypothesize that in the fluency-flexibility-originality triad, fluency and flexibility are of a developmental nature (EM-related), whereas originality is a 'gift' (G-related). Originality appeared to be the strongest component in determining creativity while the G-factor is strongly associated with differences related to solving insight-based tasks both in creativity and neuro-cognitive diminutions.

As a final comment to this presentation of our study we claim that even though mathematical ability seems not to be a part of general giftedness, rather it is a specific ability, mathematical giftedness seems to require general giftedness and can be characterized by high mathematical creativity associated with originality of mathematical thought.

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ARE WE TEACHING ROMAN NUMERALS IN A DIGITAL AGE?

Bill Ralph
Brock University

SMART PHONES AND THE NEW KNOWLEDGE ECONOMY

Do smart phones signal the end of mathematics education as we know it? The other day in a second-year course on simulation and modeling, I reached the point where we had to integrate $\sin x$ from 0 to π. A student put up her hand and said, "Dr. Ralph let me do it", and she took out her phone and spoke the integration aloud. She then presented me with the phone showing the complete calculation including all of the intermediate steps. I could tell she was wondering why she had worked so hard in first-year calculus, learning to do what her phone could already do better than she could.

As a second example, a student in a colleague's office asked why she had to learn how to graph functions using calculus when her phone did it so easily. My colleague asked to see the graph of a challenging function and was surprised that the display on the phone contained just about everything you would ever want to know and also offered the ability to zoom in and out. What does it mean when our students are actively questioning our curriculum because they have a better understanding than we do of the expert tools available to do mathematics? How can we ask students to master algorithms for long division, the quadratic formula or the integration of rational functions when they can simply speak questions into their phones and get the answers immediately?

This immediate access to information is the cultural norm for students growing up with the vast online repository of knowledge and skills the internet has become. Given the constant presence of this resource, our students have shifted away from memorizing facts to simply remembering where to find them. Members of this new online culture have no interest in copying portions of the internet into their own brains! As a consequence, *whether we like it or not*, the current generation is in the process of retiring certain older approaches to knowledge and information like the following:

1. Memorization of simple concrete facts beyond a common cultural core.
2. Unassisted hand calculation.
3. Machine correctable information such as grammar and spelling.
4. Transmission of information in physical documents or face to face.
5. Recording information by hand.
6. Working with purely textual information that is not interactive or supported by media.

The retirement of these long-standing skills opens up our mathematics curriculum to the inclusion of proficiencies like the following that are more appropriate to the new knowledge-based economy:

1. Understanding of the conceptual underpinnings and applications of a wide variety of mathematical models and the ability to adapt this knowledge to new situations.
2. The ability to assess a technical situation and correctly select the right conceptual framework and associated expert systems.
3. The ability to find and learn specialised mathematical information.
4. The ability to manipulate, visualize and analyze large data sets.
5. The ability to create and handle computer programs in a variety of contexts to solve mathematical problems and process information.

These new skills are a practical necessity in a trans-textual world where information is conveyed by multimodal expert systems capable of simulating intelligent interactivity. In the presence of this wealth of technology, we have to wonder why the mastery of pencil and paper algorithms is still the primary measure of mathematical achievement in our schools. Our continuing struggle to update our traditional classrooms can be compared with the difficulty Europe had in making the transition from Roman Numerals to the Hindu-Arabic system.

ROMAN NUMERALS: A CAUTIONARY TALE

While Europe was under the spell of Roman Numerals, the Muslim mathematician Al-Khwarizmi in 825 AD was publishing efficient algorithms for calculation based on the Hindu-Arabic system. These new methods so impressed Pope Gerbert as a young man studying in 10th century Spain that he tried to leverage his status as Pope to convince Europe to give this obviously better system a try. But even a Pope couldn't shift European mathematics and nothing changed until Fibonacci published a user-friendly account of the Hindu-Arabic approach in his great book the Liber Abaci in 1202. His book was so methodical and understandable that it began to be taken seriously and eventually served as the principal textbook in mathematics for centuries. It only goes to show what can be achieved by a great mathematics teacher with a few good stories about frisky rabbits! Leonardo Da Vinci's teacher, Pacioli, wrote about finding Fibonacci's book later in his life, around 1490, so we know that the transition to the Hindu-Arabic system took several hundred years.

Table 1 is a comparison of two transitions: the historical transition from Roman Numerals to the Hindu-Arabic system and the transition we are currently making from no technology to what I will call *Technology Assisted Problem Solving and Information Retrieval* (TAPSIR for short).

The transition to TAPSIR thinking coincides with the extraordinary transformation in human culture that is happening around us and happening so quickly that it's hard to know what we should be teaching in mathematics. I think it's possible that by 2040 there could even be a third column in the table below and the transition time could be just a few short years. For now, it's important to not overreact and to cautiously consider what some of the elements of the new curriculum might look like.

	Transition 1 **Roman → Hindu-Arabic**	**Transition 2** **No Tech → TAPSIR**
Time Period	900 -1500 AD	NOW
Old System	Roman Numerals	No Tech. Individuals master hand calculation and memorize portions of the knowledge base with no assistance from technology.
New System	Calculation using the Hindu-Arabic System	TAPSIR (Technology Assisted Problem Solving and Information Retrieval)
Advantages of the Old System	Roman Numerals look good on monuments.	Individuals are able to work more independently of technology. A certain amount of hand calculation and memorization helps conceptual understanding.
Advantages of the New System	Much faster and more accurate. More conceptually clear.	Extremely fast and able to handle the complexity of our current culture. Ends the need for humans to calculate by hand or memorize large amounts of information. Allows students to spend more time learning concepts and how to apply them. It is trans-textual in the sense that it uses media to transform information into visual and auditory forms Enables large scale projects by groups of people.
Disadvantages of the New System	Major investment of time to learn to write and calculate with the new numbers.	Loss of conceptual understanding as the knowledge of algorithms for calculations and much of the knowledge base is off-loaded to computers. Individuals are no longer autonomous but must work in conjunction with technology. Loss of the textual and computational culture core that we all had in common. Already, many university students don't know the area of a circle or the volume of a box.
First adopters of the New System	Traders around the Mediterranean.	Businesses, engineers and young people (who grew up immersed in online social networks).
Factors Inhibiting the Adoption of the New System	Records all kept in Roman Numerals. Education system entrenched in centuries of Roman Numerals.	Cultural emphasis on creating completely autonomous individuals. Education system entrenched in centuries of hand calculation and memorization. Many teachers are still illiterate in the sense that they can't write or even handle computer programs that solve mathematical problems or process information. TAPSIR pedagogy is in its infancy (the MICA program at Brock University is an attempt in this direction).
% of Population Impacted During Transition	< 10%	> 90%
Time Taken for Transition	Centuries	Decades
Eventual Outcome	Roman Numerals are an historical curiosity.	By 2040, TAPSIR will be the norm— the internet will have evolved into a nearly sentient expert system that has completely devoured our current educational culture.

Table 1

‘TAPSIR’ THINKING

We often award high marks to students who are adept at turning the handles of trusty algorithms to get “the correct answer”. But haven’t we reached a turning point in mathematics education where such skills have largely been rendered obsolete by the simple possession of a smart phone? When skills are no longer needed by our culture, what justification can we offer for retaining them? A standard answer to these questions is that in each educational context there is a common cultural core of skills that we expect students to have that might include the 5-times table, long division, the area of a circle, the quadratic formula, etc. A better justification might be that a certain amount of proficiency with these ancient algorithms can promote students’ conceptual understanding and enable them to make the best use of technology in handling mathematical problems.

In a future that accommodates both the traditional and technological flavours of mathematics, students might be encouraged to learn two complementary types of knowledge: *autonomous knowledge* consisting of skills and memorized facts that can be demonstrated without the assistance of technology, and *linked knowledge* that is contingent upon access to expert systems. Autonomous knowledge would concentrate on traditional elements of mathematics like theorems and proofs, as well as solving problems by hand. Linked knowledge would revolve around *Technology Assisted Problem Solving and Information Retrieval* or TAPSIR for short. TAPSIR pedagogy is in its infancy and is so different from traditional teaching that most of us, including myself, are nervous about handling its implementation. However, society’s need for this approach is so great that eventually mathematics teaching will have to be extended to include activities like the following:

1. Using expert systems to explore concepts and perform calculations
2. Visualizing and analyzing large data sets
3. Writing and using computer programs to build models, create simulations and investigate mathematical problems
4. Investigating systems that depend upon one or more parameters and learning how to describe the changes in the system’s behavior as the parameter(s) change
5. Making and testing conjectures.

The MICA program at Brock University is an attempt to realize a TAPSIR-style program that implements objectives 1 – 5. The level of commitment and creative engagement of students in these courses have been remarkable to watch. Students learn *VB.NET* and even in first year are asked to make conjectures and then test them by writing computer programs. Over the course of the MICA program, they use technology to investigate a wide range of mathematical concepts such as RSA encryption, stochastic models, chaos, the stock market, epidemics, warfare, traffic light synchronization, predator-prey models, and in their third year they learn C++ and use it to study models based on partial differential equations. In other courses like calculus, they learn to do certain calculations with MAPLE which frees up time for learning concepts and applications. Students are also expected to work singly and in groups to create several original technology-based projects.

MICA and, more generally, TAPSIR, begin to address some of the emerging requirements of our new knowledge-based economy that are shown on the right hand side of Table 2.

Current Culture in Mathematics	Extra Pieces the World Needs Now
Study particular well-defined beautiful problems	Study systems that may be messy and extremely complex
Exact solutions	Approximate solutions
Visualizing functions with graphs	Visualizing complex information with a variety of media
Use technology as sparingly as possible	Leverage technology as much as possible
Data analysis of a handful of variables often with less than a 1000 sample points	Large scale data analysis of hundreds of interconnected variables—millions of sample points
Work alone or in small groups	Large teams of specialists—think tanks
Not very dependent on social and communication skills	Social and communication skills critical to success of project
Glory of the individual	Glory of the group

Table 2

CONCLUSIONS

The children of this Brave New Mathematical World will grow up in effortless communion with the entire store of the world's information which will respond like a friendly puppy in its eagerness to fetch things for them. We can no longer justify the teaching of ancient algorithms that can now be trivially handled by friendly expert systems that we carry with us at all times. As mathematics educators, it is our job to define the role of teachers and students within this new paradigm. The revolution is upon us and we have to move quickly to assert and clarify the role of mathematics in the presence of extraordinary technological tools. We want to retain the tremendous accomplishments and culture of mathematics, but we are under siege these days as mathematics departments are closed or cut back around the world. Part of our problem is that mathematics departments have pursued a somewhat isolationist approach to the teaching of mathematics and have not actively pursued relationships with other specialties. In order for mathematics to continue to thrive, we must continue to demonstrate its relevance to modern times. We can achieve this goal by developing a mathematics curriculum that openly engages with other disciplines and also expands the palette of mathematics graduates to include the mastery of technological tools. With these thoughts in mind, I make the following recommendations.

RECOMMENDATIONS

1. Develop pilot projects for the exploration of TAPSIR pedagogy in mathematics.
2. At each grade level and in university, develop a mathematics curriculum for *Linked Knowledge* which would consist of skills and information that require the use of computers, smart phones, the internet, etc.
3. In several different contexts, students should learn to write sequences of instructions for computer programs to execute. For example they might learn a programming language, write *html* for a webpage, create a *Flash* animation, create an *EXCEL* macro, retrieve information from a database, etc.
4. Beginning in high school and continuing into university, students should acquire mastery of a program like *EXCEL* and learn to visualize, analyze and model data drawn from a wide variety of real world situations.

5. Every student graduating from high school should be able to make a webpage. In both high school and university, some assignments should require the creation of online interactive webpages that might include auditory and visual media.

6. Encourage students to creatively explore mathematical ideas and problems using technological tools.

SUMMARY OF THE MEDIA AND TECHNOLOGY USED IN THE TALK

My plenary talk used a variety of media to make its points concerning the role of technology in mathematics education. Here is a list of some of the examples I used to support my ideas:

1. I demonstrated computer programs written in *vb.net* by my 2nd year MICA class that illustrated how they used technology to explore the role of changing various parameters in predator-prey models and in the spread of epidemics. One of the themes of this course is the interplay between theory and practice so we try to give solid mathematical explanations for the phenomena students observe in the lab.

2. One of themes emerging from 20th century mathematics and its applications is that enormous complexity can arise from very simple mathematical feedback rules. Furthermore, it is difficult and at times impossible for traditional mathematics to fully explain these emergent phenomena. My art is based on the mathematics of discrete dynamical systems and illustrates some of the remarkable structure that can arise from simple feedback loops. Some of this art can be found on my website at www.billralph.com. I showed the video *Ice Flow* at http://vimeo.com/31485933, which is also based on very simple rules.

3. We listened to a piece by the Canadian artist Janet Cardiff that is based on a 40 part motet, *Spem in Alium*, by Thomas Tallis at http://www.youtube.com/watch?v=W0_FQ6FER74. As I mentioned in the talk, the interplay of the 40 voices made me wonder if I could find a way to mathematically estimate the number of truly independent stocks in the New York Stock Exchange. This number is now updated weekly on my website at www.portfoliomath.com and is an example of how mathematics can be used to analyze very large and complex sets of data.

4. We watched my animation *Brain Vibes* that can be seen at http://vimeo.com/58847910. I was asked by the Neuropsychology department at Brock University to create a visualization of the data they were collecting from 38 sensors attached to the scalp. They stipulated that the animation had to run in real time as the data was being acquired. With so much data being output so fast, I had to restrict myself to just drawing rectangles in order to keep up with the flow. I used the pair-wise correlations of the 38 data streams to determine the position and colour of each rectangle. Absolutely everything you see in this video, such as colour, size and movement, comes from the data.

Elder Talk

La parole aux anciens

THROUGH A CMESG LOOKING GLASS

A TRAVERS UN MIROIR DU GCEDM[1]

Eric Muller
Brock University

I will explore a few themes that, to me, are of central importance to the future of mathematics education in Canada. It is my hope that at least one of these will resonate with your interest.

Je vais explorer quelques thèmes qui, pour moi, sont d'une importance capitale pour l'avenir de l'enseignement des mathématiques au Canada. C'est mon espoir qu'au moins un d'entre eux résonnera avec vos intérêts.

Lewis Carroll (1871) in his book, *Through the Looking Glass and What Alice Found There*, provides me with insightful and fun ways to illustrate my talk.

I have used illustrations from *Animator*.

Le livre de Lewis Carroll (1872) *Alice de l'autre côté du miroir* me donne des moyens perspicaces et amusants pour illustrer ma présentation.

J'ai choisie des illustrations d'*Animator*.

"When you say 'hill,'" the Queen interrupted, "I could show you hills, in comparison with which you'd call that a valley."

"No, I shouldn't," said Alice,..: "a hill *can't* be a valley, you know. That would be nonsense …"

(Carroll, 1871, p. 29)

Animator, p. 4

– « Puisque tu parles de 'colline' », reprit la Reine, « moi, je pourrais te montrer des collines auprès desquelles celle-ci ne serait qu'une vallée pour toi ».

–« Certainement pas », déclara Alice, qui finit par se laisser aller à la contredire. « Une colline ne peut pas être une vallée. Ce serait une absurdité… » (Carroll, 1872, p. 26)

[1] Je remercie Lucie DeBlois et Caroline Lajoie pour leur aide à la traduction.

Normally a looking glass reflects what is happening behind us, but that is not the direction of the journey I have planned. My intention is to follow Alice and penetrate the CMESG looking glass in search of important research and experimental issues that will support a successful evolution of mathematics education in Canada.

Before we start the journey I share with you some of the baggage that I am carrying with me:

- my philosophical approach to progress in mathematics education; and
- my intertwined research and teaching experiences in mathematics and mathematics education.

Normalement, regarder dans un miroir reflète ce qui se passe derrière nous, mais ce n'est pas la direction du voyage que j'ai prévu aujourd'hui. Mon intention est de suivre Alice et de pénétrer le miroir GCEDM pour chercher à préciser les recherches et les expériences qui appuient une évolution réussie de l'enseignement des mathématiques au Canada.

Avant de commencer notre voyage, je partage avec vous quelques éléments du bagage que j'apporte avec moi :

- mon approche philosophique de ce qu'est progresser en enseignement des mathématiques; et
- l'interrelation de mes expériences en recherche et en enseignement des mathématiques dans ces deux domaines.

PHILOSOPHICAL OUTLOOK

André Brink (1996), in his book *Reinventing a Continent*, recounts that

> *the Greeks had a very different way, compared to ours, of assessing progress in time: we tend to think of time as a river running between its banks, carrying us with it, on a boat or raft perhaps, looking forward, the past behind our backs. But precisely this image presupposes that the future already exists, that we face it as we move towards it. Whereas the Greeks, using the same river language, suggested that we find ourselves on our little raft facing the past as the present sweeps past us and becomes visible, intelligible; while we move backwards into the future which does not exist yet.* (p. 65)

I don't visualize myself either in the front or at the back of the boat. I am drawn to the side of the boat, a position that does not feature in either the Greek's or the modern view of progress. A reason for this is that my philosophy of progress is grounded in the present, wherever the boat happens to be now, not where it may be in the future nor where it has been in the past. My efforts and energies in research and teaching have been directed at implementing progress in mathematics education within the constraints of the present.

RESEARCH AND TEACHING EXPERIENCES

My second piece of luggage contains my intertwined research and teaching experiences in mathematics and mathematics education. These have been strongly influenced by the advent and exponential growth of computer technologies.

- From 1964 to 1966, in my PhD studies at Sheffield University, I programmed numerical solutions of integral equations using an early Pegasus computer.
- In 1980 at ICME 4, held in Berkley, I participated in a hands-on session using a beta version of the MacSyma CAS.
- In 1985, I contributed to the first ICMI Study—*The Influence of Computers and Informatics on Mathematics and its Teaching* (Churchhouse, 1986).

- Both CMESG (http://cmesg.ca/) and MAA (http://www.maa.org/) have been important sources of ideas and inspirations, as I integrated technology in undergraduate mathematics education.

When I came to Brock in 1967, the university was barely three years old, and as a young faculty member I had the privilege to be given free rein to develop the service courses for the fledgling Math Department. From the beginning, I strove to integrate evolving technologies in all these courses (Muller, 2001). My horizon of technology in mathematics education was greatly expanded by Bill Ralph's (1999) *Journey Through Calculus* and MICA, the Department's innovative core mathematics program (Ralph, 2001), now in its 12th year.

The contents of my baggage provide a glimpse into my career in mathematics, which naturally has strongly influenced the issues that I have selected for exploration—issues for research and experimentation which, to me, are essential to ensure the future vitality of mathematics education in Canada. Gazing into the future, at the front of the boat, is not a comfortable position for me, and I can readily associate my situation to that of the red king.

'I see nobody on the road,' said Alice.

'I only wish I had such eyes,' the King remarked in a fretful tone. 'To be able to see Nobody! And at that distance too! Why, it's as much as I can do to see real people, by this light!'

(Carroll, 1871, p. 89)

Animator, p. 6

« Eh bien, que vois-tu ? » « Personne », répondit Alice. « Moi, je voudrais bien avoir des yeux comme les tiens », dit le Roi d'une voix chagrine. « Être capable de voir Personne ! Et à une si grande distance, par-dessus le marché ! Tout ce que je peux faire, moi, c'est de voir les gens qui existent réellement !»

(Carroll, 1872, p. 96)

THEME 1: TECHNOLOGY IN MATHEMATICS EDUCATION

Of the many areas for future research and experimentation in the use of technology in mathematics education, I have chosen two areas that I see rising above the rest in importance:

- developing and using transcendent technologies;
- rethinking the what of mathematics.

Because technology is evolving so rapidly we can expect, like the white knight, to experience an uneven ride.

THÈME 1 : TECHNOLOGIE EN ÉDUCATION DES MATHÉMATIQUES

Parmi plusieurs domaines à investiguer dans les recherches et les expériences futures quand on utilise les technologies en éducation des mathématiques, j'ai choisi deux domaines qui me semblent prendre une grande importance :

- le développement d'une utilisation transcendante (pérenne) des technologies;
- repenser ce que sont les mathématiques.

Parce que la technologie se transforme rapidement, nous pouvons prévoir, comme le cavalier blanc, une expérience inédite.

Whenever the horse stopped (which it did very often), he (the White Knight) fell off in front; and whenever it went on again (which it generally did rather suddenly), he fell off behind. Otherwise he kept on pretty well, except that he had a habit of now and then falling off sideways….

(Carroll, 1871, p. 105)

Animator, p. 6

Toutes les fois que le cheval s'arrêtait (ce qui arrivait très fréquemment), le Cavalier tombait en avant ; et toutes les fois que le cheval se remettait en marche (ce qu'il faisait avec beaucoup de brusquerie), le Cavalier tombait en arrière. Ceci mis à part, il faisait route sans trop de mal, sauf que, de temps en temps, il tombait de côté...

(Carroll, 1872, p. 116)

WHAT IS A TRANSCENDENT TECHNOLOGY?

As a means to differentiate and select cognitive technologies for mathematics education, Pea (1987) suggested the use of transcendent functions, namely,

> *We would like the functions to be transcendent in the sense that they apply not only to arithmetic, or algebra, or calculus, but potentially across a wide array, if not all, of the disciplines of mathematical education, past, present, and future.* […] *The transcendent functions to be highlighted are those presumed to have great impact on mathematical thinking. They neither begin nor end with the computer but arise in the course of teaching, as part of human interaction*. (p. 98)

For this presentation I suggest a parallel to Pea's notion by replacing the word *functions* by *technologies* and by changing the last sentence to read, "They neither begin nor end with the computer but arise in the course of student learning, as part of the student-computer interaction."

QU'EST CE QU'UNE TECHNOLOGIE TRANSCENDENTE?

Pour donner un sens à la question de différencier et de sélectionner les technologies cognitives pour l'éducation des mathématiques, Pea (1987) suggérait d'utiliser les fonctions transcendantes,

> *Nous aimerions des fonctions transcendantes dans le sens où elles s'appliqueraient non seulement à l'arithmétique, l'algèbre ou au calcul différentiel mais potentiellement à un large éventail, sinon toutes, les disciplines de l'éducation des mathématiques passées, présentes et futures…». Les fonctions transcendantes sont éclairantes, elles sont présumées avoir une grande influence sur la pensée mathématique. Elles ne commencent ni ne finissent avec l'ordinateur, mais contribuent au développement de l'enseignement comme partie de l'interaction humaine.* (p. 98)

Pour cette présentation, je suggère de faire un parallèle avec la notion de Pea en remplaçant le mot *fonction* par *technologie* et en changeant la dernière phrase pour lire «Ils ne commencent pas plus qu'ils ne finissent avec l'ordinateur mais contribuent au processus d'apprentissage de l'élève, comme partie de l'interaction élève-ordinateur».

Examples of transcendent technologies

- *Computer Mathematical Systems.* Most commonly known as CAS software, these continue to evolve by including an increasing number of mathematical topics and by embracing different educational levels, goals and teacher preferences. When the use of a CAS is systemic throughout the programs of a mathematics department, the software becomes transcendent in the students' mathematics learning. Experiences with the software arise in different courses, thereby providing an integration of mathematics across topic boundaries, and the student is able to develop an "intelligent partnership" (Martinovic, Muller, & Buteau, 2013) with the technology.

- *Programming/coding*. Programming has generally been kept on the fringes of mathematics education; however it can offer many advantages to the student learning mathematics. It can provide a problem solving experimental environment; it offers the possibility to explore generalizations, wherein the student poses "what if?" mathematical questions, etc. Programming delivers built-in self-assessment. For the teacher, it provides a window into the student's thinking, in terms of logic, approach to the problem, etc. As programming continues to evolve, (is simplified and made increasingly more user-friendly), it should be taken out of the closet and integrated into mathematics education through experimentation and research.

RETHINKING THE WHAT OF MATHEMATICS

Hoyles and Noss (2008) write:

> *Like Kaput, we noted that the incorporation of technologies into mathematical learning almost inevitably brings to the fore a range of key questions – particularly those concerned with transformation of the what of mathematics rather than merely the how – precisely because digital technologies disrupt many taken-for-granted aspects of what it means to think, explain and prove mathematically and to express relationships in different ways.* (p. 87)

Pea (1987) puts it another way.

> *Applied to mathematics education, this socio-historical perspective highlights not the constancy of the mathematical understandings of which children are capable at particular ages, but how what we take for granted as limits are redefined by the child's use of new cognitive technologies for learning and doing mathematics.* (p. 94)

REPENSER CE QUE SONT LES MATHEMATIQUES

Hoyles et Noss (2008) écrivent

> *Comme Kaput, nous observons que l'intégration des technologies dans l'apprentissage des mathématiques est aussi inévitable que les questions clés qui émergent de ce travail – particulièrement celles concernant les transformations de ce que sont les mathématiques plutôt celles portant sur le comment –précisément parce que les technologies digitales rompent avec plusieurs «évidences» de ce que signifie penser, expliquer, prouver mathématiquement et exprimer des relations de différentes façons.* (p. 87)

Pea (1987) situe cela d'une autre façon,

> *Appliqué à l'éducation des mathématiques, la perspective socio-historique n'éclaire pas la compréhension des mathématiques ce que les enfants sont capables de manifester de façon constante à différents âges, mais comment nous prenons pour acquis comme limites est redéfini par l'utilisation que l'enfant fait des nouvelles technologies pour apprendre et faire des mathématiques.* (p. 94)

At the 2002 CMESG conference I posed a question that goes to the heart of rethinking the *what* of mathematics,

> *...what would sequencing within mathematics courses and within mathematics programs look like if it was based on a conceptual hierarchy, and the requirements of the technical hierarchy were left to technology?* (Muller, 2003, p. 155)

Unfortunately, after more than ten years, I don't even have a partial answer to this question.

It is worth mentioning Brock's MICA program, established in 2002. In their major MICA projects students select the mathematics they will investigate in their *Exploratory Object* (EO), which they design, program and implement themselves, thereby transferring some of the technical hierarchy to the technology. Muller, Buteau, Ralph, and Mgombelo (2009) identify an EO as "an interactive and dynamic computer-based model or tool that capitalizes on visualization and is developed to explore a mathematical concept or conjecture, or a real-world situation (p. 64).

For examples of student Exploratory Objects, see (http://www.brocku.ca/mathematics/studentprojects). In 2011 and 2012 and in the first and second year courses, nearly 100 original EOs were created.

The MICA program illustrates how a mathematics department can rethink the *what* of mathematics, as it attends to the mathematics that is of interest to individual students and supports them in the development of their understanding through their explorations that would be out of reach without technology.

In the company of Alice we continue our journey:

En 2002, à la conférence du GCEDM, je posais une question qui visait l'action de repenser les mathématiques:

> *… quelle forme prendrait une séquence de cours de mathématiques et de programmes de mathématiques si elle était basée sur une hiérarchie conceptuelle et sur la nécessité d'utiliser une hiérarchie techniquement guidée par la technologie?* (Muller, 2003, p. 155)

Malheureusement, après plus de dix ans je n'ai pas encore une réponse partielle à cette question.

Cela vaut la peine d'indiquer le programme MICA de Brock établit en 2002. Dans leurs projets majors les étudiants choisissent les mathématiques qu'ils investigueront dans un *Objet Exploratoire* (OE) qu'ils organisent, programment et exécutent eux-mêmes, c'est-à-dire où ils transfèrent la hiérarchie techniquement guidée par la technologie. Muller, Buteau, Ralph, et Mgombelo (2009) formulent l'OE comme « un modèle informatique interactif et dynamique, ou un outil qui exploite la visualisation, et qui est développé pour explorer un concept mathématique ou une conjecture mathématique ou encore une situation de la vie réelle (p. 64).

Exemplaires de OEs, voir (http://www.brocku.ca/mathematics/studentprojects). En 2011 et 2012 et dans les cours de deuxième année, près de 100 OEs originaux ont été créés.

MICA, le programme, démontre comme un département de mathématiques peut repenser ce que sont les mathématiques à partir des intérêts mathématiques personnels des étudiantes et des étudiants de manière à supporter le développement de leur compréhension à travers leurs explorations, qui ne seraient pas possibles sans les technologies.

En compagnie d'Alice nous continuons notre voyage :

They stood so still that she quite forgot they were alive … she was startled by a voice coming from the one marked DUM.

"If you think we're wax-works," he said, "you ought to pay, you know. Wax-works weren't made to be looked at for nothing. Nohow!"

"Contrariwise," added the one marked DEE, "if you think we're alive, you ought to speak."

(Carroll, 1871, p. 50)

Animator, p. 3

Ils gardaient une immobilité si parfaite qu'elle oublia qu'ils étaient vivants. ... elle sursauta en entendant une voix qui venait de celui qui était marqué BONNET… « Si tu nous prends pour des figures de cire », déclara-t-il, « tu devrais payer pour nous regarder. Les figures de cire n'ont pas été faites pour qu'on les regarde gratis. En aucune façon! »... « Tout au contraire » ajouta celui qui était marqué BLANC « si tu crois que nous sommes vivants, tu devrais nous parler ».

(Carroll, 1872, p. 46)

The two characters Alice meets on her journey illustrate, for me, how the human mind can develop good skills to compare two concepts but finds it to be far more challenging to compare three or more concepts. For example, in Statistics, students perceive the tests for the difference of two means as straight-forward, however, they find ANOVA for the analysis of three or more means very taxing. There are many different pairs of ideas that are important for the future of mathematics education in Canada. From my point of view there are two notions that rise to the top of a list of importance. These are *communication* and *creativity*.

THEME 2: COMMUNICATION

Self-assessment is a communication skill, an internal communication that has to be learnt, as it is of primary importance for one to progress in mathematics.

Assessment of student's knowledge of mathematics is a communication skill to be developed between student and teacher.

Popularization, promotion, dissemination are communication skills that inform the general public. But, shouldn't the general public already have been informed through the many school mathematics courses they have taken?

As part of a 2004 CMESG panel, I pointed to an issue of communication with the

THÈME 2 : LA COMMUNICATION

L'auto-évaluation est une habileté de communication, une communication interne, de première importance parce qu'elle permet de progresser en mathématiques.

L'évaluation des connaissances mathématiques des étudiants est une habileté de communication à développer entre les étudiants et l'enseignant.

La popularisation, la promotion, la dissémination sont autant d'habiletés de communication qui informent le public en général. Le public ne devrait-il pas, déjà, être informé dans les cours de mathématiques vécus?

Comme participant à la table ronde du

general public and its possible consequences:

> *…the fact that the great majority of the population sees mathematics as not for them, nevertheless they are prepared to support a compulsory mathematics education in schools. How long can this support last? Could a small change in beliefs and opinions see mathematics moved from the compulsory to the optional side of the school's discipline ledger?* (Muller, 2005, p. 165)

In recent years my concern about this possible shift in emphasis on mathematics in the school curriculum has been heightened. This is because the few areas where the Canadian public was used to seeing mathematics in action have now been taken over by digital technologies.

Throughout my career I have been involved in popularising mathematics, however I now believe that a different and more significant response should also come from the second of the pair of issues, namely creativity as a focus in mathematics teaching and learning at all levels.

GCEDM 2004, j'ai indiqué qu'il y avait un défi à l'égard de la communication avec le grand public et qu'il y avait des conséquences possibles :

> *(le) fait que la plus grande partie de la population considère que les mathématiques ne sont pas pour elle. Néanmoins, les canadiennes et les canadiens sont prêts à soutenir l'idée des mathématiques obligatoires à l'école. Est-ce qu'un petit changement dans les croyances et les opinions pourrait permettre de voir les mathématiques passer d'un caractère obligatoire à un caractère optionnel dans le parcours académique des élèves?* (Muller, 2005, p. 165)

Au cours des dernières années, mon inquiétude face à cette évolution possible a été renforcée. C'est parce que les quelques zones où le public canadien a été habitué à voir les mathématiques en action ont été prises en charge par les technologies numériques.

Depuis longtemps je suis impliqué dans la vulgarisation des mathématiques mais je crois maintenant qu'une approche plus importante devrait provenir de la seconde de la paire de questions, c'est à dire la créativité dans l'enseignement des maths.

THEME 3: CREATIVITY

In their 2004 study, Kauffman and Bauer surveyed 241 university students and found that,

> *[i]n general, if students viewed themselves as generally creative, they also viewed themselves as creative in different areas. The only area that was not correlated with general creativity ratings was mathematics.* (p. 143)

To me this is an alarming outcome. However, it is my view that if more creative activities are built into mathematics education at all levels, the views about mathematics of a greater number of individuals will change to be more positive.

THÈME 3 : CRÉATIVITÉ

Dans leur étude en 2004, Kauffman et Bauer ont fait une enquête auprès de 241 étudiants universitaires et ont trouvé que,

> *En général, si les étudiants se voient eux-mêmes comme étant créatifs d'une manière générale, ils se voient aussi eux-mêmes comme créatifs dans différents domaines. Le seul domaine où il n'y a pas de corrélation avec cette créativité générale est les mathématiques.* (p. 143)

Pour moi, c'est un résultat alarmant, cependant on espère que si plus d'activités créatives étaient construites en éducation des mathématiques à tous les niveaux, la conception de plusieurs sur les mathématiques changerait pour être plus positive.

This is a major challenge for me as a teacher because, however innovative and creative I believe my great mathematics activity to be, the students are the ones who need to have the creative experience. My own perception is that creativity is a personal awareness and my students can react differently in the same situation.

En tant que professeur, c'est un défi majeur pour moi parce que, tel que je pense que soit l'activité innovante et créative de mathématiques que j'offre à mes étudiants, ce sont eux qui ont besoin d'avoir l'expérience créative. De mon point de vue, la créativité est une perception individuelle et mes étudiants pourront avoir des réactions différentes dans la même situation.

To experience creativity while learning and doing mathematics is a complex individual reaction whose origins and causes are very difficult to trace. The earliest record of a study on mathematical creativity appeared as a questionnaire in *L'Enseignement Mathematique* (Naud, 1902). Many treatises have been published since that time; some of these are referenced in the report of the 2013 CMESG Working Group, *Exploring creativity: From the mathematics classroom to the mathematicians' mind*, in this volume. In general, much of the focus has been on the product of mathematical creativity or on views of productive creative mathematicians. As a mathematics educator it would be helpful to understand the characteristics of mathematics teaching environments that may trigger a creative mathematical response by students. The Brock MICA courses provide an example of one of many possible such environments.

Buteau and Muller (2009) have proposed a student development model (shown in Figure 1 below as modified by Marshall (2012)) that details the sequence of a student's activities when creating his/her own *Exploratory Object*. The most important influence throughout the creation of his/her EO is that the mathematics learner proceeds in a way that is self-directed. From the beginning of the project the student is responsible for asking the mathematical question, that is his/her own conjecture or real world situation to explore and analyze. Earlier in the course, students had been guided through the process of asking mathematical questions that could be explored using technology. The question posing is followed by a period of, mainly on-line, research to determine what is known about the problem; this in turn may produce a modification of the original question. As the diagram shows, this is followed by a period of designing and programming an exploratory platform and finally reporting the results of their explorations and analyses. Throughout these activities there are many opportunities for students to feel that they are being creative.

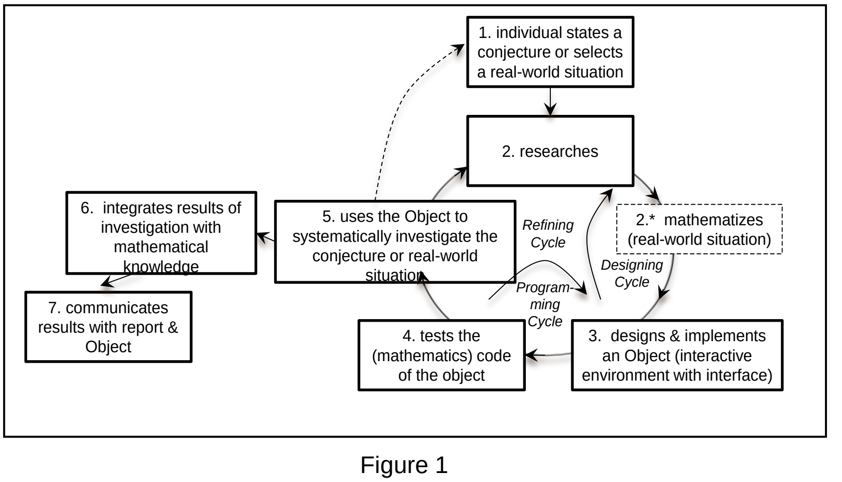

Figure 1

MY CLOSING THEME – SIMULATION

…said Alice. “It’s dreadfully confusing!” “That’s the effect of living backwards,” the Queen said… “but there’s one great advantage in it, that one’s memory works both ways.”

“I’m sure mine only works one way,” Alice remarked. “I can’t remember things before they happen.” “It’s a poor sort of memory that only works backwards,” the Queen remarked.

(Carroll, 1871, p. 66)

Animator, p. 7

MON THÈME DE CLÔTURE – LA SIMULATION

(Alice) « Je ne vous comprends pas. Tout cela m’embrouille les idées ! » « C’est toujours ainsi lorsqu’on vit à reculons », fit observer la Reine d’un ton bienveillant. « Au début cela vous fait tourner la tête ». (Alice) « Je n’ai jamais entendu parler d’une chose pareille ! » (la Reine) « Mais cela présente un grand avantage : la mémoire opère dans les deux sens ».

« Je suis certaine que ma mémoire à moi n’opère que dans un seul sens », affirma Alice. « Je suis incapable de me rappeler les choses avant qu’elles n’arrivent ». « Une mémoire qui n’opère que dans le passé n’a rien de bien fameux », déclara la Reine.

(Carroll, 1872, p. 66)

From the Report of the Working Group on Simulation, in the *2011 CMESG Proceedings* (Muller, Villeneuve, & Etchecopar, 2012), I note the following four significant points:

- Simulation has become one of the most important and widely used scientific methods for the analysis of complex systems.
- When mathematics students are exposed to and use simulation they are connecting to an important scientific approach.
- Experience in simulation prepares for future employment in the numerous disciplines that depends on simulation for analysis of complex systems.
- Simulation procedures should be one of the mathematical tools that students have at their disposal for analysing and solving mathematical problems.

J’extrais quatre points importants du rapport du groupe de travail sur la Simulation des *Actes du GCEDM 2011* (Muller, Villeneuve, & Etchecopar, 2012):

- La simulation est devenue une des méthodes scientifiques les plus importantes et les plus largement utilisées pour l’analyse de systèmes complexes.
- En se familiarisant avec la simulation, et en l’utilisant eux-mêmes, les étudiants de mathématiques prennent contact avec une approche scientifique importante.
- Une expérience en simulation prépare l’étudiant à plusieurs domaines d’emploi qui dépendent de la simulation pour l’analyse de systèmes complexes.
- Les procédures de simulation devraient être un des outils dont les étudiants disposent pour analyser et résoudre des problèmes mathématiques.

In 2011, the European Mathematical Society published a position paper on the European Commission's contributions to European research. In it we find:

> *Together with theory and experimentation, a third pillar of scientific inquiry of complex systems has emerged in the form of a combination of modeling, simulation, optimization and visualization.* (p. 2)

How will the mathematics education communities at all levels respond to the opportunities in mathematics generated by simulation?

En 2011, la Société Mathématique Européenne a publié un texte (Position Paper) sur les contributions en recherche de la Commission Européenne. On peut y lire :

> *Avec la théorie et l'expérimentation, un troisième pilier de l'étude scientifique des systèmes complexes a pris la forme d'une combinaison entre modélisation, simulation, optimisation et visualisation.* (p. 2)

Comment la communauté en éducation mathématique (tous niveaux confondus) répondra-t-elle aux nombreuses opportunités générées par la simulation ?

In the MICA program, designing, programming and implementing simulations is a principal mathematical activity for all math majors, co-majors and BSc/BEd students.

The end to our journey—the future starts now

(Caroll, 1871, p. 55)

"The time has come" the Elder said
"To act on many things:
On technology and simulation,
On creativity and communication.
And why our math is everywhere
But for most nowhere is found."

"with a tip of the hat to the walrus"

(Carroll, 1871, p. 56)

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Working Groups

Groupes de travail

MOOCS AND ONLINE MATHEMATICS TEACHING AND LEARNING

George Gadanidis, *University of Western Ontario*
Philippe R. Richard, *Université de Montréal*

PARTICIPANTS

Laura Broley	Matt Klompmaker	Robin Ruttenberg-Rozen
Gord Doctorow	Donna Kotsopoulos	Anna Sierpinska
Corina Georgescu	Kim Langen	Elaine Simmt
Susan Gerofsky	Peter Lee	Mary Stordy
Taras Gula	Anne LeSage	Marta Venturini
Jennifer Hall	Steven Pileggi	Dave Wagner
Jeff Irvine	Geoff Roulet	Kristina Wamboldt

INTRODUCTION

MOOCs (Massive Online Open Courses) are a recent development, building on an emerging trend for educational institutions (such as MIT, Harvard and Stanford) to make course content freely available on the Web. While online courses have been part of the educational landscape for a couple of decades, MOOCs offer two innovations. First, a MOOC is open, meaning that anyone with Internet access, and without having to meet any course prerequisites, can register and complete the course. Second, a MOOC is massive, meaning that there is no limit to the number of people who can take the same course. Devlin (2012) notes:

> *Stanford president John Hennessy has described the current changes in higher education initiated by technological innovations as an approaching tsunami. His remark was prompted largely by the emergence and rapid growth of MOOCs (massively open online courses), first from Stanford itself, joined soon afterwards by MIT and Harvard.* (para. 1)
>
> *But when you look a bit more deeply at the way MOOCs are developing, you see that the real tsunami is going to be a lot bigger than that. It's not just higher education that will feel the onslaught of the floodwaters, but global society as a whole.* (para. 3)
>
> *Forget all those MOOC images of streaming videos of canned lectures, coupled with multiple-choice quizzes. Those are just part of the technology platform. In and of themselves, they are not revolutionizing higher education. We have, after all, had distance education in one form or another for over half a century, and online education since the Internet began in earnest over twenty-five years ago. But that*

> *familiar landscape corresponds only to the last two letters in MOOC ('online course'). The source of the tsunami lies in those first two letters, which stand for 'massively open'.* (para. 4)
>
> *Right now, the most popular MOOCs draw student enrollments of about 50,000 to 100,000. In this it's not unreasonable to expect those numbers to increase by at least a factor of 10, once people realize what is at stake.* (para. 5)

Given these innovations, the goal of our Working Group was to better understand the MOOC phenomenon and its implication for mathematics teaching and learning.

The initial plan of our Working Group was to explore the following questions: What is a MOOC in mathematics education? What are the pedagogical and didactical issues in a MOOC? Where/how do other tools and approaches fit in? Where do we go from here? Although we did tackle these questions, as the three days of our discussion unfolded, we found ourselves focusing more on the MOOC phenomenon in general terms and less on specific implications for mathematics education. This is understandable given that MOOCs are a very recent phenomenon with very limited application in mathematics education.

MOOCS THROUGH THE LENSES OF FOUR CURRICULUM COMMONPLACES

Below we summarize our Working Group discussion using the four commonplaces of education: teacher, student, subject-matter, and milieu (Schwab, 1969).

MOOCS AND THE TEACHER

In many ways, the delivery of a MOOC resembles a large lecture, where the teacher talks and explains with no teacher-student interaction. For example, the second running of Keith Devlin's MOOC, *Introduction to Mathematical Thinking* in 2013, had an initial enrollment of 27 930 (Devlin, 2013), and Sebastian Thun's 2011 MOOC course on *Introduction to Artificial Intelligence* had an enrollment of 160 000. Even if the teacher wanted to create a two-way dialogue with students, the class size makes it impossible. F. G. Martin (2012) notes that, "In many ways, the carefully crafted online lectures, peppered with probing questions that are auto graded for correctness and then explained further, are indeed an improvement over a conventional lecture" (p. 27). However, it is difficult to imagine how a teacher in a MOOC could facilitate teacher-student interactions similar to that of a small, research-oriented, project-based course. Martin adds, "When we individually mentor each student on his or her own ideas, we are doing something that can never be performed by an auto grader." (p. 28). However, MOOCs are a very recent innovation, and we can expect that they will evolve in perhaps surprising ways. Parish (Chamberlin & Parish, 2011) notes that "The very structure of MOOC's is rapidly evolving as facilitators learn from each iteration, methods used by other facilitators, and feedback provided by the participants" (p. 1).

Our Working Group raised a number of issues that require discussion:

1. MOOCs use a social networking environment for discussion, where students can discuss ideas, self-organize into study groups, and collaborate to develop a better understanding of course content. In this context, we wondered how the teacher could possibly moderate this discussion. Will students use a social spaces style of discourse, with all the challenges that it might present, such as the use of rude, sexist, or aggressive language? How will course discussion maintain a 'safe' environment, and a scholarly environment?

2. How will the teacher use the results of auto-graded quizzes and tests? For example, if there seems to be a common misconception, will the teacher be able to provide timely feedback or elaboration of ideas by creating another video?

3. Can MOOC content be used in other, non-MOOC courses to create a flipped classroom delivery model? For example, students in a traditional face-to-face class may be asked before coming to their next class to view relevant videos from one or more MOOC courses, thus giving them a set of core content and experiences they can follow up on in in-class discussions and activities. In this sense, could MOOCs offer a library of ‘readings’ for non-MOOC courses?

4. What are other assessment options to auto-graded quizzes and tests? Could peer assessment be used and how might this be facilitated with a class of say 100 000 students? Rees (2013) notes that

   > *For me at least, the primary problem with peer grading lay in the comments. While I received five comments on my first essay, for every subsequent essay I received number grades with no comments from a minimum of two peers and as many as four. In one case, I got no peer grades whatsoever. That meant that the only student who evaluated my essay was me. Every time I did get a comment, no peer ever wrote more than three sentences. And why should they? Comments were anonymous so the hardest part of the evaluative obligation lacked adequate incentive and accountability.* (p. 29)

5. What other learning experiences might MOOCs offer? Are they limited to lecture videos? Can interactivity be incorporated? Can we imagine a multi-sensory MOOC?

6. What is the aesthetic/intellectual attraction to a MOOC for the teacher? Certainly new innovations typically have an enthusiasm that accompanies the early adopters. But what will be the long-term appeal for the teacher? Mitchell Duneier, “calls his non-credit Coursera class, which reached 40 000 students from 113 countries during its run in the summer of 2012, ‘one of the greatest experiences of my career’.” (Parry, 2013, para. 5).

7. Are MOOCs only for teachers who already have established a strong reputation in their field, like Keith Devlin in mathematics education and Sebastian Thrun in computer science? Are MOOCs opportunities for these established, prestigious ‘teachers’ to attain ‘rock star’ status? Is it possible for new or less known teachers to successfully offer a MOOC?

MOOCS AND THE STUDENT

With minimal access to the teacher in a MOOC, there is a need for students to collaborate and form learning/discussion groups. Questions in MOOCs often are pushed off into the larger discussion, where the student community does the tutoring, and offers general support. This environment offers students the opportunity to take ownership. Will this result in ‘academic discourse’ or ‘pooled ignorance’? McFedries (2011) suggests that “The MOOC becomes a kind of network of learners who spontaneously form new connections and even help direct the course and its objectives” (p. 30). Cormier and Siemens (2010) add that, “The community-as-curriculum model allows the curriculum to diverge on a learner-by-learner basis” (p. 5). Others add support to this view:

> *An interesting development in MOOC’s that might help develop the more social side of software engineering is the spontaneous creation of study groups and self-appointed teaching assistants.* (Ardis & Henderson, 2012, p. 14)

> *Our findings point to a maturing of e-learning users, who are now creating both personal learning networks and affordances, rather than just being consumers or even 'content creators'.* (Mak, Williams, & Mackness, 2010, p. 283)

> *A MOOC builds on the active engagement of several hundred to several thousand 'students' who self-organize their participation according to learning goals, prior knowledge and skills, and common interest.* (McAuley, Stewart, Siemens, & Cormier, 2010, p. 4)

> *The results of a MOOC collaboration may extend far beyond the MOOC itself: the network negotiated is just as important as the topic covered.* (McAuley et al., 2010, p. 5)

However, Tschofen and Mackness (2012) caution that

> *A paradox here is that acknowledging and accepting the importance of individual and psychological diversity, autonomy, connectedness, and openness may well result in some learners choosing (in appearance or actuality) very limited engagement with networked learning environments such as MOOCs.* (p. 138)

Kop adds,

> *For networked learning to be successful, people need to have the ability to direct their own learning and to have a level of critical literacies that will ensure they are confident at negotiating the Web in order to engage, participate, and get involved with learning activities. People also have to be confident and competent in using the different tools in order to engage in meaningful interaction. It takes time for people to feel competent and comfortable to learning in an autonomous fashion, and there are critical literacies, such as collaboration, creativity, and a flexible mindset, that are prerequisites for active learning in a changing and complex learning environment without the provision of too much organized guidance by facilitators. Especially at the start of the learning journey, support by more knowledgeable others proved to be helpful in this.* (Kop, Fournier, & Mak, 2011, p. 34)

Currently, the completion rate of MOOCs is quite low. For example, Devlin (2013) notes that in his offering of *Introduction to Mathematical Thinking*, only about 4000 of the over 27 000 registrants were active in the final week of lectures, and 870 submitted the course exam. He also suggests that, for most students, a MOOC may be less of a course and more of a resource: a MOOR rather than a MOOC.

> *We already know from the research we've done at Stanford that only a minority of people enroll for a MOOC with the intention of taking it through to completion. (Though that 'minority' can comprise several thousand students!) Most MOOC students already approach it as a resource, not a course! With an open online educational entity, it is the entire community of users that ultimately determines what it primarily is and* how it fits in the overall educational landscape. *According to the evidence, they already have, thereby giving us a new (and more accurate) MOOC mantra:* resources, not courses. *(Even when they are courses and when some people take them as such.)* (para. 26)

Devlin (2012) notes that universities also have a vested interest in identifying and attracting top students, and MOOCs create an environment where searches for such students can be conducted on a massive scale.

> *Right now, the media focus on MOOCs has been on their potential to provide (aspects of) Ivy League education for free on a global scale. But an educational system does more than provide education. It also identifies talent - talent which it in part helps to develop. That makes a MOOC the equivalent of Google, where it is not the right information you want to find but the right people.* (para. 11)

> *One crucial talent in particular that successful MOOC students possess is being highly self-motivated and persistent. Right now, innate talent, self-motivation, and*

> *persistence are not enough to guarantee an individual success, if she or he does not live in the right part of the word or have access to the right resources. But with MOOCs, anyone with access to a broadband connection gets an entry ticket. The playing field may still not be level, but it's suddenly a whole lot more level than before. Level enough, in fact. And as with Google search, in education, 'level enough' is level enough.* (para. 14)
>
> *Make no mistake about it, MOOC education is survival of the fittest. Every student is just one insignificant datapoint while the course is running. Do well, do poorly, struggle, drop out - no one notices. But when the MOOC algorithm calculates the final ranking, the relatively few who score near the top become very, very visible. Globally, talent recruiting is a $130BN industry (Forbes.com, 2.12.12). It's 'Google search for people' in action.* (para. 15)

King and Nanfito (2013) also suggest that MOOCs can be used to build relationships with potential students, as well as alumni.

> *Think, for example, of connecting students in AP calculus courses with your campus's introductory curriculum as part of the admissions recruitment culture. You can generate innumerable relationships between your faculty, your flagship programs and potential students. You can create spaces where secondary school students can interact with one another as they negotiate their college choice decision. The opportunity here for the small liberal arts college lies in the potential to encourage engaged discussion across networks, thus building awareness of what makes your campus special. Similarly, the MOOC platform and model can be used to deepen alumni relations in the context of lifelong learning.* (p. 22)

MOOCs are currently primarily designed for post-secondary students. Is it possible or likely that MOOCs will be used with younger students? Bell (2012) suggests that "The MOOC trend is so popular and successful that it is sure to be extended to instruction for younger students" (p. 24). Norris and Soloway (2012) add, "If Higher Ed can change – and do so overnight – then we in K-12 can be sure that our transformative event will occur very soon" (p. 96). Some examples with K-12 MOOC potential already exist. The Khan Academy, which falls in Keith Devlin's MOOR category, is a massive, open, online resource that addresses K-12 topics in mathematics (see www.khanacademy.org). As the content at the Khan Academy grows, it could also conceivably be organized in the form of courses. George Gadanidis has created a set of open, online courses primarily for K-8 teachers, students and parents (see www.mathclinic.com). These courses offer math experiences that are part of on-ongoing classroom-based research, and their content includes videos from documentary classrooms. The courses also have the option of teachers getting a Certificate of Completion from the Fields Institute for a minimal fee ($30/course).

MOOCS AND MATHEMATICS

Assuming the MOOC phenomenon sustains itself, there will exist significant potential to influence student experience with, and attitudes towards, mathematics. MOOCs are currently touted as giving students access to "free curricula from top-drawer professors" (Farell, 2012). As already noted, MOOCs in their current state are primarily video lectures with a pre-determined instructional design. There is no guarantee that "top-drawer" professors can offer a "top-drawer" mathematics education experience through video lectures. Historically, we have tended to associate new innovations with our highest hopes for mathematics education. For example, the *learning objects* movement promised to offer interactivity to mathematics, and many learning objects have been created world-wide. However, their quality is questionable, both from a pedagogical and from an interface design perspective (Gadanidis, Sedig, & Liang, 2004).

Devlin (2013) notes that, "By professional standards, many of the instructional video resources you can find on the Web (not just in mathematics but other subjects as well) are not very good" (para. 18). However, Devlin also notes that this does not mean that poor quality instructional videos are not effective.

> *As a professional mathematician and mathematics educator, I cringe when I watch a Khan Academy video, but millions find them of personal value. Analogously, in a domain where I am not an expert, bicycle mechanics, I watch Web videos to learn how to repair or tune my (high end) bicycles, and to assemble and disassemble my travel bike (a fairly complex process that literally has potential life and death consequences for me), and they serve my need, though I suspect a good bike mechanic would find much to critique in them. In both cases, mathematics and bicycle mechanics, some sites will iterate and improve, and in time they will dominate.* (para. 18)

MOOCs, although in their present form may not offer better 'lectures', they do make public what used to be private lectures. If the pervasiveness of the *Youtube* phenomenon is any indication, mathematics education MOOCs may similarly transform how easily private knowledge becomes public. What effect might this public sharing of mathematical knowledge have on the mathematics studied in school? Borba (2009) suggests that the pervasive access of mathematical knowledge on the Internet could (and should) change the type of mathematics that is studied and the types of problems with which students engage.

> *What kind of problems will be posed to collectives of humans-with-Internet? It is very likely they will be quite different from those posed to collectives that include only paper-and-pencil as media, not only because the answer can be easily found on the Internet, but because the Internet is also shaping the way humans organize and see things.* (p. 458)

This humans-with-media disruption may also occur due to the nature of student participation in a MOOC, where students use a social network style of discussion forum, to self-organize into learning groups. Kop et al. (2011) note that, "The MOOC acts as an environment in which new forms of distribution, storage, archiving, and retrieval offer the potential for the development of shared knowledge and forms of distributed cognition" (p. 78). Fini (2009) suggests that MOOCs are "examples of shifting from a content-centred model towards 'socialization as information objects'" (p. 79).

MOOCS AND THE WIDER MILIEU

McLuhan (McLuhan & McLuhan, 1988; McLuhan & Powers, 1989) summarized his ideas about the effect of the adoption of a medium on society by asking four questions: (1) What does the medium enhance? (2) What does the medium make obsolete? (3) What does the medium retrieve that had been obsolesced earlier? And (4) What does the medium flip into when pushed to extremes? Below we hypothesize answers to these questions, for the MOOC phenomenon. Figure 1 below shows a record of some of our discussion on this theme.

<u>What do MOOCs enhance?</u>

MOOCs enhance access to mathematics education. Anyone with Internet access can register for a MOOC, with no course prerequisites and with no time scheduling or geographical limitations. "Now we might be heading into a golden age of virtual education, where high-quality courses are available to everyone and not just those who can afford US $40 000 a year for tuition" (McFedries, 2011, p. 30). They also enhance the possibility of getting a 'bubbling up' of insights as an emergent phenomenon through social networking.

> *When looking at the shift in learning which is happening as a result of the rise in social media, ubiquitous cloud computing and new technologies, a MOOC*

> *complements all these changes and mLearning offers the devices and characteristics to realize such changes.* (de Waard et al., 2011, p. 112)

> *The digital economy is participatory, and it is participation that MOOCs enable on a grand scale.* (McAuley et al., 2010, p. 32)

> *MOOC's, or similar open transparent learning experiences that foster the development of citizens' confidence to engage and create collaboratively, are important for Canada's future as a leader in the digital economy.* (McAuley et al., 2010, p. 56)

MOOCs may promote pedagogical and programmatic innovation in traditional institutions. N. Martin (2012) notes, "It will positively disrupt traditional thinking within universities by encouraging them to focus on how they can provide education in innovative ways" (p. 32).

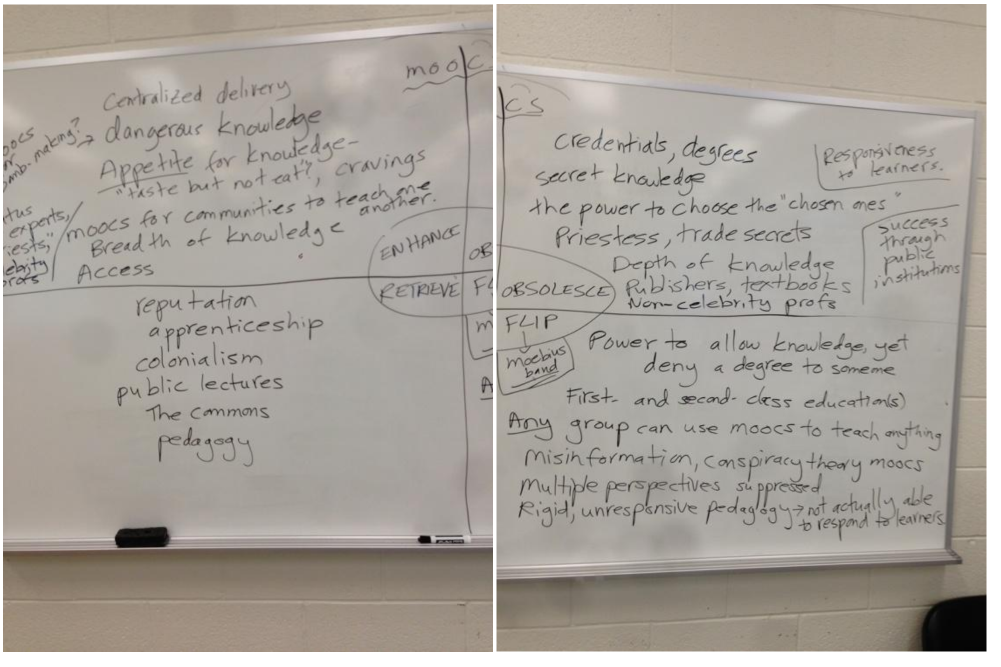

Figure 1. A record of our discussion.

What do MOOCs make obsolete?

MOOCs make physical classrooms obsolete. Beyond the production of lectures, the teacher is mostly obsolete, as it is impossible for one person to meaningfully interact with thousands of students. With course content publicly available, course textbooks and their publishers may become obsolete. MOOCs in their current state use prominent professors as teachers, thus potentially making less well-known professors obsolete. MOOCs may offer their own certificates of accomplishment which, depending on the field of study and the MOOC's reputation in the field, may make some university programs obsolete. MOOCs also make the closed, paid-membership-only nature of a university course experience obsolete.

What do MOOCs retrieve that had been obsolesced earlier?

MOOCs retrieve the lecture-based model of content delivery. They also retrieve the idea of 'public lectures' and University of the Air programs that used to be offered over radio and TV. The ability to repeat a course or part of a course retrieves the mastery-learning model. MOOCs are currently dominated by US-based start-up companies, initiated by professors who have left their university positions. The single country and single language focus of MOOCs that are offered to a worldwide audience retrieves a colonial paradigm for education. A MOOC is closer to a traditional idea of a university, with people coming together to share ideas rather than a specific program.

What do MOOCs flip into when pushed to extremes?

MOOCs promise access to 'top-drawer' professors, but the popularity of MOOCs make interactive access to the professor impossible. The promise of sharing of knowledge gets flipped into hundreds of thousands of students in a MOOC being exposed to a single professor and a singular point of view. The promise of easy access to a university degree may undermine the university institution. "[T]he enormous buzz about MOOC's is not due to the technology's intrinsic educational value, but due to the seductive possibilities of lower costs" (Vardi, 2012, p. 5).

> *MOOCs are generating a great deal of excitement in higher education circles. With their free curricula from top-drawer professors, they offer the prospect of dramatically lowering the cost of delivering a high-quality undergraduate education, perhaps even to millions of students worldwide. Less appreciated is how the MOOCs could also change employee skill development and lifelong learning at work.* (Farrell, 2012)

"Will it revolutionise education forever or will it ultimately be another dotcom bubble?" (N. Martin, 2012, p. 32). At least one professor, Mitchell Duneier from Princeton, has now ceased teaching his sociology MOOC, which had 40 000 students from 113 countries in 2012.

> *The change of heart happened, he says, after Coursera approached him about licensing his course so other colleges could use the content in a blended format, meaning a mix of online and face-to-face instruction. That could save the colleges money.*
>
> *"I've said no, because I think that it's an excuse for state legislatures to cut funding to state universities," Mr. Duneier says. "And I guess that I'm really uncomfortable being part of a movement that's going to get its revenue in that way. And I also have serious doubts about whether or not using a course like mine in that way would be pedagogically effective.* (Parry, 2013, para. 3-4)

Duneier's defection is part of a wider debate over college courses using MOOC content.

> *The issue gained attention in May after philosophy professors at San Jose State University refused to teach a course produced by edX, the MOOC platform of Harvard and MIT. In an open letter to the edX course's creator—Michael Sandel, a Harvard University government professor and political philosopher—the San Jose professors warned of "replacing faculty with cheap online education."* (Parry, 2013, para. 7)

In these early days of the MOOC phenomenon, there is also caution about their 'openness'. Rivard (2013) notes, "If you wonder why your university hasn't linked up with *Coursera*, the massively popular provider of free online classes, it may help to know the company is contractually obliged to turn away the vast majority of American universities" (para. 1).

CONCLUDING REMARKS

Il paraît évident que les cours ouverts et massivement distribués sont un succès d'envergure mondiale dont on peut facilement s'enthousiasmer avec gourmandise, en oubliant toute la lucidité et le recul critiques pourtant nécessaires à une compréhension minimale des enjeux pour l'enseignement et la formation en mathématiques. Après tout, ils semblent incarner les classes du futur, ils donnent un argument pragmatique aux administrateurs ou politiciens qui cherchent légitimement à faire baisser les droits de scolarité et ils font rêver les partisans d'une éducation accessible à tous, qui donne l'impression de diminuer les disparités sociales tout en assurant une juste répartition géographique des étudiants. Comme le dit ironiquement Bonod (2013) :

> *Finis les vieilles universités aux murs défraîchis, les professeurs soporifiques, les amphithéâtres pleins à craquer et le vieux modèle «présentiel» : vive la modernité sur écran plat, l'université à haut débit et mondialisée, bref l'école enfin dématérialisée et ramenée à son essence de pur apprentissage.*

Si notre groupe de travail a bien tenté de découvrir les avantages de ce type de cours, ce sont plutôt les inconvénients qui ont animé la plupart des discussions expertes. Pour ce que nous avons abordé, certains arguments sont généraux, comme le modèle incontournable de l'enseignement individualisé de type « one-to-one », atténué quelque peu avec la présence de forums de discussion libres censés s'autoréguler, ou le caractère déterministe des activités d'apprentissage et d'évaluation proposées. Ces inconvénients sont bien connus, contrevenant à la créativité sociale, au questionnement du milieu ou à l'adaptation aux situations non prévues. Ainsi, peut-on véritablement développer des compétences mathématiques avec les *Moocs*?

S'il faut répondre timidement par l'affirmative, esquissons brièvement une nuance à partir du cadre des *Espaces de Travail Mathématique* (ETM) formulé par Kuzniak et Richard (2013). Dans celui-ci, un *Mooc* serait un environnement pensé et organisé pour permettre le travail des individus qui résolvent des problèmes de mathématiques et dont les compétences disciplinaires sont en pleine évolution. L'espace de travail se pose en deux niveaux, un premier de nature épistémologique, en lien avec les contenus mathématiques étudiés, et un second de nature cognitive, qui touche l'action du sujet qui résout des tâches mathématiques. Le travail mathématique résulte alors d'un processus qui va permettre de donner progressivement un sens, d'une part, à chacun des niveaux épistémologique et cognitif et, d'autre part, d'articuler ces deux niveaux grâce à différentes genèses (sémiotique, instrumentale et discursives). Lorsque l'accent est mis sur le processus d'apprentissage de l'élève dans une situation didactique, ce plan épistémologique peut aussi se considérer comme un milieu épistémologique, le vis-à-vis du milieu étant alors un sujet épistémique qui interagit avec lui (Coutat & Richard, 2011). Dans ce cadre, l'interaction des plans par les genèses permet le développement de trois compétences mathématiques, c'est-à-dire de communication, de raisonnement et de découverte (Figure 2). Si nous analysions un Mooc particulier à partir de ses problèmes et des interactions potentielles que suscitent les cours, nous pourrions ainsi rendre compte du travail mathématique rendu possible. Mais si nous en restons à l'information principale qui se dégage du groupe de travail :

- La compétence de découverte semble se restreindre à une adaptation de l'étudiant au système préconçu, l'élève peu donc difficilement interroger le milieu et même s'exercer lors de situations où l'aboutissement n'est pas du tout contrôlé d'emblée par le système;
- La communication demeure essentiellement asynchronique, même au sein des forums, et elle ne permet pas facilement la coordination de plusieurs systèmes de représentation sémiotique typique du traitement mathématique (avec les registres des figures, des représentations graphique, analytique ou tabulaire de fonctions, des expressions algébriques, des graphes, arbres ou diagrammes de probabilités, des représentations non standards, etc.);
- Les raisonnements ou les calculs ne peuvent s'exprimer que difficilement à l'interface, et même si un module de conversion est prévu (retranscription numérique d'une phrase mathématique écrite à la main par exemple), il faudrait prévoir un mécanisme d'analyse discursivo-graphique pour détecter les failles dans le traitement

mathématique, surtout lorsque l'élève réussit apparemment à produire une bonne réponse.

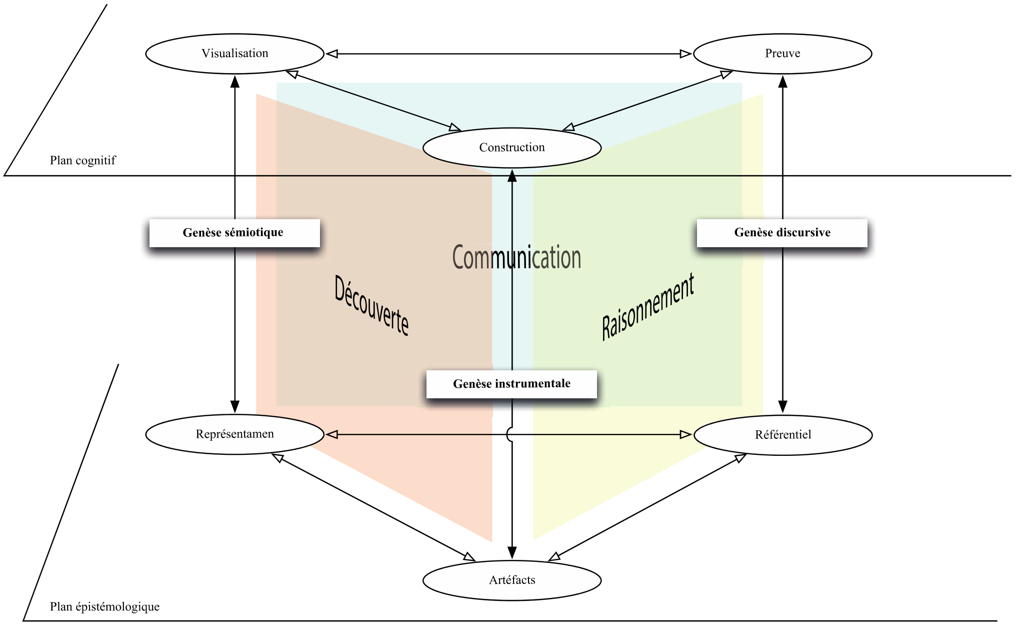

Figure 2. Les compétences mathématiques cognitives dans l'ETM (Kuzniak & Richard, 2013).

Nous pourrions prolonger aisément l'analyse critique sur les *Moocs* suivant l'ensemble des composantes du cadre des ETM, ou même confronter toute analyse a priori avec des résultats expérimentaux, mais pour cela il faudrait très certainement amorcer une étude qui dépasse largement ce que nous avons pu accomplir au cours des discussions de notre groupe. Quoi qu'il en soit, si les compétences mathématiques peuvent s'y développer, l'état actuel des connaissances didactiques et de la technologie montre que la connexion entre les concepts et les processus mathématiques, qui déborde la logique de chaque problème, ou la réflexion mathématique sur des problèmes complexes n'est pas suffisamment avancée pour conclure à la perspective du développement de compétences de haut niveau. Au fond, les *Moocs* ne permettent-ils que la reproduction de connaissances, en tant que conduite si décriée dans l'enseignement traditionnel?

Nos remarques sur les *Moocs* rejoignent d'autres critiques qui ont été formulées par le groupe de travail à la fois sur la formation en ligne et sur la conception d'environnements informatiques d'apprentissage humain. La plupart du temps, on valide de tels environnements en comparant les résultats d'un pré-test et d'un post-test, tout en obligeant l'usager à se conformer au système tel qu'il a été conçu. Cette attitude permet sans doute d'aboutir assez rapidement à des réalisations technologiques concrètes, encore faut-il que ces réalisations soient des aides efficaces au développement de compétences cognitives et à l'acquisition du savoir mathématique. Selon le groupe de travail, il semble que ce soit plutôt des modèles épistémologiques ou informatiques qui sous-tendent l'orientation du plus grand nombre de plateformes d'apprentissage en ligne, sans nécessairement s'interroger sur le sens des objets que celles-ci modélisent pour celui qui apprend. Les discussions ont plutôt insisté sur l'importance du principe qui consiste à devoir considérer d'emblée une modélisation du comportement humain dans une perspective d'enseignement des mathématiques, afin de concevoir par la suite un système informatique qui tient compte de cette modélisation (Richard et al., 2011). On a aussi souligné que l'amélioration de tels systèmes doit intégrer les particularités du processus d'instrumentation entre un élève et un dispositif technologique. Autrement dit, on exige que la conception se poursuive dans l'usage (Rabardel & Pastré, 2005), en ce sens qu'au cours du processus de conception et d'expérimentation, le dialogue entre concepteurs et usagers se substitue à la consigne initiale de faire «approprier» le système par les utilisateurs. En fin de compte, il faudrait que le système soit intelligent, c'est-à-dire qu'on lui demande une adaptation au comportement instrumenté de l'apprenant (Richard, Gagnon, & Fortuny, 2013). Une telle ambition dépasse les limites de l'enseignement traditionnel qui pose des problèmes en série, sans égard à la proximité des problèmes déjà résolus par l'apprenant ni aux connaissances acquises en cours d'apprentissage. Qu'elles s'adressent à des petits groupes ou à des groupes gigantesques, il faut arrêter d'aménager des plateformes qui ne font que reproduire des cours comme les autres sur un support sophistiqué, sans que ce que soit l'enseignement des mathématiques et ses enjeux qui trônent indiscutablement en amont de la conception.

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EXPLORING CREATIVITY: FROM THE MATHEMATICS CLASSROOM TO THE MATHEMATICIANS' MIND

EXPLORER LA CRÉATIVITÉ : DE LA CLASSE DE MATHÉMATIQUES À L'ESPRIT DES MATHÉMATICIENS

Ann Kajander, *Lakehead University*
Dominic Manuel, *McGill University*
Bharath Sriraman, *University of Montana*

PARTICIPANTS

Darien Allan	Richard Hoshino	Eric Muller
Nathalie Anwandter	Limin Jao	Leah Payerl
Jean-Philippe Bélanger	Mimi Kam	Bill Ralph
Carol Carruthers	Roza Leikin	Tina Rapke
Dalia Challita	Peter Liljedahl	Tim Sibbald
Chiaka Drakes	Ami Mamolo	Kevin Thomas
Michael Dubé	Dragana Martinovic	Jo Towers
Josh Farkas	John Mason	Chanakya Wijeratne
Viktor Freiman	Asia Mathews	Rina Zazkis

(Texte en français suit.)

Learners of mathematics do not typically experience mathematics as a creative subject, yet research mathematicians often describe their field as a highly creative endeavour (Burton, 2004). The term creativity *has sometimes come to imply eminent acts/products/achievements, yet research suggests that creative thinking is an everyday occurrence (Craft, 2002). In this working group we sought to capture the essence of mathematical creativity as seen through the eyes of mathematicians and described by current research, and express it in ways that might also be applicable to learners of mathematics, including, but not restricted to, students described as highly able. Our initial questions for consideration included: What is mathematical creativity? Does it differ from other kinds of creativity? How can we observe it in learners? Is creativity necessary for mathematics research? How can creativity be* *enhanced in classroom mathematics learning? Are some students more mathematically creative than others?*

Time was allowed during the working group meetings for those participants who wished to be generative; in particular, the construction or sharing of classroom tasks that had potential for occasioning creative behaviour was a focus for some participants. Such tasks could be illustrative to teachers who wish to provide potentially rich learning environments to students, and samples are included in this report.

BACKGROUND AND SUPPORTING LITERATURE

DEFINITIONAL ISSUES

There have been many definitional challenges on what constitutes *mathematical creativity* which arose in the group discussions. Previous examinations of the literature have concluded that there is no universally accepted definition of either *creativity* or *mathematical creativity* (Mann, 2005; Sriraman, 2005). Nevertheless, there are certain agreed upon parameters in the literature that help narrow down the concept of creativity (Sriraman, Haavold, & Lee, 2013). In a nutshell, *extraordinary creativity* (or big 'C') refers to exceptional knowledge or products that change our perception of the world (Feldman, Csikszentmihalyi, & Gardner, 1994). *Ordinary* or *everyday creativity* (or little 'c') is more relevant in a regular school setting. Feldhusen (2006) describes little 'c' as an adaptive behaviour whenever the need arises to make, imagine, produce or design something new that did not exist before in the immediate context of the creator. Finally, the relationship between giftedness and creativity has been the subject of much controversy (Leikin, 2008; Sternberg & O'Hara, 1999), as some see creativity as part of an overall concept of giftedness (Renzulli, 2005), whereas others hypothesize a relationship between the two (Sriraman, 2005; Haavold, under review). Whether or not creativity is domain specific or domain general, or if one looks at ordinary or extraordinary creativity, most definitions of creativity include some aspect of usefulness and novelty (Sternberg, 1999; Plucker & Beghetto, 2004; Mayer, 1999) depending on the context of the creative process and the milieu of the creator. There are also minority positions of viewing creativity as an affective experience (Liljedahl, 2013).

SOME RECENT STUDIES IN MATHEMATICS EDUCATION

Closely related to conceptual relationships between mathematical creativity and other concepts, such as academic ability, visualization or verbal abilities, mathematical background, etc., is the question of "who is mathematically creative?". Kattou, Kontoyianni, Pitta-Pantazi, & Christou (2013) clustered students into three subgroups: low, average and high mathematical ability. The high ability students were also highly creative students; the average ability students had an average performance on the mathematical creativity test; while low ability students had a low creative potential in mathematics. Pitta-Pantazi, Sophocleous, & Christou (2013) classified prospective teachers as *spatial visualizers*, *object visualizers* or *verbalizers*. The *spatial visualizers* scored higher on the mathematical creativity test than both other groups.

Lev-Zamir and Leikin (2013) suggest that different mathematical backgrounds of teachers affect beliefs related to mathematical creativity. Teachers with stronger mathematical backgrounds tend to have deeper beliefs regarding mathematical creativity. These recent studies distinguish individuals into different levels of mathematical creativity according to some other quality or ability (Sriraman et al., 2013).

In summary, mathematical creativity is linked to and influenced by ability, beliefs, cognitive style and the classroom environment (Lev-Zamir & Leikin, 2013; Pitta-Pantazi et al., 2013; Kattou et al., 2013). These findings are analogous to the research on general creativity and giftedness. Kattou et al. (2013) point out how mathematical creativity is essential for the growth of overall mathematical ability (or giftedness), while Lev-Zamir and Leikin (2013) show how challenging mathematical problems and flexible teaching can help the development of mathematical creativity.

In the working group challenging mathematical problems/tasks were used to investigate mathematical creativity, and participants were invited to contribute others, a few of which are included to follow. Examples of other problems, examples or tasks for use in teacher education as well as with students can be found in the literature (for example, Mason & Watson, 2001; Zazkis, 2008).

SUMMARY OF GROUP DELIBERATIONS

The initial working group description referenced three recent journal articles, chosen partly to illustrate a breadth of work around mathematical creativity (Liljedahl, 2013; Sinclair, de Freitas, & Ferrara, 2013; Sriraman, 2009). These readings were chosen primarily because they encompassed much of the classical literature on creativity, such as the Gestalt model. Participants were invited to peruse these articles in advance if they chose to, which also provided some initial background to participants newer to the field. These articles were offered as a starting point to our discussions. In addition during the working group days, participants were provided with copies of a recent special issue of *ZDM* (Leikin & Pitta-Pantazi, 2013) that contained investigations on the construct of mathematical creativity as it relates to mathematics education. In the literature review above, some of the pertinent findings from articles in this special issue were summarized.

OUR QUESTIONS

We described the working group as an opportunity to deliberate the following questions:

- What is mathematical creativity?
- Does it differ from other kinds of creativity?
- How can we observe it in learners?
- Is creativity necessary for mathematics research?
- How can creativity be enhanced in classroom mathematics learning?
- Are some students more mathematically creative than others?

It turned out that some questions were more of a focus than others; in particular much discussion related to supporting the occasioning of creativity in classrooms.

OUR DELIBERATIONS

As is typical for our conference, the group time began with introductions. Due to the size of the group (27 people plus the three leaders), participants were encouraged to write down one or two main questions of interest; participants were organised into small groups based on these statements of interest, and the small groups met during designated portions of our working time. Since the participants included a number of noted scholars in the field, an attempt was made to include a few contributing facilitators, and in this regard we are particularly grateful to Peter Liljedahl and Roza Leikin. During Day 1, we were invited to explore an initial mathematics task.

<u>Sample Problem One</u>

Peter shared the following problem:

> *You have a 4 minute timer and a 7 minute timer. How can you use them to cook a 9 minute egg the fastest way?* (It was asked and clarified that these were sand timers).

Participants began to work on the problem, as we began to ponder our questions about mathematical creativity. While a number of participants felt they had found ‘the’ solution to the problem, we were reminded to “keep working”! It turned out the fastest possible time is 9 minutes! (The challenge of solving the problem optimally is left to the reader). After engaging with the problem, a number of initial ideas were shared with respect to our questions around mathematical creativity (and whether it is different from other kinds of creativity), as well as the creative process. A few of the points made are summarised below, informally grouped under sub-headings. Where available, authors of the comments are noted in brackets.

Mathematical Creativity

It was suggested that creativity is not an adjective to describe a type of problem (John). We can make distinctions among the concepts of a creative *person*, a creative *product*, or a creative *process* (Peter). We might have a whimsical image of a creative person as an image of Einstein riding on a beam of light. But is there such a thing as a creative person?

Creative Products

Creative products might be those involving the quality or novelty of the idea. One view might be that products are only creative if they are completely novel to the world. It was suggested that it’s only a difference in degree—the creation process is the same whether a product is novel to the world or only to the individual. The ideas of *relative* versus *absolute* creativity were discussed.

The Creative Process

The sample egg-timer problem has the potential to occasion this sort of ‘creative’ experience (Peter). But you are robbed of the creative process if someone blurts out the answer before you have had enough time to engage with the problem. Peter felt this creative experience, while cognitive, is very much an affective one.

Several participants subscribed to the (Hadamard) process of the 4-stage model: initiation, intense deliberate work, incubation (time away), and illumination (solution comes from unconscious to conscious, and comes with an affective charge); then verification (Is it really right?—checking). The question of adequate working time was again noted. Chiaka added that she felt the incubation phase is important—in particular, for students: How much time is enough time? How long should we give students? If the solution is given, or given too soon, some students lose the opportunity to be creative.

Ami noted that we can be creative in the ways we approach a task, in the way we talk about a task, or in the way we work on a task. What ways of being creative are more legitimate? Some ways seem like ‘cheating’ such as putting the egg timer on its side. Perhaps the creative step comes when you are pushed further than you think is/was necessary (Dragana).

It was wondered, what are different *legitimate* ways to think creatively about a task? One small group suggested the following list for discussion:

- Avoidance creativity—being cute
- Changing a parameter that allows you to make progress

- Flipping the 'seven' (in the egg problem) with one minute gone feels like an oversight
- Fleeting insight—you have something and then you lose it
- The aesthetic sieve—when your subconscious promotes an idea that has possibility

Getting Stuck and the Moment of Illumination

While it may be that the moment of illumination is significant, you don't always know right then if it is significant. It is the affective charge that makes it appear so powerful. Poincaré would say, you have to fill your mind with ideas and let them rest (rather than consciously try to mash ideas together). What is the nature of certainty, significance, level of affect, and the amount of time you are stuck? The bigger the charge, the more you remember the experience. We continue to keep crashing ideas out until one strikes the subconscious as significant. Asia added that you can have a creative moment when you are doing something else. Contemplation is the moment between being fully engaged and backing off to give your mind space to grasp the idea.

<u>Sample Problem Two</u>

A contrasting problem, chosen to further explore our sense of a more and less creative process, was next presented by Bharath. The problem might have particular application to teacher education contexts.

It was proposed that there are some fractions for which the following is true:

> *We could argue that 16/64 = 1/4 by crossing out ('cancelling') the 6s. A second instance is 19/95 = 1/5 by crossing out the 9s. Another example is 13/325 = 1/25 if we cancel the 3s. The task is: Can we find more examples for which this is true?*

As we worked on the problem in small groups, many results and observations were noted. Another 'solution', 26/65 = 2/5, was found by Mike, and Richard shared his method of streamlining the choices via an algebraic method, indicating one step he felt was more 'creative', and also noting 49/98 = 4/8. John extended our thinking by opening the door to (negative) integer cases (see Figure 1). This problem prompted a further discussion on the nature of the creative process, with some points made summarized below.

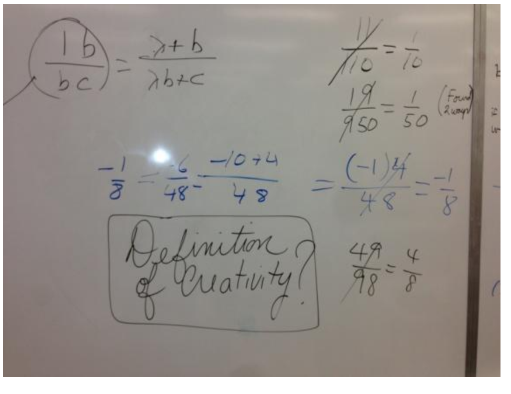


Figure 1. In-progress thinking about the fractions task.

Further Discussion on the Problem

We found the fractions problem encouraged further debate; for example, it was suggested we had many instances of creative ideas (Asia), and alternatively it was also suggested that all techniques were known acts, and thus that our work was *not* creative (Ami). Indeed, Roza argued that the problem illustrated an interpretation of creativity in the worst sense. Richard defended the need for efficiency in creative solutions, finding none found so far creative, and Asia responded that creativity can moderate our efficiency, but not the other way around. Bharath suggested that optimizing a solution indeed relates to creativity.

However it was noted that our board work does not characterise our non-linear thinking, so we couldn't really tell if the thinking was creative or not (Peter). What we saw on the board came from our knowledge and experience, but the process we went through was still argued by some as creative—discussing and trying different paths. Some participants felt we might still have had a creative process, even if not a creative result (Dragana).

Tim clarified that, to him, the creative part is going sideways, and looking for a different representation. Incubation is required, going through a process of unlearning. Ami noted that we are all schooled in seeking out representations, and didn't feel this task occasioned her to be creative. She was rooting around with known methods, and hadn't found an unpredictable aspect.

Carol also felt she was being more creative with yesterday's (egg-timer) problem, as did Dragana. Viktor on the other hand, felt more involved today. He explained he loves algebra, so he found it more fascinating and more pleasing, a more affective experience—thus creative. But he noted that this was not true for others.

Are There Creative Problems?

If you can solve a problem directly it's not a problem—you must get stuck. The process of getting un-stuck, requiring an AHA, is what makes something a problem. So that would be creative (Peter). Chiaka felt that a 'trivial' solution, such as 11/11 = 1/1 for the fractions task, may help because it gets you in to the problem.

As teachers we want tasks that promote creativity, but *tasks* cannot be creative—creativity is more of an attitude (Dalia). Does creativity always have to be exciting and fun? "The muse will inspire you"—is creativity supposed to be a-musing? (Chiaka). What are the verbs—is there a general way to pose problems? Some verbs might be: construct, discover, find, tell what you see (Dalia).

Eric, Leah, Limin and Mimi asked: Would students use the tools they are given, or will they go beyond to the space where they can be creative? How do teachers provide opportunities for students to be creative? In what situations do students feel that they are being creative? Do they overlap? These ideas are represented in this group's model, provided below.

It was suggested there must be multiple representations for something to be creative (Dragana). As a teacher, she is looking for more creative ways of teaching—and can't anticipate what students would come up with. But a different routine in the solving process might make something more creative too. Richard pointed out that there could sometimes be several solutions—a mundane one, and a more creative one. A 'creative' problem might be one for which there is such an 'alternate' solution.

Roza noted that children must be creative to construct new knowledge. When are we constructing new knowledge? The answer depends on participants' current knowledge. Some people may creatively solve a problem while for others it may not be new knowledge. This depends on your starting point and background. John agreed, saying that, as an example, for him the notion of non-standard digits is very familiar, hence using that process wasn't creative.

The Role of Prior Knowledge

John asked: The *prior experience* (he prefers to avoid the term knowledge), does it come to you when you need it? What brings it to mind? If you don't have prior knowledge coming in, you have no way to start the process (Josh). When does a certain type of problem become uncreative? In every instance of a certain type of problem, it becomes more routine (Michael). We find with elementary school students, half the students will be really engaged in a problem, and the other half don't even understand. Prior knowledge, wisdom, and experience are essential to being able to get a start (Eric).

<u>Multiple-Solution Tasks</u>

Many mathematical problems can only be solved in routine and mundane ways. However, if students see problems that can be solved in both a routine way and a surprising innovative way, then many unexpected benefits arise: a greater confidence in doing mathematics, a deeper appreciation for the beauty of mathematics, and of course, a development in one's creativity (Richard, Rina, Chanakya). The argument was again made that mathematical problems are not creative, in and of themselves. If a mathematical problem has a creative solution, then the solution is creative, but not the problem. This opening, this opportunity for discovery, is what we as educators should incorporate into our teaching.

Roza asked: "Are there tasks that require creative processes?" She alluded to the guiding questions of how much guidance to provide, and what is the knowledge base versus new learning.

<u>Sample Geometry Tasks</u>

Roza shared her work of using a number of geometric proof tasks, which students were asked to solve in more than one way. Proofs could be evaluated in terms of complexity, elegance, and so on. It was noted that out of 20 solutions, five were really creative—she had never seen these properties before. It was a real discovery, even for the teacher. But everything is relative. When we are working with students, or new teachers, on a relative level, it gives them this effect of excitement, discovery, that they did it themselves. Mathematicians discover theorems on a higher level, but on a school level, it's new. She truly believes students perform some creative acts. Peter noted that when they have to prove it, it becomes their own theorem.

Roza pushed us again to think, whether there are ultimately creative tasks (that don't depend on the level of the participants)? In her plenary talk, Roza shared examples of Multiple-Solution Tasks (see Leikin, this volume).

<u>Supporting Teachers</u>

Many conversations during our working time, as well as small group time, were focused on ways to support teachers in their work. Dragana argued it is about pushing students, and teachers that keep pushing cause students to exhaust known methods and try new ones. For example, with the egg-timer problem, after exhausting ideas, we might try different things with the timer. The shift is a creative moment. JP added, when we learn, it's always in a context. If we switch contexts, we have to create a link to the new context.

The small group of Richard, Rina and Chanakya discussed the idea of *breaking constraints* as a key idea in the creative process (also see Zazkis, 2008). We do not expect our students to reproduce, on their own, the work of mathematicians who advanced the subject through their creative insights. And yet, as teachers of mathematics, we can provide students with problems that lend themselves to non-standard solutions: where a short insightful solution exists by breaking a constraint in a surprising and novel way, or by applying a technique from a seemingly-unrelated area of mathematics.

The focus of Tim, Nathalie and Viktor was on the hindrances that exist in teachers' utilizing and, therefore modeling, creativity in their classrooms. Common experiences included teachers having difficulty with inquiry-based and problem-solving approaches. These difficulties were seen as having two relevant dimensions. One was the difficulty of identifying appropriate problems. Secondly, often it is seen that even with rich learning tasks, teachers adopt more prescriptive approaches in their instruction, rather than fostering creativity. There was a sense that providing rich questions might not be sufficient for long-term sustained promotion of student creativity (the teacher will get used to the question, perhaps, and may not maintain a creative stance in utilizing the question). The processes component of the curriculum could be used to encourage creativity. Teachers may act prescriptively, but there may be an opportunity to encourage different processes that would encourage students to generate novel approaches and techniques. These points led to discussion of good tasks, however, it was noted that in some respects all tasks can be taken as good tasks. The example was given of '3 + 5 = ?' being a valid question for promoting student thinking in kindergarten that might lead to discovery, such as '3 + 5 = 5 + 3'. The use of manipulatives was used as an example of a tangible kindergarten strategy for encouraging student creativity. This led to discussion about known methods for developing creative approaches to instruction, such as collaboration between teachers and researchers through co-teaching. However, the challenge is that such work tends to address relatively small groups of teachers and there is a substantial challenge to scale this up to address schools and boards.

Other participants posed related questions. Can mathematics in high school be consistently creative and still 'meet' the curriculum goals? Perhaps we can use creativity to expand students' views of mathematics. We need to let students play with math. What are the connections between creativity, play, and imagination? Can anyone be creative in any domain? Is it important that they are? As mathematics educators, we might try to pinpoint or understand *why* we would want to prompt creativity. For example, if it is for personal pleasure and satisfaction, then perhaps it is enough that the act is creative for the individual and not necessarily for the society. On the other hand, if the aim is to make a contribution to the environment of the individual, then perhaps it is not enough that the creative act is only creative *subjectively* (Dalia).

Emergent Thoughts on the Creative Process—Math and the Arts

One of our guiding questions was about the relationship of mathematical creativity to other types of creativity. Bill shared with us about his time involved in the arts, working with dancers and music. In the arts "there is no goal". In math there is often a very well defined goal, such as doing something in the shortest time, or finding an algorithm. In the arts world, it is the opposite. Artists want to do something original, something they have never done before. Also Bill noted that the people he worked with made it fun. In art, working with other people, and bouncing off ideas, makes it fun, which can be true in mathematics also. When you work with other artists and you start to build something, it is exciting. But in the arts something must be more than just 'novel' to be original.

Bharath argued that art is not as unbounded as a lot of people claim. Even art has boundaries. Creativity is not a free-for-all; there are tools and gate-keepers. Eminent acts at the fringes of the field may be creative, but if a person is seeing something for the first time, and they create the tools, then that is creative too. Every academic domain has jurors. Even art is not unbounded; painters historically had to first satisfy the church and the rulers. Perhaps we are confusing creativity with really eminent acts by prominent people. In every domain there are people who judge it—so no domain is as unbounded as it may seem.

Mathematics has boundaries, which makes it challenging to transcend them—perhaps this is harder in math than the arts? We need to be creative within the boundaries of the problem (Kevin).

Creativity as a Social Process

As we worked on the example problems, many of us engaged in a social process, so this topic was a natural one for discussion. It was noted that for collaboration to happen, we need a task that makes people want to talk and lends itself to collaborative work.

Tim elaborated that one aspect of creative work can involve the social construct. This fits with the cognitive dissonance criteria—working with others who may notice something different jogs your thinking and may cause you to go in a new direction from where you are. This is unlearning—you forget what you have done and you take a new path. Otherwise you get blinkered because you are driving at a goal.

Roza asked, "What is insight?" If you are staying within a known area, it is not creative. In any creative effort you are coming to something which is new to you; you are probably getting there with help from others. When you are discovering something, you certainly could get there with help from others who know this.

Creativity in the Field of Mathematics

While less discussion centered on the initial question of mathematical discovery, Richard shared an experience of how he used graph theory to solve a scheduling problem. He described having a profound illuminating experience when he realised that the scheduling problem could be solved with graph theory. But then it took him a month to work out the details. We are reminded of the idea of removing boundaries and making unexpected connections as creativity characteristics.

Emergent Questions and Ideas about Insight

John connected the experience to the notion of insight, describing Richard's experience as "casting in a fresh light", or "a new way of seeing". The issue is: "Does something come to mind?" Is there an affective component—an AHA? But what triggers what it is that gets brought to the surface? Peter suggested that Richard's work was very unbounded. So perhaps the creative generative process is like building new bridges and seeing new connections.

It was wondered, if you don't have prior knowledge, how do you start the process? Does it come from something we've learned before? Or do we tackle it any which way? Josh summed it up, noting it's so subjective; what is emphasised is so different. We might have a great creative problem in a classroom which some students are working on and enjoying—but often the other half of the students don't understand the problem. Eric sees prior knowledge as fundamental—yet we spend "too much time on knowledge and not enough time on wisdom". We need enough knowledge to choose alternatives, but we don't discuss the development of choice enough. This internal self-assessment is very important, but we don't teach it, we only focus on external assessment.

<u>Our Early Emergent – and Time Limited – Definitions/Descriptions</u>

As an (unpopular!) and provocative working group task, participants were given two minutes to write their own description or definition of creativity. Some participants naturally refused! Others worked in a small group. A few samples of our thoughts follow.

- Peter et al.: Creativity is the ability to generate something inconceivable.
- John: A particular quality of energy that comes from outside. It flows from the outside.
- Richard: A mathematically creative solution implies novelty, the breaking of constraints. It might be realising that we can connect a problem to another field or the realisation of connections, and the discovery of a new insight.
- Dragana and Carol: Novel use of standard tools to extend personal knowledge in surprising and joyful ways, generating multiple scenarios that are flexible and open to many new points. It involves the effect it has on a person.
- Limin: An individual's perspective of novelty based on their own experience. (It is relative to the individual).
- Tim: The individual moving beyond the observer's view of that observer's zone of proximal development; the observer can be oneself. It involves breaking the rules.

<u>Discussion / Critique of Our Initial Descriptions</u>

Richard asked if tools used in non-standard ways would be creative. John liked the notion that creativity is in the eye of the beholder. He added that he has no experience of working in a situation in which he had no idea what to do. One must have some sense of something doable, or one won't engage.

Roza noted the connection to the zone of proximal development. If you are staying within a given area, it is not creative. In any creative act, when you are coming to something new, you might get there with the help of others. At some point, if the ideas still don't exist, you are trying to do something new. John noted that Roza's geometry examples are about the language of seeing—stressing some things and ignoring others, recognising that there might be a relationship or property, or looking to find some relationships that are actually properties. He asked, "Seeing differently—is that creative act or not?"

Peter elaborated: "When I sit down to do a problem, I may not see a path all the way to the finish. At every path, a new idea comes to me. I don't see this as creative. I can back up and correct a wrong move. Where do ideas come from? Hadamard talks about this—not being able to see the end from the beginning does not make it a creative experience. What is creative is when I can't see the next step, and then suddenly I can see it. It could be a casting about, or it could be an illumination. It's a very subtle difference. I like the idea of seeing. It's not about not being able to see the finish from the beginning; it's about being stuck. At some point something comes that I couldn't do directly."

The creativity is about moving beyond, not just seeing, but moving. There is an internal force that drives you, whether you know the next step or not (Viktor). The stages can be initiation, incubation, illumination, and verification (Wallas, 1926). There might be a fifth stage of evaluation, such as in the arts. This speaks to an absolute creativity. In unbounded situations like the arts, Bill's choreography example, or Roza's students' work, an evaluative stage is needed. Is it entertaining or new? In the unbounded, evaluation is important (Peter).

Later Definitions and Models

We closed our deliberations by inviting participants to create and share a statement, model, or other product or outcome of their thinking and time together. A few of these statements, models, and suggested problems are offered below, as space allows. We chose to close our discussions by sharing the breadth of our perceptions and ideas, rather than striving for any sort of unified conception.

CONCLUDING IDEAS

Definitions and Descriptions

Creativity is ...

- The ability to generate something inconceivable.
- Most helpfully thought of as a particular quality of energy that comes from outside. It flows through our psyche and provides the experience of a lightening of spirit (an AHA! moment?). There is usually a thrust of energy into affect; sometimes this drains the energy away from being able to follow through on the details; sometimes this is experienced as an insight; other times it is an act of creation; and sometimes both. Creative energy is probably always available, but it is easily blocked by other energies flowing through or activating our selves. So accessing it is as much about letting go as about acting differently.
- Use of standard tools to extend current personal knowledge in surprising and joyful ways, generating multiple scenarios that are flexible and open to new viewpoints.
- Individual's perspective of novelty based on their own experience.
- The ability to generate something that is inconceivable.
- A process (Hadamard: preparation, incubation, illumination, verification; evaluation); not along a regular train of thought. Subjective and personal. Something new, original or unusual.
- An online definition for the term *creativity* is: "The use of the imagination or original ideas, esp. in the production of an artistic work". For me, the first part of this phase is fairly unambiguous, and I agree that creativity includes an imagination component. The second part of the phrase is not as clear to me. Must creativity *produce* a work? Is this work a concrete object or can it be a new way of thinking? In whose perspective must it be 'artistic'?
- Nathalie Sinclair's definition: a creative act introduces the new in an unpredictable way that transcends current habits of behaviour and exceeds existent meanings (relative originality).
- A mathematically creative approach is one that involves breaking of constraints, realizing connections, new insights.

Models

Some sub-groups also offered models, and shared these at the end of our time together. A few follow.

Model 1

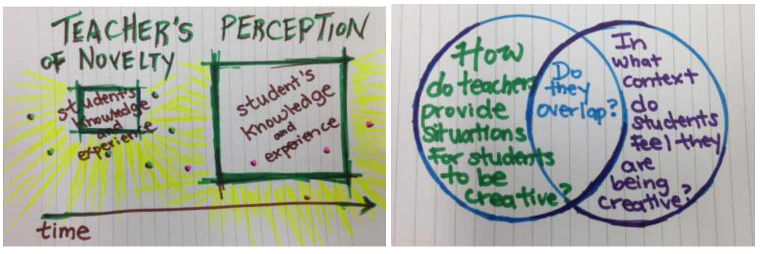


Figure 2. Teachers' Perceptions (Eric, Leah, Limin and Mimi).

Model 2

We tried to identify the opportunities for changing teacher approaches. This led to Figure 3 which maps the curriculum in terms of creativity.

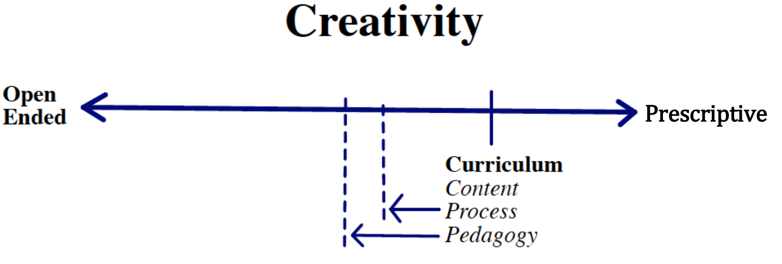


Figure 3. Where is the curriculum in terms of creativity?

In Figure 3, the horizontal line represents a spectrum of creative opportunity. Prescriptive approaches were interpreted as fostering minimal creativity, while open-ended questions are seen as having creative opportunity. The open-ended approaches were discussed in terms of the classroom and interpreted as student-accessible problems with multiple approaches or methods of solution. In terms of the curriculum, there are commonly different components with different levels of prescription. This model is considering overall curriculum enactment in general terms. In this group, curriculum content was seen as the most prescriptive, however, it needs to be emphasized that it is the minimum a teacher has to teach, so there is opportunity, as time affords, for teachers to be creative beyond the set curriculum. Processes described in the curriculum tend to be less prescriptive but there are requirements. For example, the requirement to use problem solving is somewhat prescriptive but not nearly as prescriptive as teaching a specific exponent law. Lastly, pedagogy tends to be the least prescriptive in the curriculum, but is constrained by some degree of norms of professional practice (Tim, Nathalie, Viktor).

Model 3

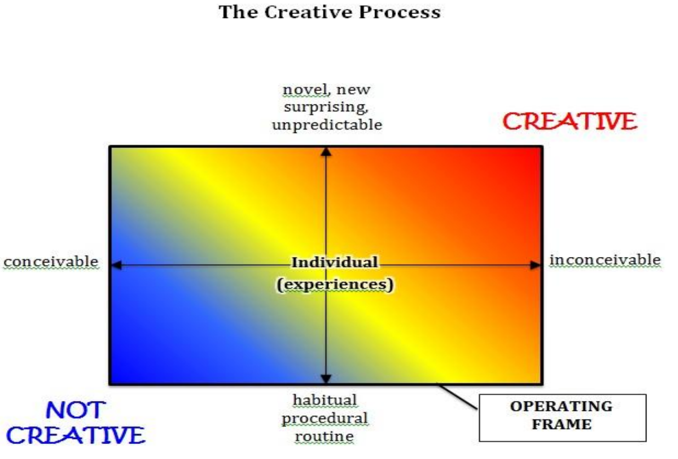


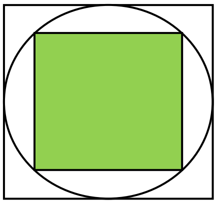
Figure 4. The Creative Process: the more creative is moving right and upward.
(Kevin, Michael, JP, Josh, Jo)

<u>Sample Problems</u>

Problem Set 1

(Richard, Rina,Chanakya)

Problem 1:



In the diagram, a circle is inscribed in a (large) square, and a (small) square is inscribed in the circle.

What is the ratio of the areas of the two squares?

Figure 5. The Circle Square Problem.

Typically, students let the small square have side length 1, use the Pythagorean Theorem to find the radius of the circle, and recognize that the side length of the large square is equal to the diameter of the circle. From there, they conclude that the area of the large square has to be twice the area of the small square. Richard described an experience working with a shy undergraduate who offered an alternate solution, although he was initially reticent to share it, feeling that what he did was “not math”. The student rotated the inner square, and drew a vertical line and a horizontal line passing through the centre of the circle. He then explained his thinking.

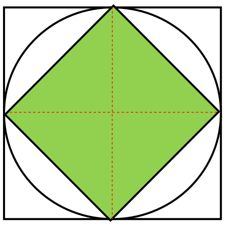


Figure 6. A new solution to the circle problem.

"If you look at it like this, it's obvious. The area of the large square has to be twice the area of the small square." The rest of the students stared at the solution. Some saw it in twenty seconds, while others took a few minutes. One by one, they came to the same conclusion. While the other students simply assumed that the inner square had to remain in a fixed position, this student broke that *self-imposed constraint* and chose to rotate the inner square, which clearly preserved the area of the small square while making the problem so much simpler. This student, who had failed a basic Math Diagnostic test upon entering the university, had significant problems with basic skills such as multiplying fractions and factoring equations. Despite his keen intellect and obvious work ethic, he had much anxiety about the course and lacked confidence, but after this experience his confidence skyrocketed. He realized that he could do math because mathematics isn't about memorizing and applying formulas.

Problem 2:

Two trains are 20 miles apart on the same track heading towards each other at 10 miles per hour, on a collision course. At the same time, a bee takes off from the nose of one train at 20 miles per hour, towards the other train. As soon as the bee reaches the other train, it turns around and heads off at 20 miles per hour back towards the first train. It continues to do this until the trains collide. How far did the bee travel?

Figure 7. The Bee Problem.

The routine mundane solution requires the calculation of an infinite series, to measure the distance traveled by the bee every time it hits one of the trains and heads back. However, if we consider the problem from the perspective of the bee, we can simply ask the question: "How long does the bee travel before it gets squished?" The answer is clearly one hour, since the two trains, 20 miles apart, approach each other at 10 miles per hour. Since the bee travels 20 miles per hour for exactly one hour, the answer to the problem is just 20 miles.

Problem Set Two – University Level Task

(John, Chiaka, Dalia, Asia, Tina)

Problem 1:

i. Given a non-singular linear transformation T from a vector space to itself, which matrices can be used to represent T?
ii. Construct a function from the reals to the reals which requires k uses of L'Hôpital's rule in order to find the limit as x approaches a for some a.
iii. Sketch a smooth function f. Define the tangent power of a point P in the plane with respect to f as the number of tangents to f that pass through P. Explore the tangent

power of points in the plane. Is there a general method for finding all the points with a given tangent power?

iv. How large a difference can there be for a function differentiable at least k times, between the number of local maxima and the number of local minima?

Some sample tasks from Mason & Watson (2001):

i. A Mean Problem: Observe that $\int_0^2 (1-x)\,dx = 0$. Generalise!

ii. A Divisory Problem: Find all the positive integers which have an odd number of divisors.

iii. Another Square Problem: Given two distinct straight lines L_1 and L_2 and a point E not on them, construct a square with one vertex at E, and one vertex on each of L_1 and L_2.

iv. Rolle Points: Rolle’s Theorem tells us that any function differentiable on an interval has a point in the interior of that interval at which the slope of the function is the same as the slope of the chord between the points on the curve at the ends of that interval. Where on that interval would you expect to look for such a point? For example, are there any functions for which the Rolle point of every interval is the midpoint? A natural question to ask is whether there are any functions for which the Rolle point on any interval is, say, 2/3 of the way along the interval, or more generally, □ of the way along.

v. Inflection points: A common method for finding inflection points of a curve which is at least twice differentiable, is to differentiate twice and set equal to zero to find the abscissa. Sometimes this gives a correct answer for a correct reason, sometimes it gives a correct answer for a wrong reason, and sometimes it gives an incorrect answer. Construct examples which exemplify these three situations, and also a family of examples which include all three in each member, and thus might bring students up against these different possibilities. Must a function be twice differentiable to have an inflection point? What about members of the family $x^k \sin\left(\frac{1}{x^2}\right)$?

A task sequence:

- Sketch the graph of a function on the interval [0, 1].
- Sketch the graph of a continuous function on the interval [0, 1].
- Sketch the graph of a differentiable function on the interval [0, 1].
- Sketch the graph of a continuous function on the interval [0, 1], with one of its extremal values at the left end of the interval [0, 1].
- Sketch the graph of a continuous function on the interval [0, 1], with both its extremal values at the end points of the interval [0, 1].
- Sketch the graph of a continuous function on the interval [0, 1], with its extremal values at the end points, and with a local maximum in the interior of the interval [0, 1].
- Sketch the graph of a continuous function on the interval [0, 1], with its extremal values at the end points, and with a local maximum and a local minimum in the interior of the interval [0, 1].

Now comes the interesting part! Work your way back through the examples, making sure that at each stage your example does not satisfy the constraints which follow! Thus your first example must be a function but must not be continuous; your last but one example must have a local maximum in the interior but not a local minimum. Finding that a set of constraints seem mutually incompatible is an excellent way to generate a conjecture leading to a little theorem. The structure of this kind of task forces students to become aware of a more general class of examples than they may have considered the first time.

Function grids task:

Construct continuous functions with the following domains and ranges:

Domain/Range	$\mathcal{R}$	[0, 1]	[0, 1)	[0, ∞)	(0, ∞)
$\mathcal{R}$					
[0, 1]					
[0, 1)					
[0, ∞)					
(0, ∞)					

[Note: References follow the French version.]

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RÉSUMÉ

Les apprenants[1] *des mathématiques ne vivent pas typiquement les mathématiques comme une matière créative, bien que les mathématiciens décrivent leur champ de recherche comme étant hautement créatif (Burton, 2004). Le terme* créativité *est malheureusement associé à des actes/produits/réussites éminents, quoique les recherches suggèrent plutôt la pensée créative comme étant une occurrence de tous les jours (Craft, 2002). Dans le cadre de ce groupe de travail, nous tenterons de capter l'essentiel des notions associées à la créativité mathématique, comme l'envisagent les mathématiciens et comme elle est décrite dans les recherches courantes, ainsi que chercher à l'exprimer de manière à ce qu'elle puisse être appliquée par les apprenants de mathématiques incluant, sans toutefois nous restreindre, ceux vus comme étant hautement capables. Les questions que nous pouvons étudier incluent les suivantes : Qu'est-ce que la créativité mathématique? Diffère-t-elle d'autres types de créativité? Comment peut-on l'observer chez les apprenants? Est-elle nécessaire pour la recherche en mathématiques? Comment la créativité peut-elle être mise en valeur dans l'apprentissage des mathématiques en salle de classe? Certains apprenants sont-ils plus créatifs comparativement à d'autres?*

Un objectif qui peut occuper une part du temps de travail consisterait à développer une série de tâches pour les enseignants qui pourront être publiées dans un journal professionnel. Ces tâches pourront servir de balises pour les enseignants qui désirent créer un environnement d'apprentissage mettant l'accent sur les habiletés créatives décrites par les mathématiciens pour ses élèves.

[1] Seul le masculin est employé dans le seul but d'alléger le texte.

CONTEXTE ET RECENSION DES ÉCRITS

QUESTIONS DE DÉFINITION

De nombreux défis définitionnels en ce que constitue *la créativité mathématique* se posent dans des discussions de groupes. Des consultations antérieures des écrits ont conclu qu'il n'existe pas de définition universelle acceptée de la créativité ou la créativité mathématique (Mann, 2005; Sriraman, 2005). Néanmoins, il y a certains paramètres convenus dans la littérature qui aident à préciser le concept de créativité (Sriraman, Haavold, & Lee, 2013). En un mot, la créativité dite *extraordinaire* (ou gros 'C') se réfère aux connaissances exceptionnelles ou aux produits qui peuvent changer notre perception du monde (Feldman, Csikszentmihalyi, & Gardner, 1994). *La créativité ordinaire* ou *la créativité quotidienne* (ou petit 'c') est plus pertinente dans le contexte de l'école. Feldhusen (2006) décrit le petit 'c' comme un comportement adaptatif chaque fois que le besoin se fait sentir de faire, d'imaginer, de produire ou de concevoir quelque chose de nouveau qui n'existait pas auparavant dans le contexte immédiat du créateur. Enfin, la relation entre la douance et la créativité a été l'objet de beaucoup de débats (Leikin, 2008; Sternberg & O'Hara, 1999) puisque certains voient la créativité comme faisant partie du cadre d'un concept global de douance (Renzulli, 2005) tandis que d'autres font des hypothèses d'une relation entre les deux (Sriraman, 2005; Haavold, en cours de révision). Indépendamment, la créativité est reliée à un domaine spécifique ou à un domaine général, ou si l'on considère la créativité comme ordinaire ou extraordinaire, la plupart des définitions de la créativité comprennent certains aspects de l'utilité et de la nouveauté (Sternberg, 1999; Plucker & Beghetto, 2004; Mayer, 1999) selon le contexte du processus créatif et le milieu du créateur. Il y a aussi des positions minoritaires de visualisation de la créativité comme étant une expérience affective (Liljedahl, 2013).

CERTAINES ÉTUDES RÉCENTES SUR L'ENSEIGNEMENT DES MATHÉMATIQUES

Étroitement liée aux relations conceptuelles entre la créativité mathématique et d'autres concepts tels la capacité académique, la visualisation, ou des aptitudes verbales et les connaissances mathématiques, etc. est la question de « qui est mathématiquement créatif? ». Kattou, Kontoyianni, Pitta-Pantazi, & Christou (2013) ont regroupé les étudiants en trois sous-groupes : basses, moyennes et hautes habiletés mathématiques. Les étudiants de hautes habiletés étaient également des étudiants extrêmement créatifs; les étudiants d'habiletés moyennes avaient un rendement moyen aux tests mesurant la créativité mathématique; tandis que les étudiants ayant de faibles habiletés avaient un faible potentiel de créativité en mathématiques. Pitta-Pantazi, Sophocleous, & Christou (2013) a classé les futurs enseignants comme des *visualiseurs de l'espace*, *visualiseurs objectivants* ou *verbalisateurs*. Les *visualiseurs de l'espace* ont eu un score plus élevé sur le test de la créativité mathématique que les deux autres groupes.

Lev-Zamir et Leikin (2013) suggèrent que les différentes expériences antérieures des enseignants en mathématiques ont un effet sur leurs croyances relatives à la créativité mathématique. Les enseignants de mathématiques ayant une formation plus solide ont tendance à avoir une plus grande conviction par rapport à la créativité mathématique. Ces études récentes distinguent les individus à différents niveaux de la créativité mathématique selon certaines autres qualités ou capacités (Sriraman et al., 2013).

En résumé, la créativité mathématique est liée et influencée par la capacité, les croyances, le style cognitif et l'environnement de la salle de classe (Lev-Zamir & Leikin, 2013; Pitta-Pantazi et al., 2013; Kattou et al., 2013). Ces conclusions sont similaires à la recherche portant sur la créativité de façon générale et la douance. Kattou et al. (2013) soulignent à quel point la créativité mathématique est essentielle pour la croissance globale de la capacité mathématique (ou douance), tandis que Lev-Zamir et Leikin (2013) montrent que des problèmes mathématiques contenant de grands défis et l'enseignement flexible peuvent aider au développement de la créativité mathématique.

Dans le groupe de travail, des problèmes mathématiques contenant beaucoup de défis/tâches ont été utilisés pour étudier la créativité mathématique, et les participants ont été invités à participer à la création d'autres problèmes dont quelques-uns sont à suivre. Des exemples d'autres problèmes et de tâches à être utilisées pour la formation des enseignants ainsi qu'avec les étudiants peuvent être trouvés dans les écrits (par exemple, Mason & Watson, 2001; Zazkis, 2008).

RÉSUMÉ DES DÉLIBÉRATIONS DU GROUPE

La description de travail du groupe initial a été centrée autour de trois articles récents de journaux choisis en partie afin d'illustrer l'étendue de travail fait autour de la créativité mathématique (Liljedahl, 2013; Sinclair, de Freitas, & Ferrara, 2013; Sriraman, 2009). Ces articles ont été choisis principalement parce qu'elles englobaient les écrits classiques sur la créativité, comme le modèle de Gestalt. Les participants ont été invités à lire ces articles à l'avance s'ils le souhaitaient, permettant ainsi de fournir quelques notions initiales pour les participants dont le domaine était assez nouveau pour eux. Ces articles ont été proposés comme le point de départ de nos discussions. De plus, durant les périodes des groupes de travail, les participants ont reçu des copies d'un numéro spécial récent de *ZDM* (Leiken & Pitta-Pantazi, 2013) qui contenait les enquêtes sur la construction de la créativité mathématique en ce qui a trait à la didactique des mathématiques. Dans la recension des écrits présentée ci-dessus, certaines constatations pertinentes par rapport aux articles de ce numéro spécial ont été résumées.

NOS QUESTIONS

Nous avons décrit le groupe de travail comme une occasion de délibérer sur les questions suivantes :

- Qu'est-ce que la créativité mathématique ?
- Diffère-t-elle d'autres types de créativité ?
- Comment peut-on l'observer chez les apprenants ?
- Est-elle nécessaire pour la recherche en mathématiques ?
- Comment la créativité peut-elle être mise en valeur dans l'apprentissage des mathématiques en salle de classe ?
- Certains apprenants sont-ils plus créatifs comparativement à d'autres ?

Il s'est avéré que certaines questions avaient une importance plus grande que d'autres, et en particulier beaucoup de discussions ont été reliées au soutien et aux occasions d'implantation de la créativité dans les salles de classe.

NOS DÉLIBÉRATIONS

Le tout a commencé avec la présentation des membres, ce qui est typique à cette conférence. En raison de la taille du groupe (27 personnes en plus des trois leaders), les participants ont été encouragés à écrire une ou deux questions principales d'intérêt. Les participants ont été regroupés en petits groupes de travail en fonction de ces déclarations d'intérêt, et ces groupes se sont réunis durant des temps désignés lors de nos périodes de travail. Puisqu'un certain nombre de chercheurs dans le domaine étaient présents dans l'équipe, nous les avons invités à présenter durant les rencontres. Nous sommes particulièrement reconnaissants à Peter

Liljedahl et Roza Leikin. Pendant le premier jour des rencontres, nous avons été invités à explorer une première tâche en mathématiques.

<u>Problème 1</u>

Peter a partagé le problème suivant :

> *Vous avez une minuterie de 4 minutes et une de 7 minutes. Comment pouvez-vous les utiliser pour faire cuire un œuf pendant 9 minutes en utilisant le moyen le plus rapide ?* (Il a été demandé et clarifié que ces minuteries étaient des sabliers).

Les participants ont commencé à travailler sur le problème, en même temps que nous avons commencé à réfléchir à nos questions sur la créativité mathématique. Même si un certain nombre de participants pensait avoir trouvé « la » solution au problème, il a été rappelé de « continuer de travailler » ! Il s'est avéré que le temps le plus rapide possible est de 9 minutes ! (Le défi de résoudre le problème de manière optimale est laissé au lecteur). Après un engagement sur le problème, un certain nombre d'idées initiales ont été partagées à l'égard de nos questions autour de la créativité mathématique (et si elle est différente des autres types de créativité) ainsi que le processus créatif. Quelques-uns des points soulevés sont résumés et à suivre, officieusement regroupés en sous-titres. Lorsqu'ils sont disponibles, les auteurs des observations sont identifiés entre parenthèses.

La créativité mathématique

Il a été suggéré que la créativité n'est pas un adjectif pour décrire un type de problème (John). Nous pouvons faire des distinctions entre *les concepts* d'une personne créative, *un produit* créatif, ou *un processus* créatif (Peter). Nous avons peut-être une image fantaisiste d'une personne créative comme une image d'Einstein circonscrit sur un faisceau de lumière. Mais existe-t-il vraiment des personnes créatives ?

Produits créatifs

Les produits créatifs pourraient être ceux qui impliquent la qualité ou la nouveauté de l'idée. Un point de vue peut être que les produits sont seulement créatifs s'ils sont complètement nouveaux pour le monde. Il a été suggéré que ce n'est qu'une différence de degré — le processus de création est le même si un produit est une nouveauté pour le monde entier ou seulement pour l'individu. Les idées de la créativité relative versus la créativité absolue ont été discutées.

Le processus créatif

L'exemple de l'œuf et de la minuterie a le potentiel d'occasionner ce genre d'expérience « créative » (Peter). Mais vous êtes démotivés par le processus créatif si quelqu'un dévoile la réponse avant que vous ayez suffisamment de temps pour vous engager sur le problème. Peter estimait que cette expérience créative, bien que cognitive, est très affective.

Plusieurs participants souscrits au processus du modèle de (Hadamard) comprenant quatre phases : l'initiation, le travail intense de délibération, l'incubation (temps éloigné), l'éclairage (la solution vient de partir de l'inconscient, au conscient, et est livrée avec une charge affective), puis la vérification (est-ce vraiment correct? — contrôlé), et la question du temps adéquat de travail est à nouveau notée. Chiaka a ajouté qu'elle estimait que la phase d'incubation est importante — en particulier, pour les étudiants, combien de temps est suffisamment de temps ? Combien de temps faut-il donner aux étudiants ? Si la solution est donnée, ou donnée trop tôt, certains étudiants perdront l'occasion d'être créatifs.

Ami a noté que l'on peut être créatif dans la manière dont nous approchons une tâche, dans la façon dont nous parlons d'une tâche, ou dans la façon dont nous travaillons sur une tâche. Quels sont les moyens les plus légitimes pour faire preuve de créativité ? Certaines méthodes semblent de la « tricherie » comme mettre le sablier sur le côté. Peut-être que l'étape créative vient lorsqu'on est poussé plus loin qu'on pense que c'est nécessaire (Dragana).

Il a été demandé : quels sont les différents *moyens légitimes* à penser de façon créative sur la tâche ? Un petit groupe a proposé les items de la liste suivante pour discussion :

- Éviter la créativité — étant mignon
- Le changement d'un paramètre qui permet de faire des progrès
- L'inversion du « sept » (dans le problème de l'œuf) avec une minute de passée donne le sentiment d'un oubli
- Aperçu éphémère — on a quelque chose, puis qu'on le perd
- Le tamis esthétique — lorsque le subconscient favorise une idée qui a de la possibilité

Se coincer et le moment de l'illumination

Même si le moment d'illumination est significatif, on ne reconnaît pas toujours s'il est significatif. Il s'agit de la charge affective qui le rend tellement puissant.

Poincaré disait que vous avez à remplir votre esprit avec des idées et les laisser reposer (plutôt que d'essayer consciencieusement de fusionner des idées). Quelle est la nature de la certitude, de la signification, du degré d'atteinte et de la quantité de temps à être coincé ? Plus la charge est grande, plus on se souvient de l'expérience. Nous continuons à écraser les idées une à une jusqu'à ce qu'une frappe le subconscient comme étant importante. Asia a ajouté que vous pouvez avoir un moment créatif lorsqu'on est en train de faire autre chose. La contemplation est le moment entre le plein engagement et le recul pour donner à votre esprit l'espace pour saisir l'idée.

<u>Problème 2</u>

Un problème contrasté, choisi afin d'explorer davantage notre sens d'un plus et moins grand processus créatif, a été ensuite présenté par Bharath. Le problème pourrait avoir une application particulière à la formation contextuelle des enseignants.

Il a été proposé qu'il existe des fractions pour lesquelles l'affirmation suivante est vraie :

> *Nous pourrions soutenir que 16/64 = 1/4 et nous obtenons ce résultat en biffant (« annuler ») les 6. Un deuxième exemple est 19/95 = 1/5 en biffant les 9. Un autre exemple est 13/325 = 1/25 si nous annulons les 3. La tâche est : pouvons-nous trouver d'autres exemples pour lesquels ceci est vrai ?*

Puisque nous avons travaillé sur le problème en petits groupes, de nombreux résultats et de nombreuses observations ont été relevés. Une autre « solution », 26/65 = 2/5 a été trouvé par Mike, et Richard a partagé sa méthode de rationalisation des choix en utilisant une méthode algébrique, indiquant une démarche qui était selon lui plus « créative », et a également trouvé que 49/98 = 4/8. John a étendu notre pensée en ouvrant la porte aux nombres négatifs (cas des nombres entiers (voir la Figure 1). Ce problème a incité la poursuite de la discussion sur la nature du processus créatif, avec quelques points résumés à suivre.

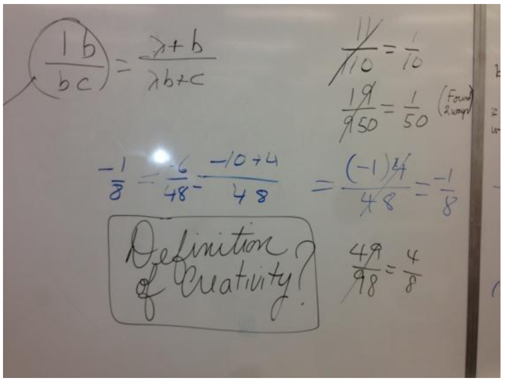


Figure 1. Réflexions en progression sur la tâche des fractions.

Poursuite de la discussion sur le problème

Nous avons trouvé que le problème des fractions a encouragé la poursuite des débats. Par exemple, il a été suggéré que nous avions de nombreuses instances d'idées créatives (Asia), et subsidiairement, il a également été suggéré que toutes les techniques étaient des actes connus et que notre travail *n'était pas* créatif (Ami). En effet, Roza a fait valoir que le problème a illustré une interprétation de la créativité dans le pire des sens. Richard a défendu le besoin de l'efficacité de solutions créatives, n'en trouvant aucune créative jusqu'à présent, et Asia a répondu que la créativité peut modérer notre efficacité, mais le contraire n'existe pas. Bharath a suggéré que l'optimisation d'une solution est en effet reliée à la créativité.

Il a été noté toutefois que notre idée ne travaille pas à caractériser notre pensée non linéaire, donc on ne peut pas vraiment dire si nos pensées étaient créatives ou non (Peter). Ce que nous avons vu sur le tableau venait de nos connaissances et nos expériences, mais le processus que nous avons traversé était encore soutenu par certains comme étant créatif — en discutant et en essayant différents chemins. Certains participants ont estimé que nous pourrions encore avoir eu un processus créatif, même si ce n'est pas un résultat créatif (Dragana).

Tim a précisé que, selon lui, la partie créative va de côté, et est à la recherche d'une représentation différente. L'incubation est requise pour passer par le processus de « désapprendre ». Ami a noté que nous avons tous été appris à être à la recherche de représentations et n'a pas l'impression que cette tâche l'a provoqué à être créative. Elle a tourné autour de méthodes connues et n'avait pas trouvé un aspect imprévisible.

Carol a également estimé qu'elle était plus créative avec le problème d'hier (sablier) tout comme Dragana. Viktor d'autre part sent qu'il était plus engagé aujourd'hui. Il a expliqué qu'il aime l'algèbre, donc il a trouvé la tâche plus fascinante et plus agréable, une expérience plus affective — donc créative. Mais il a noté que ce n'était pas vrai pour d'autres.

Y a-t-il des problèmes créatifs?

Si vous pouvez résoudre un problème directement ce n'est pas un problème — vous devez être coincés. Le processus de déblocage, nécessitant *un AHA*, c'est ce qui fait quelque chose un problème. Ce serait donc créatif (Peter). Chiaka estime que les solutions « non signifiantes », telles que 11/11 = 1/1 pour la tâche des fractions, peut aider parce qu'il nous plonge dans le problème.

En tant qu'enseignants, nous voulons des tâches pour promouvoir la créativité, mais les tâches ne peuvent pas être créatives — la créativité est plus d'une attitude (Dalia). La créativité doit-elle être toujours passionnante et amusante ? « La muse vous inspire » — la créativité est-elle censée être amusante? (Chiaka). Quels sont les verbes — il y a-t-il une façon générale de poser des problèmes ? Certains verbes pourraient l'être, construire, découvrir, trouver, dites ce que vous voyez (Dalia).

Eric, Leah, Limin et Mimi ont demandé : les étudiants utilisent-ils les outils qu'ils y sont donnés ou vont-ils aller au-delà de l'espace où ils peuvent être créatifs ? De quelle manière les enseignants offrent-ils des possibilités pour que les étudiants soient créatifs ? Dans quelles situations les élèves sentent-ils qu'ils font preuve de créativité ? Est-ce qu'ils se chevauchent ? Ces idées sont représentées dans ce modèle de groupe, à condition de suivre.

Il a été suggéré qu'il doit y avoir plusieurs représentations, pour que quelque chose soit créatif (Dragana). En tant que professeure, elle est à la recherche de moyens plus créatifs dans son enseignement — et ne peut pas anticiper ce que les étudiants pourraient ressortir. Mais une autre routine dans le processus de résolution pourrait aussi faire quelque chose de plus créatif. Richard a fait remarquer qu'il pourrait parfois y avoir plusieurs solutions — une solution banale et une solution plus créative. Un problème « créatif » pourrait être l'un de ceux pour lesquels il y a une telle solution « alternative ».

Roza a noté que les enfants doivent faire preuve de créativité pour construire de nouvelles connaissances. Quand construisons-nous de nouvelles connaissances ? La réponse dépend des connaissances actuelles des participants. Certaines personnes peuvent résoudre un problème de façon créative alors que pour d'autres, ce ne sont pas de nouvelles connaissances. Cela dépend de votre point de départ et de votre formation. John a approuvé en disant que, par exemple, pour lui, la notion de chiffres non standardisés est très familière et ce processus n'était pas créatif.

Le rôle des connaissances antérieures

John demande: *l'expérience préalable* (il préfère éviter le terme connaissance), viendra-t-il à vous lorsque vous aurez besoin d'elle ? Qu'est-ce qui l'amène à l'esprit ? Si vous n'avez pas connaissance au préalable, vous n'avez aucun moyen de démarrer le processus (Josh). Quand un certain type de problème devient-il non créatif ? Dans chaque instance d'un certain type de problème, il devient de plus en plus routinier (Michael). Nous trouvons qu'avec les élèves des écoles primaires, la moitié de ceux-ci sera vraiment engagée dans le problème, et l'autre moitié ne comprendra même pas. Les connaissances antérieures, le bon sens et l'expérience sont indispensables pour pouvoir débuter (Eric).

Tâches à solutions multiples

De nombreux problèmes mathématiques peuvent seulement être résolus par la routine et par des moyens banals. Toutefois, si les étudiants voient des problèmes qui peuvent être résolus par routine et par un moyen innovateur surprenant, alors de nombreux avantages inattendus surviennent : une plus grande confiance en faisant des mathématiques, une appréciation plus profonde de la beauté des mathématiques, et bien sûr, un développement de sa créativité (Richard, Rina, Chanakya). L'argument a de nouveau été fait que les problèmes mathématiques ne sont pas créatifs, en eux-mêmes et pour eux-mêmes. Si un problème mathématique a une solution créative, alors la solution est créative, mais le problème n'y est pas. Cette ouverture, cette possibilité de découverte, est ce que nous, les éducateurs devraient incorporer à notre enseignement.

Roza a demandé : est-ce que ce sont les tâches qui nécessitent un processus créatif ? Elle fait référence aux questions d'orientation, de la quantité d'indications à fournir et à la base des connaissances par rapport aux nouvelles connaissances.

Tâches géométriques avec des solutions multiples

Roza a partagé avec nous son travail de l'utilisation d'un certain nombre de tâches de preuves géométriques, pour lesquelles les étudiants ont été demandés de les résoudre en utilisant plus d'une manière. Les preuves pouvaient être évaluées en manière de complexité, d'élégance, et ainsi de suite. Il a été noté que, sur 20 solutions, cinq étaient vraiment créatives — et qu'elle n'avait jamais vu ces propriétés auparavant. C'était une véritable découverte, même pour l'enseignante. Mais tout est relatif. Lorsque nous travaillons avec des étudiants ou de nouveaux enseignants, sur un niveau relatif, il leur donne cet effet d'excitation, de découverte, qu'ils ont pu le faire eux-mêmes. Les mathématiciens découvrent des théorèmes de niveau supérieur, mais au niveau scolaire, c'est nouveau. Elle croît vraiment que les étudiants ont exécuté certains actes créateurs. Peter a noté que, lorsqu'ils ont à le prouver, il devient leur propre théorème.

Roza nous pousse encore une fois à réfléchir: y a-t-il définitivement des tâches créatives (qui ne dépendent pas du niveau des participants) ? Durant sa présentation plénière, Roza a partagé des exemples de tâches avec multiples solutions (voir Leikin, ce volume).

Soutien aux enseignants

De nombreuses conversations pendant notre groupe de travail, ainsi que dans les petits groupes de travail ont été concentrées sur les façons de soutenir les enseignants dans leur travail. Dragana soutient qu'il s'agit de pousser les étudiants et les enseignants qui poussent toujours à entraîner les étudiants à épuiser les méthodes connues et d'en essayer des nouvelles. Par exemple, avec le problème du sablier, après l'épuisement des idées, nous pourrions essayer différentes choses avec la minuterie. Le décalage est un moment créatif. JP a ajouté : lorsque nous apprenons, c'est toujours dans un contexte. Si nous voulons changer les contextes, nous devons créer un lien vers le nouveau contexte.

Le petit groupe de Richard, Rina et Chanakya a discuté de l'idée de *briser les contraintes*, comme une idée-clé dans le processus créatif (voir également Zazkis, 2008). Nous ne nous attendons pas à ce que nos étudiants puissent reproduire, de leur propre initiative, les travaux des mathématiciens qui ont avancé le sujet grâce à leurs habiletés créatives. Et pourtant, comme professeurs de mathématiques, nous pouvons offrir aux élèves des problèmes qui se prêtent à des solutions non standards où une solution perspicace courte existe par la rupture d'une contrainte dans une manière surprenante et novatrice, ou par l'application d'une technique d'un domaine indépendant du domaine des mathématiques.

Le focus de Tim, Nathalie et Viktor était sur les obstacles et la gêne qui existent chez les enseignants qui utilisent et, par conséquent, modélisent la créativité en salles de classe. Des expériences communes incluent les enseignants qui ont de la difficulté avec l'enquête fondée et l'approche de la résolution de problèmes. Ces difficultés ont été considérées comme ayant deux dimensions pertinentes. L'une était la difficulté d'identifier des problèmes pertinents. Deuxièmement, il est souvent vu que même avec des tâches riches d'apprentissages, les enseignants adoptent une approche plus normative dans leur enseignement plutôt que de valoriser la promotion de la créativité. Il y avait un sentiment qu'en fournissant de nombreuses questions pertinentes ne seraient pas suffisant pour la promotion à long terme pour soutenir la créativité chez les étudiants (l'enseignant va peut-être s'habituer à la question et ne pourra probablement pas maintenir une *position créative* en utilisant la question). La composante du processus dans le curriculum pourrait être utilisée afin d'encourager la créativité. Les enseignants peuvent jouer un rôle normal, mais il peut y avoir une occasion d'encourager différents processus qui pourraient encourager les étudiants à générer des approches et des techniques originales. Ces points ont permis de discuter de bonnes tâches, toutefois, il a été noté qu'à certains égards, toutes les tâches peuvent être prises comme bonnes tâches; on a donné l'exemple du '3 + 5 = ?' comme étant une question valable pour promouvoir la pensée chez les étudiants de la maternelle qui pourrait mener à la découverte, tels que '3 + 5 = 5 + 3'. L'utilisation de matériel de manipulation était employée comme un exemple tangible de stratégies pour encourager la créativité des étudiants de la maternelle. Ceci a conduit à la discussion des méthodes connues pour développer des approches créatives pour l'enseignement, telles que la collaboration entre enseignants et chercheurs avec *co-enseignements*. Toutefois, le défi à relever est que de tels travaux tendent à adresser relativement des petits groupes d'enseignants et il y a un défi important d'étendre ce défi aux écoles et aux conseils scolaires.

D'autres participants ont posé des questions reliées. Les mathématiques à l'école secondaire peuvent-elles être toujours créatives et conséquemment « rencontrer » les objectifs du programme d'études ? Peut-être qu'on pourrait utiliser la créativité pour augmenter la vision des étudiants dans le domaine des mathématiques. Nous avons besoin de permettre aux étudiants de jouer avec les mathématiques. Quels sont les liens entre la créativité, le jeu, et l'imagination ? Est-ce que quelqu'un peut être créatif dans n'importe quel domaine ? Est-il important qu'ils le soient ? Comme enseignants de mathématiques, nous pourrions essayer d'identifier ou de comprendre pourquoi nous voulons les inciter à la créativité. Par exemple, si c'est pour le plaisir et la satisfaction personnelle, alors il suffit peut-être que l'acte soit créatif pour l'individu et non pas nécessairement pour la société. D'autre part, si l'objectif est d'apporter une contribution à l'environnement de l'individu, alors, il n'est peut-être pas suffisant que l'acte soit *subjectivement* créatif (Dalia).

<u>Réflexions émergentes sur le processus créatif — les mathématiques et les arts</u>

L'une de nos questions d'orientation était entre la relation de la créativité mathématique et d'autres types de créativité. Bill a partagé avec nous son implication dans les arts, en travaillant avec des danseurs et dans la musique. Dans le domaine des arts « il n'y a pas d'objectif ». En mathématiques il y a très souvent des objectifs biens définis tels que faire quelque chose dans les plus brefs délais, ou trouver un algorithme. Dans le monde des arts, c'est le contraire. Les artistes veulent faire quelque chose d'original, quelque chose qu'ils n'ont jamais fait avant. Aussi Bill a noté que les personnes avec qui il travaille le rendent plaisant. Dans le domaine de l'art, travailler avec d'autres personnes, et rebondir des idées le rend amusant, ce qui peut être vrai aussi en mathématiques. Lorsque vous travaillez avec d'autres artistes et que vous commencez à construire quelque chose, c'est passionnant. Mais, dans le domaine des arts, quelque chose doit être plus que de simples « nouveautés » pour être originales.

Bharath a ressorti que l'art n'est pas illimité, comme beaucoup de gens le prétendent. Même l'art a ses limites. La créativité n'est pas libre pour tous; il existe des outils et des gardiens de barrières. Des actes éminents au début du champ peuvent être créatifs, mais si une personne voit quelque chose pour la première fois et qu'elle crée les outils, à ce moment-là, c'est créatif. Chaque domaine académique a des jurés. Même l'art n'est pas illimité. Les peintres historiques devaient d'abord convaincre l'Église et les dirigeants. Nous embrouillions la créativité avec les actes vraiment éminents de personnes importantes. Dans chaque domaine, il y a des gens qui la jugent — donc aucun domaine n'est illimité comme il peut le sembler.

Les mathématiques ont des frontières, ce qui rend la tâche difficile de les franchir — peut-être, est-ce plus difficile en mathématiques qu'en arts? Nous devons être créatifs dans les limites du problème (Kevin).

La créativité comme un processus social

Comme nous avons travaillé sur l'exemple des problèmes, beaucoup d'entre nous se sont engagés dans un processus social, de sorte que ce thème était un thème naturel de discussion. Il a été noté que pour arriver à travailler en collaboration, nous avons besoin d'une tâche qui amène les personnes à vouloir discuter et se prêter à travailler ensemble.

Tim a précisé que l'un des aspects du travail créatif peut impliquer la construction d'un processus social. Il s'intègre aux critères de la dissonance cognitive — vous travaillez avec d'autres personnes qui ont peut-être remarqué quelque chose de différent modifiant ainsi votre réflexion et vous risquez de passer par une nouvelle direction. C'est désapprendre — vous oubliez ce que vous avez fait et vous prenez un nouveau chemin. Sinon vous serez borné parce que vous êtes en train de conduire à un autre but.

Roza a demandé : qu'est-ce que l'éclair de génie ? Si vous séjournez dans une zone connue, ce n'est pas créatif. Dans tout effort de créativité, vous découvrez quelque chose qui est nouveau pour vous, et vous vous y êtes probablement rendu avec l'aide des autres. Lorsque vous découvrez quelque chose, vous pourriez certainement y arriver avec l'aide d'autres personnes qui savent déjà cela.

La créativité dans le domaine des mathématiques

Bien que le débat se soit moins centré sur la question initiale de découverte mathématique, Richard a partagé une expérience dont il a utilisé la théorie des graphiques pour résoudre un problème d'horaires. Il a décrit une expérience d'illumination profonde lorsqu'il s'est rendu compte que le problème de programmation pourrait être résolu par la théorie des graphes. Mais ensuite, il lui a fallu un mois pour travailler les détails. On nous rappelle de l'idée de supprimer des frontières, et des connexions inattendues, comme caractéristiques de la créativité.

Questions et idées émergentes sur l'*insight*

John lie la notion de l'éclair de génie en décrivant l'expérience de Richard comme « un moulage dans une nouvelle lumière », ou « une nouvelle façon de voir ». La question est la suivante : « quelque chose me vient-il à l'esprit », y a-t-il une dimension affective — un AHA ? Qu'est-ce qui déclenche ce qui est apporté à la surface ? Peter a suggéré que le travail de Richard était très illimité. Donc, peut-être que le processus de créativité général est la construction de nouveaux ponts et voir à de nouvelles connexions.

On s'est demandé : si l'on n'a pas les connaissances antérieures, comment fait-on pour démarrer le processus ? Viendra-t-il de quelque chose que nous avons déjà appris ? Ou, allons-nous l'attaquer n'importe comment ? Josh résume, notant qu'il est tellement subjectif, ce qui est souligné est tellement différent. Nous avons peut-être un grand problème de créativité dans une salle de classe que certains étudiants sont en train de travailler et de jouir — mais souvent, l'autre moitié des élèves ne comprend pas le problème. Eric voit les connaissances antérieures comme fondamentales — et pourtant nous dépensons « trop de temps sur la connaissance et pas assez de temps sur la sagesse ». Il nous faut suffisamment de connaissances pour choisir des alternatives, mais le développement des choix n'est pas assez discuté. Cette auto-évaluation interne est très importante, mais nous ne l'enseignons pas, nous nous concentrerons uniquement sur l'évaluation externe.

Nos premières émergences — et tâches dans un temps limité — définitions et descriptions

En tant que tâche provocatrice (non populaire!) de travail de groupe, les participants ont été donnés deux minutes pour écrire leur propre description ou une définition de la créativité. Certains participants ont naturellement refusé. D'autres ont travaillé dans un petit groupe. Voici quelques échantillons de nos pensées :

- Peter et coll. : La créativité est la capacité de générer quelque chose d'inconcevable.
- John: Une qualité particulière d'énergie qui provient de l'extérieur. Il découle de l'extérieur.
- Richard : Une solution mathématiquement créative implique la nouveauté, la rupture des contraintes. Il serait peut-être la réalisation que nous puissions connecter un problème à un autre champ ou la réalisation de connexions et de la découverte d'un nouvel éclair de génie.
- Dragana et Carol : Nouveauté dans l'utilisation d'outils standards afin d'étendre leurs connaissances personnelles de manière surprenante et joyeuse, moyens de générer de multiples scénarios qui sont souples et ouverts à de nombreux nouveaux points. Elle implique l'effet qu'elle a sur une personne.
- Limin : Le point de vue individuel de la nouveauté est basé en fonction de leur propre expérience. (Il est relatif à l'individu).
- Tim : L'individu qui va au-delà du point de vue de l'observateur, de cette zone proximale de développement; l'observateur peut être soi-même. Elle consiste à briser les règles.

Discussion / Critique de nos descriptions initiales

Richard a demandé si des outils utilisés dans des manières en *non-standards* seraient créatifs. John a apprécié l'idée que la créativité soit dans l'œil de celui qui regarde. Il a ajouté qu'il n'a pas d'expérience à travailler dans une situation dans laquelle il n'avait aucune idée de ce qu'il doit faire. Il faut avoir un certain sentiment que la tâche est faisable, ou il n'y a pas d'engagement.

Roza a noté la connexion à la zone proximale de développement. Si vous séjournez dans une région donnée, il n'est pas créatif. Dans tout acte créatif, lorsque vous arrivez à quelque chose de nouveau, il se peut que vous y arriviez avec l'aide des autres. À un certain point, si les idées n'existent pas encore, vous essayez de faire quelque chose de nouveau. John a noté que les exemples de géométrie de Roza sont des exemples de langage de vision en soulignant certaines choses et en ignorant d'autres choses, tout en reconnaissant qu'il pourrait y avoir une relation ou des propriétés, ou qui cherchent à trouver certaines relations qui sont effectivement des propriétés. Il a demandé : « En voyant différemment, est-ce que l'acte est créatif ou non ? »

Peter a précisé : « Je m'assois pour faire un problème — je ne peux peut-être pas voir tout le chemin jusqu'à la fin. À chaque chemin, une nouvelle idée me vient. Je ne vois pas ça comme créatif. Je peux reculer et corriger une erreur. D'où viennent les idées ? Hadamard parle de cela — ne pas pouvoir voir la fin du début n'en fait pas une expérience créative. Ce qui est créatif, c'est lorsque je ne peux pas voir l'étape suivante, et puis soudain je peux la voir. Il pourrait s'agir d'un moulage, ou elle pourrait être une illumination. C'est une différence très subtile. J'aime l'idée de voir. Il ne s'agit pas de ne pas être en mesure de voir la fin du début, il s'agit d'être bloqué. À un moment donné, quelque chose que je ne pouvais pas faire directement a surgi. »

La créativité est d'aller au-delà, non seulement en voyant, mais en mouvement. Il y a une force interne qui vous pousse même si vous connaissez la prochaine étape ou non (Viktor).

Les étapes peuvent être l'initiation, l'incubation, l'illumination et la vérification (Wallas, 1926). Il y a peut-être une cinquième étape de l'évaluation, telle que dans le domaine des arts. Ceci parle d'une créativité absolue. Dans des situations illimitées comme les arts, l'exemple de la chorégraphie de Bill, ou le travail des élèves de Roza — une étape d'évaluation est nécessaire. Est-ce divertissant ou nouveau ? Dans les situations illimitées, l'évaluation est importante (Peter).

Plus tard les définitions et modèles

Nous avons fermé nos délibérations en invitant les participants à créer et de partager une déclaration, un modèle, ou de tout autres produits issus de leurs réflexions et du temps ensemble. Quelques-unes de ces déclarations, des modèles, et des problèmes suggérés sont offerts et à suivre, si l'espace le permet. Nous avons choisi de terminer nos discussions par le partage de la largeur de nos perceptions et idées, plutôt que de lutter pour obtenir toute sorte de conception unifiée.

IDÉES FINALES

Définitions et descriptions

La créativité est ...

- La capacité de générer quelque chose d'inconcevable.
- Fort utilement pensé comme une qualité particulière de l'énergie qui provient de l'extérieur. Il s'écoule à travers notre psychisme et apporte l'expérience d'illumination de l'esprit (un moment AHA?). Il y a habituellement une poussée d'énergie à atteindre ; parfois, cela vide l'énergie de pouvoir suivre à travers des détails ; c'est parfois expérimenté comme un éclair de génie ; d'autres fois, c'est un acte de création ; et parfois les deux. L'énergie créatrice est probablement toujours disponible, mais elle est facilement bloquée par d'autres énergies circulant à travers ou par l'activation sur soi-même. C'est pourquoi son accès est autant une question de laisser aller et d'agir différemment.
- Utiliser des outils standards pour étendre les connaissances personnelles en manières surprenantes et joyeuses, en générant plusieurs scénarios qui sont flexibles et ouverts aux nouveaux points de vue individuels — du point de vue de la nouveauté basée sur leur propre expérience.
- La capacité de générer quelque chose qui est inconcevable.
- Un processus (Hadamard : préparation, incubation, illumination, vérification et évaluation) ; pas le long d'un train ordinaire de pensée. Subjectif et personnel. Quelque chose de nouveau, original ou inhabituel.
- Une définition en ligne pour le terme *la créativité est* : « l'utilisation de l'imagination ou d'idées originales, en particulier dans la production d'un travail artistique ». Pour moi, la première partie de cette phrase est assez claire, et je suis d'accord que la créativité comprend une composante de l'imagination. La deuxième partie de la phrase n'est pas claire pour moi. La créativité doit-elle *produire* un travail ? Est-ce un objet concret ou peut-il être une nouvelle façon de penser ? En quelle perspective doit-elle être « artistiques »?
- Définition de Nathalie Sinclair : un acte créatif introduit le nouveau par une manière imprévisible qui transcende les habitudes courantes de comportement et dépasse les significations existantes (originalité relative).
- Mathématiquement, une approche créative est celle qui consiste à briser des contraintes, à réaliser les connexions, à de nouveaux aperçus.

<u>Modèles</u>

Certains sous-groupes ont également proposé des modèles, et les ont partagés à la fin de notre temps ensemble. Quelques-uns suivent.

Modèle 1

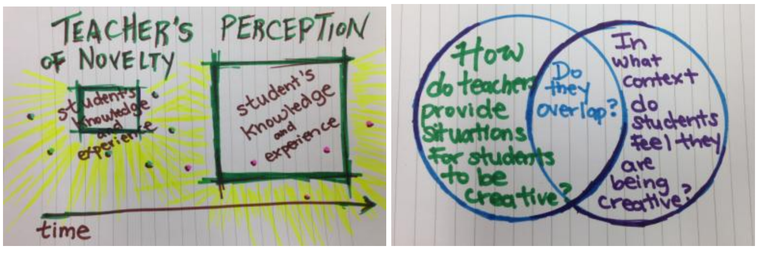


Figure 2. Les perceptions des enseignants (Eric, Leah, Limin et Mimi).

Modèle 2

Nous avons essayé d'identifier les possibilités de modifier les approches des enseignants. Cela a conduit à la Figure 3 qui trace le curriculum en termes de créativité.

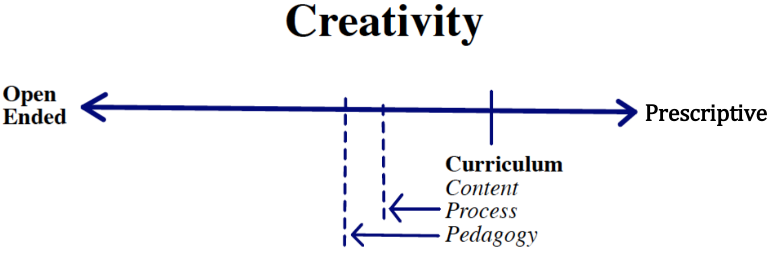


Figure 3. Où est le programme d'études en termes de créativité?

Dans la Figure 3, la ligne horizontale représente un spectre de possibilités créatives. Les approches prescrites ont été interprétées comme favorisant la créativité minimale, tandis que les questions ouvertes sont vues comme une occasion de créativité. L'approche des questions ouvertes a été discutée en termes de classe et interprétée comme étant des problèmes avec plusieurs approches ou méthodes de solutions accessibles aux étudiants. En termes de curriculum, il y a couramment différentes composantes avec différents niveaux de prescription. Ce modèle envisage de faire apprécier globalement le curriculum en termes généraux. Dans ce groupe, le contenu du curriculum était considéré comme le plus contraignant, cependant, il faut souligner que c'est le minimum qu'un enseignant doit enseigner, lui donnant la possibilité, tant que temps lui permet, pour qu'il soit créatif au-delà des curricula. Les processus décrits dans les programmes d'études ont tendance à être moins prescriptifs, mais il y a des exigences. Par exemple, l'obligation d'utiliser la résolution de problèmes est quelque peu prescrite mais pas autant que l'enseignement spécifique de la loi des exposants. Enfin, la pédagogie tend à être moins prescriptive dans le curriculum, mais est limitée par un certain degré de normes de la pratique professionnelle (Tim, Nathalie, Viktor).

Modèle 3

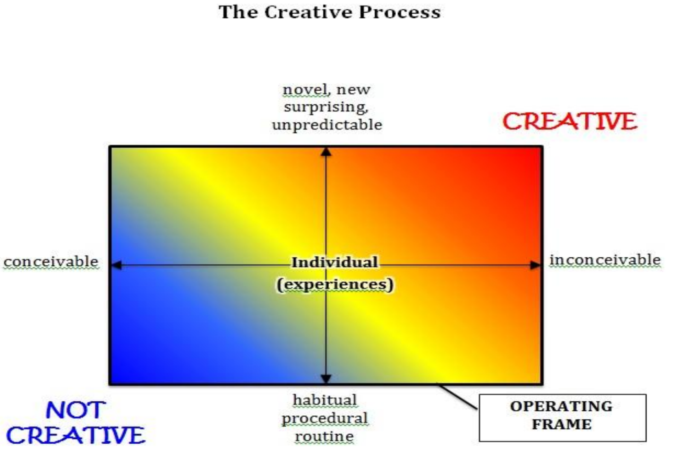


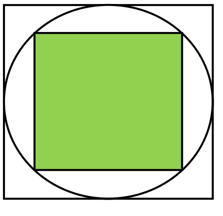
Figure 4. Le processus créatif. Le plus créatif se déplace vers la droite et vers le haut. (Kevin, Michael, JP, Josh, Jo)

Exemples de problèmes

Problèmes 1

(Richard, Rina,Chanakya)

Problème 1 :



Dans le diagramme, un cercle est inscrit dans un (grand) carré, et un (petit) carré est inscrit dans le cercle.

Quel est le rapport des aires des deux carrés?

Figure 5. Le problème cercle-carré.

Généralement, les étudiants laissent le petit carré avoir une longueur de côté de 1, utilisent le théorème de Pythagore pour trouver le rayon du cercle, et reconnaissent que la longueur du côté du grand carré était égale au diamètre du cercle. De là, ils en arrivent à la conclusion que l'aire du grand carré doit être deux fois l'aire du petit carré. Richard décrit une expérience de travail d'un étudiant timide du premier cycle du postsecondaire qui a offert une solution de rechange, bien qu'il ait été initialement réticent à partager le sentiment que ce qu'il a fait n'était « pas des maths ». L'étudiant a fait pivoter le carré intérieur et a dessiné une ligne verticale et une ligne horizontale passant par le centre du cercle. Il a ensuite expliqué son raisonnement.

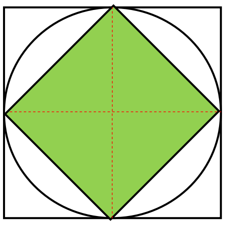


Figure 6. Une nouvelle solution pour le cercle problème.

« Si vous le regardez comme cela, c'est évident. L'aire du grand carré doit être deux fois l'aire du petit carré ». Le reste des étudiants regardent la solution. Certains l'ont vu en vingt secondes, tandis que d'autres ont pris quelques minutes. Un par un, ils sont venus à la même conclusion. Tandis que les autres étudiants ont tout simplement présumé que le carré intérieur devait rester dans une position fixe, cet étudiant a rompu cette *contrainte auto-imposée* et a choisi de faire tourner le carré intérieur, qui a clairement préservé l'aire du petit carré tout en rendant le problème tellement plus simple. Cet étudiant, qui n'avait pas réussi le test diagnostic de base en math dès son entrée à l'université, avait d'importants problèmes de compétences de base, notamment en multiplication des fractions et en facturation d'équations. Malgré sa vive intelligence et son éthique évidente du travail, il avait beaucoup d'anxiété au sujet du cours et de son manque de confiance, mais après cette expérience sa confiance a grimpé en flèche. Il s'est rendu compte qu'il pouvait faire des mathématiques parce que la mathématique n'est pas le fait de mémoriser et d'appliquer des formules.

Problème 2 :

Deux trains sont distants de 20 milles sur le même rail en se dirigeant l'un vers l'autre à 10 milles à l'heure sur une trajectoire de collision. En même temps, une abeille décolle du nez d'un train à 20 milles à l'heure vers l'autre train. Dès que l'abeille atteint l'autre train, il change de direction et retourne à 20 milles à l'heure vers le premier train. Il continue de le faire jusqu'à ce que les trains entrent en collision. Quelle distance l'abeille a-t-il voyagé ?

Figure 7. Le problème de l'abeille.

La routine d'une solution banale nécessite une série infinie de calculs, pour mesurer la distance parcourue par l'abeille chaque fois qu'il frappe l'un des trains et retourne. Toutefois, si l'on considère le problème du point de vue de l'abeille, nous pouvons simplement poser la question suivante : « combien de temps l'abeille voyage-t-elle avant d'être écrasée ? » La réponse est clairement une heure, puisque les deux trains, distants de 20 milles, s'approchent l'un de l'autre à 10 milles à l'heure. Étant donné que l'abeille voyage à 20 milles à l'heure pendant exactement une heure, la réponse à ce problème est 20 miles.

Problèmes 2 — Tâches de niveau universitaire

(John, Chiaka, Dalia, Asia, Tina)

Problème 1 :

i. Étant donné une transformation non singulière linéaire T à partir d'un espace vectoriel sur lui-même, quelles matrices peuvent être utilisées pour représenter T ?

ii. Construire une fonction de $\mathcal{R}$ à $\mathcal{R}$ qui nécessite l'utilisation k des règles de L'Hôpital afin de trouver la limite quand x s'approche de a pour certains a.
iii. Trace un graphique d'une fonction lisse *f*. Définis la puissance d'une tangente d'un point P dans le plan en respect de *f* comme le nombre de tangentes à *f* qui passent par P. Explore la puissance de tangente de points dans le plan. Y a-t-il une méthode générale pour trouver tous les points avec une puissance de tangente ?
iv. Quelle est la taille de la différence qu'il peut y avoir pour une fonction différentiable au moins k fois, entre le nombre de points maximaux et le nombre de points minimaux ?

Certains exemples de tâches de Mason & Watson (2001):

i. Un problème de moyenne: Observe que $\int_0^2 (1-x)\,dx = 0$. Généralise!
ii. Un problème de division : Trouvez tous les nombres entiers positifs qui ont un nombre impair de diviseurs.
iii. Un autre problème de carré : Étant donné deux droites distinctes lignes L_1 et L_2 et un point E pas sur eux, de construis un carré avec un sommet à E, et un sommet sur chacun des L_1 et L_2.
iv. Points Rolle : Le théorème de Rolle nous dit que toute fonction dérivable sur un intervalle a un point à l'intérieur de cet intervalle dans lequel la pente de la fonction est la même que la pente de la corde entre les points de la courbe à la fin de cet intervalle. Où, dans cet intervalle, vous attendez-vous à rechercher un tel point ? Par exemple, y a-t-il des fonctions pour lesquelles le point de Rolle de chaque intervalle est le point médian ? Une question naturelle à se poser est de savoir s'il existe des fonctions pour lesquelles le point de Rolle sur n'importe quel intervalle, est, disons, 2/3 du chemin de l'intervalle, ou plus généralement □ du chemin de l'intervalle.
v. Points d'inflexion : Une méthode commune pour diagnostic les points d'inflexion de la courbe, qui est au moins deux fois dérivable, est de différencier deux fois et fixer pour égaler à zéro pour trouver l'abscisse. Parfois, cela donne une réponse correcte pour une bonne raison, parfois, il donne une réponse correcte pour une mauvaise raison, et parfois, il donne une réponse incorrecte. Construire des exemples qui illustrent ces trois situations, et également une famille d'exemples qui incluent tous les trois dans chaque membre, et qui pourrait ainsi amener les étudiants contre ces différentes possibilités. Faut-il une fonction deux fois différentiable pour avoir un point d'inflexion? Qu'en est-il des membres de la famille $x^k \sin\left(\frac{1}{x^2}\right)$?

Une séquence de tâches :

- Trace le graphique d'une fonction sur l'intervalle [0, 1].
- Trace le graphique d'une fonction continue sur l'intervalle [0, 1].
- Trace le graphique d'une fonction dérivable sur l'intervalle [0, 1].
- Trace le graphique d'une fonction continue sur l'intervalle [0, 1] avec l'une de ses valeurs extrêmes à l'extrémité gauche de l'intervalle [0, 1].
- Trace le graphique d'une fonction continue sur l'intervalle [0, 1] avec ses deux valeurs extrêmes aux points d'extrémité de l'intervalle [0, 1].
- Trace le graphique d'une fonction continue sur l'intervalle [0, 1] avec ses valeurs extrêmes aux points d'extrémité, et avec un maximum à l'intérieur de l'intervalle [0, 1].
- Trace le graphique d'une fonction continue sur l'intervalle [0, 1] avec ses valeurs extrêmes aux points d'extrémité, et avec un maximum et un minimum à l'intérieur de l'intervalle [0, 1].

Vient maintenant la partie intéressante ! Refais ton chemin de retour à travers les exemples, en veillant à ce que, à chaque stade votre exemple ne satisfait pas les contraintes qui suivent ! Ainsi, votre premier exemple doit être une fonction, mais ne doit pas être une fonction continue; votre dernier exemple doit avoir un maximum à l'intérieur, mais pas un minimum. Trouver qu'un ensemble de contraintes semblent mutuellement incompatibles est un excellent moyen de générer une conjecture menant à un petit théorème. La structure de ce genre de tâche force les étudiants à devenir conscients d'une classe plus générale d'exemples qu'ils ont peut-être considérée la première fois.

Grille des tâches de fonctions:

Construis des fonctions continues avec les domaines et les images suivants :

Domain/Range	$\mathcal{R}$	[0, 1]	[0, 1)	[0, ∞)	(0, ∞)
$\mathcal{R}$					
[0, 1]					
[0, 1)					
[0, ∞)					
(0, ∞)					

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MATHEMATICS OF PLANET EARTH 2013: EDUCATION AND COMMUNICATION

MATHÉMATIQUES DE LA PLANÈTE TERRE 2013 : FORMATION ET COMMUNICATION

Doug Franks, *Nipissing University*
Kathleen Pineau, *École de technologie supérieure*
Walter Whiteley, *York University*

PARTICIPANTS

Yasmine Abtahi
Richard Barwell
France Caron
Stewart Craven
Egan Chernoff
Krista Francis-Poscente
Frédéric Gourdeau
Jennifer Holm
Margaret Karrass
Dianne Kenton
Steven Khan
Dave Lidstone
Minnie Liu
Miroslav Lovric
Nenad Radakovic
Peter Taylor
Diane Tepylo

INTRODUCTION: A NOTE FROM THE CO-LEADERS

The first challenge we had after being invited to lead a Working Group on Mathematics for the Planet Earth 2013 was picking a meaningful topic that would be narrow enough to be manageable over the course of three days—9 hours—and yet not so narrow that we took the stuffing out of this potentially rich and very relevant subject. The MPE2013 website (http://mpe2013.org/) identifies four major themes:

- A Planet to Discover
- A Planet Supporting Life
- A Planet Organized by Humans
- A Planet at Risk

And three major mission statements:

- Encourage research in identifying and solving fundamental questions about planet earth.
- Encourage educators at all levels to communicate the issues related to planet earth.

- Inform the public about the essential role of the mathematical sciences in facing the challenges to our planet.

There was much here from which to choose—too much to simply leave wide open.

In 2011, Barwell, Craven, and Lidstone led a CMESG working group on climate change and mathematics teaching, and although we felt that climate change was a critical issue, we did not want simply to duplicate the experience of two years before. We chose to focus on mathematics education and communication, examining the challenges of effectively communicating mathematics-related issues related to planet earth. This frequently took us to topics in *A Planet at Risk*, including climate change, with frequent links also to the other three themes, especially *A Planet Organized by Humans*. In preparing, we drew heavily on the graphical communication work of Howard Wainer (1997, 2009), Edward Tufte (1983, 1997), and Hans Rosling, creator of *Gapminder*.

Our Working Group Abstract captured our planned focus:

> *Variability, uncertainty, modeling and risk are central mathematical concepts at the core of the investigations. How these are presented has a major impact on what is communicated and what decisions are made. Examining both the scientific literature and what appears in blogs and public discussion, graphic displays and visually presented simulations are how people choose to present their 'information'. One theme for the working group will be probing such displays, to ask 'where's the math' and 'what's the math' in different choices of graphic presentations. These types of questions are a central issue of mathematics education. Given the importance of 'rhetorical communication' on the vital debates involving Planet Earth, we will consider 'graphical rhetoric'. How do we put mathematical arguments into these displays and how do people extract mathematical reasoning from such graphic displays?*

We began the discussion on Day 1 by reviewing the general theme we selected, and sharing with the group what we personally saw as questions on the visual representation issues that came to mind for each of us. This then led to an introduction to powerful historical examples of the early use of graphic representations of quantitative data intended not only to represent but to advocate, as well as other graphic resources the group was invited to explore.

Day 2 started with an introduction by Walter to the work of Tversky, particularly her seminal paper, *Cognitive Principles of Graphic Display* (1997), and sharing the principles of graphic representation offered by Tufte. This was followed by an extended discussion of items raised in Day 1. We concluded Day 2 with an exploration of data using Rosling's *Gapminder World* program, led by Kathleen.

Day 3 focused on education for the Mathematics for Planet Earth, the graphic representation of quantitative MPE data, and "Where's the math, what's the math?" Doug introduced courses by two US-based mathematicians that attempted to address mathematics issues in climate change at an early undergraduate level, as well as some University of Cambridge resources on modeling risk, epidemics, etc. He also presented a schematic of what he saw as the shape of the issues and the focus of the first two days. Working group members then split into two groups to discuss topics that were of particular interest: one we might call "context, good data, and good mathematics" and the other, "the mathematics (education) of risk and impact".

This report will now present the events and discussions of the three days in more detail.

DAY 1: GRAPHIC REPRESENTATION—CONTEXT, CONTENT, AND STORY

Because our working group was so 'visual' and depended on everyone being able to access, show, and discuss graphic data, whether from websites or *PowerPoint* slide shows, as well as to be able to add new resources over the three days, we shared information using memory keys and established a *Dropbox* location. For our first day, Walter and Kathleen had provided *PowerPoint* presentations of "FAQs" and "Issues KP" respectively.

WALTER'S PRESENTATION

Walter's slides featured climate change-related visual representations taken from various websites. He focused on seven themes related to climate change issues. The content of Walter's "Some Climate Change FAQs" presentation is given here. The specific links for the various themes are provided in Appendix 1.

<u>Theme I: Increased Carbon Dioxide</u>

- Keeling Curve (Carbon dioxide concentration at Mauna Loa Observatory)
- Questions Walter asked based on the Increase in CO_2 evident in this graphic representation:
    - Is this man-made?
    - Is it higher than anything in the last 800,000 years?
    - What are the impacts on ocean, atmosphere?
    - How long will it last?
- Sources: Walter's recommendation to explore the Keeling Curve at the given site on different time scales. What one notices in these graphic representations of the data is that each has a different message of chaos and pattern.

<u>Theme II: Ocean Acidification</u>

- Slide is from the IPCC report (2007). It is a double graph showing: (a) Ocean CO_2 levels over a 20 year period (1985-2005); and (b) Ocean water acidity over the same period. Placed side by side (using scales appropriate, respectively, to CO_2 levels and acidity), the graphs are roughly a reflection of each other. That is the power in the display—it seems clear that as CO_2 levels have increased, the global ocean pH level has decreased (and thus become more acid). The eye in this case is helped by the presentation of a drawn (non-linear) line of best fit to the data.
- "CO2 enters Oceans (makes acid)." Walter's comments and questions—
    - More acid oceans change key parts of the ecology (coral, shells)...
    - Changes in species in particular environments (extinction)
    - How long does the new equilibrium last?

<u>Theme III: Continuing Carbon Emissions? [Global Warming]</u>

- Slide of the 'hockey stick' graph: "Carbon pollution set to end era of stable climate" (period: 10 000 BCE to 2000 CE)
    - Reveals the start of an upward change around the year 2000 in "temperature change relative to 1961-1990 mean" and a projected very large and rapid upward change subsequent to that. Scale is used to dramatically illustrate relative stability over a very long period of time, and strength of projected change.
- Walter's comments and questions—

- Do carbon emissions (and other emissions—methane, etc.) warm the planet?
- Are there other sources that can compensate/dominate these human causes?
- There are differences (more change in the north, less at the equators).
- What are the risks if we are still uncertain?
- Is temperature alone (including ocean temperature) the key problem?

- Walter then asked: Do continuing carbon emissions imply Global Warming?
  - How sensitive is the average temperature to CO2 levels?
    - Skeptics say atmospheric warming has slowed.
    - Scientists say same total heat but more went into the ocean.

Theme IV: Global Warming and Extreme Weather

- A series of slides of graphic displays—
  - "5 year average precipitation categories" relative to 20^{th} Century in 2085.
    - Reveals wetter polar and northern/southern temperate regions, roughly stable equatorial region, and dryer subtropical regions. Display is based on colour change.
    - Source: NOAA
  - View of global northern hemisphere showing colder/warmer than average regions for November 2010 (Polar area shown to be 4 to 10º C higher). Warmer indicated by increasing deeper shades of red.
    - Source: NASA

- Walter' s comments (drawing upon the implications of precipitation and temperature changes visually displayed)—
  - More humidity stronger events.
  - Amplification (bigger waves) and slower movement.
  - Dry gets dryer, wet gets wetter.
  - Systems can stall—many snow storms, flooding.
  - ... then hot drought in summer …

Theme V: Sea Level Rise

- Slide showing global change in "Sea Level Trend 1993-01/2012-12 (mm/Year)"
  - Regional trends illustrated by change in colour—darker (thus more emphasis) indicates greater change (drop or rise in SL).

- Walter's concerns—
  - Melting ice sheets, glaciers.
  - Expansion of water due to warming (1 m?)
  - Risk of extreme storm surges.

Theme VI: A Budget for Carbon Emissions?

- Slide: Graphic of "Oilsands vs. Global CO_2 Budget"
  - A graphic of inset (but not concentric) circles offering a number of CO_2 emissions comparisons, and questioning the claims of Federal Government Natural Resources Minister Joe Oliver's claims about the limited impact of the Oilsands, in contrast to the "Game Over" claims of scientist Jim Hansen.

- Walter's comments and questions—

- When I was younger—concern was over peak oil (scarcity). Now concern about too much oil, coal …
- Is there a maximum safe limit for carbon sources we can burn to create CO_2?
- How would this be determined?
- If there is a budget—whose resources will be left 'in the ground'?
- If there is a budget, who will have the right to use the associated energy?
- Will your pension plan go bust investing in oil/coal?

Theme VII: Communication Barriers

- Slide from *thinkprogress.org*—histogram of "Public Perception of [Climate Scientist] Consensus"
  - Graph suggests that only 30-50% of climate scientists agree on human-caused global warming, while in reality (based on review of peer-reviewed literature) there is 97% consensus.
  - Indicates that there is confusion, or unwillingness to accept that there is an issue, on the part of the public.
- Slide illustrating that scientists need to change the way they communicate with the public about climate change significantly from the way they communicate with each other. A graphic display indicating an "inverted" approach to communicating with the public compared to that with each other.
- Walter's comments and questions—
  - Rhetorical devices: repetition.
  - Story (narrative) or formulae / graphics.
  - Metaphors to live by: e.g. "Climate is like body temperature: there is a safe range beyond which there is major risk."
  - Graphics, sequences of graphics, animation.
  - Mathematics is a barrier to public communication: numerically and visually.
  - Does this illustrate a fundamental failure of Math Ed?
  - What does research show works/does not work?
  - Hope works better than fear.

Walter focused on major climate change issues in his presentation. Among the features of his presentation that raised thoughtful questions were: the nature (form and structure) of visual displays themselves; the relationship between context, mathematics, and interpretation; the potential story or narrative embedded in the graphic display—what story does the display tell (indeed, does it tell a story?), to whom, and are the stories 'read' by the creator and the reader/viewer consistent with each other; has mathematics education (and science education) failed in its goal of supporting the development of an informed citizenry—or, does it even fully understand that as a goal it ought to have?

KATHLEEN'S PRESENTATION

Kathleen then presented her issues-based set of slides [My issues/Mes préoccupations].

She started by stating that she felt as though she was drowning in a sea of information. She posed the comment and questions: "*I am wary of media coverage; I sense that there is an underlying agenda. What's good? What's not? How do I differentiate?*" She then gave an example of what she meant by 'good' and 'bad' presentations. Again, the specific links are provided in Appendix 1.

> The Financial Post, on April 15, 2013, stated that Lawrence Solomon—"one of Canada's leading environmentalists" according to his biography—claimed that "Arctic sea ice was back to 1989 levels, and now exceed the previous decade."

Kathleen asked: "*Vraiment?*" She presented the graphic representation of the NSIDC data that Solomon reportedly used to make his claim (approximate time period, 1978 to 2013). It became clear immediately that Solomon was being extremely selective in the choice of data points in order to make this claim.

This example again raised the question of how the 'story' that the data tells, in its visual representation, is highly dependent on the perspective and intentions of the story teller (in this case, Solomon). Kathleen again asked, "*What's good? What's not? What can I do to counter reporting à la Solomon?*"

Implicit in both Kathleen's and Walter's presentations was the issue of advocacy. Visual representations have been used to support arguments covering a range of views and competing agendas. How does one counter what is just 'bad mathematics', or evidently misleading interpretations of 'good mathematics'?

Kathleen took this opportunity to introduce the "Wall of Advocacy/*Le mur de l'action réfléchie*". She invited working group members to:

> Take a look at what is posted and add to it when the mood strikes.
> *Jetez-y un coup d'œil et mettez-y du vôtre quand l'envie vous prend!*

She noted the expression: "*Une image vaut 1000 mots* – A picture is worth 1000 words." She concluded by adding to her earlier expression of concern as a citizen, her concerns as a teacher: "*How do I integrate MPE themes in my teaching? I'm not an expert…where do I get the data? What can I do with it without tainting it?*"

She offered the following suggestion: "*A good start is looking at graphic displays*."

DOUG'S PRESENTATION

Doug took a somewhat different opening approach, seeing a significant issue in the mere presence of tens, often hundreds, even thousands, of graphic images on a topic, available on the resource that most turn to now, the internet. On many MPE-related topics, one might consider one can find visual representations of a wide variety of related data. They vary by time of creation, nature of the data collected, period over which the data were collected, geographic region in which the data were collected, choice of scale, choice of graph or other graphic, table parameters, colour scheme, and other factors, often particular to the intentions and interests of the person(s) presenting the display. He showed the example of the rather esoteric but critical to the planet topic of *nitrogen fertilizer*—its overuse around the world. A quick *Google* search of "nitrogen fertilizer overuse – images" produced approximately 900 images—ranging from photos of people, crops, pollution caused by fertilizer overuse, manufacturing plants and the like, to scientific data tables and the graphic displays of such data, to graphic cartoons warning of the dangers.

How does one make sense of such a collection? What does it take to sift through a collection of images like these to identify what might be of particular interest (and perhaps more important, of greater social and ecological value), to identify particularly significant contexts, and separate 'bad' graphics from 'good' graphics, for example? As a grandfather of young children, Doug said he was concerned for their future, but noted that it took thoughtfulness and (mathematical) understanding to make sense of what the images portray, let alone move people to take action. Mathematics education, he thought, needs to pay attention to this weave of contextual and mathematical sense-making.

DISCUSSION

During the latter part of Day 1, working group members were invited to explore the data, graphic representations, and contexts already shown in Walter's and Kathleen's slides, the mathematics (including the graphical displays) of the MPE issues that were of particular interest to themselves and not yet discussed, or to investigate the "Graphical Resources" set of slides that were also located in the *Dropbox*. Briefly, this set of slides included images of five early (19^{th} C) graphic displays of quantitative data (William Playfair, "Price of Wheat," 1821; John Snow, "London cholera map," 1854; Florence Nightingale's "rose" graph of British military deaths due to disease in the East, 1858; E. J. Marey's graphical French train schedule, 1885; and C. J. Minard's graphic of Napoleon's invasion of, and retreat from, Russia, 1869). Tufte, of whom more will be said shortly, has described Minard's chart as "probably the best statistical graphic ever drawn". These displays are especially marked by their quality, their contextual particularity, and, for several of them, their explicit political or advocacy nature. Also among the slides was Andy Lee Robinson's "Canary in the Coalmine" image of the decline of Arctic ice, and an animated GIF offering a view of the difference between how 'skeptics' and 'realists' view global warming (see Appendix 1 for links to both these websites.)

DAY 2: PRINCIPLES OF GRAPHIC DISPLAY, A DISCUSSION OF ISSUES, AND AN EXPLORATION OF DATA WITH *GAPMINDER*

This day began with a presentation of "Some Principles of Graphic Design" we compiled from the works of Tufte (1983, 1997) and Tversky (1997). It was noted that these guidelines were written to refer to static and individual displays, not the animated, sometimes-interactive displays that one often finds now on the internet. Nevertheless, they represent important principles by which to judge visual displays of quantitative data, wherever one finds them.

What follows is a brief summary, with attributions where possible, of that full group discussion, first on the issues raised by day 1 topics, and second, by the Tversky and Tufte functions and principles.

THOUGHTS ON DAY 1 TOPICS

- Nenad suggested we need to differentiate between mathematics, science, and social science.
- France raised the topic of *risk*—how might we visualize *risk*? She suggested the work of economists and mathematicians such as Graciela Chichilnisky (*risk*) and Doyne Farmer (*complex systems*).
- Miroslav asked what mathematics is needed for risk studies.
- Dave suggested that we face challenges when working from a corporate model that places profit at the top of the list. These are based on a carbon economy. We need to have an alternative model—what is available that we might consider?
- Richard remarked that mathematics has limitations—it cannot do all things, such as model human experience in an ecosystem, or model Peter's concern for his grandchildren. Some important parameters cannot be modeled.
- Miroslav acknowledged this while observing that mathematics is distinct from reality, offering the quote, "All models are wrong, just some are useful." What is important to understand are the assumptions on which the model is based.

- Richard offered the view that it was dangerous and egocentric thinking to assume that through mathematics we can control and predict, and thus change conditions to suit us.
- Walter offered the example around the launch of the space shuttle Challenger where the use of a flawed graphic (with the irrelevant independent variable—date of launch—rather than the critical independent variable—temperature at launch) did not support deciding not to launch, even though those creating the visual wanted to communicate the engineers' concerns.
- Context and the issues associated with context were important: Jennifer commented that there had to be much more to both context and mathematics than, for example, simply displaying a graph in class on Arctic ice changes.
- Richard saw a need to rethink the teaching of mathematics—for example, not starting with the mathematics.
- Peter felt that teaching a 'named' course (e.g., "Linear Algebra") could be too confining, and forced one into a lock-step programmatic approach, rather than a more inclusive approach which he would prefer.
- Frédéric offered something of a reality check, noting that there often were real difficulties with trying to do something different in mathematics class, something that would make a difference. The (varied) level of students' mastery of mathematics was a problem, a barrier.
- Stewart raised the point that classroom contextual discussions need to be meaningful to the students: they have to see themselves in the class, and be engaged. "Why care if you don't see yourself?" On the other hand, as a good story teller, Stewart could convince the students that studying environmental issues in math was appropriate, but the issue was, "Where's the math?"
- France observed that the discussion had made clear for her the tension (and challenge) for mathematics educators between (mathematics) content and context (e.g., social, ecological) when it came to the classroom.

THOUGHTS ON DESIGN FUNCTIONS AND PRINCIPLES

- Frédéric questioned why *understanding* was not one of the functions listed by Tversky regarding the functions of graphic design. Graphic representations ought to deepen the understanding of the context for both the designer and the viewer. [Tversky's list of functions included: *attract attention and interest*, *serve as models of actual and theoretical worlds, serve as a record of information, facilitate memory*, and *facilitate communication*.]
- Some took issue with Tufte's claim that a good information display should be "causal", while Dianne asked where *relationship* was in the list. [Tufte claimed that information displays should be *documentary*, *comparative*, *causal and explanatory*, *quantified*, *multivariate*, *exploratory*, and *skeptical*.]
- The topic shifted somewhat to a need to understand who the intended viewers of a display were. Richard noted that we have been thinking in terms of the public being the audience for the graphs we have looked at, while in fact (for example), the IPCC 2007 graphs were intended for scientists, not the public.
- Steven referred to the distinction between 'sensitized' and 'non-sensitized' viewers, and used the expression "visual connoisseurship".
- Richard noted that *rhetoric* was missing from Tufte's list of intentions for a good display: presenting an argument to a particular audience.
- France concluded that it was necessary to think of graphs as "living things", evolving and subject to change.

A NOTE ON ETHICS AND *THE WALL OF ADVOCACY*

As an attempt to give a ubiquitous shape to advocacy as an underlying issue to be addressed by our group, the *Wall of Advocacy* was set up on Day 1 and was accessible throughout. The wall served to post articles, website addresses, and documents testifying to different forms of advocacy. As a subset of a quite large resource file that was provided to participants on Day 1, posted items included advocacy websites, letters to journalists denouncing flawed graphic displays—showing how they could be designed to better reflect the data, as well as interesting graphics and telling images gleaned off the internet.

Markers, post-its and sticky gum were available for anyone of the group to share ideas. Though many read and took note of what was posted on the *Wall* on Day 1, little was added to it over the course of the three days. The *Wall* did, however, bring about a short but interesting discussion on what form advocacy should or should not take in the classroom. In their professional role as mathematicians and mathematics educators, participants felt the need to be cautious and not distort the information contained in data. This perspective can be seen in the comments shown above made on Day 2, as well as on Day 3. Summing up, we all come to teaching with our personal set of biases and must thread a mighty fine line.

L'EXPLORATION DE DONNÉES AVEC *GAPMINDER*

Pour poursuivre la réflexion, nous avons introduit les graphiques interactifs. En particulier, nous avons exploré des données à l'aide du gratuiciel Gapminder*, un outil de visualisation Internet pour l'étude de données statistiques.*

Comme courte introduction au gratuiciel Gapminder*, nous avons présenté la vidéo* 200 Years that Changed the World, http://www.gapminder.org/videos/200-years-that-changed-the-world-bbc/*, dans laquelle Hans Rosling commente un graphique qui lie l'espérance de vie au revenu par personne selon le pays.*

After the video, participants were provided with the following two *Gapminder* graphs:

- www.bit.ly/b9p3dA—linking CO_2 emissions per person with Income per person (GDP/capita, PPP$ inflation adjusted) where the size of the bubbles shows total emissions/year for the country.
- www.bit.ly/13UnIhm—linking Water withdrawal (cubic meters per person) and Income per person (GDP/capita, PPP$ inflation adjusted) where the size of the bubble is the total water withdrawal/year for the country.

They were invited to address the *Gapminder* questions: "The USA or China, who emits the most CO_2?" and "Does income matter?"; and to play with variables and scales, create their own graphs, and explore the available data, noting as they went what works, what doesn't, where they were lead to, etc.

Some of the comments posted:

- 5 dimensions is very rich
    - Trailing one country to notice patterns of growth & relating to history generates conjectures.
- Great for teaching critical thinking and generating questions.
- Can be used in secondary curriculum. What students' projects would it support?

- Produces effective graphics: Leads to exploration & development of critical questioning & critical thinking (must source background socio-political & economic activities to provide explanation).
- Use of animated graphs always requires knowledge of historical events & content.
- Lends itself to the need for definitions (e.g. How do we compare 'incomes' over time? What does inflation-adjusted mean? Are tonnes in metric or imperial?)
- Caveats.
- Can be challenging to master all parameters.
- Data predetermined but abundant.
- Predetermined data limits the scope of the questions.

Riche, le gratuiciel Gapminder World, *disponible sous l'onglet* Gapminder World *du site http://www.gapminder.org/, permet de jouer avec cinq variables, est facile d'approche et les données disponibles sont abondantes. L'interaction a amené un questionnement plus audacieux et profond que ce qu'a l'habitude de provoquer les graphiques statiques. De plus, ayant accès à Internet, les données aberrantes, souvent reléguées aux oubliettes, ont donné lieu à davantage d'exploration afin d'en comprendre la signification d'un point de vue économique et sociopolitique, mariant ainsi le contenu mathématique au contexte social.*

One item of discontent did arise from the fact that only the indicators available in *Gapminder World* can be displayed. However, anyone interested in displaying their own data sets *à la Gapminder* can do so with a *Google Docs* spreadsheet (previously known as *Motion Chart Gadget*).

DAY 3: A VISUAL INTERPRETATION, AND MATHEMATICS AND MATHEMATICS EDUCATION

At the start of the day, Doug presented a rough sketch of his interpretation of the discussion themes the first two days: context and content figured significantly. A slightly refined version of the diagram is shown in Appendix B.

A short discussion of the previous day followed, and then Doug presented some educational resources:

- descriptions from two US mathematicians of their courses developed on climate change and mathematics (one calculus-based, the other data-based);
- Cambridge University's educational resources for middle and high school students on modeling health and risk (*Motivate Maths*);
- the *Carbon Mitigation Initiative* at Princeton University; and
- *My World 2015*.

Following this, the working group formed two smaller groups to pursue mathematics topics of particular educational interest to them, essentially, context and mathematics, and risk. An outline of these discussions is presented below.

GROUP 1: DATA, GOOD MATHEMATICS, AND MEANING

Six people were in this group (Egan, Frédéric, Jennifer, Margaret, Richard, and Yasmine), and their general focus was on the resources necessary to develop the good mathematics to support an informed citizenry, and the challenges that represented. The following description is based on notes taken at the time. Because only some comments noted at the time were attributed to specific speakers, the decision here is to avoid any attribution.

The resources considered included data sources, as well as software such as *GapMinder*. As was noted, if an instructor and students are to engage with these contexts—such as climate and economic change, for example—in a mathematically significant way, then good data are needed. Data now are to a high degree coming from online sources. Although this in turn suggests a mathematics that is increasingly computer-based, questions related to the reliability of the data, their general availability, how and when they were collected—all connected to the consistency, integrity, and context of the data—are critical.

In a teaching environment, to start with, as one member mentioned, the mathematics teacher must become familiar with the data and any software that he or she will use for instructional purposes. When using *Gapminder*, for example, it is important that the teacher first gain comfort and comprehension of the program and how it analyzes the data and presents its very graphic representations of the results of that analysis. After having spent some time exploring data with *Gapminder*, one member noted how the program offered opportunities to develop proportional reasoning skills, concepts and skills related to analytic geometry, and, to some extent, transcendental functions (because *Gapminder* makes extensive use of the logarithmic scale).

A pre-service teacher context was offered by another member as an example, with the intention of supporting these candidates in the process of developing plausible and substantial questions based on *Gapminder*. One progression-oriented classroom strategy might look like the following:

- Have teacher candidates begin with an initial exploration (i.e. play with *Gapminder*).
- The instructor (as the more experienced person) then selects questions for teacher candidates to further investigate.
- This in turn leads to more in-depth statistical investigations and understanding of the meaning of the data.
- And finally, the conclusion with an investigation or project with the intention of producing graphic representations, with a particular audience in mind. Here one might consider, for example, both a more sophisticated scientific audience, and an audience composed largely of the parents of the students that one might be teaching. What might be the similarities and differences in how the data are represented to these two audiences?

This then represents a linkage between the mathematics, the context, and the audience. The context serves as a reason for undertaking the mathematics, while the mathematics helps inform the particular audience about the contextual circumstances.

There was in fact substantial discussion of context among many of the group's members—for example, climate change and the visual representation of climate-change-related data. One person noted that a contextually based discussion allowed for the intention that teachers—and students—have the opportunity to learn to ask good questions. Visual representations of data in context need to be created and interpreted at deeper levels than they often are, in order to take greater advantage of the learning opportunities they present. Having noted that, however, it was observed that visual representations are very audience dependent. Creating visual representations is audience dependent; unpacking existing visual representations is audience dependent. When given displays—and one finds many thousands when exploring the internet—it is important to know who they were created for, and who created them. 'Insider' audiences, for example scientists, may be able to make assumptions about the data, and consequently their graphic representations, that the 'public' community or audience are not able to make (and thus may misinterpret representations intended for an 'insider' audience).

The relationships among context, the mathematics (including the creation and presentation of graphic representations), and audience meaning are in fact complex, with an inherent tension, as suggested by the preceding comments. Data and their representations are not inherently imbued with meaning. Data points on a visual display are in fact quite abstract objects, potentially leaving 'viewers' cold and uncertain how to react. Finally it was also noted that in the evolving contexts being discussed (those to do with 'planet earth'), uncertainty was an unavoidable fact of life, so to speak, and teachers, students, and audiences generally must accept that their mathematical analysis (modeling) will not produce a final solution.

GROUP 2: RISK, PROBABILITY, AND IMPACT

The second, larger group of Working Group members (Dave, Diane, Dianne, France, Krista, Minnie, Miroslav, Nanad, Peter, Steven, and Stewart) focused on the question of *risk*, which had arisen as a thread of discussion in talking about 'change' when it is relatively rapid and substantial—climate change (e.g., global warming), health change (e.g., epidemics), economic change (e.g., financial collapse), etc.—and the impact of that change on the planet. How do we measure the risk associated with these events and changes—what is the impact? And, how do we support the development of an educated citizenry that understands these issues? What follows is a summary of rough notes made while the group members shared their thoughts. The focus shifted somewhat throughout as different group members interjected with what was on their mind at that moment. Again, there is no attribution of comments to particular persons.

'Black swans'—that is, one-off catastrophic events—represented the extreme of these changes, but may in fact be most impactful. Probability concepts, although important to develop and understand, do not do a good job of accounting for 'black swans', and certainly not of understanding the risk associated with them in terms of their impact. One discussion thread looked at winning the lottery. There is a very low probability of winning (which could be calculated), but purchasers do not care about the low likelihood of winning because the impact of not winning is not going to affect their lives (in most cases)—ticket prices are too low. So there is little or no risk involved in purchasing a lottery ticket. [This discussion considered the risk associated with losing; the impact of winning the lottery apparently was not discussed.] "Lotteries are Pink Swans," one person observed.

It was noted that students often have difficulty grasping probability concepts, so some time needs to be devoted to their development; but to discuss risk and impact, students need to go beyond conceptual development, and engage with real data. Questions of risk associated with past and potential events—even catastrophic—represented by the data, and the impact, need to be examined. This would add relevance and serve as motivation. Some examples:

- 'False positives' – e.g., in health-related issues:
    - risks and impacts
    - side effects
    - effect on living
- Coal versus nuclear—which is safer?
- Swain vs. US court case:
    - Brief background:
        - 1964—black man convicted of rape, sentenced to death, Alabama
        - Appealed to US Supreme Court on basis that there were no black jurors and that potential black jurors were 'struck' from jury duty by the prosecution strictly on the basis of race. Supreme Court turned down the appeal.
    - A group member asked: What is the probability of never having a black juror in 20 years?

- North Sea rising and Holland's dikes.
- Actuarial tables and insurance rate changes.

The educational intention was not to create actuaries, but to support the development of informed citizens who could understand the risk associated with a potential event by being able and prepared to sift through information and make an informed decision about the risk to them. Currently, it was noted, this is not in the [mathematics] curriculum. An outline of the mathematical processes involved was suggested:

- Generate the questions to be examined or answered
- Decide on the data
- Analyze the data
- Summarize the results of the analysis
- Revisit the questions

Modelling was a theme that weaved through the discussion, in part because of the challenging relationships between probability, risk, and impact. One suggestion was to address only impact and not talk about probability at all, in such situations. Another suggested way of thinking of risk was that sometimes it's about probability, while at other times it is impact that must be considered. As an example of 'a way in', perhaps one could picture risk as change in insurance rates. But modelling the impact function was difficult—and we do also need the probability function. The *STELLA* dynamic systems modelling software was mentioned—which raised the question, should *dynamic systems* be taught earlier than it is currently? The hope was that students would come to university and college already with the ability to calculate and interpret numerical results [the implication being that presently they often do not arrive with these capabilities]. The comment was made that it was also important to be aware that the probabilities of events [that take place on or happen to Earth] change over time, and therefore we cannot rely exclusively on 'old' data.

Modelling based on data also raised the important question of the need to understand the assumptions embedded in the tables of secondary data that one might use with students. It was also critical that as an instructor one needed to be clear on the mathematics that students were to learn from the experience. In a reference to the impact of the North Sea rising on Dutch dikes, the question was asked, "How would one model the impact?" For example, would it be stepwise? Exponential? Modelling, it was noted, was about making decisions and "going with it"—and then discussing and refining the model. [One might ask: On what basis?]

Finally, this mathematics, it was also noted, was being affected or influenced by science (biology, and physics, for example), with a focus on functions. But discrete mathematics is addressed in a significant way in high school, and notwithstanding the question of how to handle missing values in a table of values, complex systems are accessible through discrete methods, and iterative processes are highly suitable to computer analysis.

The discussion was lively, varied, and complex, reflecting closely the nature of the topic itself.

CONCLUSION

Mathematics of planet earth is rich in data, modeling and in questions that generate lively debates. This was true for the participants in the working group, and we predict it would be true for classrooms. Assessing risk and communicating to support decision-making highlight the importance of statistical work with large data sets, of stochastic modeling with both uncertainty and enough certainty to act. They also highlight visual displays as essential tools of communication—and supports for debate. Both the reading of information from graphical displays and the development of effective honest graphical displays are important, and learnable. These tools and these discussions have an important place in mathematics and statistics classrooms.

APPENDIX A: LINKS TO WEBSITES DESCRIBED IN THE REPORT

WALTER'S DAY 1 "FAQS" PRESENTATION

Theme 1: Increased Carbon Dioxide

- Keeling Curve: http://keelingcurve.ucsd.edu/

Theme 2: Ocean Acidification

- A link to investigate: http://en.wikipedia.org/wiki/Ocean_acidification
- Interactive resources:
  - http://i2i.stanford.edu
  - http://i2i.loven.gu.se/AcidOcean/AcidOcean.htm

Theme 3: Continuing Carbon Emissions?

- "Hockey stick" graph: http://thinkprogress.org/climate/2013/04/23/1903001/the-hockey-stick-lives-new-study-confirms-unprecedented-recent-warming-reverses-2000-years-of-cooling/?mobile=nc
- Scientists and global warming: More heat has gone into the ocean: http://www.realclimate.org/index.php/archives/2013/04/the-answer-is-blowing-in-the-wind-the-warming-went-into-the-deep-end/
- Skeptics and global warming: atmospheric warming has slowed: http://www.economist.com/news/science-and-technology/21574461-climate-may-be-heating-up-less-response-greenhouse-gas-emissions

Theme 4: Global Warming and Extreme Weather

- See NOAA and NASA sites.

Theme 5: Sea Level Rise

- Sea level trend: http://en.wikipedia.org/wiki/Current_sea_level_rise
- Risk of storm surges (extremes): http://oceanservice.noaa.gov/facts/sealevel.html
- A student's guide to global warming:
  - http://www.epa.gov/climatestudents/impacts/signs/sea-level.html
  - This EPA slide reveals the upward, roughly linear trend in sea level rise for the period 1870-2010—a change of approximately 9 inches. [Units of measure— e.g., inches, millimetres, Fahrenheit, Celsius—as well as scale choice are factors to consider in these graphic representations of scientific data.]

Theme 6: A Budget for Carbon Emissions?

- Oilsands vs. Global CO_2 Budget: http://www.vancouverobserver.com/blogs/climatesnapshot/do-oilsands-threaten-our-safe-climate-hansens-game-over-vs-olivers-minuscule
- Do the Math: http://math.350.org

Theme 7: Communication Barriers

- A need for scientists to change the way they communicate—suggested links:
    - http://www.physicstoday.org/resource/1/phtoad/v64/i10/p48_s1?bypassSSO=1
    - http://www.wilsoncenter.org/dialogue-program/severe-weather-and-climate-change-there-connection
    - http://www.climatechangecommunication.org/
    - http://www.climatecentral.org/
    - http://environment.yale.edu/climate-communication/article/sixAmericasMay2011

- What does research show works / does not work?
    - http://climateandcapitalism.com/2009/11/23/naomi-klein-on-climate-debt/
    - Hope works better than fear: http://www.nature.com/nclimate/journal/v2/n8/full/nclimate1610.html

- Adapting or mitigating … (prepare for it or prevent it)
    - http://en.wikipedia.org/wiki/Adaptation_to_global_warming
    - http://www.global-greenhouse-warming.com/climate-mitigation-and-adaptation.html

KATHLEEN'S DAY 1 "ISSUES KP" PRESENTATION

Lawrence Solomon's Arctic sea ice-related links:

- Financial Post: http://opinion.financialpost.com/2013/04/15/lawrence-solomon-arctic-sea-ice-back-to-1989-levels-now-exceeds-previous-decade/
- Challenging the claim: http://tamino.wordpress.com/2013/04/16/worth-more-than-a-thousand-words/

"GRAPHIC RESOURCES" LINKS

- Andy Lee Robinson's "Canary in the Coalmine" Arctic sea ice: http://thinkprogress.org/climate/2013/02/14/1594211/death-spiral-bombshell-cryosat-2-confirms-arctic-sea-ice-volume-has-collapsed/
- Animated GIF on Global Warming (skeptics and realists): http://thinkprogress.org/climate/2013/03/28/1785461/as-scientists-predicted-global-warming-continues/

ADDITIONAL ONLINE RESOURCES IDENTIFIED BY THE WORKING GROUP

- Cambridge University: "Motivate Maths": http://motivate.maths.org/content/

- "Carbon Visuals" illustration (from Stewart): http://artthreat.net/2012/11/carbon-visuals/

- David McCandless: "The Beauty of Data Visualisation" http://www.ted.com/talks/david_mccandless_the_beauty_of_data_visualization.html

- Graciela Chichilnisky: "Catastrophic Risks" (from France) http://chichilnisky.com/pdfs/catastrophic-risks.pdf

- J. Doyne Farmer: "Economics Needs to Treat the Economy as a Complex System" http://ineteconomics.org/sites/inet.civicactions.net/files/farmer_berlinpaper.pdf

- Hans Rosling & *Gapminder*:
    - http://www.gapminder.org/videos/

  - 200 Countries, 200 Years, 4 Minutes: http://www.gapminder.org/videos/200-years-that-changed-the-world-bbc/
  - Let my data set change your mindset: http://www.gapminder.org/videos/ted-us-state-department/

- Princeton University: “Carbon Mitigation Initiative”(CMI): http://cmi.princeton.edu/
- Thomas Goetz: It’s time to redesign medical data: http://www.ted.com/talks/thomas_goetz_it_s_time_to_redesign_medical_data.html [Blood work, CRP and others tests are ‘rewritten’, inspired by the nutritional value info on cereal boxes.]
- Visual Learning for Science and Engineering http://old.siggraph.org/education/vl/vl.htm

APPENDIX B: ADDITIONAL INDIVIDUAL AND SMALL GROUP COMMENTARY

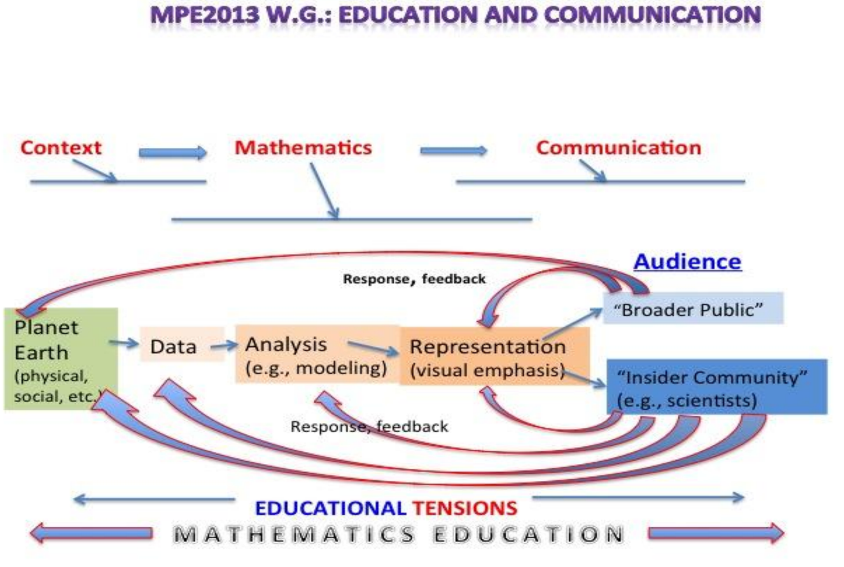
France, Krista, and Minnie’s comments on the *Ocean Acidification* graphic, comparing CO_2 level graph with ocean pH level graph:

- Challenging the necessity of including zero. Telling a story is drawing attention to something. Thin line with manipulation.
- The variability in the trend line: What’s the story? The meaning? The purpose? To make it look more authentic? Respectful of the data? Scientific? Less naïve?
- Choosing which variables to display. Only two at a time? Playing with size, colour or dynamism… What would you gain if you added a graph of pH as a function of CO_2? Or a dynamic version of that with respect to time?
- The mirror image with different variables:
  - Tells the story well
  - Strong aesthetic appeal
  - Seems too perfect to be true
- The hidden information:
  - Where do the points come from? All over the earth? Are they collected on a regular basis?
  - Requires expert knowledge to know that pH is a logarithmic indicator (pH).
- What does this linear trend, on a logarithmic indicator mean in terms of the relation?

France, Krista, and Minnie comments on *Oil Sands vs. Global CO_2 Budget* graphic:

- What’s the story? What’s the message?
- The choice of form is misleading: Diameter or areas? Could they be spheres?
- Canada fossil fuels ever burned vs. world global CO_2 budget: too much going on…

Doug Franks' Day 3 Graphic:



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WHAT DOES IT MEAN TO UNDERSTAND MULTIPLICATIVE IDEAS AND PROCESSES? DESIGNING STRATEGIES FOR TEACHING AND LEARNING

Lorraine M. Baron, *University of Hawai'i at Mānoa*
Izabella Oliveira, *Université Laval*

PARTICIPANTS

Lisa Lunney Borden	Caroline Lajoie	Jamie Pyper
Annette Braconne-Michoux	Manon LeBlanc	Miranda Rioux
Lucie DeBlois	Martha Mavor	Mina Sedaghat-Jou
Nadia Hardy	Elena Polotskaia	Amanjot Toor
Gaya Jayakody		

We "understand something if we see how it is related or connected to other things we know" (Hiebert et al., 1997, p. 4).

INTRODUCTION

From an early age, children apply multiplicative thinking while solving mathematical tasks or problems; for example, when they are called upon to find one-half of an object or quantity. Despite this, multiplicative thinking (Vergnaud, 1983) and its related ideas, including division, fractions and proportions, are not formally introduced until late primary or early intermediate grades. Students' conceptual understanding of multiplication is developed throughout their schooling. These ideas, first explored through arithmetic, are adapted by students during their algebraic learning. One of the goals during late primary grades is "to develop students' algebraic thinking, building a foundation of understanding and skills while they are young so that they can be successful in their later, more formal study of algebra" (Burns, Wickett, & Kharas, 2002, p. xii). Empson, Levi, and Carpenter (2011) argued also:

> *[...] that relational thinking is a critical precursor – perhaps the most critical – to learning algebra with understanding, because if children understand the arithmetic that they learn, then they are better prepared to solve problems and generate new ideas in the domain of algebra.* (p. 426)

This paper describes the discussions and outcomes of a working group during the 2013 Canadian Mathematics Education Study Group (CMESG) meeting at Brock University. Participants included university professors and graduate students belonging to various faculties of education or mathematics across Canada. Our goal was to investigate multiplicative thinking in 9- to 15-year olds and to explore how student understanding of ideas and procedures could be improved in the primary and middle grades. The process of our discussions eventually brought us to ask how we could best prepare pre-service teachers to understand multiplicative ideas and processes more deeply and in a more connected way so that common errors and misconceptions in school children's understanding might be diminished.

OUR GROUP'S TOPICS AND INTENTION

In general, this working group sought to reflect upon the following:

- What does it mean to understand?
- What does it mean to understand multiplicative concepts?
- How does multiplicative thinking evolve through schooling given the particular strategies taught within our schooling systems?
- What are the particular difficulties or misunderstandings experienced by students during their learning journey from primary to middle grades?
- How can we better support educators (teachers and pre-service teachers) to help them understand the complexities of multiplicative thinking so that primary students will have fewer difficulties and misunderstandings with respect to those concepts?

Multiplicative thinking begins for students at an early age, and is linked to new learnings throughout schooling. Government issued curriculum documents typically describe the prescribed learning trajectory that students are intended to follow (e.g., Province of British Columbia, 2007). But what is taught in classrooms and what individual students actually learn varies greatly (e.g., Handal & Herrington, 2003; van den Akker, 2003). Why then do practicing teachers often find what they consider to be 'common misconceptions' in student thinking around mathematical concepts (e.g., Phillips & Wong, 2010)? Are there schooling practices that might be causing confusion in learning multiplicative ideas and processes? What can be done to clarify these concepts for school-aged children?

The problems we hoped to address included the following:

- Primary students who do not learn to understand the patterns and rules of mathematics are often left behind. Their learning gaps are unique to each student, and educators must be able to judge which 'new' or 'improved' way of explaining that understanding to students is most appropriate for that student at any particular time in their learning.
- When working to help a student move her multiplicative understanding forward, an educator (teacher or pre-service teacher) may not always be mindful of the vast number of concepts that are linked or pre-requisite to multiplicative understanding, and may have difficulty considering all possible background misconceptions that may be causes of the students failure to work successfully through questions and problems they encounter in later grades.
- The limited knowledge of educators (teachers and pre-service teachers) is likely to be hindering the learning of primary students in the field.
- Moreover, curriculum documents may be limiting what is taught, which impacts students' exposure to ideas which they may be ready for at an earlier or different time.

DAY 1

TASK #1: WHAT DOES IT MEAN TO UNDERSTAND?

Our work began by asking what it means to understand something in mathematics. We opened the discussion with the quote from Hiebert et al. (1997): We “understand something if we see how it is related or connected to other things we know” (p. 4). This simple question turned out to be a very engaging professional learning task for our group. Conversations revolved around *what* it was that was being understood, and included the importance of students being able to handle many different types of tasks in many contexts, and being able to represent and communicate her/his learning in multiple ways. The group also indicated that the learning should be constructed by the child, and that there was a difference between being successful in mathematics and truly understanding it. We collected everyone’s ideas. Our summative definition for understanding appears in Figure 1.

- Understanding is… always evolving. It requires the learner to relate her/his ideas given a variety of tasks, utilizing different mathematical objects or representations, and in different contexts.
- To understand what?
    - a concept
    - a procedure
    - the milieu/place in which s/he is working (task and expectations… Brousseau: What the student learns depends on the classroom teaching and learning culture in which s/he is situated.)
    - mathematical culture: heuristics, conventions, etc.
- Understanding belongs to the child
- Success vs. understanding

♦♦♦♦♦♦♦♦♦♦

- *La compréhension c’est… en constant mouvement. Elle exige de l’apprenant l’établissement de relations entre différentes tâches, objets mathématiques, représentations, dans des contextes différents.*
- *Comprendre quoi?*
    - *un concept*
    - *une procedure*
    - *le milieu (les attentes et la tâche… Brousseau)*
    - *la culture mathématique*
- *La compréhension est celle de l’étudiant*
- *Réussite ou compréhension*

Figure 1. Group definition: What does it mean to understand?

Our group was inspired to explore this question for quite some time, and this allowed us to describe our somewhat ‘utopian’ views of what it means to understand. In our vision, learners were making sense of concepts and were able to recognize and apply these understandings in multiple ways and in various contexts. Accuracy and correctness were a part of the discussion, but our group described a much deeper and richer connected view of understanding.

TASK #2: EXEMPLAR OF STUDENT'S WORK—WHAT DOES THIS CHILD UNDERSTAND?

Next on the agenda, participants were presented with a page of student work (see Figure 2). The page originated from a grade 8 student's test that intended to assess the learning outcomes regarding algebraic expressions in the Quebec curriculum. Participants in our working group were asked to investigate this student's work in order to describe what this student understood and misunderstood about multiplication in the context of multiplying polynomials, and how this might be linked to prerequisite concepts for these tasks.

This activity—to study a child's work—also led the working group to engage very deeply in personal consideration of the topic and in conversations with their colleagues. What was this student able to do? What did 'he' understand? What was he doing rightly or wrongly, and why might he be doing that? (To simplify our group conversations, we agreed to speak of the student as a male even though it was not determined whether the student was male or female.)

Example 1:

$$5ab^2 - 3 - 7a^2b + (b^2a + 2ba^2) - 12$$

$5ab^2 . 5a^2b . 15$

$= 15a^2b^2$

Example 2:

$$7 - \frac{x}{8} = -6$$

$\frac{56}{8} - \frac{x}{8}$

$56 - 1x = -6$

$-1x + 56 = -6$

$-56 \quad -56$

$\frac{-1x}{-1} = \frac{-56}{-1}$

$x = 56$

Validation :

$-1 . 56 + -56$

$56 + -56$

0

Example 3:

$$-(2x+3) + (5y-10) - (-4x-5y)$$

$2x - 3 + 5y - 10 + 4x + 5y$

$2x - 13 + 10y$

$= -1xy$

Example 4:

$$5 - 2(-6x^2y + 7y^2x) + 3(-xy^2 + 3yx^2)$$

$5 - 2 - 12x^2y + 14y^2x - 3xy^2 + 9yx^2$

$3 - 3x^2y + 11y^2x$

$= 11x^2y^2$

Example 5:

$$\frac{5a}{2} + \frac{7y}{5} - \frac{8a}{5} + \frac{3y}{10}$$

$\frac{5a^{\times 5}}{2_{\times 5}} - \frac{8a^{\times 2}}{5_{\times 2}} + \frac{7y^{\times 2}}{5_{\times 2}} + \frac{3y^{\times 0}}{10_{\times 0}}$

$\downarrow$

$\frac{25}{10} - \frac{16a}{10} + \frac{14y}{10} + \frac{3y}{10} = \frac{26ay}{10} = 2\ \frac{6ay}{10}$

Figure 2. Working Group's task: Examples of a student's work.

The working group was given time to discuss what this child understood. This was followed by small group presentations to describe what each sub-group believed were this student's understandings and misunderstandings.

This became yet another very engaging task for our working group. Many statements were made about this student, about what he understood or didn't, and about why this might be the case. Although there were many similar observations, not all interpretations were the same. Some members of our working group explained that the student understood certain concepts or procedures, some argued that there were inconsistencies, whereas others saw consistencies in the student's work.

WHOOPS—EVALUATING UNDERSTANDING…A CHALLENGE

Once all descriptions of understanding had been fully discussed, the facilitators decided to offer a challenge to the group. Participants were asked to then "grade the paper". What became clear is that this working group of professionals generally rejected the task of evaluating (grading) the paper.

It was a tricky choice to introduce evaluation into the assessment of this student's work. Although the group had clearly enjoyed the work of assessing the student's understanding at a theoretical level, we were not willing to ascribe a grade of any form to that student's work. And yet, teachers must evaluate students' work on a continuous basis. It is a regular and necessary part of the work that is expected in school settings. Why then were we so averse to assigning a mark to this student's work? We had had plenty of time to consider his understanding in depth, and we had enjoyed professional discussions around this student's particular knowledge. Should it not have been a fairly simple task to grade these five questions on a single page of a student's test? This conversation had opened a proverbial 'can of worms', and our working group eventually agreed to "not discuss evaluation", because it would have taken time away from our original goals.

The following day, we wondered out loud together why that part of the discussion had been so difficult. There are a number of implications for our difficulties here that would be interesting to explore in a future working group or in a future paper. Why is evaluation (grading) often omitted from pedagogical discussions? This also begs the question of whether we spend enough time on evaluation with pre-service teachers.

The goal of this task had been to investigate common misconceptions in multiplicative thinking that would present in middle school students. Though the working group had stumbled on evaluation, we tentatively chose to move in a different direction and on to the task of creating a concept map for multiplicative thinking, which had been an original goal of the working group.

DAY 2—TASK #3: BUILD A CONCEPT MAP FOR MULTIPLICATIVE THINKING

The next task was to break into small self-selected groups and develop a concept map that would include the understanding needed by school children to be successful in multiplicative thinking based on test questions such as the ones used in the exemplars. Some groups included the student's particular misunderstandings within their map. The following figures show the maps created by the members of our working group. Each group focused on particular aspects of the necessary concepts for multiplicative understanding.

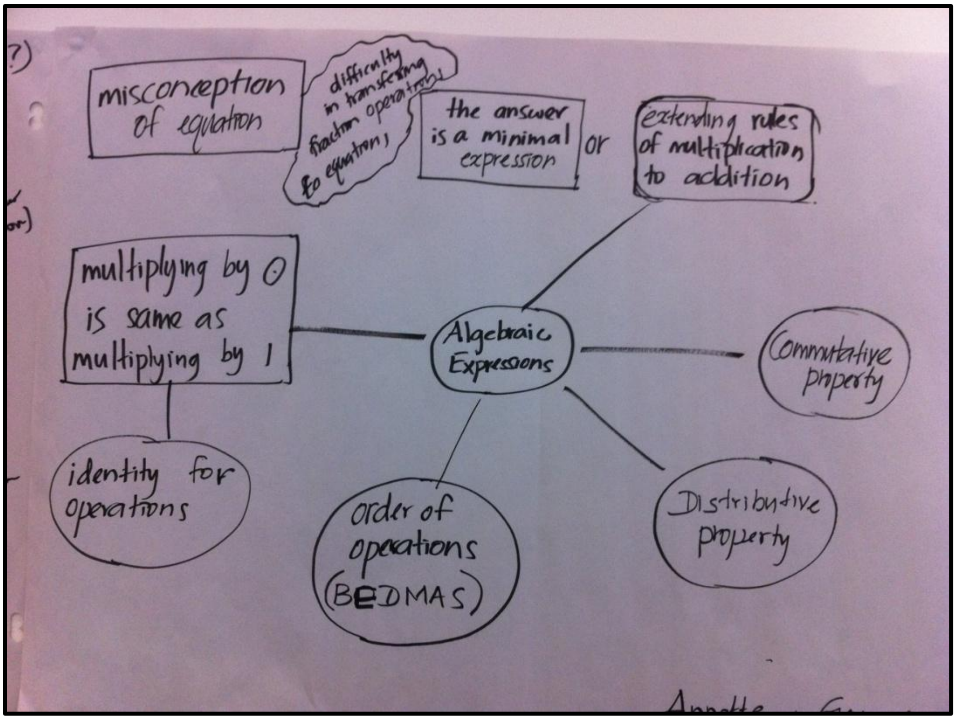

Figure 3. *Difficultés liées aux expressions algébriques* / Principle tool or concept or skill identified as the student's difficulties.

DISCUSSION OF FIGURE 3

Within the context of the worked exercises, we had studied the principle tool/concept/skill that was causing difficulties for the student during the manipulation of algebraic expressions. We used rectangular boxes to enclose the misconceptions the student seemed to have, such as envisioning multiplication by 0 as the same as multiplying by 1. The circles identified other related rules/concepts/ideas that were needed to complete the given exercises.

Les difficultés liées aux expressions algébriques:

- *d'une part, dans les bulles rondes nous avons fait un bref relevé des difficultés que je qualifierais d'ordre mathématique voire épistémologique;*
- *d'autre part, dans les bulles rectangulaires, les erreurs classiques des élèves, qui peuvent avoir une origine didactique voire ontogénique.*

L'équipe a essayé de dresser un portrait de la situation des connaissances mathématiques requises dans les activités proposées (soit la commutativité, la distributivité, les règles de priorités dans les opérations et l'élément neutre des opérations) et les difficultés que rencontraient les élèves à partir soit des réponses qu'ils ont données soit des difficultés les plus connues chez les élèves : conceptions erronées et autres à savoir que les multiplications par 0 ou par 1 sont équivalentes, qu'une réponse en algèbre est toujours une expression minimale (un monôme), que, dans le cas des produits avec exposants, il y a confusion entre le produit des monômes et l'addition des exposants.

Toutes ces difficultés se retrouvent évidemment dans les résolutions d'équations; et la notion d'équation n'est pas bien établie dans la tête de certains élèves. Il est bien évident que ce schéma est loin d'être complet et ne fait pas le tour de toutes les difficultés que les élèves peuvent avoir avec la manipulation des expressions algébriques.

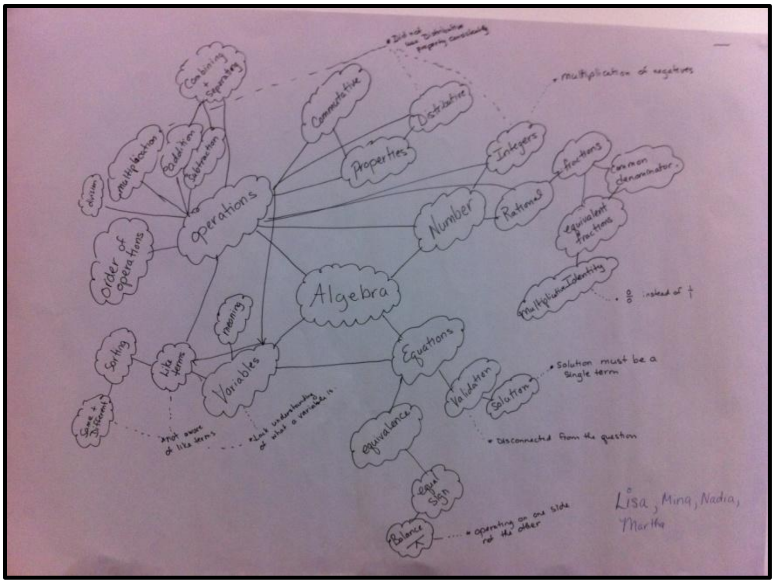

Figure 4. Map of prerequisite and co-requisite mathematical concepts.

DISCUSSION OF FIGURE 4

In this figure we tried to map most of the prerequisite and co-requisite mathematical concepts that are involved in multiplication of the polynomial (based on the provided student's test). Also, we pointed out the misconceptions that had been aroused in the student's paper and we showed their links to the related math concepts.

The graph seems to indicate that the relationship between many of the key concepts is not linear. It was an interesting process to keep drilling deeper to investigate which were all the key understandings necessary for the algebraic work set for this student.

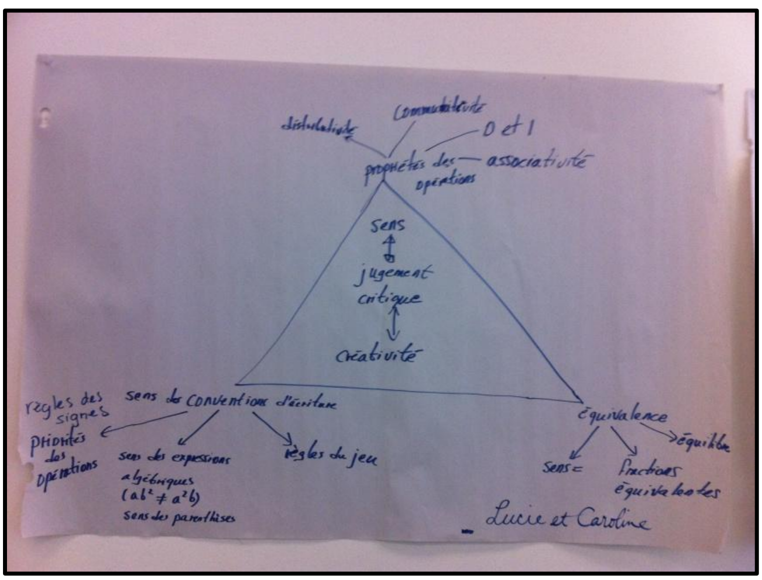

Figure 5. *Les composantes contribuant au développement d'une pensée multiplicative* / The factors that lead to the development of multiplicative thinking.

DISCUSSION OF FIGURE 5

Ce diagramme vise à illustrer les composantes contribuant au développement d'une pensée multiplicative. Ainsi, trois pôles ont émergés : le sens des conventions d'écriture, l'équivalence et les propriétés des opérations. Chacun de ces pôles est défini par certains savoirs mathématiques. Par exemple, le pôle « propriété des opérations » regroupe les savoirs liés à la commutativité, l'associativité, la distributivité et le sens du 0 et du 1 dont le rôle est différent pour l'addition et la multiplication. Le pôle « sens des conventions d'écriture » regroupe les savoirs liés aux priorités des opérations, au sens des expressions algébriques et règles du jeu. Enfin, le pôle « équivalence » regroupe différentes idées liées à l'équivalence comme l'égalité (=), les fractions équivalentes et l'équilibre. Cet ensemble de savoirs en interaction permet le développement d'interactions entre la créativité et le sens mathématique afin de susciter le développement du jugement critique des élèves.

This diagram intends to illustrate the components contributing to the development of multiplicative thinking. Thus, three clusters have emerged: an appreciation of writing conventions, equivalency and properties of operations. Each of these clusters is defined by particular mathematical knowledge. For example, the cluster indicated as *property operations* includes knowledge related to commutativity, associativity, distributivity and the meaning of 0 and 1, whose roles are different for addition and multiplication. The cluster referred to as *appreciation of writing conventions* includes knowledge related to operational priorities (such as BEDMAS) within the meaning of algebraic expressions and rules. Finally, the *equivalency* segment comprises various ideas related to equality (=), equivalent fractions and balance. These three clusters of knowledge combine to impact students' ability to navigate between creativity and mathematical understanding that, in turn, leads to critical thinking.

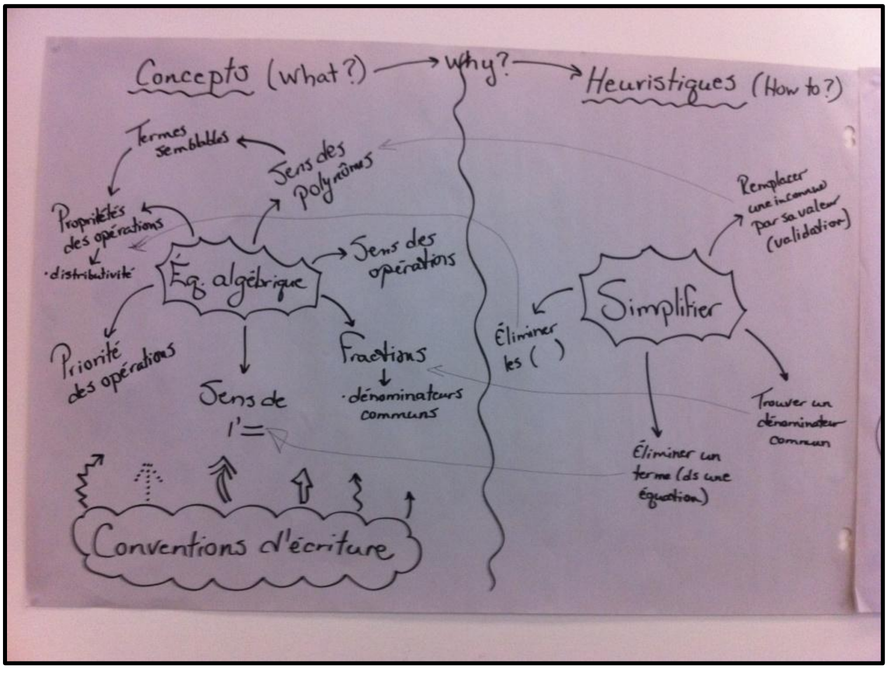

Figure 6. What is an algebraic equation, and what can we do with it?

DISCUSSION OF FIGURE 6

> *Notre idée ici est de préciser et distinguer les deux aspects : le concept ou la compréhension de ce qui est une équation algébrique et la connaissance de ce qu'on peut* faire *pour la résoudre. Le concept peut inclure : égalité, termes algébriques, termes semblables, sens des opérations…. Les heuristiques peuvent inclure: ouvrir les parenthèses, factoriser, simplifier, regrouper les termes semblables. Il y a aussi les conventions d'écriture associées.*

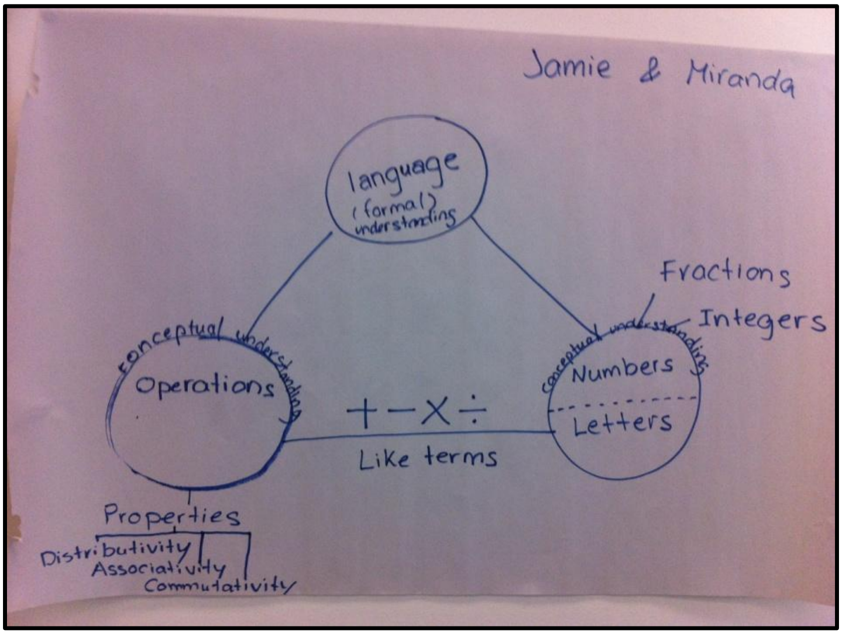

Figure 7. The role that language plays during the development of multiplicative thinking/ *Le rôle que joue le langage dans le développement de la pensé multiplicative*.

DISCUSSION OF FIGURE 7

> *La figure 7 met en relief le rôle que joue le langage dans le développement de la pensée multiplicative. À l'instar de Vergnaud (1991), les auteurs croient que le langage naturel et les autres formes de symbolisation permettent une représentation formelle des relations multiplicatives et accompagnent ainsi la pensée dans son élan de conceptualisation. En effet, selon Vergnaud:*
>
> [...] les activités langagières en situation et les activités cognitives sur le langage mettent nécessairement à contribution des conceptualisations spécifiques sur le contenu de la pensée, qu'elles sont de ce fait conditionnées par le contenu des connaissances, et qu'en retour elles jouent un rôle dans le fonctionnement de la pensée et notamment dans le processus de conceptualisation. (p. 85)
>
> *Voilà pourquoi le langage constitue le premier pôle de la figure 7. Or si le langage joue un rôle important dans le processus de conceptualisation, il ne transfère toutefois pas les significations (von Glasersfeld, 1994), lesquelles sont à rechercher du côté de la représentation du réel. Le développement d'une pensée multiplicative requiert ainsi, sur le plan conceptuel, la capacité à reconnaître les situations qui confèrent un sens à la multiplication et la capacité à traiter ces situations en mobilisant et en utilisant efficacement des schèmes d'action appropriés. C'est ce que les auteurs ont voulu signifier par « compréhension conceptuelle des opérations », le deuxième pôle de la figure 7. En précisant certaines propriétés de ces opérations, les auteurs ont simplement voulu mettre en relief certains des invariants opératoires sur lesquels reposent ces schèmes. Enfin, le troisième et*

> *dernier pôle de la figure renvoie à la conceptualisation des termes impliqués dans une multiplication et ce, que ces termes soient numériques ou non numériques. En effet, à titre d'exemple, on ne saurait multiplier 3 par ½ sans avoir au préalable développer une compréhension du concept de fraction.*

This work brought us to the end of the second session, and, at this point, we had to acknowledge that we had only one more session to go. We decided as a group how we would spend our last hours together. We agreed that we wanted to produce something tangible and practical from our discussions together. Our goal then became to create a pedagogical plan that would be designed to engage pre-service teachers in activities that could increase their understanding of multiplicative thinking. We also hoped to help pre-service teachers become more aware of the connections between skills and understanding in mathematics, and also become more mindful of the future learning needs of their students.

DAY 3—TASK #4: WHAT CAN WE DO TO BETTER PREPARE PRE-SERVICE TEACHERS?

On our third day together, the larger working group reorganized itself into new subgroups with new partners to delve deeper into the following questions as we described them.

1. What can we do with our pre-service teachers to help them understand what the specificities of multiplicative structures are, and at the same time understand how it connects to the other structures that the students know—and how it connects to the future role of multiplicative structures?
2. Ask students to answer questions themselves and consider all prerequisite concepts before showing them students' work. (Separate into primary and secondary group.)
3. Unpack the concepts:
   - Knowing the conceptual field – math – curriculum
   - Appreciating the relations and separations between concepts
   - Obstacles
   - Common mistakes
   - Conceptual/psychological development of learners

Notre tâche le troisième jour :

1. *Que pouvons-nous faire avec nos futurs enseignants pour les aider à comprendre quelles sont les spécificités des structures multiplicatives, et en même temps à comprendre comment elles se lient aux autres structures que les élèves connaissent—et comment elles se lient au futur rôle des structures multiplicatives.*
2. *Demander aux élèves eux-mêmes de répondre aux questions et d'envisager touts les préconcepts avant de leur montrer le travail des étudiants. (Séparé en groupe primaire et secondaire).*
3. *Déballer les concepts:*
   - *En prenant compte du champ conceptuel – math – curriculum*
   - *En appréciant les relations et les séparations entre les concepts*
   - *Obstacles*
   - *Erreurs communes*
   - *Développement conceptuel / psychologique des apprenants*

The final products included posters and *PowerPoints* that were presented by each group. The figures below show the resulting products shared by the group.

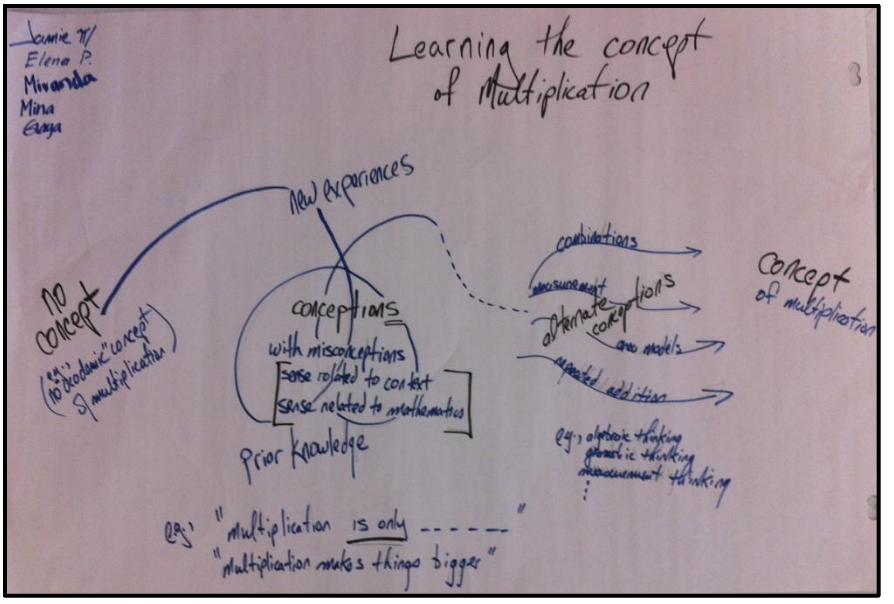

Figure 8. Conceptions: No conception, alternate conceptions and more.

DISCUSSION OF FIGURE 8

We envisioned the learning of a concept here as a journey from a point of having *no concept* (here we meant not having the scientific or 'academic' sense of the concept), to the point where the learner constructs the concept along a number of stages through various experiences. A learner might be at any given stage along this process at any given point in time. In learning the concept of multiplication in particular, we have highlighted the initial stage where the learner constructs his own 'conceptions', which may or may not be in accordance with the mathematical concept. We have shown towards the end how, by having exposure to different ways of viewing multiplication, the learner can make connections between them to build a more complete and coherent concept of multiplication.

Learning the concept of multiplication seems like a spectrum, without any bright and well defined borders. We are not sure when and how learning starts, but many experiences/skills are needed to achieve understanding of the concept. Having *no concept* does not mean that the student has *no* understanding of the multiplication, rather it means she/he might use and understand the multiplication (e.g. doubling), but it is not a deep understanding.

There were semantics and definition issues when discussing the poster in Figure 8. The idea of *alternate conceptions* caused some angst and confusion. The categories defined as *alternate conceptions* included different contexts, applications, representations, and ways of understanding which were all correct conceptions of multiplication. The map shows the 'voyage' of the student's learning experience, from having no formal mathematical understanding of multiplication, to have simple 'one of' understandings that may not be linked together, and/or may be rudimentarily related to immediate contexts or manipulatives, to a more complex understanding, where students can connect the 'big idea' of multiplication to various contexts, applications, representations, or other ways of understanding that we called *alternate conceptions*. At this stage, you would say that students understand multiplicative structures.

The purpose of the poster is to connect our professional language to the different phases of the learning of multiplication (in students). We start with the state where student have no formal knowledge about multiplication. Acting in different contexts (which educated people usually treat via multiplicative concepts) a student can make some sense of the situation using her previous knowledge and/or create some conceptions related to the context and mathematics. These first conceptions can be almost completely false ("multiplication is playing with digits") or somehow applicable in some contexts in a partial way ("multiplication makes the thing bigger"). With more learning experience, these conceptions can become more complete, more formal, and more mathematically coherent with contexts. However, at some point, different students can have different conceptions of multiplication that are mathematically coherent but associated with different contexts or based on different models. It is also possible that the same student can use different conceptions of multiplication for different contexts and have some difficulty coordinating them. This situation can be described as *alternative conceptions*. Once the student is able to coordinate many models of multiplication and use them in connection with many different contexts, we can say that she has developed the concept of multiplication.

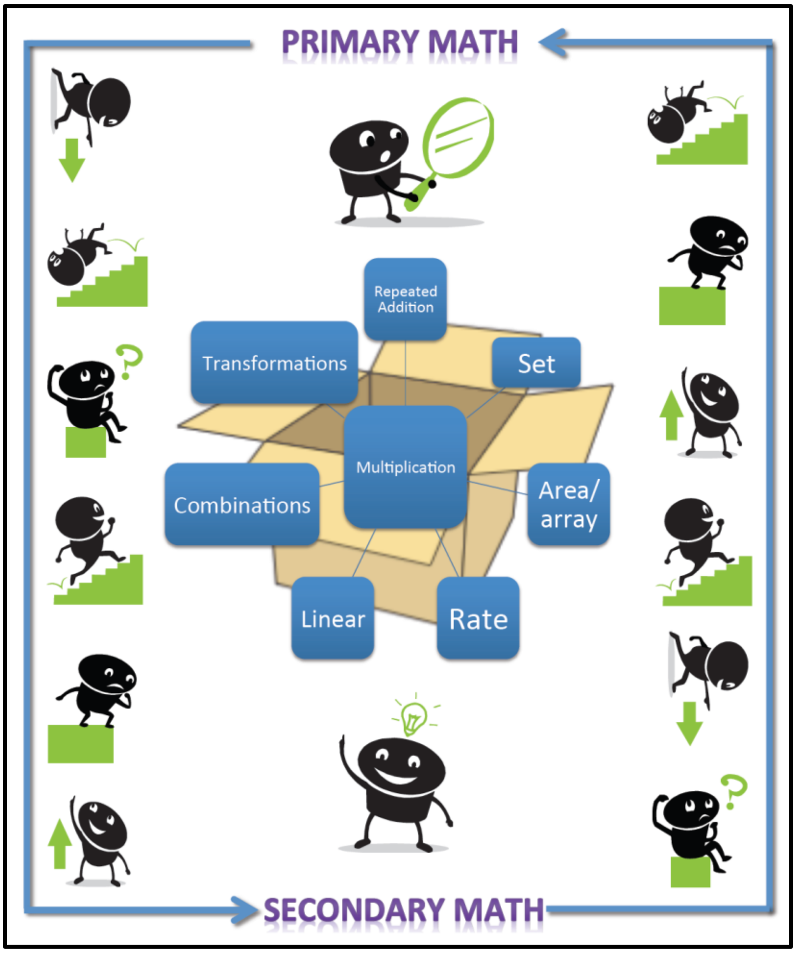

Figure 9. Unpacking multiplicative thinking.

DISCUSSION OF FIGURE 9

Context:

- The arrows around the perimeter show the importance for learners to understand what came before and what is coming next. For example, it is crucial for elementary pre-service teachers to understand the impact of what they teach on secondary school mathematics; and for secondary pre-service teachers to understand what students have done before. This helps to better understand where students' misconceptions arise.

- *Les flèches autour du périmètre représentent l'importance, pour les apprenants, de comprendre ce qui est venu avant et ce qui s'en vient. Par exemple, il est crucial que les futurs maîtres du primaire comprennent l'impact de ce qu'ils enseignent sur les mathématiques du secondaire et pour les étudiants du secondaire de comprendre ce que les élèves ont fait avant. Ceci aide à mieux comprendre d'où peuvent provenir les difficultés que nos élèves rencontrent.*

Step 1/ *1ère Étape*

To help pre-service teachers to understand multiplicative structures, we feel that different steps are important:

- First, they need to realize the knowledge that they have. They must first try to resolve the sample problem, in order to judge their own knowledge before they can judge the knowledge of others.

Afin d'amener les futurs maîtres à saisir les structures multiplicatives, différentes étapes nous semblent importantes:

- *Dans un premier temps, il faut les placer face à la connaissance qu'ils ont. Ils doivent donc tout d'abord tenter de résoudre le problème en jeu, afin de poser un jugement sur leurs propres connaissances avant de pouvoir le faire sur les connaissances des autres.*

Step 2/ *2e Étape*

- The box in the middle represents the unpacking of a concept. It is important to give pre-service teachers the opportunity to analyze in depth the concepts involved in a problem, to identify all the concepts underlying them. Without this analysis, it can be very difficult to really understand the reasoning of a student.
- It is also important to expose pre-service teachers to different types of problems (repeated addition, area, etc.) and get them to think about the relationship between these different problems, the impact of presenting only one type of problem, etc.

- *La boîte au centre représente l'éclatement d'un concept. Il est important de donner la chance aux futurs maîtres d'analyser en profondeur les concepts en jeu dans un problème afin de déceler tous les concepts qui leur sont sous-jacents. Sans cette analyse, il peut être très difficile de réellement comprendre le raisonnement d'un élève.*
- *Il est également essentiel d'exposer les futurs maîtres à différents types de problèmes (addition répétée, aire, etc.) et à les amener à réfléchir sur les liens qui existent entre ces différents problèmes, sur les répercussions de ne présenter qu'un type de problème, etc.*

Step 3/ *3^e Étape* 

- After trying to solve the problem and after an analysis of that same problem, students are ready to analyze schoolchildren's copies and suggest courses of action for those schoolchildren (depending on the mistakes they made).

- *Après avoir tenté de résoudre le problème et après en avoir fait une analyse, les étudiants sont prêts à analyser des copies d'élèves et à proposer des pistes d'intervention pour ces élèves (en fonction des erreurs qu'ils ont commises).*

Closing Thoughts on Figure 9

There are many ways to learn and that's what we tried to portray with the images that are along the arrows. These images represent the diversity of learners in our classrooms, and therefore the diversity of entries into knowledge. Some go in one direction, others go in another direction. Some are stopping and thinking, some are confident, others fearful, etc.

Il y a plusieurs façons d'apprendre et c'est ce que nous avons tenté de représenter avec les images qui se trouvent le long des flèches. Ces images représentent donc la diversité des apprenants qui se retrouvent dans nos salles de classe et, par le fait même, la diversité d'entrée dans la connaissance. Certains vont dans une direction, d'autres vont dans une autre direction. Certains sont en arrêt et réfléchissent, certains sont confiants, d'autres craintifs, etc.

***PowerPoint* Slide 1**
Que pouvons-nous faire…?

Spécificités

- Une grande variété de problèmes (transition primaire-secondaire)
- Une grande variété de modèles mais qui ne se généralisent pas à tous les problèmes

Certaines conceptions deviennent des obstacles à l'apprentissage des problèmes ayant une structure multiplicative :

- Addition répétée
- Multiplier correspond à grossir
- Comparer correspond à trouver une différence
- Le nombre ne peut plus être vu comme une juxtaposition de chiffres, 2 opérations simultanées sont nécessaires, la distributivité est nécessaire
- Le rôle du 0 est nouveau, le rôle du 1 est différent

***PowerPoint* Slide 2**
Intervenir pour les futurs maitres :

- Nommer ces conceptions
- Créer des tâches qui permettent de voir les limites de ces conceptions
- Faire prendre conscience aux futurs maîtres qu'il est possible de discuter de ces obstacles avec les élèves plutôt que de les contourner, de les éviter ou de donner les explications avant que l'erreur ne se produise

***PowerPoint* Slide 3**
Des tâches pour les futurs maitres qui pourraient permettre de voir les limites des conceptions des élèves

- Repérer la variété de problèmes dans les manuels scolaires et discuter si ce sont des vraies tâches multiplicatives
- Identifier, pour chaque catégorie de problèmes, les connaissances que les élèves pourraient utiliser compte tenu des apprentissages précédents
- Comment ces connaissances peuvent faire obstacles aux exigences de la tâche
- Analyser des erreurs d'élèves

Figure 10a. Que peut-on faire pour adresser les connaissances des futurs maîtres par rapport à la pensée multiplicative?

***PowerPoint* Slide 4**
<u>Des tâches pour les futurs maîtres qui pourraient permettre de transiter vers un raisonnement proportionnel</u>

Spécificités

- Une relation en jeu plutôt qu'une opération
- Recherche d'un terme manquant
- On réfléchit sur les relations plutôt que sur des opérations
- La relation s'exprime par un nombre

Obstacles

- L'objet du raisonnement est la relation plutôt que sur l'opération.
- La relation devient un « pattern » qui se répète ce qui est différent des opérations.

Figure 10b. Que peut-on faire pour adresser les connaissances des futurs maîtres par rapport à la pensée multiplicative?

***PowerPoint* Slide 1**
<u>What can we do?</u>

Issues that need to be addressed:

- Within the school curriculum, there are a wide variety of mathematical tasks (in the primary-secondary transition).
- There are also many multiplicative procedures and models that don't necessarily generalize to all problems.

Some *ways of understanding* and generalizations made by primary and middle school students may become obstacles to solving multiplicative problems; for example:

- if students choose to continue to use repeated addition as a strategy;
- if students believe that multiplying always means "a bigger result";
- if they believe that "compare" always means "to find a difference";
- if students cannot see a two digit number as a juxtaposition of tens and ones. For a deep understanding of this idea, two concurrent operations are required: the distributive property is a combination of multiplication and addition.
- The roles of 0 and 1 are different in multiplication and in addition.

***PowerPoint* Slide 2**
<u>Intervention for future teachers:</u>

- Identify these *ways of understanding* and generalizations that primary and middle school students make.
- Create tasks that allow pre-service teachers to see the barriers that primary and middle school students may face because of their *ways of understanding* and generalizations.
- Develop an awareness that teachers can explicitly discuss these obstacles with students rather than circumvent them, avoid them, or try to give explanations before the error occurs.

PowerPoint Slide 3
<u>Tasks for future teachers that could lead them to see the limits of the students' conceptions</u>

- Examine the variety of problems in textbooks and discuss whether they are multiplicative tasks.
- Identify, for each type of problem, the prior knowledge that students must bring to the task.
- How might this knowledge be an obstacle to the requirements of the task?
- Analyze student errors.

Figure 11a. What can we do to address pre-service teachers' understanding of multiplicative thinking?

***PowerPoint* Slide 4**
Tasks for future teachers that could help to transfer to proportional reasoning:

- look for relationships and connections rather than for an operation or procedure;
- seek the missing term in a proportional situation;
- see that the proportional relationship can be expressed using a numerical equation or expression;
- discuss and seek barriers to student understanding.
- Pre-service teachers must understand that the bigger idea is the relationship that becomes the "pattern" that can be generalized (this is different from simply performing operations, such as "cross-multiply").

Figure 11b. What can we do to address pre-service teachers' understanding of multiplicative thinking?

SUMMARY

There are a number of recommendations and activities that we offer for pre-service and practicing teachers that might help to understand the complexity of the curriculum that supports multiplicative understandings. Some key results of our group discussions include:

Recommendations for Pre-Service and Practicing Teachers

Please make sure you understand:

- the prerequisite skills that lead to understanding a concept;
- that the connections and relations between mathematical ideas are much more important than the procedures you teach;
- that you need to understand the breadth of applications and other ideas related to that concept;
- the learning trajectory and future applications of the concept you are teaching;
- the barriers and difficulties, including misunderstandings, false generalizations, that young learners may adopt;
- that you can and often should explicitly discuss common misconceptions with your class when they occur.

Ways that methods professors can work together with pre-service teachers to help them create a classroom that seeks understanding of multiplicative tasks:

- Discuss what it means to understand something in mathematics.
- Use student exemplars to see where primary and middle school students make common errors.
- Examine and explore why students are making errors.
- Create a concept map to show which background skills are necessary for students to understand an idea.
- Make a list of future school curriculum topics that relate to the concepts currently being taught.
- Examine various textbook questions to understand the complexity of how concepts are addressed.

In trying to answer the question of what it means to understand multiplicative ideas and processes, and by studying exemplars of one student's work, this working group developed ideas and classroom tasks that could be used with pre-service teachers to help deepen their understanding of the important connections between ideas and concepts related to multiplicative thinking, prerequisite skills and future applications of those same understandings. These recommendations are not limited to the topic of multiplicative thinking, but could be applied to any conceptual topic in mathematics.

Rich professional discussions emerged from fairly simple tasks, such as defining 'understanding', studying student work samples, producing concepts maps, and designing professional learning for pre-service teachers. Several suggestions and examples were given to help pre-service teachers become more proficient with multiplicative or other concepts. Our experiences have shown us repeatedly that pathways for learning are complex and not as absolute as might be expected. The work of educators is interpretivist and the activities suggested here are as important for their process as is the precision of the resulting products. This variety is seen in the posters and other artefacts produced by this group. We offer these suggestions in this paper as tools for course design and pre-service or adult learning, and we hope you can take the opportunity to try our suggestions for yourselves.

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MATHEMATICS CURRICULUM RE-CONCEPTUALISATION

Brent Davis, *University of Calgary*
Kathy Kubota-Zarivnij, *Toronto Catholic District School Board*

PARTICIPANTS

Lyla Alsalim	Martha Koch	Susan Oesterle
Andy Begg	Jill Lazarus	David Reid
Priscilla Bengo	Geri Lorway	Pat Rogers
Iain Brodie	Masomeh Jamshid Neja	Chris Suurtaam
Bev Caswell		

INTRODUCTION

Mathematics curriculum revision has become a more-or-less constant project for ministries of education, often framed in terms of adding to, deleting from, or re-situating topics in the K–12 trajectory. In this working group, we tried to think about other ways to think about curriculum and efforts to 'update' curriculum.

Specifically, the intention of this working group was to envision a model of mathematics curriculum design that is informed by contemporary *reconceptualist curriculum* literature (e.g., Pinar, 1999) and that embodies what is currently known about mathematics knowledge, learning, and teaching. In terms of specific objectives, we aimed: (1) to articulate a set of principles that orient efforts toward mathematics curriculum design, and (2) to exemplify these principles using one or two mathematical case examples.

Engagements in the working group were structured around what we called "considered contributions". For each of their contributions, participants were required to formulate an assertion, support it with an argument, and consider some of its implications and entailments. These contributions were formatted as 3-slide *PowerPoint* presentations, and the bulk of this report comprises modestly re-formatted versions of those documents.

Our other organizational strategy was to pose three key questions, one for each day of our meeting:

- Day 1: Why change/reconceptualise mathematics curriculum?
- Day 2: What might be changed?
- Day 3: How might we think about curriculum structures?

We use these questions as section headers in the balance of the report.

DAY 1: WHY CHANGE/RECONCEPTUALISE MATHEMATICS CURRICULUM?

Among the most prominent criticisms of current mathematics curricula is that they tend to embody some rigid, linearized assumptions on human development (e.g., the oft-noted teen-years transition from concrete operations to abstract thinking). While there is some utility to such constructs, they can also present problems. For example, as devices to orient curriculum development, entrenched assumptions about human development can contribute to troublesome misreadings of the conceptual abilities of young learners and oversimplified (and therefore debilitating) interpretations of mathematical concepts.

This is just one reason for rethinking mathematics curriculum. As elaborated below, Day 1 was devoted to identifying and explicating others

CHRIS, JILL, & MARTHA ON "WHY?"

<u>Assertion</u>

Curriculum needs to be reconceptualised to encourage educators to see it as a 'dynamic framework' that's open to possibilities, as an opportunity, rather than as a static checklist that must be followed/implemented because:

- educators tend not to see themselves and their students as active agents/co-constructors of curriculum; and
- educators often understand curriculum as a written, mandated document.

<u>Argument</u>

The assertion is supported by observations from two projects:

- "What Counts in Math" (a two-year project involving 42 Grades 4 - 12 math teachers)—Many teachers reported feeling expected to 'cover' the curriculum expectations and that this was a challenge for them.
- "Curriculum Implementation in Intermediate Math" (CIIM; case studies where students solve problems in different ways using different tools)—Several encouraging practices were noted: e.g., teachers did not impose algebraic solutions; they didn't approach the curriculum as a checklist.

Why move toward a dynamic curriculum framework?

- Enriching understandings of mathematics—mathematics is not a static subject; new connections can be made; the textbook is not the authority.
- Supporting learning—when teachers and students are active co-constructors, mathematical ideas have more relevance and meaning (e.g., a teacher working with a group of Grade 9 students in an applied course where students help to construct the task, use their iPods to find paper airplane designs, etc.). To support their students' learning, teachers need to understand where they're at mathematically, socially, and with regard to learning processes, interests, etc.
- Empowering teachers—teachers would likely feel greater efficacy if there were a shift in thinking away from 'covering the curriculum' toward opportunity to create. Such a shift would support the development of professional judgment, as teachers manage the unique and complex realities of their classroom, rather than following a prescribed approach.

Implications

The following would be among the major implications of a more dynamic, engaged mathematics curriculum. There would be needs:

- to reconceptualise the roles of students and teachers (pre-service, in-service, teacher educators, principals, etc.);
- to work with educators (principals, teachers, etc.) to support them in moving between a dynamic curriculum framework and their classroom practice (a checklist may seem easier to implement);
- to find ways to make the change less threatening/overwhelming for everyone (students, teachers, etc.).

DAVID & ANDY ON "WHY?"

Assertion

We see a lack of such connections and attention to significant ideas in existing implemented curricula. There is thus a need to change curriculum to re-conceptualise the learned curriculum so that students learn the connections between significant ideas (e.g., infinity, relation, proof, creativity, autonomy, collaboration) within and beyond mathematics.

Argument

The following might be better supported through a revised curriculum:

- students seeing connections to what they know and value—which will address lack of success, alienation, and lack of motivation ("Why do I need to know this?");
- a more 'bottom up' approach focused on reaching the learner.

Implications

- *Dynamic nature* of connections and significant ideas need to be critically analysed and clarified by mathematics educators and teachers.
- Ways to *describe* connections and significant ideas need to be clarified by mathematics educators and teachers.
- How to *assess* learning of connections between significant ideas need to be clarified (case studies, self assessment).
- Ways to diagnose what is *known* need to be refined so that it can be used as a basis of connections.
- Immediately accessible *experiences* need to be identified to connect to.
- Curriculum documents and implementation will have to be *restructured* to accommodate an emphasis on connections and significant ideas.

GERI & IAIN ON "WHY?"

Assertion

The learning of mathematics cannot be anything but co-constructed by students and teachers.

Argument

- Mathematics is a way of thinking.
- We need to see students *and* teachers reasoning and thinking.
- Curriculum has to make evident that mathematics is about thinking.

Implications

- Current mathematical resources do not support the teacher to learn this type of learning.
- Explicit examples need to be freely and publicly available for all (students, teachers, parents) to learn from.

SUSAN, LYLA, & MASOMEH ON "WHY?"

Assertion

Curriculum needs to change in order to respond to changing society:

- Students need to be equipped with the mathematics they need for the modern world. (But not just this!)
- The current curriculum does not prepare students in terms of WHAT is being taught as well as HOW.

Argument

What is considered 'basic knowledge/skills' has changed. It used to be the ability to do your work, be your own tool. Now that we have these tools everywhere (they are portable, user-friendly, etc.), the key skills we really need are to identify key aspects of problems and select appropriate tools to solve the problem. Deeper conceptual understanding and problem-solving skills are needed.

Implications

- For curriculum: incorporate more problem solving; more project-based learning; reduce number of topics to allow time for depth and investigations; for high school, less emphasis on prep for calculus.
- For teachers: need a shift in thinking; need PD.
- For students: see the relevance of the mathematics; will be empowered.

PRISCILLA, PAT, & BEV ON "WHY?"

Assertion

Students are not being taught to persevere.

Argument

- Math ability is so highly prized that those who aren't given opportunities to succeed in math in their own way suffer from identity issues.
- Student voice is not present in the way the curriculum is enacted—students are not given enough opportunities to engage in meaningful problem-solving activities (e.g., teachers deconstruct difficult problems for students).
- Following Gutiérrez (2012), curriculum entails more than what is explicitly identified as 'content' (see Figure 1). We must be more mindful of the implicit or hidden curriculum of school mathematics.

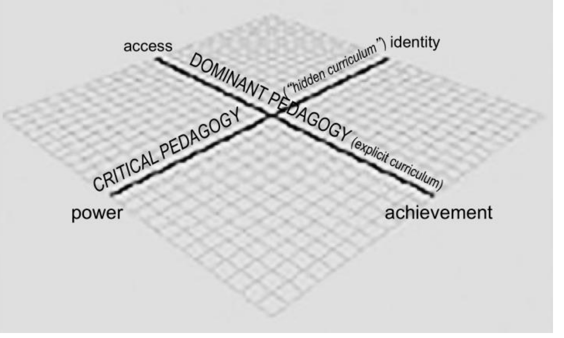

Figure 1. Attending to the explicit and implicit dimensions of curriculum (based on Gutiérrez, 2012).

Implications

We need to consider not just matters of *content*, but how that content is presented and engaged. Some strategies that might help interrupt and surface the current hidden curriculum include:

- Start with a difficult problem.
- Structure the start-up through key questions and let them struggle.
- Let the students find out what they don't know.
- Teach what they need as they need it.
- Teach Fermi problems in all the grades.

DAY 2: WHAT MIGHT BE CHANGED?

One of the most common criticisms of contemporary school mathematics is that its contents are increasingly out of step with the times. The curriculum, it is argued, comprises many competencies that have become all but useless, while it ignores a host of skills and concepts that have emerged as indispensible.

The issue of accelerating irrelevance is particularly obvious around the notion of 'basics'—or, more cogently, 'basic operations'. As a *Google* search will confirm, this phrase is almost universally understood as a reference to a four-member set that includes addition, subtraction, multiplication, and division. These operations are the mainstays of mathematics for most. Indeed, the 'basic operations' and 'mathematics' might be argued to be coterminous for a significant portion of the population. But in what ways are these operations basic?

Certainly not in the sense of irreducible fundaments, starting premises, or irrefutable axioms. It seems to be that what they are basic to is not mathematics or mathematical understanding, but the needs of a minimally numerate human in an industrialized society. Grumet (1995) makes this point in a critique of the habit of freezing competencies that are situationally specific into elements that are treated as eternal and universal. As she notes, what is essential is not a concept itself, but "the relation of … histories of human action and interpretation to the lives of the children studying them" (p. 19). Within school mathematics, the 'basics' are basic because of their necessity to a group of people at a particular time, not because of their role within a body of knowledge.

What then are the 'basics' of school mathematics in this current historical moment? Such was the question we addressed on Day 2.

CHRIS, JILL, & MARTHA ON "WHAT?"

Assertion

The overarching mathematical themes/ideas and connections need to be more explicit …

- to highlight the important mathematical ideas that students need to engage in and explore;
- to provide both direction and latitude for teachers to explore these concepts in multiple ways that are relevant to the students they teach.

By way of example: How does proof develop over K-12? What might this look like? How do we develop notions of properties of functions?

Argument

- When mathematical ideas are not connected, students and teachers view mathematics as a set of isolated facts and procedures.
- Students develop a richer understanding of mathematics by connecting mathematical ideas.
- Teachers could benefit from seeing how important mathematical ideas are developed across grades and within their course.
- Neither the value of connecting nor the way ideas can be connected is explicit in current curricula (in Ontario).

Examples include:

- proof suddenly 'appears' in Grade 12 without any grounding;
- connections between numeric, graphical, and algebraic forms of functions (e.g., quadratic) strengthens their understanding;
- connections to previous year's work increases that understanding (t-charts, linear functions).

Implications

- Students and teachers will be able to see how mathematical ideas connect within a course and from one year to the next.
- This shift enables teachers to assist students by making connections to prior learning (e.g., this can be done whole class, small group, or individually).
- Focusing on the mathematical ideas and connections has implications for the structure of curriculum documents—what might this look like?

DAVID & ANDY ON "WHAT?"

Assertion

In general the curriculum should be the connections between significant ideas within and beyond mathematics. Specific examples include:

- the connections between $+$, $-$, $\times$, $\div$ (as relations), equality, infinity, proof, everyday objects, and symmetry;
- in concrete proofs of commutativity and non-commutativity.

Argument

- Cognitive science tells us that teaching should be based on embodied experience (e.g., with objects).

- Ideas like symmetry, infinity, proof, relation are fundamental to mathematics, hence are more 'basic' than basic operations.
- A connected understanding of why things work in mathematics leads to confidence and interest, enhances retention and transfer, and reduces the need for memorization.
- Connections lead to surprise and ideas like big numbers, infinity and symmetry seem to be interesting in their own right.
- Flexible use of representations is a mark of conceptual understanding.

Implications

Teacher education needs to be different. So do schools, governments, parents, etc.

SUSAN, LYLA, & MASOMEH ON "WHAT?"

Assertion

Logic and reasoning (e.g. *if-then* statements) should be incorporated into the curriculum across grade levels—e.g., conjecture/'proof' activities, even in the lowest grades. It is important that this be recognized as mathematics.

Argument

This sort of shift …

- will give students a more complete picture of what mathematics is (not just calculation, or solving others' problems; can be created/discovered by individuals);
- will prepare students for more rigorous proof later (e.g., What is needed to be able to understand and appreciate proofs of the Pythagorean Theorem?);
- will support the development of reasoning skills that are important for other disciplines (e.g., computer programming).

Implications

- Teachers would need to be provided with lots of examples of activities and ideas for how to bring out these ideas in the classroom.
- Teacher preparation programs would need to ensure that teachers have an understanding of, and appreciation for, this broader view of mathematics, and be comfortable with the openness and uncertainty of 'playing' with mathematical ideas.

DAY 3: HOW MIGHT WE THINK ABOUT CURRICULUM STRUCTURES?

We began Day 3 by highlighting two contrasting meanings of the word *structure*. Most commonly, the word is used to refer to planned edifices of one sort or another. Less commonly, *structure* is used in reference to living entities to refer to their evolving forms—and, in particular, the manner in which those forms embody their histories. It turns out that the latter usage is more reflective of the word's origins. Derived from Sanskrit roots, the term originally had to do with spreading or stretching out. These original senses continue to be preserved in such cognates as *strew* and *construe*. Overwhelmingly, it is the former, more rigid meaning that is intended when the word *structure* is used in reference to mathematics curriculum, whereas much of current research and theorizing in curriculum favours attitudes that lean toward the latter meaning. (See Davis & Sumara, 2000, for an extended examination.)

It turns out that a very similar contrast in meaning is present in contemporary uses of the word *curriculum*. Most commonly, the word is used in reference to programs of study—that is, akin to planned edifices, relatively stable forms that are crafted prior to implementation. A second, less frequently encountered meaning is hinted in the phrase *curriculum vitae*. In this case, the curriculum is one's career history, an ever-elaborated narrative that is, by necessity, crafted *after* engagements.

And so, we wondered at the start of Day 3, might the *curriculum* of school math be understood in more emergent, less prescriptive, and more retrospective terms? To assist in this discussion, David Reid drew our attentions to some contrasts between a few prominent curriculum documents in Canada, namely the "K-9 Mathematics" portion of the *Western and Northern Canadian Protocol Common Curriculum Framework* (WNCP, 2006) and the mathematics components of Québec's education program (Government of Québec, n.d.). Two dramatic differences present themselves immediately when reviewing these documents. First, as indicated in Table 1, the WNCP document devotes considerably more space to "content outcomes", which are more parsed, more detailed, and more engineered than the manner of presentation in the Québec documents. Second, the WNCP document assumes a linear model of curriculum, with virtually no discussion of structures and organization. In considerable contrast, large portions of the Québec documents are devoted to rethinking the metaphors and images that are used—and that might be used—to think about how ideas are presented, connected, and organized. (URLs for PDFs of these documents are included in our reference list.)

	WNCP CCF	QC Primary	QC Sec Cycle 1
Total pages	63	18	34
Other front matter	3+4 (11%)	2 (11%)	10+2 (35%)
Processes/competencies	3 (5%)	8 (44%)	7 (21%)
Content outcomes	50 (79%)	4.5 (25%)	12 (35%)
Other back matter	3 (5%)	3.5 (19%)	3 (9%)

Table 1. A comparison of the contents and emphases of some major Canadian mathematics curriculum documents.

CHRIS, JILL, & MARTHA ON "HOW?"

<u>Assertion and Argument</u>

Curriculum might be construed as the co-creation/emergence of a dynamic, rhizomatic network of connected ideas, presenting the need for images and metaphors that foreground notions of emergence and growth (see Figure 2).

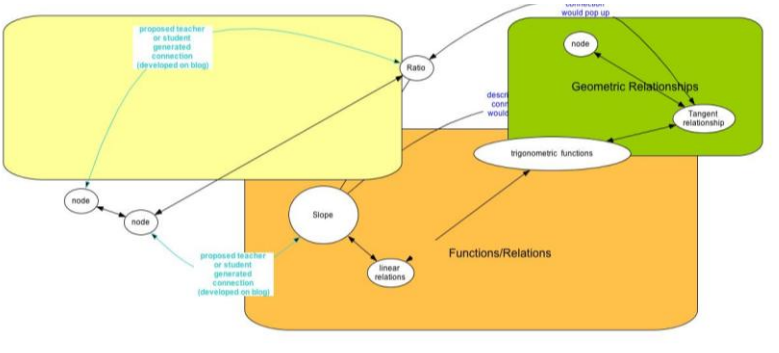

Figure 2. Curriculum as the co-creation of a dynamic, rhizomatic network of connected ideas.

In terms of strategies or means to effect this,

- a web-enabled network with hot links could be used to show layers connected to individual concepts (see Figure 3);
- blogs could be used to discuss the mathematical ideas and their connections, creating the potential for bloggers to add ideas/connections/approaches/contexts.

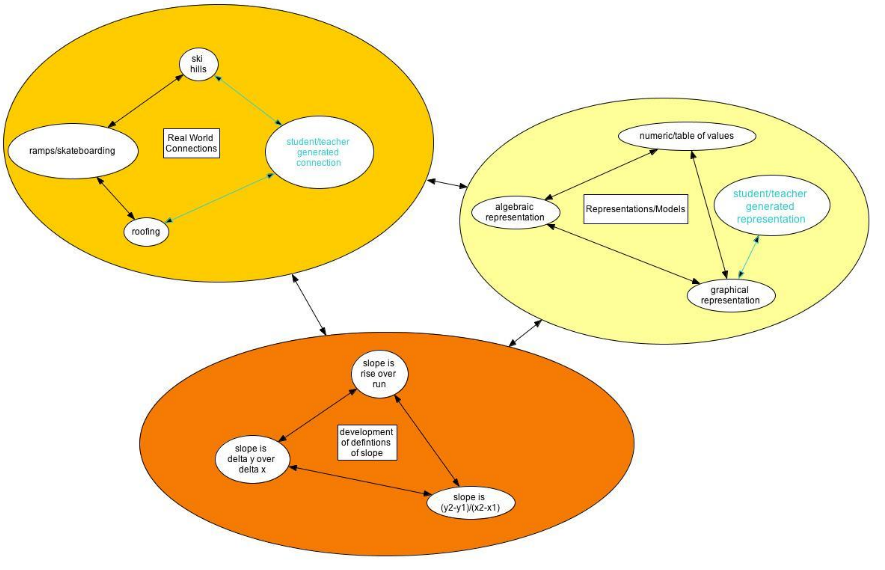

Figure 3. One means to show how layers might be connected to individual concepts.

<u>Implications</u>

- Teachers and students are co-creating the curriculum and knowledge.
- Knowledge is a complex web of interconnected ideas.
- Learning is the ongoing creation of the network/web.

IAIN, DAVID, & ANDY ON "HOW?"

<u>Assertion and Argument</u>

Curriculum might be construed as a path through a thing with multiple overlapping facets (competencies, content strands, big ideas, representations—see Figure 4). This metaphor offers the following advantages:

- It can be viewed with any facet fore-grounded.
- Every point in every facet is connected to every other point in that and every other facet.
- The path traverses the dual of this thing, in which the connections are points and the points are connections.
- For every student the path is different, but some points must be visited at least once.

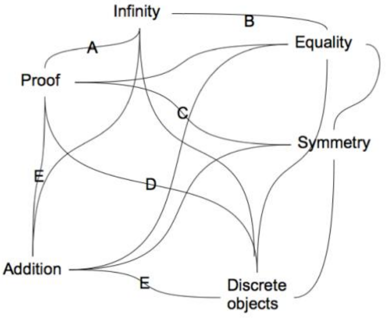

Figure 4. A representation of a curriculum-as-path metaphor.

Implications

- Teachers, students, and policy makers may need to step outside their current thinking and experience that mathematics is more than just a series of topics.
- A mathematics experience is not just one thing; it is the connections between all of these things.

SUSAN, LYLA & MASOMEH ON "HOW?"

Assertion and Argument

Curriculum might be construed as a tree (see Figure 5).

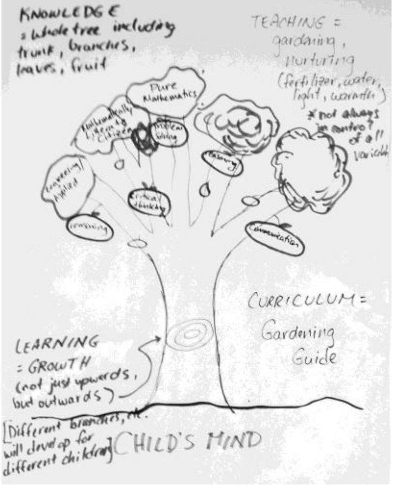

Figure 5. A representation as a curriculum-as-tree metaphor.

Implications

- *Knowledge* = a whole tree, including trunk, branches, leaves, fruit
- *Learning* = growth (not just upwards, but outwards) [Different branches, etc. will develop for different children)
- *Teaching* = gardening, nurturing (fertilizer, water, light, warmth) [not always in control of all variables]
- *Curriculum* = gardening guide

CONCLUDING REMARKS

Recent insights into the complex, emergent natures of both the discipline of mathematics and the learning of mathematics should, we believe, compel very different models and approaches to curriculum and teaching. We summarize our 3-day engagement on these matters with the following questions:

HOW HAS MATHEMATICS ITSELF BEEN USED TO ORGANIZE CURRICULUM? AND WHAT OTHER OPTIONS MIGHT BE WORTH CONSIDERING?

Assumptions about the structure of the discipline have long been used to format curriculum (e.g., when knowledge is construed in Euclidean terms as a logical edifice, attentions veer toward firm foundations and efficient construction; in contrast, when knowledge is construed fractally as, e.g., a decentralized network, attentions turn to hub-concepts and self-similarities). To what extent should we continue to look to mathematics for advice on this matter?

WHICH CONCEPTS MIGHT OR SHOULD SERVE AS BASICS/HUBS?

Centuries ago, it made sense to place arithmetic at the core of mathematics instruction, to select +, –, ×, and ÷ as the 'basics', and to gear schooling toward preparing learners for the industrialized world. Might it be time to move to different clusters of notions—such as spatial fluency and exponentiation—to serve as 'essential' mathematical competencies? More profoundly, perhaps, might the 'basics' of mathematics be reconstrued in terms of such core concepts?

HOW MIGHT A MORE PARTICIPATORY CURRICULUM BE DESIGNED?

How might students, teachers, educational researchers, mathematicians, and others be involved in ways that move beyond opinion-based contributions?

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Topic Sessions

Séances thématiques

FORUM CANADIEN SUR L'ENSEIGNEMENT DES MATHÉMATIQUES

CANADIAN MATHEMATICS EDUCATION FORUM

Ann Arden, *Osgoode Township District High School and University of Ottawa*
Richard Hoshino, *Quest University Canada*
Kathleen Pineau, *École de technologie supérieure*

(English translation follows.)

Le Forum canadien sur l'enseignement des mathématiques (FCEM) vise toujours à réunir des participants venant des quatre coins du Canada qui partagent une préoccupation et une passion pour l'enseignement des mathématiques : enseignants du primaire et secondaire, coordonnateurs de commissions scolaires, enseignants de mathématiques au collégial et à l'université, didacticiens, étudiants des cycles supérieurs et représentants du gouvernement et du monde de l'édition.

Dans cette séance thématique Ann, Kathleen et Richard ont partagé leur vision du prochain Forum pour ensuite recueillir les idées et les commentaires des participants. Ce qui suit résume ce qui a été convenu à la suite de ces discussions.

Le prochain Forum canadien sur l'enseignement des mathématiques (FCEM) se tiendra à Ottawa, Ontario, du jeudi 1[er] mai au dimanche 4 mai 2014 à la Faculté d'éducation de l'Université d'Ottawa.

Inspiré des activités décrites dans les références ci-dessous, le FCEM 2014 mettra l'accent sur le partage d'expériences en enseignement. Ces expériences prendront la forme de « vignettes ». Au FCEM 2014, on désigne par vignette un texte destiné aux enseignants de mathématiques dans une ou plusieurs des catégories :

- Une activité mathématique qui aide les élèves à comprendre un sujet ou un concept important. Une telle activité sera riche en contenu, ouverte à de nombreuses méthodes de recherche, posera un défi tout en étant accessible aux étudiants et visera plus d'un sujet du curriculum mathématique.
- Une réflexion pédagogique sur une question importante de l'enseignement des mathématiques. Une telle réflexion sera amenée par une tension ou un dilemme important de l'enseignement des mathématiques (par exemple, le rôle de la technologie, le défi de l'évaluation) et évoquera les efforts déployés afin de la ou le résoudre.

- Une innovation curriculaire qui aide les élèves à découvrir les mathématiques sous un jour nouveau. Une telle innovation relatera une histoire de réussite en enseignement des mathématiques qui a amené un changement: localement, au niveau régional, provincial ou national.

Pour plus d'informations au sujet du Forum de 2014, voyez le site http://cms.math.ca/Reunions/FCEM2014/.

♦♦♦♦♦♦♦♦♦♦

The purpose of the CMEF is to bring together, from all parts of Canada, a full spectrum of participants who share a concern and a passion for mathematics education: elementary and high school teachers, school board coordinators, college and university mathematicians and math educators, graduate students, and representatives from government and publishing.

In this Topic Session, Ann, Kathleen and Richard shared their vision for the next Forum and welcomed comments and further ideas for the theme and structure of the 2014 meeting. The following summarizes what was decided as a result of these discussions.

The 2014 Canadian Mathematics Education Forum will be held in Ottawa, Ontario, from Thursday, May 1st, to Sunday, May 4th, at the Faculty of Education (University of Ottawa).

Inspired by the activities described in the references below, CMEF 2014 will focus on sharing experiences in teaching. These experiences will take the form of 'vignettes'. At CMEF 2014, a vignette is a text intended for teachers of mathematics in one or more of the following categories:

- A mathematical activity that helps students understand an important topic or concept. An ideal activity will be rich in content, open to numerous methods of investigation, challenging, yet accessible to students, and cover two or more topics in the mathematics curriculum.
- A pedagogical reflection on an important issue in mathematics education. An ideal reflection will be motivated by an important tension or dilemma in math education (e.g. the role of technology, the challenge of assessment) and discuss the steps that were attempted to address or resolve this issue.
- A curricular innovation that has helped students experience mathematics in a new light. An ideal innovation will share a 'success story' of mathematics education that has inspired change: locally, regionally, provincially, and/or nationally.

For more information on the 2014 Forum, see http://cms.math.ca/Events/CMEF2014/.

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SOCIAL MEDIA AND MATHEMATICS EDUCATION: WHENEVER THE TWAIN SHALL MEET?

Egan J. Chernoff
University of Saskatchewan

During this Topic Session, an overview of the current state of social media for mathematics education was presented (in four parts). First, the different types of social media being used for mathematics education were highlighted and detailed (e.g., social networks, blogs, microblogs, social bookmarking, media sharing, aggregators and discussion forms). Second, for each of the different types of social media, I detailed 'who' (e.g., individuals and organizations) is using 'what' (e.g., Facebook, Google+, Twitter, Tumblr, Delicious, StumbleUpon, YouTube, Instagram, Pinterest, RSS, LinkedIn, Academia, *listserves and others). Third, I discussed my use of (and varying attempts to manage) social media over the past five-plus years. With a better picture of the social media for mathematics education landscape I, lastly, discussed when social media and mathematics education will truly coexist. The purpose of this article is to highlight certain aspects of this Topic Session.*

DISCLAIMER

In the interest of full disclosure, social media for mathematics education is not my area of research. With that said, I have, over the past five years, come to fully embrace my use of social media for mathematics education. Initially, I adopted a few social media platforms, such as *Facebook* and *Twitter* to bolster my engagement in one of the three pillars required for a successful career as a faculty member at a research institution: service. As I continued to adopt other platforms, such as *Academia(.edu), Delicious*, *Google+*, *Instagram*, *LinkedIn*, *Pinterest*, *Second Life*, *StumbleUpon*, *Tumblr* and *YouTube*, my service expanded beyond faculty and university service to academic community service and public and community service. Five years on, I now categorize my continued efforts as 'digital service', which I contend, in the future, will be added to faculty, university, academic and public and community service required of faculty members at research institutions. While my use of social media for mathematics education has, more recently, resulted in some of the more traditional activities associated with research expertise (e.g., invited lectures, conference presentations, articles in journals and conference proceedings, and visiting scholar invitations), I must reiterate: social media for mathematics education is not my area of research. Further compounding the issue, I am not a historian, ethnographer or anthropologist, nor am I well versed in narrative inquiry. Having presented my disclaimer, the purpose of this article—as was the case with this Topic Session and as is the case with my use of social media for mathematics education—is simple: to share information related to mathematics education.

INTRODUCTION

Mathematics education researchers, in general, have been quite slow in adopting social media. For example, part of the excitement associated with getting my own *Twitter* account (follow me @MatthewMaddux) in May of 2009 (considered late adoption given that *Twitter* was launched in July 2006), was the potential use of *Twitter* during mathematics education conferences. With a *Twitter* account and the right hashtag, I could potentially keep track of what was happening in sessions I was not attending, attend Twitter organized social functions and network with individuals I may have not have traditionally crossed paths with. However, looking back at my conference activities for (the annual meetings of) the:

- Canadian Mathematics Education Study Group (CMESG/GCEDM) 2009;
- Canadian Mathematics Education Forum (CMEF) 2009;
- The North American Chapter of the International Group for the Psychology of Mathematics Education (PMENA) 2009;
- CMESG/GCEDM 2010;
- PMENA 2010;
- The International Group for the Psychology of Mathematics Education (PME) 2011;
- PMENA 2011;
- The Congress of the European Society for Research in Mathematics Education (CERME) 2011; and,
- Research in Undergraduate Mathematics Education (RUME) 2012

I and a (scant) few others were the only ones doing any Tweeting.

In attempt to confirm my conference recollections, I recently looked back through the archive of my first 10 000 Tweets (which are available for download, for some reason, on my website: eganchernoff.com). Social media activity during mathematics education research conferences has been extremely limited, but, lately, things are getting ‘better’. For example, recent conferences, such as the annual meeting of the American Education Research Association (AERA) 2012, the International Congress on Mathematical Education (ICME-12) and PMENA 2012 have all had a much stronger *Twitter* presence. (Update: PMENA 2013, with respect to Twitter, was a bust and, for the record, I have ‘Storified’ the limited social media activity during the conference here: storify.com/matthewmaddux). The late adoption of social media by mathematics education researchers (as ‘demonstrated’ above) was a key point that I wanted to make during this Topic Session.

THE #CMESG2013POPQUIZ

With a room full of people (this is actually true and I was quite pleased) well versed in mathematics, I had no issue in starting my talk with a pop quiz. Below are the ten (plus) questions, which I denoted the #CMESG2013PopQuiz.

1. According to a recent *EA SPORTS* simulation, who will win the 2013 Stanley Cup? **Bonus point opportunity**: What did the simulation predict would happen to the Toronto Maple Leafs in the first round? (Too soon?)
2. Which mathematician is alleged to have invented *Bitcoin*? **Bonus point opportunity**: What else is she/he famous for?
3. Which fast food company did Yitang Zhang work at prior to his recent contribution to the twin primes conjecture?
4. Which insect utilizes prime numbers as a way to protect themselves from predators?
5. Why has eminent biologist E. O. Wilson been under attack recently? **Bonus point opportunity:** Why has Andrew Hacker been under attack recently?

6. Mathematics education research journals were recently graded. What grade did FLM get?
7. Where is (David) Wheeler Island located?
8. *Numberphile* produces....
9. *Zequals* is a...
10. "#MTT2k" was a...

Bonus question. What is the one common thread to all these questions?

In case you were wondering, no, I will not be providing the answers to the #CMESG2013PopQuiz in this article. Instead of making the answers readily available, my hope is that, if interested, you will seek out the answers by either embracing, adopting or utilizing a particular social media platform (e.g., the answers are on *Twitter*) or, to use a *Facebook* term, 'poking' one of your colleagues who attended the talk either via email or the next time you run into them. (If neither of those approaches appeal to you, then you can always find the answers on my website.) By not presenting the answers in this article, my hope is that there exists a social component, digital or not, to your seeking out the answers to the #CMESG2013PopQuiz, which was another central tenet to my Topic Session.

You have probably noticed my message, thus far, has been a bit contradictory. First, I explicitly declare that the purpose of this article is "to share information". Second, I do not provide any answers to the #CMESG2013PopQuiz. A contradiction. Well, not really. I am still sharing the information, that is, the answers to the quiz, but I am attempting to change *how* and *where* you access this information. Why? Well, another central tenet to my Topic Session is that how we access (mathematics education) information is changing.

Actually, I will provide the answer to one of the #CMESG2013PopQuiz questions in this article. The answer to the Bonus question, that is, the one common thread to all of the #CMESG2013PopQuiz questions, is that I obtained the (what I consider and hope you agree) interesting mathematics education information (that I would not have come across through more traditional means), required to create and answer the questions, via social media. In using the term 'social media' I am, at this point, painting with a rather broad brush. (The different types of social media were discussed during my Topic Session and interested individuals are able to download the presentation slides here: eganchernoff.com/conference-presentation-slides.) Worthy of note, and another central tenet to the Topic Session: I did not seek out the (interesting) information (that I traditionally would not have come across if not for my use of social media) for the #CMESG2013PopQuiz—the information came to me.

A TECHNOLOGICAL TWIST TO 'PASSIVE RECIPIENT'

In mathematics education, the phrase 'passive recipient' has been around for some time. Traditionally, the phrase is associated with the theory of *constructivism* and can be interchanged for the more colloquial phrase, "filling an empty vessel". With the advent of Web 2.0, however, I argue that the notion of 'passive recipient' and 'empty vessel' must be parsed. Essentially, I was a 'passive recipient' of the mathematics education information which comprised the #CMESG2013PopQuiz. Further, I am a 'passive recipient' of the deluge of mathematics education information—some of which (more on this in a moment) I share through my various social media platforms. To be clear, 'I' means me and my technology (i.e., Web 2.0 tools).

Whenever I run into an individual who is familiar with my use of social media for mathematics education, but not necessarily familiar with social media or Web 2.0 tools, the questions or statements are usually the same. "Do you sleep?" "You have no life." (Similar questions came up at the end of my Topic Session.) Inevitably, the questions always come around, at some point, to the same one: “How do you do it?” I am more than willing to share my ‘secret’. (Worthy of note, there is no secret.)

The key to my use of social media for mathematics education lies in my creation of an ‘empty vessel’ that, after the initial set up process, is continually filled up and then emptied. In other words, I am consistently filling an empty vessel with mathematics education information—that vessel is an RSS reader. Once the vessel is ‘full’ (a relative term), the process of emptying the vessel is how I go about sharing a variety of mathematics and mathematics education information. This raises the related questions of how the vessel is filled and emptied. Essentially, my RSS reader is an inbox for the internet. Through a variety of website RSS subscriptions, *Google* alerts and social media aggregators I am able to curate the majority of mathematics education information that is found on the web. The beauty of setting up an inbox for the internet, during the Web 2.0 era, is that every time a website is updated (e.g., a journal puts out a new issue, someone has a new blog post, etc.) my RSS reader, that is, the vessel, fills up with the change. In this manner, I, that is, my technology and I, are passive recipients of mathematics education information. As I go about my day, driving to work, teaching classes, walking the dog and many other activities, my RSS reader, simply, ‘fills up’. Key here is that I am able to decide when I wish to open up my inbox for the internet. Also key, long gone are the days of me checking individual websites one at a time in order to actively seek out mathematics education information, which is a task I still complete in order to make my inbox for the internet continually better at its job. Yes, the initial set up took some time, but, for me, it was worth it. My heavy lifting, for the most part, is over. I am now able to sit back, relax and whenever I feel so inclined, I open up my inbox for the internet and in a matter of minutes am up to date with nearly everything mathematics education related.

After five years of this practice, I consider myself more of a *prosumer* of mathematics education information, that is, I have an active consumer role that is more involved in the process, than a consumer. Clearly, obtaining mathematics education information is required for a successful career as a faculty member at a research institution. However, as a *prosumer*, I made the decision (a few years ago) to curate the mathematics education information I obtain and, in doing so, share with others.

DIGITAL REPOSITORIES FOR MATHEMATICS EDUCATION SIGNALS

As mentioned, my inbox for the internet is, on a daily basis, continually filled and continually emptied. Described above are some of the particulars associated with the ‘filling’ of the vessel. In what follows, I detail some of the particulars associated with the ‘emptying’ of said vessel.

Every morning, when I check my inbox for the internet (usually at the same time I am checking my email), it is ‘full’ of mathematics education information. Emptying said inbox could be quite simple: just delete everything there and wait for it to fill up again, which I have done when things get a bit overwhelming. However, for the most part, I take a slightly different approach to emptying my RSS reader.

As I navigate through the ‘feeds’ in my RSS reader, I am looking for ‘signals’ amongst the ‘noise’. (Note: I do not imply, nor should you infer, a negative connotation to the term ‘noise’). Signals, or information that I have decided to curate on one of my many social media platforms, are determined by whether or not the information resonates with me. If it does, then I will curate that information. If not, then I will not curate the information. Further, if I deem information a signal, I then have the second added step of determining whether the signal is 'clear' or 'noisy'. Clear signals end up on *Tumblr* (matthewmadduxeducation.com), and noisy signals end up on *Twitter* (twitter.com/MatthewMaddux). As a result of this process, *Tumblr* has a better signal to noise ratio than Twitter.

This raises a harder question to answer: What mathematics education information is of interest or what resonates with me? Topics such as probability, popularization, math wars, that is, topics dear to me, are automatically curated. However, through the continual process of filling and emptying my RSS reader, I am further learning about which mathematics education topics are of interest or resonate with me. Arguably, I am learning about my mathematics education interests as I go through my inbox for the internet. In fact, as I continue to curate mathematics education 'signals', as I continually 'wade' through the mathematics education 'noise', I will, in the process, be creating a number of digital repositories which will paint a better picture for myself (and the world) of my mathematics education interests.

Amongst the different digital repositories I am responsible for, I have implemented a signal to noise ratio hierarchy. For example, all of my *Tumblr* posts are automatically 'pushed' to all of the other forms of social media I have adopted. In other words, all of my *Tumblr* posts show up on my *Twitter*, *Facebook*, *Google+* and *LinkedIn* accounts. This way, followers can 'pick their poison'. However, there are some tweets, that is, noisy signals, which only get pushed to *Facebook* and not up to *Tumblr*. From this hierarchical perspective, I am most 'active' (i.e., sheer number of posts) on *Twitter*, but my clearest signals are found on my *Tumblr* (which get pushed to *Twitter*, *Google+* and *LinkedIn*). Utilizing my hierarchy, I am able to choose (what I consider) some of the strongest signals that I have curated (e.g., numberphile.com) and share this information with those who may not (yet) be part of the social media landscape, but are involved with more traditional manners of sharing mathematics education information, such as a Topic Session at CMESG/GCEDM 2013. My hope is that sharing some of the strongest signals I have encountered may whet someone's social media appetite.

As you will have noticed, I have, throughout this article, pointed you 'outside' this article, which has been done on purpose. I have done so with the hopes that you venture, in some way, into the social media for mathematics education landscape. As mentioned, I have made slides for my Topic Session—which categorize the types of social media I discussed during my Topic Session (e.g., Discussion Forums, Listservs, Social Bookmarking, Media Sharing, Aggregators, Social Networks and Blogs/Microblogs)—at eganchernoff.com/conference-presentation-slides. Do not stop there—I encourage you to spend an afternoon finding out who is on *Twitter* or *Facebook* or *Google+* or any of the other platforms. I think you will be pleasantly surprised. Further, I think you will find that adopting social media for mathematics education will supplement your current wealth of mathematics education information. Remember to share. I have no doubt that you will find something that, someday, we can discuss at a future @CMESG 'Tweetup', which (before you start laughing) is a real thing and, actually, has many similarities to the famous CMESG/GCEDM pizza run. I look forward to it.

ACKNOWLEDGEMENTS

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LE JEU DE RÔLES DANS UN COURS DE DIDACTIQUE DES MATHÉMATIQUES: UN OUTIL POUR LA FORMATION OU UN OUTIL POUR LA RECHERCHE SUR LA FORMATION ?

ROLE-PLAY IN A MATHEMATICS METHODS COURSE: A TOOL FOR MATHEMATICS TEACHER EDUCATION OR A TOOL FOR RESEARCH ON MATHEMATICS TEACHER EDUCATION?

Caroline Lajoie
Université du Québec à Montréal

In one of our mathematics methods courses at Université du Québec à Montréal (UQAM), students, through 'role-play', become active actors in different teaching situations (involving teacher/pupil interactions) instead of simply imagining or analyzing such situations. Developed at first as a pedagogical approach in the course «didactique de l'arithmétique au primaire», 'role-play' now also provides us with an approach to research on mathematics teacher education.

This Topic Session was devoted to role-play. With the help of a few examples, I was able to stress some of our intentions at UQAM regarding mathematics teacher education at the primary level. Also, participants were invited to explore with me the potential and the limits of this approach for mathematics teacher education, as well as for research on mathematics teacher education.

♦♦♦♦♦♦♦♦♦♦

À l'Université du Québec à Montréal (UQAM), dans le cours « didactique de l'arithmétique au primaire », à travers des 'jeux de rôles', les futurs enseignants du primaire se glissent à chaque semaine dans la peau d'enseignants et d'élèves vivant une situation de classe donnée (impliquant une interaction enseignant/élève(s)) et agissent comme ils croient que ces personnes le feraient en situation réelle. Développés au départ comme approche de formation, les 'jeux de rôles' sont actuellement utilisés aussi comme outil de recherche sur la formation à l'enseignement des mathématiques.

Au cours de cette séance thématique, à travers quelques exemples, nous avons exploré ensemble cette approche. Ces exemples m'ont permis de mettre en lumière certaines intentions que nous avons à l'UQAM en regard de la formation à l'enseignement de *l'arithmétique au primaire. Ils nous ont permis aussi de discuter ensemble du potentiel et des limites que présente une telle approche, tant pour la formation à l'enseignement des mathématiques que pour la recherche sur celle-ci.*

CONTEXTE DE LA FORMATION DES MAÎTRES DU PRIMAIRE AU QUÉBEC, EN PARTICULIER À L'UQAM

Les didacticiens du département de mathématiques à l'Université du Québec à Montréal (UQAM) qui interviennent dans le programme de baccalauréat en éducation préscolaire et primaire (EPEP) le font principalement par le biais de trois cours obligatoires, soit les cours *Activité mathématique, Didactique de l'arithmétique au primaire* et *Didactique de la mesure et de la géométrie au primaire*. Le premier est un cours de mathématiques alors que les deux autres sont, comme leur titre l'indique, des cours de didactique.

Au début des années 2000, les universités québécoises, dont l'Université du Québec à Montréal (UQAM), ont dû introduire dans leurs programmes de formation des maîtres une approche par compétences dites *professionnelles*. Suivant un document ministériel intitulé *La formation à l'enseignement: les orientations, les compétences professionnelles* (MEQ, 2001), une compétence professionnelle se déploie en contexte professionnel réel; elle se situe sur un continuum qui va du simple au complexe; elle se fonde sur un ensemble de ressources; elle se situe dans l'ordre du savoir-mobiliser en contexte d'action professionnelle; elle se manifeste par un savoir-agir réussi, efficace, efficient, récurrent; elle est liée à une pratique intentionnelle et elle est un projet, une finalité sans fin. Aussi, toujours suivant le même document ministériel, la compétence professionnelle « exige que, dans le vif de l'action, la personne compétente sache interpréter les exigences et les contraintes de la situation, sache identifier les ressources disponibles » et les utiliser (pp. 57-58). Quelques unes des compétences attendues des futurs enseignants sont : concevoir et piloter des situations d'enseignement-apprentissage pour les contenus enseignés, évaluer la progression des apprentissages et le degré de maîtrise des compétences des élèves pour les contenus enseignés et adapter ses interventions aux besoins et aux caractéristiques des différentes clientèles (MEQ, 2001, p. 56).

Au sein de l'équipe de didacticiens des mathématiques au primaire alors en place au département de mathématiques de l'UQAM, un problème s'est rapidement posé : celui de trouver un contexte proche de celui de l'enseignement, et dans lequel les futurs enseignants du primaire développeraient des compétences professionnelles, des habiletés à faire face à l'imprévu en utilisant les ressources à leur portée (Lajoie & Pallascio, 2001). Comment espérer contribuer au développement de compétences professionnelles chez des futurs enseignants du primaire dans des salles de classes à l'université ? Les didacticiens intervenant dans le cours de didactique de l'arithmétique au primaire ont alors pensé plonger les étudiants dans un contexte *simulé* mais tout de même *réaliste* d'intervention en classe du primaire à travers les *jeux de rôles*.[1]

Depuis, au fil des ans et des formateurs, les jeux de rôles, de même que la manière de les mettre en place en classe, peuvent varier. L'approche qui sera décrite dans ce qui suit est celle que je mets en place dans mes classes. Le lecteur intéressé à en savoir plus sur des adaptations possibles des jeux de rôles pourra consulter Marchand, Adihou, Lajoie, Maheux, et Bisson (2012) et Lajoie, Maheux, Marchand, Adihou, & Bisson (2012).

[1] Les jeux de rôles avaient été intégrés dans le cours de didactique de l'arithmétique au primaire au milieu des années 1990, comme activité complémentaire au cours, dans le but d'exercer les futurs enseignants à enseigner. Au début des années 2000, ils ont été complètement remodelés et ils sont devenus l'activité principale du cours (Lajoie & Pallascio, 2001; Lajoie, 2010).

LE JEU DE RÔLES DANS LE COURS DE DIDACTIQUE DE L'ARITHMÉTIQUE AU PRIMAIRE

Le jeu de rôles est la mise en scène d'une situation problématique impliquant des personnages ayant un rôle donné. Le jeu de rôles peut être utilisé à des fins thérapeutiques, de formation personnelle, de formation professionnelle, ou encore comme méthode pédagogique (Mucchielli, 1983). L'idée derrière le jeu de rôles est que des personnes, par exemple des étudiants, doivent se glisser dans la peau de personnages plongés dans une situation donnée et agir exactement comme ils croient que ces personnages pourraient agir. L'objectif du jeu de rôles, lorsque utilisé dans l'enseignement, est d'amener les étudiants-acteurs, de même que tout le reste de la classe, à rapprendre quelque chose à propos des personnages eux-mêmes et/ou de la situation (van Ments, 1989).

Le cours de didactique de l'arithmétique s'articule autour d'une dizaine de jeux de rôles. Chaque jeu de rôles est structuré de la même manière. En tout premier lieu, les étudiants, placés en équipes, sont informés des principaux objectifs du jeu de rôles. Puis, une mise en situation est posée, laquelle présente une situation-problème de nature didactique (et non de nature mathématique) qui implique un ou des élève(s) de même qu'un enseignant (du primaire). Une fois la situation posée, toutes les équipes se préparent pour le jeu de rôles, en ne sachant pas à l'avance si un de ses membres devra jouer le rôle d'un élève ou d'un enseignant devant toute la classe. Par la suite, le professeur choisit les équipes qui devront envoyer une personne pour jouer le rôle d'un enseignant ou d'un élève, et fait en sorte que les différents acteurs proviennent d'équipes différentes, de manière à éviter que le jeu devienne un sketch où tous les acteurs sont arrangés entre eux (ce qui ne reflèterait aucunement le contexte d'une classe du primaire). Enfin, le jeu a lieu et un retour est fait sur la prestation de chacun, sur la situation, de même que sur les apprentissages réalisés par tous les étudiants grâce au jeu de rôles en question.

À travers l'ensemble des jeux de rôles, les étudiants sont appelés à développer des compétences professionnelles. Ils sont aussi appelés par le fait même à réfléchir aux contenus arithmétiques à être enseignés au primaire, à juger de la pertinence d'une situation d'enseignement-apprentissage face à l'enseignement d'un sujet mathématique donné et proposer des améliorations s'il y a lieu, à juger de la pertinence de certaines approches pédagogiques et de certains matériels didactiques face à l'enseignement d'un sujet mathématique donné et à proposer des améliorations s'il y a lieu, à analyser des productions d'élèves et élaborer des stratégies d'intervention qui tiennent compte de ces productions, à anticiper des réactions d'élèves dans une situation donnée et intervenir en respectant ces réactions, etc. Aussi, les étudiants sont amenés à travailler en collaboration, à prendre des décisions, à débattre leurs idées dans leurs équipes et devant toute la classe, à communiquer « mathématiquement », à faire face à l'imprévu, etc.

Chaque jeu de rôles se vit en quatre temps.

PREMIER TEMPS : MISE EN SITUATION

Les étudiants sont informés du jeu de rôles à l'étude et des intentions visées par celui-ci.

DEUXIÈME TEMPS : PRÉPARATION EN ÉQUIPES DE QUATRE

Les équipes se préparent, en ne sachant pas à l'avance lesquelles seront sollicitées pour le jeu devant toute la classe. Elles se soumettent à une activité liée à la situation. Généralement, les étudiants ont lu un ou des article(s) (portant sur un ou des concepts mathématiques, sur des conceptions d'élèves, sur des erreurs, etc.) avant la rencontre.

Pendant le travail de préparation dans les équipes, le professeur circule dans la classe, répond aux questions et en pose, pousse les étudiants à approfondir leur réflexion, etc. Par la suite, il choisit les équipes qui devront envoyer une personne pour jouer le rôle d'un enseignant ou d'un élève, et fait en sorte que les différents acteurs proviennent d'équipes différentes, de manière à éviter que le jeu devienne un sketch où tous les acteurs se seraient préalablement entendus sur le déroulement (ce qui ne reflèterait pas le contexte d'une classe du primaire).

TROISIÈME TEMPS : LE JEU DEVANT LA CLASSE

Un « enseignant » se présente devant la classe avec un ou des « élève(s) » et le jeu commence. S'amorce alors une réflexion dans l'action pour l'« enseignant », mais aussi pour les « élèves ». L'« enseignant » est particulièrement sollicité puisque, dans le feu de l'action, des réactions (hypothèses, questions, réponses) qu'il n'avait pas prévues de la part des « élèves » le forcent à prendre des décisions immédiates. Les « élèves » aussi sont sollicités puisqu'ils doivent en quelque sorte suivre l'« enseignant » (faire ce qu'il demande, répondre à ses questions, etc.) mais ils doivent aussi s'investir dans le jeu (par exemple en posant des questions, en formulant des commentaires, comme le feraient des élèves) tout en ne prenant pas la place de l'« enseignant ».

Pendant ce temps, une réflexion a lieu aussi chez les spectateurs, i.e. chez les autres étudiants : Que se passe-t-il ? Pourquoi l'« enseignant » aborde-t-il la situation de cette façon ? Pourquoi prend-il ces décisions ? Qu'aurais-je fait à sa place ? Qu'attend-il de ses « élèves » ? Le résultat sur lequel s'entendent finalement les acteurs est-il réellement le « bon » résultat ? etc.

QUATRIÈME TEMPS : RETOUR EN GRAND GROUPE

Un retour réflexif auquel participe l'ensemble de la classe, incluant le formateur, est fait sur la prestation de chacun, sur la situation, de même que sur les apprentissages réalisés par tous les étudiants grâce au jeu de rôles en question. Des commentaires sont alors formulés sur les interventions des différents acteurs, des comparaisons sont faites, mais surtout plusieurs questions sont posées et des réponses sont offertes.

<u>Premier exemple: La calculatrice comme source de questionnement et de réflexion</u>

Vos élèves de troisième cycle [10-12 ans] explorent avec l'aide de leur calculatrice personnelle [ils ne disposent pas tous de la même calculatrice] quelques problèmes mathématiques. Ce faisant, ils doivent effectuer un certain nombre de calculs. Or, ils n'obtiennent pas tous les mêmes résultats aux différents calculs, ce qui semble les choquer puisqu'ils affirment tous avoir utilisé correctement leur calculatrice!

Que se passe-t-il? Vous souhaitez amener vos élèves à répondre eux-mêmes à cette question!

Les calculs qui posent problème sont les suivants :

1. $2 \times 12 + 3 \text{ x } 10 = ?$
2. $123\,456 \times 456\,789$
3. $(4 \div 3) \times 3 = ?$
4. $500 – 8\% = ?$
5. $5\% + 2\% = ?$

Notes:

Chaque *enseignant* désigné aura quelques minutes pour traiter un de ces calculs avec trois élèves désignés.
Chaque *élève* désigné aura en mains sa calculatrice.

<u>Deuxième exemple : L'intervention face à des erreurs liées à des algorithmes de calculs</u>

Cette semaine, vous avez accepté de consacrer quelques heures de votre temps à l'aide aux devoirs dans votre école. Alors que vous vous promenez à travers les tables, vous remarquez que certains élèves ont commis quelques erreurs de calculs en voulant utiliser les algorithmes traditionnels. Vous souhaitez les aider à ne plus commettre ces erreurs et vous êtes bien entendu soucieux de ne pas régler les problèmes en surface seulement, mais plutôt en profondeur.

Chaque *enseignant* désigné aura quelques minutes pour identifier une erreur commise par un *élève* au tableau, pour identifier son raisonnement et pour débuter son intervention (en partant de l'erreur et du raisonnement de l'élève et non en partant à neuf!). Il est possible que les erreurs traitées pour le jeu de rôles soient différentes de celles traitées dans les équipes.

LE JEU DE RÔLES COMME APPROCHE DE RECHERCHE SUR LA FORMATION À L'ENSEIGNEMENT DES MATHÉMATIQUES AU PRIMAIRE

Lorsque le jeu de rôles a été intégré au cours de didactique de l'arithmétique au primaire à l'UQAM au milieu des années 1990, il était utilisé à des fins de formation seulement (Lajoie, 2010). Petit à petit, il nous est apparu aussi comme un dispositif original pour la recherche sur la formation à l'enseignement des mathématiques. Le contexte dans lequel il plonge les futurs enseignants, sans être un contexte réel d'enseignement, ni même celui de la formation pratique (les stages), s'en rapproche suffisamment pour permettre au chercheur d'observer les futurs enseignants alors qu'ils :

- collaborent entre pairs à préparer des interventions et à anticiper des scénarios d'interactions;
- manipulent certains concepts mathématiques en vue de les enseigner;
- sont en pleine interactions élèves/enseignant;
- identifient et utilisent les ressources disponibles pour la préparation et pour l'enseignement;
- reviennent sur les interventions réalisées en temps réels par leurs pairs; etc.

Le lecteur intéressé à en savoir plus sur les travaux de recherche en cours en lien avec les jeux de rôles pourra consulter Lajoie et Maheux (2013) et GREFEM (en préparation).

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New PhD Reports

Présentations de thèses de doctorat

LE NOUVEAU STATUT DES GRANDEURS DANS LE PROGRAMME DES MATHÉMATIQUES EN FRANCE ET SES RÉPERCUSSIONS SUR LES PRATIQUES ENSEIGNANTES

THE NEW STATUS OF MAGNITUDES IN THE CURRICULA OF MATHEMATICS IN FRANCE AND ITS IMPACT ON TEACHING PRACTICES

Nathalie Anwandter-Cuellar
Université du Québec en Outaouais

The epistemological upheavals caused by the evolution of the field of mathematics produced a series of changes in the school curricula in France, one of particular interest being in the teaching of magnitudes. We wanted to analyse the place and role of magnitudes in the 2005 restructure of the curricula and to see the impact of these changes on the teaching conditions and the restrictions teachers must face when teaching magnitudes. In order to do this, we conducted a study, based on the Anthropological Theory of the Didactic *developed by Yves Chevallard. We examined teaching practices by looking at the inter-relationships between magnitudes, the functional and the numeric for the case of proportionality, and the internal functioning of magnitudes for the notion of area. Our results show how the new status of magnitudes in official documents creates difficulties for teachers to integrate the new curricula knowledge for an adequate teaching of magnitudes.*

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Suite aux bouleversements épistémologiques provoqués par l'évolution des mathématiques, le programme scolaire du collège[1] *en France a connu différents changements relatifs à l'enseignement des grandeurs. Je me suis questionnée sur la place et le rôle des grandeurs en analysant la récente structuration du programme de 2005 comme génératrice des nouvelles conditions et contraintes auxquelles les enseignants doivent faire face. De ce fait, j'ai réalisé une recherche dans le cadre de la* Théorie Anthropologique du Didactique *développée par Yves Chevallard. Plus particulièrement, j'ai réalisé une étude clinique des pratiques enseignantes en regardant les interrelations entre les grandeurs, le fonctionnel et le numérique pour le cas de la proportionnalité, et le fonctionnement interne des grandeurs pour la notion d'aire. Elle révèle que le nouveau statut des grandeurs dans les documents officiels*

[1] École secondaire au Québec.

crée chez les professeurs des difficultés au niveau de l'intégration des nouveaux savoirs curriculaires pour un enseignement adéquat des grandeurs.

INTRODUCTION

À une certaine époque, les constructions des nombres étaient basées sur les grandeurs; elles font l'objet du livre V d'Euclide et des nombreux travaux de mathématiciens ultérieurs, tels que Descartes ou Stevin. Cependant, depuis la fin du XIX^e siècle, on assiste, dans l'histoire des mathématiques, à un renouvellement de la pensée mathématique. On voit dans cette discipline le modèle de toute connaissance scientifique qui s'appuie sur l'étude des structures et sur le langage mathématique formel. Ainsi, avec ce courant formaliste, les grandeurs disparaissent des mathématiques, elles ont été envoyées au domaine de la physique et les constructions mathématiques des nombres réels vont s'appuyer sur l'ensemble des entiers naturels ou sur le corps de nombres rationnels.

La rupture entre grandeurs et nombres apparue à la suite de cette réforme scientifique a eu des répercussions dans l'enseignement d'aujourd'hui : le statut et la place accordés aux grandeurs dans l'enseignement secondaire restent très ambigus (Chevallard & Bosch, 2002). En effet, en 1970, on observe une rupture entre le numérique et les grandeurs, ce qui semble avoir réduit leur place dans les différents domaines. Quelques années après, on retrouve le retour des grandeurs dans les programmes de 1995. Cette présence devient plus insistante quand ces documents institutionnels les placent au même niveau que les domaines des nombres, des fonctions et de la géométrie dans les programmes de 2005.

Dans mon travail de thèse (Anwandter-Cuellar, 2012), je me suis proposé d'étudier la place et le rôle des grandeurs dans la construction de différents domaines mathématiques au niveau de leurs interrelations au collège, ainsi que la nouvelle constitution d'un domaine des grandeurs. L'objectif était d'analyser les pratiques enseignantes relativement à ces objets en prenant en compte les contraintes institutionnelles qui pèsent sur leur enseignement et d'étudier les connaissances actuelles des élèves. La méthodologie utilisée était du type clinique et s'insère dans les travaux développés par Larguier (2009, 2012).

Dans ce texte, je présenterai quelques résultats concernant la vie des grandeurs en tant que domaine et leurs interrelations avec d'autres domaines dans les programmes et les pratiques de deux enseignants.

OUTILS THÉORIQUES ET MÉTHODOLOGIQUES

Le cadre théorique global dans lequel je me suis placée est constitué des éléments de la *Théorie Anthropologique du Didactique* (TAD) développée par Chevallard (1992, 1999). De plus, je me suis inspirée des travaux de Bronner (1997, 2007) sur les domaines du numérique et de l'algébrique pour créer le *filtre des grandeurs* en identifiant différents composants.

LES ORGANISATIONS MATHÉMATIQUES ET LES NIVEAUX DE CODÉTERMINATION

Une organisation mathématique est un quadruplet T, τ, θ, Θ : T qui désigne un type de tâches (formé d'un ensemble de tâches spécifiques); τ désignant une technique que l'on peut appliquer pour la réalisation des tâches appartenant à T; θ une technologie, un discours qui justifie l'adéquation de la technique à la réalisation des tâches de T. Enfin Θ désigne une théorie qui elle peut servir à justifier le discours technologique. En reprenant cette notion et la classification des situations-problèmes relatives au concept d'aire des surfaces planes proposée par Moreira Baltar (1995), j'ai proposé un premier volet de genre de tâches relatives aux grandeurs pour le secondaire (et leurs techniques respectives) : comparer des grandeurs, calculer une grandeur, étudier les effets des déformations et des transformations géométriques et numériques sur l'une des grandeurs d'un objet, produire un objet d'une grandeur donnée, produire un objet de grandeur plus grande ou plus petite que la grandeur d'un objet donné, donner la mesure d'une grandeur dans une autre unité, mesurer une grandeur.

Une organisation mathématique, telle qui est décrite auparavant est appelée praxéologie ponctuelle, et elle est rencontrée rarement de manière isolée (Chevallard, 2002). Pour modéliser le questionnement de l'existence d'organisations mathématiques, Chevallard élargit le cadre en intégrant ce qu'il appelle les niveaux de codétermination didactique (Chevallard, 2002). À l'intérieur de la discipline des mathématiques, à chaque organisation mathématique ponctuelle lui correspond un sujet d'étude relatif au type de tâches dans l'enseignement. Ce type de tâches fait partie des tâches prescrites dans un thème d'étude, auquel lui correspond une organisation mathématique locale formée des organisations mathématiques ponctuelles ayant même technologie. Cette organisation mathématique est à la fois partie d'une organisation mathématique régionale, un secteur d'étude, qui est l'amalgamation des organisations locales ayant la même théorie. On trouve comme dernier niveau une organisation mathématique globale relative à un domaine d'étude. Les domaines se regroupent autour d'une discipline, dans ce cas, les mathématiques.

LES OBJETS

Je présente ici les objets de l'univers de la mesure identifiés par G. et N. Brousseau (1992) :

a. Le système d'objets S(Oi) est constitué par des objets Oi mesurable de deux types : les objets (ostensifs) concrets (un ballon, une bouteille) et les objets (non-ostensifs) géométriques (un triangle, un cylindre);
b. La grandeur est un ensemble de propriétés communes à plusieurs espèces de grandeurs particulières. Chaque espèce de grandeur est déterminée par l'ensemble d'objets mesurables pour laquelle il existe une propriété d'addition et une relation d'équivalence entre les objets par rapport au type de grandeur;
c. La valeur particulière est assignée à chaque objet sans tenir compte du système utilisé pour la quantifier. La valeur particulière d'un objet Oi est la classe d'équivalence que définit un type de grandeur sur le système d'objets;
d. Les fonctions-mesure sont des applications additives de l'ensemble d'objets dans l'ensemble des nombres réels positifs. À chaque unité correspond une fonction-mesure différente relative au même (type de) grandeur;
e. La valeur d'une fonction-mesure ou l'image d'une fonction-mesure est le nombre réel positif qui corresponde à la mesure à chaque objet Oi;
f. La mesure ou nombre concret est le couple formé par l'image et l'unité de mesure.

LES DIMENSIONS OUTIL ET OBJET DES GRANDEURS

À partir de la distinction générale introduite par Régine Douady (1986) entre la dialectique outil/objet pour les concepts mathématiques, j'ai proposée une organisation du savoir de grandeurs autour de deux dimensions principales :

- Premièrement les grandeurs ont une dimension outil, car elles servent de moyens pour résoudre des problèmes émergeant de plusieurs contextes : la vie quotidienne, les mathématiques et des disciplines autres que les mathématiques.
- Deuxièmement les grandeurs sont un objet d'étude pour les mathématiques, car elles forment un ensemble structuré autour d'objets, d'opérations et de comparateurs dotés de propriétés, de modes de traitement (règles algébriques, découpage-recollement, …), de modes de représentations permettant ces traitements (figures, nombres concrets, graphiques, …).

LES DYNAMIQUES INTERNE ET INTER-DOMAINES

Dans mon filtre j'ai défini trois dynamiques particulières :

Une dynamique interne au domaine des grandeurs

On peut étudier les relations entre les objets en restant dans le domaine des grandeurs. Par exemple, les activités relatives aux grandeurs géométriques peuvent être liées, en regardant l'aire comme une grandeur produit. Le principe est que l'aire d'un rectangle est proportionnelle à chacune de ses dimensions et ainsi, on peut définir l'aire du rectangle comme produit de sa longueur par sa largeur, noté $L \times l$.

Les dynamiques inter-domaines

On trouve trois domaines clairement en relation avec celui des grandeurs au secondaire en France : les fonctions à travers la proportionnalité, la géométrie avec l'étude de figures et le numérique à travers le calcul sur les nombres.

Une dynamique extra-mathématique

Aujourd'hui, le citoyen doit faire face à des grandeurs plus complexes en réponse à une évolution socio-économique, comme le signale le document ressource « Grandeurs et mesures » (DGESCO, 2007). Les grandeurs font le lien entre les mathématiques et le monde réel et les mathématiques de la vie quotidienne s'intéressent à l'étude des grandeurs fondamentales.

L'ÉVOLUTION DES PROGRAMMES DEPUIS 1995

Dans ma thèse, j'ai étudié les grandeurs dans les programmes de 1995 à 2005 en fonction des différents niveaux de codétermination didactique. J'ai analysé des organisations mathématiques qui englobent les genres de tâches relatives aux grandeurs que j'ai identifiées avec le filtre des grandeurs.

Dans la période 1995-2005 qui précède les programmes parus en 2005, il existe dans l'enseignement des mathématiques au collège trois domaines d'étude, clairement exposés par les programmes (CNDP, 1996; 1997; 1998) : travaux géométriques, travaux numériques, organisations de données, fonctions. Les grandeurs sont présentes explicitement dans deux de ces domaines : « Travaux géométriques » et « Organisation de données, fonctions ». Pour comprendre l'évolution des programmes, je m'appuierai sur quelques résultats concernant la place et le rôle de l'aire dans le domaine de la géométrie au secondaire. Alors, regardons pour la classe de 5^{e} du secondaire l'enseignement de l'aire en termes de niveaux de codétermination (Tableau 1, ci-dessous).

En général, dans les programmes de la période 1995-2005, les grandeurs sont présentes au niveau du thème d'étude dans le domaine géométrique et dans celui des fonctions. Du point de vue des organisations mathématiques, elles vont constituer des organisations locales. Cette structuration des contenus en termes de niveaux de codétermination révèle un positionnement des grandeurs qui met en avant l'aspect outil des grandeurs. En effet, les grandeurs servent à mettre en place des organisations mathématiques relatives à d'autres notions appartenant à différents domaines. Dans notre exemple, les aires sont un outil pour l'étude des figures géométriques comme le triangle et le disque, et un exemple de situation d'application pour l'étude de la proportionnalité.

Domaine	Secteurs	Thèmes	Sujets d'étude
Travaux géométriques	Prismes, cylindres de révolution	Aires	Calculer l'aire latérale d'un prisme droit.
			Calculer l'aire latérale d'un cylindre de révolution
	Parallélogramme	Aires	Calculer l'aire d'un parallélogramme
	Triangle	Aires	Calculer l'aire d'un triangle
	Disque	Aires	Calculer l'aire d'un disque de rayon donné
Organisation de données, fonctions	Exemples des fonctions	Proportionnalité	Effectuer de changements d'unités

Tableau 1. Structuration en termes de niveaux de codétermination en 5e pour la grandeur aire dans la période 1995-2005.

À partir de 2005, le programme (Ministère de l'Éducation Nationale, 2008) divise l'étude en 4 domaines : organisation et gestion de données, fonctions, nombres et calculs, géométrie et grandeurs et mesures. De plus, un document ressource *Grandeurs et mesures* (DGESCO, 2007) définit une théorie et des technologies d'un possible domaine des grandeurs.

En comparaison avec les programmes de la période 1995-2005, à partir de 2005, on trouve deux nouvelles caractéristiques dans l'enseignement des grandeurs. D'une part, on voit dans l'enseignement au collège l'apparition du domaine d'étude « Grandeurs et mesures ». D'autre part, les grandeurs géométriques enseignées sont les mêmes que dans la période 1995-2005, néanmoins on ajoute à cela un travail sur les grandeurs quotients et produits. De nouveaux objets viendront s'établir, mais surtout une nouvelle place est née pour les grandeurs. Pour comprendre ces changements, on va reprendre l'exemple de la grandeur aire en classe de 5e à l'aide du Tableau 2 :

Domaine	Secteurs	Thèmes	Sujets d'étude
Grandeurs et mesures	Aires : mesure, comparaison et calcul d'aires	Comparer des aires	Comparer géométriquement des aires
		Déterminer une aire	Déterminer l'aire d'une surface
		Différencier périmètre et aire	Différencier périmètre et aire
		Calcul d'aires	Calculer l'aire d'un rectangle dont les dimensions sont données
			Calculer l'aire d'un triangle rectangle
			Calculer l'aire **d'un triangle quelconque dont une hauteur est tracée*
			Connaître et utiliser la formule donnant l'aire d'un disque
		Changements d'unités d'aire	Effectuer pour les aires des changements d'unités de mesure

Tableau 2. Niveaux de codétermination pour les aires en 5e dans la période après 2005.

Premièrement, on peut observer que les espèces de grandeurs deviennent des secteurs d'étude du domaine « Grandeurs et mesures » et ils sont structurés selon des genres de tâches que j'ai identifiés dans mon filtre : comparer des grandeurs, calculer des grandeurs et changer d'unités. Chaque grandeur géométrique est ainsi un secteur d'étude où devrait se constituer une organisation mathématique régionale autour de cette grandeur, alors les grandeurs deviennent des objets d'étude et non seulement des outils. Un deuxième aspect très important est la réunion des genres de tâches relatifs aux grandeurs dans un seul domaine. Si dans la période 1995-2005 les genres de tâches étaient répartis dans les domaines « Travaux géométriques » et « Organisation de données, fonctions », ils sont rassemblés dans la période actuelle.

Bref, l'étude des programmes scolaires a révélé que ce qui se présente de façon fragmentée dans la période 1995-2005 est réuni en formant tout un domaine à partir de 2005. Ainsi les grandeurs deviennent dans la deuxième période un domaine central du travail pour l'enseignement au collège. Leur statut change, elles seront davantage considérées en tant qu'objet plutôt qu'en tant qu'outil, même si elles continuent à servir de support à l'enseignement d'autres notions.

Deuxièmement, dans cette restructuration, on retrouve une nouvelle hiérarchie des grandeurs que j'ai analysée avec l'échelle des niveaux de codétermination didactique. Les grandeurs forment un domaine d'étude et à chacune des grandeurs correspond un secteur d'étude. Ce statut donné aux grandeurs va déterminer leur fonctionnement dans le système d'enseignement.

La montée des grandeurs dans les niveaux de codétermination et l'apparition de nouveaux types de grandeurs engendrent de nouvelles conditions et contraintes, au niveau des technologies et théories, auxquelles les enseignants doivent faire face, ce qui aura des répercussions au niveau de l'apprentissage des élèves.

LES PRATIQUES DES ENSEIGNANTS : LE CAS DE L'AIRE

Dans cette partie, je me centre sur l'aspect objet (Douady, 1986) des grandeurs en étudiant une grandeur spécifique, l'aire. Je présente quelques résultats relatifs à l'étude des pratiques dans une classe de 6^{e} et une classe de 5^{e} chez un même enseignant, le professeur Sylvain.

UNE DYNAMIQUE INTERNE AUX GRANDEURS ET UNE DYNAMIQUE INTER-DOMAINES

J'ai considéré une relation d'équivalence « avoir même aire » sur un ensemble d'objets, les surfaces. J'ai établi ainsi un lien entre le cadre géométrique et celui des grandeurs. Du point de vue numérique, on choisit une unité d'aire pour mesurer les aires des surfaces. La grandeur aire est ainsi au cœur d'une dynamique géométrique-grandeur-numérique que je schématise en m'inspirant des travaux de Moreira Baltar (1999) :

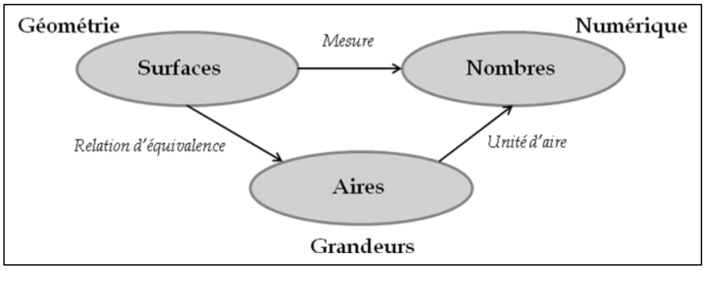

Figure 1. Dynamique inter-domaines autour de l'espèce de grandeur aire.

Dans cette dynamique la différenciation entre l'objet, la grandeur et la mesure n'a pas été véritablement prise en charge par le professeur Sylvain. Le passage vers le numérique est réalisé à travers le dénombrement en classe de 6e, et en classe de 5e, le lien entre le numérique et le géométrique-grandeurs se construit avec la mesure, mais il existe un amalgame entre l'objet et la grandeur. Par exemple, des tâches concernant la connaissance, « deux figures de formes différentes peuvent avoir même aire », ne sont jamais proposées. Je résume la dynamique établie dans les classes dans la figure suivante :

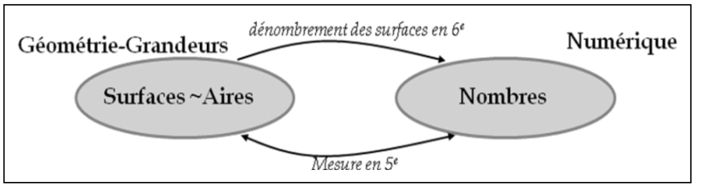

Figure 2. Schéma de la dynamique mise en place par Sylvain.

LA PLACE ET LE TRAITEMENT DES UNITÉS

Pour résoudre les exercices, l'enseignant Sylvain accepte plusieurs représentations des unités comme : « □ », « unités », « cm^2 » ou des réponses et des traitements sans unités.

De plus, les analyses que j'ai menées m'ont également permis de montrer que le traitement et la place des unités restent confus dans les enseignements du professeur Sylvain. Les résultats m'ont permis de repérer un manque de savoir pour le professeur (Cirade, 2008a, 2008b) au niveau technologique et théorique pour enseigner les grandeurs en tant que domaine d'étude chez cet enseignant. Effectivement, l'incorporation de l'étude des aires en tant que grandeur de manière à faire vivre le domaine grandeurs et mesures demande la construction de nouvelles organisations mathématiques permettant de le faire (Anwandter-Cuellar, 2012). De fait, le seul moyen de faire vivre les aires en tant que grandeur est de proposer une organisation régionale construite autour des problèmes pour différencier, d'une part, la notion d'aire de celle de surface et, d'autre part, la notion d'aire de la notion de mesure. Le besoin de technologies pertinentes au niveau du traitement de l'aire et au niveau du traitement des unités explique la difficulté à faire vivre cette notion en tant que grandeur dans les classes de 5e du professeur Sylvain.

LES PRATIQUES DES ENSEIGNANTS : LA PROPORTIONNALITÉ

Hersant (2001) et Comin (2002) identifient deux théories sur lesquelles peut s'appuyer l'enseignement de la proportionnalité : la théorie des proportions et la théorie relative à la fonction linéaire. Comin signale que, dans l'enseignement, l'étude de la proportionnalité à l'aide des proportions prend place dans le cadre arithmétique et la fonction linéaire dans le cadre algébrique. Le programme de la période actuelle (2005-…) met en avant la fonction linéaire comme outil implicite pour l'étude des situations de proportionnalité. Dans cette partie, je montrerai comme ces deux théories apparaissent dans les pratiques des enseignants observés, Sylvain et Marc.

L'ENSEIGNEMENT DE MARC

L'étude des organisations, mathématique et didactique, m'a aidé à situer l'enseignement du professeur Marc d'une manière générale. Après l'analyse de la pratique de cet enseignant, j'ai pu observer que le secteur « proportionnalité » est divisé en deux chapitres dans l'enseignement de Marc. Le premier chapitre est consacré aux notions de proportionnalité et pourcentage, et le deuxième à la notion d'échelle. En regardant l'organisation didactique régionale et l'organisation mathématique régionale, on constate que le professeur introduit la notion de proportionnalité en proposant des situations qui mettent en relation deux grandeurs, et que, ces situations sont étudiées à l'aide d'une technologie relative à la notion de fonction linéaire. Une fois que le coefficient de proportionnalité est institutionnalisé, il devient l'élément technologique principal des techniques utilisées pour résoudre les problèmes relatifs aux pourcentages et aux échelles. Ces problèmes sont, en général, étudiés dans le cadre des grandeurs, en utilisant les unités et des relations entre les grandeurs dans les situations de proportionnalité. Par exemple, il s'agit toujours de calculer le pourcentage d'une grandeur (ex. 30% de 40$) et non d'un nombre (ex. 30% de 40). L'enseignement du professeur Marc traverse ainsi les domaines des grandeurs, du fonctionnel et du numérique. De plus, le professeur Marc détermine une dynamique qui met en place des raisonnements sur les grandeurs facilitant la compréhension des élèves. Il semble que l'utilisation des propriétés de la fonction linéaire favorise l'apprentissage de la proportionnalité quand elles servent d'appui aux raisonnements dans le cadre des grandeurs. Par exemple, on trouve chez les élèves des raisonnements du type « si j'achète le double de bonbons je paierai le double d'argent ».

L'ENSEIGNEMENT DE SYLVAIN

L'enseignant Sylvain insère l'enseignement de la proportionnalité dans le secteur d'étude « quotients et applications ». Il apparaît que l'organisation mathématique locale relative à la proportionnalité se présente de manière désarticulée dans le secteur « quotients et applications », car l'enseignant fait le passage des calculs sur les nombres aux situations de proportionnalité mettant en lien deux grandeurs. En effet, la progression du professeur Sylvain commence par l'étude des quotients et des rapports dans le cadre du numérique. Le professeur Sylvain veut utiliser les éléments technologiques de la théorie des quotients pour enseigner cette notion. Cependant, selon les programmes du collège, l'enseignement de la proportionnalité s'appuie sur l'étude des situations mettant en jeu deux grandeurs proportionnelles. Ce travail doit mettre en avant les propriétés de linéarité qui préparent à l'enseignement de la fonction linéaire en classe de 3^{e}. Ainsi, l'enseignant Sylvain propose de nouveaux éléments technologiques, comme la propriété multiplicative, pour enseigner la proportionnalité en utilisant comme représentation principale le tableau de proportionnalité, où les raisonnements sur les grandeurs sont peu présents. Cela peut s'expliquer par la volonté du professeur Sylvain d'investir les connaissances sur les quotients et les rapports dans les situations de proportionnalité. La conception de l'enseignement de la proportionnalité dans le cadre des quotients mise avant par le professeur Sylvain est confrontée à une contrainte institutionnelle, celle d'étudier cette notion dans le cadre des fonctions. Ainsi des raisonnements à l'aide des propriétés de la linéarité dans le cadre des grandeurs sont négligés au profit de l'étude des relations numériques dans un tableau de proportionnalité, comme le produit en croix. Dans la pratique de l'enseignant Sylvain, la notion de proportionnalité en tant que fonction linéaire rencontre les anciens éléments théoriques relatifs aux proportions numériques. Lorsque les propriétés de linéarité sont traitées seulement dans le cas particulier d'un tableau de proportionnalité, elles peuvent réduire l'étude des situations de proportionnalité à un travail sur des relations numériques :

> *La disparition des grandeurs et, subséquemment, des rapports comme objets d'enseignement en mathématiques réduit la proportionnalité à l'étude de relations numériques et rend difficiles les explications qui permettent de distinguer la nature des nombres et leurs fonctions dans différentes situations.* (Comin, 2002, p. 146)

Il apparaît ainsi qu'en classe de 6^{e} les grandeurs peuvent donner du sens aux objets et connaissances relatifs à la proportionnalité, mais des difficultés relatives à l'enseignement de la proportionnalité apparaissent dans les interrelations entre les cadres fonctionnel, grandeurs et numérique. Les éléments technologiques et théoriques associés à ces différents cadres peuvent se présenter de manière désarticulée, ce qui provoque une réduction de la place des grandeurs et un traitement inadéquat des grandeurs mesurées, comme le signale Comin :

> *La coexistence géographique de plusieurs cultures conduit à une hétérogénéité des pratiques de résolution : la coexistence épistémologique de différentes organisations mathématiques semble constituer un obstacle à l'acquisition des connaissances sur la proportionnalité.* (Comin, 2002, p. 140)

CONCLUSION

Dans cette communication, j'ai montré que les contraintes apparues avec la mise en place d'un nouveau programme en 2005 engendrent de nouveaux besoins au niveau des organisations mathématiques et didactiques pour l'étude de ce domaine.

L'analyse des pratiques des enseignants révèle que la création d'un domaine des grandeurs provoque une évolution des technologies et théories entre les périodes étudiées. Par exemple, dans le cas de mon étude de la grandeur aire, j'ai observé au niveau technologico-théorique que la différenciation entre grandeur-mesure-objet n'est pas abordée et le traitement et la place des unités restent confus dans les classes des enseignants observés. Ceci s'explique par le fait que l'introduction des grandeurs dans le système d'enseignement en tant que domaine d'étude nécessite l'intégration des nouveaux savoirs dans les pratiques enseignantes. Par ailleurs, pour la proportionnalité, l'enseignement mis en avant par l'enseignant Sylvain dans le cadre des proportions est confronté à une contrainte institutionnelle, celle d'étudier la proportionnalité comme la relation entre deux grandeurs dans le cadre des fonctions. Ceci peut provoquer une désarticulation des organisations mathématiques au niveau de la construction de ces différents domaines mathématiques à l'aide des grandeurs (Anwandter-Cuellar, 2012).

J'ai ainsi montré que les difficultés de l'enseignement des grandeurs se trouvent notamment dans les éléments technologiques et théoriques relatifs à ces notions. Les professeurs présentent un manque technologique et théorique pour l'enseignement adéquat des grandeurs au collège, cela se traduit par une grande diversité de choix didactiques relativement aux grandeurs. Effectivement, incorporer l'étude des grandeurs en tant que domaine demande la construction de nouvelles organisations mathématiques ainsi que l'articulation entre les anciens et nouveaux savoirs (Anwandter-Cuellar, 2012). Cependant, l'institution a ignoré les savoirs existants ce qui a entrainé une absence de cohérence globale dans les pratiques vis-à-vis du programme en vigueur pour donner du sens à l'enseignement des grandeurs au secondaire.

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MATHEMATICS COACHING TO IMPROVE TEACHING PRACTICE: THE EXPERIENCES OF MATHEMATICS TEACHERS AND COACHES

Priscilla Bengo
Ontario Institute for Studies in Education, University of Toronto

The study explores the relationship between teacher emotions during mathematics educational reform, teacher learning, and support from a mathematics instructional coach. Using a case study approach, it shows that: a) mathematics reforms produce negative and positive emotions; b) the emotions are a result of not knowing how to implement the mathematics reforms, beliefs about teaching and learning mathematics, the nature of coaching, gains in student achievement and engagement, and positive in-school factors; c) coaching may not help teachers build their professional self-understanding when it fails to address their self-image issues; d) teacher learning or the correct use of reform strategies can occur even when teacher beliefs are inconsistent with reform initiatives; and e) reform strategies are modified by teachers even with the support of a coach. Coaches experienced positive and negative emotions based on how well the reforms were implemented by teachers. As a result, they require support during reforms. The directions for future research are described.

INTRODUCTION

There exists research that notes that teacher coaching leads to improvements in mathematics instruction and student achievement during mathematics reforms (e.g., Clarke, Thomas, & Vidakovic, 2009; Driscoll, 2008). The research has studies that acknowledge the emotions that are evoked during reform initiatives and shows that coaching can help the teacher implement the reforms (e.g., Clarke et al., 2009). The emotions that have been considered in mathematics studies have been categorized as positive and negative. For example, positive emotions were connected to improvements in student achievement. Negative emotions may occur when reform efforts challenge a teacher's role, identity or professional understanding (e.g., Cross & Hong, 2009). Teachers may hinder reform implementation if the negative emotions are not addressed (e.g., Kelchtermans, 2005). Some of these studies show that coaching support can help change negative emotions to positive ones (e.g., Driscoll, 2008). The importance of studying teachers' emotions and their connection to professional understanding is because mathematics reform aims to improve instruction and student achievement. This paper contributes to the research on teacher emotions by promoting the awareness of the impact of emotions on teachers' work. The research questions are:

1. What are secondary school mathematics teachers' specific emotions during mathematics education reform initiatives?
2. What factors are associated with the emotions that teachers experience?

3. What factors facilitate teacher learning during mathematics education reform given these emotions?
4. How does coaching help secondary school mathematics teachers learn during mathematics education reform?

REVIEW OF LITERATURE

The review outlines work on the relationship between coaching, teacher learning and the emotions that result from mathematics reform initiatives: it is based on these themes. The review shows the factors that affect the effectiveness of coaching during mathematics reform and the relationship between beliefs and emotions.

MATHEMATICS REFORM

Mathematics reform is educational change. As a result, it has behavioural, emotional and value-based components (Hoffman, 2010). Behavioural changes are the most challenging changes to implement (e.g., Hoffman, 2010). The 2003-2004 school year was the first year for the new Grade 12 courses in Ontario. This revision stressed the use of technology in courses. In 2005, new mathematics curricula were implemented for Grade 9 and 10. Their most striking features were the integration of technology into courses and enabling students to solve problems in real life situations. They were based on a constructivist approach to teaching and included a focus on conceptual understanding. However, procedural knowledge was also important. The new documents suggested that the new instructional practices a teacher had to learn focused on mathematical processes and included more literacy. Teachers also had to learn new assessment and evaluation policies to support the new instructional practices. In 2007, Grade 11 and 12 mathematics curricula were implemented. They were based on the same principles as the Grade 9 and 10 curricula and were seen as a continuation of them.

Studies have indicated that the reforms have not been implemented as intended (e.g., Goldin, Rösken, & Törner, 2009; Kajander, Zuke, & Walton, 2008). Many have noted that the visions of the reform imply great challenges for teachers (e.g., Manouchehri, 2003) such as developing proficiency in mathematical content and requiring pedagogical content knowledge to implement the reforms. Manouchehri and Goodman (1998) outlined other factors hindering reform initiatives in mathematics, such as lack of time for sufficient planning for teachers. Teachers' lack of necessary characteristics to support innovative teaching has also been noted as a reform effort's challenge (Manouchehri, 2003). The necessary characteristics follow:

> (1) The teachers were confident in their ability to control student learning and possessed a detailed vision of the type of teaching that could advance student learning.
> (2) They held strong philosophical views on the role of education in general and of mathematics in particular as agents for social change.
> (3) They assumed teaching as a moral and ethical act and themselves as change agents.
> (4) They perceived teaching as a learning process and were reflective about their practice.
> (5) They expressed strong respect for children's thinking and believed in students' ability to achieve in the presence of innovative instruction. (p. 78)

The above research gives reasons why the reform strategies may not be used by teachers as intended.

TEACHERS' MATHEMATICS BELIEFS

Goldin et al. (2009) argue that beliefs, like emotions, are elements of the affective domain and that beliefs and the stability of beliefs are frequently impacted by affective factors. Therefore, when discussing beliefs, emotions, attitudes and values must also be included. The affective domain consists of beliefs, attitudes, values and emotions. In addition, it is important to consider not only how beliefs interact with emotional feelings but also "how the person feels about having those feelings" (p. 12). Beliefs are connected to objects of belief and the objects "can be domain specific, and can be personal, social or epistemological in nature" (p. 3). They can be thought of as groupings of mental states, "are highly subjective" and have various functions. Beliefs are difficult to change.

MATHEMATICS TEACHERS' EMOTIONS

The role of emotions in mathematics reform initiatives can be framed using research on teacher emotions from studies of teaching and learning in mathematics (e.g., Clarke et al., 2009; Cross & Hong, 2009), from general studies on teaching and learning, and studies in other subject areas (e.g., Darby, 2008; Kelchtermans, 2005; Leithwood & Beatty, 2008; Schmidt & Datnow, 2005; van Veen, Sleegers, & van de Ven, 2005). Emotions are a dimension of teaching (e.g., Kelchtermans, 2005). Kelchtermans (2005) argues that "teachers' emotions have to be understood in relation to the vulnerability that constitutes a structural condition of the teaching job" (p. 995) and self-understanding using narrative biographical work on teacher development and the micro political analysis of changes in schools and teaching. Self-understanding consists of self-image, job motivation (the reasons for joining and not leaving the profession), future perspective (future expectations), self-esteem (how well a teacher thinks they are doing their job), and task perception (the everyday tasks). Emotions are defined, therefore, as more than psychological, intrapersonal phenomena. The research suggests that, if teachers do not agree with reforms, they may hinder their implementation.

The research has identified negative and positive emotions and the causes of the emotions. The negative emotions are fear, intimidation, terror, anger, anxiety, guilt, shame, loss of self-esteem, reduction in positive emotions and loss of harmony (e.g., Darby, 2008; Cross & Hong, 2009). The positive emotions are pride, excitement, and joy (e.g., Darby, 2008). Schmidt and Datnow (2005) found that teachers attach little emotion to reforms they have made sense of. Leithwood and Beatty (2008) have argued that emotions come from acquiring knowledge and skills for reform initiatives; when teachers are challenged by educational reform, they may experience loss of self; personal traits (e.g., teachers' locus of control or demographics such as the teachers' age, experiences and education) can evoke emotions. Personal traits have a smaller impact on teacher emotions than school leadership and working conditions (Leithwood & Beatty, 2008). School leadership and working-condition factors vary directly with positive emotions, teacher learning and performance during reforms. Balancing conditions of work with the demands of their private life and their personal career trajectories is another cause of emotions (Leithwood & Beatty, 2008). Darby (2008) argues that emotions are a result of meeting the needs of many different groups of people. Anger, anxiety, guilt, shame and hopelessness can be associated with lack of time, lack of sufficient support from subject colleagues, school management and governments (van Veen et al., 2005). Negative emotions can also result from the scrutiny that teachers can be subjected to during reforms (Darby, 2008). Also, the number of years a teacher has taught may have an impact on how the teacher responds to educational change (Hargreaves, 2005). For example, veteran teachers may be emotionally drained and too tired to respond to educational reforms as expected. Changing instructional practices, even with coaching, can take time and can be difficult given the emotions teachers experience during reforms (e.g., Cross & Hong, 2009).

THE ROLE OF COACHING

Different definitions exist of coaching (Grossek, 2008). It is most commonly known as a type of professional development for teachers, involving an expert or content coaching (Brown, Stroh, Fouts, & Baker, 2005). Different theoretical models exist of coaching. Some examples are appreciative coaching, cognitive coaching and peer coaching. Appreciative coaching and cognitive coaching are based on behavioural science. The teachers in this study were involved in co-teaching during coaching. During co-teaching, the coach and teacher jointly plan and coordinate lessons. The teacher decides the components of the lesson and evaluates the lesson. During such exercises, the teacher is able to learn new strategies from the coach. When teachers joined The Learning Consortium, they worked with university faculty, coaches from various boards and collaborated with other Grade 9 applied teachers about important issues in their classrooms. The teachers in this study had been part of The Learning Consortium for four years.

TEACHER LEARNING

Teacher learning has been studied in relation to participation in professional development activities that aid with the implementation of reform strategies. According to Graven (2003), it has five learning components: meaning, practice, identity, community and confidence. Teacher learning has occurred if all teachers provide evidence of increased "ownership of 'new' ways of talking about teaching and the new curriculum; use of learner-centred methodologies and engagement with mathematical meaning; participation in a wide range of education activities; status and personal identity as a competent professional; confidence" (p. 29). I adopt this view of learning because it has been used to analyze teacher learning during a mathematics reform initiative.

Based on the research on teacher learning during educational reforms, teachers learn from other teachers (e.g., Graven, 2003), formal professional and on-the-job training (e.g., Parise & Spillane, 2010) and when the professional development evokes pedagogical curiosity in teachers (Olson & Barret, 2004). Teachers also learn from coaching but it must be effective for learning to occur (McClymont & da Costa, 1998). Effective coaching depends on the qualifications of the coach, the particular strategies that the coach employs to improve instruction, partnerships between the principal and/or university faculty and the coach and protecting the coaching relationship. Effective coaching also depends on having sufficient time to work with teachers, professional development for instructional coaches, trust between the coach and teachers, immediate feedback and a focus on vital conversations (McClymont & da Costa, 1998). In addition, teachers learn if their current beliefs about teaching and learning mathematics no longer lead to student success (Goldin et al., 2009).

METHOD

I employed a qualitative multiple case study because I was exploring the impact of reform initiatives on teacher emotions. The design is suitable for impact studies (Fraenkel & Wallen, 2003). Four secondary school mathematics teachers and two secondary mathematics coaches participated in the study. I purposefully selected them because they had worked together to implement the Grade 9 applied mathematics in The Learning Consortium activities. The teachers were Robert, Helen, James and Andrew. They all taught in a school in Ontario consisting of grades 9-12 with a diverse student body of approximately 2000 students. At the time of the study, Robert had taught for 34 years, Helen was in her sixth year of teaching, James had taught for 25 years and Andrew was in his eleventh year. They were at different stages of changing their practice. They therefore offered different insights into the pattern of implementation. The coaches were Theresa and Christina. Both coaches had been heads of mathematics departments and mathematics teachers in secondary schools. Theresa had taught for 16 years and had been a mathematics coach for five years. Christina had taught for 17 years and had been a coach for two years. I had collaborated with the coaches previously in other PD settings and knew their work.

DATA COLLECTION AND DATA ANALYSIS

The study took place from September 2011 to June 2012. The primary sources of data were surveys, interviews, observations and archival data – these were used to answer the research questions. A self-assessment survey (McDougall, 2004) was administered at the beginning of the study to all participants to assess beliefs and self-reported practices regarding teaching and learning. Critical incident interviews involved all participants and invited them to tell a story about the emotions they experienced as they implemented reforms. They were conducted once to determine the specific emotions during reform initiatives and probable causes of the emotions. The semi-structured interviews allowed me to determine the nature of teacher learning during the reforms, the educational backgrounds, and the teaching experiences of teachers. The Teacher Confidence Survey (Manouchehri, 2003) was administered to participants once at the beginning of the study to determine their confidence with the instructional roles and techniques. It could also show the emotions of participants as they implemented reform strategies. The archival data helped me know aspects of the reforms. Robert and Helen were observed a total of 17 times. I imported the data into *NVivo 9*, a data analysis software program. I examined the data from each participant separately, identifying emerging themes. The themes became the codes. I then conducted a cross-case analysis to see the codes that were relevant to all teachers or coaches. Member checks were conducted. I discuss the findings as they relate to the research questions.

FINDINGS

TEACHERS' SPECIFIC EMOTIONS DURING MATHEMATICS REFORMS

The negative emotions teachers specifically experienced in the study were: feeling out of control and unhappy; drained; ineffective; and loss of harmony, pedagogical values and self-esteem. The negative emotions occurred at the beginning of reform implementation, without instructional support and remained, in some cases, after instructional support. The positive emotions were feeling effective again, proud, excited, confident and happy about the improvements in student achievement. Robert and Helen experienced the positive emotions after the coaching. The finding is consistent with (Darby, 2008) who noted that teachers experienced positive emotions after coaching. James and Andrew experienced negative emotions even after the coaching. The emotional responses to mathematics reforms were strong as indicated by Darby (2008) and Cross and Hong (2009). The findings in the study are also consistent with research that linked negative and positive emotions to reforms (e.g., Darby, 2008). The critical incident analysis demonstrated a temporary loss of self for all teachers and the reconstruction of self-understanding for Robert and Helen. Since all of the teachers in the study continued to have significant emotions as they implemented the reforms, the results are inconsistent with Schmidt and Datnow (2005) who concluded that, as teachers made sense of reforms at the school, most attached little emotion to them.

THE FACTORS ASSOCIATED WITH THE EMOTIONS THAT TEACHERS EXPERIENCE

Robert and Helen had negative emotions due to their lack of knowledge of how to teach Grade 9 Applied Mathematics and experienced positive emotions when student achievement improved. This is consistent with Darby (2008), for example, who argued that when teachers acquire knowledge to increase student achievement, they experience positive emotions. Coaching helped Helen and Robert improve their instructional practices because it introduced them to instructional strategies that made them more effective. The coaching resulted in positive emotions for Robert and Helen as they were able to reconstruct their professional understanding. This professional support led to improvements that they could see in their classrooms that made them proud and happy. The results are therefore consistent with Darby (2008) and van Veen et al. (2005) for example, who found that teachers experienced pride and joy when they reconstructed their professional self-understanding.

The four teachers in the study were supported by their department and the administration as they learned the new methods. For example, the administration supported them in terms of days off to participate in The Learning Consortium activities. Yet James and Andrew continued to have negative emotions. The reforms had an impact on how they saw themselves as teachers. Specifically, James' coaching role had been reduced with the curriculum revisions and Andrew had to implement reform strategies that he did not think were effective for students. He complained about not being permitted to modify the reform strategies. The finding suggests that in-school factors such as leadership and support for reforms did not have a positive impact on teachers because they failed to address the issues teachers had with the reforms. Therefore, it may not be enough to support teachers in their work in order to help them adopt reform strategies. Personal and demographic factors such as the teacher's locus of control, experiences and education must also be considered. The results are inconsistent with Leithwood and Beatty (2008) who argued that personal factors and demographic factors were not as important in bringing about teacher emotions as in-school factors.

THE FACTORS THAT FACILITATE TEACHER LEARNING DURING REFORMS

Robert and Helen seemed to have benefited significantly from their participation in The Learning Consortium activities. James and Andrew reported smaller gains from their participation. The veteran teacher, Robert, explained that the collaborations with university faculty and other teachers, and working with the coaches, were positive experiences. He mentioned that the supports were important because "*teachers needed to be shown that these methods actually worked*" [October, 2011]. Similarly, Helen and James mentioned that the sessions and the materials were very useful because student achievement improved. The growth in professional understanding, combined with the students' academic gains and attendance improvements, made these teachers feel more effective. Andrew mentioned that he obtained some useful information from other teachers involved with The Learning Consortium.

Robert and Helen sought professional development through The Learning Consortium because their beliefs about how students learn best in Grade 9 Applied Mathematics could not lead to student success. The teachers felt ineffective and wanted to improve student achievement. Therefore, the results in this study are consistent with Goldin et al. (2009) who argue that teachers will change if there is evidence that their current practices are ineffective. Based on the study results, professional development is considered useful if it increases student achievement. This finding is consistent with many studies on reforms (e.g., Manouchehri, 2003). In addition, the impediments to teacher learning during mathematics reform may not be associated with the number of years of teaching. For example, the most committed teacher to reform initiatives was the veteran teacher who had taught for 34 years.

TEACHER COACHING AND LEARNING DURING MATHEMATICS REFORM

Coaching had positive effects for three teachers: Robert, Helen and James. It was effective because sufficient time was allotted to work with teachers, the coaches were qualified, teachers obtained immediate feedback and conversations that could lead to improvements in instruction were emphasized. Specifically, the instructional coaches were former mathematics teachers and had received specific training to address their issues. One of the instructional coaches said "*as the instructional leader for the east region, I have received training in co-*

teaching, co-planning and facilitation" [Interview, January, 2012]. Therefore the strategies she helped teachers implement were informed by her experiences in the classroom as a mathematics teacher and her training.

An important part of the coaching program was co-teaching which allowed for immediate feedback, sufficient time to work with teachers and important conversations between teachers and coaches. The results suggest that teachers willing to help their students will find support that includes coaching on use of effective instructional practices. The results also suggest that there must be continual support to show teachers if the new methods actually work. Though Robert and Helen used reform-based practices, they adapted some of them in their classrooms because of their beliefs. Helen mentioned, for example, that the materials she needed to use to implement reform strategies were not suitable for her students. As a result, some reform strategies were not used by teachers as expected with coaching. The nature of the coaching efforts also limited the effectiveness of coaching as mentioned in the literature. For example, the coaches did not examine and address the issues that Andrew had with the reforms. His interviews indicated that this was the case. He thought that he might have found the new methods ineffective because he did not use them long enough or modify them. The coach needed to trace where meaning got lost for Andrew in terms of the effectiveness of the new methods and use this knowledge to improve Andrew's understanding. The coaching efforts also did not address James' self-image issues. The coaches experienced negative emotions when they were unable to implement reforms with some teachers and positive emotions when they were successful with reform implementation.

DIRECTIONS FOR FUTURE RESEARCH

This study is one of a few studies using Kelchtermans' (2005) definition of professional self-understanding instead of teacher identity. More research needs to be conducted to identify the strengths and weaknesses of this framework. More research is needed on how professional development can be designed to help teachers who cannot see the usefulness of the new and proven methods. In addition, the role of in-school factors in the implementation of mathematics reforms and the supports that coaches need as they experience negative emotions while helping others implement reforms must be studied more.

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MATHEMATICAL MODELLING: FROM NOVICE TO EXPERT THESIS SUMMARY

Chiaka Drakes
Simon Fraser University

INTRODUCTION

Mathematical modelling is an important aspect of the applied mathematics curriculum. It provides students, particularly graduate students, with the skills to succeed professionally in industry. It gives these students the tools to analyze, understand and forecast based on data that, in this age, is easily accessible to them. Modelling helps students to transfer the knowledge that they have learned in their less open-ended classes, to real-world problems. This transfer of knowledge is a skill that students can go on to use in the workplace and other areas of study. Since most mathematics and applied mathematics students will not go on to be mathematicians, these skills had better prepare them for whatever they aspire to do next. In the 1970's McLone (1973) reported that mathematics graduates had difficulty when moving from the classroom to the workplace:

> *Good at solving problems, not so good at formulating them, the graduate has a reasonable knowledge of mathematical literature and technique; he has some ingenuity and is capable of seeking out further knowledge. On the other hand the graduate is not particularly good at planning his work, nor at making a critical evaluation of it when completed; and in any event he has to keep his work to himself as he has apparently little idea of how to communicate it to others.* (p. 33)

However mathematical modelling courses have several problems that a classic mathematics course (such as an introductory differential calculus course) would not have. One problem with teaching mathematical modelling is that we want to use precise mathematics to fit imprecise problems for which there may be no well-defined solution at all:

> *It is the nature of real-world problems that they are large, messy and often rather vaguely stated. It is very rarely worth anybody's while to produce a 'complete solution' to a problem which is complicated and whose desired outcome is not necessarily well specified (to a mathematician). Mathematicians are usually most effective in analysing a relatively small 'clean' subproblem for which more broad-brush approaches run into difficulty.* (Howison, 2005, p. 4)

To address this problem, this work begins with the fundamental question: "What is modelling?" In order to answer that question, I looked at the literature and the experts. It was also interesting to find out what those who were not experts thought modelling was. Was it the same, a simplified version, or a completely different idea altogether?

The second question of interest to me was "How do you get 'unstuck' when modelling?" This was motivated by my own experience of modelling. In terms of teaching and learning, mathematical modelling can be different from other classic mathematics classrooms in that the method required to solve the problem is not clear. So how do we deal with this problem? *Is* this similar to that of problem-solving strategies or is something more needed here?

My third research question was "What is the difference between the expert and novice modeller?" In order to get our students properly prepared for the world of work, we must know what the expert skills are and set them on the path to achieving them. This first requires an understanding of these expert skills. Are they purely cognitive? Is there a difference in the attitude about, beliefs in, and approach to, modelling problems? If so, is time and maturity the only thing that is necessary, or can we aid students in developing expert-like behaviour?

LITERATURE

Mathematical modelling has a short but rich history in the literature, which highlights several of the issues of the mathematical modelling process. In order to start looking at the broad issue of what exactly mathematical modelling is, I looked at two different types of literature on modelling: textbooks and the *International Community of Teachers of Mathematical Modelling and Applications (Ictma)* journal articles. These two types of literature encompass two different perspectives: those who do and teach modelling and those who research modelling education and culture. I began with a look at mathematical modelling textbooks, as these are the primary tools for teaching modelling and are often a student's first introduction to the field.

The first thing of note was the vagueness in the definition of mathematical modelling. Illner, Bohun, McCollum, and van Roode (2005) give a very broad definition of modelling: "Mathematical modelling is a subject without boundaries in every conceivable sense. Wherever mathematics is applied to another science or sector of life, the modelling process enters in a conscious or subconscious way" (p. xi).

This definition certainly covers the breadth of modelling but is not a working definition.

Howison (2005) explains that modelling should not be precisely defined: "There is no point in trying to be too precise in defining the term mathematical model: we all understand that it is some kind of mathematical statement about a problem originally posed in non-mathematical terms" (p. 4).

Interestingly, Gershenfeld (1999) does not explicitly define what a mathematical model is. He describes issues that arise when building a model, but never specifically says what it is that he is building:

> *To build a model, there are many decisions that must be made, either explicitly or more often, implicitly [...] Each of these is a continuum rather than a discrete choice. This list is not exhaustive, but it's important to keep returning to it: many efforts fail because of an unintentional attempt to describe either too much or too little. These are meta-modeling questions. There are no rigorous ways to make these choices, but once they've been decided there are rigorous ways to use them. There's no single definition of a 'best' model, although quasi-religious wars are fought over the question.* (pp.1-2)

Fowler (1997) also describes some of the issues that are associated with mathematical modelling, in particular the teaching of it:

> *Mathematical modeling is a subject that is difficult to teach. It is what applied mathematics (or to be precise, physical applied mathematics) is all about, and yet there are few texts that approach the subject in a serious way. Partly, this is because one learns it by practice: There are no set rules, and an understanding of the 'right' way to model can only be reached by familiarity with a wealth of examples*. (p. 3)

Otto and Day (2007) also avoid defining mathematical modelling but raise several issues involved in doing modelling. These issues touch more on the feelings that might be experienced when modelling.

This lack of precision when defining modelling is understandable but begs the question, are all modellers describing and focusing on the same thing when they use the term mathematical modelling? While in many cases these authors have addressed some of the difficulties of the student, what they have not addressed are the skills necessary to move from novice to expert. It is therefore necessary to continue on to the *Ictma* modelling articles to see how the spectrum from novice to expert is addressed.

Haines and Crouch (2007, 2010) outline several cognitive and meta-cognitive differences between experts and non-experts (Haines, Crouch, & Fitzharris, 2003); two of them will be highlighted here. The first difference between experts and novices is their approach to problems. Experts begin with analysis and a plan. They constantly return to re-examine the problem and re-define variables. Novices, on the other hand, tend to plunge in, go straight to equations and stick to their original thoughts regardless of where these thoughts lead (Schoenfeld, 1987; Galbraith & Stillman, 2001; Heyworth, 1999). Another difference is a meta-cognitive one. Experts not only have better domain-specific knowledge, but this knowledge is also better inter-connected. This superior knowledge causes experts to focus on underlying principles. Novices, on the other hand, have knowledge that is loosely connected and tend to focus on the surface features of the problem rather than the underlying principles (Sternberg & Horvath, 1998; Chi, Feltovich, & Glaser, 1981). It must be noted that these skills of expertise described here take a relatively long time to acquire (Glaser, 1996).

This look at the textbooks and the literature on modelling education provides a general idea of the modelling process and the differences we expect between the novice and the expert modeller on a cognitive and meta-cognitive level. However, looking at them together still does not provide a novice modeller with enough information to move along the path to expertise. There are various definitions of modelling available, making it difficult to ensure that experts and novices are thinking of the same process when talking about mathematical modelling. The cognitive deficits of the novice modeller are cited, but as these aspects of expertise take time to develop, the novices cannot force themselves to be more expert at organising their thoughts, for example. There is also no discussion evidenced of how to get unstuck, except via simplification of the model. This creates problems in the case when it is uncertain how to simplify the model, and also if the model already seems to be in its simplest form.

METHODOLOGY

I tailored this study in order to target different groups of expertise, and answer the three research questions that have emerged. I conducted a qualitative study with 78 people who do modelling at some level. A qualitative as opposed to quantitative approach was used as I was trying to establish a fundamental understanding of mathematical modelling and the people who partake in it. The aim of the project is to understand different variations in modelling and to understand the nuances in the modelling world. This requires a qualitative approach.

Observations, interviews and questionnaires were my tools of data gathering. In an effort to answer the research question regarding the differences between the expert and the novice modeller, I needed to understand the modelling process from the point of view of the expert, the novice and those who are in-between. How do we identify the different groups and find out what modelling entails for them? In order to do this, I needed to determine who would qualify as an expert or a novice in the field of modelling. This led me to look at the Dreyfus model of expertise (Dreyfus & Dreyfus, 1980, 2005) for a description of the mental skills expected as expertise increases. Using this as a base, I split the participants of the study into four major groups: *expert*, *intermediate*, *novice* and *complete novice*, each of which represents a different level of expertise.

EXPERT

The obvious choice for *experts* was professors of modelling. These professors had to be active mathematical modellers, preferably well-recognised in the field, as this is an indicator of their expertise. If they also taught mathematical modelling, this would be an added bonus, as they would have insight into their novice counterparts. The *experts* that participated came from prestigious universities in Canada, the United States and the United Kingdom, including UCLA, Oxford and Duke.

In the case of the *experts*, the data consisted of their responses to ten interview questions. Interview questions seemed appropriate as they gave some flexibility in being able to ask follow-up questions. This is pertinent as Dreyfus and Dreyfus (1980) warn us that *experts* may have difficulty explaining what they do, as it has become automatic. The interview questions were informed by Hadamard's (1945) survey, but were adapted to address my own research questions. The interview addresses different aspects of the modelling process, and questions were ordered to mimic the order of the steps in the modelling process.

INTERMEDIATE

The *intermediate* participants were made up of 11 of SFU's applied mathematics graduate students and post-doctoral fellows. These participants had varied backgrounds, coming to SFU from six different countries, including Canada, the United States and China. They were also varied in their applied mathematics interests. I requested interviews with the *intermediates* in person, as I had direct access to them. Oftentimes, after an *intermediate* was interviewed, they would suggest another possible participant for the study whose work they were more familiar with than I. While several of my colleagues were willing to help, a few declined, explaining that they had not done any mathematical modelling. (It is unlikely that they had no modelling experience at all, which makes this an interesting response. I assume that their definition of modelling was not the same as mine, since we have seen that modelling has varying definitions.)

I realised that although *intermediates* did not necessarily have the wealth of experience of the *experts*, I still needed to interview them to get more in-depth responses about their various modelling experiences. I therefore used the same interview questions used with the *experts* in the field to gather data from this group, which would allow me to compare and contrast responses from the two groups.

COMPLETE NOVICE

To complete the spectrum from *novice* to *expert*, I first chose a set of participants who were clear *novices* on the Dreyfus (1980) scale: having no skill, needing rules, lacking self-confidence, interested in completing as opposed to learning, and progressing by relying only on rules. These students came from two separate FAN X99 classes held at SFU. The FAN

X99 class is a 'foundation of numeracy' course. In other words, these participants had issues with all aspects of mathematics, not only mathematical modelling, with many of them not having done high school mathematics past grade 10. For these students, mathematical modelling consisted of solving word problems. This is the most basic of modelling problems and is what Briggs (2005) calls "modelling or story problems". The work I did with these *complete novices* was influenced by the work of Schoenfeld (1985) and Liljedahl (2008), who both observed and worked with novices in classroom settings while addressing research on problem solving.

The data for this group consisted of responses to questionnaires distributed at the end of the semester. Students were again reminded that they were under no obligation to participate. I chose questionnaires instead of interviews here because I was asking them to relate a specific experience. This was in contrast with the *experts* and *intermediates* where there was the need to probe into varied experiences which a questionnaire might not cover. The use of questionnaires yielded a large number of responses (45 students responded). The questionnaire contained some questions that paralleled the *expert* and *intermediate* interview. Other questions were included to establish *novice* students' preconceptions.

NOVICE

Although the FAN X99 students qualify as *complete novices*, they are not expected to progress along the spectrum to modelling expertise in this course, as modelling is not the primary focus of the course. (This is not to say that the data from the FAN X99 students was useless—far from it!) This led me to SFU's fourth-year undergraduate modelling course, Math 461, in which students are encouraged to participate in the *Mathematics Contest in Modelling* (MCM). Although this was a fourth-year course, many of the students had very little or no modelling experience. This made them an appropriate group of *novice* modellers, as their lack of experience qualified them as *novices*, but their decision to take the class and participate in the MCM showed a likelihood of moving along the spectrum towards expertise in modelling.

Those who were participating in the MCM were asked via email to fill out a questionnaire about this experience. The contest is a weekend long modelling competition, in which students work together in groups of three on a given problem. Students have a choice of two problems to work on and are not allowed to consult anyone outside of their group for help. Eight of the MCM participants agreed to complete the questionnaire. The data for this group of *novice* modellers are their responses to the questionnaires. This questionnaire was different from that for the complete novice. Students were asked to comment on what made their chosen problem a modelling problem, as well as being asked several questions that paralleled the interview of *experts* and *intermediates*.

ANALYSIS

The data in this study are the interview and questionnaire responses. While no discourse analysis was done, pauses and exclamations that highlight a particular point were included. Observations were used as a backdrop to the analysis of the transcribed and questionnaire data. As the data is primarily spoken words, it was edited to allow the reader to follow the train of thought of the speaker.

After the expert and intermediate interview data was recorded and transcribed, I transferred the data to an *Excel* spreadsheet. This allowed comparison by question as well as by person. Individual responses to each question were then coded using line-by-line coding informed by Grounded Theory (Charmaz, 2006). To do this, I looked at each line in a given response to a question and summarized it. I then looked at these lined summaries and identified any themes of interest within them. I noted recurring themes as well as outliers and compared the results across groups, contrasting the responses of the intermediates with those of the experts. In many cases new codes had to be created specifically for the intermediates.

While the coding was done using the principles found in Grounded Theory, the analysis of the themes was not. Charmaz (2006) describes Grounded Theory as developing theory as it emerges from the data, thus the theory comes from the data as opposed to the data being analysed using existing theories. In place of this method of analysis, I made use of Patton's (2002) *principle of analytic induction*. Sriraman (2004) explains that Patton's principle works well when studying "an extremely complex construct involving a wide range of interacting behaviours" (p. 25). Since the literature had motivated my study, common themes that emerged were compared to the existing literature using Patton's principles, as opposed to developing a theory from the ground up as Grounded Theory suggests.

Questionnaire responses for both groups of novices were also transferred to spreadsheets to allow comparison by question and by person. These responses were succinct and did not require coding. Responses of the MCM student-novices were compared and contrasted with those of the experts and intermediates. For the FAN students I focused more on their preconceptions of mathematics. However I contrasted them with the experts to highlight the differences here, and in some cases, the similarities.

RESULTS

EXPERTS

For the experts there was a dichotomy in the definition of *mathematical modelling*. For some experts, mathematical modelling is a description of the real world problem, that is, the formulation of a real world problem into a mathematical framework. For others, modelling is a process encompassing not only the formulation of the model, but also the solution of that model, verification of the solution, refining and predictions.

Upon first encountering a mathematical model, experts focus on understanding the problem, particularly if they are stuck initially. Experts collaborate with the person who brought the problem, as well as their colleagues, in order to better understand the problem. Experts stated simplification of the problem initially as one of their main heuristics. While prompting was often necessary to get the experts to begin discussing their feelings, several of them spoke of experiencing excitement, curiosity and interest when first faced with a modelling problem. Others spoke of initially feeling worry or anxiety, but were able to move past those feelings to tackle the problem. A question about what makes problems difficult or easy revealed that the difficult problems were the interesting ones for the experts.

The experts interviewed revealed several aspects that go into successful modelling. They have autonomy and choose problems that they are interested in. They deal with being stuck in the middle of modelling by collaborating, simplifying and trying to understand the problem better. They always check that their solution is sensible, usually by comparing it to the data, but also by comparing it to solutions of other methods. They deem several skills important to modelling, including a breadth of knowledge and an understanding of the background of the problem. Among the non-cognitive skills valued by the experts are patience, collaborative skills, persistence, maturity and passion.

INTERMEDIATES

A look at the group of intermediates showcases several differences and similarities between the intermediates and the experts. The intermediates tended to provide more detailed responses on many of the topics, whereas experts were more succinct. The dichotomy in the definition of modelling was not evident as with the experts, with intermediates defining modelling as the use of mathematics to solve a real world problem. There is a shift in focus here from formulation or the entire modelling process to the solution step of modelling.

Intermediates were more forthcoming with their feelings on modelling than experts and had more feelings of persistent self-doubt. Intermediates also discussed trying to understand the problem initially, however they make use of research primarily to do this as opposed to collaborating with others. Intermediates recognised that the complexity of the problem often leads to it being difficult, as well as a lack of clarity and the openness of the problem.

Several of the themes mentioned by the experts re-emerged here. While intermediates have less autonomy, they are still usually interested and motivated in their particular area of study. They recommended asking questions when stuck, highlighting that taking a step back or articulating your difficulties often helps you to overcome them. They named several mathematical areas of knowledge that were seen in the expert responses and recognized breadth as opposed to depth of knowledge as being important. Intermediates valued non-cognitive skills of perseverance, good collaboration and taking a break.

NOVICES

The novice modellers who participated in the MCM have a basic definition of mathematical modelling. However, some of them assumed that modelling problems must have more than one solution method which, based on the expert responses, is not a necessary condition for a problem to be a modelling problem. The novices, like the intermediates, used research more than collaboration when stuck initially.

There was a wide range of emotions discussed upon seeing the modelling problems, with some novices feeling confident and others feeling completely clueless. There was much less autonomy than the experts on possible choices of the problems they were to work on. The novices did not always have a realistic idea of the time that would be required to work on the problem and did not have great collaboration skills, although they did recognise that collaboration could be useful. They did however indicate in their responses that they were willing to defer to others in the group and change strategies if they were stuck.

Many novices either stuck to one solution method regardless of the outcome, or switched completely with no effort made to understand why they were experiencing difficulty. They quoted several mathematical topics that they deemed necessary to solve the MCM problem they chose to work on. There was also no mention of checking that their solution was correct, but this may be due to the time constraint of the MCM.

COMPLETE NOVICES

The focus of the FAN X99 class is not mathematical modelling per se and so the students were not asked for a definition of mathematical modelling. When asked about the time frame for solving modelling or word problems, most students stated that these problems should be solved in the order of minutes. However, two students explained that the time taken to solve the problems is problem dependent. Students were very forthcoming about their feelings when solving word problems, for the most part expressing fear, dread, panic and anxiety, although several of them also expressed feelings of interest.

It should be noted that these students have little to no autonomy. For many of them the course is compulsory for their degree, and in class they do not get to choose which problems they prefer to work on. Unlike the experts, these complete novices tended to find difficult questions frustrating as opposed to interesting, with the theme of frustration being evident throughout most of the students' responses.

The complete novice participants tended to plunge in rather than plan their solutions, but the majority of them thought of themselves as organised. The suggestion of giving up completely when stuck was only mentioned by members of the complete novice participants. Interestingly, several complete novices discussed expert heuristics such as simplifying the problem and drawing a picture to get access into the problem. Also of interest was the transition for some of the students, where they noticed that they became less anxious and more willing to try to do problems by the end of the semester.

CONCLUSIONS

We have seen in the modelling literature that there is no agreed upon definition of modelling. Among the participants of this study there are also differences. The experts exhibit a dichotomy in their responses, with some viewing modelling as the formulation of the model and others viewing modelling as the entire process including verifying and refining the model. (This dichotomy in definition may explain the dichotomy in approach towards teaching mathematical modelling.) The intermediates focus on the solution step of the modelling process but expressed a similar definition to that of the experts. On the other hand, some novices misunderstood what modelling is, assuming that modelling problems are ambiguous by definition, as opposed to being ambiguous as a consequence of coming from real-world problems.

When dealing with being stuck, the experts tend to collaborate with others around them, those who have brought the problem, colleagues, and even those who have not worked on the problem at all. This is not seen in the modelling literature but was raised by almost every expert interviewed, and highlights the fact that for many of the expert participants, mathematical modelling is a group exercise. Most important for the experts is understanding the problem in order to become unstuck. The intermediates turned primarily to the literature to increase understanding as opposed to collaboration. The novices spoke of switching strategies when stuck without discussing trying to understand why they are stuck. Complete novices were the only group to mention giving up completely when stuck. They also tended towards more passively asking for help or waiting, as opposed to active collaboration.

There are several other differences as we traverse the landscape from novice to expert. There is an increase in autonomy as we move along the spectrum, with the complete novices having little or no autonomy and the experts having almost complete autonomy. There is also a decrease in persistent self-doubt or anxiety as we travel along the spectrum from novice to expert. Experts do speak of feeling some anxiety, although they are able to distance themselves from these emotions in order to address the modelling problem. Finally, experts described difficult problems as interesting, while complete novices saw them as frustrating.

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TWO PERSPECTIVES REGARDING THE PEDAGOGICAL FILM *ALL IS NUMBER*: CRITICAL AND MAROON

Steven Khan
University of British Columbia

INTRODUCTION

Like many people who choose to do research in mathematics education, I began with/in the amorphous discourse of a 'concern about the quality of mathematics education' that acts to code multi-national anxieties, insecurities, and fears of vulnerability about present and future economic competitiveness. At the same time, I could not help but notice the wide and increasing diversity of artefacts of mathematics popularization which are often produced under the auspices of powerful stakeholders, promoted as a response to these concerns, and the limited sustained critical scholarly attention that these artefacts have received to date, which, as Kelecsenyi (2009) notes, may be the result of the significant methodological challenges involved in studying popularizations and popular culture. Following cultural theorist Stuart Hall (as cited in Storey, 2009), I take popular culture as a site where "collective social understandings are created: a terrain on which 'the politics of signification' are played out in attempts to win people to particular ways of seeing the world" (p. 4).

After viewing a short mathematics popularization artefact produced in the Caribbean—the pedagogical film, *All is Number* (Haque & Sampson, 2010)—I believed that this could serve as an interesting and informative object of analysis or interpretation for beginning to open more critical conversations around the pedagogy of mathematics popularization artefacts. Later on in the research process, I added another goal: to develop concepts for Caribbean Curriculum Theorizing.

The scholarly literature on mathematics popularization identifies some concerns. The most frequently debated issues are: when simplifications of mathematical ideas become significant distortions; stereotypical representations of mathematicians, minorities and mathematics; the challenge of adapting material for different genres; the use of powerfully placed advocates, who at times engineer a climate of crisis with respect to mathematics education; the actual markets for popularizations; and the moral appropriateness of some educational content in school-settings. In this paper, I aim to contribute to conversations about the representations of mathematics and mathematicians in popularization artefacts via consideration of a single short film, *All is Number*, which was produced in the Caribbean by professors at the University of the West Indies, was aired on regional television stations and was intended for secondary school and non-specialist audiences.

In particular, one of these ways of seeing of concern in the popularization of mathematics literature is around stereotypical representations. In simplifying mathematics for a 'popular' audience, popularisers sometimes resort to stereotypes. Such stereotyping, however, may also come into conflict with another major goal of popularizations, viz. to interrupt and challenge negative stereotypes associated with mathematics and mathematicians. For example, there has been much concern in the mathematics education community about the stereotypical representations and associations of mathematics and gender.

UK based researchers Heather Mendick and her colleagues Moreau and Hollingsworth (2008), in their report on representations of mathematics and mathematicians in popular culture (films, websites, books, radio, television), recommend that producers of popular culture develop,

> *more representations of women doing mathematics and particularly more adult women whose abilities are independent of the men in their lives; representations of women doing mathematics who are classically attractive, feminine and engaged in heterosexual relationships and of those who are not; a greater diversity of people doing mathematics, in particular, people from different ethnicities, nationalities, sexualities, ages, social class backgrounds and with different bodies…[particularly] ones that go against the clichés*. (p. iv)

In many ways the film *All is Number*, coming from the Caribbean, can be seen as a response to those sorts of recommendations.

METHODOLOGY

I have worked closely with Gillian Rose's (2007) descriptions of *critical approaches to visual materials and culture*, which entail thinking "about the visual in terms of cultural significance, social practices and power relations in which [they are] embedded…" (p. xv) and with Elizabeth Ellsworth and Marianne Whatley's (1990) related ideas around *ideological analysis of educational media*. The latter note that this is a type of critical hermeneutic practice whose objective is to "expose the underlying operations of a text by means of a symptomatic reading" (p. 4). More explicitly, they explain that this usually involves "a commentary on each segment in turn" that may "concentrate on specific moments…[that seem] to condense ideological processes" (p. 4).

After multiple repeated viewings, I partitioned the film *All is Number* into 15 segments based on the mathematical content or function. This resulted in what I consider to be a medium grain size for analysis, with segments ranging from 14 to 155 seconds. I also transcribed the audible narrative content of each segment and juxtaposed these with screen captures of salient images to attend to the aural and visual dimensions of the film. In examining the ideologies in the film, I attended to how these were represented and communicated via word and image.

In the next section, I describe some of my findings with respect to the construction of *mathematical authority*—one of the means through which the ideology of the film is communicated.

MATHEMATICAL AUTHORITY

In understanding what the ideological emphases are in this film it became important to investigate the strategies used in constructing mathematical authorities. The film constructs an identity of mathematical authority for men and women of various ages, nationalities and ethnicities, and can be seen as responding to the specific recommendation in the literature by Mendick et al. (2008) mentioned earlier, "for more representations…of a greater diversity of people doing mathematics…people from different ethnicities, nationalities, sexualities, ages, [and] social class backgrounds…" (p. iv).

Two strategies employed in constructing these authorities include the use of captions and the contextual positioning of individuals. Captioning serves to establish individuals as authorities by identifying them by name and professional designation—Scientist, Professor of Physics, Mathematician, Agricultural Meteorologist and Director. The strategy is clearer when other individuals whose activities are described in the film as having a mathematical component—musicians and a sidewalk vendor—are not identified in a similar way through captioning.

Secondly, the location of individuals in these scenes also extends the authority instituted by captioning or lack thereof. By positioning certain individuals within specific backgrounds, the film/images also convey a sense of where these specific mathematical authorities exert an influence—in the natural world (beach, river, outdoors), in classrooms/lecture halls/seminar rooms, in managerial positions (indoors), and in economic transactions. A contrast is again evident, however, with the images of the guitarist and the harpist, where the backgrounds are darkened and the individuals are removed, isolated from any recognizable social and cultural context. This positioning situates the type of work with which these individuals are engaged differentially, with artistic and creative expression (handicraft and music) not being given the same kinds of authorizing agency as science and business in the film. Thus, while the film is attentive to its representation of mathematical authorities in a way that perhaps positively works to interrupt some of the stereotypical representations of mathematical authorities, the film does not give the same attentiveness and care to the representation of those others whose activities the film also chooses to inscribe within a mathematical locus.

As I offered earlier, following Stuart Hall, one of the things that particular practices of signification do is attempt to construct and communicate a particular view, or ideology, to win people to particular ways of seeing the world. Following Ellsworth and Whatley (1990), I concentrate on a specific moment that condenses some of the ideological constructs at work in the film.

Representations are not neutral. As argued above, one of the things that particular practices of signification do in this film is to construct and communicate a particular view, or ideology, of mathematical authority. Such authority itself is not neutral. In this section, I examine the clip (01:40-02:25) which introduces Professor Leo Moseley, and the ideology in his utterance, and ask what mathematical authority is being used for in this segment. In particular, I want to suggest that mathematical authority is being used here to generate difference, to mark *otherwise*, an *otherwise* that is not merely ‘in relation to’, but is ‘less than’. In the clip, Professor Moseley states,

> *All Science depends very heavily on mathematics, which is the language of Science. Now* ***you may think that*** *as a biologist you do not need that much mathematics,* ***but when you think*** *of the sophisticated statistics which are used by biologists as they examine the natural world, then* ***you will see that you perhaps*** *have a quality of mathematics which is* ***beyond many other people*****.** ***Even in Social Sciences*** *where* ***they tend to think of themselves as somehow different,*** *they depend very heavily on graphs and analysis of graphs which is of course mathematics. Everywhere we go there is mathematics.* [emphases mine] (Haque & Sampson, 2010)

Consider the following rhetorical sequence: i)“you may think that”; ii) “but when you think of”; iii) “then you will see perhaps”. The first part of this sequence is an attempt to call some viewers’ beliefs about the role of mathematics in Science, Life Science in particular, into question. It is simultaneously permissive yet derisive. The second part of the sequence begins to use the mathematical authority of the presenter to introduce another thought to the viewer. Finally, the third part suggests that only when one begins to think in the way identified in the second part of the sequence that one will be able to see, and even then, the inclusion of "perhaps" is an acknowledgement that there is no guarantee. According to the rhetoric in the sequence, in such a case the cause lies in uninformed thinking.

In this clip, the constructed mathematical authority is used to direct viewers' thinking away from a perceived belief, towards a different one. However, the specific conclusion to which viewers are being directed through the deployment of mathematical authority is another potentially crippling belief, namely, that the type of mathematics used by Biologists is something "beyond many other people". This conclusion may function to reinforce beliefs that mathematics is a special gift/quality/capacity which only some people possess. The research literature in mathematics education, however, has repeatedly demonstrated that such beliefs often function as self-fulfilling prophecies.

Now in the second part of the utterance, the mathematical authority of Professor Moseley is used to further position Biology (Life Sciences) and Social Sciences, in relation to mathematics, as dependents. The use of the word "even" in this utterance, in relation to the role of mathematics in social scientific disciplines, functions to make those subjects seem simultaneously exceptional and non-exceptional. It can be read as being somewhat dismissive of those aspects of the social sciences which do not quantify as being *less than* 'real' science. Indeed, the example offered that social scientists who analyse graphs are doing mathematics is probably somewhat offensive to both mathematicians and social scientists—a gross over-simplification of what professionals in both domains are up to.

What these utterances do in the very short timeframe of the clip is to identify, label and mark Biology and Social Sciences as being different from mathematics, yet scribed within an inescapable locus of mathematical authority, as suggested by the final statement that, "everywhere we go there is mathematics". What Professor Moseley accomplishes here is an act of *othering* of independent but related disciplines.

WHAT IS MATHEMATICS?

Consider that the final utterance of a pedagogical film often serves the purpose of reiterating and attempting to reinforce the main idea or concept, functioning as a metonymic signifier for the message that the film-makers have attempted to communicate to the imagined audience. It is the last words that the audience will hear in the film and will likely be among the things recalled. At one level, my dissertation work can be thought of as a *response* to this final utterance of the film *All is Number*, in which the female narrator opines,

> *The beauty of mathematics is that it does not matter if you are at a river's edge on a Caribbean island or in the far reaches of outer-space. It does not depend on a place or time, a people or culture. It is universal in its relevance. It describes that which we can and cannot see, choreographs the dance of the atoms on the tiniest scales, while describing the universe on the grandest. Nature speaks and if we listen carefully enough, it speaks…mathematics.* (Haque & Sampson, 2010, 16:45)

In this closing utterance, as in the succinct Pythagorean-inspired title of the film—*All is Number*—a complex epistemological ideology is summarized and valorized: to enumerate and to quantify is to *know* and such knowing transcends people and places. My immediate reaction to this final utterance the first time I saw and heard it was that "this statement is untrue". Where the beauty of mathematics is concerned I believe it *does* matter if you are at a river's edge in the Caribbean, the far-reaches of outer-space, tattooed on a train to Auschwitz, or shackled and packed-for-profit in the belly of a slave-ship to a 'New' world. It *does* depend on place, time, peoples and cultures. It is *not* universal in its relevance, and nature does *not* 'speak' mathematics. These differences of opinion were already well-rehearsed in the philosophy and histories of mathematics and mathematics education, as for example, in the work of Reuben Hersh, Paul Ernest, and Ubiratan D'Ambrosio, as well as many others.

In this final utterance, the political dimension, if not motive, of a pedagogical film like *All is Number* is rendered explicit. This moment then, despite offering an invitation to closure, also offered an important opportunity to contest. And it is that opportunity that I take up.

The particular set of privileged beliefs about what mathematics is that prevails during a given period is strongly associated with what those in positions of authority and responsibility in a society choose to teach, and not teach, *as* mathematics. The ontological status of mathematics is thus linked to epistemological beliefs about what knowledge is of most worth (curriculum), how such knowledge can be represented and communicated to the next generation (pedagogy), and decisions about who can or cannot do mathematics (politics).

The alternative perspective, a mythopoetic re-imagination and orienteering towards something that might be called a *mindful* mathematics, is no less ideological, but is less well-rehearsed in the present moment. Perhaps the resistance and recalcitrance of this educational problem resides in the very term, that contested subject, object of disgust and admiration, of love and hate, shame and pride, that is 'mathematics' itself, and the quality and nature of its relations to its proliferating, hyphenated, and alienated descendants, including mathematics-education.

And this really is what my work, which I have situated in the past, at the intersection of mathematics education, aesthetics and ethics (Khan, 2010), is about in the end—the need for responsible and ethical dialogue among the disciplines and for a larger set of awarenesses in our pedagogies, whether they occur in classrooms or via artefacts of mathematics popularization.

PART 2: MATHEMATICS EDUCATION AS MAROON NARRATIVE

Earlier, I stated that a concurrent goal of this work was to contribute to developing concepts for *Caribbean Curriculum Theorizing*. In this part of the work, I experiment with the expanded concept of *Maroon Narrative*, as articulated by Caribbean theorist Cynthia James (2002), in engaging with the film.

CONCEPTUAL DEVELOPMENT

Historically, the term *maroon* in Caribbean history and literary theory is generally used to refer to primarily African slaves in the 17th and 18th centuries who, having escaped plantation slavery, formed independent communities in mountainous and forested areas. Tracing the different articulations of the concept, James (2002) offers evidence for an origin in the Spanish word *cimarron* which "in the New World originally referred to domestic cattle that had taken to the hills in Hispaniola and soon after to [Amer]Indian slaves who had escaped from the Spanish as well" (p. 11). By the 17th and 18th century, the term *maroonage* had entered into Anglophone Caribbean vocabulary via its French usage and was transformed into a referent with connotations of *shipwrecked* or being *isolated*. This feeling foreshadowed the psychological ambivalence and anomie associated with the term in 19th century literature about the Caribbean when,

> *'to be marooned' meant to be psychologically placed in the condition of a Caribbean runaway with all its attendant connotations of deprivation, brutality, withdrawal, and separation from ancestral culture… harbor[ing] connotations of pleasuring for a period in the wilds like the natives…[and suggesting] connotations of 'tourist' behavior.* (James, 2002, p. 13)

As the concept evolved over time, "depictions of physical confrontation in a plantation context" associated with plantation runaways waned and emphases on "psychological confrontations mainly in villages and urban yards" increased, so that as James notes, "Maroon becomes less associated with open rebellion…and more associated with self-analysis, rootlessness, and identity formation within the context of ethnic diversity and ethnic estrangement" (p. 55). The concept of psychological *maroonage* as "withdrawal, with flight…manifesting itself in internal dis-ease" (James, 2002, p. 8) introduced by Barbadian poet Kamau Braithwaithe is developed further by Gordon Rohlehr in exploring the terms "'self-in-maroonage' and 'the submerged self'…terms connotat[ing] inner resistance and self-affirmation" (p. 8).

Cultural maroonage is perhaps the conceptual articulation that will be most resonant. René Dépèstre defines it as, "an artistic mission of resistance—in postcolonial terms, artistic effort that stakes its distinction on writing against the grain of the European and European depiction of the Caribbean" (as cited in James, 2002, p. 9). In maroon narratives, postcolonial emphases on 'writing back' and 'contrapuntal readings' represent "an emergent voice that seeks to refashion English. [It] no longer wishes to be destabilized by the English pentameter…[and is] burdened with an 'urge to interrupt the text'" (James, 2002, p. 6). This mission of resistance manifested in 'writing back' and 'contrapuntal readings' can be seen as an example of polyphony at play in these works in the way that the multiple ideas of *maroon* and *maroonage* inter-relate, the way they *shape* each other through *dialogue*.

What distinguishes a maroon narrative from other literatures and descriptions of exile and migration, for me, is the coming to awareness of an oppressive situation, the active revolt against the oppressive situation/formulation, the conscious decision not to return to that state, the psychological wrestling with these choices, and the desire and activity to create something new, and hopefully less oppressive, *with* others.

Having described/explained the features/characteristics of the concept of maroon narratives I turn now to the specific object of analysis—the film *All is Number*—as I attempt to demonstrate how the concept is useful in interpreting the film, as well as critically engaging with the film as a maroon narrative within the context of mathematics popularization.

ALL IS NUMBER AS MAROON NARRATIVE

I want to claim that the film *All is Number* is an example of a maroon narrative. In order to justify this claim, I will need to demonstrate that it shares the necessary qualities identified previously. I argue that *All is Number* can be viewed as a response to the undesirable situation in mathematics education in the Caribbean, creating something new with (and for) others.

All is Number stands as perhaps the first attempt at mathematics popularization via the medium of documentary film in the Anglophone Caribbean. It is something new, a form of "indigenous adaptation" (James, 2002) whose destination is cultural enrichment, as well as survival in the modern world. In producing the film, the film-makers engage with a diverse set of individuals, including those who appear in the film, as well as those who served as consultants and who are credited. This, however, is not the most important way in which the film can be considered a maroon narrative.

All is Number is a Response to an Undesirable Situation

All is Number can be considered to be a response to the undesirable situation, euphemistically described as 'a concern for the quality of mathematics education' in which, over a period of seven years, more than 400 000 (or approximately three-fifths of all) examination candidates in the Caribbean region's experiences of mathematics included that of failure on the regional

Mathematics examination. Such massive failure and concerns with social, political and economic well-being often serve as a prompt for mathematics popularization projects. Although the producers of *All is Number* are not reported as explicitly citing the regional failure in mathematics as an incentive for their popularization project, it is worth noting that the film emerged from within a regional higher educational institution where the context of estrangement and separation from mathematics is an ongoing concern.

The film can be read, following Dépèstre (as cited in James, 2002) as an "artistic mission of resistance" (p. 9) on several fronts/levels. Firstly, it provides alternative ways of looking at mathematics that are rendered in sharp relief to traditional modes of presentation and elaboration often found in school mathematics. Some of the main ideas communicated about mathematics through the aural narrative are that school mathematics is not all there is to mathematics, and that mathematics is useful and is associated with beauty. These ideas are meant to offer an alternative position and some resistance to beliefs about school mathematics which often make the discipline seem tedious, cold and distant to the everyday aspirations and lived-experiences of learners.

Part of its resistance, too, is accomplished through the visual elements presented which are intended to be familiar to members of the imagined audience of secondary/high-school students, such as the beach, trees, a river-bank, and flowers. At another level, the film also can be construed as a deliberate and artistic 'writing back' to practices of education and mathematics education, in particular in the Caribbean and elsewhere, that are not inviting and cut-off learners from sources of knowledge and understanding that might enable them to access and deploy the 'culture of power' that is mathematics.

All is Number Wrestles with Tensions of Accommodation

The film *All is Number*, while being made in the Caribbean, is positioned in relation to other films and artefacts within the genre of science and mathematics popularization that are made elsewhere. The choice of topics—Pythagoras' Theorem, Fractals, Chaos, Fibonacci, Golden Ratio—follows some of the more successful areas that have been the focus of repeated popularization efforts in this genre. In this choice of topics, there is an attempt to wrestle with and come to a suitable accommodation that is attendant to the limits imposed, both by the formal curriculum structures, as well as the need to start from within these structures and move outwards.

Another area in which the film demonstrates a negotiation with 'insides' and 'outsides' concerns the visual representation of the landscape of the Caribbean. In attempting to illustrate that "mathematics is all around us" the film-makers show familiar images from within the Caribbean, such as sea, sand and vegetation juxtaposed with images from outside, such as the space-shuttle, the Parthenon and the Mona Lisa. In presenting images and ideas that are simultaneously proximal and distal, this visual polyphony poses an unstated challenge to viewers—namely, "to find ways to deal with a sense of distance, inferiority, and loss in evolving new identities and new societies" (James, 2002, p. 56) through the narrative and ideological bridges that run throughout the film, viz. that "mathematics is everywhere" and "all is number".

An early image of a triangle, traced with a stick in sand on a beach, is an invitation to begin in one familiar place, and through mathematics, traverse unfamiliar realms. As a metaphor, the image of the triangle on the beach also evokes the idea of a mathematical palimpsest which will eventually be erased by the actions of wind and tide. It thus offers a moment for reflection and wonder as to what other mathematical traces might remain yet hidden but present in one's environment and what forms of new inscriptions one might write for oneself by choosing mathematics.

Self-analysis and Identity Formation in *All is Number*

Perhaps the most important attribute in describing the pedagogy of the film as a maroon narrative is the space it opens up, engaging with questions of individual and collective identity and individual and cultural analysis. Dr. Haque, one of the producers, states in an interview that her motivation for producing her previous film and this one was to offer an expanded narrative of what the Caribbean is known for, beyond the stereotypes of the beach, Carnival, and Laureates in literature. Seen from this perspective, *All is Number* can be construed as an attempt to offer students in the Caribbean region an opportunity for self-analysis and identity construction in relation to images of mathematics that are perhaps not often presented as being available to many.

The film, however, can only *offer* the opportunity; it cannot guarantee that the opportunity will be taken up, or predict when and how this self-analysis and identity formation might occur, or even in which directions it may unfold. What is important, though, is taking this first-step towards an affirmation that 'we' too in the Caribbean, despite the fact that we have not often thought of ourselves in this way, might take on the particularly privileged label of scientist and mathematician for ourselves and come to notice and value mathematical competencies as well as challenge as part of the work that one does.

The film, as a maroon narrative, addresses those who continue to experience conflict in the entanglement of estrangement, hurt and dislocation in mathematics, and wrestles with the need to accommodate difference and otherness from the perspective of the previously colonized.

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A CASE STUDY OF THE MULTIPLE-USE OF A GRADE 9 MATHEMATICS ASSESSMENT: IMPLICATIONS FOR THE VALIDATION PROCESS

Martha J. Koch
University of Manitoba

ABSTRACT

Using questionnaire data, document analysis, school-based interviews and interviews with test development personnel, this study examines the multiple-use of the Education Quality and Accountability Office (EQAO) Grade 9 Assessment of Mathematics, administered in Ontario, Canada. The study focuses on two uses of this assessment: the use within Ontario's accountability program and the use by teachers as part of students' grades. Evidence of interactions between these uses is provided and the limitations of applying an argument-based model of validation, given these interactions, are discussed. The study suggests new ways of approaching validation that better address the practice of multiple-use. Proposed approaches draw on hermeneutics and the use of boundary objects as analytic tools for enriching the validation process. In closing, the contributions of this research to mathematics education and educational measurement are presented. The article is a summary of the author's doctoral dissertation. More detail about each aspect of the study can be found in Koch (2010).

INTRODUCTION

In theory, large-scale educational assessments are usually designed for one specific purpose, often referred to as the intended use (APA, AERA, & NCME, 1999). In practice, the results from assessments are routinely used for several purposes at the same time. I suggest the term *multiple-use* to refer to the practice where results from a single administration of an assessment are used for their intended use and for one or more additional uses (Koch, 2010). For instance, the results of an assessment designed and administered to measure student achievement in mathematics may also be used as a measure of teacher effectiveness within a merit pay program. Multiple-use is neither a recent phenomenon nor an uncommon practice. By the early 1900s, results from high school entrance exams designed to screen students for admission were also being used to measure teacher performance in some US states (Tyack, as cited in Koretz & Hamilton, 2006). More recent examples can be found in Canada (Koch, 2010), the US (Miller, 2008), England (Stobart, 2009), and Sweden (Wolming & Wikstrom, 2010). And yet, to date, little research on the practice of multiple-use has been conducted.

The purpose of the dissertation, which is the basis of this article, is to investigate the implications of the multiple-use of large-scale assessments for the process of validation.

Validation is a fundamental aspect of measurement because it provides evidence that the inferences that are made from an assessment can be justified (APA, AERA, & NCME, 1999). The article begins with a brief description of the conceptual frameworks on which the dissertation is based. This is followed by a theoretical discussion of two previously unidentified challenges for validation that result from multiple-use. To further explore these challenges, an empirical study of the multiple-use of a Grade 9 mathematics assessment is described and some innovative approaches to the validation process in the context of multiple-use are presented. In closing, a summary of the contributions of this research to mathematics education and educational measurement is offered.

CONCEPTUAL FRAMEWORKS

In this study, both large-scale and classroom assessment are viewed from a socio-cultural perspective (Delandshere, 2001; Gipps, 1999; Shepard, 2000). From this perspective, assessment is seen as inherently value-laden and culturally situated. Assessment contributes to the classroom culture but also responds to that culture. For instance, a classroom may be highly collaborative or strongly competitive; this culture both influences and is influenced by the assessment practices taking place. At the same time, other activities occurring at the school, district and provincial levels influence and are influenced by assessment policies and practices. Thus, assessment is seen as a dynamic process involving many individuals and groups, and multiple interpretations of assessment practice are possible. Considering these interpretations is an important dimension of research based on a socio-cultural framework.

A second conceptual framework that is central to this study is the approach to validity and the process of validation advocated by many measurement theorists and promoted in the standards that guide measurement practice (APA, AERA, & NCME, 1999). Although theorists disagree on some aspects of validity and validation, there is general consensus on several basic tenets. To begin, most theorists acknowledge that it is the inferences made from an assessment that must be validated rather than the assessment itself (Kane, 2006; Messick, 1989; Moss, 2007; Shepard, 1997). In addition, validation is seen as an ongoing process that requires the integration of multiple sources of evidence. Validation is also generally considered to be a responsibility that should be shared by test developers and test users. The most widely accepted approach to validation at this time is Kane's argument-based model (Kane, 1992, 2006).

While acknowledging the value of Kane's approach, several researchers identify the need to re-conceptualize validity and validation to better address certain practices (Brookhart, 2003; Moss, 2007; Shepard, 1997). To address this need, a number of researchers suggest that drawing on disciplines outside educational measurement may be worthwhile. In particular, Moss offers a well-articulated discussion of the ways that hermeneutics and socio-cultural concepts such as boundary objects can build on current understandings of validity (Moss, 2007; Moss, Girard, & Haniford, 2006). This discussion became the impetus for a key part of the analysis conducted in the empirical part of the dissertation.

THE PRACTICE OF MULTIPLE-USE FROM A THEORETICAL PERSPECTIVE

Considering multiple-use from a theoretical perspective reveals two major concerns for the process of validation that have not been previously identified or investigated: increased stakes and interactions among uses.

INCREASED STAKES

The stakes associated with a large-scale assessment may be quite low when only the intended use of an assessment is considered. For example, an assessment may have minimal impact on individual students, teachers and school administrators if it is only used to provide aggregate information about student achievement in a subject area. However, where multiple-use occurs, the stakes associated with an assessment can become much higher. If the results are also used as a graduation requirement, for example, or as the basis of a merit pay program for teachers and/or to rank schools, an increase in stakes may occur. The increase in stakes that can result from multiple-use is a serious concern because, as several measurement researchers have argued, high-stakes assessments require different kinds of validity evidence (Koretz & Hamilton, 2006). Thus, each multiple-use must be factored in to accurately determine the stakes of an assessment. The increase in stakes resulting from multiple-use has not been previously acknowledged in the measurement literature and empirical studies of this issue are needed.

THE IMPACT OF INTERACTIONS BETWEEN MULTIPLE-USES

The second problem that emerges from multiple-use is that uses cannot be assumed to be independent of one another. Since all of the uses come from a single administration of the assessment, practices associated with one use may have an impact on the validity of the inferences that can be made for other uses. In other words, multiple-uses may interact with one another. Nolen, Haladyna and Haas (1992) provide an example of this problem. Though they do not use the term *interaction*, these authors show that the practices teachers use to administer a statewide achievement test differ when teachers believe the assessment is also used to evaluate their performance. Teachers reported using practices such as giving rewards for test completion, providing extra time, and other non-standard testing procedures when they thought the achievement test would also be used to evaluate their performance. Nolen et al. state that the “test score pollution” (p. 9) that results from these practices makes the scores useless for any purpose.

More broadly, I maintain that the possibility of interactions among multiple-uses changes the approach needed to gather validity evidence. The validation process traditionally focuses on gathering evidence to support each inference made from a test score (AERA, APA, & NCME, 1999). In effect, each proposed use is validated separately as though the other uses were not taking place. This approach is not adequate in the context of multiple-use because one use may impact the validity of the inferences that can be made for another use. Thus, the validation process must include a consideration of the ways multiple-uses interact. Exploring the impact of interactions on validation is the central concern of the empirical part of the dissertation.

AN EMPIRICAL STUDY OF THE PRACTICE OF MULTIPLE-USE

Having explored the theoretical implications of multiple-use for validation, I conducted a case study of one instance of multiple-use to gather additional insights. The Education Quality and Accountability Office (EQAO) Grade 9 Assessment of Mathematics is a mandatory assessment administered annually to approximately 140 000 students (EQAO, 2013). The assessment includes multiple-choice and open-response items and was developed as part of Ontario’s accountability program. While students are required to take the assessment, they need not obtain a minimum score to get credit in their Grade 9 math course nor to graduate from high school. Thus, on the basis of the intended use alone, this assessment can be considered a low-stakes assessment.

RESEARCH DESIGN

The design and rationale for the case study can be summarized as a sequence of six stages: (a) determine the uses of the EQAO Grade 9 assessment, (b) choose two uses to focus on, (c) gather evidence of how each use takes place, (d) analyse the evidence to find interactions between the two uses, (e) consider how Kane's (1992, 2006) approach to validation might account for these interactions, and (f) explore some analytic tools that may better address multiple-use in the validation process. An overview of the data collection, analysis and findings for each stage is provided.

METHOD AND FINDINGS

Determining Uses of the EQAO Grade 9 Assessment

To determine the uses of this assessment, I conducted an extensive analysis of EQAO documents and published research related to the Grade 9 assessment. This analysis revealed that the intended use of the assessment is as part of Ontario's accountability program (EQAO, 2009) with results for schools, districts, and the province released to the public each year (EQAO, 2013). However, several other multiple-uses take place. For instance, the Fraser Institute uses the assessment to develop and publish a ranking of secondary schools across Ontario (Cowley & Easton, 2008). EQAO does not support the use of the assessment for ranking but they do suggest a number of other benefits and uses for the assessment such as: providing individual student score reports, school and district improvement planning and target setting, encouraging the implementation of the curriculum, and improving teachers' and parents' assessment literacy (EQAO, 2009). EQAO also permits teachers to mark some or all of the items on the assessment before returning the booklets to EQAO for official scoring. Moreover, the Ontario Ministry of Education (OME) permits teachers to include these teacher-derived scores in their students' grades (OME, 2010).

Choosing Uses and Gathering Evidence of How Each Use Takes Place

Investigating the potential interactions among all the multiple-uses of the EQAO Grade 9 assessment was beyond the scope of the dissertation. Accordingly, I decided to focus on two uses: the intended use as part of Ontario's accountability program and the use of the assessment by teachers as part of their students' grades.

To better understand the practices teachers engage in when they use the assessment as part of students' grades, data was constructed from several sources: a document analysis; a province-wide teacher questionnaire (n=272); interviews with Grade 9 math teachers, math department heads, and principals in three schools in districts where teachers use the assessment as part of grades (n=14); and interviews with EQAO personnel involved in the design and administration of the Grade 9 assessment (n=6). Interviews were audio-recorded and transcribed for analysis.

Responses to the teacher questionnaire indicate that 91% of teachers across the province use the assessment as some part of their students' grades. However, considerable variation takes place across teachers, schools and districts in terms of the percentage the assessment contributes to students' grades (i.e., the weighting), the items teachers choose to score, and the procedures they use for scoring. With regard to weighting, the interviews reveal a range of practices including individual teachers and schools who do not include the assessment in grades, cases where the EQAO assessment is included as 5%, 10% or 15% of grades and cases where the assessment is used in lieu of an in school final exam constituting 20% of each student's grade. Variations were also observed within classrooms with some teachers including the assessment in grades for most students but not for those students who have difficulty demonstrating their mathematics understanding on pencil-and-paper tests.

With regard to choosing items to score, some teachers mark only the multiple-choice items while others mark a selection of multiple-choice and open-response items. Most teachers indicate they do not mark items that include content they did not cover or that are worded in ways their students might find confusing. Teachers must develop scoring guides for the items they mark because item-specific scoring guides are not released by EQAO at the time the assessment is written. Interestingly, several teachers indicate they base these scoring guides on the approach they use to score similar items on unit tests. More detail about the procedures teachers use to score the assessment as well as their reasons for deciding to include the assessment in students' grades and their views of the tensions created by this practice can be found in Koch (2010).

Interactions Between Multiple-Uses

An interaction occurs when practices associated with one use have an impact on the validity of the inferences that can be made for another use. Analysis of the interviews and EQAO documents reveals four interactions between the two uses of this assessment. One interaction relates to the weighting of the assessment in grades. Allowing teachers to decide how much the assessment contributes to students' grades results in non-standard testing conditions; some students approach the assessment with considerably more motivation than others. As a result, the comparability of the official scores across classrooms, schools and districts is compromised. Thus, a practice associated with the use of the assessment in students' grades impacts the validity of the inferences that can be made for accountability purposes.

A second interaction emerges from the approaches teachers use to score open-response items. Since teachers create their own scoring guides for these items, students have to decide if they should structure their response to align with their teachers' criteria or with the EQAO criteria. Students' official EQAO scores may be lower if they structure their responses according to their teachers' criteria. Again, a practice associated with one use has an impact on the validity of inferences made for the other use.

A third interaction emerges from the discrepancies that exist between teacher-derived and official EQAO scores. Since students and parents receive their teacher-derived score several months in advance of their official score, the teacher-derived score creates an expectation of what the official score will be. Moreover, differences between the scores undermine the perceived meaning and validity of each score.

In the fourth interaction, the case study observations show that characteristics of the EQAO assessment, such as the strong reliance on multiple-choice items and the requirement to keep the assessment items secure, impact the validity of the inferences that can be made when the test is used as a classroom assessment. Essentially these characteristics of the EQAO test enhance its validity as an accountability measure but detract from the validity of the test when it is used in students' grades.

Applying Kane's Model of Validation

Given the observed interactions, I considered how Kane's argument-based model could be used to validate the two uses of the EQAO assessment. In Kane's (2006) model, validation begins with building an interpretive argument that is based on a clear articulation of the meaning ascribed to the test score. Specifying the meaning of the EQAO score is challenging since it has one meaning in contexts where the assessment counts for students' grades and a different meaning in contexts where it does not count. In addition, the teacher-derived score and the official score have distinct meanings because each is based on a slightly different set of items.

The next step in Kane's approach is to determine the assumptions that are inherent in the interpretive argument and to collect validity evidence for each assumption. This process is also quite difficult when there are interactions between uses because the assumptions made for one use are not the same as the assumptions made for the other use. More importantly, the assumptions being made differ when the assessment is used for both grades and accountability than when it is only used for one of these purposes. These sorts of challenges led to the conclusion that Kane's approach is not well suited to contexts where evidence of multiple, interacting uses has been found.

Analytic Tools to Better Address Multiple-Use in the Validation Process

The challenges for validation that emerge from multiple-use may require the use of new analytic tools. Drawing on Moss' suggestion (2007; Moss et al., 2006), I demonstrate some ways that the wider interpretive scope of hermeneutics and the socio-cultural concept of *boundary objects* can contribute to validation in the context of multiple-use. Adopting a hermeneutic stance can facilitate the integration of a variety of types of validity evidence, provide a means of incorporating multiple interpretations of an assessment, and help ensure that the individuals involved in each use understand those interpretations. The concept of a *boundary object* comes from the work of Star and Griesemer (1989). A boundary object is an object, practice or abstract concept that is shared by two or more groups of people. The object acts as a bridge or translator across the groups even though it is interpreted and used differently by each group. In the dissertation, I present evidence that the EQAO Grade 9 assessment functions as a boundary object when teachers use the items as part of their students' grades.

CONTRIBUTIONS OF THE RESEARCH

In terms of contributing to the field of mathematics education, this study provides a portrait of how Grade 9 teachers incorporate a large-scale assessment into their classroom assessment practice. Analysis of the teacher interviews reveals how these teachers use professional judgment to decide on an appropriate weighting for the assessment in students' grades, choose which items to mark, create scoring guides, and ensure their use of the assessment reflects the mathematics learning that took place in their classroom. The dissertation also summarizes teachers' views of the limitations of multiple-choice items as measures of students' mathematics understanding. In addition, their unease with the use of large-scale assessments as measures of mathematics learning for students not taking academic level courses is documented. Papers exploring these aspects of teachers' assessment practice have been presented at conferences held by the National Council for Teachers of Mathematics and the Canadian Society for the Study of Education.

With regard to measurement, the identification and definition of the practice of multiple-use and the theoretical problems that emerge from this practice have not been previously explored. The study highlights the increase in stakes associated with multiple-use and the difficulties that emerge from interactions between multiple-uses. Some ways to further develop the process of validation using hermeneutics and boundary objects are presented using data from the empirical study. These approaches to validation could be used with any large-scale assessment where multiple-use takes place. Finally, the dissertation demonstrates how case study methodology can be used to frame the process of validation. This work is proving to be of interest to the measurement community and a paper is scheduled to appear in a forthcoming issue of *Educational Measurement: Issues and Practice*, a leading North American measurement journal.

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CREATING, UNDERSTANDING, AND TEACHING MATHEMATICS: COMPLEMENTARY PROCESSES

Tina Rapke
University of Calgary

ABSTRACT

My doctoral research, part of a fully joint PhD in mathematics and mathematics education, explored the relationships among understanding, creating, and teaching mathematics. My dissertation included two pairs of complementary mathematics and education papers. The papers are complementary in the sense that the education papers drew on the experiences of creating the mathematics that appeared in the mathematics papers and in doing so, provided insight into how the mathematical concepts, theorems, and proofs were constructed. The education papers not only provided insight into one mathematician's creating of mathematics, but mined the experiences and insights for their pedagogical implications.

OVERVIEW

The purpose of my doctoral research was to explore the relationships among understanding, creating, and teaching mathematics. The research came from a love of mathematics and a love of education and not being able to abandon one for the other. The work is about marrying the two disciplines and exploring their relationship. My dissertation included two pairs of complementary mathematics and education papers/chapters, which will be published in peer-reviewed journals of their respective disciplines. The mathematics papers follow the standard definition-theorem-proof structure of mathematics (Davis & Hersh, 1981; Thurston, 1994; Weber & Alcock, 2009). The education papers draw on the experiences of creating the mathematics that appears in the mathematics papers and in doing so provide insight into how the mathematical concepts, theorems, and proofs were constructed. A reader of my dissertation can choose to read a mathematics paper and its complementary education paper alongside one another or may choose to read the beginning of the mathematics paper and then the education paper and return to the mathematics paper or the other way around. Each pair is truly complementary in the sense that the mathematics paper and education paper can be read synchronously or one after the other.

The first mathematics paper in my dissertation is entitled *Oriented and Injective Oriented Colourings of Grid Graphs* and includes new bounds for the injective oriented chromatic numbers of grid graphs. Finding good upper bounds for oriented chromatic numbers of special classes of planar graphs has proven to be challenging (Ochem & Pinlou, 2006). The bounds for the injective oriented chromatic number of grid graphs, in this chapter, are deemed to be important to the mathematical community, as a manuscript containing the bounds has been accepted for publication in the *Journal of Combinatorial Mathematics and Combinatorial Computing*.

The accompanying education paper to the first mathematics paper uses the Pirie and Kieren model of dynamical growth of mathematical understanding (Martin, 2008; Pirie & Kieren, 1994) to conceptualize the phenomena of understanding. I trace experiences of a mathematician using examples for understanding, creating, and verifying mathematics. I illustrate some of the processes of creating the mathematics contained in the paper *Oriented and Injective Oriented Colourings of Grid Graphs*. The unpacking that occurred during the process of understanding oriented colourings is detailed.

The second mathematics paper is about injective-nice graphs and describes the specific structure of tournaments that are injective-nice. This chapter includes new definitions that have allowed me to describe a new class of digraphs. The companion education paper to the second math paper describes the unpacked mathematics of injective-nice tournaments. I describe the mathematics that the mathematical notions are built upon and illustrate some of my workings within the concept of injective-nice graphs. I trace the creation of the definitions to an example of a tournament that I created that is not injective-nice.

METHODOLOGY

For the education chapters of my dissertation, I chose to use an adaption of duoethnography because I wanted to go beyond telling a story of creating mathematics. Duoethnography is a research methodology that takes the form of a dialogue or play script and juxtaposes two or more different perspectives (Norris, 2008). A typical duoethnography involves researchers engaging in dialogue to explore their histories and identities. Authors and readers of duoethnographies learn about themselves through others' stories and opinions. The strength of the methodology lies in its acknowledgment that there is the potential to reconceptualise perspectives and beliefs through being open to the opinions and stories of others (Sawyer & Norris, 2013). The strength of duoethnography is in the unexpected shifts in perspectives and understandings. In line with Jardine (1998), it's not about controlling, predicting and manipulating the reoccurrence of the experiences. It's about "understanding being provoked by something unwittingly" (Jardine, 1998, p. 39). Duoethnography allowed me to push my perspectives and encourage readers to be provoked to insert their own stories within the dialogue to push their understanding.

As I am the sole author of my dissertation, I have adapted duoethnography by introducing fictional dialogue. In the education papers, I have not replayed the events in their exact details, but through fictional dialogue, I have mined, re-created, blended, and interrogated to provoke understandings and perspectives. Fictionalizing has also allowed me to infuse the dialogues with other voices of mathematicians and educators, to step beyond my inner world to represent two different perspectives. Other mathematicians' and educators' perspectives were also infused in the dialogue, as it is not the goal to disentangle two of my identities (i.e., mathematician and educator) but to create a space where motives, perspectives, and understandings can be interrogated.

Duoethnography is an emerging qualitative research methodology but its gestation can be found in other qualitative research genres. Duoethnography, and my adaptation of it, are consistent with ethnographic principles (Creswell, 1998; Ellen, 1984; Fetterman, 1989). Similar to autoethnography (Ellis & Bochner, 2000; Ellis, 2004) in which the researcher is the site, not the topic, of the research (Oberg & Wilson, 1992), I engaged daily in mathematical and educational research. The duoethnographic methodology moves away from an autoethnographic approach by having a dialogic form and emphasizing the reconceptualization of past experiences through the dialogic process. Duoethnography steps away from the meta-narrative style found in autoethnography as "readers are released from the hegemonic expectation of aligning with a protagonist" (Norris, Sawyer, & Lund, 2012, p. 10). This is the strength that duoethnography offers mathematicians and educators—it invites both mathematicians and educators to be provoked by other perspectives and to reconceptualise their own to open the possibility of new collaborative projects.

My work is developed in the form of a fictionalized conversation between two characters: Georgia and Lucas. Georgia represents a mathematician's perspective and Lucas represents an educator's perspective. Lucas is not only representing my identity that aligns with an educator, but draws on and includes the voices of other educators. Georgia's dialogue not only represents my identity as a mathematician, but draws on and represents opinions and conversations that I have had with mathematicians and graduate students in mathematics. The dialogue representing Lucas is derived from experiences with mathematicians with interests in education, graduate students in education, faculty members in education, and school teachers.

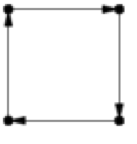
A SAMPLE

Some of the key findings of my work concern mathematical definitions and unpacking. One of the key findings of Ball and Bass' (2003) research involves the notion that teachers must deal with mathematics in its growing, unpacked form. This notion is paradoxical, as compression/packing is a key feature of advanced mathematics. I share a sample of the dialogue from my dissertation that involves definitions and the unpacking of a definition. Unlike mathematics papers, here I share examples upon which the mathematical notions were built.

Lucas: Hi Georgia! I took a look at your paper that you sent me. It's the one that you just had accepted for publication. I couldn't get past the definition of oriented colourings. Can you help me?

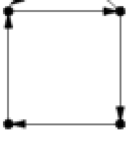
Georgia: Yes, of course. Let's first consider oriented colourings by talking about oriented graphs. Remember the graphs that I research consist of a set of vertices (or dots) and a set of edges (or lines connecting the dots). Also, two vertices are called adjacent if there is an edge between them and if an edge has a direction assigned to it then it's called an arc.

Actually, let's start by constructing an oriented graph. We begin with graphs that have no loops or multiple edges, called simple graphs, and then give each edge one of two directions. Here's an example of an oriented graph:



Lucas: What is the difference between oriented graphs and directed graphs?

Georgia: Here's an example of a graph that is directed but is not an oriented graph:



Notice that if we take away the directions of the edges there are two edges, i.e., multiple edges, between two of the vertices.

Okay, now we can talk about oriented colourings. Here's the formal definition:

An *oriented colouring* of an oriented graph $\vec{G}$ is a mapping c:V($\vec{G}$)→C where V($\vec{G}$) is the vertex set of $\vec{G}$, A($\vec{G}$) is the arc set of $\vec{G}$ and C is a set of colours such that:

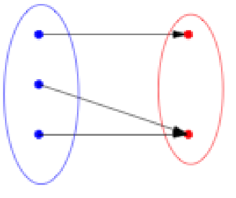
i. no two adjacent vertices receive the same colour and

ii. if the vertices x, y, w, z are assigned colours c(x), c(w), c(y), c(z), respectively such that c(x) = c(w), c(y) = c(z), and $\overrightarrow{xy} \in A(\vec{G})$, then $\overrightarrow{zw} \notin A(\vec{G})$

The game I play is trying to find the minimum number of colours required for a colouring that follows those rules, i.e., finding the oriented chromatic number of a graph.

Lucas: Huh? What are you talking about? I don't get what you just said….

Georgia: The first part of the definition for oriented colourings seems clear. If there is an arc between any two vertices, then they must receive different colours. The second condition is really saying that all the arcs between any two colours have the same direction. Here's what I really mean:



Notice there are no arcs between the blue vertices and there are no arcs between the red vertices. Also, ALL of the arcs are pointing from blue to red.

Lucas: So, can we work through some examples of oriented colourings? Like some of the mathematicians in Parameswaran's (2010) study, I find examples very useful when I'm trying to understand definitions and learn mathematics.

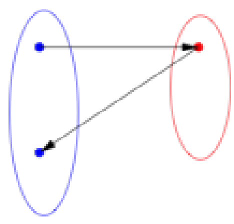
Georgia: Alright, let's start by looking at paths. Some paths of length two need three colours. Here's one:

This is a directed path, i.e., it is a path where all the edges are pointed in the same direction. Actually, all directed paths of length two need three colours.

Lucas: Why?

Georgia: We need three colours because if we try to colour it with two colours we end up with:

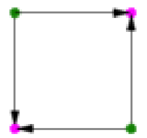
And then the path would look like:



There is an arc pointing from blue to red and another one pointing from red to blue. They need to ALL be pointing from red to blue or blue to red. Not both directions!

Lucas: Okay. I see what you're saying. Can we try another example?

Georgia: Sure. Here is a graph whose oriented chromatic number is two:



Lucas: You could have coloured that graph with four colours right? But you didn't because you're trying to find the minimum number of colours…

And Georgia, how about you give me an oriented graph and let me figure out how many colours it needs? There is a big difference between you telling me the answer and me figuring it out for myself. It's like Papert's (1972) distinction between teaching children about mathematics and having them do mathematics.

I'm not alone in thinking this either. Some of the mathematicians that Liljedahl (2004) interviewed also talked about having to work out examples for themselves. Mason (2010) talks about how 'in his retrospective regret', he used to show students how to do a problem.

You've already interpreted the definition for me and provided great visuals. I think I have a good concept image of oriented colourings. So let me play a bit….

FINDINGS/CONCLUSION

The education papers add to the education literature surrounding the practices of mathematicians (Burton, 1999a, 1999b, 2004; Ricks, 2009), examples and the learning of mathematics (Antonini, 2006; Edwards & Alcock, 2010; Iannone, Inglis, Mejia-Ramos, Siemons, & Weber, 2009; Watson & Mason, 2005), and students' difficulty with the role of mathematical definitions (Alcock & Simpson, 2002; Edwards & Alcock, 2010; Edwards & Ward, 2004).

In one of the education papers, I explored the processes of mathematicians creating original definitions and educators making definitions suitable for students at particular grade levels, noting that the processes have remarkable similarities. I believe that more research that focuses on the practices of mathematicians can provide new insights into how teachers can craft definitions that are appropriate for students.

In both the education papers, I detailed how I obtained results from wrong turns or mistakes. This indicates that teachers should be encouraged to help students fold back and "inquire whether there is something more behind it" (Pólya, 1971, p. 65) when they make errors. In my case, I inquired into my errors to see if there was more behind them. These inquiries led me to create original mathematical contributions in both papers. More research into mathematicians creating and including the trails to their results may greatly encourage teachers to value and pursue 'mistakes' in the classroom and help shape mathematics curricula.

Considering the education papers together indicates a relationship between compressing and unpacking mathematical understanding. Unexpectedly, by exploring how I compressed mathematics, it seems that in the second education paper, I had actually unpacked the mathematics for the reader. In the first education paper, I expected that through demonstrating my process of unpacking, I would unpack the mathematics in its companion mathematics paper. I did not predict that describing how I compressed mathematical notions in order to write its companion mathematics paper would result in the same type of unpacking for the reader. More research about mathematicians creating mathematics may indicate that compressing and unpacking are 'two sides of the same coin'. If this is the case, we can learn more about one process from the other. Thus, we should focus on researching how mathematicians create mathematics because this type of research can inform the research about the unpacking process that is necessary for teaching.

I am not aware of any academic materials where a mathematics paper has an accompanying paper that can be read alongside to help explain or unpack the mathematics by providing insight into how the mathematics concepts, theorems, and proofs were constructed. Presenting pairs of papers, like the ones found in my dissertation, might aid in communicating research mathematics to a larger population. Krantz (2008) says that communicating research is essential for advancing the discipline of mathematics. In addition, Tomlin (2005) shares that Gowers believes that "most mathematics papers are incomprehensible to mathematicians" (p. 622). Mathematicians sharing these experiences would not only inform the teaching and learning of mathematics but may advance the discipline of mathematics. Moreover, I hope that the format and methodology of my dissertation will act as an example of how other researchers can explore the processes of mathematicians. In the end, I feel that my dissertation speaks to some of the ways that documenting cases of creating mathematics and interpreting them for the learning of mathematics advances both disciplines, and I am confident that there are many more advances that can occur with similar research.

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ÉVOLUTION DES PROJETS DE FORMATION DE FUTURS ENSEIGNANTS DU PRIMAIRE AU CONTACT DE SITUATIONS PROBABILISTES

EVOLUTION OF PROSPECTIVE TEACHERS TRAINING PROJECTS IN CONTACT WITH PROBABILISTIC SITUATIONS

Miranda Rioux
Université du Québec à Rimouski

This communication unveils the results of our doctoral study, which stands at the crossroads of research in mathematics education and teacher training. It was conducted among 58 undergraduate students in special education at Université du Québec à Rimouski. Two general objectives were formulated: the first aims at describing student training projects while the second addresses the development of a sequence of situations to help enrich their initial projects. It was highlighted that although most students want to develop knowledge of techniques and teaching methods, the sensitivity to complexity shown in some projects does not allow the reduction of students' expectations regarding their training to the building of a repertoire of teaching techniques deemed effective.

DES VISIONS DISTINCTES DE L'ENSEIGNEMENT

Notre problème de recherche s'inscrit dans le contexte de la formation initiale à l'enseignement des mathématiques au primaire. À l'instar de DeBlois et Squalli (2002), nous avons noté que les étudiants avaient tendance à adopter une vision déterministe de l'enseignement, laquelle impliquerait que l'on puisse les former, à l'avance, pour répondre efficacement à tout problème potentiel d'enseignement. Les étudiants inscrits dans nos cours nourrissent ainsi, envers leur formation, des attentes que nous jugeons difficiles à combler puisque nous mettons plutôt en relief, pour paraphraser Lester et Wiliam (2000), la sensibilité des phénomènes éducatifs aux changements induits par le contexte dans lequel ils s'inscrivent. Or selon Sowder (2007), le développement d'une vision partagée de l'enseignement des mathématiques est l'un des principaux mandats de la formation initiale à l'enseignement des mathématiques. Nous nous sommes donc questionnée sur les moyens à déployer pour répondre à ce mandat.

VERS UNE VISION COMMUNE

Comment développer une vision partagée de l'enseignement des mathématiques? Si la question s'énonce clairement, la réponse, elle, se conçoit plus difficilement. Selon Caron (2010), la communication d'une vision constitue en elle-même tout un défi:

> *[...] s'il est relativement facile de communiquer une passion, il est beaucoup plus difficile de communiquer une vision: d'une part, parce que chaque vision est une construction individuelle qui s'appuie sur un ensemble d'expériences qu'il serait utopique et vain de vouloir partager dans le cadre d'un cours; d'autre part, parce que l'interprétation de cette vision par un tiers repose sur son propre répertoire d'expériences et sur ce que ce répertoire lui permet d'imaginer.* (p. 172)

En effet, nous avons souvent tenté, dès le début des cours, d'engager un dialogue avec les étudiants, avec pour intention manifeste de mettre en commun nos visions respectives et de favoriser leur rapprochement. Or cet exercice de mise en commun n'a que très rarement suscité le rapprochement escompté et nous pourrions même dire, pour paraphraser Bednarz (2010), que la prise de conscience du caractère peu viable de cet exercice nous a vite sauté aux yeux. C'est n'est qu'après avoir essuyé plusieurs échecs que nous avons finalement compris; pour favoriser le rapprochement de nos visions, il faut d'abord et avant tout regarder le même horizon.

EN PASSANT PAR L'IDÉE DE PROJET

Les attentes que nourrissent les étudiants à l'égard de leur formation ne sont pas fortuites; elles sont notamment tributaires 1) des expériences qu'ils ont vécues en tant qu'élèves et 2) des expériences qu'ils souhaitent vivre en tant qu'enseignants. Or s'il est impossible de changer les premières, il est toutefois possible d'orienter les secondes. C'est ainsi que nous en sommes venues à nous intéresser au concept de projet, lequel, selon Roegiers (2007), est élaboré dans le but ultime d'amenuiser l'écart entre le vécu et le souhaité:

> *Le projet anticipe une situation future. Anticiper, c'est se projeter dans le futur. C'est suspendre momentanément le cours des choses pour chercher à savoir comment ce cours va évoluer, pour tenter le cas échéant d'infléchir la suite des évènements. Une anticipation n'est donc pas passive. Il y a dans la notion de projet le désir de maîtriser ce futur, voire même de le modifier.* (p. 181)

Cette projection dans le futur est une condition nécessaire, mais non suffisante à l'établissement d'un projet. Jonnaert (2000) disait d'ailleurs du projet qu'il « ne peut pas s'exprimer uniquement en termes de rêverie » (p. 119). Il doit absolument comporter une planification des étapes à franchir, des actions à poser pour que le projet se concrétise. Les attentes que nourrissent les étudiants à l'égard de leur formation découlent de cette planification, plus ou moins explicite, des apprentissages à effectuer pour être en mesure d'enseigner les mathématiques aux élèves des classes primaires. S'intéresser au concept de projet, dans le contexte que nous avons esquissé, c'est ainsi prendre en compte à la fois la vision qu'ont les étudiants de l'enseignement des mathématiques et leurs attentes à l'égard de la formation à son enseignement. Nous avons donc choisi 1) d'explorer les projets de formation des étudiants et 2) de concevoir et de mettre à l'essai une séquence de situations susceptible de favoriser leur évolution.

DES CONCEPTS POUR ANALYSER LES PROJETS

L'élaboration de notre cadre théorique nous a permis de fixer les modalités d'analyse des projets de formation. Dans le cadre de cet article, nous n'avons pas l'espace pour effectuer une présentation détaillée de chacune d'entre elles. Nous en offrons toutefois une description succincte et invitons le lecteur à consulter notre thèse pour obtenir une présentation plus détaillée de ces modalités.

LES PROJETS 'VISÉE' ET PROGRAMMATIQUE

Les projets 'visée' et programmatique correspondent, de la façon dont ils ont été théorisés par Ardoino (1977, 1984, 1985), à des degrés différents d'élaboration du projet, le projet 'visée' correspondant à l'intention de mener une action, le projet programmatique correspondant plutôt aux opérations devant être menées afin de réaliser cette action (voir tableau 1). Ainsi, dans l'analyse du projet de formation des étudiants, il semble opportun de distinguer 1) l'anticipation des connaissances et des compétences professionnelles qui doivent être développées durant la formation initiale à l'enseignement des mathématiques (projet 'visée') et 2) l'anticipation des activités de formation requises pour développer ces compétences professionnelles (projet programmatique).

Projet 'visée'	Projet programmatique
• Anticipation plus ou moins détaillée • Formulée au présent • D'une finalité (situation ou état visé) • Devant advenir dans un futur plus ou moins lointain	• Planification des ressources /des actions • Jugées utiles ou nécessaires • Pour réaliser un projet 'visée'

Tableau 1. Caractéristiques distinctives du projet 'visée' et du projet programmatique.

LES MODES D'ANTICIPATION DU PROJET

Tout projet implique une anticipation du futur. Il ne peut en effet en être autrement, la situation ou l'état visé ne correspondent pas, pour l'auteur du projet, à une situation ou à un état actuel. Cette anticipation du futur prévoit selon nous au moins trois scénarios possibles, que nous baptiserons *futur anticipé, futur projeté* et *futur idéalisé*. Il s'agit de :

1. L'anticipation d'un futur proche ou lointain où le projet n'est pas réalisé (*futur anticipé*);
2. L'anticipation d'un futur proche ou lointain où le projet est réalisé (*futur projeté*);
3. L'anticipation d'un futur idéal (*futur idéalisé*).

La notion de projet est liée à une volonté de changer ou de maîtriser un futur proche ou lointain que l'acteur anticipe (1), afin qu'au terme du projet, le futur projeté (2) corresponde davantage au futur que l'acteur a idéalisé (3). Selon Roegiers (2007), il y a trois modes d'anticipation du projet: *le mode adaptatif*, *le mode prévisionnel* et *le mode prospectif*. Dans *le mode adaptatif*, la probabilité de réalisation du futur anticipé (1), bien qu'estimée de manière plutôt subjective, est estimée très forte et est réputée indépendante de la volonté du sujet. Dans *le mode prévisionnel*, le sujet cherche à se préparer à un futur dans lequel il sait que certaines variables auront changé et pour lesquelles il anticipe la direction du changement. Enfin, dans *le mode prospectif*, le sujet anticipe un avenir lointain, à propos duquel on ne connaît pas grand-chose. La frontière entre le futur anticipé (1), le futur projeté (2) et le futur idéalisé (3) est donc moins distincte pour celui qui anticipe grâce à un mode prospectif.

DES SITUATIONS POUR APPRÉHENDER LA COMPLEXITÉ

Comment amener les étudiants à développer, dans le cadre de leurs cours en didactique des mathématiques, des projets qui s'appuient sur une vision moins déterministe et, par conséquent, plus complexe de l'enseignement des mathématiques? Comment faire en sorte qu'ils appréhendent la complexité de cet enseignement?

LES HEURISTIQUES DE JUGEMENT

Tversky et Kahneman (1983) ont étudié les jugements émis dans des situations non déterministes et ont identifié deux types de cognitions pouvant servir d'assise à ces jugements: 1) les modèles stochastiques, qui sont des modèles mathématiques autorisant un traitement statistique des situations aléatoires et 2) les heuristiques de jugement, qui sont des types de raisonnement intuitif permettant de porter rapidement des jugements dans des situations d'incertitude. Kahneman et Frederick (2002) associent les heuristiques de jugement à un processus de substitution d'attributs. Ainsi, un jugement découle d'une heuristique s'il se base sur l'évaluation non pas de la caractéristique ciblée par le jugement, mais sur l'évaluation d'une autre caractéristique, plus simple à évaluer. Par exemple, pour évaluer la distance qui sépare une personne d'un objet, une personne pourrait examiner la netteté de l'objet qu'elle perçoit. Elle jugera ainsi qu'un objet est distant si son image est floue et proche si son image est nette. Cette manière de juger donnera, la plupart du temps, de bons résultats. Cela dit, des facteurs environnementaux tels que la neige ou le brouillard pourraient affecter l'image de cet objet et un objet qui semble distant pourrait ainsi se révéler être plus proche qu'il n'y paraît.

LE JUGEMENT DE PROBABILITÉ

Dans la vie de tous les jours, les individus ont tendance à délaisser l'utilisation des modèles stochastiques afin de privilégier l'utilisation d'heuristiques de jugement (Tversky & Kahneman, 1971, 1974). Bien qu'il arrive que les jugements émis grâce à ces heuristiques coïncident avec ceux qui auraient été inférés à partir de modèles stochastiques, il y a toutefois plusieurs situations où ce n'est pas le cas (Konold, 1989), notamment lors du traitement de situations probabilistes, situations dont la contre-intuitivité n'est plus à démontrer. C'est ainsi qu'il nous a semblé pertinent de nous intéresser au jugement de probabilité et d'examiner l'apport de ce cadre sur la rencontre des projets; nous pensons en effet que ce cadre situationnel pourrait se révéler particulièrement propice au développement d'un système de significations qui permette aux étudiants non seulement de répondre de manière optimale aux problèmes que nous leur posons, mais également d'apprendre 1) à raisonner de façon moins déterministe et 2) à composer avec l'incertitude des phénomènes éducatifs.

MÉTHODOLOGIE

Cette recherche a été menée auprès de 58 étudiants du baccalauréat en enseignement en adaptation scolaire et sociale de l'UQAR, lesquels entamaient leur formation initiale à l'enseignement des mathématiques. Ces étudiants avaient été recrutés sur une base volontaire et n'avaient jamais suivi de cours de didactique des mathématiques.

EXPLORER LES PROJETS DE FORMATION

Trois instruments nous ont permis d'explorer les projets de formation des étudiants : un questionnaire individuel sur leur vision de l'enseignement des mathématiques, une discussion de groupe sur le sujet de même que des entretiens individuels avec certains participants (8).

Les questionnaires individuels

Les questionnaires individuels ont été auto administrés et ont été complétés par les étudiants au tout début de leur formation initiale. Dix questions visaient à explorer le projet de formation des participants à l'étude. Ces questions ont été posées afin d'explorer les deux faces des projets de formation des futurs enseignants, soit leur projet 'visée' et leur projet programmatique. Exception faite de la question 6, l'ensemble des questions figurant dans ce questionnaire étaient ouvertes et engageaient la production d'une réponse à court développement.

La discussion de groupe

Une discussion de groupe a été menée afin de recueillir des données supplémentaires sur les projets de formation des futurs enseignants. Cette discussion s'est inscrite dans le cadre des activités normales de leur formation et s'est tenue dans la salle de cours. Pour guider cette discussion, nous avions préparé un canevas comportant différentes questions sur leur projet de formation ainsi que sur leur façon de voir l'enseignement des mathématiques. Ce canevas reprenait grosso modo les items du questionnaire individuel et a été utilisé de manière à susciter, entre les étudiants, les échanges les plus riches possibles.

<u>Les entretiens individuels</u>

Après la réalisation de notre séquence, huit étudiants ont été choisis pour être reçus en entretien individuel. Durant chacun des entretiens, nous avons repris les réponses émises dans le questionnaire afin de vérifier si les participants étaient toujours en accord avec ce qu'ils avaient répondu. Nous les avons également invités à raffiner l'expression de leur pensée et à préciser, le cas échéant, le rôle qu'a joué la séquence de situations dans la modification de leur projet initial.

FAVORISER L'ÉVOLUTION DES PROJETS

Trois problèmes ont été présentés aux étudiants: le problème de l'hôpital (problème 1), le problème des pièces de monnaie (problème 2) et le problème des jetons (problème 3).

<u>Problème 1</u>

Le premier problème est une traduction d'un problème posé par Fischbein et Schnarch (1997), lui-même inspiré du problème initialement proposé par Kahneman et Tversky (1982). En voici l'énoncé :

> Dans une ville, il y a 2 hôpitaux. Il y a un petit hôpital où il y a, en moyenne, 15 naissances par jour, et un grand hôpital où il y a, en moyenne, 45 naissances par jour.
>
> La probabilité de donner naissance à un garçon est environ de 50%. Il y a toutefois des jours où il y a plus de 50% des nouveau-nés qui sont des garçons et d'autres jours où il y a moins de 50% des nouveau-nés qui sont des garçons.
>
> Dans le petit hôpital, on tient un registre des jours dans l'année où le nombre de nouveau-nés de sexe masculin est supérieur à 9, ce qui représente plus de 60% des naissances dans le petit hôpital. Dans le grand hôpital, on tient également un registre des jours dans l'année où le nombre de nouveau-nés de sexe masculin est supérieur à 27, ce qui représente plus de 60% des naissances dans le grand hôpital.
>
> Selon vous, peut-on inférer à partir de ces informations qu'un des deux hôpitaux doit compter plus de jours où le nombre de nouveau-nés de sexe masculin est supérieur à 60%? Si oui, précisez lequel. Si non, expliquez pourquoi il n'est pas possible d'avancer une telle affirmation.

<u>Problème 2</u>

Le deuxième problème est lui aussi la traduction d'un problème posé par Fischbein et Schnarch en 1997. En voici l'énoncé :

> Selon vous, quel événement est le plus probable : l'événement « lancer 3 pièces de monnaie et obtenir au moins 2 faces » ou l'événement « lancer 300 pièces de monnaie et obtenir au moins 200 faces » ?

<u>Problème 3</u>

Le troisième problème de la séquence a été expérimenté dans les années '70 par Nadine et Guy Brousseau, à l'école Jules Michelet de Talence (Brousseau, Brousseau, & Warfield, 2002). En voici l'énoncé :

> J'ai envie de vous proposer un défi, dont la solution nous fournira peut-être des pistes pour valider les réponses émises pour le problème précédent. Le voici :
>
> Voici une boîte contenant un grand nombre de jetons noirs et de jetons blancs. Sans regarder, je pige 5 jetons dans cette boîte et les dépose, toujours sans regarder, dans un sac opaque. En touchant au sac de l'extérieur, il est possible de constater qu'il y a bel et bien 5 jetons à l'intérieur. Je vous propose comme défi de déterminer la composition du sac, mais en tâchant de respecter les deux contraintes suivantes : 1) personne n'a le droit de regarder dans le sac; et 2) vous avez le droit de piger et de regarder juste un jeton à la fois, en prenant soin de remettre le jeton dans le sac immédiatement après l'avoir pigé.
>
> Je vous invite à vous placer en équipe de 4 personnes. Il vous faudra consigner dans ce document les traces de votre raisonnement. Nous comparerons ensuite les résultats obtenus par chaque équipe.

En empêchant graduellement l'utilisation d'une heuristique de jugement, ces trois problèmes devaient inciter les étudiants à porter un jugement de probabilité qui prenne en compte la complexité des situations proposées. Il s'agissait ici d'émettre un jugement qui considère la taille de l'échantillon. Le tableau 2 rend compte de leur articulation.

Problème 1 - Hôpital	**Problème 2 – Pièces**	**Problème 3 – Jetons**
Forte probabilité d'une réponse intuitive qui néglige la taille de l'échantillon	Probabilité importante, mais moins forte qu'au problème 1, d'une réponse qui néglige la taille de l'échantillon	Faible probabilité d'une réponse qui néglige la taille de l'échantillon
Utilisation d'une heuristique de représentativité possible, mais non congruente avec celle d'un modèle stochastique	Utilisation d'une heuristique de représentativité possible, mais non congruente avec celle d'un modèle stochastique	Utilisation plus difficile d'un processus de substitution d'attributs (pas d'attributs heuristiques) Nécessité d'utiliser un modèle stochastique pour trouver la réponse

Tableau 2. Articulation des trois problèmes de la séquence.

RÉSULTATS

LES PROJETS AVANT LA SÉQUENCE

Nos résultats tendent à démontrer que les futurs enseignants visent tout d'abord à se former eux-mêmes, et qu'ils souhaitent ensuite évoluer suffisamment pour être en mesure d'enseigner les mathématiques et de faire apprendre ou comprendre les mathématiques aux élèves. Ils nourrissent certes des visées concernant les élèves et les mathématiques, mais ces visées passent après celles qui les concernent directement. Cela se reflète dans les programmes qu'ils comptent utiliser pour atteindre ces visées, puisque les programmes recensés le plus fréquemment sont centrés sur les connaissances concernant l'enseignement et de façon plus spécifique, sur les connaissances relatives aux techniques et aux méthodes d'enseignement. Présentés comme tels, ces résultats pourraient nous porter à croire que les futurs enseignants nourrissent une vision très déterministe de l'enseignement, et que leur soif de techniques et de méthodes d'enseignement reflète une quête de recettes d'enseignement présumées à toute épreuve. Toutefois, lorsque ces résultats sont confrontés aux modes d'anticipation des projets des étudiants, il appert que les connaissances qu'ils souhaitent développer correspondent plutôt à des enjeux de contrôle d'un futur auquel ils devront s'adapter. Ils anticipent en effet un futur à l'intérieur duquel les élèves ont des visions distinctes des problèmes et des concepts mathématiques et souhaitent par conséquent apprendre différentes techniques et méthodes d'enseignement. L'idée n'est donc pas d'apprendre une technique à toute épreuve, mais bien d'être en mesure de rejoindre tous les élèves en développant et en utilisant un répertoire de techniques et de méthodes variées, ce qui constitue le signe d'une vision qui se situerait entre déterminisme et appréhension de la complexité.

LES PROJETS APRÈS LA SÉQUENCE

Avant la réalisation de la séquence, trois étudiants nourrissaient une vision déterministe, deux nourrissaient une vision intermédiaire alors que trois témoignaient d'une sensibilité à la complexité des phénomènes et des problèmes d'enseignement. Il est également à noter qu'à l'exception d'une étudiante, tous les étudiants avaient initialement formulé un projet programmatique ciblant des connaissances concernant uniquement l'enseignement. Après la réalisation de la séquence, les projets 'visée' des trois étudiants témoignant d'une vision déterministe ont gagné en complexité, se rapprochant dorénavant de la quête d'une variété de stratégies d'enseignement. En ce qui a trait aux projets des étudiants qui témoignaient d'une vision intermédiaire, on note un léger gain de complexité chez une étudiante, ainsi qu'une légère perte de complexité chez une autre. Enfin, en ce qui a trait à l'évolution globale des projets des étudiants témoignant d'une sensibilité à la complexité, exception faite d'une étudiante, on note un élargissement des projets 'visée' et/ou programmatique. Ils ont par ailleurs tous les trois maintenu leur sensibilité à la complexité. En somme, après la réalisation de notre séquence de situations, les projets 'visée' de quatre étudiants se sont complexifiés, trois sont demeurés stables et un a régressé.

Il convient maintenant de se demander si notre séquence de situations a favorisé une certaine reformulation des modes d'anticipation des projets des étudiants. Nous avons constaté que tous les étudiants sélectionnés pour des entretiens réalisaient des anticipations grâce aux modes adaptatif et prévisionnel. Quand ils anticipent leur futur, ils réfléchissent surtout à ce qui « doit être » et à ce qui « ne doit pas être ». Contrairement à ce que nous avions observé avant la réalisation de notre séquence, des anticipations se rattachant à tous les modes d'anticipation ont cette fois-ci été relevées. Des anticipations réalisées grâce à un mode prospectif ont ainsi été émises et permettent notamment d'identifier des zones d'incertitude et de liberté sur lesquelles il est possible d'agir, en tant que formateur, afin d'accroître la sensibilité à la complexité des situations et des pratiques professionnelles que les futurs enseignants devront maîtriser.

CONCLUSION

En utilisant le jugement de probabilité comme cadre situationnel, nous pensions que les étudiants allaient apprendre à raisonner de façon moins déterministe et que cette évolution allait ultimement favoriser le développement d'une vision partagée de l'enseignement des mathématiques. Est-ce que ce fut le cas? Certes, les étudiants en sont venus à émettre des jugements de probabilité qui prennent en compte la complexité des situations. Le problème des jetons leur a notamment permis de réaliser que s'il est impossible de prédire le résultat d'un tirage, il est néanmoins possible d'observer, en effectuant un grand nombre de tirages, la stabilisation de la fréquence relative d'obtention de chaque résultat. Ce ne sont toutefois pas tous les étudiants qui ont appliqué ce raisonnement aux problèmes d'enseignement et qui ont réalisé que s'il est impossible de garantir l'efficacité d'une intervention, il est toutefois possible d'observer, sur la longue durée, que certaines interventions sont plus fécondes que d'autres. Dans notre thèse, nous croyons néanmoins avoir fait valoir qu'en cherchant à faire vivre aux enseignants des situations qui enrichissent leur répertoire d'anticipations, il est possible de penser l'organisation de la formation à l'enseignement dans une direction qui soit plus féconde.

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Ad Hoc Sessions

Séances ad hoc

REALLY BIG IDEAS: ADVENTURES IN DATA MANAGEMENT

Iain Brodie
Western University

"*Look, Ma! No hands!*"

Parents have heard this phrase and never failed to shudder. Their child, having mastered the basics of locomotion on only two wheels is pushing the boundaries of the possible. And what is it that the parent inevitably says to the child?

"*Put both hands back on those handlebars before you get hurt!*"

What if the parent did not accept the inevitability of a fall, but recognized the achievement and encouraged more?

"*You know, if you lean a tiny bit to the right or left, you can turn your bicycle with no hands.*"

A parent tries to limit the risk-taking of their children. The limits they place upon them are frequently based on their own perceived limits, but the truth of the matter is that each generation of children is ever so slightly more capable than their parents' generation.

This is the quandary that faces us in mathematics education, especially at the elementary level. We have designed a curriculum that is a 'both hands on the handlebars' type of curriculum supposedly to safely take our children from one concept to another. All the different strands are carefully and neatly separated so that each can be learned without the intrusion of related concepts. The only problem with this is that while it is safe, it is also particularly well designed to beat the natural curiosity and risk-taking out of the students by the time they have reached their teenage years.

There is a better way. A way that mixes concepts. A way that encourages risk-taking. A way that allows for children to construct ideas, concepts and knowledge, and to make connections between big, important concepts. It is a way that involves really big ideas.

This is the mathematical story of a class of grade three children who inadvertently discovered the normal curve when trying to answer the question, "How long is a worm from our class vermicomposter?" They search all over their school for another group of things that they can measure to produce another normal curve. Their failures and successes were all taken in stride, and along the way they internalized how to measure accurately, to really understand mode, to flirt with the law of large numbers, to understand that a normal curve is high in the middle and low at both ends, and to become extremely good and efficient at graphing large sets of data.

This is a hopeful tale that points us in a new direction of how we might make up mathematical activities for young children. Rather than the disaggregation and simplification of mathematical concepts, these grade three children have shown that rich, complex, really big mathematical ideas can be understood using the concepts they already know.

USING LEARNING OBJECTS IN THE MATHEMATICS CLASSROOM

Carol Carruthers
Seneca College

There are multiple resources freely available on the internet that can be used to augment materials presented in the mathematics classroom. Wiley (2000) defines a *learning object* as "any digital resource that can be reused to support learning" (p. 7). Janson and Janson (2009) determine that digital learning objects "challenge students to question, investigate, analyze, synthesize, problem solve, make decisions, and reflect on their learning" (p. 1). The purpose of this *ad hoc* was to demonstrate to the audience the benefits of the learning objects consistently used in my mathematics classroom. The participants were polled to determine which, if any, learning objects they have used in their practice. The anticipated outcome was to develop a catalogue of learning objects mathematics educators could potentially find useful. The list generated was emailed to all participants and a brief review of some examples is provided below.

In 2007, John Breen developed the online game *Free Rice* (http://freerice.com/category). Mathematics students can open the website and practice their skills with the multiplication tables or basic algebra. For each correct answer, sponsors donate 10 grains of rice to the United Nations World Food Program. I use this website as an icebreaker activity to help students to become familiar with the educational resources on the web. For developmental classes, its use gives students an understanding that although we will be reviewing basic mathematical concepts, our approach will be quite different and novel from any previous learning experience.

John Page developed the *Math Open Reference Project* (http://www.mathopenref.com) to provide high-quality digital mathematics content to serve as an enhancement over paper textbooks. For first-year college students, this website is used for learning geometry. As students at this level have a variety of learning experiences, this website can provide a differentiated approach. For example, regarding the page entitled "Angle" (http://www.mathopenref.com/angle.html), those who have studied geometry since elementary school can simply use the applet to refresh their understanding of angle terminology. For students trained in other languages, complete definitions are provided as an explanation. Finally, for those that have not taken geometry previously, this website gives an interactive format to learn the subject matter in detail. In all cases, students can learn at their own pace, at the level that suits their needs.

Math is Fun (http://www.mathsisfun.com) is maintained by Rod Pierce and consists of mathematical activities for all levels from Kindergarten to Grade 12. According to the website, each page is edited by three reviewers and receives constant input from user feedback. Students find this resource an easy website to learn from due to its simple page-like construction and directed animation. Each page concludes with a short quiz to consolidate learning. Teachers can demonstrate these quizzes in class to provide instantaneous feedback of concepts. Finally, as the site has its own search engine, both teacher and students can easily navigate directly to desired content.

The *Math Worksheet Generator* (http://www.math-aids.com) allows Kindergarten to Grade 12 teachers, students or parents to print PDF format worksheets. The advantage of a worksheet generator is that students can customize their learning by choosing the characteristics of the mathematics concepts they want to learn. Each user determines the type and how many questions are required. A new worksheet is generated for students to complete. Students can compare their work with the stepwise solutions provided and decide for themselves whether additional work is needed.

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MATHEMATICS THERAPY: ENGAGING PRE-SERVICE TEACHERS IN RICH MATHEMATICAL EXPERIENCES

Martha Mavor and Leah Payerl
Western University

When confronted with the question, "*How do you feel about mathematics?*" most adults openly share a dislike, aversion, or negative perspective about the subject (Gadanidis, 2005). This negative perception about mathematics also holds true for pre-service teachers. Gadanidis (2004) explored this phenomenon using a simple demonstration of raising hands, asking approximately 440 elementary pre-service teachers to indicate whether they loved math, or hated math. An overwhelmingly negative response to math was observed. This distaste for mathematics is a challenge faced by faculties educating the province's newest teachers.

Gadanidis (2004) has found that while many teachers enter their faculties of education with an aversion to mathematics, he has also discovered that those who do enjoy math, generally reflect on having positive school mathematics experiences as a child, engaging in problem-solving opportunities, and/or enjoying mathematics as a family activity through daily immersion in games, real-life problems, and mathematical thinking. This suggests that perhaps the enjoyment of mathematics may be a necessary element in pre-service teacher education programs.

Mathematics Therapy, as coined by Gadanidis and Namukasa (2005) is an approach to generalist elementary school teacher preparation in mathematics that allows participants an opportunity to experience mathematics in a markedly different way than they may have in their own education. This allows for the possibility of repairing the damaged relationship they have developed with the subject over time. This therapeutic approach involves participating in and reflecting on a series of rich mathematical learning tasks, and culminates with writing a mathematics essay at the end of the term.

During the thought-provoking discussion of this *ad hoc* session, several members suggested that there appears to be a disconnection between mathematics as a subject matter and mathematics in everyday life, asserting that perhaps those who do not participate in math-related work fields often have misconceptions or skewed views of what mathematicians, or those who study or teach mathematics actually do. This suggests that perhaps *Mathematics Therapy* has a role in demystifying the subject area.

Similarly, many members recounted instances where they had interacted with people who have negative affective responses to mathematics and often openly shared their distaste for the subject. Apprehension towards mathematics was also likened to apprehension to other subjects such as music, visual art, and other languages, subjects that are seen as requiring specific expertise or technical content knowledge. It was suggested that pre-service teachers need an access point to mathematics, permission to make mistakes, and experience with problems that instil confidence. Rich, open tasks such as those used in *Mathematics Therapy* may be one outlet for pre-service teachers to experience mathematical success.

Also, the process of *Mathematics Therapy* was likened to those of cognitive and behavioural therapies, and similarly others proposed potential connections to counselling psychology. It was suggested that perhaps these approaches might provide insight into the process that the pre-service teachers move through as they experience mathematics in this rich, problem-based format.

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STUDENTS DOING RESEARCH IN A MATHEMATICS EDUCATION COURSE… WHY NOT?

DES ÉTUDIANTS QUI FONT DE LA RECHERCHE DANS UN COURS DE DIDACTIQUE DES MATHÉMATIQUES… POURQUOI PAS?

Manon LeBlanc
Université de Moncton

In order to integrate research in the initial teacher training of students enrolled in a bachelor in secondary education program with a first or second concentration in mathematics, some changes were made to the mathematics education course at the Université de Moncton (N.-B.). The assignments were modified in order to allow students to achieve all the learning outcomes through the implementation of didactic engineering (Artigue, 1996). Such an approach allowed university students to make links between theory and practice while working with students in the school system and to write an article in order to share their experience with other teachers. This *ad hoc* session took the form of a discussion primarily aimed at sharing the learning experience lived in the mathematics education course, but also to see if this experience could be replicated in a class with more students.

EXPERIENCE LIVED IN THE MATHEMATICS EDUCATION COURSE

First and foremost, it should be noted that only three students were enrolled in this particular mathematics education course during the winter session of 2013. It is largely the low number of students that led us to try a different experience with them. A local high school teacher agreed to open the door of his classroom to us and chose two key concepts on which our students could work. Each student chose the concept he or she preferred did an *a priori* analysis of that concept and subsequently built an assessment tool based on the preliminary analysis. The study of the results obtained from these tools allowed students to identify the concept with which they wanted to work until the end of the session. They chose trigonometry, mainly because of the number and the types of errors made by students. As a team, they developed a teaching-learning scenario on trigonometry, taking into account both the preliminary analysis and the errors found in the assessment tool. They tested their scenario in a Grade 10 class and, subsequently, did a micro-didactic and macro-didactic *a posteriori* analysis. Finally, they shared their experience in a text that was submitted to the professional journal *Envol* (see no. 162, pp. 17-20).

COMMENTS FROM PARTICIPANTS

The participants in the *ad hoc* session recognized the value of such an experience for students enrolled in initial teacher training. However, several questions remain. The low number of students allowed us to make formative assessment several times during the academic session and to monitor students closely (without ‘choking’ them in their learning). Is it realistic to expect to do the same with a class of 25 or 50 students? A second limitation lies in managing partnerships with teachers (finding enough teachers who welcome university students in their classroom). In short, this experience seems very rewarding for our students, but could be potentially difficult to manage for professors. How could we enable more students to live this learning experience?

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Dans le souci d'intégrer la recherche à la formation initiale des étudiants inscrits au baccalauréat en enseignement secondaire avec une première ou une deuxième concentration en mathématiques, certains changements ont été amenés au cours de didactique des mathématiques de l'Université de Moncton (N.-B.). Les travaux ont été modifiés de façon à atteindre l'ensemble des résultats d'apprentissage à travers la réalisation d'une *ingénierie didactique* (Artigue, 1996). Une telle façon de faire a permis aux étudiants de faire des liens entre la théorie et la pratique en travaillant auprès d'élèves et de rédiger un article afin de partager leur expérience avec d'autres enseignants. Cette session *ad hoc* a donc pris la forme d'une discussion visant, premièrement, à partager l'expérience vécue dans le cours de didactique des mathématiques et, dans un deuxième temps, à voir si cette expérience pourrait être reproduite dans une classe avec plus d'étudiants.

EXPÉRIENCE VÉCUE DANS LE COURS DE DIDACTIQUE DES MATHÉMATIQUES

D'emblée, il importe de préciser que seuls trois étudiants étaient inscrits au cours de didactique des mathématiques au secondaire pendant la session d'hiver 2013. C'est en grande partie ce nombre peu élevé d'étudiants qui nous a amenée à tenter une expérience différente avec eux. Un enseignant a accepté de nous ouvrir la porte de sa salle de classe et a ciblé deux concepts clés sur lesquels nos étudiants pouvaient travailler. Chaque étudiant a choisi le concept qu'il préférait, a fait des analyses préalables de ce dernier et a par la suite construit un instrument d'évaluation en fonction des analyses préalables réalisées. L'étude des résultats obtenus sur ces instruments a permis aux étudiants de cibler le concept avec lequel ils désiraient travailler jusqu'à la fin de la session, soit la trigonométrie. Leur choix a été influencé par le nombre et le genre d'erreurs commises par les élèves. En équipe, ils ont développé un scénario d'enseignement-apprentissage en prenant en compte à la fois les analyses préalables et les erreurs commises sur l'instrument d'évaluation. Ils ont expérimenté leur scénario dans une classe de 10^{e} année et ont par la suite fait une analyse *a posteriori* microdidactique et macrodidactique. Enfin, ils ont relaté leur expérience dans un texte qui fut soumis à la revue professionnelle *Envol* (voir no. 162, pp. 17-20).

COMMENTAIRES DES PARTICIPANTS

Les participants à la session *ad hoc* ont reconnu la pertinence de faire vivre une telle expérience d'apprentissage à des étudiants inscrits à la formation initiale en enseignement. Or, plusieurs questions demeurent. Le nombre peu élevé d'étudiants nous a permis de faire de l'évaluation formative à plusieurs reprises pendant la session et de suivre les étudiants de près (sans toutefois les « étouffer » dans leur apprentissage). Serait-il réaliste de penser en faire autant avec une classe de 25 ou de 50 étudiants? Une seconde limite réside dans la gestion des partenariats avec les enseignants (trouver suffisamment d'enseignants pour accueillir les étudiants). Bref, cette expérience semble très enrichissante pour les étudiants, mais possiblement difficile à gérer pour les professeurs. Comment faire pour permettre à plus d'étudiants de vivre une telle expérience?

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HINDRANCES AND AFFORDANCES IN TEACHER-AS-RESEARCHER

Tim Sibbald
Nipissing University

This *ad hoc* session was a small discussion group that discussed teacher-as-researcher as it might be interpreted for practicing teachers. The stance of a practicing teacher trying to implement a research activity was taken. The discussion framework, shown in Figure 1, guided the discussion. In this framework, the teacher who is trying to implement a research activity has various considerations, with each varying in terms of both practical-theoretical position and the amount of hindrance-affordance that is available to address the consideration. Each aspect of activities necessary for the teacher to be successful corresponds to a point on the discussion framework. The discussion was exploratory in the sense of allowing a wide variety of aspects to be discussed and the position on the framework to be discussed.

In the initial phase of the discussion, an issue around the interpretation of the teacher-as-researcher came to light. While the term could be interpreted as the perception of how a practicing teacher's activities might be interpreted as analogous to research activities, the intention was to have a definition that was more in keeping with academic research. In the former sense, one could interpret a teacher implementing a new teaching strategy as research because it is new to them. However, the intention of the discussion was to focus on research in the sense of being enabled to contribute, in a meaningful way, to a broader community of teachers.

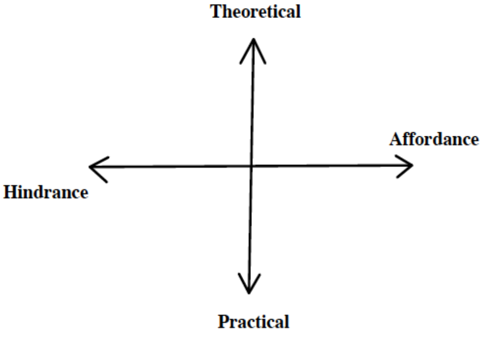

Figure 1. Discussion Framework.

Much of the discussion was focused on the left side of the diagram. Particular note was made of the level of knowledge of methodology needed to meet the requirements of academic muster. Where teachers have sufficient knowledge of methodology, there remain issues with achieving the needs of methodology, for example, completing a literature review within the confines of limited access to academic resources. Changes arising from the internet, such as open access journals, were mentioned as a possible affordance. Some projects facilitating teacher research were mentioned but they typically involved academic researchers and there was a question of funding and power relationships as a trade-off with the literature access and methodological knowledge of the academic researcher. It was also noted that in the absence of cooperation with an academic researcher, teacher-based research may have difficulties developing sufficient scope for generalisability.

Overall the discussion focused on the practical details of a practitioner engaging in research. While many issues were brought to light in a relatively short time, it was clear that the affordances aspect would require substantially more time. A few affordances were brought to light, but they were specific in nature. It was unclear how specific instances could be generalized and, at the end of the discussion, that was perhaps the most promising avenue to continue the discussion.

Mathematics Gallery

Gallérie Mathématique

INTERPRÉTER LA CRÉATIVITÉ MANIFESTÉE DANS LES PRODUCTIONS D'ÉLÈVES EN MATHÉMATIQUES[1]

INTERPRETING CREATIVITY MANIFESTED IN STUDENTS' PRODUCTION IN MATHEMATICS[2]

Jean-Philippe Bélanger, *Université Laval*
Lucie DeBlois, *Université Laval*
Viktor Freiman, *Université de Moncton*

La résolution de problèmes mathématiques vue comme activité créative de la part de l'élève attire dernièrement l'attention de chercheurs (Leikin & Lev, 2007; Van Harpen & Sriraman, 2013). Cette vision se traduit dans le courant de récentes réformes de l'enseignement des mathématiques. En considérant l'imagination comme le moteur du raisonnement de l'élève, ce dernier serait une forme de jeu en fonction des contraintes considérées et du but fixé. Nous nous intéresserons à la créativité comme processus : « *un saut qui génère de nouvelles relations entre des connaissances, mais à l'occasion ce saut part du connu vers l'inconnu* » (Tammadage, 1979, p. 148). Ainsi, elle correspond à des relations élaborées en fonction d'un besoin, dans un contexte donné, à partir de l'ensemble des connaissances acquises au fil de ses expériences.

Nous avons mené une étude sur la créativité présente dans 50 productions d'élèves issues du site Internet CAMI (Freiman, Langlais, & Vézina, 2005) pour deux problèmes de transformation et trois problèmes de taux (Marchand & Bednarz, 1999) en lien avec les travaux de DeBlois (2003). En retenant dix productions d'élèves pour chaque problème en fonction de leur diversité, nous avons repéré 4 créativités : dominance du système de connaissances personnelles de l'élève qui oriente les calculs, dominance des éléments du problème qui oriente les calculs, mise en relation des contraintes à partir du système de connaissances personnelles de l'élève, mise en relation des contraintes à partir d'un équilibre entre les éléments du problème et le système de connaissances personnelles de l'élève.

Les problèmes de taux font intervenir une plus grande variété de créativité que les problèmes de transformation. En outre, même si la démarche des élèves montre l'utilisation de toutes les contraintes du problème, rien n'assure l'obtention d'une réponse culturellement plausible par des processus culturellement plausibles. Ce sont plutôt les relations entre les contraintes arrimées aux éléments du problème qui contribuent à ce que la démarche de l'élève soit culturellement plausible. Enfin, c'est la sensibilité des élèves de l'élève au milieu (Brousseau, 1988) qui semble une composante déterminante du processus de créativité.

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Mathematics problem solving as a creative activity has been recently investigated by Leikin and Lev (2007) and Van Harpen and Sriraman (2013), among others. This view is also promoted across Canadian reform-based school curricula. Considering the imagination as the

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[1] Recherche menée grâce au Ministère des affaires intergouvernementales canadiennes.
[2] Research conducted thanks to the Department of Canadian Intergovernmental Affairs.

key element of the student's reasoning, allowing her to play with the constraints considered in relation to the goal to be achieved, we focus on creativity as a process, "a leap which establishes new relationships, usually between areas of existing knowledge but sometimes from the known to a completely new area" (Tammadage, 1979, p. 148). Thus, it corresponds to the relationships between the needs in a given context and the knowledge gained from previous experiences.

We conducted a study on creativity on 50 students' solutions submitted to the CAMI Internet website (Freiman, Langlais, & Vézina, 2005) for two *transformation* problems and three *rate-of-change* problems (Marchand & Bednarz, 1999) in connection with the work of DeBlois (2003). Ten student productions for each problem were selected according to their diversity. We identified four types of creativity: dominance of the system of personal knowledge of the student guides calculations, dominance of part of the problem guides the calculations, linking constraints starting with the system of personal knowledge of the student, linking constraints from equilibrium between the elements of the problem and the system of personal knowledge of the student.

Rate-of-change problems involve a wider range of creativity than *transformation* problems. Moreover, even if the students demonstrate that they use all the problem's constraints, this does not guarantee that they obtain culturally plausible solutions with culturally plausible processes. It is rather the relationship between the constraints linked to the elements of the problem that enables the student to produce culturally plausible outcome. Our findings show that the students' sensitivity to the student's "milieu" (Brousseau, 1988) appears to be a critical component of the process of creativity.

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E-BROCK BUGS©[1] : THE CREATION AND ANALYSIS OF AN EPISTEMIC MATHEMATICS COMPUTER GAME

Laura Broley
Brock University

While seriously under-represented in our current education system, many argue that video games are an ideal medium for the teaching and learning of mathematics. To set the platform for his arguments of why this is the case, mathematician Keith Devlin (2011) explains that on one hand, “Teachers complain that many students appear uninterested in [mathematics] ...” (p. 45). On the other, a study by the Pew Research Center (2008) found that 97% of American teens aged 12-17 play video games. And yet, thus far, there have been very few (successful) attempts at creating a ‘good’ mathematical video game.

This, together with my own interest in video games and passion for mathematics education, led me to take on the challenge of designing and implementing a ‘good’ mathematics computer game that would later be entitled *E-Brock Bugs* (Broley, 2013). Building on a board game that was created by Dr. Eric Muller in the early 1980s, *E-Brock Bugs* seeks to bring players to learn basic probability concepts, many of which are encountered in the Grade 12 Data Management course of the Ontario mathematics curriculum, in a personalized, interactive, animated and fun way. Within the context of the game, a player’s mission is to save Bug City, the once-placid community that has been transformed into a wasteland by the all-powerful Dr. P and his evil Band of Bullies. To do this, players must work their way through six different districts, each of which entails a new environment, probabilistic game and Bully. Along the way, they meet an interesting cast of characters, including their guide, Bugzy, and Smarty, the extremely intelligent bug who has developed the theory behind each Bully’s scheme. With their help, the player is guided towards the defeat of Dr. P’s empire and the restoration of Bug City to a peaceful state.

In addition to taking inspiration from Muller’s work, the design of *E-Brock Bugs* carefully stems from the principles of a good math video game outlined by Devlin (2011); e.g., the selection of an in-game avatar, an environment where mathematics arises in a natural and meaningful way, and a structure that provides mathematical knowledge both on-demand and just-in-time. Ultimately, after a long dynamic cycle of creation and analysis that still continues today, *E-Brock Bugs* has become an educational tool that, I suggest, doesn’t just teach basic facts and skills, but rather encourages the development of mathematical thinking; in the words of Devlin (2011), “The game is no longer about learning how to do math; it is about learning how to be a (better) mathematician” (p. 126).

The game can be accessed online at: www.brocku.ca/mathematics/e-brock-bugs-game

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COULD 'IT' BE AN IMPLEMENTABLE FORM/ALTERNATIVE TO MICROWORLDS?

Chantal Buteau, Eric Muller & Neil Marshall
Brock University

In the core undergraduate *Mathematics Integrated with Computers and Applications* (MICA) program at Brock University, students learn to design, program, and use *Exploratory Objects*, that are "interactive and dynamic computer-based model[s] or tool[s] that capitalise on visualisation and [are] developed to explore a mathematical concept or conjecture, or a real-world situation" (Muller, Buteau, Ralph, & Mgombelo, 2009, p. 64). A MICA student creates at least 12 *Exploratory Objects* during his/her studies, three of which necessitate an original topic selected by the student. For example, in 2011-12, "471 assigned *Exploratory Objects* (i.e., topic and exploration questions provided to students in the assignment guidelines) and approximately 98 original *Exploratory Objects* (i.e., topics selected by students) were created" (Marshall, Buteau, & Muller, 2013, p. 192).

A second-year MICA student, participant in a survey study (ongoing), describes her understanding of the three specialized MICA courses as, "*[y]ou learn about many mathematical concepts and apply them and explore them on computers. You create programs that use these concepts and bring them to life to create a more concrete understanding of math and what it accomplishes*".

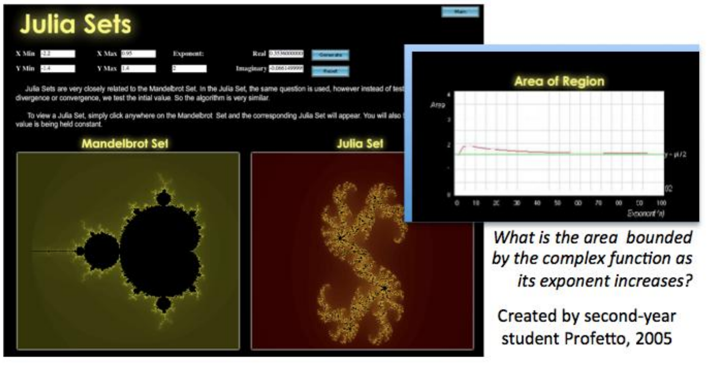

Figure 1. Example of an *Exploratory Object*.

We suggest that the *Exploratory Objects* are actually closely connected to the well-researched *microworlds*, i.e., "[c]omputational environments embedding a coherent set of scientific concepts and relations designed so that with an appropriate set of tasks and pedagogy, students can engage in exploration and construction activity rich in the generation of meaning" (Healy & Kynigos, 2010, p. 64). *Microworlds* are not merely the digital objects but rather the objects "in association with the kinds of activities emerging from their use" (p. 64). It is in terms of these activities that *Exploratory Objects* were connected to *microworlds* (Marshall & Buteau, in press). But since it is commonly agreed that "[t]he ideas behind the microworld culture have not yet been presented in a form readily acceptable not only to school systems, but also to other stakeholders in education" (Healy & Kynigos, 2010, p. 68)… could it be that *Exploratory Objects* are an implementable form or alternative to *microworlds*?

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MATHEMATICS AND SOCIAL JUSTICE: CONSIDERING SCHOOL AND COMMUNITY LEARNING

Indigo Esmonde
University of Toronto

The purpose of this analysis is to juxtapose data from two studies: a study of social justice activities in a mathematics class, and a study of the use of mathematics in an activist community. Both studies are considered through the theoretical lens of *figured worlds* (Holland, Lachicotte, Skinner, & Cain, 1998).

MATHEMATICS FOR SOCIAL JUSTICE IN SCHOOL

In a study of school mathematics, I analysed data from a classroom activity in which students were instructed to work in groups to compute each continent's population and GDP as a % of the global total and as a proportion of students in the class. After computing the mathematics, they held a lively discussion about whether the distribution of wealth is fair.

Multiple *figured worlds* were at play, including school, global wealth, individual finance, and an intermediate *figured world* in which cookies represented wealth and students represented continents. Analogies between the four different *figured worlds* guided student discussions about the social justice issue of global wealth disparity. The mathematics underlay their discussion but the disagreement was largely based on ethical issues. The storylines about fair share differed across the different figured worlds, and thus in the debate, students took on different positions and leveraged different *figured worlds* to support their point of view.

MATHEMATICS FOR SOCIAL JUSTICE IN COMMUNITY ACTIVISM

In a study of community activism, I used participation observation, artefact collection, and interviews, to understand how social justice activists incorporated mathematics into their work. This was a single *figured world,* that of city politics, including the mayor and city councillors, and city residents (especially immigrants, the elderly, homeless people, low income families).

This was a real-world task with real-world consequences, and the activists engaged in many forms of mathematical argumentation and reasoning. There was differentiated participation, with some activists (including a Research Committee) taking on the bulk of specialized mathematics. However, mathematics was never disconnected from the *figured world* of lived experience, and many activists played a role in making sense of what should be mathematized and how. One similarity to the school analysis is that participants with different theories of how the world works (*figured worlds*) came to different mathematical conclusions.

DISCUSSION

Research in out-of-school mathematics has already shown that people engage in flexible, open-ended approaches to mathematics problem solving as they go about their daily lives. Activist math is particularly interesting because it is intended to investigate issues of inequality, and is often intended to educate (both in, and outside of schools). This juxtaposition of school and community math is meant to encourage the reader to consider how an understanding of activist math in and out of school can inform teaching and learning practices in both locations.

ACKNOWLEDGEMENTS

Thanks to Joe Curnow, Lesley Dookie, Jennifer Langer-Osuna, Dominique Riviere, Tesha Sengupta-Irving, and Jessica Thompson for their feedback. The research was supported in part through funding from the Knowles Science Teaching Foundation as well as the Social Sciences and Humanities Research Council of Canada.

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ELEMENTARY TEACHERS' VIEWS OF AND EXPERIENCES WITH MATHEMATICS, MATHEMATICIANS, AND MEDIA

Jennifer Hall
University of Ottawa

This poster shared findings from a portion of a larger research project that investigated Ontario elementary students' views of mathematics and mathematicians, and the ways that students' views may be influenced by parents' views, teachers' views, and popular media representations of mathematics and mathematicians. Specifically, I reported on findings from interviews with Grade 4 and 8 teachers, wherein I investigated their experiences with and views of mathematics and mathematicians. The interviews took place during the 2010-2011 school year with 10 teachers from Ontario public schools.

The teachers reported limited exposure to *mathematicians* in the media, and nearly all of the examples cited were very stereotypical in nature (e.g., 'math geek', mathematician with a mental illness). In contrast, the teachers reported a great deal of exposure to *mathematics* in the media; however, the examples tended to narrowly focus on statistics and financial mathematics. With regard to 'real life' mathematicians, the teachers' exposure was quite varied, and was highly linked to the manner in which they defined who a mathematician is (e.g., university professor, financial professional).

The teachers' views of mathematicians tended to be rather stereotypical in nature, particularly with regard to appearance (e.g., an old 'white' man with glasses). Although the teachers recognized that such views were stereotypes, they did not have any alternative representations with which to challenge these views. Several teachers expressed uncertainty regarding who a mathematician is. Regardless of their stance on mathematicians, the teachers tended not to think of themselves as mathematicians—even though some participants considered their colleagues (i.e., other elementary teachers) to be mathematicians. In general, the teachers' views of mathematics were narrow and limited in nature. Mathematics—and the importance placed on learning mathematics—was frequently linked to arithmetic and numbers, and/or to financial mathematics.

The teachers generally reported having positive relationships with mathematics in terms of their views of themselves as learners and teachers of the subject area, and of mathematics' importance and utility. The Grade 8 teachers tended to have always had such a relationship with mathematics, whereas the Grade 4 teachers tended to develop such a relationship through re-learning mathematics in a new way when they became teachers. The teachers' enthusiasm for mathematics was evident in their interviews, and arguably would be related to their students. The teachers reported incorporating a real-world focus in mathematics class and discussing mathematics' utility on a regular basis with their students. However, like the teachers' descriptions of mathematics in general, the types of mathematics addressed in such discussions were often limited and narrow.

Overall, the teachers' experiences with and views of mathematics and mathematicians were shown to be both complex and contradictory in nature.

SEARCHING FOR PROFOUND UNDERSTANDING: UNPACKING PRESERVICE TEACHER MATHEMATICS KNOWLEDGE

Jennifer Holm & Ann Kajander
Lakehead University

At our mid-sized northern Ontario faculty of education, the *junior-intermediate* cohort (for teaching grades four to ten) receives a total of 36 course hours in mathematics curriculum and instruction ('methods'), which may or may not be preceded by any mathematical background beyond high school. Here, we explored the effect of recent school mathematics curricular revisions on the conceptual understandings of these beginning teacher candidates.

METHODS

For our study, we examined different cohorts of junior/intermediate preservice teachers, (PSTs) to explore potential differences in their mathematics knowledge, as possibly attributable to the new Ontario curriculum (Ontario Ministry of Education, 2005). Using the *Perceptions of Mathematics Survey* (Kajander, 2007), we chose two questions that dealt with modeling and explaining elementary curriculum concepts for further analysis: explain $5 - ({}^{-}3)$ and explain $1\frac{3}{4} \div \frac{1}{2}$ (see the framework in Kajander & Holm, 2013). We then examined response samples from the beginning of the methods course in 2008 and in 2012 from PSTs who had experienced the new elementary curriculum.

RESULTS

In comparing the responses from incoming PSTs between the two years, minimal differences were discernible. For integers, the most frequently used response category in both cohorts was simply stating a rule such as "two negatives make a positive". In both sets of samples, very few individuals could provide a correct explanation for the question, and no one was able to give more than one, despite the need for teachers to have multiple representations for questions (Small, 2009). For the fraction question, in both years, the most frequent response was to either leave the question blank or give an incorrect answer. Again, very few could give an explanation, and no one was able to give multiple explanations for the solution method.

CONCLUSION

Despite our use of a framework to separate response-types into different categories, we remain unable to identify increases in conceptual understandings of elementary mathematics content based on curriculum reform between cohorts of PSTs. While we were previously disappointed not to be able to identify evidence of conceptual understanding in PSTs who had studied during the early years of the elementary curriculum implementation, we had hoped that, four years later, evidence of a more conceptual approach as described in our provincial curriculum document (Ontario Ministry of Education, 2005) would have been observable. Our work continues to underscore the serious need for emphasis on mathematics content for teaching.

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THE MATH OLYMPIAN

Richard Hoshino
Quest University Canada

In March 2010, I moved to Tokyo after my wife Karen landed her dream job at a highly regarded Japanese university. As an unemployed house-husband starting a new life in a new country, I wondered how I could best apply my passion and experience to contribute to society.

Having a love for expressing myself through writing (my passion), as well as possessing twenty years of experience doing and writing math contests (my experience), I was inspired to write a fictional novel about a shy and insecure teenager who dreams of representing her country at the International Mathematical Olympiad, and thanks to the support of innovative mentors, combined with her own relentless perseverance, discovers meaning, purpose, and joy.

Mathematics has changed my life—it has brought me opportunities and privileges beyond my wildest dreams, and through my journey of studying this subject, I have found a deep clarity of purpose, discovering the arena in which I can serve society and live life to the full.

It is this journey that I hope to share in my novel, with the hope of inspiring tens of thousands of young people with the message that with inspired teaching and mentorship, anyone can succeed in mathematics and develop the confidence, creativity, and critical-thinking skills so essential in life. Here is the half-page description of my novel, *The Math Olympian*.

> As a small-town girl in Nova Scotia bullied for liking numbers more than boys, and lacking the encouragement of her unsupportive single mother who frowns at her daughter's unrealistic ambition, Bethany MacDonald's road to the International Math Olympiad has been marked by numerous challenges.
>
> Through persistence, perseverance, and the support of innovative mentors who inspire her with a love of learning, Bethany confronts these challenges and develops the creativity and confidence to reach her potential.
>
> In training to become a world-champion mathlete, Bethany discovers the heart of mathematics – a subject that's not about memorizing formulas, but rather about problem-solving and detecting patterns to uncover truth, as well as learning how to apply the deep and unexpected connections of mathematics to every aspect of her life, including athletics, spirituality, and environmental sustainability.
>
> As Bethany reflects on her long journey and envisions her exciting future, she realizes that she has shattered the misguided stereotype that only boys can excel in math, and discovers a sense of purpose that through mathematics, she can and she will make an extraordinary contribution to society.

TIMSS: WHAT SHOULD WE FOCUS ON IN MATHEMATICS TEACHING?

Zhaoyun Wang
OISE, University of Toronto

The *Third International Mathematics and Science Study* (TIMSS) (Mullis, Martin, Foy, & Arora, 2012) identified students' performance on strands and cognitive domains. However, TIMSS reports did not examine students' performance on mathematics conceptual knowledge and application.

This study addresses students' performance on understanding mathematics basics such as concepts, formulas, theorems, rules and properties, and applying mathematics to solving problems. Hence, for this study, I cluster the 38 countries, identify the characteristics in mathematics performance, determine correlations of these components, and find regression equations of students' achievement.

The hypotheses of this study are that similar performing countries may have similar characteristics in teaching and learning. I use the TIMSS 2003 participants' mathematics test scores as my sample. I will define a few variables such as: CON indicates concept knowledge including definitions, principles and ideas; FTRP indicates formulas, theorems, rules and properties; APP refers to applying mathematics knowledge; RT is routine problems; and NRT is non-routine problems. *Excel* and *SPSS* are used for data coding and analysis.

The results show that the averages of CON, FRTP, APP, RT, and NRT are 52.21, 46.06, 42.69, 47.78 and 34.14 respectively. Students are weakest in non-routine (NRT) problem solving, and applying (APP) mathematics knowledge to solve real-life problems. The ratio of APP/FTRP is less than 1. Top-performing countries are consistently high on mathematics components such as: CON, FTRP, APP, RT and NRT. Intermediate-performing countries are unevenly developed on these components. Low performing countries are low in each component.

By using *hierarchical cluster analysis* I was able to group the thirty-eight countries into six groups. The English-speaking countries are in the same group, and East Asian countries are in the same group. In the East Asian group, the ratio of APP/FTRP is less than 1. The group of intermediate countries and a couple of other TIMSS countries have a high ratio of APP/FTRP (> 1); the ratio in the other countries is less than 1. Correlation analysis shows that the correlation coefficients between each pair of CON, FTRP, APP, RT, NRT and total score (Tscore) are greater than 0.9. The regression equations of students' achievement: Tscore = 227.15 + 2.348FTRP + 2.797CON, and Tscore = 235.582 + 5.142RT. I noted that, due to the high correlation of variables, the independent variables in regression equations may represent the combined effect of the other variables.

The average scores and regression question results suggest that mathematics teaching should focus on conceptual understanding of basic concepts, formulas, theorems, rules, properties, and routine problem solving. The ratio of APP/FTRP in the clustered high performing country group is lower than 1 while the ratio in the English speaking group and a couple of other countries is greater than 1. This indicates that the English-speaking group (USA, England, New Zealand, Australia, and Sweden) and a couple of other countries may not address enough basic mathematics knowledge in contrast to the high-performing countries. They may examine their curriculum standards or curriculum materials.

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Appendices

Annexes

Appendix A / Annexe A

WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

· Assessing mathematical thinking
· From theory to observational data (and back again)
· Bringing Ethnomathematics into the classroom in a meaningful way
· Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

· Information technology and mathematics education: What's out there and how can we use it?
· Applied mathematics in the secondary school curriculum
· Elementary mathematics
· Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

· Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
· Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
· Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
· Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

· Considering how linear algebra is taught and learned
· Children's proving
· Inservice mathematics teacher education
· Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

· Mathematics and the arts
· Philosophy for children on mathematics
· The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
· Mathematics, the written and the drawn
· Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

· L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
· Teacher research: An empowering practice?
· Images of undergraduate mathematics
· A mathematics curriculum manifesto

2004 *Université Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke, Québec*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 *York University, Toronto, Ontario*

- Mathematically gifted students / Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d'enseignement
- Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010 *Simon Fraser University, Burnaby, British Columbia*

· Teaching mathematics to special needs students: Who is at-risk?
· Attending to data analysis and visualizing data
· Recruitment, attrition, and retention in post-secondary mathematics
Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
· Beauty in applied mathematics
· Noticing and engaging the mathematicians in our classrooms

2011 *Memorial University of Newfoundland, St. John's, Newfoundland*

· Mathematics teaching and climate change
· Meaningful procedural knowledge in mathematics learning
· Emergent methods for mathematics education research: Using data to develop theory / Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories
· Using simulation to develop students' mathematical competencies – Post secondary and teacher education
· Making art, doing mathematics / Créer de l'art; faire des maths
· Selecting tasks for future teachers in mathematics education

2012 *Université Laval, Québec City, Québec*

· Numeracy: Goals, affordances, and challenges
· Diversities in mathematics and their relation to equity
· Technology and mathematics teachers (K-16) / La technologie et l'enseignant mathématique (K-16)
· La preuve en mathématiques et en classe / Proof in mathematics and in schools
· The role of text/books in the mathematics classroom / Le rôle des manuels scolaires dans la classe de mathématiques
· Preparing teachers for the development of algebraic thinking at elementary and secondary levels / Préparer les enseignants au développement de la pensée algébrique au primaire et au secondaire

2013 *Brock University, St. Catharines, Ontario*

· MOOCs and online mathematics teaching and learning
· Exploring creativity: From the mathematics classroom to the mathematicians' mind / Explorer la créativité : de la classe de mathématiques à l'esprit des mathématiciens
· Mathematics of Planet Earth 2013: Education and communication / Mathématiques de la planète Terre 2013 : formation et communication (K-16)
· What does it mean to understand multiplicative ideas and processes? Designing strategies for teaching and learning
· Mathematics curriculum re-conceptualisation

Appendix B / Annexe B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe
	N. BOULEAU	La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Art, culture et mathématiques en pays d'Islam (IX^{e}-XV^{e} s.)
	A. WATSON	Adolescent learning and secondary mathematics
2009	M. BORBA	Humans-with-media and the production of mathematical knowledge in online environments
	G. de VRIES	Mathematical biology: A case study in interdisciplinarity
2010	W. BYERS	Ambiguity and mathematical thinking
	M. CIVIL	Learning from and with parents: Resources for equity in mathematics education
	B. HODGSON	Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective
	S. DAWSON	My journey across, through, over, and around academia: "...a path laid while walking..."
2011	C. K. PALMER	Pattern composition: Beyond the basics
	P. TSAMIR & D. TIROSH	The Pair-Dialogue approach in mathematics teacher education
2012	P. GERDES	Old and new mathematical ideas from Africa: Challenges for reflection
	M. WALSHAW	Towards an understanding of ethical practical action in mathematics education: Insights from contemporary inquiries
	W. HIGGINSON	Cooda, wooda, didda, shooda: Time series reflections on CMESG/GCEDM
2013	R. LEIKIN	On the relationships between mathematical creativity, excellence and giftedness
	B. RALPH	Are we teaching Roman numerals in a digital age?
	E. MULLER	Through a CMESG looking glass

Appendix C / Annexe C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242

Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	ED 431624
Proceedings of the 1999 Annual Meeting	ED 445894
Proceedings of the 2000 Annual Meeting	ED 472094
Proceedings of the 2001 Annual Meeting	ED 472091
Proceedings of the 2002 Annual Meeting	ED 529557
Proceedings of the 2003 Annual Meeting	ED 529558
Proceedings of the 2004 Annual Meeting	ED 529563
Proceedings of the 2005 Annual Meeting	ED 529560
Proceedings of the 2006 Annual Meeting	ED 529562
Proceedings of the 2007 Annual Meeting	ED 529556
Proceedings of the 2008 Annual Meeting	ED 529561
Proceedings of the 2009 Annual Meeting	ED 529559
Proceedings of the 2010 Annual Meeting	ED 529564
Proceedings of the 2011 Annual Meeting	*submitted*
Proceedings of the 2012 Annual Meeting	*submitted*
Proceedings of the 2013 Annual Meeting	*submitted*

NOTE

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.