Pre-Calculus California Content Standards

Standards Deconstruction Project

2008

Version 1.0

Version 1.0 produced by Bruce Arnold, Karen Cliffe, Judy Cubillo, Brenda Kracht, Abi Leaf, Mary Legner, Michelle McGinity, Michael Orr, Mario Rocha, Judy Ross, Terrie Teegarden, Sarah Thomson, Geri Villero

For information please contact

Dr. Shelly Valdez
Director of Regional Collaboration
svaldez@calpass.org
619-219-9855

Dr. Eden Dahlstrom
Associate Director of Regional Collaboration
edahlstrom@calpass.org
530-204-7129

A note to the reader: This project was coordinated and funded by the California Partnership for Achieving Student Success (Cal-PASS). Cal-PASS is a data sharing system linking all segments of education. Its purpose is to improve student transition and success from one educational segment to the next.

Cal-PASS is unique in that it is the only data collection system that spans and links student performance and course-taking behavior throughout the education system—K–12, community college, and university levels. Data are collected from multiple local and state sources and shared, within regions, with faculty, researchers, and educational administrators to use in identifying both barriers to successful transitions and strategies that are working for students. These data are then used regionally by discipline-specific intersegmental faculty groups, called "Professional Learning Councils," to better align curriculum.

Cal-PASS' standards deconstruction project was initiated by the faculty serving on the math intersegmental councils after reviewing data on student transition. A deconstruction process was devised by the participating faculty with suggestions from the San Bernardino County Unified School District math faculty (Chuck Schindler and Carol Cronk) and included adaptations of the work of Dr. Richard Stiggins of the Assessment Training Institute and Bloom's *Taxonomy of Educational Objectives* (B. S. Bloom, 1984,. Boston: Allyn and Bacon).

The Algebra II, Geometry, and Pre-calculus deconstruction projects followed using the same procedure that was used for deconstructing Algebra I standards. The following document represents a comprehensive review by K–16 faculty to deconstruct and align Pre-calculus standards.

In order to continue the collaboration on these standards, thus improving on the current work, we invite and encourage the reader to provide feedback to us. Please contact Dr. Shelly Valdez at: **svaldez@calpass.org.**

Regarding the Precalculus standards, many high schools combine some of the California mathematics content standards from trigonometry, linear algebra, and mathematical analysis to form a precalculus course. Precalculus courses in community colleges and four-year institutions also combine the content of these standards. This integrated approach reflects the fact that these topics do not exist in isolation, but rather work in unity with each other. It is also important to note that the order of the standards, or even groups of standards, may not reflect the order in which they might appear in standard textbooks or course outlines.

Table of Contents

Standards are presented in numerical order; each includes the following:

- Standard as written in the California DOE Pre-calculus Content Standards
- Deconstructed standard
- Prior knowledge needed by the student to begin learning the standard
- New knowledge student will need in order to achieve mastery of the standard
- Categorization of educational outcomes
- Necessary physical skills required
- Assessable result of the standard

Assessment Items

- Computational and procedural skills
- Conceptual understanding
- Problem solving/application

California Trigonometry Content Standards

Retrieved from CDE website (8-29-08): http://www.cde.ca.gov/be/st/ss/mthtrig.asp

TRIGONOMETRY **Grades Eight Through Twelve - Mathematics Content Standards.**
Trigonometry uses the techniques that students have previously learned from the study of algebra and geometry. The trigonometric functions studied are defined geometrically rather than in terms of algebraic equations. Facility with these functions as well as the ability to prove basic identities regarding them is especially important for students intending to study calculus, more advanced mathematics, physics and other sciences, and engineering in college.
Standard Set 1.0: Students understand the notion of angle and how to measure it, in both degrees and radians. They can convert between degrees and radians.
2.0: Students know the definition of sine and cosine as *y*- and *x*- coordinates of points on the unit circle and are familiar with the graphs of the sine and cosine functions.
3.0: Students know the identity $\cos^2(x) + \sin^2(x) = 1$: 3.1 Students prove that this identity is equivalent to the Pythagorean theorem (i.e., students can prove this identity by using the Pythagorean theorem and, conversely, they can prove the Pythagorean theorem as a consequence of this identity). 3.2 Students prove other trigonometric identities and simplify others by using the identity $\cos^2(x) + \sin^2(x) = 1$. For example, students use this identity to prove that $\sec^2(x) = \tan^2(x) + 1$.
4.0: Students graph functions of the form $f(t) = A\sin(Bt + C)$ or $f(t) = A\cos(Bt + C)$ and interpret *A, B,* and *C* in terms of amplitude, frequency, period, and phase shift.
5.0: Students know the definitions of the tangent and cotangent functions and can graph them.
6.0: Students know the definitions of the secant and cosecant functions and can graph them.
7.0: Students know that the tangent of the angle that a line makes with the *x*- axis is equal to the slope of the line.
8.0: Students know the definitions of the inverse trigonometric functions and can graph the functions.
9.0: Students compute, by hand, the values of the trigonometric functions and the inverse trigonometric functions at various standard points.
10.0: Students demonstrate an understanding of the addition formulas for sines and cosines and their proofs and can use those formulas to prove and/ or simplify other trigonometric identities.
11.0: Students demonstrate an understanding of half-angle and double-angle formulas for sines and cosines and can use those formulas to prove and/ or simplify other trigonometric identities.
12.0: Students use trigonometry to determine unknown sides or angles in right triangles.

13.0: Students know the law of sines and the law of cosines and apply those laws to solve problems.
14.0: Students determine the area of a triangle, given one angle and the two adjacent sides.
15.0: Students are familiar with polar coordinates. In particular, they can determine polar coordinates of a point given in rectangular coordinates and vice versa.
16.0: Students represent equations given in rectangular coordinates in terms of polar coordinates.
17.0: Students are familiar with complex numbers. They can represent a complex number in polar form and know how to multiply complex numbers in their polar form.
18.0: Students know DeMoivre's theorem and can give *n*th roots of a complex number given in polar form.
19.0: Students are adept at using trigonometry in a variety of applications and word problems.

California Math Analysis Content Standards

Retrieved from CDE website (8-29-08): http://www.cde.ca.gov/be/st/ss/mthanalysis.asp

MATH ANALYSIS **Grades Eight Through Twelve - Mathematics Content Standards.**
This discipline combines many of the trigonometric, geometric, and algebraic techniques needed to prepare students for the study of calculus and strengthens their conceptual understanding of problems and mathematical reasoning in solving problems. These standards take a functional point of view toward those topics. The most significant new concept is that of limits. Mathematical analysis is often combined with a course in trigonometry or perhaps with one in linear algebra to make a year-long precalculus course.
Standard Set 1.0: Students are familiar with, and can apply, polar coordinates and vectors in the plane. In particular, they can translate between polar and rectangular coordinates and can interpret polar coordinates and vectors graphically
2.0: Students are adept at the arithmetic of complex numbers. They can use the trigonometric form of complex numbers and understand that a function of a complex variable can be viewed as a function of two real variables. They know the proof of DeMoivre's theorem.
3.0: Students can give proofs of various formulas by using the technique of mathematical induction.
4.0: Students know the statement of, and can apply, the fundamental theorem of algebra.
5.0: Students are familiar with conic sections, both analytically and geometrically: 5.1 Students can take a quadratic equation in two variables; put it in standard form by completing the square and using rotations and translations, if necessary; determine what type of conic section the equation represents; and determine its geometric components (foci, asymptotes, and so forth). 5.2 Students can take a geometric description of a conic section - for example, the locus of points whose sum of its distances from (1, 0) and (-1, 0) is 6 - and derive a quadratic equation representing it.

6.0: Students find the roots and poles of a rational function and can graph the function and locate its asymptotes.
7.0: Students demonstrate an understanding of functions and equations defined parametrically and can graph them.
8.0: Students are familiar with the notion of the limit of a sequence and the limit of a function as the independent variable approaches a number or infinity. They determine whether certain sequences converge or diverge.

California Linear Algebra Content Standards

Retrieved from CDE website (8-29-08):
http://www.cde.ca.gov/be/st/ss/mthlinearalgebra.asp

LINEAR ALGEBRA **Grades Eight Through Twelve - Mathematics Content Standards.**
The general goal in this discipline is for students to learn the techniques of matrix manipulation so that they can solve systems of linear equations in any number of variables. Linear algebra is most often combined with another subject, such as trigonometry, mathematical analysis, or precalculus.
Standard Set 1.0: Students solve linear equations in any number of variables by using Gauss-Jordan elimination.
2.0: Students interpret linear systems as coefficient matrices and the Gauss-Jordan method as row operations on the coefficient matrix.
3.0: Students reduce rectangular matrices to row echelon form.
4.0: Students perform addition on matrices and vectors.
5.0: Students perform matrix multiplication and multiply vectors by matrices and by scalars.
6.0: Students demonstrate an understanding that linear systems are inconsistent (have no solutions), have exactly one solution, or have infinitely many solutions.
7.0: Students demonstrate an understanding of the geometric interpretation of vectors and vector addition (by means of parallelograms) in the plane and in three-dimensional space.
8.0: Students interpret geometrically the solution sets of systems of equations. For example, the solution set of a single linear equation in two variables is interpreted as a line in the plane, and the solution set of a two-by-two system is interpreted as the intersection of a pair of lines in the plane.
9.0: Students demonstrate an understanding of the notion of the inverse to a square matrix and apply that concept to solve systems of linear equations.
10.0: Students compute the determinants of 2 x 2 and 3 x 3 matrices and are familiar with their geometric interpretations as the area and volume of the parallelepipeds spanned by the images under the matrices of the standard basis vectors in two-dimensional and three-dimensional spaces.

11.0: Students know that a square matrix is invertible if, and only if, its determinant is nonzero. They can compute the inverse to 2 x 2 and 3 x 3 matrices using row reduction methods or Cramer's rule.
12.0: Students compute the scalar (dot) product of two vectors in *n*- dimensional space and know that perpendicular vectors have zero dot product.

Trigonometry Standard #1

Standard Set 1.0
Students understand the notion of angle and how to measure it, in both degrees and radians. They can convert between degrees and radians.

Deconstructed Standard
1. Students understand the notion of angle.
2. Students understand how to measure an angle in degrees.
3. Students understand how to measure an angle in radians.
4. Students can convert angle measures from degrees to radians.
5. Students can convert angle measures from radians to degrees.

Prior Knowledge Necessary
Students should know how to:
- recognize and identify rays.
- define angles in triangles and other simple geometric figures.
- determine the measures of the angles of a triangle.
- identify complementary and supplementary angles.
- perform unit conversions.

New Knowledge
Students will need to learn to:
- identify the initial side of an angle.
- identify the terminal side of an angle.
- identify the standard position of an angle.
- measure an angle in degrees.
- measure an angle in radians.
- recognize if an angle has a negative or positive measure.
- identify angles that exceed one revolution.
- recognize that $180^\circ = \pi$ radians.
- convert an angle measure in degrees to radians by multiplying by $\pi / 180^\circ$.
- convert an angle measure in radians to degrees by multiplying by $180^\circ / \pi$.

Categorization of Educational Outcomes
Competence Level: Knowledge
1. Students will identify the components of an angle.
2. Students will describe the basis of the measurement of an angle.

Competence Level: Comprehension
1. Students will classify an angle as having a positive or negative measure.
2. Students will understand the concept of coterminal angles.
3. Students will describe an angle greater than one revolution.

Competence Level: Application

1. Students will determine the radian measure of an angle expressed in degrees.
2. Students will determine the degree measure of an angle expressed in radians.

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will identify the initial and terminal sides of an angle.
2. Students will measure an angle in degrees and radians.
3. Students will convert angle measures from degrees to radians.
4. Students will convert angle measures from radians to degrees.

Trigonometry Standard #1 Model Assessment Items

Computational and Procedural Skills

1. Find the radian measure of the angle with the given degree measure.

A. 72°	B. 54°	C. -45°
D. 1080°	E. -300°	F. 202.5°
G. -3960°	H. -60°	I. 7.5°

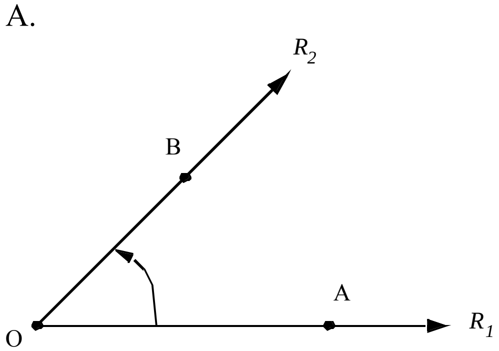
2. Find the degree measure of the angle with the given radian measure.

A. $\frac{7\pi}{6}$	B. $\frac{11\pi}{3}$	C. $\frac{-5\pi}{4}$
D. 3	E. -2	F. $\frac{\pi}{10}$
G. $\frac{5\pi}{18}$	H. $\frac{-2\pi}{15}$	I. $\frac{-13\pi}{12}$

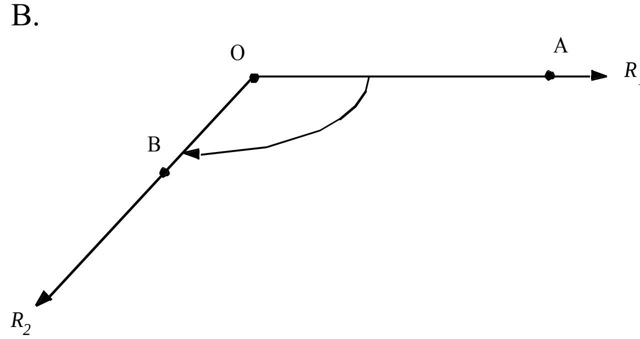
Conceptual Understanding

1. Given $\angle AOB$ consisting of the two rays R_1 and R_2. Label the initial and terminal sides of $\angle AOB$. Identify the angle as a positive or negative angle.

A.



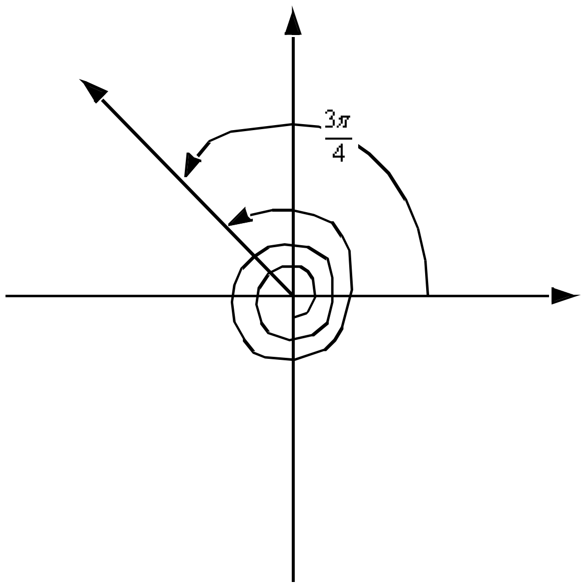
B.



2. Sketch a unit circle and draw the given angle.

A. $-\frac{\pi}{2}$	B. $-\frac{17\pi}{6}$	C. $\frac{23\pi}{4}$
D. 2π	E. $\frac{2\pi}{3}$	F. $\frac{3\pi}{4}$

3. Write the angle shown below in terms of its total measure. Also write the negative angle that is coterminal to this one.



<u>Problem Solving/Application</u>

1. You strike a tether ball causing the ball to revolve around the pole 2 ¼ times. How many radians is this?

2. Through how many radians does the minute hand of a clock rotate from 12:40 PM to 1:30 PM.

Trigonometry Standard #2

Standard Set 2.0
Students know the definition of sine and cosine as *y*-and *x*-coordinates of points on the unit circle and are familiar with the graphs of the sine and cosine functions.

Deconstructed Standard
1. Students know the definition of sine as the *y* coordinate of points on the unit circle.
2. Students know the definition of cosine as the *x*coordinate of points on the unit circle.
3. Students are familiar with the graph of the sine function.
4. Students are familiar with the graph of the cosine function.

Prior Knowledge Necessary
Students should know how to:
- identify the *x*-value of an ordered pair.
- identify the *y*-value of an ordered pair.
- recognize the graphical representation of the unit circle $\left(x^2 + y^2 = 1\right)$.
- locate a particular ordered pair on the unit circle.
- identify radian measures on the unit circle.
- convert between degrees and radian measure.
- understand the definition of function, including domain and range.
- interpret transformations of functions as related to the original functions.

New Knowledge
Students will need to learn to:
- recognize that a point $(x,\ y)$ on the unit circle lies on the terminal side of an angle, t, measured counter-clockwise from the *x*-axis.
- given a point $(x,\ y)$ on the unit circle corresponding to an angle measure, *t*, identify $\sin(t)$ as *y*.
- given a point (x, y) on the unit circle corresponding to an angle measure, *t*, identify $\cos(t)$ as *x*.
- identify the domain of the sine function as all real numbers.
- identify the domain of the cosine function as all real numbers.
- recognize the range of the sine function as the interval [-1, 1].
- recognize the range of the cosine function as the interval [-1, 1].
- identify the graphical representation of $y = \sin x$ as the graph of the sine function.
- identify the graphical representation of $y = \cos x$ as the graph of the cosine function.
- identify the amplitude of the sine function.
- identify the amplitude of the cosine function.
- identify the frequency of the sine function.
- identify the frequency of the cosine function.

- identify the period of the sine function.
- identify the period of the cosine function.
- interpret transformations of the original sine function.
- interpret transformations of the original cosine function.

Categorization of Educational Outcomes
Competence Level: Knowledge

1. Students will define sine as the *y* coordinate of a point on the unit circle.
2. Students will define cosine as the *x* coordinate of a point on the unit circle.
3. Students will identify the graph of the sine function.
4. Students will identify the graph of the cosine function.

Necessary New Physical Skills
None

Assessable Result of the Standard

1. Students will identify $\sin(t)$ given a point *t* on the unit circle.
2. Students will identify $\cos(t)$ given a point *t* on the unit circle.
3. Students will identify the graph of the sine function.
4. Students will identify the graph of the cosine function.
5. Students will determine the point (x, y) on the unit circle corresponding to an angle measure *t* by using $\sin(t)$ and $\cos(t)$.
6. Students will explain how transformed sine functions relate to the original.
7. Students will explain how transformed cosine functions relate to the original.

Trigonometry Standard #2 Model Assessment Items

Computational and Procedural Skills

1. Evaluate the sine and cosine functions at each real number.

   A. $t = \frac{\pi}{6}$ B. $t = \frac{5\pi}{4}$ C. $t = 0$ D. $t = \pi$

2. Find the point (x, y) on the unit circle that corresponds to the real number *t*.

   A. $\frac{\pi}{4}$ B. $\frac{5\pi}{3}$

Conceptual Understanding

1. Find the amplitude of the function and describe how the graph of the function is related to the graph of $y = \sin x$ or $y = \cos x$.

   A. $y = 2\sin x$ B. $y = -4\cos x$

2. Find the period and frequency of the given functions.

   A. $y = -3\sin\ x$ B. $y = \frac{1}{2}\cos x$

Problem Solving/Application

1. Let (x_1, y_1) and (x_2, y_2) be points of the unit circle corresponding to $t = t_1$ and $t = t_1 + \pi$, respectively. Identify any symmetry between the two points. Make a conjecture about any relationship between $\cos(t_1)$ and $\cos(t_1 + \pi)$.

Trigonometry Standard #3

Standard Set 3.0

3.0: Students know the identity: $\cos^2(x)+\sin^2(x)=1$.

3.1: Students prove that this identity is equivalent to the Pythagorean theorem (i.e., students can prove this identity by using the Pythagorean theorem and, conversely, they can prove the Pythagorean theorem as a consequence of this identity).

3.2: Students prove other trigonometric identities and simplify others by using the identity $\cos^2(x)+\sin^2(x)=1$. [For example, students use this identity to prove that $\sec^2(x)=\tan^2(x)+1$.]

Deconstructed Standard

1. Students know the identity $\cos^2(x)+\sin^2(x)=1$.
2. Students prove the identity $\cos^2(x)+\sin^2(x)=1$ using the Pythagorean theorem.
3. Students prove the Pythagorean theorem as a consequence of the identity $\cos^2(x)+\sin^2(x)=1$.
4. Students prove trigonometric identities by using the identity $\cos^2(x)+\sin^2(x)=1$.
5. Students simplify trigonometric identities by using the identity $\cos^2(x)+\sin^2(x)=1$.

NOTE: Since the proof of identities is included in no other standard, Standard 3.0 has been expanded to include all trigonometric identities.

Prior Knowledge Necessary

Students should know how to:

- identify and use the Pythagorean theorem.
- perform basic operations on equations.
- factor equations.
- simplify equations.
- use deductive reasoning to establish a conclusion from a given statement.
- write a point on the unit circle in polar (trigonometric) form.
- interpret the quantity $\cos^2(x)+\sin^2(x)$ as equal to 1.

New Knowledge

Students will need to learn to:

- use the Pythagorean theorem to prove that $\cos^2(x)+\sin^2(x)=1$.
- use the identity $\cos^2(x)+\sin^2(x)=1$ to prove the Pythagorean theorem.
- derive the other Pythagorean identities from $\cos^2(x)+\sin^2(x)=1$.
- use deductive reasoning to establish the equivalence of trigonometric identities.
- prove trigonometric identities using other identities such as $\cos^2(x)+\sin^2(x)=1$.

- simplify trigonometric expressions using other identities such as $\cos^2(x)+\sin^2(x)=1$.

<u>Categorization of Educational Outcomes</u>

Competence Level: Knowledge

1. Students will define the quantity $\cos^2(x)+\sin^2(x)$ as equal to 1.

Competence Level: Application:

1. Students will construct a logical proof of the identity $\cos^2(x)+\sin^2(x)=1$ using the Pythagorean theorem in polar coordinates.
2. Students will show that the Pythagorean theorem results as a consequence of the identity $\cos^2(x)+\sin^2(x)=1$.
3. Students will establish the equivalence of other Pythagorean identities using the identity $\cos^2(x)+\sin^2(x)=1$.
4. Students will prove trigonometric identities.
5. Students will simplify trigonometric expressions.

<u>Necessary New Physical Skills</u>

None

<u>Assessable Result of the Standard</u>

1. Students will recognize that $\cos^2(x)+\sin^2(x)$ as equal to 1.
2. Students will produce a proof of this identity using the Pythagorean theorem.
3. Students will produce a proof showing that the Pythagorean theorem is a consequence of this identity.
4. Students will produce proofs of trigonometric identities.
5. Students will simplify trigonometric expressions.

Trigonometry Standard #3 Model Assessment Items

Computational and Procedural Skills

1. Using the Pythagorean theorem $x^2 + y^2 = r^2$ with $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$, prove that $\cos^2\theta + \sin^2\theta = 1$.

2. Show that $\sec^2\theta = \tan^2\theta + 1$.

3. Simplify $\frac{1}{\sec x + 1} - \frac{1}{\sec x - 1}$.

Conceptual Understanding

1. Show that the identity is not true for all values of θ. (More than one answer is possible).
   $\csc\theta = \sqrt{1 + \cot^2\theta}$

2. Give a counterexample to show that the conditional statement is false:
   If a figure is part of a line, then it is a segment.

Problem Solving/Application

1. Let θ be an acute angle such that $\cos\theta = 0.8$. Using trigonometric identities, find the values of $\sin\theta$ and $\tan\theta$.

Trigonometry Standard #4

Standard Set 4.0

Students graph functions of the form $f(t) = A\sin(Bt + C)$ or $f(t) = A\cos(Bt + C)$ and interpret *A*, *B*, and *C* in terms of amplitude, frequency, period, and phase shift.

Deconstructed Standard

1. Students graph functions in the form $f(t) = A\sin(Bt + C)$.
2. Students graph functions in the form $f(t) = A\cos(Bt + C)$.
3. Students interpret the amplitude of the function using *A*.
4. Students interpret the frequency of the function using *B*.
5. Students interpret the period of the function using *B*.
6. Students interpret the phase shift of the function using *B* and *C*.

Prior Knowledge Necessary

Students should have the computational and conceptual knowledge outlined in Trigonometry Standard #2: "Students should know the definition of sine and cosine as *y*- and *x*-coordinates of points on the unit circle and are familiar with the graph of the sine and cosine functions."

Students should know how to:

- evaluate sine and cosine for any given angle.
- graph sine and cosine functions.
- translate functions and their graphs.
- shrink/stretch functions and their graphs.
- reflect functions and their graphs across the *x*-axis.
- recognize the definition of the amplitude.
- recognize the period of a function.
- recognize the frequency of a function.
- recognize the relationship between a translation of a function and a phase shift.

New Knowledge

Students will need to learn to:

- identify the relationship between *A* and the amplitude of the function.
- identify the relationship between *B* and the period and frequency of the function.
- identify the relationship between *B* and *C* and the phase shift of a sine/cosine function.
- graph functions of the form $f(t) = A\sin(Bt + C)$ or $f(t) = A\cos(Bt + C)$.

Categorization of Educational Outcomes

Competence Level: Application:

1. Students will graph $f(t) = A\sin(Bt + C)$ or $f(t) = A\cos(Bt + C)$.

2. Students will estimate the amplitude of a graph.
3. Students will estimate the phase shift of a graph.
4. Students will estimate the period of a graph.
5. Students will determine the amplitude, period, frequency, and phase shift of a given function or graph.

<u>Necessary New Physical Skills</u>

None.

<u>Assessable Result of the Standard</u>

1. Students will produce a graph of the function $f(t) = A\sin(Bt + C)$ or $f(t) = A\cos(Bt + C)$.
2. Students will find the amplitude, period, frequency, and phase shift of a given graph of the form $f(t) = A\sin(Bt + C)$ or $f(t) = A\cos(Bt + C)$.
3. Students will produce a graph and an equation of the form $f(t) = A\sin(Bt + C)$ or $f(t) = A\cos(Bt + C)$ given the amplitude, period/frequency, and phase shift.
4. Students will identify *A*, *B*, and *C* given a function of the form $f(t) = A\sin(Bt + C)$ or $f(t) = A\cos(Bt + C)$.

Trigonometry Standard #4 Model Assessment Items

Computational and Procedural Skills

1. State the amplitude, phase shift, and period of the sinusoid $y = -2\sin\left(3x - \frac{\pi}{4}\right)$.
2. Graph $f(x) = 3\cos 2x$.
3. Given the amplitude is 3, the period is π, and the graph passes through the point (0,1), write an equation in the form of $f(t) = A\sin(Bt + C)$ and graph the function.

Conceptual Understanding

1. Given the following two functions, $f(x) = 3\cos 2x$ and $g(x) = -3\cos\left(2x + \frac{\pi}{3}\right)$, identify the relationship between these two functions (i.e. what translations are involved in producing the second function from the first).
2. What is values of *A*, *B*, and *C* in $f(t) = A\sin(Bt + C)$ are needed to produce the graph that is the same as $-2\cos\left(t + \frac{\pi}{4}\right)$?

Problem Solving/Application

1. A Ferris wheel 50 ft in diameter makes on revolution every 40 seconds. If the center of the Ferris wheel is 30 feet above ground, how long after reaching the low point is a rider 50 feet above the ground?
2. On a particular Labor Day, The high tide in southern California occurs at 7:12 a.m. At that time you measure the water at the end of the Santa Monica Pier to be 11 feet deep. At 1:24 p.m. It is low tide, and you measure the water to be only 7 feet deep. Assume the depth of the water is a sinusoidal function of time with a period of half a lunar day, which is about 12 hours, 24 minutes.
   A. At what time on that Labor Day does the first low tide occur?
   B. What was the approximate depth of the water at 4:00 a.m. and at 9:00 p.m.?
   C. What is the first time on that Labor Day that the water is 9 feet deep?

Trigonometry Standard #5

Standard Set 5.0
Students know the definitions of the tangent and cotangent functions and can graph them.

Deconstructed Standard
1. Students know the definition of the tangent function.
2. Students graph the tangent function.
3. Students know the definition of the cotangent function.
4. Students graph the cotangent function

Prior Knowledge Necessary
Students should know how to:
- use the vocabulary of functions (i.e.: asymptotes, continuity, increasing/decreasing, domain and range, end behavior, etc.).
- use the measure of angles in radians and degrees.
- evaluate sine, cosine, and tangent.
- recognize the period of a function.
- define the tangent using sine and cosine.

New Knowledge
Students will need to learn to:
- identify the definition of cotangent as a reciprocal of tangent.
- evaluate the cotangent function.
- find the x-intercepts of the cotangent function.
- find the period of tangent.
- find the period of cotangent.
- identify where the tangent function is undefined (vertical asymptotes).
- identify where the cotangent function is undefined (vertical asymptotes).
- determine the behavior of the graph of tangent near the vertical asymptotes.
- determine the behavior of the graph of cotangent near the vertical asymptotes.
- graph the tangent function.
- graph the cotangent function.

Categorization of Educational Outcomes
Competence Level: Knowledge
1. Students will define the cotangent function.
2. Student will determine values of the tangent functions for various angles.
3. Student will determine values of the cotangent functions for various angles.

Competence Level: Application
1. Students will graph the tangent function.
2. Students will graph the cotangent function.

Necessary New Physical Skills
None

Assessable Result of the Standard
1. Students will produce the graph of the tangent function.
2. Students will produce the graph of the cotangent function.
3. Students will define the tangent function.
4. Students will define the cotangent function.

Trigonometry Standard #5 Model Assessment Items

Computational and Procedural Skills

1. Graph two periods of the function $y = \tan x$.
2. Graph two periods of the function $y = \cot x$.
3. Find the period and graph the following:

   A. $y = \tan 2x$ B. $y = -3\cot x$ C. $y = \tan\left(x + \frac{\pi}{4}\right)$

Conceptual Understanding

1. Graph the function $f(x) = -\cot x$ on the interval $(-\pi, \pi)$. Explain why it is correct to say that *f* is increasing on the interval $(0, \pi)$, but it is not correct to say that *f* is increasing on the interval $(-\pi, \pi)$.

Problem Solving/Application

1. The beam from a lighthouse completes one rotation every two minutes. At time *t*, the distance *d* is $d(t) = 3\tan \pi t$, where *t* is measured in minutes and *d* in miles.

   A. Find $d(0.15)$, $d(0.25)$, and $d(0.45)$.

   B. Sketch a graph of the function *d* for $0 \le t \le \frac{1}{2}$.

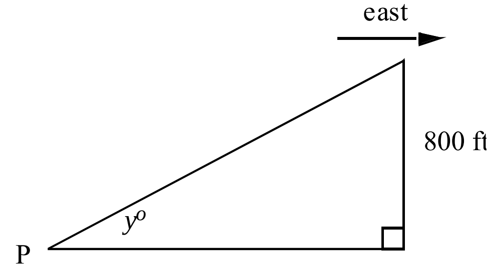
   C. What happens to the distance *d* as *t* approaches $1/2$?

2. A hot air balloon over Albuquerque, New Mexico, is being blown due east from point *P* and traveling at a constant height of 800 feet. The angle, *y*, is formed by the ground and the line of vision from *P* to the balloon. This angle changes as the balloon travels.

   A. Express the horizontal distance *x* as a function of the angle *y*.

   B. When the angle is $\pi/20$ radians, what is the balloon's horizontal distance from *P*?

   C. What happens to the value of *y* as the balloon continues east?



Trigonometry Standard # 6

Standard Set 6.0
Students know the definition of the secant and cosecant functions and can graph them.

Deconstructed Standard
1. Students know the definition of the secant function.
2. Students graph the secant function.
3. Students know the definition of the cosecant function.
4. Students graph the cosecant function.

Prior Knowledge Necessary
Students should know how to:
- use the vocabulary of functions (i.e.: asymptotes, continuity, increasing/decreasing, domain and range, end behavior, etc.).
- use the measure of angles in radians and degrees.
- evaluate sine, cosine, and tangent.
- identify the period of simple trigonometric functions.

New Knowledge
Students will need to learn to:
- recognize the definition of secant as the reciprocal of cosine.
- find the period of secant.
- evaluate the secant function.
- Recognize definition of cosecant as the reciprocal of sine.
- evaluate the cosecant function.
- find the where the secant function is undefined (vertical asymptotes).
- determine the behavior of the graph of secant near the vertical asymptotes.
- find the where the cosecant function is undefined (vertical asymptotes).
- determine the behavior of the graph of cosecant near the vertical asymptotes.
- graph the secant function.
- graph the cosecant function.

Categorization of Educational Outcomes
Competence Level: Knowledge
1. Students will define the secant function.
2. Students will define the cosecant function.
3. Student will determine values of the secant functions for various angles.
4. Student will determine values of the cosecant functions for various angles.

Competence Level: Application
1. Students will graph the secant function.
2. Students will graph the cosecant function.

Necessary New Physical Skills
None

Assessable Result of the Standard
1. Students will produce the graph of the secant function.
2. Students will produce the graph of the cosecant function.
3. Students will define the secant function.
4. Students will define the cosecant function.

Trigonometry Standard #6 Model Assessment Items

Computational and Procedural Skills

1. Graph two periods of the function $y = \sec x$.

2. Graph two periods of the function $y = \csc x$.

3. Graph both of the following functions on the same coordinate plane over the interval $-2\pi \le t \le 2\pi$:

$$y = \sin\left(t - \frac{\pi}{4}\right) \text{ and } y = \csc\left(t - \frac{\pi}{4}\right).$$

4. Find the period and graph the following:

   A. $y = \csc 3x$ B. $y = 3\sec 4x$ C. $y = 3\csc\left(x + \frac{\pi}{2}\right)$

Conceptual Understanding

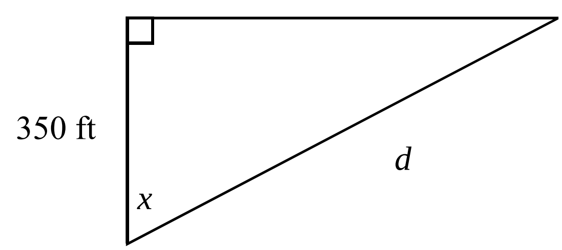
1. Explain why the following formula is true: $\sec\left(x - \frac{\pi}{2}\right) = \csc x$.

Problem Solving/Application

1. The Bolivar Lighthouse is located on a small island 350 feet from the shore of the mainland as shown in the figure below.

   A. Express the distance *d* as a function of the angle *x*.

   B. If *x* is 1.55 radians, what is *d*?



Trigonometry Standard #7

Standard Set 7.0

Students know that the tangent of the angle that a line makes with the *x*-axis is equal to the slope of the line.

Deconstructed Standard

1. Students know how to find the tangent of an angle.
2. Students know how to determine the angle a line makes with the *x*-axis.
3. Students know how to find the slope a line.
4. Students verify the tangent of the angle is equal to the slope of the line.

Prior Knowledge Necessary

Students should know how to:

- graph a linear equation.
- compute slope of a line.
- identify the *x*-intercept of a line.
- write the equation of a line.
- recognize the definition of tangent as the ratio of the opposite side divided by the adjacent side.
- calculate the tangent of an angle.

New Knowledge

Students will need to learn to:

- recognize slope as a rate of change of *y* in relation to *x*.
- identify the angle of inclination a line makes with the *x*-axis is measured from the positive *x*-axis, where $0^\circ \le \theta \le 180^\circ$.
- determine the tangent of the angle a given line makes with the *x*-axis.
- verify the tangent of the angle formed by the given line and the *x*-axis is equivalent to the slope of the line.

Categorization of Educational Outcomes

Competence Level: Application

1. Students will compute the slope of a line.
2. Students will calculate the tangent of an angle.
3. Students will demonstrate that the tangent of the angle is equivalent to the slope of the line.

Necessary New Physical Skills

1. Use of scientific calculator or trigonometric table.

Assessable Result of the Standard

1. Students will calculate the slope of a given line.
2. Students will calculate the tangent of the angle of inclination a given line makes with the x-axis.
3. Students will calculate the angle of inclination of a given line to the x-axis.
4. Students will show the equivalency of the slope and the tangent of the angle.

Trigonometry Standard #7 Model Assessment Items

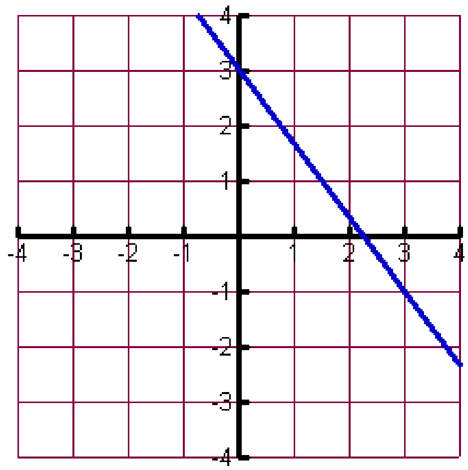
Computational and Procedural Skills

1. Given a line joining $(-1,2)$ and $(4,1)$.
   A. Find the slope of the line.

   B. Find the tangent of the angle of inclination of the line.

   C. Find the angle of inclination of the line.

2. Find the tangent of the angle of inclination formed by the given line and the *x*-axis.



3. Find the slope of a line which forms a 124° angle with the *x*-axis.

Conceptual Understanding

1. Find the equation of a line which forms a 38° angle with the *x*-axis and contains the point $(3,5)$.

2. Find the equation of a line which forms a 114° angle with the *x*-axis and has an *x*-intercept of -3.

3. Find the equation of a line which forms a 72° angle with the *x*-axis and has a *y*-intercept of 6.

4. Write an equation of a line for which the tangent of the angle that line makes with the *x*-axis is undefined.

5. Describe the relationship of lines which form the same angle to the *x*-axis. Give examples as well as a sketch to explain your answer.

Problem Solving/Application

1. The cost of producing T-shirts is given by the following formula: $y = 2.5x + 8000$, where x represents the number of T-shirts produced and y represents the total cost of producing x number of T-shirts. Each shirt uses $2.50 in materials and there is a fixed cost of $8,000.
   A. Draw a sketch of the graph which represents the cost of producing T-shirts. Be sure to include scale on your graph.

   B. Find the slope of the line. Explain the real world significance of the slope.

   C. Find the angle of inclination of the line to the positive x-axis.

   D. Describe how increasing the material cost of the T-shirts would affect the angle of inclination of the line.

Trigonometry Standard # 8

Standard Set 8.0

Students know the definitions of the inverse trigonometric functions and can graph the functions.

Deconstructed Standard

1. Students know the definition of the inverse sine trigonometric function.
2. Students know the definition of the inverse cosine trigonometric function.
3. Students know the definition of the inverse tangent trigonometric function.
4. Students know the definition of the inverse cotangent trigonometric function.
5. Students know the definition of the inverse secant trigonometric function.
6. Students know the definition of the inverse cosecant trigonometric function.
7. Students graph the inverse sine trigonometric function.
8. Students graph the inverse cosine trigonometric function.
9. Students graph the inverse tangent trigonometric function.
10. Students graph the inverse cotangent trigonometric function.
11. Students graph the inverse secant trigonometric function.
12. Students graph the inverse cosecant trigonometric function.

Prior Knowledge Necessary

Students should know how to:

- define the six trigonometric functions, sine, cosine, tangent, cotangent, secant and cosecant.
- graph the six trigonometric functions, sine, cosine, tangent, cotangent, secant and cosecant.
- find the domain and range of the six trigonometric functions, sine, cosine, tangent, cotangent, secant and cosecant.
- use the vertical line test to determine if a relation is a function.
- use the horizontal line test to determine if a function is one-to-one.
- define an inverse function.
- determine if a function has an inverse.
- solve for an inverse function.

New Knowledge

Students will need to learn to:

- define the six inverse trigonometric functions, inverse sine, inverse cosine, inverse tangent, inverse cotangent, inverse secant and inverse cosecant.
- distinguish between the inverse function and the inverse relation of the six trigonometric functions.
- identify the appropriate quadrants in which an inverse function is defined.
- graph the six inverse trigonometric functions.
- define the domain and range of the six inverse trigonometric functions.

Categorization of Education Outcome

Competence Level: Knowledge

1. Students will define the six inverse trigonometric functions.

Competence Level: Comprehension

1. Students will explain the difference between an inverse trigonometric function and an inverse trigonometric relation.
2. Students will explain why the domain of an inverse trigonometric relation must be restricted in order to create a function.
3. Students will describe how to select the appropriate domain for an inverse trigonometric function.

Competence Level: Application

1. Students will demonstrate their understanding of inverse trigonometric functions with restricted domains by identifying the appropriate range in which the function is defined.
2. Students will graph the six inverse trigonometric functions.

Necessary New Physical Skills

1. Use of scientific or graphing calculator.

Assessable Result of the Standard

1. Students will define of the six inverse trigonometric functions.
2. Students will identify the restricted domain of the six trigonometric functions.
3. Students will graph the six inverse trigonometric functions, stating the domain and range of each.

Trigonometry Standard #8 Model Assessment Items

Computational and Procedural Skills

1. Write each equation in the form of an inverse trigonometric function:
   A. $\sin 30^\circ = 0.5$

   B. $\tan 45^\circ = 1$

   C. $\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$

   D. $\csc\frac{5\pi}{6} = -\frac{2}{\sqrt{3}}$

2. State the range of the following functions:
   A. $Sin^{-1}(x)$

   B. $Cos^{-1}(x)$

   C. $Cot^{-1}(x)$

3. Graph the function $y = Cos^{-1}(x)$. State the domain and range.

4. Graph the function $y = Csc^{-1}(x)$. State the domain and range.

Conceptual Understanding

1. Draw triangle ABC where $\angle C = 90^\circ$. Label the sides *a*, *b*, and *c*. Write an inverse sine function relating $\angle A$ and the appropriate sides.

2. Find the exact value of $\tan\left(Cos^{-1}\left(\frac{4}{5}\right)\right)$. Use a diagram to explain your thinking.

3. Find the exact value of $\cot\left(Sin^{-1}\left(-\frac{5}{13}\right)\right)$. Use a diagram to explain your thinking.

4. Explain why the function $f(x) = Cos^{-1}(x)$ has a unique solution while the equation $y = \cos^{-1}(x)$ has multiple solutions. Use a graph or diagram to help explain your answer.

5. Determine whether each equation is true or false. If false, give a counterexample.

   A. $\tan\left(Tan^{-1}x\right) = x$

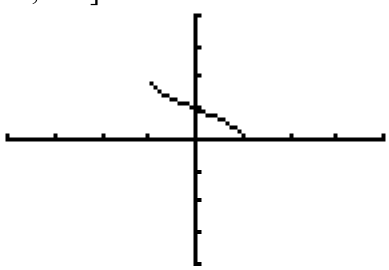
   B. $Tan^{-1}\left(\tan x\right) = x$

   C. $Sin^{-1}\left(\sin x\right) = x$

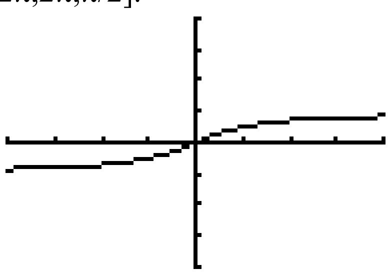
   D. $\sin\left(S\text{in}^{-1}x\right) = x$

   E. $Cos^{-1}\left(-x\right) = -Cos^{-1}\left(x\right)$

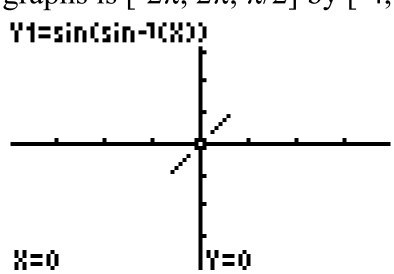
6. Identify the function graphed below. The *x*-scale is [-4,4,1] and the *y*-scale is [-2π, 2π, π/2].

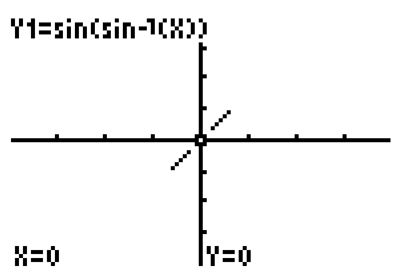


7. Identify the function graphed below. The *x*-scale is [-4,4,1] and the *y*-scale is [-2π,2π,π/2].



8. Explain why there is a difference in the two graphs below. The window of both graphs is [-2π, 2π, π/2] by [-4, 4, 1].





Problem Solving/Application

1. A barge is pulled into the dock by means of a winch located on the dock, which is 15 feet above the deck of the barge where the rope is attached. Let θ be the angle of elevation from the barge to the winch and let $r =$ the length of the rope from the winch to the barge.
   A. Write θ as a function of r.

   B. Find θ when $r = 50$ feet.

   C. Find θ when $r = 20$ feet.

   D. The winch stops effectively pulling when the angle is more than 65°. At that point, how far is the barge away from the dock?

Trigonometry Standard #9

Standard Set 9.0
Students compute, by hand, the values of the trigonometric functions and the inverse trigonometry functions at various standard points.

Deconstructed Standard

1. Students compute, by hand, the values of the trigonometric function of sine at the standard points $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and the multiples of those points spanning the unit circle.
2. Students compute, by hand, the values of the trigonometric function of cosine at the standard points $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and the multiples of those points spanning the unit circle.
3. Students compute, by hand, the values of the trigonometric function of tangent at the standard points $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and the multiples of those points spanning the unit circle.
4. Students compute, by hand, the values of the trigonometric function of cotangent at the standard points $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and the multiples of those points spanning the unit circle.
5. Students compute, by hand, the values of the trigonometric function of secant at the standard points $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and the multiples of those points spanning the unit circle.
6. Students compute, by hand, the values of the trigonometric function of cosecant at the standard points $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and the multiples of those points spanning the unit circle.
7. Students compute, by hand, the values of inverse sine at the same standard points.
8. Students compute, by hand, the values of inverse cosine at the same standard points.
9. Students compute, by hand, the values of inverse tangent at the same standard points.
10. Students compute, by hand, the values of inverse cotangent at the same standard points.
11. Students compute, by hand, the values of inverse secant at the same standard points.
12. Students compute, by hand, the values of inverse cosecant at the same standard points.

Prior Knowledge Necessary

Student should know how to:

- perform arithmetic computations with rational and irrational numbers, including rationalization of the denominator.
- recognize the sine and cosine as *y*- and *x*-coordinates of points on the unit circle.
- compute sides of a 30-60-90 triangle.
- compute sides of a 45-45-90 triangle.
- convert radian angles to degrees.
- use the definitions of the six trigonometric functions.
- use the definitions of the six inverse trigonometric functions.

New Knowledge

Students will need to learn to:

- identify the coordinates of the unit circle at the quadrantal angles of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, and\, 2\pi$.
- evaluate the trigonometric functions at the quadrantal angles of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, and\, 2\pi$.
- recognize some standard points on a unit circle can be evaluated using special triangles like 30-60-90 and 45-45-90 triangles.
- understand similar triangles in each quadrant will have the same trigonometric values, with the exception of the sign of the answer.
- use a reference angle to evaluate a trigonometric function.
- identify angles and/or points utilizing the inverse trigonometric functions.
- evaluate the inverse trigonometric functions using the point of intersection of the unit circle and the terminal side of an angle.

Categorization of Educational Outcomes

Competence Level: Knowledge

1. Students will define the coordinates of the unit circle at the quadrantal angles of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, and\, 2\pi$.
2. Students will identify the y-coordinate is the sine value, the *x*-coordinate is the cosine value, and the ratio *y/x* is the tangent value at the quadrantal angles of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, and\, 2\pi$, if defined.

Competence Level: Comprehension

1. Students will describe how a triangle utilizes a point on the unit circle to evaluate a trigonometric function.
2. Students will understand how a special triangle like a 30-60-90 triangle or a 45-45-90 triangle can be used to evaluate the trigonometric values of special angles in a unit circle.
3. Students will explain why trigonometric functions are undefined for some angles.

Competence Level: Application

1. Students will determine trigonometric values using a 30-60-90 triangle and a 45-45-90 triangle at the quadrantal angles of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, and\, 2\pi$, if defined.
2. Students will extend the trigonometric values of angles in quadrant I into quadrants II – IV.
3. Students will determine the appropriate sign of the trigonometric value for angles in all four quadrants.
4. Students will compute the inverse trigonometric functions associated with the various standard points in the unit circle

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will compute, by hand, the values of the trigonometric functions at various standard points.
2. Students will compute, by hand, the values of the inverse trigonometric functions at various standard points.

Trigonometry Standard #9 Model Assessment Items

Computational and Procedural Skills

1. Complete the following table, showing the ordered pair or triangle used to find each answer.

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
0°						
π/2						
270°						

2. Complete the following table, showing the ordered pair or triangle used to find each answer.

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
π/6						
45°						
π/3						

3. Complete the following table in degrees, showing the ordered pair or triangle used to find each answer. Leave the box blank if there is no angle with the given answer.

x	$Sin^{-1}x$	$Cos^{-1}x$	$Tan^{-1}x$	$Cot^{-1}x$	$Sec^{-1}x$	$Csc^{-1}x$
0						
1						
-1						

4. Complete the following table in radians, showing the ordered pair or triangle used to find each answer. Leave the box blank if there is no angle with the given answer.

x	$Sin^{-1}x$	$Cos^{-1}x$	$Tan^{-1}x$	$Cot^{-1}x$	$Sec^{-1}x$	$Csc^{-1}x$
0						
1						
-1						

5. Find the exact value of each expression in radians

A. $Sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ B. $Cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ C. $Tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$ D. $Sin^{-1}\left(-\frac{1}{2}\right)$

Conceptual Understanding

1. Complete the following table, showing the ordered pair or triangle used to find each answer.

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
135°						
210°						
300°						

2. Complete the following table, showing the ordered pair or triangle used to find each answer.

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
$\frac{\pi}{6}$						
$-\frac{5\pi}{6}$						
$\frac{7\pi}{4}$						

3. Give a reference angle for the following angles:

   A. 150° B. $-\frac{3\pi}{4}$ C. 210° D. $\frac{7\pi}{6}$

4. Identify the angles where sec θ is undefined. Explain your reasoning.

Problem Solving/Application

None

Trigonometry Standard #10

Standard Set 10.0
Students demonstrate an understanding of the addition formulas for sines and cosines and their proofs and can use those formulas to prove and/or simplify other trigonometric identities.

Deconstructed Standard
1. Students demonstrate an understanding of the addition formula for sines.
2. Students demonstrate an understanding of the addition formula for cosines.
3. Students demonstrate an understanding of the proof of the addition formula for sines.
4. Students demonstrate an understanding of the proof of the addition formula for cosines.
5. Students can use the addition formula for sines to prove other trigonometric identities.
6. Students can use the addition formula for cosines to prove other trigonometric identities.
7. Students can use the addition formula for sines to simplify other trigonometric identities.
8. Students can use the addition formula for cosines to simplify other trigonometric identities.

Prior Knowledge Necessary
Students should know how to:
- define the basic trigonometric functions.
- write a coordinate proof.
- apply the distance formula.
- apply the fundamental trigonometric identities.
- locate points on a unit circle.

New Knowledge
Students will need to learn to:
- use the addition formula for sines.
- use the addition formula for cosines.
- apply the addition formula for sines to simplify trigonometric identities.
- apply the addition formulas for cosines to simplify trigonometric identities.
- understand the proof of the addition formula for sines.
- understand the proof of the addition formula for cosines.

Categorization of Educational Outcomes
Competence Level: Comprehension
1. Students can explain the proof of the addition formula for sines.
2. Students can explain the proof of the addition formula for cosines.

Competence Level: Application

1. Students will compute the sine of angles using the addition formula.
2. Students will compute the cosine of angles using the addition formula.
3. Students will construct the proof of other trigonometric identities using the addition formula for sines.
4. Students will construct the proof of other trigonometric identities using the addition formula for cosines.
5. Students will utilize the addition formula for sines to simplify trigonometric identities.
6. Students will utilize the addition formula for cosines to simplify trigonometric identities.
7. Students will calculate sine and cosine angle measures using the addition formulas.

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will use the addition formula for sines to calculate the sines of angles using the sum and difference of special angles.
2. Students will use the addition formula for cosines to calculate the cosines of angles using the sum and difference of special angles.
3. Students will recognize the steps in the proof the addition formula for sines.
4. Students will recognize the steps in the proof the addition formula for cosines.
5. Students will prove other trigonometric identities using the addition formula for sines.
6. Students will prove other trigonometric identities using the addition formula for cosines.
7. Students will simplify other trigonometric identities using the addition formula for sines.
8. Students will simplify other trigonometric identities using the addition formula for cosines.

Trigonometry Standard #10 Model Assessment Items

Computational and Procedural Skills

1. Apply the addition formula for sine to find the exact numerical value of $\sin\left(120^\circ + 45^\circ\right)$.

2. Apply the addition formula for cosine to find the exact numerical value of $\cos\left(\frac{7\pi}{6} - \frac{\pi}{3}\right)$.

3. Find the exact value of $\cos\left(165^\circ\right)$.

4. Find the exact value of $\sin\left(\frac{5\pi}{6}\right)$.

5. Find the exact trigonometric function given that $\sin u = \frac{5}{13}$ and $\cos v = -\frac{3}{5}$.

   A. $\cos\left(u + v\right)$

   B. $\sin\left(v - u\right)$

Conceptual Understanding

1. For the following, insert the correct symbol in the box:

   A. $\sin\left(135^\circ \square 30^\circ\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$

   B. $\cos\left(\frac{\pi}{4} \square \frac{\pi}{3}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$

2. Prove: $\sin\left(a+b\right) + \sin\left(a-b\right) = 2\sin a \cos b$.

3. Simplify $\sin\left(\frac{3}{2}\pi + \theta\right)$.

Problem Solving/Application

None

Trigonometry Standard #11

Standard Set 11.0

Students demonstrate an understanding of half-angle and double-angle formulas for sines and cosines and can use those formulas to prove and/or simplify other trigonometric identities.

Deconstructed Standard

1. Students demonstrate an understanding of the half-angle formula for sines.
2. Students demonstrate an understanding of the half-angle formula for cosines.
3. Students demonstrate an understanding of the double-angle formula for sines.
4. Students demonstrate an understanding of the double-angle formula for cosines.
5. Students can use the half-angle formula for sines to prove other trigonometric identities.
6. Students can use the half-angle formula for cosines to prove other trigonometric identities.
7. Students can use the double-angle formula for sines to prove other trigonometric identities.
8. Students can use the double-angle formula for cosines to prove other trigonometric identities.
9. Students can use the half-angle formula for sines to simplify other trigonometric identities.
10. Students can use the half-angle formula for cosines to simplify other trigonometric identities.
11. Students can use the double-angle formula for sines to simplify other trigonometric identities.
12. Students can use the double-angle formula for cosines to simplify other trigonometric identities.

Prior Knowledge Necessary

Students should know how to:

- use the definitions of the trigonometric functions.
- apply the sum and difference formulas and the Pythagorean identities.
- evaluate the trigonometric functions for special angles.

New Knowledge

Students will need to learn to:

- apply the half-angle formula for sine to simplify and/or prove other trigonometric identities.
- apply the half-angle formula for cosine to simplify and/or prove other trigonometric identities.
- apply the double-angle formula for sine to simplify and/or prove other trigonometric identities.
- apply the double -angle formula for sine to simplify and/or prove other trigonometric identities.

Categorization of Educational Outcomes
Competence Level: Evaluating

1. Students will use the half-angle and double-angle formulas of sine and cosine to verify trigonometric identities.

Competence Level: Application

1. Students will evaluate trigonometric expressions using the half-angle formulas for sine and cosine.
2. Students will evaluate trigonometric expressions using the double-angle formulas for sine and cosine.
3. Students will simplify trigonometric expressions using the half-angle formulas for sine and cosine.
4. Students will simplify trigonometric expressions using the double -angle formulas for sine and cosine.

Necessary New Physical Skills
None

Assessable Result of the Standard

1. Students will use the half-angle formula for sine to simplify other trigonometric identities.
2. Students will use the half-angle formula for cosine to simplify other trigonometric identities.
3. Students will use the half-angle formula for sine to prove other trigonometric identities.
4. Students will use the half-angle formula for cosine to prove other trigonometric identities.
5. Students will use the double-angle formula for sine to simplify other trigonometric identities.
6. Students will use the double-angle formula for cosine to simplify other trigonometric identities.
7. Students will use the double-angle formula for sine to prove other trigonometric identities.
8. Students will use the double-angle formula for cosine to prove other trigonometric identities.
9. Students will use the double-angle and half-angle formulas to evaluate trigonometric functions for sine and cosine.

Trigonometry Standard #11 Model Assessment Items

Computational and Procedural Skills

1. For the following, insert the correct symbol in the box:

   A. $\sin\frac{u}{2} = \pm\sqrt{\frac{1 \boxed{} \cos u}{2}}$

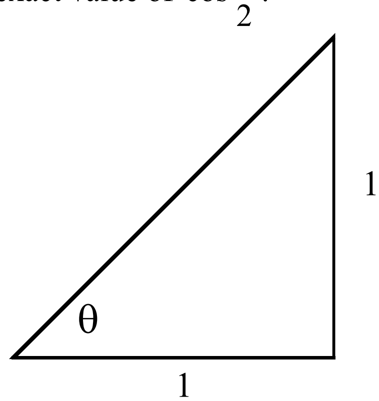
   B. $\cos\boxed{} = \cos^2 u - \sin^2 u$

2. Use a half-angle formula to fine the exact value of $\sin 105^\circ$.

3. Use a double-angle formula to rewrite: $6\sin x \cos x$.

Conceptual Understanding

1. Use the figure to find the exact value of $\cos\frac{\theta}{2}$.



2. Verify the identity: $\cos^2 2\theta - \sin^2 2\theta = \cos 4\theta$

Problem Solving/Application

None

Trigonometry Standard #12

Standard Set 12.0
Students use trigonometry to determine unknown sides or angles in right triangles.

Deconstructed Standard
1. Students use trigonometry to determine unknown sides in right triangles.
2. Students use trigonometry to determine unknown angles in right triangles.

Prior Knowledge Necessary
This standard is an extension of Geometry Standard #19: "Students use trigonometric functions to solve for an unknown length of a side of a right triangle, given an angle and a length of a side."

Students should know how to:
- use the Pythagorean Theorem to find the missing lengths of sides of right triangles.
- determine the third angle in a right triangle.
- use trigonometric functions to solve for an unknown length of a side of a right triangle, given an angle and the length of a side.

New Knowledge
Students will need to learn to:
- apply the inverse trigonometric functions to find a missing angle of a right triangle when given the lengths of two sides.
- compute inverse trigonometric functions on a scientific calculator.

Categorization of Educational Outcomes
Competence Level: Application
1. Students will compute the angle of a right triangle using inverse trigonometric functions on a scientific calculator.
2. Students will demonstrate their ability to correctly use the trigonometric functions to find the missing side or angle of a right triangle.
3. Students will solve right triangles for all missing sides and angles.

Necessary New Physical Skills
1. Use of a scientific calculator.

Assessable Result of the Standard
1. Students will be able to solve a right triangle given an angle and a side using trigonometric functions.
2. Students will be able to solve a right triangle given two sides using inverse trigonometric functions.
3. Students will use trigonometry functions to solve real life word problems involving right triangles.

Trigonometry Standard #12 Model Assessment Items

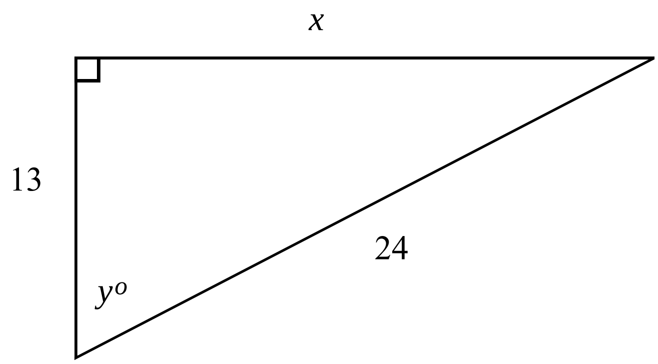
Computational and Procedural Skills

1. Given $\triangle ABC$ where $\angle C = 90^\circ$, $\angle A = 50^\circ$ and $AB = 4$, solve the triangle.

2. Given $\triangle ABC$ where $AB = 4$, $BC = 12$, and $\angle C$ is a right angle, solve the triangle.

Conceptual Understanding

1. Find *x* and *y*.



Problem Solving/Application

1. A television camera at ground level is filming the lift-off of a space shuttle at a point 750 meters from the launch pad. When the shuttle is 1200 meters in the air, what is the angle of elevation to the shuttle?

Trigonometry Standard #13

Standard Set 13.0
Students know the law of sines and the law of cosines and apply those laws to solve problems.

Deconstruction of Standard
1. Students know the law of sines.
2. Students know the law of cosines.
3. Students apply the law of sines to solve problems.
4. Students apply the law of cosines to solve problems.

Prior Knowledge Necessary
Students should know how to:
- classify triangles by angle size.
- solve ratio problems.
- find sine of angles.
- find cosine of angles.
- find the measure of the third angle of a triangle given the measure of the other two angles.
- identify the relationship between angle and side (largest side opposite largest angle).
- determine the reference angle given its sine or cosine.
- interpret the sign of an angle's cosine to determine if the angle is obtuse.
- determine the measure of a third angle given the measure of the other two angles.
- interpret that existence of an obtuse angle in a triangle to mean that the other two angles in the triangle must be acute.
- find square roots of real numbers

New Knowledge
Students will need to learn to:
- determine whether it is appropriate to use the Law of Sines or the Law of Cosines.
- solve a triangle given two angles and any side (AAS or ASA).
- determine if none, one or two triangles exist when given two sides and one opposite angle.
- solve a triangle given two sides and an angle opposite one of them (SSA).
- solve a triangle given three sides (SSS).
- solve a triangle given two sides and their included angle (SAS).
- solve real world problems using the Law of Sines and/or the Law of Cosines.

Categorization of Educational Outcomes
Competence Level: Application
1. Students will use the Law of Sines to calculate the measure of the remaining sides and angle of a triangle given two angles and any side.

2. Students will use the Law of Sines to calculate the measure of the remaining side and angles of a triangle given two sides and an angle opposite one of them.
3. Students will use either Law of Sines or the Law of Cosines to calculate the measure of all angles of a triangle given the measure of the three sides of the triangle.
4. Students will use the Law of Cosines and the Law of Sines to calculate the angles and the remaining side of a triangle given the measure of two sides and their included angle.
5. Students will demonstrate their ability to determine whether to use the Law of Sines or the Law of Cosines depending on what information is given.
6. Students will determine if zero, one, or two triangles can be formed when given two sides and an angle (SSA).
7. Students will utilize the Law of Sines and/or the Law of Cosines to solve real world problems.

Necessary New Physical Skills

None

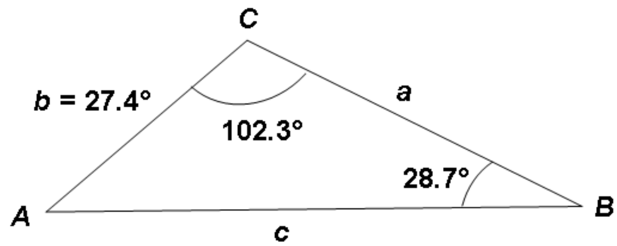
Assessable Result of the Standard

1. Students will produce the measure of the remaining side of any triangle given two angles and any side.
2. Students will produce the measure of the third side of any triangle given two sides and an opposite angle.
3. Students will produce the measure of all angles of any triangle given the measure of the three sides of the triangle.
4. Students will produce the measure of the remaining side of any triangle given the measure of two sides and their included angle.
5. Students will correctly state and verify if zero, one, or two triangles exist when given two sides and an angle (SSA).
6. Students will find solutions to real world problems using the Law of Sines and/or the Law of Cosines.

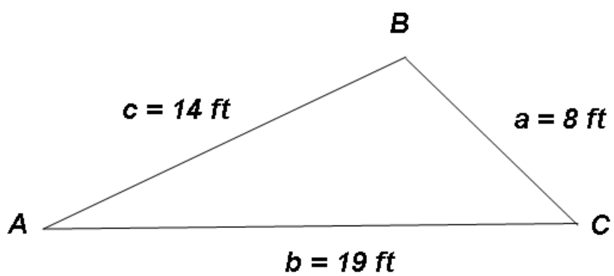
Trigonometry Standard #13 Model Assessment Items

Computational and Procedural Skill

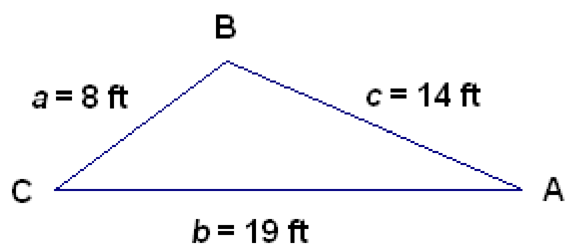
1. For the triangle in the figure below, calculate the measure of the remaining angle and sides.



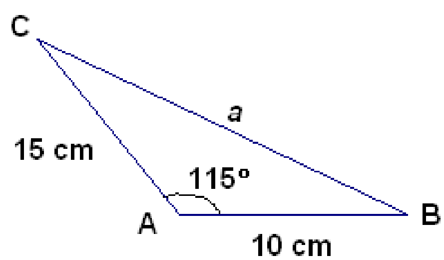
2. Calculate the measure of all the angles of the triangle shown below.



3. Calculate the three angles of the triangle shown below.



4. Solve the triangle shown below.



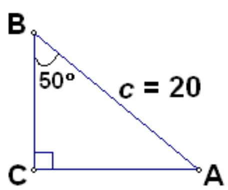
Conceptual Understanding

1. Show that there is no triangle for which $a = 15,\ b = 25,$ and $A = 85°$.

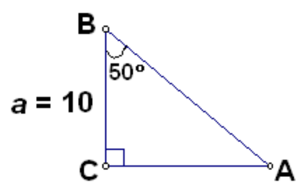
2. Solve the triangle for which $a = 12,\ b = 31,$ and $A = 20.5°$. Is the solution to this triangle unique? Explain.

3. You have learned how to solve an oblique triangle using the Law of Sines. Can the Law of Sines also be used to solve a right triangle? If so, write a short paragraph explaining how to use the Law of Sines to solve each triangle shown below. Is there an easier way to solve these triangles?

a. (AAS)



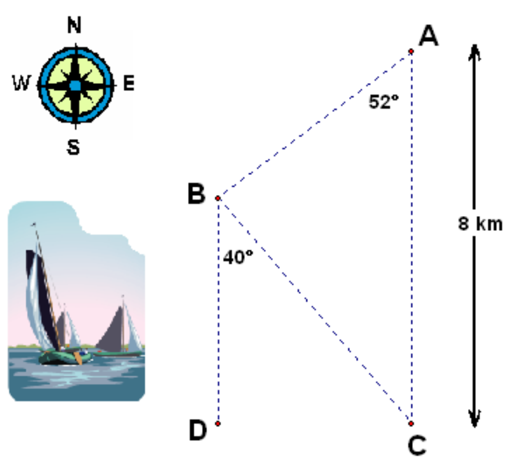
b. (ASA)



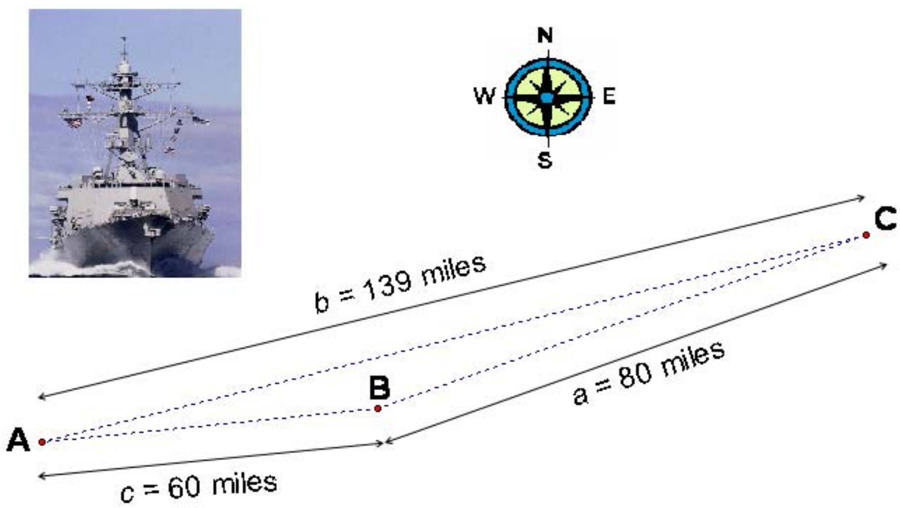
Problem Solving and Application

1. A vertical pole sits between two points that are 60 feet apart. Guy wires to the top of that pole are staked at the two points. The guy wires are 40 feet and 35 feet long. How tall is the pole?

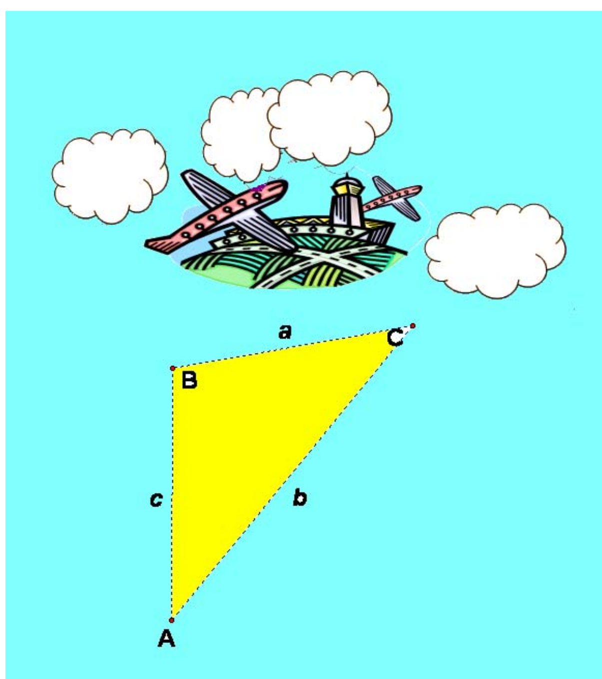
2. The course for a boat race starts at point A and proceeds in the direction S 52° W to point B, then in the direction S 40° E to point C, and finally back to A, as shown in the figure below. Point C lies 8 km directly south of point A. Approximate the total distance of the race course.



3. A ship travels 60 miles due east, then adjusts its course northward, as shown in the figure below. After traveling 80 miles in that direction, the ship is 139 miles from its point of departure. Describe the bearing from point B to point C.



4. To determine the distance between two aircraft, a tracking station continuously determines the distance to each aircraft and the angle A between them as shown below. Determine the distance *a* between the planes when $A = 42^\circ$, $b = 35$ miles, and $c = 20$ miles.



Trigonometry Standard #14

Standard Set 14.0
Students determine the area of a triangle, given one angle and the two adjacent sides.

Deconstructed Standard
1. Students determine the area of a triangle, given one angle and the two adjacent sides.

Prior Knowledge Necessary
Students should know how to:

- find the area of a triangle using $A = \frac{1}{2}bh$.
- determine which side of a triangle is longest given the measure of the largest angle.
- find a reference angle for $\angle\theta$ using $180° - \angle\theta$.
- apply the difference formula to a reference angle in an obtuse triangle to determine the height of the obtuse triangle.
- find sine and cosine of the angles in a triangle using ratios of lengths of appropriate sides to hypotenuse of the triangle.
- determine if the value of a ratio is within the acceptable ranges for sine or cosine.
- determine that in order for a triangle to exist, the sine and cosine values for all angles of the triangle must lie between negative one and positive one.
- solve a triangle using the Law of Sines.
- solve a triangle using the Law of Cosines.

New Knowledge
Students will need to learn to:
- determine the area of a triangle when they are given the measure of one angle and two adjacent sides.
- derive the area formula of a right triangle from the area formula of an oblique triangle.

Categorization of Educational Outcomes
Competence Level: Application
1. Students will calculate the area of a triangle given the measure of one angle and two adjacent sides.
2. Students will demonstrate their ability to use an appropriate strategy to calculate the area of a triangle given one angle and two adjacent sides.

Necessary New Physical Skills
None

Assessable Result of the Standard

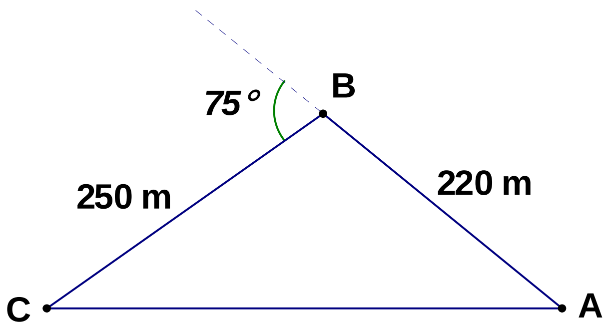
1. Students will produce the area of an acute triangle given the measure of one angle and two adjacent sides.
2. Students will produce the area of an oblique triangle given the measure of one angle and two adjacent sides.
3. Students will produce the area of a right triangle given the measure of one angle and two adjacent sides.
4. Students will calculate area of triangles to find solutions to real world problems.

Trigonometry Standard #14 Model Assessment Items

Computational and Procedural Skills

1. Find the area of a triangle having two sides of lengths 70 meters and 32 meters and an included angel of 102°.

2. Find the area of the following triangle:

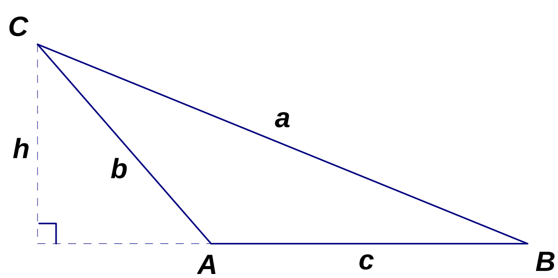


Conceptual Understanding

1. Suppose in $\triangle ABC$ and $\triangle A'B'C'$, the sides of $\overline{AB}$ and $\overline{A'B'}$ are congruent, as are $\overline{AC}$ and $\overline{A'C'}$, but $\angle A$ is larger than $\angle A'$. Which triangle, $\triangle ABC$ or $\triangle A'B'C'$, has a larger area? Prove that your answer is correct.

2. The area of the following triangle can be found using the formulas:

$$Area = \frac{1}{2}(base)(height) = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$$



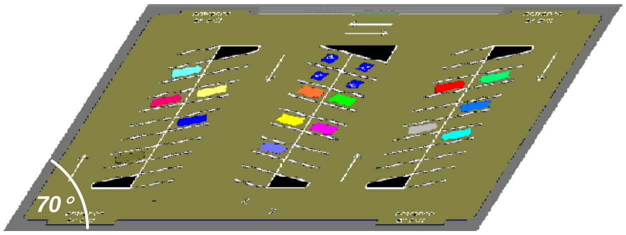
Show how these formulas can be derived using the Law of Sines.

3. The following formula is called Heron's Area Formula:

   Given any triangle with sides of lengths *a*, *b*, and *c*, the area of the triangle is $Area = \sqrt{s(s-a)(s-b)(s-c)}$. Show how this area formula can be derived using the Law of Cosines.

Problem Solving/Application

1. A parking lot in Jaymesville Shopping Center has the shape of a parallelogram as shown below. The lengths of two adjacent sides are 80 meters and 110 meters. The angle between the two sides is 70°. What is the area of the parking lot?



2. A study (Dorege and Hall, 2000) examined factors that affect maximum production from corn crops. Some of the factors considered were temperature, speed of seed disbursement, depth of planting and spacing between seeds. Consistent spacing of seeds was found to be an important factor that yields profitable increases in more than 80% of the crops that were studied. The recommended spacing between seeds within a row was 5.12 inches and the recommended spacing between rows was 30 inches. A farmer has a triangular field with sides 120 yards, 170 yards, and 220 yards. He plans to fertilize the **entire** field at a cost of $1.30 per square yard. How much will it cost the farmer to fertilize the triangular field? What are some challenges the farmer will face in planting corn on his triangular field?

3. Farmer Ethan wants to sell a parcel of land on his farm. He directs a surveyor to mark off a parcel of land using the following directions. Place a post into the ground at a certain spot as a starting point. From the post, walk due east for 310 feet, then proceed S 40° E for another 140 feet. Turn direction again S 60° W and continues walking for 420 feet. Finally, walk back to the post in a straight line. Find the area enclosed by the surveyor's path.

Trigonometry Standard # 15

Standard Set 15.0
Students are familiar with polar coordinates, in particular, they can determine polar coordinates of a point in rectangular coordinates and vice versa.

Deconstructed Standard
1. Students are familiar with polar coordinates.
2. Students can determine polar coordinates of a point in rectangular form.
3. Students can determine rectangular coordinates of a point in polar form.

Prior Knowledge Necessary
Students should know how to:
- graph ordered pairs in the rectangular coordinate system.
- convert angle measurements from radian to degree and vice-versa.
- produce multiple representations of an angle, θ, using the format $(\theta \pm 2n)$ or $(\theta \pm 2n+1)$.
- use the Pythagorean Theorem to find a missing side of a triangle.
- use trigonometry to determine unknown sides or angles in right triangles.
- use inverse trigonometric functions to determine lengths of sides or angles in right triangles.
- compute the values of the trigonometric functions and the inverse trigonometric functions at various standard points.
- interpret the slope of a line as the tangent of the angle that the line makes with the *x*-axis.
- sketch a right triangle from a point P with rectangular coordinates (x, y), by sketching a segment from P to the origin, a perpendicular segment from P to the *x*-axis and a segment from the origin along the *x*-axis to the perpendicular segment.
- find the angle of a triangle when given the sine, cosine, or tangent of the angle.

New Knowledge
Students will need to learn to:
- interpret the pole as coinciding with the origin.
- interpret the polar axis as coinciding with the positive *x*-axis.
- interpret θ as a directed angle extended counterclockwise from the polar axis to a line or line segment.
- interpret polar coordinates as having the format (r, θ).
- interpret *r* as a directed distance from the pole or origin such that $(r,\theta) = (r, \theta \pm 2n\pi)$.
- interpret $-r$ as a directed distance from the pole or origin such that $(r,\theta) = (-r, \theta \pm (2n+1)\pi)$.
- plot points on a polar coordinate system as coordinate pairs, (r, θ).

- interpret a given coordinate pair as the vertex of an acute angle of the right triangle which is formed by a segment extending from the origin to the vertex, a perpendicular segment from the vertex to the *x*-axis, and a segment along the *x*-axis from the origin to the perpendicular line.
- establish trigonometric, numeric and algebraic relationships between polar and rectangular coordinates.
- determine polar coordinates of a point given in rectangular coordinates.
- determine rectangular coordinates of a point given in polar coordinates.

Categorization of Educational Outcomes

Competence Level: Application

1. Students will determine polar coordinates of a point given in rectangular coordinates.
2. Students will determine rectangular coordinates of a point given in polar coordinates.

Necessary New Physical Skills

None

Assessable Results of the Standard

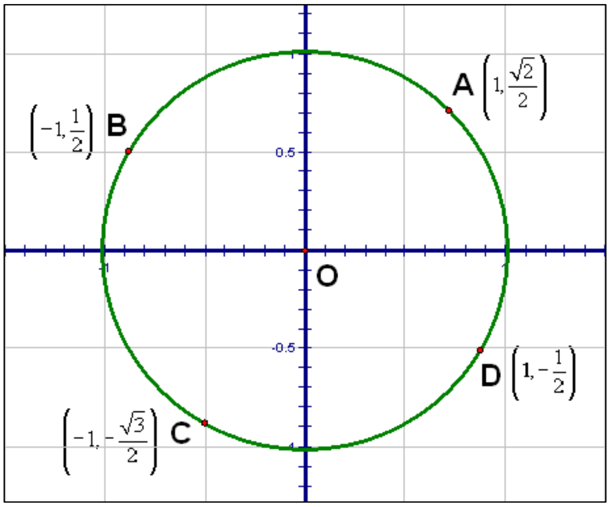
1. Students will produce the polar coordinates of a point given in rectangular coordinates.
2. Students will produce the rectangular coordinates of a point given in polar coordinates.
3. Students will convert rectangular coordinates to polar coordinates to find solutions to real world problems.
4. Students will convert polar coordinates to rectangular coordinates to find solutions to real world problems.

Trigonometry Standard # 15 Model Assessment Items

Computational and Procedural Skills

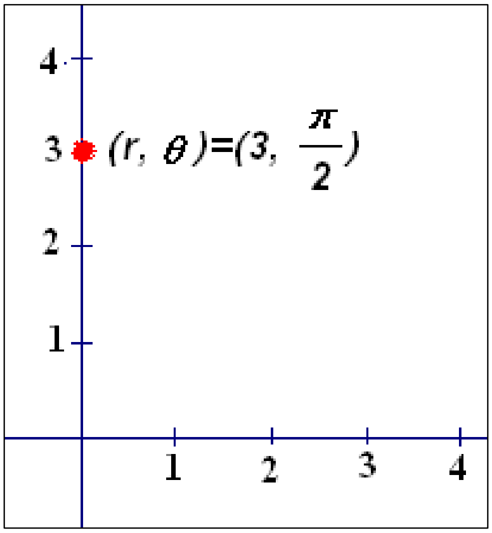
1. Determine the rectangular coordinates of the point having polar coordinates $\left(2, \frac{\pi}{3}\right)$.

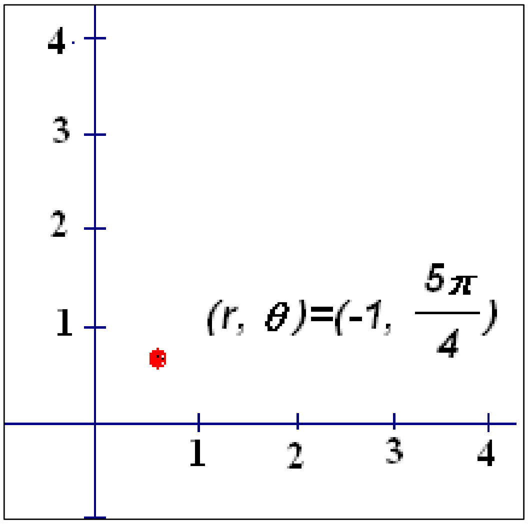
2. Determine the polar coordinates of each of the points A, B, C and D shown in the sketch below:



3. Use a graphing utility to find one set of polar coordinates of the point whose rectangular coordinates are (3, -2).

4. A point in polar coordinates is given in each grid below. Convert the points to rectangular coordinates





Conceptual Understanding

1. Determine if the following statement is TRUE or FALSE and justify your answer. If $\theta_1 = \theta_2 + 2\pi n$ for some integer n, then (r, θ_1) and (r, θ_2) represent the same point on the polar coordinate system.

2. If r_1 and r_2 are not 0, and if (r_1, θ) and (r_2, θ) and $(r_2, \theta + \pi)$ represent the same point in the plane, then $r_1 = -r_2$. Justify your answer.

3. TRUE or FALSE? Every point in the plane has exactly two polar coordinates. Justify your answer.

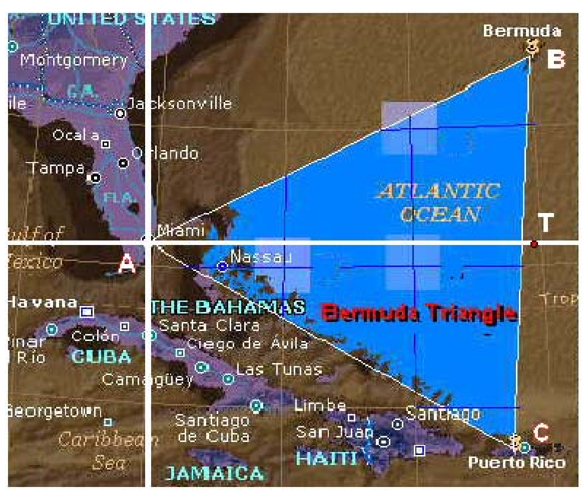
4. If $r \neq 0$, which of the following polar coordinate pairs represent the same point as the point with polar coordinates (r, θ)?
   A. $(-r, \theta)$ B. $(-r, \theta + 2\pi)$ C. $(-r, \theta + 3\pi)$ D. $(r, \theta + \pi)$ E. $(r, \theta + 3\pi)$

5. Show that $\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$ is the distance between the points (r_1, θ_1) and (r_2, θ_2).

Problem Solving/Application

1. There is an area off the eastern coast of Florida known as the Bermuda Triangle which is illustrated in the figure below. According to many reports, ships and aircraft have mysteriously disappeared after venturing into the area of the Bermuda Triangle. The boundaries of the Bermuda Triangle are not universally agreed upon, however,

one version places point B with polar coordinates $(1000 \text{ km}, \frac{\pi}{6})$. A well-equipped fishing vessel sets off from Miami traveling due east into the Atlantic Ocean heading for Point T which is an excellent fishing point. It will take the vessel about 2 days to arrive at Point T where it will remain stationary for three days while the fishermen indulge in excellent fishing. The vessel will then head due north from Point T to Point B. How many total miles will the fishing vessel travel on the journey from Miami to Bermuda (assuming it survives the Bermuda Triangle and that it does not venture off course from due east to Point T and then due north to Point B)? How many travel hours will it take the vessel to go from Point T to Bermuda assuming the vessel cruises at 25 km/hour?



2. The location of planes is monitored by traffic control towers to assure safe distances and timings between landings. The (polar) coordinates of two planes approaching the Ryansville airport are registered as (4 mi, $12°$) and (2 mi, $72°$). How far apart are the airplanes at the time of these readings?

3. The location of two ships from Port O'Ryan Landing are (3 mi, $170°$) and (5 mi, $150°$). How far apart are the ships?

Trigonometry Standard #16

Standard Set 16.0

Students represent equations given in rectangular coordinates in terms of polar coordinates.

Deconstructed Standard

1. Students can represent an equation given in rectangular coordinate form in terms of polar coordinate form.

Prior Knowledge Necessary

Students should know how to:

- represent coordinates in rectangular form.
- represent coordinates in polar form.
- use the conversion equations to convert rectangular coordinates to polar coordinates.

New Knowledge

Students will need to learn to:

- to convert equations in rectangular form to polar form.

Categorization of Educational Outcomes

Competence Level: Comprehension

1. Students will convert equations in rectangular coordinate form to polar coordinate form.

Necessary New Physical Skills

None

Assessable Results of the Standard

1. Students will be able to produce the equation in polar coordinate form given an equation in rectangular coordinate form.

Trigonometry Standard #16 Model Assessment Items

Computational and Procedural Skills

1. Convert the rectangular equation $x = 8$ to polar form.

2. Convert the rectangular equation $y = 4x + 10$ to polar form.

3. Convert the rectangular equation $y = 2x^2$ to polar form.

4. Convert the rectangular equation $x^2 + y^2 = 16$ to polar form.

Conceptual Understanding

1. Use what you know about converting equations from rectangular to polar form, to convert the polar equation $r = 4$ to rectangular form.

2. Use what you know about converting equations from rectangular to polar form, to convert the polar equation $r^2 = \sin 2\theta$ to rectangular form.

Problem Solving

1. Two airplanes depart at the same time from the same destination and are on paths that have a possible mid-air collision point. Given the following polar equations that describe their travel paths:

   Airplane 1 could be modeled by $r = \sec\theta\left(4 - \tan\theta\right)$ &
   Airplane 2 by $r = \sec\theta\left(5 - \frac{1}{4}\tan\theta\right)$.

   Find the approximate time and height when and where these planes will meet so that the traffic controller could redirect the airplanes avoiding an accident. Use the conversion equations: $t = r\cos\theta$ and $h = r\sin\theta$, where t is the time of the airplane in hours and h is the height of the airplane in miles.

Trigonometry Standard #17

Standard Set 17.0

Students are familiar with complex numbers. They can represent a complex number in polar form and know how to multiply complex numbers in their polar form.

Deconstruction of Standard

1. Students are familiar with complex numbers.
2. Students can represent a complex number in polar form.
3. Students know how to multiply complex numbers in their polar form.

Prior Knowledge Necessary

Students should know how to:

- multiply complex numbers in rectangular form.
- convert rectangular coordinates to polar coordinates.
- use trigonometric identities.

New Knowledge

Students will need to learn to:

- convert complex numbers to polar form.
- multiply complex numbers in polar form.

Categorization of Educational Outcomes

Competence Level: Application

1. Students will represent complex numbers in polar form.
2. Students will multiply complex numbers in polar form.

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will represent complex numbers in polar form.
2. Students will multiply complex numbers in polar form.

Trigonometry Standard #17 Model Assessment Items

Computational and Procedural Skill

1. Convert the complex number $z = -4 - 4i$ into polar form.

2. Convert the complex number $z = \frac{1}{2} + \sqrt{\frac{3}{2}}\, i$ into polar form.

3. Find the product of the complex numbers in polar form and leave the answer in polar form: $z_1 = 5(\cos 60° + i\sin 60°)$ and $z_2 = 2(\cos 45° + i\sin 45°)$.

Conceptual Understanding

1. Show how the complex number $z = a + bi$ in rectangular form is represented by $z = r(\cos\theta_1 + i\sin\theta_2)$ in polar form. (Hint: use a graph of z in rectangular form).

2. Multiply the complex numbers: $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Verify using trigonometric identities that their product yields: $z_1 z_2 = r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$.

3. Show that the complex number in polar form: $z^c = r\left[\cos(-\theta) + i\sin(-\theta)\right]$ is the conjugate of $z = r\left[\cos\theta + i\sin\theta\right]$. (*Recall that if* $z = a + bi$ *and its conjugate is* $z^c = a + bi$ *then* $zz^c = a^2 + b^2$ *.).*

Problem Solving and Application

None

Trigonometry Standard #18

Standard Set 18.0

Students know DeMoivre's theorem and can give n^{th} roots of a complex number given in polar form.

Deconstructed Standard

1. Students know DeMoivre's theorem.
2. Students can give n^{th} roots of a complex number in polar form.

Prior Knowledge Necessary

Students should know how to:

- convert a complex number into polar (trigonometric) form.
- multiply complex numbers in polar form.
- equate radian measures on the unit circle.
- evaluate sine, cosine, and tangent for various angles on the unit circle.
- use the inverse of the tangent function to find an angle θ.
- take n^{th} roots of whole numbers.
- use the laws of exponents.
- recognize that complex roots exist in conjugate pairs.

New Knowledge

Students will need to learn to:

- identify and calculate powers of complex numbers using DeMoivre's theorem.
- define the n^{th} root of a complex number.
- use the formula for finding the n^{th} roots of a complex number
  $\sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n}+\sin\frac{\theta+2\pi k}{n}\right)$ where $k=0,1,2,...,n-1$.

Categorization of Educational Outcomes

Competence Level: Knowledge

1. Students recall DeMoivre's theorem.

Competence Level: Application

1. Students will apply the formula for n^{th} roots to a complex number to calculate the n^{th} root of a complex number in polar form.
2. Students will apply DeMoivre's theorem to calculate the powers of a complex number.

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will know DeMoivre's theorem.
2. Students will calculate powers of a complex number given in polar form.
3. Students will produce the n^{th} roots of a complex number given in polar form.

Trigonometry Standard #18 Model Assessment Items

Computational and Procedural Skills

1. Use DeMoivre's theorem to find $\left(-1+\sqrt{3}i\right)^{12}$.

2. Find all the 4^{th} roots of $16\left(\cos\frac{4\pi}{3}+i\sin\frac{5\pi}{6}\right)$.

3. Find the 5^{th} roots of $32\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$.

Conceptual Understanding

1. Determine whether the statement is true or false. Justify your answer.

   $\frac{1}{2}\left(1-\sqrt{3}i\right)$ is a 9^{th} root of -1.

Problem Solving and Application

None

Trigonometry Standard #19

Standard Set 19.0
Students are adept at using trigonometry in a variety of applications and word problems.

Deconstructed Standard
1. Students use trigonometry to solve application problems.
2. Students use trigonometry to solve word problems.

Prior Knowledge Necessary
Students should know how to:
- find missing sides in a right triangle using the Pythagorean Theorem.
- find missing sides in a right triangle using trigonometry.
- find missing angles in a right triangle using trigonometry.
- find missing sides in a triangle using Law of Sines.
- find missing angles in a triangle using Law of Sines.
- find missing sides in a triangle using Law of Cosines.
- find missing angles in a triangle using Law of Cosines.
- graph the sine, cosine, and tangent functions.
- measure angles in both degrees and radians.

New Knowledge
Students will need to learn to:
- analyze the trigonometric relationships in a given situation/context..
- relate and apply trigonometric knowledge in given situations.

Categorization of Educational Outcomes
Competence Level: Application
1. Students will use previously learned methods of finding missing sides of triangles in new and concrete situations.
2. Students will use previously learned methods of finding missing angles of triangles in new and concrete situations.
3. Students will utilize trigonometric graphs to solve problems.

Necessary New Physical Skill
None

Assessable Result of the Standard
1. Students will produce solutions to new problems that utilize trigonometry.

Trigonometry Standard #19 Model Assessment Items

Computational and Procedural Skills

None

Conceptual Understanding

None

Problem Solving/Application

1. Two towers face each other separated by a distance *d* = 30 m. A man standing on the top of the first tower looks down at the base of the second tower with an angle of depression of 40^{o} and down at the top of the second tower with an angle of depression of 15^{o}. What is the height (in meters) of the second tower?

2. The captain of a ship is planning to sail to a port that is 600 miles away and $20°$ due east of north. The captain first sails the ship 100 miles due north to a refueling station. He then turns the ship and sails to the port.
   A. How many miles is it from the refueling station to the port?

   B. In what direction (what angle) should the captain turn the ship after refueling to sail in a direct line to the port?

3. In a certain city, the number of daylight hours $H(d)$ on a particular day *d* of the year of the year is given by $H(d) = A\sin\left[\frac{2\pi}{365}d + c\right] + K$, where *d* = 0 corresponds to January 1^{st}. The maximum number of daylight hours occurs on June 21^{st} (or day 171) and is equal to 15 hours. The minimum number of daylight hours is equal to 11 hours.
   A. Determine values for the constants *A, c,* and *K*.

   B. Sketch a graph of this function.

   C. On what day will the minimum number of daylight hours occur? Explain how you know.

Math Analysis Standard #1

Standard Set 1.0

Students are familiar with, and can apply, polar coordinates and vectors in the plane. In particular, they can translate between polar and rectangular coordinates and can interpret polar coordinates and vectors graphically.

Deconstructed Standard

1. Students will be able to translate between polar and rectangular coordinates.
2. Students will be able to interpret polar coordinates graphically.
3. Students will be able to interpret vectors graphically.

Prior Knowledge Necessary

Students should know how to:

- find missing sides in a right triangle using the Pythagorean Theorem.
- find missing sides in a right triangle using trigonometry.
- find missing angles in a right triangle using trigonometry.
- measure angles in both degrees and radians.
- express points in a plane in rectangular coordinates.

New Knowledge

Students will need to learn to:

- translate points in the coordinate plane into polar coordinates.
- express vectors in a plane in component form and unit vector form.
- translate points expressed in polar coordinates into rectangular coordinates.
- represent vectors given in component form or unit vector form graphically.
- represent vectors given graphically in component form, unit vector form.

Categorization of Educational Outcomes

Competence Level: Application

1. Students will convert between polar and rectangular coordinates.
2. Students will determine polar and rectangular coordinates for points in a plane.
3. Students will produce graphs of vectors.
4. Students will relate various representations of vectors to one another.

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will produce the graphical representation of a vector.
2. Students will produce the component and unit vector forms of a vector.
3. Students will produce the polar coordinates of a point given in rectangular coordinates.
4. Students will produce the rectangular coordinates of a point given in polar coordinates.

Math Analysis Standard #1 Model Assessment Items

Computational and Procedural Skills

1. Give the polar coordinates (in both degrees and radians) for the following points given in rectangular form. Decimals should be given accurate to two decimal places.
   A. $(1,0)$

   B. $(0,2)$

   C. $(-4,4\sqrt{3})$

   D. $(-2,-5)$

   E. $(3,-4)$

2. Give the rectangular coordinates for the following points given in polar form.
   A. $(2;\pi)$

   B. $(5;\frac{7\pi}{4})$

   C. $(6;120°)$

   D. $(1;80°)$

3. Graph the following vectors:
   A. $\vec{v}=\langle 3,5\rangle$ B. $\vec{w}=-2i+3j$ C. $\vec{r}=-5i$

4. Given points A(2, –5) and B(4, 1), express $\overline{AB}$ in both component and unit vector form.

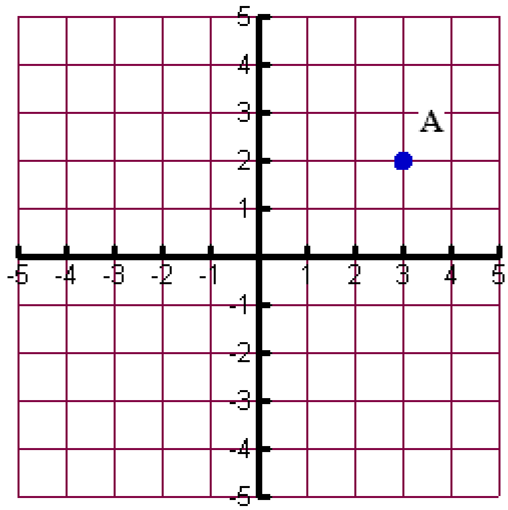
Conceptual Understanding

1. Consider point A shown in the grid to the right.

   A. Give the coordinates of point A in both rectangular and polar form.

   B. Give the component form of the vector $\overrightarrow{OA}$, where O is the origin.

   C. Is the component form of $\overrightarrow{OA}$ more closely related to the rectangular or polar coordinates for point A? Explain.



Problem Solving/Application

1. John and Kathy were asked to express the point (-1, 1) in polar form. John expressed it as $(\sqrt{2};135°)$ while Kathy expressed it as $(\sqrt{2};-45°)$. Explain clearly who is correct, using diagrams to support your answer.

Math Analysis Standard #2

Standard Set 2.0

Students are adept at the arithmetic of complex numbers. They can use the trigonometric form of complex numbers and understand that a function of a complex variable can be viewed as a function of two real variables. They know the proof of DeMoivre's theorem.

Deconstructed Standard

1. Students will be able to perform arithmetic operations (add, subtract, multiply, and divide) with complex numbers given in rectangular form. (This an extension of Algebra II Standard 6.0).
2. Students will be able to perform arithmetic operations (multiply, and divide) with complex numbers given in trigonometric (polar) form.
3. Students will be able to convert between rectangular and trigonometric (polar) form of a complex number.
4. Students will be able to apply DeMoivre's Theorem to compute powers of complex numbers.
5. Students will be able to prove DeMoivre's Theorem (*for positive integer powers only*) using mathematical induction.

Prior Knowledge Necessary

This standard is an extension of Algebra II Standard #6: "Students add, subtract, multiply, and divide complex numbers."

Students should know how to:

- add and subtract complex numbers in rectangular form.
- multiply complex numbers in rectangular form.
- find the conjugate of a complex number.
- divide complex numbers in rectangular form, including rationalizing the denominator.
- graph complex numbers in the complex plane.
- find missing sides in a right triangle using the Pythagorean Theorem.
- find missing sides in a right triangle using trigonometry.
- find missing angles in a right triangle using trigonometry.
- measure angles in both degrees and radians.
- express points in a plane in rectangular and polar coordinates.
- apply the Law of Cosines.
- apply trigonometric identities.
- prove statements using math induction (Math Analysis Standard 3.0).

New Knowledge

Students will need to learn to:

- determine the modulus and argument of a complex number.
- convert complex numbers given in rectangular form to trigonometric (polar) form.
- multiply and divide complex numbers in trigonometric (polar) form.

- use DeMoivre's Theorem to compute powers (including fractional powers) of complex numbers.
- prove DeMoivre's Theorem using mathematical induction (for positive integer powers only).

Categorization of Educational Outcomes

Competence Level: Comprehension

1. Students will convert between rectangular and trigonometric form of a complex number.

Competence Level: Application

1. Students will compute the sum, difference, product, and quotient of complex numbers in rectangular form.
2. Students will compute the product and quotient of complex numbers in trigonometric (polar) form.
3. Students will be able to apply DeMoivre's Theorem to compute powers of complex numbers.

Competence Level: Evaluation:

1. Students will be able to prove DeMoivre's Theorem (for positive integer powers) using mathematical induction.

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will utilize polar form to multiply and divide complex numbers.
2. Students will convert complex numbers from rectangular to trigonometric (polar) form, and vice versa.
3. Students will use De Moivre's Theorem to compute powers of complex numbers.
4. Students will prove DeMoivre's theorem (*for positive integer powers only*) by math induction.

Math Analysis Standard #2 Model Assessment Items

Computational and Procedural Skills

1. Let $z = \sqrt{3} + i, w = -2 - 2i\sqrt{3}$.
   A. Express *z* and *w* in polar form.

   B. Compute $z \cdot w$, expressing your answer in both polar and rectangular form.

   C. Compute $\frac{z}{w}$, expressing your answer in both polar and rectangular form.

   D. Compute z^6, expressing your answer in both polar and rectangular form.

   E. Compute the cube roots of *z*.

2. Prove, using math induction, DeMoivre's Theorem:
$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta), \quad n \in Z^+$$

Conceptual Understanding

1. Graph, in the complex plane, the points $z = 3 + 3i, \quad w = 2 + 2i\sqrt{3}$.
   A. Express *z* and *w* in polar form.

   B. Using the rectangular forms, find and graph the product $z \cdot w$.

   C. Using the rectangular form you found in part b, determine the polar form of $z \cdot w$.

   D. Explain in geometric terms how the graph of $z \cdot w$ is related to the graphs of *z* and *w*. Relate this to the polar forms of *z*, *w*, and $z \cdot w$.

2. Consider $z = 1 + i$.
   A. Compute z^n, for *n* = 2, 3, 4.

   B. Graph, on the complex plane, z^n, for *n* = 1, 2, 3, 4.

   C. Discuss the relationship between the magnitude and direction of the complex numbers graphed in (B).

Problem Solving/Application

1. Prove, using trigonometric identities, that $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$.

2. Find the six roots of 64 using:
   A. Algebraic methods. That is, use algebraic techniques to solve the equation $x^6 - 64 = 0$.

   B. Trigonometric methods. That is, use polar form and DeMoivre's Theorem to find the six roots of 64.

   C. Graph the six roots of 64 on the complex plane. Explain the geometric relationship between the 6 roots.

Math Analysis Standard # 3

Standard Set # 3.0
Students can give proofs of various formulas by using the technique of mathematical induction.

Deconstructed Standards
1. Students use mathematical induction to prove formulas.

Prior Knowledge Necessary
This standard is an extension of Algebra II Standard #21: "Students apply the method of mathematical induction to prove general statements about the positive integers."

Students should know how to:
- use mathematical induction.
- use sigma notation.
- identify and determine sequences and series.
- employ the Binomial Theorem.

New Knowledge
Students will need to learn to:
- extend the Algebra II standard of using mathematical induction to prove general statements to include proofs of formulas.

Categorization of Educational Outcomes
Competence Level: Evaluation
1. Students will utilize the steps of mathematical induction to prove the validity of formulas.

Necessary New Physical Skills
None

Assessable Results of the Standard
1. Students will produce proofs of formulas using the steps of mathematical induction.

Math Analysis Standard #3 Model Assessment Items

Computational and Procedural Skills

1. Write the statements P_1, P_k, and P_{k+1} for the following:

   A. P_n: $1 + 2 + . . . + n = \dfrac{n(n+1)}{2}$

Conceptual Understanding

1. A classmate is having trouble understanding mathematical induction proofs because he does not understand the inductive hypothesis. He wants to know why if we can assume it is true for k, we can't assume it is true for n and be done with it? After all, a variable is a variable! Write a response to clear up the confusion.

Problem Solving/Application

1. Prove the formula P_n: $1 + 2 + . . . + n = \dfrac{n(n+1)}{2}$

2. Use mathematical induction to prove, for $n \geq 3$, P_n: The sum of the interior angles of an n-sided polygon is $180°(n - 2)$.

3. Use mathematical induction to prove that the binomial theorem is valid for all positive integral values of n.

Math Analysis Standard #4

Standard Set 4.0
Students know the statement of, and can apply, the fundamental theorem of algebra.

Deconstructed Standard
1. Students can state the fundamental theorem of algebra.
2. Students can apply the fundamental theorem of algebra.

Prior Knowledge Necessary
Students should know how to:
- divide polynomials using long division.
- divide polynomials using synthetic division.
- factor quadratic expressions.
- solve quadratic equations by factoring.
- solve quadratic equations by applying the quadratic formula.
- find the zeros or roots of an equation.
- use the arithmetic of complex numbers.

New Knowledge
Students will need to learn to:
- use Descartes Rule of Signs.
- use the Rational Zero Theorem.
- use the Upper and Lower Bounds Theorem.
- use the fundamental theorem of algebra.
- use the Complete Factorization Theorem.
- use the Remainder Theorem.
- use the Factor Theorem.
- recognize complex zeros in conjugate pairs.

Categorization of Educational Outcomes
Competence Level: Application
1. Students will use methods they have learned to factor a polynomial into linear and irreducible quadratic factors with real coefficients.
2. Students will use methods they have learned to factor a polynomial completely into linear factors with complex coefficients.
3. Students will calculate all the zeros of a polynomial.

Necessary New Physical Skills
None

Assessable Result of the Standard
1. Students will factor a polynomial completely into linear factors with complex coefficients.
2. Students will produce all the zeros of a polynomial.

Math Analysis Standard #4 Model Assessment Items

Computational and Procedural Skills

1. Given the following polynomials $P(x)$, $Q(x)$, $f(x)$, and $g(x)$.

   $P(x) = 2x^4 + x^3 + 17x^2 + 9x - 9$ $\quad$ $Q(x) = 2x^4 - x^3 + 17x^2 - 9x - 9$

   $f(x) = 3x^4 - 5x^3 + 10x^2 - 20x - 8$ $\quad$ $g(x) = 3x^4 + 5x^3 + 10x^2 + 20x - 8$

   A. Use Descarte's Rule of Signs to determine the possible number of (i) positive zeros and (ii) negative zeros.

   B. Use the Rational Zero Test to list all possible rational zeros of the polynomials.

   C. Determine if the following is an upper bound, lower bound, or neither: (i) $x = 1$ and (ii) $x = -1$.

   D. Find all zeros of the polynomials.

   E. Write the polynomials as the product of factors that are all real.

   F. Write the polynomials in completely factored form.

Conceptual Understanding

1. Find a third degree polynomial with the zeros 6 and $2 - 3i$.

2. Match each complex number with its corresponding conjugate pair.

Complex Number		Corresponding Conjugate Pair	
a.	$7 - 9i$	i.	$-9i$
		ii.	9
b.	$9i$	iii.	$7 + 9i$
		iv.	$2 - 9i$
c.	9	v.	-9

Problem Solving/Application

None

Math Analysis Standard #5

Standard Set 5.0

5.0 Students are familiar with conic sections, both analytically and geometrically:

5.1 Students can take a quadratic equation in two variables; put it in standard form by completing the square and using rotations and translations, if necessary; determine what type of conic section the equation represents; and determine its geometric components (foci, asymptotes, and so forth).

5.2 Students can take a geometric description of a conic section - for example, the locus of points whose sum of its distances from (1, 0) and (-1, 0) is 6 - and derive a quadratic equation representing it.

Deconstructed Standard

1. Students can rearrange a quadratic equation in two variables into standard form.
2. Students can put a quadratic equation in two variables into standard form using completing the square (i.e. $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$).
3. Students can put a quadratic equation in two variables into standard form using rotations. (NOTE: This component is currently not taught at the High School or Community College level).
4. Students can put a quadratic equation in two variables into standard form using translations.
5. Students can identify the type of conic section the equation represents.
6. Students can determine the foci of the quadratic equation in two variables, if it exists.
7. Students can determine the asymptotes of the quadratic equation in two variables, if they exist.
8. Students can determine the vertices of the quadratic equation in two variables, if they exist.
9. Students can derive a quadratic equation in two variables given a description of a conic section.

Prior Knowledge Necessary

This standard is an extension of Algebra II, Standards #16 and #17. Standard #16 is "Students demonstrate and explain how the geometry of the graph of a conic section (e.g., asymptotes, foci, eccentricity) depends on the coefficients of the quadratic equation representing it." Standard #17 is "Given a quadratic equation of the form $Ax^2 + By^2 + Cx + Dy + E = 0$, students can use the method for completing the square to put the equation into standard form and can recognize whether the graph of the equation is a circle, ellipse, parabola, or hyperbola. Students can then graph the equation." Students should have the computational and conceptual knowledge outlined in those standards.

Students should know how to:
- arrange a quadratic equation into standard form ($Ax^2 + Bx + C = 0$).
- write the equation for a parabola in the form $y = a(x-h)^2 + k$.
- use the technique of completing the square.
- use the Pythagorean theorem.
- determine the equation of lines and graph them.
- factor polynomials.
- graph quadratic equations.

New Knowledge
This standard is an extension of Algebra II Standards #16 and 17. The following knowledge reflects the extension beyond Algebra II Standards #16 and 17.

Students will need to learn to:
- recognize the standard form of the equation for a circle as $(x-h)^2 + (y-k)^2 = r^2$ with center located at (h,k) with a radius of r.
- recognize the standard form of the equation for an ellipse, centered at the origin, as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (horizontal major axis) or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (vertical major axis).
- Recognize the standard form of the equation for a hyperbola, centered at the origin, as $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (horizontal transverse axis) or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ (vertical transverse axis).
- use completing the square to recognize the translation of a conic section relative to the origin.
- identify the type of conic section that a quadratic equation in two variables represents (i.e. circle, parabola, ellipse, or hyperbola).
- identify the center of a circle.
- identify the radius of a circle given the quadratic equation in two variables.
- identify the orientation of a parabola (vertical or horizontal).
- identify the vertex, the focus, the axis of symmetry, and the equation of the directrix of a given parabola.
- identify the orientation of the major axis of an ellipse from its equation.
- identify the center of an ellipse from its quadratic equation in two variables.
- identify the length of the major and minor axes of an ellipse.
- calculate and/or identify the location of the foci of an ellipse.
- calculate the eccentricity of an ellipse.
- identify the orientation of a hyperbola.
- identify the center of a hyperbola from its quadratic equation in two variables.
- identify the vertices of a hyperbola.
- identify the foci of a hyperbola.
- identify the asymptotes of a hyperbola.
- calculate the eccentricity of a hyperbola.

- identify if a conic section is rotated (not taught at the high school or community college level).
- identify the axis of rotation for a rotated conic section (not taught at the high school or community college level).
- determine the equation of a conic section given a geometric description of the conic.
- calculate the determinant of a quadratic equation in two variables to identify the type of conic the equation represents.

Categorization of Educational Outcomes

Competence Level: Knowledge

1. Students will identify the type of conic section represented by a quadratic equation in two variables.
2. Students will identify the various components of the conic section (i.e. vertices, foci, directrix, axis of symmetry, eccentricity, and asymptotes (where applicable)).
3. Students will identify the translation of a conic section if it is translated.

Competence Level: Analysis

1. Students will recognize when a conic section is rotated from the quadratic equation.
2. Students will distinguish between types of conic sections given the quadratic equation describing a conic section.

Competence Level: Application

1. Students will determine the quadratic equation for a conic section given a geometric description of the conic.
2. Students will compute the standard form of a quadratic equation describing a conic section using completing the square.

Necessary New Physical Skills

None

Assessable Result of the Standard

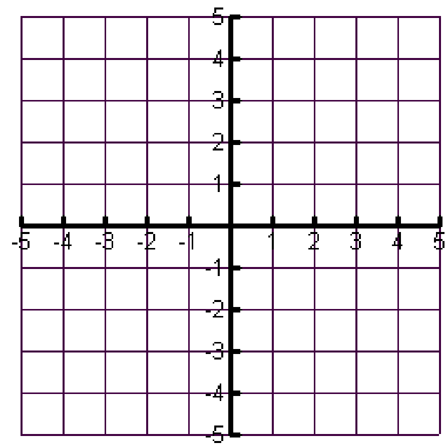
1. Students will identify the various geometric components of a conic section from the quadratic equation in two variables describing the conic.
2. Students will find the quadratic equation in two variables of a conic section given a geometric description of the conic.
3. Students will identify the translation of a conic section from its quadratic equation in two variables

Math Analysis Standard #5 Model Assessment Items

Computational and Procedural Skills

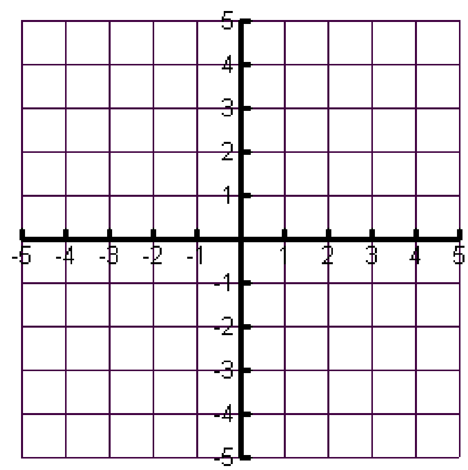
1. Graph each of the following conic sections and find the desired information.

A. $y^2 - \frac{x^2}{4} = 1$



Foci: ____________________________ Asymptotes: _____________________

B. $\frac{(x+2)^2}{4} + \frac{(y-1)^2}{9} = 1$



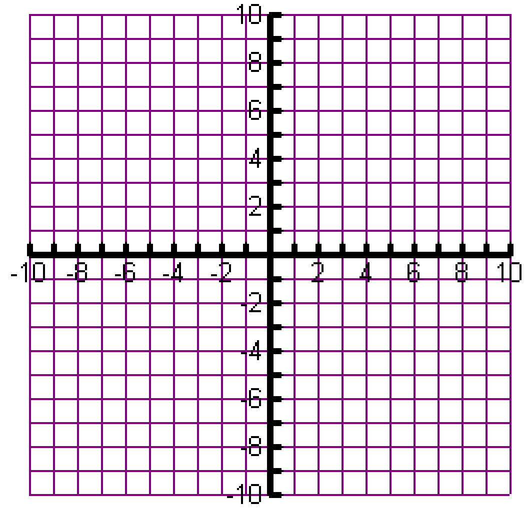
Center:____________________________ Eccentricity: _____________________

2. Find the focus and directrix of the parabola $y^2 = -8x$. Graph the parabola. Does it open up, down, left or right?

Focus ________________

Directrix ________________

Opens _________________

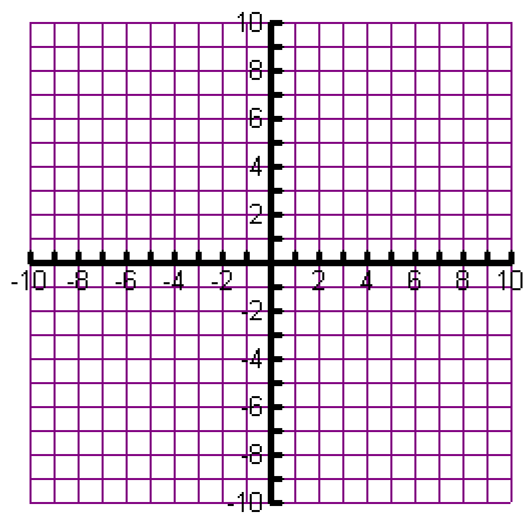


3. Complete the square to determine whether the equation represents an ellipse, a parabola, or a hyperbola. If the graph is an ellipse, find the center, foci, vertices, and length of the major and minor axes. If it is a parabola, find the vertex, focus, and directrix. If it is a hyperbola, find the center, foci, vertices, and asymptotes. Then sketch the graph of the equation.

$x^2 - 4x - 4y^2 - 24y - 48 = 0$

Type of conic:

____



4. Determine the type of curve represented by the equation. Find the foci and vertices (if any), and sketch the graph.
   A. $\frac{x^2}{12}+\frac{y^2}{144}=\frac{y}{12}$

   B. $x^2-y^2+144=0$

   C. $3x^2-6\left(x+y\right)=10$

   D. $2x^2-12x+y^2+6y+26=0$

   E. $9x^2+8y^2-15x=-8y-27$

5. Find an equation for the conic section with the given properties.
   A. The parabola with focus $F\left(0,1\right)$ and directrix $y=-1$.

   B. The ellipse with center $C\left(0,4\right)$, foci $F_1\left(0,1\right)$ and $F_2\left(0,8\right)$, and major axis of length 10.

   C. The hyperbola with vertices $V\left(0,\pm 2\right)$ and asymptotes $y=\pm\frac{1}{2}x$.

   D. The hyperbola with center $C\left(2,4\right)$, foci $F_1\left(2,1\right)$ and $F_2\left(2,7\right)$, and vertices $V_1\left(2,6\right)$ and $V_2\left(2,2\right)$.

6. Find an equation of a conic section with the given properties.
   A. The ellipse with foci $F_1\left(1,1\right)$ and $F_2\left(1,3\right)$ with one vertex on the *x*-axis.

   B. The ellipse with vertices $V_1\left(7,12\right)$ and $V_2\left(7,-8\right)$, and passing through the point $P\left(1,8\right)$.

7. Find an equation of a conic section with the given properties.
   A. The ellipse with foci $F_1\left(1,1\right)$ and $F_2\left(1,3\right)$ with one vertex on the *x*-axis.

   B. The ellipse with vertices $V_1\left(7,12\right)$ and $V_2\left(7,-8\right)$, and passing through the point $P\left(1,8\right)$.

8. An equation for the conic section is given. Use the discriminant to determine whether the graph of the equation is a parabola, an ellipse, or a hyperbola.

   A. $x^2 + 4xy + y^2 = 1$

   B. $5x^2 - 6xy + 5y^2 - 8x + 8y - 8 = 0$

   C. $9x^2 + 24xy + 16y^2 = 25$

<u>Conceptual Understanding</u>

1. Given the following general equation for an ellipse as: $\frac{x^2}{16+k^2} + \frac{y^2}{k^2} = 1$.

   A. Draw graphs of the following family of ellipses for $k = 1, 2, 4, \text{and } 8$.

   B. What do you notice regarding the foci of the ellipses in part A?

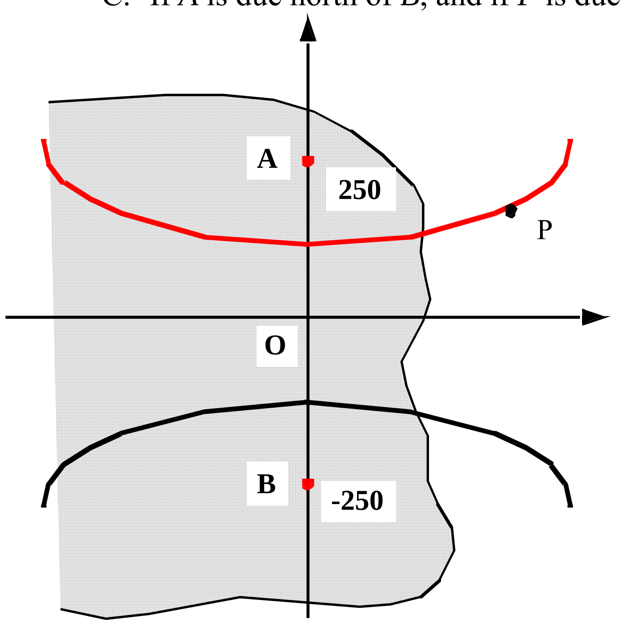
<u>Problem Solving/Application</u>

1. In the figure below (not drawn to scale), the LORAN (Long Range Navigation) stations at *A* and *B* are 500 miles apart, and the ship at *P* receives station's A signal 2640 microseconds (μs) before it receives the signal from *B*.

   A. Assuming that the radio signals travel at 980 ft/μs, find $d(P, A) - d(P, B)$.

   B. Find an equation for the branch of the hyperbola indicated in red in the figure (the upper branch near point *A* including point *P*). Use miles as the unit of distance.

   C. If *A* is due north of *B*, and if *P* is due east of *A*, how far is *P* from *A*?



2. The path of the Earth around the sun is an ellipse with the sun at one focus. The ellipse has a major axis of 186,000,000 miles and an eccentricity of 0.017.
    A. Find the distance between the Earth and the sun when the Earth is closest to the sun.

    B. Find the distance between the Earth and the sun when the Earth is farthest from the sun.

Math Analysis Standard #6

Standard Set 6.0
Students find the roots and poles of a rational function and can graph the function and locate its asymptotes.

Deconstructed standard
1. Students find roots of rational functions.
2. Students find vertical asymptotes of rational functions.
3. Students graph rational functions.
4. Students locate other asymptotes of rational functions.

Prior Knowledge Necessary
Students should know how to:
- find roots of functions.
- define the term "asymptote".
- employ the vocabulary used to describe functions (i.e.: continuity, increasing/decreasing, domain and range, end behavior, etc.).
- factor polynomials.
- divide polynomials.
- find the degree of polynomials.

New knowledge
Students will need to learn to:
- recognize a rational function.
- recognize where rational functions are undefined.
- identify the graph behavior near vertical asymptotes.
- find vertical asymptotes.
- find roots graphically and/or algebraically.
- determine end behavior of the rational function.
- find horizontal asymptotes.
- find any other asymptotes.
- determine types of discontinuity for rational functions.
- graph rational functions.

Categorization of Educational Outcomes
Competence Level: Application:
1. Students will calculate roots.
2. Students will determine the behavior of the graph near asymptotes.
3. Students will graph rational functions.

Necessary New Physical Skills
None

Assessable Results of the Standard
1. Students will find roots of the function.
2. Students will produce a graph of the function.
3. Students will determine the equation of the asymptotes when given a rational function.

Math Analysis Standard #6 Model Assessment Items

Computational and Procedural Skills

1. Find the *x*- and *y*-intercepts of the rational functions:

   A. $f(x) = \dfrac{x-1}{x+4}$ B. $r(x) = \dfrac{2}{x^2+3x-4}$ C. $h(t) = \dfrac{t^3+8}{t^2+4}$

2. Find all asymptotes:

   A. $r(t) = \dfrac{3}{t+2}$ B. $t(x) = \dfrac{(x-1)(x-2)}{(x-3)(x-4)}$ C. $f(x) = \dfrac{x^2-9}{x+1}$

3. Find the intercepts and asymptotes then sketch the graph.

   $f(x) = \dfrac{x^2+3x}{x^2-x-6}$

Conceptual Understanding

1. Give an example of a rational function that has vertical asymptote $x = 3$. Now give an example of a rational function that has vertical asymptote $x = 3$ and horizontal asymptote $y = 2$. Finally, give an example of a rational function with vertical asymptotes $x = 1$ and $x = -1$, horizontal asymptote $y = 0$, and *x*-intercept 4.

2. Explain how you can tell (without graphing it) that the function $r(x) = \dfrac{x^6+10}{x^4+8x^2+15}$ has no *x*-intercept and no vertical, horizontal, or slant asymptote. What is its end behavior?

Problem Solving/Application

1. As a train moves toward an observer, the pitch of its whistle sounds higher to the observer than it would if the train was at rest, because the crests of sound waves are compressed closer together. This phenomenon is called the *Doppler effect*. The observed pitch *P* is a function of the speed *v* of the train and is given by $P(v) = P_0\left(\dfrac{s_0}{s_0 - v}\right)$ where P_0 is the actual pitch of the whistle at the source and $s_0 = 332$ m/s is the speed of sound in air. Suppose that a train has a whistle pitched at $P_0 = 440$ Hz. Explain how the vertical asymptote of this function can be interpreted physically.

Math Analysis Standard #7

Standard Set 7.0
Students demonstrate an understanding of functions and equations defined parametrically and can graph them.

Deconstructed Standard
1. Students demonstrate understanding of functions defined parametrically.
2. Students demonstrate understanding of equations defined parametrically.
3. Students can graph functions defined parametrically.
4. Students can graph equations defined parametrically.

Prior Knowledge Necessary
This standard is an extension of Algebra II Standards #8 ("Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.") and 24 ("Students solve problems involving functional concepts such as composition, defining the inverse function, and performing arithmetic operations on functions.") and Precalculus Math Analysis Standard #5 ("Students are familiar with conic sections, both analytically and geometrically .") and Trigonometry Standards #9 ("Students compute, by hand, the values of the trigonometric functions and the inverse trigonometry functions at various standard points."). Students should have the computational and conceptual knowledge outlined in those standards.

Students should know how to:
- solve problems involving functional concepts, such as compositions.
- solve problems by performing arithmetic operations on functions.
- graph quadratic equations.
- identify geometric properties of conic sections from their quadratic equations.
- graph conic sections including translations.
- solve equations/functions involving trigonometric functions.
- simplify equations using trigonometric identities.
- evaluate trigonometric functions.

New Knowledge
Students will need to learn to:
- identify the parameter in equations defined parametrically.
- graph parametrically defined functions.
- identify the range of parameter values that need to be investigated in order to fully graph parametrically defined functions.
- recognize the effect varying the parameter has on the functions.

Categorization of Educational Outcomes
Competence Level: Comprehension
1. Students will identify the range of parameter values necessary to fully graph parametrically defined functions/equations.
2. Students will explain the behavior of the graphs of parametrically defined functions/equations.

Competence Level: Application
1. Students will compute the values of the parametrically defined function/equation.
2. Students will construct the graphs of parametrically defined functions/equations.

Necessary New Physical Skills
None

Assessable Result of the Standard
1. Students will tabulate values of parametrically defined functions/equations.
2. Students will graph parametrically defined functions/equations.
3. Students will analyze the behavior of parametrically defined functions/equations as the parameter varies.

Math Analysis Standard #7 Model Assessment Items

Computational and Procedural Skills

1. Graph the following parametrically defined equations:

   A. $x = 3\sin^3 t$, $y = 3\cos^3 t$

   B. $x = 3\sin 3t$, $y = 2\cos t$

   C. $x = 2t - 1$, $y = t^2 + 2$, $-4 \le t \le 4$

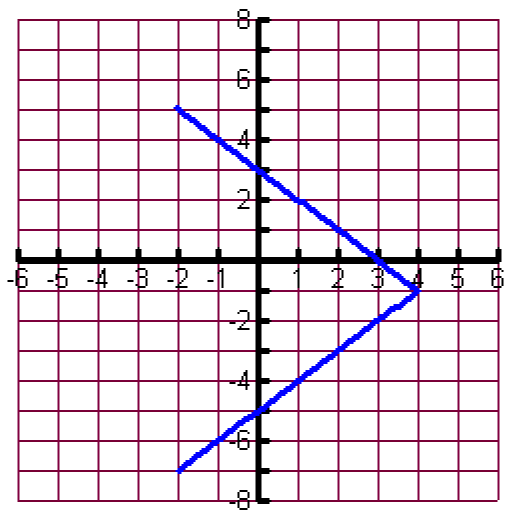
   D. $x = 7\sin t$, $y = 7\cos t$, $0 \le t \le 2\pi$

Conceptual Understanding

1. Given the following parametrically defined equations, find the values of *t* that produce the graph in the given quadrant.

   $x = 4 - |t| \qquad -6 \le t \le 6$

   $y = t - 1$



   A. Quadrant I

   B. Quadrant II

   C. Quadrant III

   D. Quadrant IV

2. Consider the curve described by $x = 3\cos t$, $y = 3\sin t$; $0 \le t \le 2\pi$. As t increases, the curve is traced in the counterclockwise direction. How can the equations be changed so that the curve is traced in the clockwise direction?

Problem Solving/Application

1. The motion of an object that is propelled upward can be described with parametric equations. It can be shown using more advanced mathematics, that neglecting air resistance, the following equations describe the path of a projectile propelled upward at an angle θ with the horizontal from a height *h*, in feet, at an initial speed v_0, in feet per second:

$$x = (v_0 \cos\theta)t, \qquad y = h + (v_0 \sin\theta)t - 16t^2.$$

We can use these equations to determine the location of the object at time *t*, in seconds.

A ball is thrown from a height of 6 ft above the ground with an initial speed of 80 ft/sec at an angle 30^o with the horizontal.

A. Find the parametric equations that give the position of the ball at time *t*, in seconds.

B. Find the height of the ball after 1 sec and 2 sec.

C. Determine how long the ball is in the air.

D. Determine the horizontal distance that the ball travels.

E. Find the maximum height of the ball.

2. Suppose that a rocket is fired at an angle of 5^o from the vertical, with an initial speed of 1000 ft/s.

A. Find the length of time the rocket is in the air.

B. Find the greatest height that it reaches.

C. Find the horizontal distance it has traveled when it hits the ground.

D. Graph the rockets' path.

3. The initial speed of a missile is 330 m/s.

A. At what angle should the rocket be fired so that it hits a target 10 km away? (You should find that there are two possible angles.) Graph the missile paths for both angles.

B. For which angle is the target hit sooner

Math Analysis Standard #8

Standard Set 8.0

Students are familiar with the notion of the limit of a sequence and the limit of a function as the independent variable approaches a number or infinity. They determine whether certain sequences converge or diverge.

Deconstructed Standard

1. Students will define the limit of a sequence or function (if it exists).
2. Students will determine the limit of a sequence.
3. Students will determine the limit of a function (if it exists) as the independent variable approaches a number.
4. Students will determine the limit of a function (if it exists) as the independent variable approaches infinity.
5. Students will determine if certain sequences converge or diverge.

Prior Knowledge Necessary

Students should know how to:

- construct a table of values by evaluating functions.
- simplify algebraic expressions by factoring, multiplying by the conjugate, multiplying by the least common denominator, etc.
- write an explicitly defined function for a given sequence of numbers.
- write a sequence of numbers when given an explicitly defined function.
- know the definitions of arithmetic and geometric sequences.

New Knowledge

Students will need to learn to:

- read and write limit notation.
- determine limits of sequences numerically; that is, estimate the limit of a sequence by examining a table of values.
- determine limits of sequences algebraically.
- determine limits of functions (if they exist)as the independent variable approaches a number or infinity numerically; that is, estimate the limit of a function by examining a table of values.
- determine limits of functions (if they exist) as the independent variable approaches a number or infinity graphically; that is, estimate the limit of a function by examining the graph of the function.
- determine limits of functions (if they exist) as the independent variable approaches a number or infinity algebraically.
- recognize when the limit of a function as the independent variable approaches a number or infinity does not exist.
- recognize when the limit of a sequence does not exist.
- define and determine convergence or divergence of a sequence.

Categorization of Educational Outcomes

Competence Level: Knowledge

1. Students will read and write limit notation.
2. Students will use notation to define sequences.

Competence Level: Application

1. Students will determine whether a sequence converges or diverges.
2. Students will determine the limit of a convergent sequence.
3. Students will determine the limit (if it exists) of a function.

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will determine whether a sequence converges or diverges.
2. Students will determine the limit of a convergent sequence.
3. Students will determine the limit (if it exists) of a function.

Math Analysis Standard #8 Model Assessment Items

Computational and Procedural Skills

1. For each sequence below, determine whether the sequence converges or diverges. If it converges, give the limiting value.

   A. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

   B. $2, 4, 6, 8, \ldots$

   C. $81, 27, 9, 3, \ldots$

   D. $1, -1, 1, -1, 1, -1, \ldots$

   E. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

   F. $8, -6, 4.5, -3.375, \ldots$

   G. $\{a_n\} = \left\{ \frac{n}{3-n} \right\}$

2. Given $f(x) = \frac{4}{x-2}$, find:

   A. $\lim_{x \to 0} f(x)$

   B. $\lim_{x \to 2} f(x)$

   C. $\lim_{x \to \infty} f(x)$

Conceptual Understanding

1. Give an example of a sequence satisfying the given condition:

   A. The sequence increases and converges.

   B. The sequence increases and does not converge.

   C. The sequence decreases and converges.

   D. The sequence decreases and does not converge.

2. Use a graph, a table, and an analytical approach to explain why $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$.

3. Is it possible for an arithmetic sequence to converge? Explain and give examples to support your conclusion.

<u>Problem Solving/Application</u>

1. The amount of interest earned on an investment can be computed using the formula $A = P(1 + \frac{r}{n})^{nt}$, where A = amount of money after t years, P = principal investment, r = interest rate, n = number of times per year the interest is compounded, and t = number of years.

   Suppose you invest $5000 for 10 years in an account that earns 4% interest. Investigate the effect an increase in n (the frequency of compounding) will have on your final amount.

n times per year	Amount of money (*A*)
1 (annually)	
2 (semi-annually)	
6 (bi-monthly)	
12 (monthly)	
52 (weekly)	
365 (daily)	

   From the table, does it appear there might be a limit to the amount of money you can accumulate? Explain.

Linear Algebra Standard #1

Standard Set 1.0
Students solve linear equations in any number of variables by using Gauss-Jordan elimination.

Deconstructed Standard
1. Students solve systems of linear equations.
2. Students apply the Gauss-Jordan elimination method.

Prior Knowledge Necessary
Students should know how to:
- solve a system of linear equations in two variables by graphing.
- solve a system of linear equations in two or three variables by substitution.
- reduce rectangular matrices to row-echelon form.
- write a system of linear equations in matrix form.
- perform basic matrix row operations.
- solve a system of linear equations in two or three variables by elimination.

New Knowledge
Students will need to learn to:
- identify the **reduced** row echelon form for a rectangular matrix.
- use matrix row operations to convert a matrix into **reduced** row-echelon form.
- determine the solution to the system of linear equations from the **reduced** row-echelon form.

Categorization of Educational Outcomes
Competence Level: Knowledge:
1. Students will be able to identify a matrix in reduced row-echelon form.

Competence Level: Application
1. Students will apply the Gauss Jordan method to solve systems of linear equations.

Necessary New Physical Skills
None

Assessable Result of the Standard
1. Students will produce matrices in reduced row-echelon form using the Gauss-Jordan method.
2. Students will produce the solution to a system of linear equations from the resulting matrix in reduced row-echelon form.

Linear Algebra Standard #1 Model Assessment Items

Computational and Procedural Skills

1. What is the reduced row-echelon form for a 2×3 matrix? 3×4 matrix?

2. The following matrix represents the final stage in solving a system of linear equations by the Gauss-Jordan elimination method. Identify the resulting solution.

   A. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

3. Given that the following matrices are the result of systems of linear equations, use the Gauss- Jordan elimination method to find the solution set.

   A. $\begin{bmatrix} 2 & 3 & 10 \\ 4 & -2 & 4 \end{bmatrix}$ B. $\begin{bmatrix} 2 & -2 & 3 & -10 \\ 1 & -4 & 5 & -21 \\ -2 & 3 & -2 & 11 \end{bmatrix}$

4. Write the following system of linear equations in augmented matrix form and solve using the Gauss-Jordan elimination method.

   A. $\begin{aligned} 2x+5y&=9 \\ x-3y&=-1 \end{aligned}$ B. $\begin{aligned} x-2y+3z&=11 \\ 2x+y-z&=3 \\ -x+3y+2z&=-2 \end{aligned}$

Conceptual Understanding

1. Given a system of linear equations, does the order in which they are input into the matrix affect the solution to the system?

Problem Solving/Application

1. The San Diego zoo sells adult and children's tickets. Mrs. Teegarden took her 3 grandchildren to the zoo and it cost her $106. Mr. and Mrs. Ross took their 2 grandchildren to the zoo and it cost them $116. How much is an adult ticket to the San Diego zoo? Solve using the Gauss-Jordan elimination method.

Linear Algebra Standard #2

Standard Set 2.0
Students interpret linear systems as coefficient matrices and the Gauss-Jordan method as row operations on the coefficient matrix.

Deconstructed Standard
1. Students interpret linear systems as coefficient matrices.
2. Students recognize the Gauss-Jordan elimination method as row operations.

Prior Knowledge Necessary
Students should know how to:
- write systems of linear equations as a matrix.
- solve system of linear equations using elimination.
- solve systems of linear equations using the Gauss-Jordan method.

New Knowledge
Students will need to learn to:
- compare the elimination method to the Gauss-Jordan method.
- recognize the relationships between the coefficients of the linear equations and the matrix entries.

Categorization of Educational Outcomes
Competence Level: Analysis
1. Students will recognize the relationship between the coefficients of the linear equations and the original matrix entries.

Competence Level: Synthesis
1. Students will compare the steps of the elimination method with those of the Gauss-Jordan method.

Necessary New Physical Skills
None

Assessable Result of the Standard
1. Students will produce a description of relationship between the coefficients of the linear equations and the original matrix entries.

Linear Algebra Standard #2 Model Assessment Items

Computational and Procedural Skills

1. Given the following system of linear equations, write the corresponding matrix. (Do not solve the system.)

   A. $\frac{1}{3}x - \frac{2}{9}y = 8$ and $\frac{1}{6}x - \frac{1}{3}y = \frac{2}{3}$

   B. $$\begin{aligned} 10x + 20y + 50z &= 4.7 \\ 35x + 0.5y + 25z &= 3.39 \\ 20x + 8y + 10z &= 2.34 \end{aligned}$$

2. Write the following matrices as a system of equations:

   A. $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -4 & 2 & 8 \\ 3 & 0 & 5 & -6 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Conceptual Understanding

1. Solve the following system of two equations using both elimination and the Gauss-Jordan method. How do the two methods compare?

   $x + 3y = 5$ and $2x - y = 3$

2. Solve the following system of three equations using both elimination and the Gauss-Jordan method. How do the two methods compare?

   $$\begin{aligned} x + y - z &= 3 \\ 2x - 3y + 5z &= -4 \\ x - y + 3z &= 9 \end{aligned}$$

Problem Solving/Application

None

Linear Algebra Standard #3

Standard Set 3.0
Students reduce rectangular matrices to row echelon form.

Deconstructed Standard
1. Students reduce rectangular matrices to row echelon form.

Prior Knowledge Necessary
Students should know how to:
- perform basic row operations on matrices.

New Knowledge
Students will need to learn to:
- use basic row operations to upper triangularize the matrix (row echelon form).
- recognize when a matrix is in row echelon form.

Categorization of Educational Outcomes
Competence Level: Knowledge
1. Students will be able to identify a matrix in row echelon form.

Competence Level: Application
1. Students will apply methods of row operations to convert a matrix to row echelon form.

Necessary New Physical Skills
None

Assessable Result of the Standard
1. Students will produce matrices in upper triangularized form.
2. Students will produce the solution to a system of linear equations using row operations.

Linear Algebra Standard #3 Model Assessment Items

Computational and Procedural Skills

1. Using basic row operations put the following matrices into row echelon form:

A. $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$

B. $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 1 & 2 & -6 & 0 \\ 3 & 1 & -2 & 3 \end{bmatrix}$

Conceptual Understanding

1. Suppose you obtain the following matrix while reducing it to row echelon form. What do you do? Why?

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 \end{bmatrix}$$

Problem Solving/Application

None

Linear Algebra Standard #4

Standard Set 4.0
Students perform addition on matrices and vectors.

Deconstructed Standard
1. Students perform addition on matrices.
2. Students perform addition on vectors.

Prior Knowledge Necessary
Students should know how to:
- use matrix notation.
- identify the dimensions of a matrix.

New Knowledge
Students will need to learn to:
- use capital letters to name matrices and use lower case letters and subscripts to identify the entries (ex. a_{23}).
- recognize that only matrices of the same dimension may be added together.
- recognize and demonstrate that addition of matrices is commutative.
- recognize that only vectors of the same dimension may be added together.
- recognize and demonstrate that addition of vectors is commutative.

Categorization of Educational Outcomes
Competence Level: Knowledge:
1. Students will be able to compute the sums of matrices.
2. Students will be able to compute the sums of vectors.

Necessary New Physical Skills
None

Assessable Result of the Standard
1. Students will produce the sum of two or more matrices.
2. Students will produce the sum of two or more vectors.

Linear Algebra Standard #4 Model Assessment Items

Computational and Procedural Skills

1. Add the following matrices:

   A. $\begin{bmatrix} 2 & -5 & 4 \\ 8 & -3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 3 & -8 \\ 1 & -6 & 2 \end{bmatrix}$

   B. $\begin{bmatrix} -3 & 5 & 1 \\ -7 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} + \begin{bmatrix} 4 & -3 & -8 \\ 6 & -5 & 1 \\ -2 & 7 & 9 \end{bmatrix}$

2. Add the following vectors:

   A. $V_1 = \langle 2, -1 \rangle$ and $V_2 = \langle -3, 2 \rangle$

   B. $V_1 = \langle 3, -2, 6 \rangle$ and $V_2 = \langle -1, 2, 5 \rangle$

Conceptual Understanding

1. Given $A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -2 & 5 & 1 \\ 3 & 2 & -4 \end{bmatrix}$

   A. Is $A + B = B + A$?

   B. Does $A + C$ exist? Why or why not?

2. Find the x and y values that make the following true.

$$\begin{bmatrix} 2x & 3 \\ -3 & -4 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ y & 2 \end{bmatrix} = \begin{bmatrix} 11 & 2 \\ 2 & -2 \end{bmatrix}$$

Problem Solving/Application

None

Linear Algebra Standard #5

Standard Set 5.0
Students perform matrix multiplication and multiply vectors by matrices and by scalars.

Deconstructed Standard
1. Students perform matrix multiplication.
2. Students multiply vectors by matrices.
3. Students multiply matrices by scalars.
4. Students multiply vectors by scalars.

Prior Knowledge Necessary
Students should know how to:
- ➢ determine the dimensions of a matrix.
- ➢ identify the position of each entries in the matrix (a_{11} , b_{23} , etc).
- ➢ recognize that matrices can be represented by upper case letters.
- ➢ determine the magnitude of a vector.

New Knowledge
Students will need to learn to:
- ➢ identify the necessary dimensions for two matrices to be multiplied.
- ➢ determine the dimension of the product matrix.
- ➢ multiply the corresponding entries of each matrix, add the appropriate products and place the solution into the corresponding position in the product matrix.
- ➢ multiply a vector by a scalar by multiplying each entries of the vector by the scalar.
- ➢ recognize that multiplying a vector by a positive scalar (not 1) affects the magnitude.
- ➢ recognize that multiplying a vector by a negative scalar affects the magnitude and direction of the vector.
- ➢ multiply a matrix by a scalar.
- ➢ write a vector in matrix form.
- ➢ multiply a vector by a matrix by using the matrix form of the vector and matrix multiplication.
- ➢ recognize that matrix multiplication is not always commutative.
- ➢ recognize the identity matrix for multiplication (I) for square matrices.
- ➢ verify that $AI = IA = A$.

Categorization of Educational Outcomes
Competence Level: Knowledge
1. Students will be able to determine the dimensions needed for matrix multiplication to exist as well as the dimensions of the product of two matrices.

Competence Level: Comprehension

1. Students will be able to explain how to change the direction and magnitude of a vector by multiplying by an appropriate scalar.

Competence Level: Application

1. Students will be able to use the multiplication algorithm to multiply matrices of compatible dimensions by scalars and by matrices.
2. Students will be able to use the multiplication algorithm to multiply vectors by scalars and by matrices.

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will produce a new vector with a different magnitude and/or direction from a given vector.
2. Students will product the product of a scalar and a matrix or two matrices with appropriate dimensions.
3. Students will product the product of a scalar and a vector or a matrix and a vector.

Linear Algebra Standard #5 Model Assessment Items

Computational and Procedural Skills

1. Multiply:

   A. $2\langle -5, 7\rangle$

   B. $3\begin{bmatrix} 4 & -2 \\ 0 & 3 \end{bmatrix}$

2. Given the matrices shown below.

   $$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix},\ B = \begin{bmatrix} 6 & 7 \\ 0 & -2 \end{bmatrix},\ C = \begin{bmatrix} -2 & 4 \\ 0 & 3 \\ 1 & 5 \end{bmatrix},\ \text{and}\ D = \begin{bmatrix} 6 & \frac{1}{2} & 0 \\ -3 & 4 & 1 \end{bmatrix}$$

   Find AB, BA, AC, CA, CD, and DC, if possible.

3. For a 2×2 matrix, $I = ?$ For a 3×3 matrix, $I = ?$

Conceptual Understanding

1. If E is a 2×3 matrix, F is a 3×2 matrix, and G is a 3×3 matrix, what are the dimensions of EF, FE, EG, GE, and G^2 (if these operations are possible)?

2. What must be true about the dimensions of matrices A and B for AB to exist?

3. Given that AB and BA both exist, are they always equal? Explain.

4. Given any vector $\vec{V}$, what would the resultant vector look like after performing the following multiplications.

   A. $3\vec{V}$

   B. $\frac{1}{2}\vec{V}$

   C. $-2\vec{V}$

Problem Solving/Application

None

Linear Algebra Standard #6

Standard Set 6.0
Students demonstrate an understanding that linear systems are inconsistent (have no solutions), have exactly one solution, or have infinitely many solutions.

Deconstructed Standard
1. Students demonstrate an understanding that some linear systems are inconsistent (have no solutions).
2. Students demonstrate an understanding that some linear systems have exactly one solution.
3. Students demonstrate an understanding that some linear systems have infinitely many solutions.

Prior Knowledge Necessary
Students should know how to:
- ➢ solve systems of linear equations in two and three variables by substitution or elimination.
- ➢ solve systems of linear equations in two and three variables using Gaussian elimination methods.

New Knowledge
Students will need to learn to:
- ➢ recognize when a system is inconsistent or consistent.
- ➢ recognize when a consistent system is dependent or independent.
- ➢ recognize that a matrix in row echelon form with a row of zeros, represents a system with an infinite number of solutions. (consistent, dependent).
- ➢ recognize that a matrix in row echelon form with a row whose final entry is the only nonzero value, represents a system with has no solution. (inconsistent).
- ➢ recognize a matrix that can be put in reduced row-echelon form represents a system with a single solution. (consistent, independent).

Categorization of Educational Outcomes
Competence Level: Application
1. From the solution set, students will identify the systems as inconsistent or consistent.
2. From the solution set, Students will identify consistent systems as independent or dependent.

Necessary New Physical Skills
None

Assessable Result of the Standard
1. Students will produce the solution to inconsistent, consistent, independent and dependent systems of equations.

Linear Algebra Standard #6 Model Assessment Items

Computational and Procedural Skills

1. Determine if the following systems are consistent or inconsistent. Identify consistent systems as dependent or independent:

   A. $\begin{cases} 2x - y = 4 \\ 4x - 2y = 8 \end{cases}$

   B. $\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$

   C. $\begin{cases} x + y + z = 6 \\ 2x - y - z = 3 \\ x + 2y + 2z = 0 \end{cases}$

Conceptual Understanding

1. Given the following matrices, determine if the corresponding systems are consistent or inconsistent. Identify consistent systems as dependent or independent. Where appropriate, give the solution.

   A. $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. If you solve a system of linear equations without using matrices, how do you know if the system is consistent or inconsistent? If consistent, how do you know if the system is dependent or independent.

Problem Solving/Application

None

Linear Algebra Standard #7

Standard Set 7.0

Students demonstrate an understanding of the geometric interpretation of vectors and vector addition (by means of parallelograms) in the plane and in three-dimensional space.

Deconstructed Standard

1. Students demonstrate an understanding of the geometric interpretation of vectors in the plane.
2. Students demonstrate an understanding of the geometric interpretation of vectors in three dimensions.
3. Students demonstrate an understanding of the geometric interpretation of vector addition in the plane.
4. Students demonstrate an understanding of the geometric interpretation of vector addition in three dimensions.

Prior Knowledge Necessary

Students should know how to:

- define a vector in two and three dimensions.
- graph in two dimensions.
- use the Law of Cosines.
- use inverse trigonometric functions.

New Knowledge

Students will need to learn to:

- graph points in three dimensional space.
- illustrate a vector.
- add vectors in two and three dimensions.
- add vectors geometrically in two dimensions using the method of parallelograms.
- add vectors geometrically in three dimensions using parallelepipeds.
- compute the direction of the resultant vector using an inverse trigonometric function.
- compute the magnitude of the resultant vector using the Law of Cosines.

Categorization of Educational Outcomes

Competence Level: Comprehension

1. Students explain vector addition using parallelograms. Students explain vector addition using parallelepipeds.

Competence Level: Knowledge

1. Students describe the different ways to add vectors in two and three dimensions using parallelograms and parallelepipeds, respectively.

Necessary New Physical Skills

None

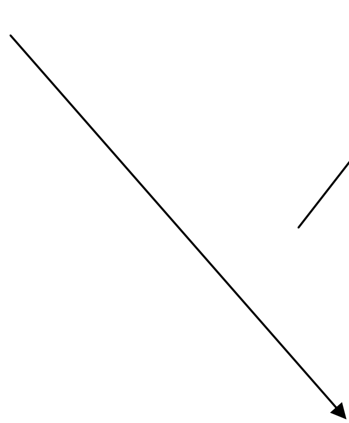
<u>Assessable Result of the Standard</u>

1. Students will add vectors geometrically in two and three dimensions.
2. Students will use the Law of Cosines and inverse trigonometric functions to determine the magnitude and direction of the resultant vector for vector addition.

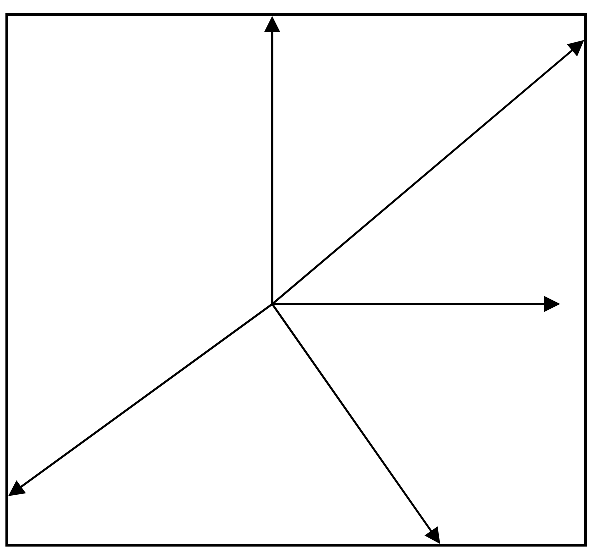
Linear Algebra Standard #7 Model Assessment Items

Computational and Procedural Skills

1. Add the given vectors using a parallelogram.



2. Add the given vectors using a parallelepiped.



Conceptual Understanding

1. Why is the addition of vectors commutative? Explain using two dimensional diagrams and complete sentences.

Problem Solving/Application

1. Two forces of magnitude 40 newtons (N) and 60 N act on an object at angles of 30° and -45°, respectively, with the positive *x*-axis. Find the direction and magnitude of the resultant force.

Linear Algebra Standard # 8

Standard Set 8.0

Students interpret geometrically the solution sets of systems of equations. For example, the solution set of a single linear equation in two variables is interpreted as a line in the plane, and the solution set of a two-by-two system is interpreted as the intersection of a pair of lines in the plane.

Deconstructed Standard

1. Students interpret the geometric representation of the solution set of a system of linear equations in two variables.
2. Students interpret the geometric representation of the solution set of a system of linear equations in three variables.

Prior Knowledge Necessary

Students should know how to:

- graph systems of linear equations in two dimensions.
- determine the nature of solutions by examining a graph of the system.

New Knowledge

Students will need to learn to:

- recognize the geometric interpretation of a solution set of a system of linear equations in two variables as a point, infinite set, or empty set.
- recognize the geometric interpretation of a single solution set of a system of linear equations in two variables as the intersection point of two lines on a plane.
- recognize the geometric interpretation of the empty set solution of a system of linear equations in two variables as representing two parallel lines on a plane.
- recognize the geometric interpretation of the infinite set solution of a system of equations in two variables two equations representing the same line on a plane.
- recognize the geometric interpretation of the single set solution of a system of linear equations in three variables as representing a common point.
- recognize the geometric interpretation of the infinite set solution of a system of linear equations in three variables as representing a common plane or a common line.
- recognize the geometric interpretation of the empty set solution of a system of linear equations in three variables as representing three planes that do not all intersect at the same point, line or plane.
- relate the geometric interpretations to the definitions of the solution sets as consistent, inconsistent, dependent, and/or independent.

Categorization of Education Outcome

Competence Level: Comprehension

1. Students will classify the geometric representation for the solution to a system of linear equations in two variables as a point, the same line, or parallel lines. Students will classify the geometric representation for the solution to a system of linear equations in three variables as a point, a line, a plane, or none of these.

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will classify solutions of systems of two dimensional systems geometrically.
2. Students will classify solutions of systems of three dimensional systems geometrically.

Linear Algebra Standard #8 Model Assessment Items

Computational and Procedural Skills

1. Identify the following systems graphically:

   A. $\begin{cases} 2x - y = 4 \\ 4x - 2y = 8 \end{cases}$

   B. $\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$

   C. $\begin{cases} x + y + z = 6 \\ 2x - y - z = 3 \\ x + 2y + 2z = 0 \end{cases}$

Conceptual Understanding

1. What does it mean graphically to have a system of 2 equations and 2 unknowns be dependent? Inconsistent?

2. What does it mean graphically to have a system of 3 equations and 3 unknowns be dependent? Inconsistent?

Problem Solving

None

Linear Algebra Standard #9

Standard Set 9.0
Students demonstrate an understanding of the notion of the inverse to a square matrix and apply that concept to solve systems of linear equations.

Deconstructed Standard
1. Students understand the notion of the inverse of a square matrix.
2. Students use inverse matrices to solve systems of linear equations.

Prior Knowledge Necessary
Student should know how to:
- perform basic row operations on matrices.
- multiply two matrices, where the product is defined.
- create the coefficient matrix for a given system of linear equations.
- compute inverse matrices for square matrices, where they exist.
- understand that some square matrices do not have inverses.

New Knowledge
Students will need to learn to:
- to write 2×2 and 3×3 systems of linear equations as matrix equations, $AX = B$, where *A* is the coefficient matrix, *X* is the unknown matrix, and *B* is the constant matrix.
- solve 2×2 and 3×3 systems of linear equations by multiplying the inverse of the coefficient matrix by the constant matrix, $X = A^{-1}B$, when A^{-1} exists.
- understand that the product of a square matrix, A, and its inverse, A^{-1}, if it exists, is an identity matrix.

Categorization of Educational Outcomes
Competence Level: Comprehension
1. Students will understand that systems of linear equations where the coefficient matrix has an inverse can be solved by a matrix multiplication. (NOTE: When the coefficient matrix does not have an inverse, this solution method cannot be used and an algebraic method (e.g., Gaussian elimination) must be used. In such cases, the system will have either no solutions or an infinite number of solutions).

Competence Level: Application
1. Students will solve 2×2 and 3×3 systems of linear equations by multiplying the inverse of the coefficient matrix by the constant matrix, $X = A^{-1}B$.
2. Students will write 2×2 and 3×3 systems of linear equations as matrix equations, $AX = B$, where *A* is the coefficient matrix, *X* is the unknown matrix, and *B* is the constant matrix.

Necessary New Physical Skills
None

Assessable Result of the Standard
1. Students will produce the solution set to a system of linear equations by using the inverse of the coefficient matrix.

Linear Algebra Standard #9 Model Assessment Items

Computational and Procedural Skills

1. If $A = \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 11 & -8 \\ -4 & 3 \end{pmatrix}$, solve the following system of equations:
   $3x + 8y = -12$.
   $4x + 11y = 0$

2. The inverse of the matrix $A = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$ is $A^{-1} = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$. Solve the following system of equations: $3x + 2y + 6z = 28$.
   $x + y + 2z = 9$
   $2x + 2y + 5z = 22$

Conceptual Understanding

1. If $A = \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}$, show that $B = \begin{pmatrix} 11 & -8 \\ -4 & 3 \end{pmatrix}$ is an inverse matrix of *A*, i.e., $B = A^{-1}$.

2. Let $A = \begin{pmatrix} 1 & -6 & 3 \\ 2 & -7 & 3 \\ 4 & -12 & 5 \end{pmatrix}$. Compute the matrix product AA. What do you observe?

3. Can the following system of equations be solved by matrix multiplication using the inverse of the coefficient matrix? Why or why not? $4x - 6y = -12$
   $-6x + 9y = 18$

Problem Solving/Application

1. Use a matrix inverse to find an equation in the form $y = ax^2 + bx + c$ for the parabola that passes through the points $(-2, 1)$, $(3, 4)$ and $(7, 2)$.

2. A shopkeeper has two types of coffee beans on hand. One type sells for \$9.20/lb, the other for \$9.80/lb. How many pounds of each type must be mixed to product 16 lb of a blend that sells for \$9.60/lb? Use a matrix inverse to solve.

Linear Algebra Standard #10

Standard Set 10.0
Students compute the determinants of 2×2 and 3×3 matrices and are familiar with their geometric interpretations as the area and volume of the parallelepipeds spanned by the images under the matrices of the standard basis vectors in two-dimensional and three-dimensional spaces.

Deconstructed Standard
1. Students compute the determinants of 2×2 and 3×3 matrices.
2. While a 2×2 determinant $\det A$ may be considered as the amount by which the area of the unit square (in the first quadrant with a vertex at the origin) is multiplied under transformation by the matrix *A* [and a 3×3 determinant may be considered as the amount by which the volume of the unit cube (in the first octant with a vertex at the origin) is multiplied under transformation by the matrix], high school students are not normally asked to be familiar with the geometric interpretations of determinants as the area of parallelograms and the volume of parallelepipeds spanned by the images under the matrices of the standard basis vectors in two-dimensional and three-dimensional spaces. Instead, students should understand the geometric properties of cross products of three-dimensional vectors **u** and **v**, i.e., that the area of the parallelogram formed by vectors **u** and **v** is equal to the length of the cross product or $\|\mathrm{u} \times \mathrm{v}\|$. Students should also understand that the cross product of **u** and **v** can computed by finding the determinant $\mathrm{u} \times \mathrm{v} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$, where $u = u_1\mathrm{i} + u_2\mathrm{j} + u_3\mathrm{k}$ and $v = v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k}$. [Note, the *triple scalar product*, i.e., the dot product of a vector **u** and a cross product $\mathrm{v} \times \mathrm{w}$ or $\mathrm{u} \cdot (\mathrm{v} \times \mathrm{w})$, is not normally covered in the high school curriculum. The absolute value of the triple scalar product is equal to the volume of a parallelepiped with vectors **u**, **v**, and **w** as adjacent edges.].

Prior Knowledge Necessary
Students should know how to:
- identify elements (entries) of a matrix by row and column.

New Knowledge
Students will need to learn to:
- understand that determinants are numerical values associated with $n \times n$ matrices.
- compute determinants of 2×2 matrices.
- identify co-factors and minors for any entry of an $n \times n$ matrix.
- compute determinants of 3×3 matrices by *expanding* about any row or column and adding the products of each row (or column) entry by its co-factor and minor.

Students will understand that the value of an n-th order determinant is defined in terms of certain determinants of order $n - 1$.

- use rules for manipulating determinants to simplify the computation of values of determinants.

Categorization of Educational Outcomes

Competence Level: Comprehension

1. Students will understand that determinants are numerical values associated with $n \times n$ matrices.

Competence Level: Knowledge

1. Students will identify co-factors and minors for any entry of an $n \times n$ matrix.

Competence Level: Application

1. Students will compute determinants of 2×2 and 3×3 matrices.
2. Students will use rules for manipulating determinants to simplify the computation of values of determinants.

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will produce the determinant of 2×2 and 3×3 matrices.

Linear Algebra Standard #10 Model Assessment Items

Computational and Procedural Skills

1. Evaluate the determinant $\begin{vmatrix} \sqrt{2}-1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2}+1 \end{vmatrix}$.

2. Use rules for simplifying the computation of determinants to evaluate the determinant $\begin{vmatrix} -6 & -8 & 18 \\ 25 & 12 & 15 \\ -9 & 4 & 13 \end{vmatrix}$.

Conceptual Understanding

1. Using rules for simplifying the computation of determinants, show that $\begin{vmatrix} a & b & c \\ a & a+b & a+b+c \\ a & 2a+b & 3a+2b+c \end{vmatrix} = a^3$.

Problem Solving/Application

1. Find all values of ***x*** for which $\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & 1 \\ 4 & 5 & 0 \end{vmatrix} = 0$.

2. Show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$.

Linear Algebra Standard #11

Standard Set 11.0

Students know that a square matrix is invertible if, and only if, its determinant is nonzero. They can compute the inverse to 2×2 and 3×3 matrices using row reduction methods or Cramer's rule.

Deconstructed Standard

1. Students know that a square matrix is invertible if, and only if, its determinant is nonzero.
2. Students will compute inverse matrices of 2×2 and 3×3 matrices using basic row operations on an augmented matrix, $A : I_n$.
3. Students will use Cramer's rule to solve 2×2 and 3×3 linear systems. [Normally high school students are not expected to use Cramer's rule to compute inverses of 2×2 and 3×3 matrices. Instead, they use basic row operations as discussed above.].

Prior Knowledge Necessary

Students should know how to:

- compute determinants of 2×2 and 3×3 matrices.
- understand the concepts of identity and inverse matrices.

New Knowledge

Students will need to learn to:

- understand that a square matrix is invertible if, and only if, its determinant is nonzero.
- compute inverse matrices of 2×2 and 3×3 matrices using basic row operations on an augmented matrix, $A : I_n$.
- solve 2×2 and 3×3 linear systems using Cramer's Rule.

Categorization of Educational Outcomes

Competence Level: Comprehension

1. Students will understand that matrices have inverses when their determinants are non-zero.

Competence Level: Application

1. Students will compute inverse matrices of 2×2 and 3×3 matrices using basic row operations on an augmented matrix, $A : I_n$.
2. Solve 2×2 and 3×3 linear systems using Cramer's Rule.

Necessary New Physical Skills

None

Assessable Result of the Standard

1. Students will produce, when appropriate, the inverse for a 2×2 and 3×3 matrix.
2. Students will produce the solution to a system of linear equations using Cramer's Rule.

Linear Algebra Standard #11 Model Assessment Items

Computational and Procedural Skills

1. Use basic row operations to compute the inverse matrix, if it exists, for the matrix $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$.

2. Use basic row operations to compute the inverse matrix, if it exists, for the matrix $\begin{pmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{pmatrix}$.

3. Use Cramer's rule to solve the following system of linear equations:
$$\begin{aligned} 2x - y &= 8 \\ 3x - 2y &= 13 \end{aligned}$$

4. Use Cramer's rule to solve the following system of linear equations:
$$\begin{aligned} 2x + 2y - 3z &= -20 \\ x - 4y + z &= 6 \\ 4x - y + 2z &= -1 \end{aligned}$$

5. Let $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 8 \\ 6 & 7 \end{pmatrix}$. Compute $(AB)^{-1}$.

Conceptual Understanding

1. Explain why the matrix $\begin{pmatrix} 2 & 6 \\ -1 & -3 \end{pmatrix}$ does not have an inverse.

2. Can Cramer's rule be used to solve the following system of linear equations? If not, why?
$$\begin{aligned} 4x + 3y - 2z &= 14 \\ x + 2y - 3z &= 5 \\ 2x - y + 4z &= 2 \end{aligned}$$

Problem Solving/Application

1. Use properties of determinants to show that the following is an equation of a line through the point $\left(x_1,\ y_1\right)$ that has slope *m*: $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ 1 & m & 0 \end{vmatrix} = 0$.

Linear Algebra Standard #12

Standard Set 12.0
Students compute the scalar(dot) product of two vectors in n-dimensional space and know that perpendicular vectors have zero dot product.

Deconstructed Standard
1. Students compute the dot product of two vectors in 2-dimensions and 3-dimensions using algebra.
2. Students know that if the dot product of two non-zero, 2-dimensional vectors is zero, then the vectors are perpendicular. If two non-zero, 2-dimensional vectors are perpendicular, then the dot product is zero.
3. Students know that if the dot product of two non-zero, 3-dimensional vectors is zero, then the vectors are orthogonal. If two non-zero, three-dimensional vectors are orthogonal, then the dot product is zero.

Prior Knowledge Necessary
Students should know how to:
- identify the coefficients of like components in either component or unit vector form.
- express vectors in 2-dimensions and 3-dimensions.
- apply the Law of Cosines.

New Knowledge
Students will need to learn to:
- perform dot product multiplication.
- interpret a zero dot product of non-zero vectors as evidence of perpendicular vectors.

Categorization of Educational Outcomes
Competence Level: Comprehension
1. Students will be able to support a geometric relationship of two vectors using the dot product.

Necessary New Physical Skills
None

Assessable Result of the Standard
1. Students will be able to compute dot products.
2. Students will determine if vectors are perpendicular.

Linear Algebra Standard #12 Model Assessment Items

Computational and Procedural Skills

1. $\langle 2,3,-4\rangle \bullet \langle 1,-2,-1\rangle =$

2. State if the vectors in the above problem are orthogonal.

3. Determine if $\langle 2,-3\rangle$ and $\langle -2,3\rangle$ are perpendicular.

Conceptual Understanding

1. Vectors $\langle 2,-1\rangle$ *and* $\langle -4,2\rangle$ are both perpendicular to $\langle 3,6\rangle$. How is that possible?

Problem Solving/Application

1. Construct a vector that is perpendicular to $\langle 3,-4\rangle$. (There are an infinite number of vectors that are perpendicular to a given vector.)

2. How are the multiple solutions to #1 related geometrically?

Appendix #1

Developing Learning Targets for Geometry Standards (Instructions given to teachers involved in the project—same as the instructions used in Algebra deconstruction projects)

Please note the following terms and/or definitions which have been agreed upon for this deconstruction project:

- ***Prior Knowledge:*** *Prior knowledge is defined as acquired knowledge that has been mastered in a previous standard.*
- ***New Knowledge:*** *New knowledge is defined as knowledge that students need to acquire and apply to the components in step #2 of this deconstruction process to create the products listed in step #7 of this process.*
- ***Introduced:*** *When a standard mentions that a concept or idea has been "introduced," this does not mean that it has been mastered.*
- ***Familiar With:*** *When a standard mentions that students should be "familiar with" certain concepts or ideas, this does not mean that students have actually mastered these ideas or concepts.*

Sample:
Deconstruction of Algebra I Standard #6

Step #1: Underline noun phrases, and box or circle the verbs.
Standard 6.0: Students [graph] a linear equation and [compute] the *x*- and *y*-intercepts. (e.g., graph $2x + 6y = 4$). They are also able to [sketch] the region defined by a linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

Step #2: Rewrite standard into short components.
1. Students graph linear equations.
2. Students compute *x*-intercepts.
3. Students compute *y*-intercepts.
4. Students use intercepts to graph linear equations.
5. Students sketch the region defined by a linear inequality.

Step #3: Identify prior knowledge students should know. *(See note above)*
1. Students must be able to perform arithmetic computations with rational numbers.
2. Students must be able to graph ordered pairs.
3. Students must be able to compute slope from the graph of a line.
4. Students must be able to compute slope when given two points.
5. Students must be able to recognize slope as a rate of change of *y* in relation to *x*.
6. Students must be able to graph a linear equation using a "t-chart."
7. Students must be able to evaluate a linear equation for a given *x* or *y* value.
8. Students must be able to solve one-variable linear inequalities.

9. Students must be able to graph the solution set for a one-variable linear inequality.
10. Students must be able to verify that any element in the solution set of a one-variable inequality satisfies the original inequality.

Step #4: Identify what new knowledge students will need to learn. *(See note above)*

1. Given the slope/intercept form of a line, $y = mx + b$, students will plot the *y*-intercept, and then use the slope to find a second point in order to complete the graph of the line.
2. Students will identify the graphical representation of $(a, 0)$ as the *x*-intercept, and $(0, b)$ as the *y*-intercept.
3. Students will be able to compute the *x*-intercept and *y*-intercept given a linear equation.
4. Students will identify that the linear equation implied by the linear inequality forms a boundary for the solution set and that this boundary may or may not be included in the final graph.
5. Students will interpret the inequality symbol to determine whether or not the boundary is solid or dashed.
6. Students will identify and shade the region of the graph that contains the solutions to the inequality.
7. Students will recognize that linear inequalities have multiple ordered-pair solutions.

Step #5: Identify patterns of reasoning using *Bloom's Taxonomy*.

Use the *Bloom's Taxonomy* handouts provided to describe the overall competence level expected of students with respect to these topics. Then highlight the "skills demonstrated" using as many of the key words and phrases provided on the handouts. See example below for Standard #6. Box in the key words or phrases taken from *Bloom's Taxonomy*.

Competence Level: Application

1. Students will use methods they have learned to graph lines, solve inequalities, and to locate and/or identify the *x*- and *y*-intercepts for a given equation or graph.
2. Students will demonstrate their ability to find and use *x*- and *y*-intercepts in the context of graphing.
3. Students will calculate *x*- and *y*-intercepts.
4. Students will solve inequalities in two variables.
5. Students will use information they have learned to graph lines, solve inequalities, and find *x*- and *y*-intercepts.
6. Students will show that they know the correct interpretation of the boundary line for the solution of an inequality by appropriately making the boundary solid or dashed.

Step #6: Identify required physical skills.

In this section we are looking for physical skills such as: use of a calculator, protractor, ruler, compass, etc.

1. Use of a ruler

Step #7: Identify assessable results of the standard.
1. Students will produce the graph of a line.
2. Students will produce the ordered pairs representing the *x*- and *y*-intercepts.
3. Students will produce a bounded and shaded region of the *x*-*y* plane representing the solution set of a linear inequality in two variables.

Model Assessment Items
In this section you will write model, or exemplar, assessment items that will serve to demonstrate the level and depth of instruction for these particular topics. For example, you would expect a lower level and depth for a topic in Basic Algebra than you would for the same topic in Intermediate Algebra.

Please be sure to include assessment items that measure abilities in the following three categories:
1. Computational and Procedural Skills
2. Conceptual Understanding
3. Problem Solving/Application

Category #1: Computational and Procedural Skills
1. Find the *x*- and *y*-intercepts for the line defined by the following equation: $2x+3y=9$.
2. Use the *x*- and *y*-intercepts to graph the line given by the equation: $2x+3y=6$.
3. Graph the following lines using the method of your choice. Identify and label the *x*- and *y*-intercepts for each graph if they exist:
   A. $3x-5y=10$

   B. $y=-\frac{2}{3}x+4$

   C. $y=2$

   D. $x=3.5$

   E. $2x+4y=3$

   F. $\frac{1}{2}x-\frac{3}{4}y=2$

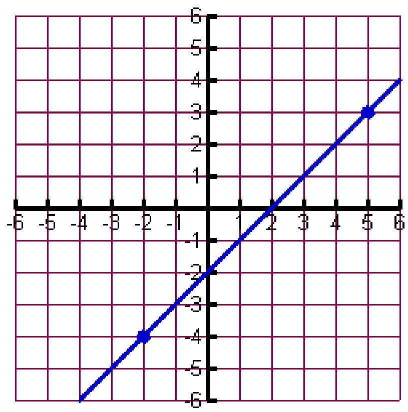
5. Graph the solution set for the following inequalities:
   A. $2x-3y<6$

B. $y \geq -\frac{3}{4}x + 2$

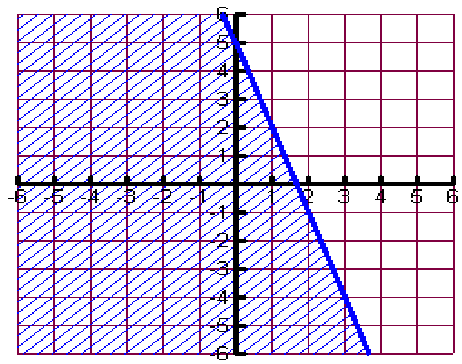
C. $\frac{1}{2}x - \frac{2}{3}y \leq \frac{5}{6}$

Category #2: Conceptual Understanding

1. Sketch the graph of a line that has no *x*-intercept.
2. Identify the *x*- and *y*- intercepts from the graph of the given line.



3. Can a line have more than one *x*-intercept? Explain your answer using a diagram.
4. The solution to an inequality has been graphed correctly below. Insert the correct inequality symbol in the inequality below to match the graph of the solution. (Everything else about the inequality is correct—it just needs the correct symbol).



$y \,\square\, -3x + 5$

Insert correct symbol in box.

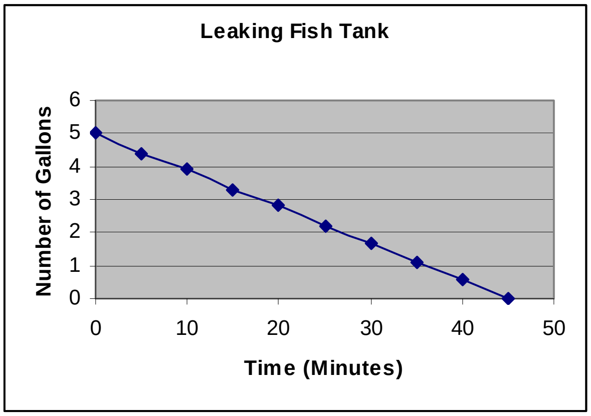
5. When is it advantageous to use the *x*- and *y*-intercepts to graph the equation of a line? When would it perhaps be easier or better to use another graphing method? Give an example to illustrate your answers to both of these questions.

Category #3: Problem Solving/Application

1. The graph displayed below is the graph of the following equation: $y = \left(\frac{-1}{9}\right)x + 5$,

   where *x* represents the amount of time that has passed since a 5 gal. fish tank sprung a leak, and *y* represents the number of gallons of water in the tank after the leak.

A. What is the significance of the x-intercept in this situation? What information is given to us by this point?
B. What is the significance of the y-intercept in this situation? What information is given to us by this point?



2. The cost of a trash pickup service is given by the following formula: $y = 1.50x + 11$, where x represents the number of bags of trash the company picks up, and y represents the total cost to the customer for picking up the trash.
   A. What is the y-intercept for this equation?
   B. What is the significance of the y-intercept in this situation? What does it tell us about this trash pickup service?
   C. Draw a sketch of the graph which represents this trash pickup service.

Exemplar Teaching Methods

Please record an example of "Lesson Plans" that demonstrate an excellent method of how the concepts in this standard could be presented to students. Be sure to give examples or illustrate any unique or creative methods that you have used that bring these concepts to life. Use as much detail as needed to communicate your ideas.

Appendix #2

Categorization of Educational Outcomes

Identifies the type of reasoning students will use to learn the skills necessary to master each standard. Teachers were asked to use *Bloom's Taxonomy* to describe the overall competence level expected of students with respect to these topics and highlight the skills demonstrated.

Major Categories in the* Taxonomy of Educational Objectives, *Bloom 1984
Categories in the Cognitive Domain: (with Outcome-Illustrating Verbs)

Knowledge—of terminology; specific facts; ways and means of dealing with specifics (conventions, trends, and sequences; classifications and categories; criteria, methodology); universals and abstractions in a field (principles and generalizations, theories and structures)—**The remembering (recalling) of appropriate, previously learned information.**

- defines; describes; enumerates; identifies; labels; lists; matches; names; reads; records; reproduces; selects; states; views.

Comprehension: **Grasping (understanding) the meaning of informational materials.**

- classifies; cites; converts; describes; discusses; estimates; explains; generalizes; gives examples; makes sense out of; paraphrases; restates (in own words); summarizes; traces; understands.

Application: **The use of previously learned information in new and concrete situations to solve problems that have single or best answers.**

- acts; administers; articulates; assesses; charts; collects; computes; constructs; contributes; controls; determines; develops; discovers; establishes; extends; implements; includes; informs; instructs; operationalizes; participates; predicts; prepares; preserves; produces; projects; provides; relates; reports; shows; solves; teaches; transfers; uses; utilizes.

Analysis: **The breaking down of informational materials into their component parts, examining (and trying to understand the organizational structure of) such information to develop divergent conclusions by identifying motives or causes, making inferences, and/or finding evidence to support generalizations.**

- breaks down; correlates; diagrams; differentiates; discriminates; distinguishes; focuses; illustrates; infers; limits; outlines; points out; prioritizes; recognizes; separates; subdivides.

Synthesis: **Creatively or divergently applying prior knowledge and skills to produce a new or original whole.**

- adapts; anticipates; categorizes; collaborates; combines; communicates; compares; compiles; composes; contrasts; creates; designs; devises; expresses; facilitates; formulates; generates; incorporates; individualizes; initiates; integrates; intervenes;

models; modifies; negotiates; plans; progresses; rearranges; reconstructs; reinforces; reorganizes; revises; structures; substitutes; validates.

Evaluation: **Judging the value of material based on personal values/opinions, resulting in an end product, with a given purpose, without real right or wrong answers.**

- appraises; compares & contrasts; concludes; criticizes; critiques; decides; defends; interprets; judges; justifies; reframes; supports.

Appendix #3
Sample Teaching Items

Trigonometry Standard #7

Anticipatory Set:
Starter problems as students enter the room:
1. Find the slope of the line through the points (-2, 3) and (5, 1).
2. Find the slope of the line through the points (-3, 0) and (-3, 5).
3. Find the slope of the line $y = \frac{-2}{5}x + 3$.
4. Find the slope of the line $4x - 3y = 6$.
5. Sketch the graph of the equation: $2x + 5y = -5$.
6. Sketch the graph of the equation: $x = 4$.

Discuss the starter problems to review the definition of slope, how to find slope using 2 points, how to find slope using the slope-intercept form of the equation of a line, how to find slope from the standard form of the equation of a line and how to sketch the graph of a line.

Statement of Objective:
Students will discover the relationship between the angle of inclination of a line and the slope of the line.

Instructional Input:
Ask students to take out graph paper and a straight edge.
1. Tell students to plot the points (-2, 0) and (2, 1).
2. Draw the line through the two points.
3. Determine the slope between these two points.
4. Now draw a vertical line from (2, 1) to the *x*-axis, which forms a right triangle with the x-axis.
5. Review the definition of tangent, which is $\tan\theta = \frac{opposite}{adjacent}$. Using the triangle drawn, determine the tangent of the angle formed by the given line and the *x*-axis. You could find the length of both the opposite and adjacent sides by counting the squares on the graph paper or by determining the change in y vertically and the change in x horizontally.

6. Now highlight the point (6, 2) on your graph. Notice that this point also lies on the line. Draw a vertical line from (6, 2) to the *x*-axis. Again determine the tangent of the angle. Compare with the answer you got from the first point (2, 1). Explain why the answers are equivalent. Discuss how you could find other points to repeat the process. Ask students in partners to repeat the process using two new points on their line which are neither *x*-intercepts nor

y-intercepts. Students are to analyze their results and predict the relationship between the slope of the line and the tangent of the angle formed by the line and the x-axis.

Discuss the predictions that students discovered. Then draw a line passing through the origin with an inclination of θ°. Draw a unit circle. Let *P* be the point where the line intersects the unit circle, so $P = (\cos\theta, \sin\theta)$. Find the slope of the line through the points *P* and the origin (0, 0), using the formula for slope: $m = \frac{y_2 - y_1}{x_2 - x_1}$. Then $m = \frac{\sin\theta - 0}{\cos\theta - 0}$ or $m = \frac{\sin\theta}{\cos\theta}$. Remind students the definition of tangent of an angle is $\frac{opposite}{adjacent}$. However, the length of the opposite side is $\sin\theta$ and the length of the adjacent side is $\cos\theta$, so $\tan\theta = \frac{\sin\theta}{\cos\theta}$ which is the same as the slope of the line.

Model:
Using the formula the class discovered, $\tan\theta = m$, to determine the angle of inclination of a line. Do the following examples to illustrate the use of the formula.

A. Find the tangent of the angle of inclination of the line $y = 2x + 4$.

B. Find the angle of inclination of the line $3x - 4y = 5$.

C. Find the slope of the line inclined at an angle of 56° to the *x*-axis.

D. Find the equation of the line inclined at an angle of 113° which goes through the point (-4, -2).

Check for Understanding:
Have students do several similar problems. Circulate, assist and check for understanding.

Trigonometry Standard #8

Anticipatory Set:
Starter problems as students enter the room:

1. Suppose a function *f* has an inverse. If $f(3) = 7$, find the following:
   a. $f^{-1}(7)$
   b. $f\left(f^{-1}(7)\right)$
   c. $f^{-1}\left(f(3)\right)$
2. Find a rule for $f^{-1}(x)$ if:
   a. $f(x) = 2x - 4$
   b. $f(x) = x^3 + 1$
   c. $f(x) = \dfrac{2x}{3} + 4$
3. Explain how you know there is *no* rule for $f^{-1}(x)$ when $f(x) = x^2 + 1$. What could you do in order to make a rule for $f^{-1}(x)$?

Go over the starter problems, reminding students that an inverse function "undoes" what the function does, that a function must be one-to-one in order to have an inverse and how you could restrict the domain in order to make a function one-to-one so that you could find its inverse (which means selecting only "half" of the parabola in example 3, where $x \geq 0$). Graphically, you would reflect a graph over the line $y = x$ in order to obtain the inverse.

Statement of Objective:
Students know the definitions of the inverse trigonometric functions and can graph them.

Instructional Input:
Review the definition of the six trigonometric functions, using a right triangle and the names of the sides: opposite, adjacent and hypotenuse. With a trigonometric function, you are given an angle and you find a ratio of sides from that angle. An inverse function "undoes" what a function does, so with an inverse trigonometric function, you are given a ratio of sides and you find the angle.
Another way to find an inverse function is to solve the equation for y, switch the domain and range (x and y) and then resolve for y. In a trigonometric function, that would look like the following:

The given function: $y = \sin x$
Switch the *x* and *y*: $x = \sin y$
Solve for *y*: $y = \sin^{-1} x$ which is the inverse of the sine function.

Another way to write this is as follows:

$\theta = Sin^{-1}x \quad \Leftrightarrow \quad x = \sin\theta$

$\theta = Cos^{-1}x \quad \Leftrightarrow \quad x = \cos\theta$

$\theta = Tan^{-1}x \quad \Leftrightarrow \quad x = \tan\theta$

Model:
Give several examples of the inverse trig functions, emphasizing that you are given the ratio of two sides and you are looking for the angle, in either degrees or radians. You might say "$S\text{in}^{-1}\left(\frac{1}{2}\right) = x$ means I am looking for an angle whose sin (opposite side over hypotenuse) is $\frac{1}{2}$. What is that angle?" Include a sketch of the triangle you are using to help students visualize. You are saying $30^\circ = Cos^{-1}\left(\frac{1}{2}\right)$ is the same as $\cos 30^\circ = \frac{1}{2}$.

Check for Understanding:
Give several examples for students to do, like $Cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$, being sure to include the sketch of the triangle.

Instructional Input:
Now display a graph of the sine curve. Discuss the vertical line test to determine if it is a function. Discuss the horizontal line test to determine if the sine curve is one-to-one, which means it will have an inverse. Since the sine is not one-to-one, this means when you are given an answer or ratio of sides and need to find the angle, there are many different answers.

On the same graph of the sine curve, add the line $y = x$. Reflect the graph of the sine curve over the line $y = x$. (*Note: It works well if each of these graphs is on a separate overhead which can be layered over each other*.) Since this new graph will not pass the vertical line test, it can't be a function; the inverse will not exist as a function. Going back to the original sine curve, find the largest portion of the curve closest to the origin that will pass the horizontal line test. Highlight that region on the graph. Now reflect it over the line $y = x$. What you see is the graph of the inverse sine. Emphasize you have interchanged the domain and range, so for the inverse sine, the domain represents the "answers" or the ratios of the sides of the triangles while the range represents the angles.

Repeat the demonstration with the graph of the cosine , reflecting it over the line $y = x$ and noting that it no longer passes the vertical line test. Go back to the cosine graph and look for the largest possible portion of the graph closest to the origin which will pass the horizontal line test. Note that if you go on either side of the origin, it will no longer pass the horizontal line test, so you must choose either the positive or negative side of the origin in order to pass the horizontal line test. Mathematicians made the choice to select the positive side for the inverse cosine function. Highlight the portion of the graph to the right of the origin that passes the horizontal line test. Now reflect this highlighted portion over the line $y = x$ and note that it does pass the vertical line test and is a function.

At this point, look at both the inverse sine graph and the inverse cosine graph. Discuss how the domain of the original graphs was restricted to obtain an inverse function, noting the difference. For the Sin^{-1} graph, we use the restricted domain of the sine curve $\left[-\pi/2, \pi/2\right]$ while for the Cos^{-1} graph, we use the restricted domain of the cosine curve $\left[0, \pi\right]$. Draw a unit circle and demonstrate the quadrants closest to the origin where the y value is either positive and negative (Quadrants I and IV) gives the domain for the Sin^{-1} graph and the quadrants closest to the origin where the x value is either positive and negative (Quadrants I and II) for the Cos^{-1} graph.

Repeat for tangent and Tan^{-1} graphs. Students should be quick to see the similarity so you will not need to go into as much detail. Repeat for cosecant, secant and tangent graphs as well as their inverses.

The key for student understanding is to visualize the graph of the trig function and select the portion of the graph closest to the origin which will make the graph pass the horizontal line test. Students need to see the graphs of the original functions and then the reflected function. You might want to review the unit circle with the acronym ASTC (all students take calculus) to represent **All** functions are positive in quadrant I, the **Sin** is positive in quadrant II, the **Tan** is positive in quadrant III and the **Cos** is positive in quadrant IV to help students recognize which quadrants close to the positive x-axis have positive or negative values. Later on, you might also suggest that students can graph the inverse function by switching the ordered pairs from (x, y) to (y, x) which is a result of switching the domain and range.

Check for Understanding:
Put up several inverse trig function graphs and ask students to identify the graph. Also ask them to identify the quadrant in which the graph is defined. Circulate, assist and check for understanding.

Trigonometry Standard #9

Anticipatory Set:

Starter problems on the board as students walk in:

1. Identify the sides of a 30-60-90 triangle.
2. Identify the sides of a 45-45-90 triangle.
3. Convert to degree measure
   A. $\frac{\pi}{3}$
   B. $\frac{5\pi}{6}$
   C. $\frac{3\pi}{2}$

Ask students to check answers with their partner. Discuss any concerns.

Statement of Objection:

Students compute, by hand, the values of the trigonometric functions and the inverse trigonometric functions at various standard points.

Instructional Input:

1. Provide students with a large template of a unit circle.
2. Identify the "corner points" on the unit circle with degree measures of 0°, 90°, 180°, 270°, and 360°. Review how to convert from degrees to radians and also write the radian measures for the corner points on the unit circle template.
3. Identify the coordinates of these corner points of the unit circle, writing each of them on the template.
4. Review the definition of sine and cosine as the *y*- and *x*-coordinates of points on the unit circle.

Model:

1. Find the values of the sine of each of the corner points using the definition as the sine is the y-coordinate of each point.

Check for Understanding:

1. Ask students to use the definition of the cosine as the x-coordinate of each point and the definition of the tangent as the ratio of the y-coordinate over the *x*-coordinate. Students will write the cosine and tangent values on their template for each of the corner points. Circulate, assist and check for understanding.

Instructional Input:

1. Review the definitions of the cotangent, secant and cosecant. Remind students that they can not divide by 0 and explain how that will effect where each of the co functions is undefined.
2. Draw a triangle in the first quadrant of the unit circle. Review the definition of the sine, cosine and tangent as a ratio of the sides of the triangle. Assume

the triangle is a 30-60-90 triangle and identify the sine, cosine and tangent ratios.

3. Show on the template all equivalent reference angles to a 30° angle, which is 150°, 210°, and 330°. Explain why the ratios of sides will all be equivalent.
4. Use a different color on the template to show all equivalent reference angles to 60°.
5. Repeat with a different color to show all equivalent reference angles to 45°.
6. Identify for each quadrant the signs of the trigonometric functions. You might use the statement "All Students Take Calculus" to help students remember that in the first quadrant, ***a***ll functions are positive. In the second quadrant, only the ***s***ine and its cofunction, the cosecant, are positive. In the third quadrant, only the ***t***angent and its co function, the cotangent, are positive. In the fourth quadrant, only the ***c***osine and its co function, the secant, are positive.

Model:

1. Use a 30-60-90 triangle to find the sine, cosine, tangent, cotangent, secant, and cosecant of a 30° angle.
2. Then demonstrate how to find the correct signs for each of the equivalent reference angles in quadrants II – IV.

Check for Understanding:

1. Students repeat for a 60° angle as well as the equivalent reference angles in the other quadrants. Circulate, assist and check for understanding.
2. Students repeat for a 45° angle. Circulate, assist and check for understanding.

Instructional Input:

1. Demonstrate how to find reference angles for radian measures. Students already know how to convert from degree to radian and vice-versa, but the following helps to rapidly find the appropriate degree measure to use.
2. $180^\circ \div 6 = 30^\circ$ so $\frac{\pi}{6}$ is a 30° angle. Any reduced radian measure with a denominator of 6 will have a reference angle of 30°. Therefore, $\frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ all are reference angles to $\frac{\pi}{6}$.
3. Similarly, $180^\circ \div 3 = 60^\circ$ so $\frac{\pi}{3}$ is a 60° angle. Any reduced radian measure with a denominator of 3 will have a reference angle of 60°. Therefore, $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ all are reference angles to $\frac{\pi}{3}$.

4. Similarly, $180^{\circ} \div 4 = 45^{\circ}$ so $\frac{\pi}{4}$ is a 45° angle. Any reduced radian measure with a denominator of 4 will have a reference angle of 45°. Therefore, $\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ all are reference angles to $\frac{\pi}{4}$.
5. Refer to the colored template to see the reference angles are all colored the same.

Check for Understanding:

1. Ask students to find the six trigonometric function values for $\frac{5\pi}{4}$. Circulate, assist and check for understanding.

Instructional Input:

1. Review the definitions of the inverse trigonometric functions. Now that students understand how to find the values of the trigonometric functions by hand at the standard points, they should recognize the answers of known angles. An inverse function "undoes" the function, which means if you know the answer, you can find the angle. Model several problems such as $Sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ or $Tan^{-1}(1)$.

Check for Understanding:

1. Give several problems like the two above. Circulate, assist and check for understanding.