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INVITED ARTICLE

Geometrical Constructions in Dynamic and Interactive Mathematics Learning Environment

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This paper concerns teaching Euclidean geometry at the university level. It is based on the authors' personal experience. It describes a sequence of learning activities that combine geometrical constructions with explorations, observations, and explanations of facts related to the geometry of triangle. Within this approach, a discussion of the Euler and Nigel lines receives a unified treatment via employment of a plane transformation that maps a triangle into its medial triangle. I conclude that during this course delivery, the role of constructions in dynamic and interactive environment was significant for students' genuine understanding of the subject. In particular, it helped them to work with concrete figures and develop their own preformal approaches before learning general theorems and proofs. At the same time it was essential to follow such strategies as gradually lead students from basic to advanced constructions, from making simple analogies to generalizations based on critical ideas and unified principles, and emphasize structural interconnectedness of the problems each of which adds a new element into a bigger picture.

Introduction

With rapid development of technology in recent years, the "dynamic and interactive mathematics learning environments (DIMLE), such as Cabri, GeoGebra, Geometer Sketchpad, Fathom and the like" (Martinovic & Karadag, 2012, p. 41) become accessible to more learners of mathematics. However, the presence of technology by itself does not necessarily "produce what is expected" (that is, improve students' understanding) and may have different cognitive influence on different students. "A deeper analysis is needed to better understand both the potential and limitations of computers" (Mariotti, 2002, p. 698).

The problem of effective use of technology in mathematical education challenges researchers for many decades (Kaput & Thompson, 1994). In particular, it is critical to know what are conditions of successful employment of DIMLE in educational processes, and what kinds of mathematical activities are beneficial for students' understanding of the subject. It is important to investigate the role of DIMLE in development of mathematical thinking, specifically, in view of mathematical problems that become affordable and thus open new dimension in learner's advancement.

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In this paper I attempt to address these questions. First, I discuss the role of experimental and explanatory phases in learning of mathematics. Then I talk about geometrical constructions as a type of problems that allow to bridge empirical and theoretical knowledge in geometry. Finally, I give an example from personal teaching practice of possible use of DIMLE: Topics of the Euler and Nigel lines are presented through construction and exploration. I conclude with a discussion of six pedagogical principles consistent with this teaching practice.

More details related to the context of this study can be found in my paper (Kondratieva, 2012), which highlights advantages of using basic geometric configurations (BGC) and dynamical invariants in teaching Euclidean geometry at the undergraduate level with assistance of DIMLE. In that paper, I conclude that “the freedom of experimentation offered by the use of DGE [dynamic geometry environments] needs to be very well structured by the instructor in order to help students to conceptualize geometrical knowledge at both intuitive and formal deductive levels” (p.212). Here I add further details based on my teaching experience gained through another year of the course delivery with focus on geometrical constructions.

Experimentation, explanations and learning

Many researchers (see e.g. Borwein, 2012) agree on the idea that modern technology such as computer algebra systems (CAS) and DGE allows to design, perform and validate experiments, which gives mathematics ‘almost empirical’ status (Arzarello et al, 2012, p. 110). Exploration of problems using physical tools is a widely known approach in natural sciences. However, while being not much developed, “exploratory approach to experiments that includes concept formation also pertains to mathematics” (Sorensen, 2010). Experimentation in CAS and DGE aims at such processes as pattern observation and generation of conjectures. While being an important part of mathematical research, learning through observation and conjecturing followed by explanations is also invaluable for student training (Jahnke, 2007) because it allows students to participate in the process of creation of knowledge and discover new mathematical relations. According to this approach, learning “starts with extended calculations, constructions and experimentations. In this way a ’quasi-reality’ is created which allows for observing phenomena, discovering patterns, formulating conjectures, and last not least for explaining, that is, proving patterns.” (Wittmann, 2009, p. 255).

Yet, teaching approaches involving experimentation require thoughtful design. “Engaging students in situations which make them aware of the constructive character of mathematical activities, especially those involving conjecture and proof, possess complex challenges”(Durand-Guerrier at al, 2012, p.364). Indeed, pattern may reveal themselves and become ‘obvious’ in an appropriate representation. Nevertheless, the choice of representation and pattern interpretation are tasks that may require solid mathematical background. However, even if certain relations are easily observable, they call for further explanation and formulation of precise mathematical statements. It appears that often finding explanations and proofs is a much more difficult problem, separate form the process of conjecturing and finding mathematical facts empirically.

On that reason, it is arguable to what extend mathematical skills should be developed and practiced before students’ engagement in mathematical experimentation takes place. Educators, who rely on more ‘traditional’ approaches, insist on extensive practice with number facts, formulas, methods and exercises prior to an experimental phase of learning. Those who favour a constructivists view, believe that necessary knowledge are better developed during such experimentations and solving problem that allow students to think, recall and sometimes to develop their own approaches to handle a question.

The role of technology in this process is also disputable. While educators observe many cases when technology generates a mathematical insight (Adams, 1997), yet 'the worriers … warned of a growing population of high school graduates, who, without their calculators would not be able to… multiply a number by 10, or make change for a dollar" (Fey, 2003, p.97), and that development of calculator-dependency or over-reliance on technology is a serious threat for learning of mathematics. Thus, the way technology is used in teaching is critical for possible outcomes.

Theoretical justification: Preformal and potential proofs

What are the conditions of a successful employment of an experimentation approach and technology in learning mathematics? One of the pioneers of computer assisted experimental learning, Seymour Papert, suggested that conducing a meaningful activity that engages students is the key defining the success of this approach. Further, it was suggested that the use of technology should be accompanied by its "internalization as psychological tools" (Arzarello et al 2012, p. 118). As well, technology should "foster cognitive unity" of empirically formed conjectures and possible links to their theoretical justifications. Otherwise, students produce "conjectures with no theoretical elements to bridge the gap between the premise and the conclusion of the conditional link" (Ibid, p. 118), which becomes an obstacle for a successful construction of an explanation. "Conjectures should go beyond merely guessing … and encompass in some way a search for a structural explanation. Only then does it involve meaningful exploration and interpretation and a genuine need for validation." (Durand-Guerrier et al, 2012, p. 357). In addition, a proof that looks for a structural reason and explanation of a phenomenon as opposed to a formal verification of it, often leads to a deductive generalization (DeVillier, 2012), that is, "generalisation of a critical idea to more general or different cases by means of deductive reasoning" (p.7).

It was observed that learners benefit from writing detailed explanations. As long as these reasoning protocols (even being incorrect) present enough details to be checked by the learner or others, they form so-called *potential* proof. Quinn (2012) suggests that focus on writing potential proofs first, followed by their verification, helps students to "routinely get correct answers" (p. 237).

Even if they are suitable in the first place, explanations arising from experimental mathematics often have status of *preformal* proofs, or a "chain of correct but not formally represented conclusions", as defined in Blum and Kirsch (1991, p 187). Preformal proofs include visual, operative (Wittmann, 2009), and generic (Tall, 1979) proofs, which carry on the same logic as formal reasoning but reduce the level of abstraction by dealing with either visual images, or 'quasi-real' objects, or ideas presented in generic examples (particular cases), respectively. While not being general, preformal proofs provide an important step on the way from empirical observation towards general proofs because they essentially capture "the main ideas of the complete proof in an intuitive and familiar context, temporarily suspending the formidable issue of full generality, formalism, and symbolism"(Leron & Zaslavsky, 2009, p. 56). Formalization, verification and exposition of these preformal arguments for an external reader lead to a formal proof.

Geometrical constructions

A process of geometrical constructions is often conjoint with preformal proofs because it results in concrete figures relying on general ideas. On that reason construction tasks were traditionally used for bridging theoretical and practical aspects of geometry. For example, theoretical notion of circle (as a collection of all points in the plane, equidistant from a given point) reveals its practical meaning in compass constructions that involve distances, e.g. construction of a triangle with 3 given segments as its sides. "Any successful construction corresponds to a specific theorem" (Arzarello et al, 2012, p.100) and it represents certain properties or relationships.

Despite their fundamental importance for learning geometry, "constructions have recently lost their centrality and almost disappeared from Geometry curriculum" (ibid, p.101). They no longer belong to "the set of problems commonly proposed in the textbooks"(ibid, p.101). This situation has been criticized by some mathematics educators and scholars (see e.g. Tymoczko, 1998), particularly in view of presence of new construction tools offered by modern technology. However, teaching of geometry still often suffers from either formal theoretical approach (e.g. two column proofs) or an utilitarian view emphasising memorization of formulas for area, volume etc. (Protasov & Sharygin, 2004).

If the goal of study of mathematics is viewed as neither production of formal statements nor exclusively immediate practical application, but rather as advancing our understanding (Thurston, 1994), the students of mathematics should be exposed to appropriate experiences allowing them to develop, demonstrate, negotiate and convey such an understanding.

In this paper I illustrate a possible scenario of how geometrical constructions may be included in lessons, especially if dynamic geometry software is involved.

From Euler to Nagel line: a practical example.

Euler line is an important object in the geometry of triangle. We will discuss the introduction of this topic using constructions in DIMLE. A discussion of less traditional object, the Nagel line, will follow to illustrate how the ideas learned in the first part can be naturally developed to form a connected view on the topic.

This section is based on my teaching experience in Euclidian geometry at the undergraduate university level. The sequence of tasks was given to students using two principles: the complexity of tasks increased from simple to advanced, and solutions of later tasks relied on ideas and properties found in earlier tasks. In addition, some hints were available upon students demand. The students were required to construct an object with given properties and explain how and why their construction works. A number of potential (Quinn, 2012) and preformal (Blum and Kirsch, 1991) proofs were produced by students during this course. These students' explanations were used by the instructor to improve and advance students' understanding of relations between elements in a triangle. As well, discussions and reflections on students' preformal proofs helped them to generalize critical ideas and transfer them from one problem situation to other cases.

Basic constructions

The first task was "using only compass and unmarked ruler, to construct an isosceles triangle". Figure 1 illustrates a student's construction and its preformal justification. The student wrote “I draw all points A’ that are distance 3 from A, and points B’ that are distance 3 from B. The point C, where A’ and B’ meet gives an isosceles triangle ABC with AC=BC=3.” Another student proposed a different approach: “Draw a circle with centre at C and radius R=1, and pick two arbitrary points A and B on the circumference. Since CA=CB is the radius, ABC is isosceles”. While being generic (the students used concrete length of 3 units or 1 unit), these constructions illustrate well the principal idea and can be easily generalized to an arbitrary case.

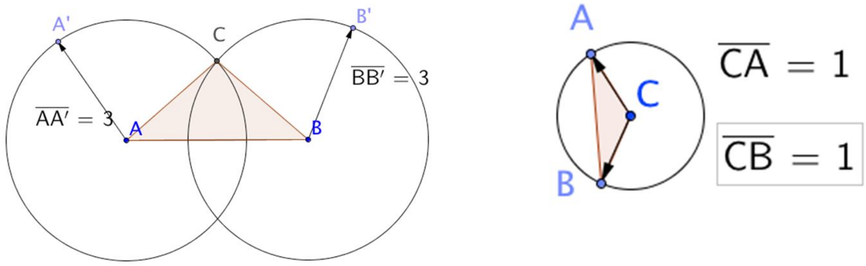

Figure 1: Two ways to construct an isosceles triangle.

While both being valid constructions of an isosceles triangle, the first method appeared to be more useful for solving the second task, namely, “to find the centre of a given segment, and then to describe the place of all points X in the plane equidistant from the two given points A and B, that is AX=XB.” The student who solved Task 1 by the first method observed: “in order to find the centre of the segment AB, I need to reduce the radii of the circles until they meet in just one point.” Figure 2 (left) illustrates this approach, which also brings the student to the realization that all points equidistant from the two given points form a straight-line.

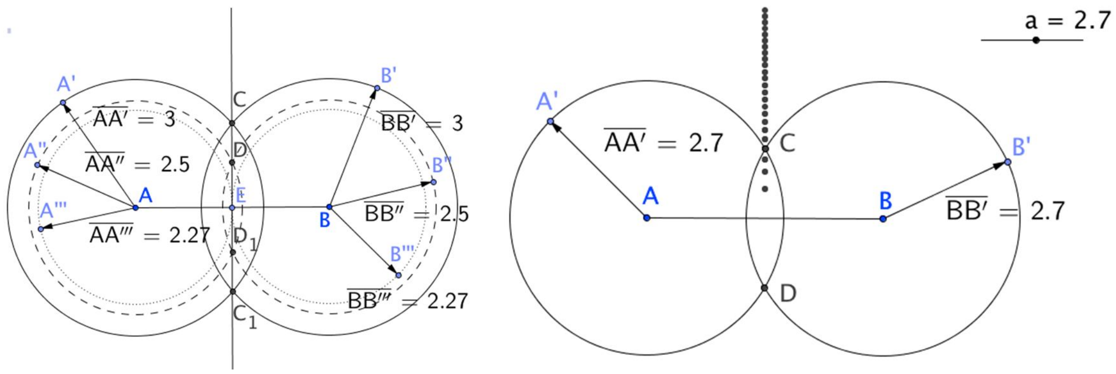

Figure 2: Two ways of finding all points equidistant from given points A and B.

Thus, “it is sufficient to draw just a pair of circles, and line connecting their two points of intersection (e.g. CC_1) will cross the segment AB exactly at its mid-point E.” Another student arrived at the same algorithm after using the ‘trace’ function available in DIMLE and revealing the position of the point of intersection of two circles with centres at A and B respectively and radii controlled by slider (see Figure 2, right). Then, the students were asked “to construct a line perpendicular to a given line L and passing through a given point X in the plane” (Task 3). The students observed that the line constructed in Task 2 must be perpendicular to the segment AB “because it is the axis of symmetry of this segment”. This is the key to construction in Task 3. “We first draw a sufficiently large circle with centre at X and mark the intersection points with L as A and B. We have AX=XB and then continue as in Task 2, constructing perpendicular bisector of AB" (Figure 3, left). An additional assignment (Task 3a) was to "construct a line passing through a given point X and parallel to given line L". This can be done by performing Task 3 twice, that is, to construct L' perpendicular to L and passing through X, and then draw L'', perpendicular to L' and passing through X. Since both lines are perpendicular to L', L'' is parallel to L (see Figure 3, right).

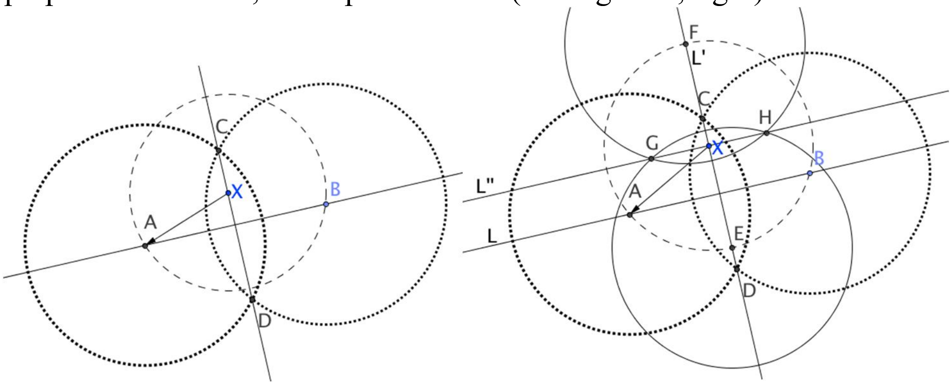

Figure 3: Line passing through X and either perpendicular or parallel to a given line.

While many Dynamic Geometry Environments have these elementary constructions build as a 'short cut' option, I insisted on disabling these software functions at first and familiarising the students with these constructions by means of compass and ruler. Students' were asked to produce drawings that depict geometrical statements along with auxiliary elements pertinent to their explanations. Such drawings are called basic geometric configurations (BGC) and discussed in (Kondratieva, 2011a). The use of basic geometric configurations associated with these elementary constructions was a deliberate choice, allowing students to internalize basic geometrical properties. Later, when the constructions became more complex and crowded, the students were allowed to use the 'short cuts' of a dynamic environment.

Our Task 4 was "to construct the centre of given circle". One participant described the process as follows: "If I had a circle cut from paper, I would fold it in halves to find the diameter. Then I would fold it in a different way to find another diameter, and the intersection of the folding lines will give me the centre. Now, what is the folding line? It is the line of symmetry of the circle. Thus, if I pick two arbitrary points on the circle, and find the line of their symmetry, I will get a diameter. My problem is as in Task 2: to draw a perpendicular bisector of a segment with end-points located on the circle. Picking two pairs of such points I will get two intersecting diameters and complete the task. Well, if I need to minimize the effort, I can actually pick only 3 points on the circle to form two pairs." Refer to figure 4 (left).

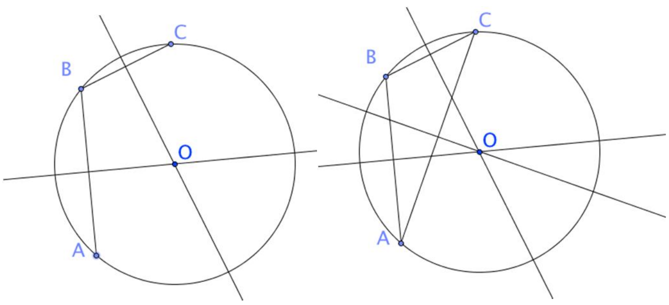

Figure 4: The centre of a circle and concurrency of perpendicular bisectors in a triangle.

With this algorithm in hand, it is straightforward "to construct the circumcenter of a given triangle", which was our Task 5 (see Figure 4, right). The students were also asked to explain the fact of concurrency of all three perpendicular bisectors in a triangle. In words of one student, "since O is equidistant from A and B by construction, as well as from B and C by construction, it must be that OA=OB=OC and so O is equidistant from A and C. But that means it lies on the perpendicular bisector to AC, and thus all three meet at O".

Medial triangle, M-transformation and the Euler line

Next, the students were requested to construct midpoints A', B' and C' (Task 2) of sides BC, CA and AB respectively, and form the medial triangle A'B'C' (Figure 5). The students were asked: "What relations exist between the medial and the original triangles?" One can observe that their sides are pairwise parallel (e.g. AB ||A'B' etc) by Thales' Intercept Theorem. Consequently, medial triangle is similar to the original one in the proportion 1:2. This and other properties of the medial triangle are central to our further discussion of the Euler line. Then students were asked to draw the altitudes in a given triangle (which can be done by using construction from Task 3).

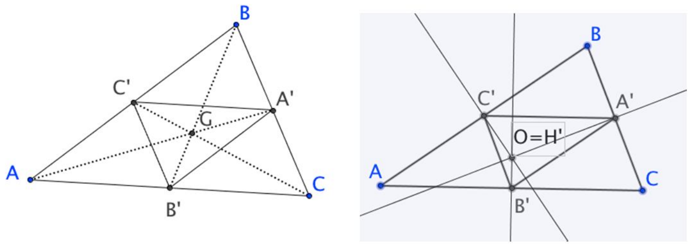

Figure 5: Medial triangle, the centre of mass G, and proof of concurrency of altitudes.

While students observed the property of their concurrency using dragging feature, this fact needed an explanation. Such an explanation does not easily emerge from the construction itself. At this point I suggested constructing an 'external medial triangle', that is, bigger triangle, for which the given triangle is the medial one. Then students observed that the altitudes of the medial triangle are perpendicular bisectors of the external medial triangle. This led to an exclamation: "But we already know that perpendicular bisectors of a triangle are concurrent, thus the altitudes of A'B'C' are concurrent as well" (Figure 5, right). This students' conclusion is an important type of indirect reasoning, recognizing, in this particular case, the circumcentre of a triangle ABC as the orthocentre of it medial triangle A'B'C' and thus proving the concurrency of altitudes in any triangle.

Next, we observed that medians of a triangle are also concurrent, intersecting at the 'centre of mass' G. That is a 'physical' explanation of this phenomenon, while the pure geometrical one usually is based on the Ceva theorem. The medians of a triangle and its medial triangle lie (pairwise) on the same lines, which explains the fact that the centre of mass of a triangle coincides with one for the medial triangle.

It was important to agree upon the notations. Let A, B, C be vertices of a triangle, O be its circumcentre, H – orthocentre and G the centre of mass. As we introduce more points, all corresponding points in the medial triangle will be denoted by prime, for example A', B', C' are vertices, H' is the orthocentre, O' is the circumcentre and G' is the centre of mass in the medial triangle. In these notations we can summarise our findings so far: (1) BA'=CA', AC'=BC', CB'=AB'; (2) G=G'; (3) O=H' (Fig. 5).

With these objects in hands, we discussed the transformation of the plane that converts a triangle into its medial triangle. For short, we will call it M-transformation. Being a homothety with centre G and coefficient -0.5, M-transformation is a composition of two transformations, namely, 180-degrees rotation around the centre of mass G and shrinking the object to produce a twice-smaller triangle. Both transformations were studied by using DIMLE. Students observed that the image A' of each point A in the plane lies on the line extension of segment AG, on the other side from G, twice closer to G: AG=2A'G. This observation immediately suggested that the centre of mass G divides all median in any triangle in the proportion 2:1, that is AG:GA'=BG:GB'=CG:GC'=2:1. From the property of M-transformation we know that H, G and H' are collinear and HG=2H'G. But from the property H'=O we immediately conclude that H, G and O are collinear and HG=2OG in any triangle. The segment HO is usually referred to as the Euler line (Figure 6).

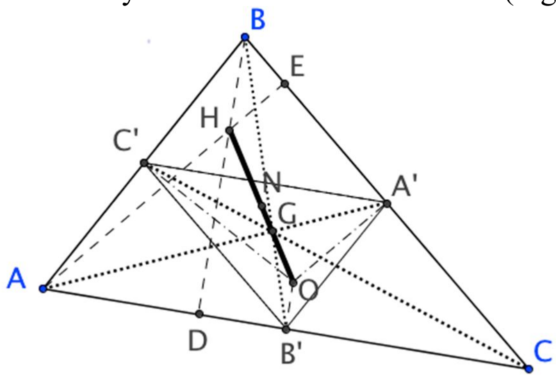

Figure 6: The Euler line connecting orthocentre H and circumcentre O.

There is one more point on the Euler line, namely, the midpoint N (HN=ON), which also is the centre of the Nine Point Circle. We are not going to discuss this circle here, just recall that it passes through the midpoints A', B', C' and thus coincides with the circumcircle of the medial triangle. This means that N=O'. Applying the same reasoning as above, we see that by the property of M-transformation, points O,G,O' are collinear and OG=2O'G. Since O'=N, we have OG=2NG, which is consistent with the proportions HG=2OG and HN=NO=2NG.

Construction of the Nagel line

There is one more important point in a triangle, the incentre I. Before constructing of this point we discussed the task of constructing a point equidistant from the sides of a given angle, and then construction of the angular bisector (Figure 7, left). Indeed, the incentre lies on the intersection of angular bisectors of a triangle (figure 7, right), and is equidistant from all three sides of the triangle (this distance is the in-radius). This recognition explains the concurrency of the angular bisectors in any triangle, in a way similar to the explanation of concurrency of perpendicular bisectors, given above.

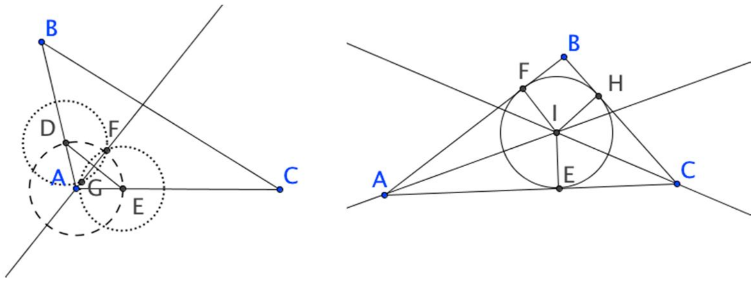

Figure 7: Angular bisector and incentre.

Then we discussed less traditional geometrical objects such as cleavers and splitters.

Cleaver is a segment that starts at a midpoint of a side and bisects the perimeter of a triangle. From this description students were asked to construct cleavers. One student proposed the following construction for cleaver adjacent to B': "Since B' is a midpoint of AC, we need to extend AB by the length of BC and split the resulting segment in two halves. This can be achieved by the construction shown on Figure 8 (left). Indeed, we see that AD=AB+BD, and J is the midpoint of AD, so B'J divides the perimeter in halves." That figure suggests that the cleaver B'J is parallel to the angular bisector of angle B. This fact can be explained as follows. First, B'J is parallel to DC, as a midline in ADC. Second, angle ABC =180-DBC=2BDC. Therefore, the angular bisector of ABC is parallel to DC and thus to B'J as well.

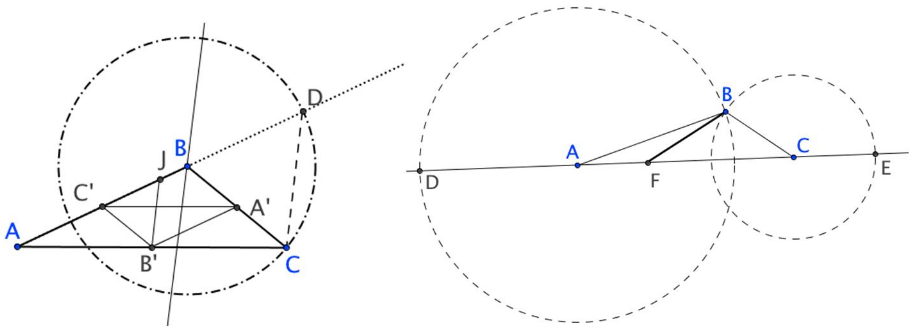

Figure 8: Constructions of clever B'J (left) and splitter BF (right).

By the same type of reasoning as before, students concluded, "since cleavers of the original triangle ABC are the angular bisectors of the medial triangle, cleavers of any triangle are concurrent". The point of concurrency is called Spieker point and denoted by S. We can express this fact in a formula: S=I'.

Since we know that I, G and I' are collinear with property GI=2GI', we conclude that I, G and S are collinear with GI=2GS. This configuration is similar to the Euler line and is called the Nagel line (Figure 9). Nagel line also contains so-called Nagel point W. For completeness of this presentation we will define the Nagel point and outline its properties. First we introduce splitters as segments starting at the vertex and bisecting the perimeter. The task is to construct splitters and observe their concurrency. One way to construct a splitter is shown on Figure 8 (right): "we draw two circles with centres A and C and radii AB and CB respectively. These circles intersect the extension of AC at points D and E. If F is the midpoint of DE, then BF is a splitter". This construction allows to find the lengths of AF=DF-AB=p-c and FC=FE-

BC=p-a in terms of triangle's sidelengths AB=c, AC=b, BC=a and semi-perimeter p=(a+b+c)/2. Constructions of splitters associated with the other two vertices leads to similar algebraic expressions, and the concurrency of splitters easily follows from Ceva theorem.

The point of concurrency of splitters W is called the Nagel point. It can be observed in DIMLE that W lands on the Nagel line passing through I, G and S. In addition, GW=2GI.

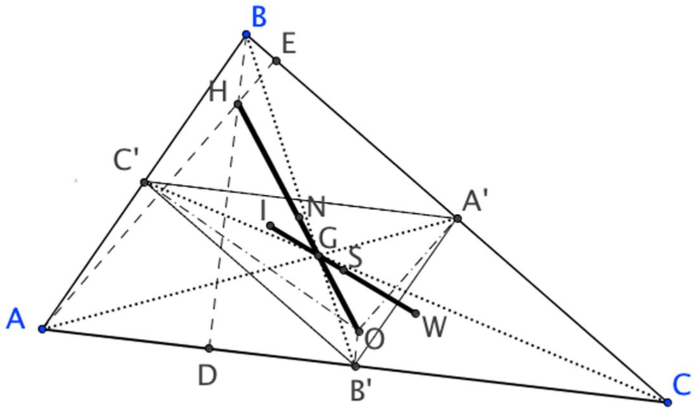

Figure 9: The Nagel line connecting the incentre and Nagel point.

The latter property can be proved by use of similar triangles and algebraic calculations. By M-transformation, this property immediately implies that I=W', that is, the Nagel point of the medial triangle coincides with the incentre of ABC.

Note that this situation differs from several previous examples because the angular bisectors of ABC have nothing to do with the splitters of A'B'C'. On the other hand, splitters correspond to the points of tangency of escribed circles of ABC with its sides: segment AT is a splitter in ABC if and only if T is the point of tangency of corresponding escribed circle with the side BC (see Figure 10, left). It appears that if R is a point of tangency of the incircle with side BC, then CR=BT. From the above construction we can also observe that if AT and BS are two splitters then AS=BT (because BT=CR=CP=AS). This observation becomes important in the following proof.

Indeed, instead of proving that GW=2GI and concluding that I=W', we can proof that two splitters A'E' and C'D' of triangle A'B'C' and angular bisector of angle ABC are concurrent (Figure 10, right). Using the property of splitters A'D'=C'E', and two pairs of similar triangles C'W'E', FW'A', and A'D'F, BC'F, we obtain that BF:BC'=A'F:A'D'=A'F:C'E'=FW':W'C'. The relation BF:BC'= FW':W'C' implies that BW' is angular bisector in C'BF and thus W' lies on the angular bisector of ABC.

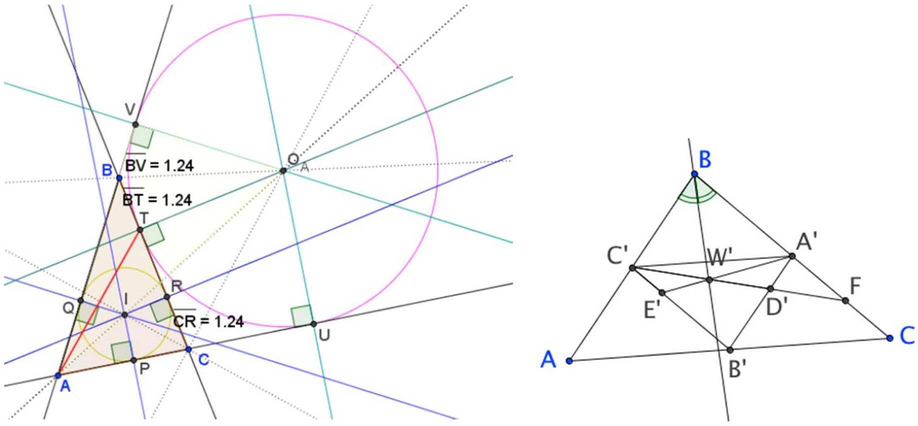

Figure 10: Splitter AT and inscribed and escribed circle (left); a proof that W’ lies on the angular bisector of ABC (right).

This argument can be applied to another pair of splitters and corresponding angular bisector proving that splitters of A’B’C’ and angular bisectors of ABC all meet at one point, that is, I=W’.

Concluding remarks

In this section I discuss six major points emerged from my observation of students engaged in geometrical constructions while learning Euclidean geometry with assistance of DIMLE.

Accuracy and confidence.

The use of constructions in dynamic environments allows students at least to make neat and accurate figures. Also, it helps them to become more confident that they correctly understood geometrical rules and statements. In words of one student, “dynamic constructions allow observing various geometric relations (such as concurrency or collinearity). This leads to obtaining more certainty”.

Exemplification, sense making, and need for theory.

More importantly, constructions in dynamic environments allow exemplifying many ideas and making students’ thinking more concrete and explicit each time when they feel that it is necessary for their understanding of a more general discussion. Instructor has an opportunity to relate new general ideas to the constructions already familiar to students through their own effort, and work on perfecting of students’ potential or preformal proofs. Thus, communication with students becomes more meaningful and purposeful. During their work in DIMLE, students asserted that in order to be implemented in a dynamic environment, “constructions actually require knowledge of geometrical facts or basic geometric configurations”. Once students realize this, they start asking sensible questions and making conjectures based on their own dynamical models; students find similarities with previous examples and activate previously learned knowledge in order to complete and explain their constructions. For example, as we see in the previous section, the notion of M-transformation relating a triangle and its medial triangle was essential for proving of several observable facts. For many students it was critical to be able to experiment with this concept in DIMLE before they grasped the theoretical argument based on its properties. Once having learned the idea of M-transformation and the essence of the logical argument applied in case of Euler's line, students started to produce deductive generalizations (DeVillier, 2012) extending this approach from explaining the Euler line geometry to exploration of the Nigel line. However, the latter case introduced its own nuances and thus added some new ideas to ones learned in the former case.

Emphasis on basic facts and constructions.

A library of basic geometric configurations could be developed for students and by students. They recognize BGC (Kondratieva, 2011a) and refer to them as they progress in their learning. Once working on drawing BGC in the form of *robust* constructions with constrained dragging domains, students may also experiment with *soft* constructions (Healy, 2000; see also discussion in Kondratieva 2012) in order to better understand the implicative nature of the statement they study. By changing constrains students receive an option to compare various cases. For example, investigate what is common and what is specific for cases of obtuse, right, or acute triangles. The advantage of dynamic BGC drawn in DIMLE compare to static paper-and-pencil constructions comes from the fact that "perception of continual change in mathematical objects may affect the users' understanding of mathematical concepts and lead them to develop a new type of learning" (Martinovic & Karadag, 2012, p. 47).

Gradual development of a big picture.

While performing geometrical constructions is helpful for some students, this assignment may make things more complicated for others. Dynamic environments gives yet another representation of Platonic geometry, different from its paper-and-pencil or analytic versions, and thus requires special way of dealing with it and interpreting the results of dynamic experimentation. If I want students to discover an explanation or proof by themselves, every such learning situation needs to be carefully designed. As I illustrated above, students benefit from discussions of a chain of interrelated construction problems such that the theory gradually builds during the process of their completion and each problem adds a new idea to the entire picture. The importance of building a genuine theoretical network as opposed to a memorization of a collection of isolated facts is discussed in the literature (see e.g. Hanna 2000, Jahnke 2007).

Interconnectivity.

As it was suggested by students, "constructions of an object in multiple ways help to connect several ideas". Moreover, many geometrical constructions can be treated as interconnecting problems (Kondratieva, 2011b). First, they often allow simple formulation available for younger learners, and can be approached at an intuitive level, as students are familiar with the geometry of the world they live in. For example, finding the centre of a circle can be achieved by folding a paper circle. Second, different construction methods may emerge from using different geometrical tools as well as application of different geometrical theorems. For instance, establishing correspondence between points of original and medial triangles gives two ways of their constructions. Also, as we discussed above a splitter can be drawn either by its definition (Figure 8, right) or using the point of tangency with escribed circle (Figure 10, left). Third, some constructions may be pertinent to other branches of mathematics either as an auxiliary problem in solving some other tasks, or requiring application of methods other than pure geometrical. For example, all points and lines in a triangle, which we discussed in the previous section, can be expressed within the coordinate and linear algebraic methods, providing a link to analytic geometry.

All these features allow mathematics teachers to use geometric constructions in various grades and courses and help students to connect their mathematical knowledge. Due to its affordances, such as dynamism and interactivity, the use of DIMLE may make these construction tasks more engaging and insightful for learners.

Clarity and assistance.

Problems should be clear and accessible but at the same time not over-simplified. In addition, in order to support students of various abilities, a system of hints and references should be incorporated in DIMLE-assisted construction tasks. More experienced and persistent students would occasionally rely on such help, while the novice learners would be given more assistance and guidance, which, however, should not be obstructive, leaving a room for students' own ideas and approaches. Organizing problems in chains of interrelated constructing-and-proofing tasks, most of which allows multiple solutions, is one such possibility illustrated in this paper.

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