Is Mathematics Indispensable and Are Pre-requisites Needed in Mathematics Courses?

1056-Z1-356

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Paper presented at the Annual Meeting of
the Mathematical Association of America
San Francisco, CA

16 January 2010

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Abstract

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The author of this paper argues "is mathematics indispensable & are pre-requisites needed in mathematics courses?" that the answer to one question is, 'yes,' and to the other, 'no;' however, the response to the questions is inverted in this sentence. Such responses might surprise some people; but, the author shall argue that learning requires doing; only through inquiry is learning achieved, and mathematical thought is one that must be focused on the process of deriving a proof, constructing an adequate model of some occurrence, or providing connection between and betwixt the two.

The questions were posed in two provocative papers, "Do We Need Pre-requisites" ([62, 1997]) and "Is Mathematics Necessary" ([24, 1997]), which the author has read (seemingly) every year and has provoked much reflection. The answers to the questions will lead to a discussion of the Committee on the Undergraduate Programme (CUPM) 2004 Recommendations ([16, 2004]).

The paper is organised in the following manner. In the first part of the paper the author reviews the literature and offers a synopsis of the major points that are addressed in the paper. In the second part the author argues the main points: we need pre-requisites in mathematics courses; but, mathematics is not a necessary area of academia for all students. In the third part of the paper the author discusses the CUPM 2004 Recommendations and their relationship to the questions.

Finally, the author proposes questions he believes should be addressed before the next set of CUPM Standards are produced.

I. Introduction, Background, and a Synopsis of the Basic Ideas Presented in the Paper.

Mathematics has developed over the centuries through processes which include many types of inquiry: applied, computational, statistical, or theoretical. Mathematics is not a unary[1] topic – there are many branches of mathematics, they are not mutually exclusive, nor is there but one way of creating, discovering, or doing mathematics. Mathematicians conjecture, analyse, argue, critique, prove or disprove, and can (hopefully) determine when an argument is valid or invalid. Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is a need for justification that is open to criticism and can withstand scrutiny - - there is a stated or understood demand for succinct argument from a logical foundation for the veracity of an assertion. Such a demand fits within the context of what the author refers to as positive constructive scepticism (or positive scepticism).[2]

In the modern academy it seems that Mathematics, along with English, is considered a fundamental area of study[3] and mathematics only entered the standard programme in primary, secondary, or higher education in the 19^{th} century. It is a standard part of a core curriculum (also known as a general curriculum) at many United States (U.S.) comprehensive universities, liberal arts colleges, research universities, junior colleges, and other institutions of higher education.[4] Much of the programme from 1980 through 2010 became 'Balkanised,'[5] that is to say a hodgepodge of courses were devised, designed, and implemented. So, instead of a linear basic course frame-work (or a few tracks) there were specific courses created such as Mathematics for Business, Mathematics for Art, Business Calculus, Statistics for Business, Statistics for Psychology, Statistics for Sociology, Mathematics for Information Science, etc. which these courses created a segregation between different majors and an explosion of courses that were introductory and overlapping. The trend continued and also there appeared upper-level 'Balkanised' programmes so that there exists courses Engineering Calculus, Biostatistics, Differential Equations for Engineering Students, Applied Calculus for Physics, etc. in some programmes (see [5] for one example and it should be noted that such seems especially to be the case at 'elite' colleges or universities where there is a large student body). The "Balkanisation' of the mathematics curriculum seems congruent with the store-customer model that some view is the role and place of higher education; that universities are in competition for 'customers' and that a 'product' is offered to the 'customers.'

Concurrent with the 'Balkanisation' of mathematics programmes came an accent on the socio-political considerations of any undergraduate programme and an emphasis on secondary or tertiary components of a programme (see [31] for a

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[1] However in the United States of America (U.S.A.) mathematics is referred to as a unary object, "math is;" whereas elsewhere it is oft referred to as a non-unary, "maths are."

[2] [51]

[3] [1], [28]

[4] [7]

[5] We shall use the term, 'Balkanised,' to mean broken apart into small pieces that are actually interrelated and possibly better considered together but are treated discretely (alluding to the break-up of Yugoslavia).

stinging critique of radical constructivism and its effect on education) and not the content of the programme (which the author opines is primary).[6] Phenomenology, hermeneutics, radical constructivism, and their kin see mathematics primarily as a social construct, as a product of culture, subject to interpretation, and it changes dependent on audience (see, for example: [30], [35], [77], [81], [41], [69], [2], [80], and [90]). The belief that mathematics is hounded and bounded by the fashions of the social group performing it or by the needs of the society financing it seems to lack merit given the permanence of mathematics. It seems that some researchers' attention has focused on the radical constructivist notion of knowledge created by the creator, that truth (as everything) is relative, that cultural considerations and critical theory inform the academy and appear to infer that content seems stagnant and not of import ([82], [45], [71]). Indeed, it seems to have come to pass that much of the consideration in research about college undergraduate mathematics education is an extension of research about primary mathematics education or secondary mathematics education and seems to reflect a National Council of Teachers of Mathematics (NCTM) -based[7] reference point.[8]

Furthermore, why should a subject be studied simply for its utility? If such were the only or primary consideration, would it not be the case that there would be a flowering of polytechnic institutes and a waning of liberal arts colleges; would it not be the case that the vaunted marketplace would cause liberal arts colleges to become more like polytechnic institutes and comprehensive universities cut-back on classical liberal arts and sciences majors and create more 'customer-friendly' majors? Since such seems not to be the case, then it seems that there is not as strong a desire on the part of the society for such utilitarian training. The idea of training in utile mathematics has been deemed 'quantitative literacy' and centres on application, computers, consumer arithmetic, etc. Some authors or practitioners advocate or advise lower-level college mathematics courses focus on said (see [17], [13], [16], or [67] for examples) and the 'audience served.'

Likewise, along with both of the aforementioned trends, there seems to have been a suggestion or a 'move' to limit pre-requisites and 'broaden' the accessibility of the mathematics curriculum (see [13], [16], and [62] for some prime examples). In the provocative paper "Do We Need Pre-requisites?" O'Shea & Pollatsek ask the question and suggest some modifications to a traditional (linear or few track) mathematics programme with pre-requisites. It cannot go without remarking that this author has read the O'Shea & Pollatsek paper (seemingly) every year and has thought rather long & hard about the ideas posited, the programme suggested, and the recommendations made.

Indeed, within the field one can find the idea of content being dependent upon a student group, for example, in the Committee on the Undergraduate Program in Mathematics (CUPM) 2004 Guide [9] as well as the idea of a minimisation of pre-requisites[10] So, it *seems* that the O'Shea & Pollatsek article had some impact and there exist faculty who opine that the answer to "Do We Need Pre-requisites?"

[6] A prime example of such a position is in [17].
[7] [59]
[8] [72]
[9] [16]
[10] [16], [4]

is, 'no.'[11] The author has colleagues who have argued quite passionately that pre-requisites are not needed in many courses (including advanced undergraduate courses) and that any background deficiency can be remediated whilst the student is enrolled in a course.

In this paper, the author will attempt to argue that undergraduate study of mathematics need not be utile; that undergraduate study of mathematics is not necessary; and that the concept of 'quantitative literacy' is unconstructive. Furthermore, the author will attempt to argue that an undergraduate's study of mathematics is better served by a well-thought, reasonable, and efficient pre-requisite design so that a student may achieve some breadth and some depth and to retort that pre-requisites are not needed in many courses & that background deficiencies can be remediated efficiently, expeditiously, without stigma, and in such a way as to not compromise the academic integrity of a course, programme, or curriculum.

[11]To be fair, for some courses the answer is, 'no,' but for others it would be, 'yes, **but** very few or as few as possible.'

II. Is Mathematics Indispensable and Must Everyone Study Mathematics in College?

In his paper "Is Mathematics Necessary?" Underwood Dudley asks, "is mathematics necessary? Necessary, that is, for citizens of the United States to function in the world of work?"[12] He answers, 'no,' and with great wit and aplomb states, "Is mathematics necessary? No. But it is sufficient."[13] His wit and wisdom toy with the idea that some in the academy presume: that in order for students to be engaged in an academic pursuit there needs to be an application to their major field, interest, or future. He tears through this assertion with an amalgam of historical, practical, and pragmatic reasons. He quotes from Pearson's, Why don't most engineers use undergraduate mathematics in their professional work? *Undergraduate Mathematics Education (UME) Trends,* **3**, 3 (1991),

> 'Why do 50% (probably closer to 70%) of engineers and science practitioners seldom, if ever, use mathematics above the elementary algebra/trigonometry level in their daily practice? My work has bought me into contact with thousands of engineers, but at this moment, I cannot recall, on average, more than three out of ten who were well versed enough in calculus and ordinary differential equations to use either in their daily work.'

He goes on to ask, "If 70% of engineers don't need calculus to do their jobs, then how many of the 500,000 or so students that we put through calculus every year will? Minutely few, so we should not tell them how tremendously useful calculus is going to be to them when they go to work. If most engineers can do quite well with only algebra and trigonometry (or perhaps even less), is it not reasonable that non-engineers can survive and flourish with arithmetic, or even less? Yes, it is."[14] He points out his contention that mathematics is not necessary but, "*Everyone* should learn algebra, but not because it is necessary to manage warehouses. Does it mean that we should stop assigning "applied" problems? Certainly not; we should assign more. Problems expressed in words are the best kind, but they should all start with 'Suppose that...' If we can't be realistic, we can at least be honest."[15]

Dudley, this author contends, masterfully argues for the study of mathematics for the sake of the study of mathematics:

> It is time to stop claiming that mathematics is necessary for jobs.
> It is time to stop asserting that students must master algebra

[12] [24, page 360]
[13] [24, page 364]
[14] [24, page 361]
[15] [24, page 361]

to be able to solve problems that arise every day, at home or at work.
It is time to stop telling students that the main reason they should
learn mathematics is that it has applications. We should not tell our
students lies. They will find us out, sooner or later.
Besides, it demeans mathematics to justify it by appeals to work,
to getting and spending. Mathematics is above that-far, far above.
Can you recall why you fell in love with mathematics? It was not,
I think, because of its usefulness in controlling inventories. Was
it not instead because of the delight, the feelings of power and
satisfaction it gave; the theorems that inspired awe, or jubilation,
or amazement; the wonder and glory of what I think is the
human race's supreme intellectual achievement?
Mathematics is more important that jobs.
It transcends them, it does not need them.[16]

We would like to build a discussion upon Dudley's work and submit a, perhaps, more controversial point, that college-level mathematics is *not indispensable.* Let us assume that there is a body of material that is of use and a literate person should know. It seems that such can be attained before college: the material through secondary level Algebra, Geometry, Applied Mathematics (modelling), and Statistics. It seems it should also include classical logic, predicate calculus, syllogistic arguments, and basic quantification instead of Calculus I.

People seemingly can live quite nicely without college-level mathematics (some studiously avoid it) and can manage[17] without it. Indeed, it seems to be the case that a business owner would hire someone else to do anything approaching real mathematics; it seems to be the case that an engineer would have on staff someone else to do anything approaching real mathematics; it seems to be the case that a financier would employ someone else to do anything approaching real mathematics; etc.

So, should everyone study mathematics in college? We opine that answer is, 'no.' The underlying rationale for everyone to study mathematics is that most literate people use 'the mathematical way of thinking' every day. However, it seems there are at least two other disciplines which could serve students to learn how to think rigorously and logically and not enrol in a mathematics course, a student could take a programming class in Computer Science or a symbolic logic class in Philosophy and successful completion of either course seems a justifiable substitute for an entry-level college mathematics class and the student would be engaged in 'a mathematical way of thinking.' We opine that said would be a superior intellectual endeavour for

[16][24, page 364]

[17]Though I know not how, but they do. Most of my family have nothing whatsoever to do with college-level mathematics but use basic arithmetic, geometry, modelling, and statistics oft. Since I am a mathematician I see mathematics in most things but for an anthropologist or a police officer that person may not see it likewise.

a student than taking a watered-down 'mathematics appreciation' course as is common at many liberal arts colleges or comprehensive universities (called, typically, Introduction to Mathematics, General Mathematics, Contemporary Mathematics, etc.). Deleting lower-level courses that are barely really college-level courses or are in actuality high school remedial courses packaged as university courses such as 'college algebra,' might have the effect of improving the academic integrity of an institution.

Evidence for such a programme (a programming class in Computer Science or a symbolic logic class in Philosophy instead of Mathematics) includes some of the following. For example, let us consider the nonsense called 'math anxiety.' If someone is 'math anxious,' instead of tormenting the person, let them take a rigorous programming class or symbolic logic class! Such would alleviate the anxiety and relieve the mathematics department of having to accommodate such. Another reason for such a programme is that it allows students who's major is not mathematics-dependent flexibility but does not compromise academic integrity. Furthermore, there has been an explosion of remedial mathematics classes at many institutions of higher learning and it seems that a programming class in Computer Science or a symbolic logic class in Philosophy in lieu of a Mathematics class would decrease the need for remedial classes in mathematics and would provide mathematics faculty more opportunity to concentrate on Calculus and beyond courses. There is a problem that a number of students enter university and are found to be deficient in one or more areas. Such students should be told the truth and should be encouraged to be remediated but the remediation should be at the junior college, secondary school, or before level. It would probably do our society, the university, and the individual more good to have remediation outside of the university or liberal arts college environment.[18] This author has encountered some students who are *extremely* gifted, can reason well, can programme, but are terrible at classical mathematics; some students who possess good common sense and can learn classical logic but have almost no ability to understand basic mathematical statements. Therefore, would it not serve said students to direct them to courses outside of mathematics?

The idea that seems tied to the idea that an undergraduate education should include 'a mathematical way of thinking,' but that has come to be (mis-)understood as being 'quantitatively literate' (see [17] and [67]) which is not a well-defined term but seems to be a mixture of the idea of being an informed 'consumer of mathematics,' and an aware 'user of mathematics.' We believe Dudley expertly argued that these are not the purpose of

[18]By doing so it might 'force' the primary and secondary schools to reform pedagogy and refocus content on pre-undergraduate mathematics. It seems that in the last 30 years the problem has been 'punted' to later and later grades so that now much of the primary school material that was to be mastered is now expected to be mastered in secondary school, much of the secondary school material that was to be mastered is now expected to be mastered in an undergraduate education, and so forth. This seems not to be optimal nor advisable for it simply postpones confronting a problem.

an undergraduate mathematics education nor of an undergraduate core curriculum education that includes mathematics and to extend the argument being an informed 'consumer of mathematics,' and an aware 'user of mathematics' should be pre-requisite to university. Basic elementary Arithmetic, Number Theory, Geometry, and Statistics should be a part of the primary and secondary school curriculum. If academe is serious about encouraging 'a mathematical way of thinking;' then true first-year mathematics or statistics classes should be a part of a core curriculum (Pre-calculus (not 'College Algebra' but a course built upon a pre-requisite of three years of secondary school mathematics;[19] Calculus I; Calculus II; Linear Algebra; Mathematical Logic; or, Introduction to Probability with Calculus I as a pre-requisite.[20]

One does not have to reinvent the wheel nor look far for a sanguine outline of beginning programme in mathematics at the college level, it exists and is still quite apropos. Such is contained within the 1965 Committee on the Undergraduate Program in Mathematics, *A General Curriculum in Mathematics for College* issued by the Mathematical Association of America (MAA) - - in many papers the author has previously written he has mentioned his contention that the CUPM 1989; 1991; and, 2004 Standards do not delineate material for courses nor focus on particular content within courses but focus more on course delivery, 'mathematics appreciation,' and computational mathematics applications with computers.[21]

The author opines that it is correct to assume ('axiomatically') that it is a fundamental principle that any policy or procedure which leads to a strengthening of a mathematics programme, leads a strong programme to become a stronger undergraduate programme than before the policy, or provides better preparation for graduate school is a good policy and should be adopted by a faculty. The author opines that it is correct to assume ('axiomatically') that it is a fundamental principle that any policy or procedure which lessens remediation in university and shifts the practice of learning pre-college material before university back to the primary, secondary, or junior college level is a good policy and should be adopted by a faculty because it frees up the university or liberal arts college faculty to concentrate on mathematics courses that are at the college-level or above and not concentrate on mathematics courses that are at the pre-college level.

[19][10, pg. 10]

[20][10, pg. 10])

[21]Post-1985 CUPM guidelines are not 'bad' or 'wrong' but do not seem to accentuate the kind of major programme course-work or strength of preparation for advanced work in mathematics that is the focus of this paper. The author opines that the 1963 - 1965 Committee on the Undergraduate Program in Mathematics (CUPM) materials delineated that which is fundamental to a strong undergraduate programme and preparation for graduate school. The 'weakening' of the scope and purpose ogf th CUPM standards seem to have been started with the 1989 CUPM Guide (see [11]).

III. Are Pre-requisites Needed in Mathematics Courses?

O'Shea and Pollatsek (1997) asked the provocative question, "do we need pre-requisites," and outlined a programme where pre-requisites were minimised. Furthermore, they issued a challenge:

> In summary, we have bought into the broader, more accessible curriculum outlined above, and we invite you to consider doing the same. For us, the ideal curriculum would consist of courses which are independent but which nourish one another by increasing the students' repertoire of mathematical examples and experience. Every course should stand alone in the sense that it provides a student with examples and tools which will remain with him or her for life. Progression through the major should provide students with access to ever wider and deeper examples, an enhanced appreciation of what can go right and what can go wrong, a progressively more sophisticated sense of argument, and a growing respect for their own ideas and those of others. It should sharpen students' sense of wonder while providing them with the critical tools to distinguish dross from miracles. For those who would argue that what we sketch here is a dumbing down of mathematics courses, we issue two challenges. The first is that you make it a principle that *every* Mathematics course should increase students' options for further study of mathematics. The second is to thoroughly reexamine [sic] the prerequisites for the courses you teach and to consider how the course you want to teach might itself teach the prerequisite topic(s).[22] Even if you retain the prerequisites, the exercise will keep you honest.[23]

So, O'Shea & Pollatsek asked the question, outlined a programme, and issued a challenge. Thus, this section of this paper will summarise their argument and retort some of the contentions made in that paper.[24] The Pollatsek & O'Shea paper is a sincere effort at proposing an undergraduate mathematics programme model.

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[22]If a course relies on pre-requisite material and the pre-requisite topics are taught in the course, then how can the requisite topics be taught? Are O'Shea and Pollatsek not inferring that the material they are referencing is co-requisite and not truly pre-requisite?

[23][62, pg. 570]

[24]Absolutely *nothing* within this paper should be construed as a criticism of O'Shea & Pollatsek - - all contained within is simply a critique of their argument and a suggestion that their argument contains errors, have unintended consequences, are not necessarily generalisable, and might need re-examination.

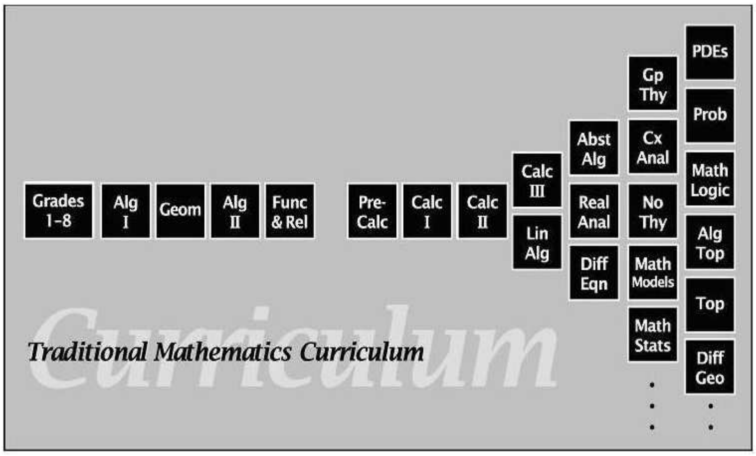

Figure 1: From O'Shea & Pollatsek, the Traditional Curriculum

It is entirely within the realm of possibility and reality that there are better instructors than the author and said better instructors could eliminate or minimise pre-requisites, could remediate elementary, secondary, Calculus, Foundations, Set Theory, and logic concepts whilst teaching, say Real Analysis, but it seems (at least to the author) to be quite improbable *without* compromising the academic integrity or rigour of the Real Analysis class.

O'Shea and Pollatsek suggest that the traditional model (from [62]) is lacking and they suggest a reform model. They contend, "as the implementation of the NCTM Standards proceeds, the dependence of K-12 mathematics courses, one upon the preceding, suggested by the diagram, [sic] is lessening." However, in the intervening years between 1997 and 2010 some primary and secondary schools have reformed to include a reimplementation of a more traditional programme and the NCTM Standards were revised to reflect more traditional topics. Therefore, O'Shea and Pollatsek may be correct that the linear nature of the pre-college programme may change over time it is not all together clear that the *content* has or will change.

However, rather than offering a stinging indictment of a linear model of pre-requisites, they note, "We pause to list a few of the defects of this curricular structure. Although obvious, they are too seldom acknowledged, and the harm they work is almost always understated. First, the whole structure creates a climate of fear. A student who does poorly in one course feels incapable of mastering further courses ..."[25] Why is there a problem? Is it not the case that when a person does poorly in one course that person's education may genuinely be compromised and it is harder to master subsequent courses' material? Is it not the case that 'higher' mathematics concepts build (unlike history where the battle of Waterloo might have very

[25] [62, pg. 564]

little to do with the battle of Trafalgar or the battle of Lecanto) upon 'lower' concepts? O'Shea and Pollatsek note, "... a student who falls off the track somewhere in high school and enters college requiring one or more precalculus [sic] courses has a very long road to negotiate before encountering the useful and appealing ideas in junior- and senior-level mathematics courses. Such students often find themselves effectively closed off from a mathematics major (or even a minor) and any other majors requiring substantial mathematics."[26] If a student has pre-requisite deficiencies, how is the (inferred) shortening of a 'long road' helpful? Does that not set students up for possible failure because foundational concepts might not be understood, known, or have been mastered to the point that said areas of inquiry are attainable - - therefore could it not be that the student is not 'empowered' but actually harmed?

In the article it is contended that, "The curriculum also cheats the very able student who has understood everything but who chooses not to pursue a mathematics major."[27] How is the student cheated? It is not at all clear. Is it that Pollatsek & O'Shea suggest no pre-requisites across the academy? If so, should not a student be able to take Organic Chemistry III without any pre-requisite? What about Advanced Sculpture, Cellular Genetics, La Literatura del Siglo de Oro without any pre-requisite? Does that not set students up for possible failure; does it not create havoc across the university; does it not 'water-down' courses and programmes? Moreover, Pollatsek & O'Shea contend, "students who major in mathematics but do not go on to graduate school in mathematics or science do not encounter the range of ideas embraced by modern mathematics until their junior and senior years."[28] One can encounter a range of ideas but if one has not the tools to really understand the ideas (prove, disprove, etc.) what good is the encounter? What good is a discussion of quantum physics without previous courses in the foundations of physics?

Further, Pollatsek & O'Shea bemoan the mathematics major is being somehow cheated by a quasi-linear programme. They state, "How else to explain that when we think of teaching some course-Galois Theory, say-we think first of the prerequisites, Linear Algebra and a semester of Abstract Algebra, that we will use to **exclude**[29] students rather than how we might explain the material to students without these courses and what opportunities might arise along the way to teach something about vector spaces, groups, and fields. (If this seems impractical, would you tell an interested colleague, a statistician or an electrical engineer, who asked you to explain the elements of Galois theory [sic] to first take the prerequisite courses?)"[30] In this part the authors seem to infer that pre-requisites are not needed

[26] [62, pg. 564]
[27] [62, pg. 564]
[28] [62, pg. 565]
[29] Emphasis added.
[30] [62, pg. 565]

in many courses (including advanced undergraduate courses) and that any background deficiency can be remediated whilst the student is enrolled in a course. They use a trite and incongruent example of a colleague inquiring as to a topic to ridicule the notion of the need for pre-requisites for Galois Theory.

To be reasonable they note, "We are not claiming that prerequisites are unnecessary. Rather, we submit that insufficient thought has been given to creating a curriculum which minimizes prerequisites and maximizes a student's options."[31] This is confusing since they seem to (previously) ridicule the entire notion of pre-requisite material before undergraduate mathematics courses; but, let us assume it is a minimisation that is recommended. Once again we return to the problem that minimisation of pre-requisites *does not necessarily maximise a student's options* because a student may enrol in a course but that student *may not be prepared for the course.* Also, a student enrolling in a course without the background needed has the unintended consequence of compromising the course for the students with the background for the course can easily get bogged down in remediation or the material might not be able to be studied by all in a deep manner. So, how is that fair to the student enrolling in a course without the background and how is that fair to the student who has the background?

It sounds nice and it is fashionable to contend that anyone can attain anything. However, is it reasonable? O'Shea & Pollatsek state, "In short, we contend that the traditional hierarchical curricular structure brutalizes [sic] most who proceed through it."[32] Indeed, is it not cruel to give false hope to (or lie to) a person telling him that they can succeed at something at which they are not adept? From a popular culture standpoint, for every Susan Boyle, there are a billion Padraig McLoughlins.[33] Perhaps it is time to bring to the discussion the notion that to *do mathematics* takes talent, different people have different abilities, some people are not adept at mathematics, mathematics is not everyone's cup of tea, and not everyone needs mathematics (see previous section). This is not an elitist, toffee-nosed, meant to be discriminatory, or exclusionary remark; it is simply a fact that even Antonio Salieri was atypical (no Mozart but still able).

Furthermore, they contend that "the curriculum also cheats the very able student who has understood everything but who chooses not to pursue a mathematics major."[34] If the student has understood everything but does not fulfill a pre-requisite special consideration could be made for him (permission of instructor) so he may take the course. Any student who is very able student who has understood everything previous to a course has truly

[31] [62, pg. 565]

[32] [62, pg. 565]

[33] Susan Boyle is a very talented singer with a wonderful voice who burst on to the entertainment world in the show, "Britain Has Talent." Padraig McLoughlin (me) has a tin ear and cannot sing, dance, sculpt, paint, bat, etc.

[34] [62, pg. 565]

fulfilled pre-requisites since it is the knowledge obtained or refined - - the content - - that is the pre-requisite, not attendance in a course (if attendance were pre-requisite then any student can flunk, for example, Abstract Algebra I and the take Abstract Algebra II). The author opines that it is much rarer to find a 'very able student' who has understood everything but who chooses *not* to pursue a mathematics major than a mathematics major; so, it seems preferable to accommodate that rare non-math major 'very able student' than a student majoring in math. O'Shea and Pollatsek seem to be of the opinion that the students planning on attending graduate school are some sort of problem; it may be incorrect to say such but it does seem to be the case when they note "however it is done, we think that it makes more sense educationally to make special arrangements for the few students who are bound for graduate school than to have the relatively few such students drive the curriculum for the vast majority of other students."[35] Indeed, this author opines that "... departments will probably find that they need to make some special arrangements for students planning to continue into graduate studies,"[36] is not ideal and should be rejected; for is it not better or ideal to try to get students to major in our disciple and not in another? The author opines that O'Shea and Polletsek have it reversed from what the author opines is the better arrangement. The author opines that it is better to make special arrangements for a (rare) non-major then to make special arrangements for the student who is planning on attending graduate school.[37]

A person genuinely interested in mathematics and wishing to study upper-level mathematics *has to begin somewhere* - - understanding and mastering most arithmetic, algebra, geometry, logic, and set theory *creates opportunities* it does not deny opportunity! Therefore, the author contends that a well thought out mathematics programme with pre-requisite *properly* set offers the student *more* not less chance to succeed and creates access rather than denies it. Further, it is in the primary and secondary levels of education where opportunities are made or denied and reform of those levels (better mathematical preparation for college) might do society more good. So, the author opines we should concentrate on those (and reform of teacher education to produce better and more knowledgeable mathematics teachers at the pre-college level).

When Pollatsek & O'Shea make their recommendations for a 'broader

[35] [62, pg. 566]

[36] [62, pg. 566]

[37] For example, when the author was in graduate school at Auburn University the first-year Real Analysis class was taught by Dr. Coke Reed and Dr. Reed made special arrangements for two students. One student was a re-entering student to graduate school; he attended the classs but did not participate since he had had the class before and it was taught via the Moore method. The other student was an 11-year old child who was (and is) a prodigy (Dr. Gregory Kuperberg). So, the author opines this is what a good use of special arrangements would be rather than for a regular undergraduate who is planning on attending graduate school.

curriculum' they summarise that they recommend, "1) Fix the precalculus/calculus [sic] sequence; ... 2) add entry-level courses that go somewhere ... 3) increase access to advanced mathematics courses ...; and, 4) add a program [sic] for students planning a Ph.D. in mathematics."[38] Let us take the points in reverse order. O'Shea and Pollatsek state, "the traditional curriculum was designed with the prospective Ph.D. student in mind, and broadening the curriculum is not without cost. In order to make sure that students bound for graduate school are as prepared as those in previous generations, departments will probably find that they need to make some special arrangements for students planning to continue into graduate studies."[39] The author has experienced such a programme and it suffers from the classes are dragged down by the students not planning on further study, the special arrangements for students planning to continue into graduate studies oft suffer because over over-burdened faculty (having to teach large sections of major and non-major courses), and, frankly, nothing substitutes for the camaraderie and competition of a student having a class rather than directed reading.

That O'Shea & Pollatsek suggest increasing access to advanced mathematics courses; but offer only, "departments should rework existing junior-senior-level mathematics and statistics courses to eliminate as many prerequisites as possible, thereby making them accessible to students in other majors."[40] As previously noted, they offer no substantive method toward doing so without 'dumbing-down' the courses or allowing the courses to be watered-down where material needs be introduced time-and-time again. The author has experienced such a situation where classes suffered by having little pre-requisites; therefore, mathematical induction was treated as a new concept in Math 154 (Pre-calculus), Math 180 (Principles of Mathematics), Math 255 (Set Theory), Math 280 (Discrete Math), Math 353 (Advanced Calculus I), and Math 371 (Abstract Algebra I) and the effect of said was to drag-down the programme and slow students' progress. The unintended consequence of the elimination of 'as many prerequisites as possible' is to significantly slow students' progress toward mathematical maturity and compromise a student's chance of studying a topic well and in depth. Once more the question is begged, does the elimination of 'as many prerequisites as possible, thereby making them accessible' *really* create accessibility or dilute content?

O'Shea & Pollatsek recommend adding entry-level courses that 'go somewhere' and this recommendation seems entirely appropriate and executable. Clearly there exists non-Calculus courses at the non-entry level and such could have prerequisites other than Calculus; such intermediate and advanced non-Calculus based mathematics courses could be offered without

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[38] [62, pgs. 565 - 566]
[39] [62, pg. 566]
[40] [62, pg. 566]

a Calculus pre-requisite (e.g.: Graph Theory, Combinatorics, Linear Algebra, Applied Statistics, etc.). As to the 'fix the precalculus/calculus [sic] sequence,' the author seems to recall that there have been appeals for said for over 30 years and none of the reforms heretofore proposed or enacted have produced much.[41]

O'Shea and Pollatsek offer recommendations with the caveat that "we do not claim that we have the answers. In fact, we do not even claim that everything we are doing is wise."[42] Likewise, such a caveat is true for this author and his observations or recommendations. It seems that those who are absolutely sure are oft absolutely wrong; we hope not to make the same such error. Pollatsek & O'Shea suggest a reform model (from [62]) which does not follow a linear track but includes many choices such that courses from one group do not necessarily precede or follow from another group.

They note that, "our department reworked seven of our junior-senior-level courses in order to reduce the prerequisites to at most two semesters of college-level mathematics: Differential Equations with Modeling, Analytic Number Theory, Mathematical Statistics, Lie Groups, Polyhedral Differential Geometry, Theory of Equations, and Symmetry Groups in Geometry and Physics."[43] These seem to be the 'FIPSE courses' in the diagramme. What is troubling is that any two courses in the same column can be taken concurrently or independently and that **one** or two course in one column serve as pre-requisite for a course in a subsequent column; when considering the material in the columns it seems a student can take 'Quantiative Reasoning,' then Data Analysis, then Probability Theory![44] They do not make it clear if such is so with specificity (Calculus I a pre-requisite for Calculus II but let us assume such). Pollatsek & O'Shea offer much praise of how their 'Lab' course functions and claim it is quite a success. However, one is left wondering how that course prepares a student for Real Analysis and with the maximisation of flexibility it seems there are problems. "Lab course, on the other hand, as an unqualified (and demonstrable) success. We require it of mathematics majors-it is a prerequisite for Real Analysis-and recommend, but do not require, that it be taken in the sophomore year. The result is that a number of majors take Abstract Algebra without first taking the Lab . . . we feel that students can hardly begin to appreciate the meaning of 'proof' unless they have had concrete experience with examples which sometimes lead to correct conjectures and sometimes to false ones. The Lab requires either Calculus I or one of our freshman courses, although most students take it after Calculus II. Very often, the computer is the tool that enables us to bring these examples to the desired level."[45]

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[41] Some 'reforms' have been counter-productive but such is a topic for another paper.

[42] [62, pg. 566]

[43] [62, pg. 567]

[44] How any student can take Probability Theory with any reasonable expectation of profundity in such a manner is mind-boggling.

[45] [62, pp. 567; 569; &, 568]

It is disconcerting to the author for use of a computer to bring, 'examples to the desired level,' seems to indicate that ideas are illustrated but not proven or disproven. A trap of too much reliance on inductive reasoning and not enough deductive reasoning seems to be a possible outcome of said endeavour. We opine the centre of the mathematical experience is proof (see McLoughlin, [48]; [49]; [50]; [51]; [52]; [53]; [54]; [55]) as, we believe, any Moore-method instructor would argue. It is probably fair to state that no student can really appreciate the meaning of "proof" (it, like much, requires time (more time than an undergraduate programme provides)). It is the author's experience that that students are better served by having concrete experience with examples along with concrete experience with proofs, counter-arguments, and counter-examples. Therefore, we take exception to Pollatsek & O'Shea position on the Lab course but cannot disregard it since we are not at their institution and have not had experiences congruent with theirs.

The contention, "however it is done, we think that it makes more sense educationally to make special arrangements for the few students who are bound for graduate school than to have the relatively few such students drive the curriculum for the vast majority of other students"[46] is disturbing and confusing. It is disturbing insofar as it would seem to this author that having 'better' students drive the curriculum, present a challenge to themselves, the other students, and the faculty is **desirable**. <u>It is not something which should be avoided</u>. Second it is confusing because Pollatsek and O'Shea do not justify segregating 'better' students nor explain how such is educationally better or optimal.

With regard to the suggested curriculum that Pollatsek and O'Shea submit in the paper, they note that, "How does all this work? Is the curriculum outlined above an improvement over the standard curriculum? The short answer is that we think so, but we really do not know. We do offer a number of tentative observations and conclusions, organized about the strategies mentioned above ... Unfortunately, students who go into these courses with weak algebra skills come out with weak algebra skills ... we offer an intensive high school algebra review during January term (in fact, since skills are easy to teach, we have senior mathematics majors run it)."[47] So, they did not have much data to support the model in 1997 but there has not been any follow-up report to assess the success of the programme or the lack thereof. Further, the comment about skills being easy to teach was stunning (to this author) and baffling. It is very difficult to comment on that statement since it seems to be self-contradictory.

[46] [62, pg. 566]
[47] [62, pp. 568- 569]

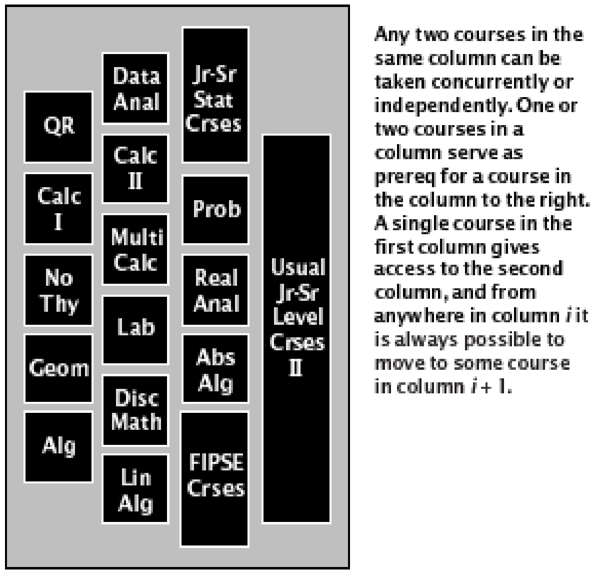

Figure 2: From O'Shea & Pollatsek,
the Pollatsek-O'Shea Curriculum

They also comment that," We think of the Lab course, on the other hand, as an unqualified (and demonstrable) success. We require it of mathematics majors-it is a prerequisite for Real Analysis-and recommend, but do not require, that it be taken in the sophomore year. The result is that a number of majors take Abstract Algebra without first taking the Lab. Comparing this group with those who take the Lab first shows clearly that, with few exceptions, those who take the Lab first do better than those who do not. This bears out the strong anecdotal evidence that suggests the Lab really does make a substantial difference in students' mathematical maturity."[48] This seems to contradict the main thrust of the paper - - that pre-requisites may not be needed, a traditional curriculum creates "fear," "disenfranchises," "cheats," and "brutalizes" [sic] and is therefore to be avoided yet they submit the Lab is a good pre-requisite for Real Analysis and is not a pre-requisite for Abstract Algebra I but "those who take the Lab first do better than those who do not." *Is that not part of why a pre-requisite is required?*[49]

Pollatsek and O'Shea also note, "Unfortunately, while we have made a number of courses much more accessible, we have so far not succeeded in

[48] [62, pg. 569]

[49] I feel I am missing something and each time I re-read the article this is one of the places where I am, frankly, baffled. I cannot understand how this fits with the essence of the article. I recall the admonishment, "We are not claiming that prerequisites are unnecessary. Rather, we submit that insufficient thought has been given to creating a curriculum which minimizes prerequisites and maximizes a student's options," [pg. 565] but there seems to be contrarians minimally in these comments.

attracting significant numbers of nonmajors [sic] to them. On the other hand, our majors are enthusiastic about these courses, and some of them credit their existence as a factor in their decision to major in mathematics."[50] Again, a central tenet of the article was to create this curriculum to "increase access and student choice," ... "by reducing the prerequisites, we hoped to attract students who were not mathematics majors but who would enjoy and be able to use some of the ideas encountered in traditional junior-senior-level courses," .... open mathematics to "the very able student who has understood everything but who chooses not to pursue a mathematics major," ... and to "rework existing junior-senior-level mathematics and statistics courses to eliminate as many prerequisites as possible, thereby making them accessible to students in other majors."[51] Ergo, the revised curriculum has not produced that which it was designed to produce; there is scant evidence of its effectiveness; and, they did not address whether or not this revised curriculum increased the number of people majoring or minoring in mathematics.

Furthermore, with regard to the mathematics majors' enthusiastic reviews, how did said courses prepare the students for graduate school or post-graduate work? Did the revised curriculum help, hurt, or was it neutral with regard to preparation for graduate school? Such was not mentioned but it was interesting to note the authors mentioned, "Finally, because of the existence of strong REU programs, we do not feel that the preparation of the few students who go on to mathematics graduate school has suffered. On the contrary, we feel that our current students who are bound for graduate schools are actually better prepared than our students a decade ago."[52] Is it this programme or summer Research Experiences for Undergraduates (REUs) that better prepare students? There is no evidence offered to clarify this point and it leaves one wondering.

Finally, there are two other points worth mentioning about the "Do We Need Pe-requisites?" article. First, the comment that "Every course should stand alone in the sense that it provides a student with examples and tools which will remain with him or her for life,"[53] is something not universally accepted. The *programme* provides a student with examples and tools which will remain with him or her for life but not necessarily each course and the idea that each can stand alone in this sense is seemingly absurd and unrealistic. Second returning to the challenge O'Shea & Pollatsek issued, "For those who would argue that what we sketch here is a dumbing down of mathematics courses,"[54] it does seem to be the case! The author's experience as an undergraduate, graduate student, and faculty member is 180° opposite of that which is outlined in the article. Pre-requisites helped the author

[50] [62, pg. 570]
[51] [62, pg. 564 - 570]
[52] [62, pg. 570]
[53] [62, pg. 570]
[54] [62, pg. 570]

progress through mathematics in a structured and ordered manner (though not always smoothly). Further, that which is recommended in the article is contrary for the observation this author has made as a faculty member. The author has witnessed a number of students harm themselves (intellectually, emotionally, and with regard to grades) by attempting to take courses without fulfilling pre-requisites. Some concepts can be superficially discussed and seemingly understood, but that, sadly, smacks of Sophistry.

This author becomes more and more convinced that R. L. Moore was right - in the competition between Sophistry and Socraticism, Socraticism is authentic, preferred, and correct. However, Sophistry is ascendant in the 21^{st} century university. It prevails in many a classroom because:
1) it is easier for the instructor–no arguments with students, parents, or administrators; complaints of things being 'hard' are almost non-existent if one employs sophistry and the instructor does not have to "think as hard;"
2) it is easier for the student–he does not have to "think as hard" (or think at all), she can "feel good," it can have its self-esteem 'boosted;'
3) it is easier for the institution– standardisation can be employed (which seems to be a goal at many institutions); students retained ('retention' seems to be a 21^{st} 'buzz word'), graduation rates increase, and accrediting agencies are mollified.

For a Socratic, we would be concerned with measuring a course, the instructor, the content, etc. of a course let us call it $\mathcal{A}$, by how well a student does in a course or courses subsequent to $\mathcal{A}$ for which $\mathcal{A}$ is an authentic prerequisite or to which it applies (say in a different field). For a Socratic, we would be concerned with measuring a programme by how well its graduates succeed in graduate school, the workforce, etc. Such is not how *it is, but how it, perhaps, should be.*

IV. On the 2004 CUPM Standards

It is not surprising or shocking to note that of the two articles mentioned in detail, "Is Mathematics Necessary?"[55] and "Do We Need Pre-requisites?"[56] that the former is hardly (if at all) reflected in the most recent CUPM Curriculum Guide[57] but many of the ideas proposed by O'Shea and Pollatsek are precisely as advocated in the O'Shea and Pollatsek article or are variations thereof since Pollatsek was on the Committee on the Undergraduate Programme in Mathematics (CUPM), chaired CUPM from 2000 - 2003, chaired the Curriculum Project Steering Committee, and chaired the CUPM Writing Team.[58]

The CUPM Standards have been published intermittently since the 1950s and the 2004 Standards[59] is but the latest iteration of extensions, revisions, or supplements. In the 2004 CUPM Standards, the first thing that 'pops out' from the guide is that it does not offer a guide to the courses that should be in an undergraduate programme in mathematics, indeed, it is stated that "it would make life easier if these CUPM recommendations could include a list of courses describing the ideal mathematical sciences major. However, such a list is neither possible nor **desirable**.[60] It is not possible because of the varying demographics and aspirations of students at diverse institutions nationally. It is not desirable because of the varied careers and fields in which mathematical sciences majors are needed and the capacities of different institutions to meet these different societal needs. Even at a single institution, providing a flexible major or a variety of tracks within the major can position the department to meet the diverse needs of its students most effectively."[61] This contention is rather bothersome since such was produced in previous version (see, for example, the 1963 and 1965 Standards) and there are *some courses* that are common to a flexible major or a variety of tracks within the major or varied careers and fields in which mathematical sciences majors are needed. Clearly the Calculus I - III, Linear Algebra, Foundations (or Set Theory), Real Analysis I, Abstract Algebra I, Senior Seminar seem rather common. The 2004 CUPM Guide is long on platitudes and short on details (this author may be wrong about this but having read them often, it seems to be the case). Why were there commonalities in mathematical programmes in the 1960s and not in the 2000s?[62] In the Executive Summary

[55] [24]

[56] [62]

[57] [16]

[58] Again it needs to be stated clearly absolutely *nothing* within this paper should be construed and a criticism of Dr. Pollatsek (or O'Shea) - - all contained within is simply a critique of their argument, of the CUPM Standards, and a suggestion that their argument may contain errors, have unintended consequences, are not necessarily generalisable, and might need re-examination.

[59] [16]

[60] Emphasis added.

[61] [16, pg. 44]

[62] One can easily suppose Sputnik, the 'science gap,' the Apollo project, etc. provided fodder for the 1963 and 1965 Standards, but how could such a distinguished panel come to such agreement

the Writing Team contends, "In particular, previous reports focused on the undergraduate program for mathematics majors, although with a steadily broadening definition of the major. *CUPM Guide 2004* addresses the *entire* college-level mathematics curriculum, for *all* students, even those who take just one course."[63] This is, unfortunately, incorrect since (at least) the 1965 Standards addressed the curriculum in general whilst the 1963 Standards addressed the preparation of students for graduate school.

There were six broad recommendations made in the 2004 Standards which are summarised as:

(1) Recommendation 1: Mathematical sciences departments should Understand the strengths, weaknesses, career plans, fields of study, and aspirations of the students enrolled in mathematics courses;
   - Determine the extent to which the goals of courses and programs offered are aligned with the needs of students as well as the extent to which these goals are achieved;
   - Continually strengthen courses and programs to better align with student needs, and assess the effectiveness of such efforts.

(2) Recommendation 2: Every course should incorporate activities that will help all students progress in developing analytical, critical reasoning, problem-solving, and communication skills and acquiring mathematical habits of mind. More specifically, these activities should be designed to advance and measure students' progress in learning to
   - State problems carefully, modify problems when necessary to make them tractable, articulate assumptions, appreciate the value of precise definition, reason logically to conclusions, and interpret results intelligently;
   - Approach problem solving with a willingness to try multiple approaches, persist in the face of difficulties assess the correctness of solutions, explore examples, pose questions, and devise and test conjectures;
   - Read mathematics with understanding and communicate mathematical ideas with clarity and coherence through writing and speaking.

(3) Recommendation 3: Every course should strive to
   - Present key ideas and concepts from a variety of perspectives;
   - Employ a broad range of examples and applications to motivate and illustrate the material;
   - Promote awareness of connections to other subjects (both in and out of the mathematical sciences) and strengthen each student's ability to apply the course material to these subjects;

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then and such a distinguished panel as produced the 2004 Standards come to so very different a position?

[63] [16, pg. 9]

- Introduce contemporary topics from the mathematical sciences and their applications, and enhance student perceptions of the vitality and importance of mathematics in the modern world.

(4) Recommendation 4: Mathematical sciences departments should encourage and support faculty collaboration with colleagues from other departments to modify and develop mathematics courses, create joint or cooperative majors, devise undergraduate research projects, and possibly team-teach courses or units within courses.

(5) Recommendation 5: At every level of the curriculum, some courses should incorporate activities that will help all students progress in learning to use technology
- Appropriately and effectively as a tool for solving problems;
- As an aid to understanding mathematical ideas.

(6) Recommendation 6: Mathematical sciences departments and institutional administrators should encourage, support and reward faculty efforts to improve the efficacy of teaching and strengthen curricula.

None of the recommendations seem particularly odd, out-of-line, or problematic. However, the main trust of the recommendations seems to be on activities, delivery of material, methods of teaching, student needs, awareness, enhancing perceptions, etc. This author suggests that such are all well and good, such are nice and 'feel good,' but such are not that which should be the primary concern of faculty in mathematical sciences departments. The accent on delivery and appreciation of mathematics woven through the 2004 Standards, sadly, carries the whiff of Sophistry. What to teach - - -the content - - - needs to precede how to teach and pre-requisites are a part of that discussion. The content is the centre of the academy, it is why a student is enrolled, it is why the faculty are at the university, it is the very core of academe. The worth of a programme, how to improve a programme, or how effective a programme is or can be **must** focus first on the content, then follow with how well a student does in a course or courses and its pre-requisites, how well the faculty are engaged in teaching, research, and service, how well its graduates succeed in graduate school, the workforce, etc.

What binds and supports mathematics is a search for truth, a search for what works, and a search for what is applicable within the constraints of the demand for justification. It is not the ends, but the means which matter the most! We in mathematics differ from other areas of academia because the process of deriving an answer, the progression to an application, the method of generalisation, the proof or disproof of a claim are what are the nucleus of our discipline. Our area demands more than mere speculative ideas; it demands reasoned and sanguine justification! For a student to understand that student must opine, hypothesise, conjecture, describe the process of deriving an answer, explain the progression to the application, justify the method of generalisation - - these actions demand more than mere ‘mathematics appreciation,’ ‘busy work’ activities, or lectures - - they require objectivity; positive scepticism; and, a willingness to be wrong. The content studied is the keystone which holds an academic endeavour together; it is the reason for academe to exist; it is the reason for inquiry – the student, (needs, awareness, perceptions, etc.) and the instructor (activities, delivery of material, methods of teaching, etc.) should be and are secondary to the material in an academic setting. Again, the fact that the 2004 CUPM Standards do not describe in detail courses and the content contained within is very problematic and the position that is such a list is neither possible nor **desirable**[64] is, in this author’s opinion, a mistake. We would suggest that such a position should be reconsidered and the next set of Standards focus on content rather than superficially superfluous considerations.

Nonetheless, one *can* envision a course or some courses where no[65] pre-requisite or a minimum of pre-requsites *might* have a positive effect. Mathemtical Logic can be studied without a pre-requisite; Set Theory or Foundations of Mathematics[66] could be learnt quite well through inquiry-based learning (IBL) methods such as the Moore method without previous college-level material[67]

Whilst discussing the 2004 CUPM Standards, it would do the topic a disservice without mentioning the David Bressoud[68] article ”Launching From the CUPM Curriculum Guide: Keeping the Gates Open.”[69] Bressoud highlights some of the worst (in this author’s estimation) components of the Standards. He begins by highlighting the recommendation that, ”Mathematical topics and courses should be offered with as few prerequisites as feasible so that they are accessible to students majoring in other disciplines or who have not yet chosen majors.”[70] So, Bressoud begins with the idea of minimal pre-requisites and even suggests mathematical topics be offered with as few prerequisites as feasible so that they are *accessible* . . . ergo, what does accessibility imply or mean? Logically, in the most extreme of cases it means *no* pre-requisites which is contra to what Bressoud asserts when he states, ”and we do need prerequisites. [sic] One of the strengths of the mathematics curriculum is that students are building on what they have learned [sic] before. But what about the students who have completed have

[64] Emphasis added.

[65] Meaning no college-level pre-requisite.

[66] Also oft titled: Bridge to Higher Mathematics; Transition to Advanced Mathematics; Principles of Higher Mathematics; or, Introduction to Higher Mathematics. It is assumed that such a course is a course designed to transition the student from a computational or mechanical understanding of mathematics to a more abstract understanding of mathematics.

[67] However, it also is probably the case that it would be best learnt in the Sophomore or above year - - the argument being a student might be more intellectually mature.

[68] Bressoud wass on the Committee on the Undergraduate Programme in Mathematics (CUPM), has chaired CUPM, serves on the Curriculum Project Steering Committee, and the CUPM Writing Team.

[69] [4]

[70] [4, pg. 1]

completed Calculus I and are reluctant to continue with calculus? Should they be forever barred from learning about number theory or combinatorics or modern geometry?"[71]

It *seems* Bressoud offers a similar argument to O'Shea and Pollatsek's that there are 'roadblocks,' 'gates,' and courses should be independent but 'nourish' one another.[72] This author is perplexed by the idea that courses or material are independent but 'nourish.' Either there is a logical, persuasive, and acceptable reason for a pre-requisite or there is not. If there is a reasonable, sanguine, and justifiable reason for a pre-requisite it should be retained; if not, it should be dispensed with. If that is what O'Shea, Pollatsek, Bressoud, and CUPM mean it is not entirely clear that is what was meant; and, if that is what they mean then this author is four-square in favour of it.

However, it *seems* that is not the case because of previous points made about the O'Shea and Pollatsek article and Bressoud states, "Discrete Mathematics is the second widest gate in our curriculum, but students who are hooked by Elementary Statistics or Introduction to Statistical Modelling can move onto Applied Multivariate Statistics, Probability, and Mathematical Statistics with no other prerequisite."[73] To say this is flabbergasting is an understatement. How any student can take Probability or Mathematical Statistics at any reasonable level without Calculus and Set Theory is beyond (my) comprehension. How does that student study, learn, or master basic probability; deterministic versus stochastic functions; &, the Kolmogorov axioms of probability? How does that student study the theory of probability; claims about probability; conditional probability; independence versus non-independence; Bayes' Theorem; discrete random variables; probability mass functions (PMF); cumulative distribution functions (CDF); moments; specific PMFs or CDFs; and, claims about them? How does that student consider continuous random variables; probability density functions (PDF); cumulative distribution functions (CDF); the Gamma function; moment generating functions (MGF); or, claims about PDFs or CDFs? What of joint distributed random variables; conditional probability; marginal probability; statistical independence versus non-independence; joint probability mass or density functions; specific JPDFs; JCDFs; and claims about joint PDFs and PMFs; covariance; correlation; statisitical independence versus zero correlation; marginal distributions; conditional distributions; &, applications? How does the student handle the theory of expectation; expectation of a sum of random variables; conditioning; and applications; limit theorems; Tchebyshev's inequality; weak law of large numbers; the Central Limit Theorem (CLT); the strong law of large numbers; bounding; and, applications? To say nothing of estimation theory; sufficiency; bias; relative efficiency; consistency; robustness; the method of maximum likelihood;

[71] [4, pg. 2]
[72] [62, pg. 570 (paraphrased)]
[73] [4, pg. 3]

the method of moments; and, applications. How about the theory of hypothesis testing; statistical hypotheses; the Neyman-Pearson lemma; point estimates; confidence intervals; inferences about μ; inferences about σ^2; inferences about comparisons of parameters; and, the F and t distributions? It is not at all clear that a student can understand or study in a rigorous manner the theory of linear regression; non-linear regression; the method of least squares; parametric inferences; non-parametric statistics versus parametric statistics; etc. with no real college-level mathematics background! Hence, "students who are hooked by Elementary Statistics or Introduction to Statistical Modelling can move onto Applied Multivariate Statistics, Probability, and Mathematical Statistics with no other prerequisite,"[74] seems to indicate that the Probability and Mathematical Statistics mentioned by Bressoud is a watered-down application course and not a true mathematics course.[75]

The de-emphasis on pre-requisites in the 2004 CUPM Standards is troubling, perhaps, most especially because it is clear that the intent of the Standards and the authors of the Standards is to assist student learning. It is absurd to posit that any MAA or American Mathematical Society (AMS) member is intent on <u>not</u> having mathematics taught or wishes to obliterate mathematical learning. The 2004 CUPM Guidelines state, "students should indeed be equipped with the specific mathematical skills they need for their academic programs . . . student needs should be determined . . . they should also be equipped with mathematical knowledge that will remain with them and be useful in their careers and in their lives as contributing citizens."[76] But then a contradiction or contrarian position forwarded, "general education and introductory courses in the mathematical sciences should be designed to provide appropriate preparation for students taking subsequent courses, such as calculus, statistics, discrete mathematics, or mathematics for elementary school teachers. In particular, departments should determine whether students who enroll in subsequent mathematics courses succeed in those courses and, if success rates are low, revise introductory courses to articulate more effectively with subsequent course."[77] Said recommendation seems in conflict with "mathematical topics and courses should be offered with as few prerequisites as feasible so that they are accessible to students majoring in other disciplines or who have not yet chosen majors."[78]

Also, the recommendation, "faculty in partner disciplines do not want a calculus pre-requisite for introductory statistics. The fundamental ideas of statistics, such as the omnipresence of variability and the ability to quantify

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[74][4, pg. 3]

[75]Quite frankly, it is my opinion that a student is better served by having Calculus I - III, Linear Algebra, Set Theory, and Real Analysis I before that student takes the Probability and Mathematical Statistics sequence.

[76][16, pg. 30]

[77][16, pg. 31]

[78][16, pg. 38]

and predict it, are important subjects that can be studied without sophisticated mathematical formulations. In particular, the notion of sampling distribution-which underlies the concepts of significance testing and confidence interval-is challenging enough on its own to justify a first course in statistics."[79] Studied without sophisticated mathematical formulations? Is that not a watering down of material? An introductory statistics class at such a level is not appropriately a part of a mathematics major but is a fine general education or service course (one supposes) since the foundations of sampling theory rest upon Calculus and upper-level mathematics.

At the university where the author is currently engaged, the Biology Department does not require of its majors any Calculus. Such harms their students because most Biology Departments in the U.S. require at least through Calculus II. Students who graduate from the biology programme at Kutztown University of Pennsylvania (KUP) are at a disadvantage when compared with their contemporaries who graduate from other biology programmes where calculus is required. So, is it logical to adhere to requests from faculty in partner disciplines or is it better to determine that which is best for the mathematics programme and for students in other disciplines and then work with partner discipline faculty to improve the mathematics programme and in so doing to perhaps assist improving partner discipline programmes?

The recommendation that, "one of the most serious consequences of overly rigid prerequisite structures is that they unnecessarily deprive mathematical sciences departments of potential students for intermediate and advanced mathematics courses. Some prerequisites are necessary, but generally not as many as are typically required."[80] Really, where is the data to support this contention? The recommendation continues, "For example, if the only real requirement for a certain course is 'mathematical maturity,' then courses other than calculus may provide it. Many versions of discrete mathematics courses stress development of students' ability to prove and disprove mathematical statements, and some even contain brief introductions to number theory, combinatorics, or matrix theory. Such a course could be accepted as a prerequisite [sic] for linear algebra, number theory, abstract algebra, or combinatorics."[81] Yes, said *could* but *should* it? There is scant evidence to suggest many versions of discrete mathematics are apropos as a prerequisite for abstract algebra. Furthermore, are there not subjects that are better studied in the junior or senior year? For some faculty, 'mathematical maturity' is a valid reason for a pre-requisite and any student can be well accommodated if a course has pre-requisites but said can be modified 'by permission of the instructor.'

Moreover, the 2004 CUPM Standards contend that, "allowing students to

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[79] [16, pg. 38]
[80] [16, pg. 38]
[81] [16, pg. 38]

use discrete mathematics courses as prerequisites for higher-level mathematics courses can draw able and interested students into courses such as linear algebra, computational algebraic geometry, abstract algebra, and number theory, and increase the likelihood that they will decide to pursue a joint or double major with mathematics."[82] Perhaps this author has missed something; but, where is the evidence for this?

However, the 2004 CUPM Standards do make a good case for the study of a single area in depth and work on a senior-level project. However, once again the ugly pre-requisite problem compromises the recommendation. At the College where the author was previously engaged (for 17 years), the Mathematics Department required of its majors that students had to compleat the programme through Abstract Algebra I and Real Analysis I <u>before</u> taking Senior Seminar. There were only two tracks at Morehouse College (MC) and each was 'rigid' if viewed through the lens of the 2004 CUPM Standards.

The students' work on a senior-level project was, by and large, quite fruitful, of substance, and clearly was in depth. Contrast that with the university where the author is currently engaged, our department (sadly) **does not even require the Calculus sequence be completed before a student takes Senior Seminar!** Clearly the mathematics programme at Kutztown University of Pennsylvania is more in line with the 2004 CUPM Standards; but one must ask is this advisable, is it doing the students a favour, is it harming them, and what consequences follow from said?[83] When there is little or no common background at the end of a programme (as at KUP) when students are working on projects in Senior Seminar presents problems for the faculty member conducting the class. It is rather difficult for a student to do a meaningful, genuine, and in depth project when he has not had many (or any) advanced courses.

There are many examples of where a lack of pre-requisite does a student harm. One case is a mathematics secondary education major who took Math 301, Probability & Statistics I who did not have the Math 273 Calculus III (on a 4 three-hour sequence of Calculus I - IV) pre-requisite so could not prove that for $X \sim Pois(x, \lambda)$ it is the case that $\mu = \lambda$ since she had not studied sequences and series. A more serious problem occurred for a mathematics secondary education major in Math 351, Real Analysis I, who did

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[82] [16, pg. 46]

[83] I can only speak for myself insofar as I have witnessed some dreadful senior-level senior-level project at both institutions; but, let us say there are 15 students per class at each institution. There were, perhaps, three terrible Senior Seminar projects, five satisfactory Senior Seminar projects, and, seven outstanding Senior Seminar projects at Morehouse. On the other hand in the past four years at KUP there were, perhaps, ten or so terrible Senior Seminar projects, two or three satisfactory Senior Seminar projects, and, one or maybe two outstanding Senior Seminar projects at KUP. I opine there is not a huge difference in ability between and betwixt the students at each school, I opine it is the expectation that differs, the lack of pre-requisites at KUP that compromise the programme, and the existence of an apparent acceptance of mediocrity at KUP versus a culture of striving for excellence at MC that make the difference.

not have Calculus I - IV completed and who had a terrible Set Theory / Foundations of Mathematics background. The student was confused as to the nature of $\aleph_0$ and how $|\mathbb{N}| = \aleph_0$. Her difficulty was she 'perceived' there to be problems with $\mathbb{N}$ and $\aleph_0$ because

(1) $0 \notin \mathbb{N}$ (not to mention she thought $\notin$ meant 'not epsilon');
(2) how could $\mathbb{N}$ be infinite but be nothing (the zero (not understanding subscripts));
(3) why was the term 'function' used for absolute value; and,
(4) what this 'bijection stuff' meant.

The student had completed the Set Theory – Foundations of Mathematics pre-requisite but had (reportedly) never studied product sets, relations, image sets, functions, injections, surjections, cardinality, etc. though all are in the course description. This case illustrates that pre-requisites on paper can be satisfied but it is the content that matters (pre-requisite content and content under study at a particular time). Obviously there is no guarantee that students will do well in a course let us call it J_1 (a first course in the junior level) that has two sophomore courses (say S_1 and S_2) and three freshman courses (F_1, F_2, and F_3); but fulfillment of the pre-requisites - - a passing grade earned in S_1, S_2, F_1, F_2, and F_3 increases the possibility the student can 'handle' the J_1 course.

A third example is a project from Senior Seminar where a student was discussing his project and was considering $\int_0^\infty \left(e^{-2x} \cdot (x/3)^{10}\right) dx$ and said it was $\frac{5}{648}$. When asked what it was and why, there was a pregnant silence; finally the student responded, "I T-I-ed it."[84] When asked how he programmed it he replied he had not. Now, this student had Calculus I - IV before Senior Seminar but not Real Analysis yet there was *no discernable understanding of what was done.* The student seemed incapable of explaining the integral, integration by parts, L'Hôpital's Rule (and how it applies), or alternately the Gamma Function, etc. All that the student could say was, "I T-I-ed it." Is that authentic learning? How is that tied (pun intended) into a study of a single area in depth when the course work that came before had little or no depth since the courses were designed like O'Shea & Pollatsek's "ideal curriculum ... courses which are independent but which nourish one another by increasing the students' repertoire of mathematical examples and experience."[85] The depth is lost when the connections between and betwixt material is ignored or de-emphasised.

The author opines that anyone who holds either the opinion or Pollatsek & O"Shea or this author's opinion WANTS to hurt the students but faculty can wind up hurting the students in the manner:
the top students – by not going far enough or deep enough;
the middle students – the title of the course not being congruent

[84] The student used a Texas Instruments calculator.
[85] [62, pg. 570]

with the material;

the bottom students – by giving false hope (for example, Secondary Education Major repeating every course twice or more).

How can one understand the nuances and proper definition of function without the classical Aristotelian logic, axioms of the reals, point-set theory, product sets, and relations? How can an adequate proof be constructed by a student who has not said pre-requisite knowledge? How can a typical student conceive of the graph and the lines $x = 3, y = -4$, and then for any two lines $y = a$ and $y = b$ such that $a < -4 < b$ there exists lines $x = h$ and $x = k$ such that $h < 3 < k$ for each x where $h < x < k$ and $x \neq 3$ it is the case that $a < x^2 - 3x - 4 < b$ when that student has been 'machined' (used calculators and watched things rather than doing graphing) to death? How can an average student understand $\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \ni 0 < |x - 3| < \delta \Longrightarrow \left|x^2 - 3x\right| < \varepsilon$ in order to do a proof when said student has little or no background (pre-requisite)? **This is, the author opines, the central problem with the 2004 CUPM argument, perspective, and approach: minimisation of pre-requisites might not be an assistance to learning – it can be a hinderance!** A strong set of pre-requisites can be a help to a student so that said student is *mathematically mature* enough for a topic and ready to tackle some of the wonderful concepts that are a part of mathematics. The author opines that this perspective may not have been considered as seriously as it might have been by members of CUPM since (at least) the 1989 CUPM Standards.

It is worth re-emphasising and bears repeating there is tension that exists within the 2004 Standards between sections of the recommendations:

(1) B.3. Critically examine course prerequisites
Mathematical topics and courses should be offered with as few prerequisites as feasible so that they are accessible to students majoring in other disciplines or who have not yet chosen majors. This may require modifying existing courses or creating new ones. In particular,
   - Some courses in statistics and discrete mathematics should be offered without a calculus prerequisite;
   - Three-dimensional topics should be included in first-year courses;
   - Prerequisites other than calculus should be considered for intermediate and advanced non-calculus based mathematics courses.[86]

(2) C.4. Require study in depth
All majors should be required to
   - Study a single area in depth, drawing on ideas and tools from previous coursework and making connections, by completing two related courses or a year-long sequence at the upper level;

[86] [16, pg. 37]

- Work on a senior-level project that requires them to analyze and create mathematical arguments and leads to a written and an oral report.[87]

The author opines that the tension between these goals would be minimised or deleted if the members of the Committee on the Undergraduate Programme in Mathematics (CUPM) revisited the recommendation on pre-requisites, revised it, and considered whether or not the goal of accessibility through minimisation of pre-requisites is not in conflict with the goal of rigour. True accessibility must be married with the opportunity to learn *authentically and meaningfully* and one must ask, is there an opportunity to learn when one does not have the necessary tools (the background *not a machine to do it for one*) to learn?

[87] [16, pg. 48]

V. A Modest Proposal

Should we try to 'cram' mathematics down the throat of every student who enters the academy no matter the major? We think not for we think that mathematics is not indispensable. However, we opine that it is wonderful, so any student who wishes to learn it (is motivated as in the sense of [57]) should be given the opportunity so long as that student has the initiative, motivation, background necessary,[88] and ability to learn the material in a course. A student with an eighth-grade education rarely is admitted into a university; a student with a sixth-grade reading education cannot reasonably do well in college; a student with no course-work (and a non-speaker of Spanish) is not prepared for a course in La Literatura de la Generación del Noventa y ocho; so, why should the faculty of a mathematics department accept the notion that someone with (seemingly) and eighth-grade mathematics educational background be 'empowered' to attempt to study, let us say, Point-Set Topology?[89]

Is mathematics a subject that everyone at the university level should be *forced* to take? Probably not; but, if faculty at university determine that a course or courses in mathematics are required then it is the mathematics faculty's responsibility to design and teach said courses *and to ensure that said courses are not 'dumbed-down.'* Terms thrown about such as "student populations," "student audiences," "student demographics," etc. give the impression that there is some sort of elitist or inequitable attitude behind said terms. Is it code to infer that the students of 2010 are less capable than the students of 1990; the students of 1990 are less capable than the students of 1970; or, the students of 1970 are less capable than the students of 1950? We do not believe any mathematician could reasonably hold such views but we do opine that in 2010 the material discussed or the exposition of the material in many undergraduate texts are considered with less rigour, less detail, more pictures and more 'fluff' than undergraduate texts of 1950. Therefore, it may be the case that book publishers; members of the National Council of Teachers of Mathematics; faculty of a College of Education; or, the staff of a state Department of Education could believe the students of 2010 are less capable than the 1950 but we <u>should not</u> and we should not accept a watering-down of the undergraduate mathematics canon.

Furthermore, is it worth the effort to 'cram' mathematics down the throat of every student who enters a university no matter the major? Is it constructive? Would it not behoove us (members of the Mathematical Association of America (MAA) and the American Mathematical Society (AMS)) to attempt to construct a set of material (a criterion) which a student at the primary level should master; a criterion which a student at the secondary

[88] A background that can be verified or checked that gives the student a reasonable opportunity to learn or succeed. No one can be certain but having a pre-requisite fulfilled gives an instructor at least some assurance that the student has a reasonable chance to grasp the material in a class.

[89] And what we mean by studying Point-Set Topology is actually studying it and not taking a 'mathematics appreciation' course or some watch-in-awe computer simulation recitation.

level should master; a standard which a student at the undergraduate level should master; a benchmark which a student at the master's graduate level should master; and, a basic standard which a student at the Ph.D. graduate level should master? Would that not be a constructive discussion which could lead to some impassioned debates, some insightful discussions, and some possible areas of agreement (and some areas of disagreement which is fine since there is not one programme for everyone). If such criterion were established, then not meeting the standard would be deemed unacceptable and a pre-requisite not fulfilled. Such would allow for a measure of students' preparation for secondary school, undergraduate school, or graduate school. Clearly such would be on an ordinal scale, but it would be better than that which exists currently.

The education that we obtained and earned was not derived effortlessly; there was hard work involved but we enjoyed (most of it) because it was that which we desired to study. We found interesting concepts, theories, applications, or things in mathematics at each level. The idea of rushing students to concepts or material beyond the students' capabilities (at the level which a student is at a given point in time) is not ideal and could be detrimental. Improper use of technology or the introduction of "contemporary topics from the mathematical sciences and their applications ... (or) presentation of contemporary topics and at least glimpses of current fields of research,"[90] could actually be less helpful than a focus on depth of topics (especially in lower-level undergraduate courses).

Adhering to pre-requsites and enforcing said does not *guarantee* a student is properly prepared for a course, but having such minimally seems to increase the possibility of a student not being prepared.[91] Is it not the case that as Bressoud noted, "one of the strengths of the mathematics curriculum is that students are building on what they have learned [sic] before."[92] What a person learnt in a course that is not used in subsequent courses oft becomes forgotten. Again the strength of the undergraduate mathematics curriculum and of mathematics in general is that concepts are built one upon another and if there is a firm understanding of, for instance, Set Theory, much (if not all) of subsequent course material can be mastered well. The author opines that most students learn a concept in a course, let us call it P_1, superfluously or tritely at best. The concepts are not *really* learnt until

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[90] [16, pg. 20]

[91] My background illustrates the problem. When I was a student at Emory University I had a fantastic professor, David Doyle, for Algebra (taught under the Moore method). I learnt much deeply but not much broadly. When I was a student at Auburn University I had a well-intended professor who lectured, where one only saw the back of his head during class as the notes were written on the board (perfect, errorless notes), who quizzed with questions such as, "state and prove theorem X," so, I memorised his notes and received and "A" each semester. Therefore for Math 631 I had the pre-requisites, for Math 632 I had the pre-prequisites, etc. but I had not the pre-requisite knowledge. So, the key point is we are building on what **we learnt** previously.

[92] [4, pg. 2]

a course, or two, or three later (let us call them C_1, C_2, and C_3)! The subsequent courses need not be in a chain but each had P_1 as a pre-requisite (and provided the courses are designed well so that the material from P_1 is used authentically). However, this learning environment is stunted or destroyed if the instructor of C_1, C_2, or C_3 does not hold the students responsible to have learnt the material in P_1. The student has the opportunity to *really* learn because the concepts introduced in P_1 are no longer 'new' but are 'familiar' and 'recalled' therefore reinforcing the ideas from P_1 whilst learning material in C_1, C_2, or C_3. Skipping over, minimising, or 'remediating on the fly' material from P_1 in order 'expose' students to the concepts in C_1, C_2, or C_3 robs the student of the opportunity to learn well and in depth! Therefore, we submit, to use some of the verbs Pollatsek & O'Shea used, the student is cheated, brutalised, and is haunted because that student knows not the foundation of the material in C_1, C_2, or C_3! The student is in fear (**and rightly so**) because the student has the nagging, aching, and irksome inkling that something is missing and are left not empowered but dependent on a machine, on another person, or on hoping they are right but not *knowing* they are correct. Minimisation or deletion of pre-requisites can have the <u>exact</u> <u>opposite</u> <u>effect</u> of the intent of O'Shea, Pollatsek, Bressoud, and CUPM - - creating a *poor rather than strong* learning environment for a student.

So, might we diffidently suggest that the members of the Committee on the Undergraduate Programme in Mathematics (CUPM) go beyond simply revisiting the recommendation on pre-requisites as suggested in the previous section of this paper but revisit the recommendations and use the 1965 and 1963 CUPM Standards as a road-map for a new set of CUPM standards in this decade? New course sequences could be developed and described (in detail) versus the idea that "a list is neither possible nor desirable."[93] Further, the members of CUPM could broaden the curriculum and involve some research mathematicians (as was the case with the 1960 CUPM Panel) by designing revisions in conjunction with a Select Panel of the American Mathematical Society (AMS). Finally, would it not behoove the faculty of every mathematics department at every college or university in the U.S.A. to have the MAA (spear-headed by CUPM) also create a suggested curriculum for primary and secondary school? The assistance that such would provide to the members of the National Council of Teachers of Mathematics (NCTM) is congruent with the assistance the American Mathematical Society (AMS) would provide the Mathematical Association of America (MAA); and, is that not a good thing to do? Is it not productive and advantageous to have the NCTM, MAA, and AMS all working together?

[93][16, pg. 44]

VI. Summary and Conclusion

It is in that spirit that a core point of the argument presented in the paper is that <u>the content studied</u> is the keystone which holds an academic endeavour together; it is the reason for academe to exist; it is the reason for inquiry – the student and the instructor should be and are secondary to the material in a university setting. The experience of **doing** mathematics rather than **witnessing** mathematics is that which we, a mathematics faculty, need encourage; for it is that which we love to do as well - math! Such cannot be done in a vacuum, such cannot be done where courses ... are independent but which nourish one another by increasing the students' repertoire of mathematical examples and experience."[94] There needs to be connections between and betwixt courses, pre-requisites or co-requisites that naturally link the concepts, principles, theory, and applications of the mathematics that are contained with the mathematics canon and most especially within an undergraduate mathematics programme.

The author of this paper is an adherent of the Moore method and attempts to use a modified Moore method (MMM) when teaching any course to actualise an inquiry-based learning (IBL) experience for the students. IBL, MMM, and the Moore method are interlinked are based on Moore's Socratic philosophy of education - - the student must master material by *doing*; not simply discussing, reading, or seeing it and that *authentic* mathematical inquiry relies on inquiry though 'positive scepticism' (or the principle of *epoikodomitikos skeptikistisis*). 'Positive scepticism' is meant to mean demanding objectivity; viewing a topic with a healthy dose of doubt; remaining open to being wrong; and, not arguing from an *a* priori perception. If one happens upon a fact but really does not know why the fact is indeed so, does he really know the thing he claims to know? The author is fond of quoting his late mother, "mean what you say and say what you mean," (perhaps that accounts for the apparent harsh tone of the critique in this paper) and, "you can put lipstick and pearls on a pig and it is still a pig" (This means 2004 CUPM Guidelines evoke, at least a whiff of, Sophistry. The author opines that the 2004 CUPM Guidelines are a watered-down version of what MAA Standards should be or are capable of being (see [9] and [10] for examples of Standards, which the author opines, *genuinely* set reasonable goals, standards, and attainable rigour for an undergraduate mathematics programme.).

One of the reasons for academia to exist is to assist an undergraduate's progress toward authentic understanding of mathematics; to encourage thought; to encourage deliberation; to encourage contemplation; and, to encourage a healthy dose of scepticism so that one does not wander too far into a position of subservience, 'give-me-the-answer'-ism, or a position of arrogant 'know-it-all'-ism. Thus, there is a need for some objectivism predicates any true learning. Unfortunately, it seems to be the case that much

[94] [62, pg. 570]

of the American education system from the primary through the secondary level is more attuned to the Sophistic rather than Socratic ideal and that much Sophistic principles and concerns are leeching into the post-secondary curriculum. Such needs to be halted and reversed.

If the Mathematical Association of America (MAA) and the Committee on the Undergraduate Programme in Mathematics (CUPM) revises the 2004 Standards and creates standards that focus on content, rigour, and authentic learning (much like the 1965 and 1963 CUPM Standards), it is the opinion of the author that said would do much to improve mathematics education and education in general.

O'Shea & Pollatsek, Bressoud, and the members of CUPM all were well-intended; but, this author opines they erred. However, it may be that I am erring in arguing that they erred; I really do not know. However, I offer this paper as, let us say, the start of a conversion on both topics touched upon in this tome. That is, is mathematics indispensable? Probably not, but math is fun; maths are fun; and, the study of said is a worthy endeavour. Are pre-requisites needed in mathematics courses? Most certainly yes because learning requires doing; only through inquiry is learning achieved, and mathematical thought is one that must be focused on the process of deriving a proof, constructing an adequate model of some occurrence, or providing connection between and betwixt the two and that builds and cannot be chopped up in such a manner that each course in the mathematics canon is independent of another.

It is facile to find fault with others' arguments and not offer alternatives as it is easy to criticise and tear down but difficult to commend and build. As for Dudley's adroit position, the author of this paper has attempted to follow-up his argument and extend it to include the idea that concern for mathematics courses before the calculus ahould be of secondary concern when comapred to the calculus and later courses. Further, this author has hopefully not only critiqued the O'Shea, Pollatsek, and CUPM position on pre-requisites but offered a propitious prelusive beginning point to build an alternative.

To close, the author tries to include the Halmos quote of Moore in each paper he writes (sometimes it is not included in the most elegant of ways - I fear such in this instance). P. J. Halmos recalled a conversation with R. L. Moore where Moore quoted a Chinese proverb. That proverb provides a summation of the justification of why pre-requisites are needed now more than ever - - it states, "I see, I forget; I hear, I remember; I *do* , I *understand*." One cannot *do* without a *firm* foundation (at any level in any inquiry).

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