Gap Thinking in Fraction Pair Comparisons is not Whole Number Thinking: Is This What Early Equivalence Thinking Sounds Like?

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Gap thinking has been categorised as one of several whole number strategies that interfere with early fraction understanding. This study showed that this claim is not supported by interview data of Grade 6 students' gap thinking explanations during a fraction pair comparison task. A correlation with equivalence performance was uncovered, leading to the suggestion that the additive nature of gap thinking may actually reveal the (erroneous) additive nature of students' early engagement with equivalence concepts.

Gap thinking is a misconception prevalent in explanations of mental comparisons of the size of two fractions. It is most striking in incorrect attributions of equivalence. For example, Clarke and Roche (2009) found that 29% of the Grade 6 children gave a gap thinking explanation when asked which was larger, 5/6 or 7/8, claiming both fractions were "equivalent, because they both require one bit to make a whole" (p. 129). Gap thinking may affect a quarter of our students and other studies show that it is present in Years 4, 6, and 8. It is worth a closer look.

Literature Review

Pearn and Stephens (2004) use the phrase gap thinking to describe a Year 8 student's comparison of 3/5 and 5/8: 3/5 is larger because "there is less of a gap between the three and the five (in the first fraction)" than there is between the "five and the eight (in the second fraction)" (p. 434). They distinguish this from comparing-to-a-whole thinking in which the students claim that 2/3 is bigger than 3/5, because 3/5 "is two numbers away from being a whole" while 2/3 "is one number away from being a whole" (p. 434). Clarke and Roche (2009) combine these two strategies in their definition of gap thinking. Post and Cramer (1987) describe the same strategy in Grade 4 children who believe 3/4 and 2/3 are equal because "the difference between numerator and denominator in each fraction was one." (p. 33). Cramer and Wyberg (2009), give the example of a child who claims 3/4 to be bigger than 5/12 because "5/12 still has 7 more to go" as opposed to 3/4 which has "one more to go. So it should be bigger." (p. 241). These examples all describe gap thinking, as it is called in Australia, and categorise it as one form of whole number thinking.

There are descriptions of other whole number thinking strategies that are not gap thinking. Clarke and Roche (2009) describe higher or larger numbers in which the student compares numerator 1 with numerator 2, and then denominator 1 with denominator 2. For example, 7/9 is larger than 3/4 because "7 and 9 are bigger than 3 and 4" (p. 131). Earlier American research describes this same strategy as whole number dominance, where 3/5 is less than 6/10 because "3 is less than 6, and 5 is less than 10" (Behr, Wachsmuth, Post & Lesh, 1984). Pearn and Stephens (2004) include higher or larger numbers in their definition of gap thinking, unlike us. Denominator (bigger/bigger) comparisons are also described as whole number dominance where the larger denominator is taken to signify the larger fraction (Behr et al., 1984).

In the literature, the three strategies; gap thinking, higher or larger numbers, and denominator (bigger/bigger), are all classified as examples of whole number thinking. This

L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia.* Fremantle: MERGA.

model of cognitive development would suggest that these strategies should be apparent before other more sophisticated fraction understandings are evident amongst students. It seems self-evident that students need to resolve these misconceptions in order to successfully integrate their increasing fraction knowledge.

Successful strategies for fraction size comparisons include residual thinking and benchmarking (see e.g. Behr et al., 1984; Clarke & Roche 2009). Residual thinking is a mathematically correct strategy useful for comparing fractions that are one away from the whole. 5/6 is one sixth away from the whole and 7/8 is one eighth away from the whole. As one eighth is smaller, 7/8 is closer to the whole. Residual thinking represents a successful transition from comparisons to the whole, in same denominator contexts, to considering the size of the pieces in pairs with different denominators. Benchmarking (transitive reasoning in the Rational Number Project research) is a strategy in which a child compares fractions to a self generated benchmark such as a half. For example, 5/8 is larger than 3/7 because 3/7 is less than a half and 5/8 is more than a half.

Benchmarking draws upon equivalence. Recognising the fraction pair 2/4 and 4/8 as equivalent can draw on either scale relationships – the bottom number is double the top number in both fractions; or functional relationships – 2 pizzas for 4 people is as much as 4 pizzas for 8 people (see e.g. Kieren, 1992). Ways of introducing equivalence include generating equivalent fractions and recognising the same fraction in different measure representations (Wong & Evans, 2007). Kieren cautions that drawing on equivalence as an internalised strategy in other tasks is not apparent in half the age cohort until age 12, and full common denominator reasoning occurs later (1992). Fraction pair comparisons, therefore, are relevant to all levels of the primary school.

Methodology

Instead of using Kieren's earlier five-part model encompassing part-whole, measure, quotient, operator and ratio sub-constructs as much of the research literature does, it is possible to reconcile the activities and strategies of fraction size comparisons with the three underpinning constructs of his later four part model – partitioning, equivalence and unit-forming. The earliest comparisons, making unit fractions and comparing them, fits into the partitioning construct. Comparing related fractions, such as 3/8 and 7/8 involves partitioning and unit-forming actions. Exploring how 2/4 is as much as 4/8 engages with quantitative equivalence. Unit-forming, seeing fractions as units, as sums of other amounts, also relates improper and proper fractions. For example, 2/4 is 2 one-quarters, and two-quarters of a whole, and 4 two-quarters are 2 (see Kieren, 1992; 1995). In this framework 4/2 is larger than 2/4 because 4/2 is four halves which is a recognisable amount to the student.

Clinical interviews provide an opportunity to gather rich data on children's descriptions of their mathematical strategies (DiSessa, 2007). For this study, 88 Grade 6 students from three metropolitan state schools in Melbourne were interviewed on a one-to-one task-based interview. Children were not told the correctness of their answer, but asked to explain their thinking. No teaching took place during the interview. All interviews were audio-taped and more than half were also video-taped. All the interviews were conducted by the first author. To be coded as correct, the child had to give the correct answer and an explanation of a mathematically correct strategy. A record sheet with dot points of strategies identified in the literature was used and any other strategy was noted during the interview. All strategies were given a code and entered into a spreadsheet. Specific tasks, for example, the fraction pairs, were double coded from the video recordings by another researcher familiar with the strategies described in the literature. For the gap thinking strategy, all instances identified by either coder, including those only audio-taped, were transcribed providing a full set of transcripts of gap thinking explanations. A further code was used, possible gap, which indicated that the explanation was possibly gap thinking but that there was not enough evidence in the child's response to be sure. All coding describes the first (or self corrected) preferred strategy. Gap thinking, possible gap (pgap), higher or larger numbers, denominator (bigger/bigger), and denominator (bigger/smaller) strategies were given separate codes.

The eight fraction pair questions are the same as used by Clarke and Roche (2009): 3/8 or 7/8, 2/4 or 4/8, 1/2 or 5/8, 2/4 or 4/2, 4/5 or 4/7, 3/7 or 5/8, 5/6 or 7/8, 3/4 or 7/9. The children were shown a card with the two fractions as symbolic inscriptions and were asked, please point to the larger fraction or tell me if they're the same. After they stated or pointed to their answer they were asked, and how did you work that out?

The interview was wide ranging and parts of four separate tasks had an equivalence component. Thirteen questions were identified as drawing on equivalence understanding. Eight of the questions required the children to recognise either a) 4/6 or 6/9 as two thirds, b) 2/12 as one sixth, or c) 3/12 and 2/8 as a quarter, and there were length, area (equal parts and non-congruent parts) and discrete contexts. Two further questions used concrete materials (golden beans) and required the child to generate an equivalent fraction (having thrown something out of six) and to rename their answer of 3/9 or 1/3 which had been modelled with the golden beans. Another question, the fraction pair 2/4 or 4/8, required the child to recognise equivalence in a symbolic inscription. Two final questions required the application of equivalence. One was the fraction pair 3/7 or 5/8. The other was the addition, using symbolic notation, of 1/2 + 1/3. Both were identified as connected to equivalence by a factor analysis, and subsequent checking of the coding and the children's inscriptions showed that all correct answers successfully used benchmarking or common denominators respectively. The fraction pair 1/2 or 5/8 was NOT included in the equivalence categorisation because it was possible for children to think of 5/8 as a half, a unit in itself, plus a bit, and not use the equivalence 1/2 = 4/8. Of course, many did use equivalence knowledge, if they had it, to solve this question. Two questions were common to both the fraction pairs and the equivalence category.

Results

The following quotations are of children's responses to the fraction pair 5/6 or 7/8.

- They're the same because five sixths has got one more to become a whole. And seven eighths it also has one more to become a whole.

- They're the same.
  [Interviewer] And how did you work that out?
  Because five out of six is one piece left and seven out of eight is one piece left.

- They're the same.
  [Interviewer] How do you know?
  Because there's both, because the top numbers are both one less than the bottom numbers.

- They're the same.
  [Interviewer] And how did you decide?

Cause they're both two thirds, that's another way to say them. Cause seven plus one is eight and five plus one is six.

- They're the same.
  [Interviewer] And how did you decide?
  Because they're like. Five sixths there's one more. There's one more sixth to make a whole. And it's one more eighth.

- They're the same.
  [Interviewer] And how did you decide?
  Because they both need one more to be coloured in.

All of these responses were coded as gap thinking in the fraction pair 5/6 or 7/8. Both a gap answer and a gap explanation were needed for positive identification. This was especially true in other pairs when a gap thinking strategy and a mathematically correct strategy would give the same fraction. For example, in the pair 3/7 or 5/8 the correct answer is 5/8 but the fraction with the smaller gap is also 5/8, and it is possible to get the right answer for the wrong reason. In contrast to the examples above, fractional language, such as one-sixth, sometimes did indicate a grappling with the size of the pieces and consideration of the numerator and denominator, and in those cases, the child was not coded as demonstrating gap thinking, nor possible gap thinking. In contrast, in our categorisations, the only children who were coded as gap thinking in the pair 3/8 or 7/8 were those who chose the larger gap with a gap thinking explanation similar to those above. As complement-to-one thinking is correct in the 3/8 or 7/8 context, similar explanations to those above, with the choosing of the smaller gap, were not coded as gap thinking. Gap thinking explanations sound like successful how-close-to-the-whole thinking misapplied to inappropriate fraction pairs, rather than whole number thinking misapplied to fractions. While one other child claimed the fractions were the same in 5/6 or 7/8 but had a faulty residual explanation, this pair, with its distinctive gap answer, enables us to see the full range of gap thinking explanations transcribed above.

We can hear in the first four explanations similar descriptions to those in the literature. There is the-complement-to-one strategy – "one more to become a whole". There is the gap as a "bit" – "one piece left". There is attention to the numerical difference between numerators and denominators – "the top numbers are both one less than the bottom numbers". Also, there is the string of equivalences – 5/6 is 7/8 is 2/3 is 3/4 is 9/10, if we include all responses to this question. What the fraction pair 5/6 or 7/8 also reveals is that using the fractional language of sixths and eighths does not automatically rule out gap thinking. The influence of part-whole counting and shading activities rather than activities framed in partitioning, unit forming and equivalence actions, may be being described by the child who says, "They both need one more to be coloured in".

Some fraction pairs were more difficult than others to compare, as shown below in Table 1. Not every pair elicited gap thinking. Around a quarter of the students did not give a correct answer to the pairs 2/4 or 4/8 nor 2/4 or 4/2, but none of their incorrect answers were gap thinking. The highest proportion of gap thinking occurred on the pair 5/6 or 7/8 where half of the students demonstrated this strategy. This is higher than the 29% reported by Clarke and Roche (2009), but this cohort was not chosen to be representative. Overall, 54% of the students demonstrated gap thinking one or more times during the eight fraction pair questions. Choosing the larger gap was uncommon and only four children did this.

Table 1
Frequency of Success on Fraction Pair Questions and the Incidence of Gap Thinking

Pair	3/8 7/8	2/4 4/8	1/2 5/8	2/4 4/2	4/5 4/7	3/7 5/8	5/6 7/8	3/4 7/9
Success	90.9%	73%	54.5%	72.2%	20.5%	13.6%	12.5%	6.8%
Gap	2.3%	0%	6.8%	0%	22.7%	21.6%	50%	23.9%
Pgap	0%	0%	0%	0%	10.2%	3.4%	0%	8%

Of the 88 students, 4 did not get any of the eight pairs correct. There were 20, 14, 17 and 16 students who were successful on only 1, 2, 3, or 4 of the fraction pairs respectively. So, around 80% of the students were successful on four pairs or less. There were 6, 3, 3, and 5 students who were successful on 5, 6, 7 or 8 pairs respectively. If we are to find whole number thinking strategies, then it is expected that this cohort would be able to show it to us with its spread of performance on fraction pair comparisons.

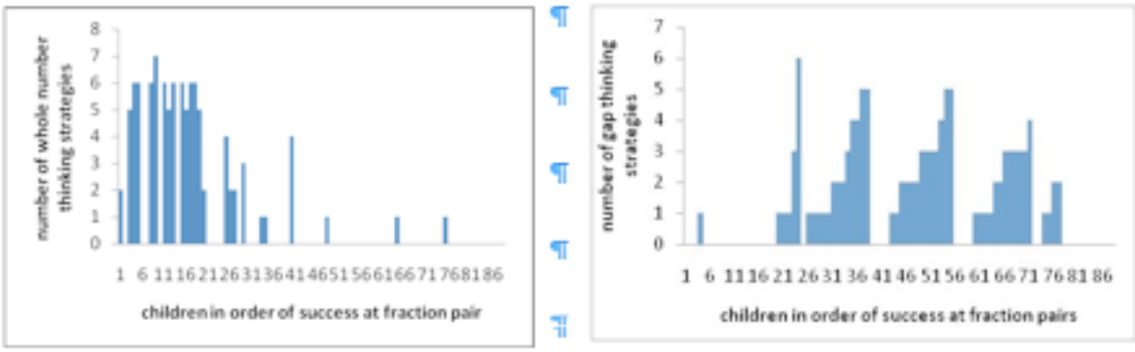

Figure 1 Higher and larger numbers, and denominator (bigger/bigger) thinking (left), gap thinking (right).

The graph of the incidence of higher or larger numbers and/or denominator (bigger/bigger) thinking (see Figure 1 above) where each bar represents one student and is ordered from lowest to highest success, shows a high frequency (many students) and a high intensity (many explanations) of these whole number thinking strategies in the 24 students who had no success or only one correct fraction pair (represented by the first 24 bars on the graph). On the gap thinking graph, however, only 6 of these same 24 children demonstrated gap thinking (represented by the bars before child 26). Higher or larger numbers and denominator (bigger/bigger) thinking is the first preference of only four students (the last four bars) who are successful on 3 or more of the pairs. Gap thinking lingers and there is a range of intensity (number of explanations per student), from no use of gap thinking to five gap thinking explanations, in the middle performers. Gap thinking, higher or larger numbers, and denominator (bigger/bigger) thinking were not often used for different fraction pairs by the same child. If we just look at the three strategies a) gap thinking and b) higher or larger numbers and/or denominator (bigger/bigger) thinking, we see that 42% used gap thinking only, 17% used the other two forms of whole number thinking only, 12.5% used both strategy types, and 28.4% used neither (used other strategies).

Using Kieren's framework of partitioning, equivalence and unit forming to describe fraction comparisons leads us to consider equivalence as a possible factor in success at fraction pair questions. As 29 out of 88 students had no success or only one equivalence question correct (see Table 2 below), it is possible to describe the incidence of whole number thinking in students with a range of performance on equivalence tasks.

Table 2
Spread of Equivalence Questions Correct

Correct	0	1	2	3	4	5	6	7	8	9	10	11	12	13
No. of students	14	15	4	8	2	3	9	10	6	6	2	4	5	0

The incidence of higher or larger numbers and denominator (bigger/bigger) thinking, see *Figure 2* below, is most prevalent in the students who have success on three or less equivalence questions.

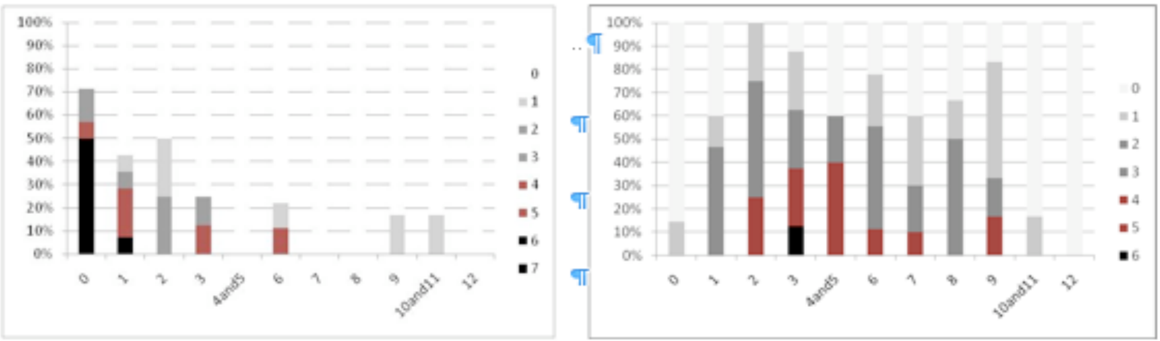

Figure 2. The correlation of higher and larger numbers, and denominator (bigger/bigger) thinking (left) and gap thinking (right) with performance on equivalence questions.

Gap thinking, on the other hand, does not match the predicted pattern for whole number thinking. It appears that at the same time that equivalence becomes a concept 'on the radar' for these children, gap thinking also emerges. Gap thinking is almost non-existent in children for whom equivalence is possibly not a concept that they have engaged with in their personal fractional understanding (despite classroom exposure). There are two children who demonstrated gap thinking and who were not correct on any equivalence questions. One we might think of as an outlier as this was her only gap thinking explanation and she chose the larger gap, a much less common variation of gap thinking. The other child, while getting none of the equivalence questions correct, did correctly identify 5/8 as larger than ½, unlike all of the other 12 students who had no success on this nor on the 13 equivalence questions. Equivalence is 'a shadow on her radar'. The other 12 students were giving many incorrect explanations for the fraction pairs but none were using a gap thinking strategy. For the students who had success with two of the equivalence questions, and so were not yet competent with all of the contexts for equivalence, all of them demonstrated gap thinking on at least one fraction pair. The highest intensity (the number of gap thinking explanations out of eight) occurred in the group of students who were successful on three equivalence questions. Gap thinking is a strategy that lingers, however, in students with more than beginning equivalence understanding. The frequency of gap thinking decreases dramatically in the group of students who are successful on 10 to 12 equivalence questions. If we include all possible gap thinkers, those coded *gap* and *pgap*, the graph looks similar (not pictured), with greater intensity at similar points to the graph above.

Discussion

The data supports the idea that higher or larger numbers and denominator (bigger/bigger) thinking may be whole number thinking strategies. This is because these strategies are used by the low performers but seem to be less preferred as the students engage with fraction ideas, as demonstrated by increasing success in either fraction pair questions or equivalence questions.

Gap thinking is not a strategy that is clearly present before other emerging fraction knowledge. It is not overly represented in students categorised as low attaining by fraction pairs performance. Their errors are mostly for other reasons. It seems to have increased frequency after the other whole number strategies wane. Only 12.5% of students use both gap thinking and higher or larger numbers and/or denominator (bigger/bigger) thinking, so these strategies do not co-exist in the students' repertoires very often. The data does not support the idea that gap thinking is a whole number thinking strategy.

Most students who do not have equivalence on their radar do not use gap thinking. At the same time as the initial engagement with equivalence thinking begins, however, gap thinking emerges strongly in both frequency and intensity. In the early stages of this engagement with equivalence ideas, by equivalence score 3, the highest intensities are found – students using gap thinking on 4, 5 or 6 questions out of 8. As students become more competent with equivalence, gap thinking lingers, but is much less prevalent by the time students are successful on most of the equivalence questions that were offered. It is timely to remember Kieren's caution about the time it takes to internalise equivalence. Success on 10 to 12 equivalence tasks is really only beginning equivalence knowledge.

It may be worthwhile to screen for gap thinking. The fraction pair 5/6 or 7/8 had the highest gap thinking response. There were only three students who demonstrated gap thinking on some other pair and not this one, so this question should identify most gap thinkers. However, while it screens for frequency, it does not offer useful data on intensity. For 15 out of the 44 students who used gap thinking on this pair, this was their only instance of this strategy. It also does not provide information on whether a child is at the beginning, middle or end of their gap thinking journey. Equating fractions that both have the numerator one less than the denominator is a misconception that appears early in gap thinking; from our transitional student described earlier with equivalence as 'a shadow on her radar' and no success on equivalence tasks, through every level of equivalence knowledge from success on one question until success on ten questions. It is the first variation of gap thinking to appear and the last to disappear.

We cannot claim to reveal the mechanism that triggers gap thinking because our data collection provides us with descriptive information, rather than evidence for cause and effect. But we can suggest explanations that fit with the data. If we attend to what the children actually say in their gap thinking explanations we can see that they find great mathematical comfort in the (erroneous) idea that 5/6 = 7/8 = 2/3 = 3/4 = 9/10. Gap thinking and equivalence are linked here.

However, it would seem that children's gap thinking explanations do not have much connection with equivalence concepts because they are about the difference between numerator and denominator, an additive relationship rather than the proportional relationship described by equivalence. But this is how experts see equivalence. Maybe, the additive explanations about difference, in all their variations, are how students think about equivalence when they are just beginning to engage with this new idea that a fraction could have more than one name. If, when equivalence first 'appears on the radar' for students, their non-integrated understanding had (erroneous) additive aspects, then these understandings would sound a lot like gap thinking. Maybe this is what the data is telling us.

Gap thinking does not appear to be a whole number thinking strategy, and it appears at the same time as the first engagement with equivalence concepts. Its application is successful in comparing fraction pairs in a common early context – fractions with the same denominator between 0 and 1, which may add to the difficulty of discarding it as a useful algorithm. Early equivalence contexts often involve doubling or halving, which can appear additive. Gap thinking explanations may provide a window into the extent of additive thinking in children's early engagement with equivalence concepts.

Acknowledgements

The authors wish to thank Anne Roche and Doug Clarke for their help in developing tasks and record sheets and double coding the data.

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