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A Framework for the Development of Mathematical Thinking With Teacher Trainees: The Case of Continuity of Functions*

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Continuity of functions appears throughout the grades in South African high school (FET (further education and training)) topics as prescribed by the final draft of the Curriculum and Assessment Policy Statement. This article reports on the use of a combined framework of APOS (action-process-object-schema) and DCT (dual coding theories) to analyze data captured in a study which investigated second-year teacher trainees' understanding of the concept of continuity. The study is qualitative in that it reports on these teacher trainees' mental constructions of the concept of continuity of single-valued functions, obtained from analysis of their responses to structured activity sheets. The 12 students in this study specialize in the teaching of mathematics for the FET high school curriculum at a South African education faculty. A two-tiered concurrent approach was attempted, one through student-collaborations and the other through an instructional design worksheet, to develop mathematical understandings of the concept of continuity.

Keywords: APOS (action-process-object-schema) theory, three worlds of mathematics, DCT (dual coding theories), single-valued function, continuity

Introduction

Previous studies, for example (Dubinsky, Weller, McDonald, & Brown, 2005) analyzed students' mathematical learning on an individual basis. This study however analyzed teacher trainees' understanding, after they carried out investigations first individually and then in a collaborative manner. This is to address the learner-centered approach which underpins Curriculum 2005 (DoE (Department of Education), 2003). We report on an investigation based on the use of activity sheets and group-work to construct the concept of continuity. To collaborate is to work with another or others. In practice, collaborative learning has come to mean students working in pairs or small groups to achieve shared learning goals (Barkley, Cross, & Major, 2005).

A new trend (at least in the European mathematics education community) is using several theories and approaches in a meaningful way (Radford, 2008). In coordinating theories, elements from the different theories are chosen and integrated to investigate a certain research problem. Tall (2004) presented a framework for mathematical thinking based on three worlds of mathematics: (1) the embodied; (2) the symbolic; and (3) formal. It is thought that as new conceptions are compressed into more thinkable concepts, individuals develop

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through these worlds hierarchically (Tall & Ramos, 2006). The embodied world containing embodied objects (Gray & Tall, 2001) is where we think about the things around us in the physical world, and it includes not only our mental perceptions of the real world objects, but also our internal conceptions that involve visual-spatial imagery (Tall, 2004, p. 30). In order for us to understand this "world of continuous function", we employed the DCT (dual coding theories) (Paivio, 1986). For the symbolic and formal "worlds of continuity", we made use of APOS (action-process-object-schema) (Dubinsky & McDonald, 2001). We found it useful to blend the frameworks used by Stewart and Thomas (2009) and Brijlall and Maharaj (2009) to facilitate our analysis in this paper. This led us to the framework given in Figure 1.

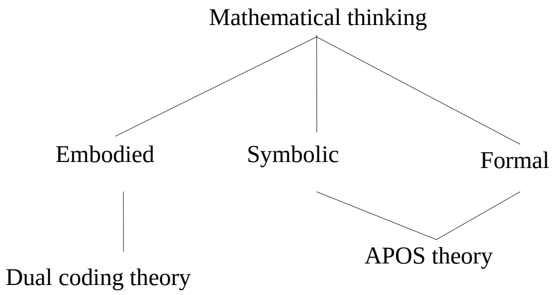

Figure 1. A modified framework for reflective abstraction.

Maharajh, Brijlall, and Govender (2008) investigated the concept image and the concept definition (Tall & Vinner, 1981) with regard to a deeper understanding of continuity in differential calculus within a Vygotskian paradigm. Vidakovic (1996; 1997) used APOS theory in the context of collaborative learning. Those investigations focused on the differences between group and individual mental constructions of the inverse function concept. Vidakovic (1997) described the construction processes for developing schema (genetic decomposition) of the inverse function. In particular, genetic decompositions which predict the mental constructions are a part of every good APOS based study.

Bezuidenhout (2001) pointed out that misconceptions relating to students' understandings of the concepts of limit and continuity are impediments to the development of deeper understandings in differential calculus. It seems that many students perform poorly in mathematics because they: (1) are unable to adequately handle information given in symbolic form which represent objects (abstract entities), for example, mathematical expressions, equations and functions; and (2) lack adequate schema or frameworks which help to organize and link different objects (Maharaj, 2005).

Background

Relevance of Study to South African Educators

Following the work by Shulman (1986; 1987) to make academics rethink what was necessary for effective classroom practice, Ball, Thames, and Phelps (2008) made it relevant to content knowledge for mathematics teaching and learning. They have introduced four sub-domains of content knowledge. This paper illustrates each of these sub-domains by features which appear in the questionnaire we employed in the data capture. The four sub-domains of pedagogical knowledge are: (1) CKK (common content knowledge); (2) SCK (specialized content knowledge); (3) KCS (knowledge of content and student); and (4) KCT (knowledge of content and teaching). The knowledge of continuity we found overlaps with the sub-domains (1) and (2). Most functions to be taught in the high, as guided by the NCS (National Curriculum Statement) (DoE, 2003) are continuous.

These functions include $\{(x;y)/y=ax^2+bx+c\}$, $\{(x;y)/y=mx+c\}$, $\{(x;y)/ax^3+bx^2+cx+d\}$, $\{(x;y)/y=\sin x\}$, etc.. However, the educators are also faced with the teaching of discontinuous functions like $\{(x;y)/y=\frac{k}{x}, x\neq 0\}$, $\{(x;y)/y=\tan x\}$, $\{(x;y)/y=2^x\}$ and $\{(x;y)/y=\log x\}$. One of the expectations of the Norms and Standards for Educators (DoE, 1999) is that the educator should be well grounded in the knowledge relevant to the occupational practice. She/he has to have a well-developed understanding of the knowledge appropriate to the specialism. Many mathematics educators find themselves in a position requiring them to implement the syllabus, which includes certain topics they are unfamiliar with. According to Adler (2002), educators with a very limited knowledge of mathematics need to develop a base of mathematical knowledge. They need to relearn mathematics so as to develop conceptual understanding. Taking this into account, we attempted to make certain that trainees' teachers leave with a base of knowledge relevant to their occupational needs. Mwakapenda (2004) concurred, when stating that a significant concern in school mathematics is learning with understanding of mathematical concepts.

The NCS emphasizes a learner-centered, outcomes-based approach to the teaching of mathematics to achieve the critical and developmental outcomes (DoE, 2003). The following question guided our inquiry into teacher trainees' understandings in their constructions of the concept of continuity.

How does the graphical representation learning approach facilitate students' learning process with regard to the construction of the concept of continuity of single-valued functions in differential calculus?

The main intention of the study was to observe how learning of mathematics content, whether effective or not, took place under these circumstances. In order to answer the above research questions, an APOS analysis of the data was conducted.

Theoretical Framework

This study was carried out in accordance with a specific framework for research and curriculum development in undergraduate mathematics education advocated by Asiala, Brown, DeVries, Dubinsky, Mathews, and Thomas (1996) which guided our systematic enquiry of how students acquire mathematical knowledge and what instructional interventions contributed to student learning. The framework consists of the following three components: instructional treatment, theoretical analysis and observations and assessment of student learning.

According to Asiala et al. (1996), the functions of APOS theory according to the paradigm illustrated in Figure 2.

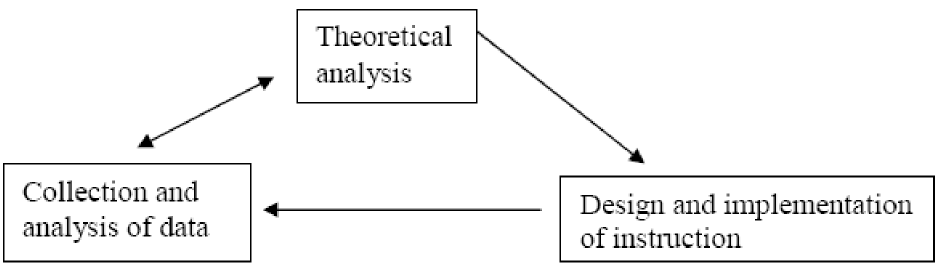

Figure 2. Paradigm: General research programme.

In this paradigm, theoretical analysis occurs relative to the researchers' knowledge of the concept in question and knowledge of the APOS theory. Our study followed the steps of this paradigm. The theoretical analysis served to propose mental constructions (the genetic decomposition) that are most likely responsible for the learning of the continuity concept by the student teachers. The instructional treatment included a collaborative worksheet design and was intended to get the student teachers to make the proposed mental constructions. They were then expected to use the mental constructions to construct an understanding of continuity and hence apply the concept to other situations. Pedagogical strategies that were used including small group-work to complete mathematical tasks, making conjectures using the model proposed by Cangelosi (1996), and a de-emphasis of lecturing in favor of cooperative learning. According to Figure 2, the analysis of data relates to the theoretical analysis in two directions as seen by the double sided arrow. The theoretical analysis provides the questions to be asked of the data which in turn, gives an indication about the effectiveness of the theoretical analysis in terms of mental constructions as well as the mathematics that each student teacher may have learned in the investigation.

Instructional Treatment

Visualization plays an important role in learning (Vygotsky, 1978) and in particular, the learning of mathematics. This idea is espoused in the old adage, "a picture is worth a thousand words". The role of graphs in the teaching of mathematics is complex and has multi-fold dimensions (Brijlall, 1997). In this regard, we adopted the DCT as discussed by Paivio (1986) to motivate our design of the structured worksheet. DCT, a theory of cognition, postulates that visual and verbal information are processed differently and along distinct channels with the human mind creating separate representations for information processed in each channel. Both visual and verbal codes for representing information are used to organize incoming information into knowledge that can be acted upon, stored and retrieved for subsequent use. We designed the worksheet to create graphical representations of continuous and non-continuous functions as visual information. Then both students (apprentices) and tutor (the experienced teacher) engaged collaboratively (Vygotsky, 1978) to provide verbal information, allowing mathematical connections for a deeper conception of continuity to emerge.

According to Paivio (1986), mental images are analogue codes, while the verbal representations are symbolic codes. Analogue codes represent the physical stimuli we observe in our environment and in this study viewed as graphical representations of continuous and non-continuous mathematical functions. These codes are a form of knowledge representation that retains the main perceptual features of what is being observed. Symbolic codes, on the other hand, are a form of knowledge representation chosen to represent something arbitrarily as in the concept definition of continuity in calculus (Tall & Vinner, 1981). Continuity has a pre-formal visual meaning in that it has elements of dynamic movement, being all in one piece, not changing suddenly (either in direction or in formula) and having no holes (Tall & Vinner, 1981; Tall & Bakar, 1992). Supporting evidence comes from research (Vygotsky, 1978) that showed that memory for some verbal information is enhanced if a relevant visual is also presented or if the student can imagine a visual image to go with the verbal information. Verbal information can often be enhanced when paired with a visual image, real or imagined (Anderson, 2005; Anderson & Bower, 1973). This paper also uses collaborative learning from Vygotsky's theory as a framework for classroom interactions to occur fruitfully (Vygotsky, 1978). Unlike traditional teaching approaches, collaborative learning rewards all individual students participating in the group by explicitly ensuring that all have achieved the intended lesson outcomes (Barkley et al., 2005).

Theoretical Analysis

Piaget, as cited in Brijlall and Maharaj (2009; 2011), expanded and deliberated on the notion of reflective abstraction which refers to the construction of logic-mathematical structures by a learner during the process of cognitive development (Dubinsky, 1991a). Two features of this concept are: (1) It has no absolute beginning but appears at the very earliest ages in the coordination of sensori-motor structures; and (2) It continues on up through higher mathematics to the extent that the entire history of the development of mathematics from antiquity to the present day may be considered as an example of the process of reflective abstraction (Dubinsky, 1991b).

We define the following four concepts that are used in APOS theory of conceptual understanding (Brijlall & Maharaj, 2009; 2011):

(1) Action: An action is a repeatable physical or mental manipulation that transforms objects;

(2) Process: A process is an action that could take place entirely in the mind;

(3) Object: The distinction between a process and an object is drawn by stating that a process becomes an object when it is perceived as an entity upon which actions and processes can be made, and such actions are made in the mind of the learner;

(4) Schema: A schema is a more or less coherent collection of cognitive objects and internal processes for manipulating these objects. A schema could aid students to "... understand, deal with, organise, or make sense out of a perceived problem situation" (Dubinsky, 1991a, p. 102).

Observations and Assessment of Student Learning

This followed the instructional treatment and allowed us to gather and analyze data. The data was used in two ways. Firstly, the results of the data analysis were used to test our initial genetic decomposition. Secondly, the data gathered was used to report on the performance of students on mathematical tasks related to the concept of continuity.

Methodology

The structured design of worksheet used an example and non-example approach. In particular, we focused on sorting, reflecting and explaining, generalizing, verifying and refining. The methodology adopted five stages: (1) design of worksheet; (2) facilitation of group-work; (3) capture of written responses; (4) interviews; and (5) analysis and findings. The data collection relied to a large extent on what students could say or write about their learning experiences. The worksheet task was completed over two double periods, each of one and a half hour duration. This included the individual work by students, the discussions in the groups, the group class presentations and the final discussion involving the tutor. The interviews were done with individuals a week later during the free periods involving both the student and tutor. All the interviews were video recorded.

Design of Worksheet

A worksheet was designed in accordance with ideas postulated by a guided problem-solving model suggested by the work of Cangelosi (1996). This work modeled how meaningful mathematics teaching could be planned with the aim of simultaneously addressing the cognitive and affective domains when students solve problems. An interpretation and modification of the guided problem-solving model (Maharaj, 2007) illustrated in Figure 3 has the following three interlinking levels or phases: (1) inductive reasoning: conceptual level processing occurs; (2) inductive and deductive reasoning: Where simple knowledge and knowledge of a process level occurs; and (3) deductive reasoning: occurring at an application level. This model has also been used in studies involving the learning of concepts in sequences by pre-service students at a South African education faculty by Brijlall and Maharaj (2011) and Maharajh et al. (2008).

In our case, to provide a structured approach in an inductive manner, we implemented the graphical representations as tools to guide the discussion in arriving at the concept definition of continuity. However, we noted that there is always interplay between inductive and deductive reasoning for the different levels. They are continuously present and constantly following each other in mathematical thinking. For example, in an inductive process, very often a preliminary generalising step is reached; the finalisation of an inductive part is the beginning of the deductive part (Maharaj, 2007). Therefore, generalising at each of the different levels implies that the deductive mode of reasoning comes into play.

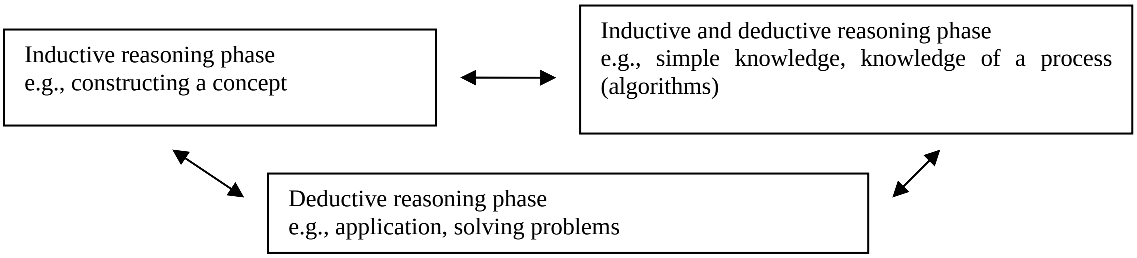

Figure 3. A guided problem-solving teaching model.

In creating constraints for the examples and non-examples in the guided worksheets, we implemented the concept of boundaries (Mason & Watson, 2004) that include the characteristics of the existence of function values and limits (see Figures 5 and 6 which show extracts from the worksheet handed to students). For the design of the worksheet, inductive learning activities were used to construct the concept of continuity of functions. This design promoted visualisation and verbalisation. These activities had the following stages within the inductive level: (1) comparison with examples and non-examples and categorising; (2) reflecting and explaining the rationale for categorising; (3) generalising by describing the concept in terms of attributes, that is, what sets examples of the concept apart from non-examples; and (4) verifying and refining the description and definition by testing and refining it. Those stages were chosen since they could be exploited to facilitate the combined framework and contribute to conceptual understanding: action, process, object and schema.

Group-Work Facilitation

Twelve second year teacher trainees engaged with the activities individually for approximately 15 to 20 minutes. This was to allow students to make contributions when working in a group setting. The groups were formed by the lecturer using the marks attained in a mathematics education module from the previous semester (Mathematics for Educators 210). The purpose was to ensure that the groups had members with different ability levels, mixed race, mixed gender and different home language. Preston and Robert (2003) noted in this regard that the teacher should carefully group students that can potentially develop in collaboration with more capable persons. When constructing the concept of continuity, they worked in four groups, comprising of three members each. Each group, after discussing and reaching a collective decision, presented their mathematical ideas to the class. The student facilitators reported on the collective ideas or thoughts of their groups. The students were given time limits set by the facilitator to encourage them to focus on the task on hand. The groups were similar in that they had members with a spread of ability levels. At the end of the group presentations, an intensive classroom discussion including responses from the lecturer led students establish the concept definition of continuity.

Written Responses

A guided activity sheet was given to each teacher trainee. When they were in groups, they were required to present the collective group response to the activities. The following five instructions appeared on the worksheets: (1) complete each worksheet on an individual basis; (2) now form groups of three; (3) discuss your findings within the group to reach consensus; (4) write down a collective response and elect a leader to discuss with class; and (5) finally conclude findings as a class with lecturers. This involved the tutor, who is a Ph.D. student and a mathematics lecturer, who clarified, using the worksheets, the mathematically acceptable definition of continuity. The group response worksheets were then collected by the lecturer for analysis of teacher trainees' constructions of the continuity concept, within a group context.

Interviews

The interviews took place after the written responses were analyzed. After categorizing them in Tables 1 and 2, it was then that we employed verification interviews.

Genetic Decomposition

Most of the second teacher trainees already had adequate knowledge of existence of limits at this stage in the course. This was verified orally by the lecturer. They also sketched graphs of piecewise functions comprising of linear, quadratic, hyperbolic, semi-circular and absolute-valued functions. This involved an inter-play between graphical illustrations and algebraic notations. The graphical approach provided a visual representation of the algebraic expression of the function. As an example when finding $\lim_{x \to 2} f(x)$ where $f(x)=2x^2$, the students normally proceed using a substitution algorithm without a possible graphical representation of the function. In this regard, when algebraic notations of functions were alone presented, the two-sided approach in the limit concept was not immediately perceived. The students had no formal prior knowledge of the concept of continuity. During the guided problem-solving activity, students were expected to develop the following definition in full sentences, for example, "There is a y-value for $x = a$", and in notation form as follows: A real single-valued function is continuous at $x = a$ if: (1) $f(a)$ exists; (2) $\lim_{x \to a} f(x)$ exists; and (3) $f(a) = \lim_{x \to a} f(x)$. So, a function f is continuous if it is continuous at every point in its domain. The thorny question of whether a function can be considered discontinuous outside its domain arises. Yes as a global gestalt because there is a hole, but none from the formal definition of a continuous function since continuity only refers to points in the domain.

Based on the above, the following genetic decomposition of the concept of continuity was used to guide our instructional treatment.

As a part of his/her functional schema, the student: (1) has developed a process or object conception of a function; and (2) has developed at least an action conception of graphs of piece-wise functions. As a part of his/her limit schema, the student: (1) has developed a process conception of the limit of a function; (2) has developed at least an action conception of the existence of a limit of function; and (3) recognizes and uses suitable notation and their respective applications to specific situations, and then coordinates previously constructed schemas of a function, limits of functions and appropriate notation to define continuity of a function (see Figure 4).

Analysis and Findings of Results

The following is an extract from the students' worksheet, labeled Stages A to D. In particular, the mathematical stages are sorting, reflecting and explaining, generalizing, verifying and refining. In Stage A (see Figures 5 and 6), the researcher demarcated the examples and non-examples and the students then compared these distinguishing features, namely, the existence of a function value, the existence of a limit and the equality of the function value with its limit that characterize continuous functions from non-continuous ones.

Extracts Taken From Students' Worksheet Covering the Four Stages

Stage A: Sorting

The following are examples of graphs of continuous functions.

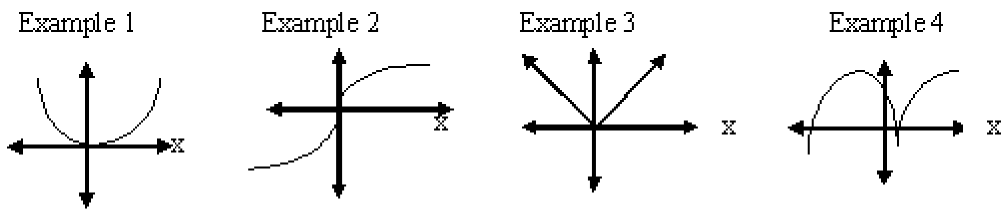

Figure 5. Examples of continuous functions.

The following are examples of graphs of functions, which are not continuous.

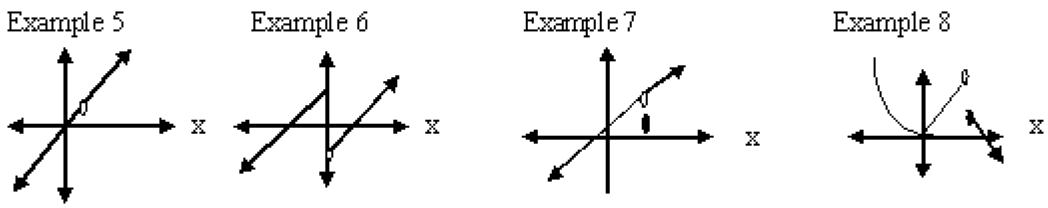

Figure 6. Examples of non-continuous functions.

Stage B: Reflecting and explaining

After interrogating the above examples and non-examples of graphs of continuous functions, explain why one would categorize them as such;

Stage C: Generalizing the description of continuous functions

Now provide mathematical conditions which a function need satisfy in order for it to be called continuous at $x = a$;

Stage D: Verifying and refining

Check whether the following functions are continuous or not by using the conditions you have derived in the generalization above:

Example 9 $f(x) = \begin{cases} x + 1 & , if \;\; x \geq 2 \\ x + 4 & , if \;\; x < 2 \end{cases}$

Example 10 $f(x) = \begin{cases} x^2 & , if \;\; x \neq 2 \\ 1 & , if \;\; x = 2 \end{cases}$

Example 11 $f(x) = \begin{cases} x^3 + 1 & , if \;\; x \geq 2 \\ 2x + 5 & , if \;\; x < 2 \end{cases}$

The summary of responses covering the above stages appears in Tables 1 and 2. Table 1 summarizes the four group responses, with the data captured from the video as well as from the group activity sheet. The reflecting and explaining stage and generalizing the concept continuity of functions are tabulated. Characterization of coded categories is as follows: (1) none was used for no response; (2) inadequate codes implied an incorrect or unclear response with features which are not in accordance with our genetic decomposition; (3) partial codes indicated gaps in description where responses had features that resembled our genetic decomposition; and (4) complete codes implied a mathematically correct response in accordance with the concept of continuity.

Table 1

Results on Group Constructions for Continuity of Functions (n = 4)

Stages	None	Inadequate	Partial	Complete
Reflecting and explaining	0	2 (groups A and B)	2 (groups C and D)	0
Generalizing	0	1 (group A)	2 (groups B and D)	1 (group C)

As seen in Table 1, two groups provided inadequate explanations of continuity after they studied the examples and non-examples provided in the guided worksheet. Group B did not consider limits or function values when reflecting and explaining the rationale for categorizing. Their responses given were:

> The first four graphs are continuous where the x and y intercepts are included and the graph passes completely through the x and y intercepts. The next four graphs are not since some parts of the graphs are excluded and included. In some cases, there is more than one sketched graph on the same set of axes, indicating that the graph is not continuous.

This has three separate sentences. The first says that the first four graphs are continuous (with some extra observations about intercepts) but does not say the reason why they are continuous. The second sentence, which seems separate from the first, says that the next four are not continuous since some parts are excluded. In other words, there are gaps in the graph contrary to the pre-conception of continuous operations going on smoothly without gaps (Tall & Vinner, 1981). The third sentence says that there is more than one sketched graph in each picture where a picture has a single set of axes. This relates to both the idea of a graph that continues and the long experience that the student will have had of a function given by the same formula that continues through its domain. It is a consequence of how the students have been previously taught functions as being given by a single formula. Thus, all the comments of group B relate to preconceptions of functions as a formula, drawn smoothly and having no gaps. In Dubinsky's work, a formula is only an action, so the above response from Group B is not even at the process level, since the students possibly think that there are two functions because each piece is defined by a different rule, which suggests an action level response in APOS.

Group B used symbolic language to generalize the definition of continuous functions as confirmed below:

Straight line graph $y = mx + c$; parabola $ax^2 + bx + c$

$f(x) = f(a)$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$, if the limit always exists:

$\lim_{x \to a} f(x) = f(a)$

Point "a" on a domain of $f(x)$, $f(a) = exist$

Firstly, this group's references to the existence of limits occur from their existing schema. It suggests that these students had an object conception of the existence of a limit of function. They correctly chose suitable notation to illustrate their conception of continuity as an object, since they wrote $\lim_{x \to a} f(x) = f(a)$. However, two instances of misuse are evident in the second and last lines of their responses. The last line implies that they meant $f(a)$ exists, so this line should replace the second line. Secondly, there is a clear difference between the group's concept image in the reflecting and explaining activity with that portrayed during generalizing the definition. In the former activity, the group responses indicate that emphasis was placed on a visual analysis rather than on the algebraic meaning, as it was the case during generalizing. This would imply that at this point the group used only one representation. Thus, the students' conception of continuity is limited, as it does not include a number of different representations. They did not integrate the symbols that are associated with an algebraic representation of continuity. However, they later resorted to graphical representations as evidenced in the first two functions in Table 2 when provided with examples of piecewise functions in the verifying stage. They provided correct responses and applied sketches of graphs. We suggest that the algebraic statements made in their arguments were not clear in their mental constructions, but when using graphs they had a better sense of what they were arguing. This seems to justify the need for visual representations when trying to understand and make sense of the concept of continuity.

From Table 1, we observe that two out of four groups when reflecting and explaining constructed partial understandings of the concept of continuity. These were:

> Group C: The continuous functions have no disturbance and their limits exist at all points. Not continuous functions have disturbance (hole) on the graph or either whose limits do not exist at certain points or $f(x)$ does not exist at some points. There are points which are not on the domain of the function.

> Group D: One can recognize the first four graphs as continuous only because at point of, for example, $x \to a$, we have the fixed *y*-value which means that at every point of *x*, there is also a fixed *y*-value. For the last four graphs, one can say that they are discontinuous because at some of the *y*-values, *x* does not exist. This means that at the point of *x*, there is no unique *y*-value.

Group C related continuity to no disturbance, at the same time referring to the existence of limits and points not in the domain, which could relate to an intuitive notion of limit as given in the course earlier. Group D used the intuitive language of limits but asserted that for the last four graphs, for some *y*-values, there are no *x* values (*x* does not exist) again referring to difficulties relating to what they may perceive as holes in the graph. We note that the response of Group C was incomplete, since this group seems to assume that the existence of $\lim_{x \to a} f(x)$ is a sufficient condition for continuity of the function *f* at $x = a$. It is likely that this option arose from previously learnt concepts that they intended to link now. On the other hand, this misconception could be sourced by an understanding that if $\lim_{x \to a} f(x) = c$, then, $f(a) = c$ so f is then continuous at $x = a$. It seems that this group did adequately reflect on example 7 in Figure 6. As a result, they were unable to satisfy part 6 of our genetic decomposition for continuity, namely then coordinate previously constructed schemas of a function, limits of functions and appropriate notation to define continuity of a function. They were unsuccessful in linking the function schema with the limit schema as illustrated in Figure 4. With reference to the DCT, note that this group could represent visual information by verbal codes since they described points of discontinuity as disturbance.

The response given by Group D may translate to a view that at $x = a$, $f(a) = c$ so the limit exists there because it has a value. This means that Group D incorrectly linked $f(a)$ or the *y*-value at $x = a$ with the $\lim_{x \to a} f(x)$. They did not consider the behavior of a function about points, but they only focused on the value of the function at $x = a$. This might also explain other shortcomings we come across when teaching calculus. An example of this is that when dealing with procedures, like using substitution to find limits algebraically, students seem to believe that the value of the function at a point (in this case $x = a$) is of greater importance, rather than how function values behave around the point. Students in this group were unable to proceed beyond the function schema in our genetic decomposition, as illustrated in Figure 4. They did not realize that the existence of the limit of a function $f(x)$ as $x \to a$, does not depend on whether $f(a)$ is defined. The response clearly shows that Group D employed a correspondence between an interval about an *x*-value and an interval about a *y*-value. However, they did not use the independent and dependent variables satisfactorily and conclude that $\lim_{x \to a} f(x)$ and $f(a)$ must be equal. It is important to help the student(s) move on from colloquial to mathematical insight in a meaningful way as argued by Tall (2003).

Group A's response when explaining and reflecting was as follows: A continuous function is a function without a break in co-ordinates or a function that goes on without end to either $-\infty$ or $+\infty$.

The response above seems to be derived from the colloquial use of the word "continuous" in phrases like "goes on" (meaning that there were no stops). It is observed that Group A viewed the visual representation of a continuous function as a graph in one piece, with domain set of real numbers. This contradicts examples 1, 2 and 4 in Figure 5. With reference to part 5 of our genetic decomposition, this group's use of symbols was limited to expressing the domain of the function in example 3 of Figure 5. This led to their inadequate generalizations.

Table 2

Results on Verifying and Refining the Concept Continuity of Single-Valued Functions

Function	No group response	Incorrect group response	Correct group response
1. $f(x) = \begin{cases} x + 1 & , if \ x \geq 2 \\ x + 4 & , if \ x < 2 \end{cases}$	0	0	4
2. $f(x) = \begin{cases} x^2 & , if \ x \neq 2 \\ 1 & , if \ x = 2 \end{cases}$	0	1 (group A)	3 (groups B, C and D)
3. $f(x) = \begin{cases} x^3 + 1 & , if \ x \geq 2 \\ 2x + 5 & , if \ x < 2 \end{cases}$	0	2 (groups A and D)	2 (groups B and C)

Even though the other three groups offered inadequate or partial explanations when responding to the reflecting and explaining processes in the construction of the concept of continuity, their generalizations displayed evidence of features included in our genetic decomposition of continuity. Group C, in particular, generalized the description of continuity concisely and in accordance with our genetic decomposition. These findings display an interplay existing between graphical and symbolic representations. This was assisted by the guided design using inductive reasoning within the framework of the guided problem-solving teaching model (see Figure 3) and the collaborative learning approach to facilitate the development of the concept of continuity.

In the verifying and refining aspect, three functions were defined in questions 1, 2 and 3 (see Table 2). The majority of students identified the first two functions correctly while the third function was identified correctly by half of the groups. What was pedagogically interesting was that groups A and D could get the first one correct but not the third. This might be a sign that the students confuse continuity (connectedness of the graph) with differentiability (smoothness of graph). Example 1 has an obvious jump in the middle so it is discontinuous (in a visually coded sense not necessarily symbolically coded). Example 3 is problematic because there is no jump in value, but it has what may be conceived as a discontinuity in the change in the formula. Thus the distinction between examples 1 and 3 is self-evident. The first is clearly discontinuous for any reason one cares to name (visual or formal), the second is mathematically continuous but may "feel" discontinuous visually and dynamically.

Group A tried to reason graphically or geometrically for the first two functions. However, this group of teacher trainees portrayed an inadequate conception of piecewise functions as they considered the second function to be two separate graphs due to the definition provided. Group D supported this notion when their response was "$f(2)$ does not exist from the second graph in question one". These groups did not comply with part 2 of our genetic decomposition of continuity. Group D used visual-pictorial processing to check whether functions 1 and 2 were continuous. Group B considered limit existing to be an adequate condition for continuity without investigating whether $f(x)$ exists or whether it was defined at $x = 2$. Therefore, with regard to Figure 4, it seems that this group was unable to link their function schema and limit schema to verify the continuity of the function.

Group A further displayed their inadequate conception of piecewise functions when looking at the second function, by stating that "$f(x)$ is two graphs". Their answers were that the one definition of the graph is continuous while the second definition is not continuous. In a separate interview, the representative for group A said "we did not know what to do with $x = 2$". This implies that the students either did not know how to read/represent the point (1; 2) or that that part of the function was not considered vital in deciding on the continuity of the function. They saw what they believed to be two graphs (actually two formulae), one is x^2 for $x \neq 2$. This clearly "continues" in the sense that it continues off to infinity in both directions. The other is $f(x) = 1$ (for $x \neq 2$) which is a single point and so stays in place and does not continue at all. The reason for the distinction in terms of global visual coding is self-evident, and does not involve the mathematical definition of continuity but the colloquial preconception of discontinuity. Group B used both symbolic or numeric and geometric reasoning to give a concise answer to the second function under verifying and refining. Group C used numerical reasoning to arrive at the same concise answer for function 2. Group D, on the other hand, considered the fact that $f(2)$ did not exist on their graphical representation of function 2 (i.e., open dot on graph), to be a sufficient condition for it not being continuous.

Groups B and C did not use visual or pictorial modeling to answer the third question, but the algebraic manipulations done to find limits and function values were correct. Using this they correctly concluded that the function was continuous. This implies that their schema for continuity satisfies the illustration in Figure 4. Groups A and D, on the other hand, did not use the generalizations they made about continuity earlier to determine whether the third function was continuous or not. Both of these groups reflect on the inequalities in the definition of $f(x)$ as shown below:

> Group D: The use of inequality disturbs movement of our graphs to infinity, therefore, the graphs do not flow (move) freely. There are restrictions; therefore, the function is not continuous. There is an open gap (dot) in the second function in exercise 3.

> Group A: not continuous because $\geq and <$ means open dot on the graph.

A beautiful expression of dynamic movement is suggested by the response of Group D. The use of inequality disturbs movement, suggesting the changing of the formula disturbs dynamic continuity. The graphs do not flow, so the function is not continuous according to their own personal concept definition. Notice that there are separate statements here. The first two sentences refer generally to disturbing changes of formula. The last sentence refers to the open gap in the third function and also the dot which gives a discontinuity.

Conclusions

The findings of this study showed that some students demonstrated the ability to make use of symbols, verbal and written language, visual models and mental images to construct internal processes as a way of making sense of the concept of continuity of single-valued functions. We also found that our modified theoretical framework was an effective one for this qualitative study and can be considered for future work.

On perceiving functions as mathematical entities, teacher trainees could manipulate these entities, which were understood as a system of operations. The study provided some valuable insights into the mental constructions of teacher trainees with regard to limit of a function and continuity at a point in calculus. These insights should be analyzed and understood keeping in mind the specific methodology that was used. The verifying and refining stages in the construction of the continuity concept required a conceptualization of the concept of continuity as a meaningful mathematical entity. This conceptualization enabled the formulation of a new mathematical idea that can be applied to a wider range of contexts. The responses received in the four stages A to D of the worksheet indicate that most of the teacher trainees were able to construct the concept of continuity and hence were in fact capable of definition-making with some degree of success. This was evident by the overlap in ideas arising from the mental constructions formulated and those that are encapsulated in the definition. The worksheet possibly nurtured their creativity by encouraging and providing opportunities for them to value, share and discuss the new concept freely without fear of being judged or embarrassed by anyone. They were able to assist each other in addressing the common misconceptions at certain stages of the worksheet. This approach offered opportunities for them to collectively recognize previous knowledge, as well as engage in alternative conceptions with group members.

It should be noted that despite accepting that the initial learning of continuity involved the three worlds of mathematical thinking, students at university education faculty are required to be taken to higher levels of learning. We realized that this lesson involved the continuity of a single-valued function at an interior point in the domain of the function and the domain in the case of this worksheet involved the set of real numbers. In follow-up lessons we provided other experiences on continuity to students. We highlight two examples that were discussed in the follow-up tasks.

Investigation of the continuity of the functions is given below:

In example 1, we require students to arrive at an understanding of continuity at end points. They need then to rework condition two of the formal definition arrived at in the worksheet of this study. In example 2, we allow the teacher trainees to experience functions with more than one point of discontinuity.

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