Teaching areas of polygons: An alternative approach

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Abstract: Measurement is an integral component of the PK-12 mathematics curriculum. At various grade levels, much attention is given to the areas of geometric shapes. However, there is research evidence that measurement is problematic for US students. In this paper, the merits and demerits of two possible sequences (conventional and alternative) in teaching areas of polygons are discussed in the context of current reforms in mathematics education. Although both sequences are viable, the alternative sequence is suggested because it seems to be more consistent with the process standards (problem solving, reasoning and proof, communication, connections, and representation) of the National Council of Teachers of Mathematics (NCTM).

Key words: measurement; area; polygon; inductive and deductive reasoning

1. Introduction

Unquestionably, measurement is a significant component of the PK-12 mathematics curriculum. It is one of the content standards of the National Council of Teachers of Mathematics (NCTM) and the Georgia Performance Standards (GPS). NCTM (2000) states that "The study of measurement is important in the mathematics curriculum from prekindergarten through high school because of the practicality and pervasiveness of measurement in so many aspects of everyday life" (p. 44). NCTM goes on to say that a study of measurement involves the application of other aspects of mathematics (for example, number and operations) and the connection to other subject areas (for example, science).

In particular, much attention is devoted to determining the areas of geometric shapes at various grade levels. In the GPS, the area of simple geometric figures is introduced in Grade 3 (M3M4) and it is extended in Grade 5 (M5M1) and Grade 6 (M6M2, M6M3) (Georgia Department of Education website). NCTM also recommends that the area of simple shapes be addressed in Grades 3-5 and in the middle grades. Unfortunately, there is research evidence which shows that measurement is a problematic concept for US students. For example, according to the National Assessment of Educational Progress (NAEP), "Students across all three grade levels had difficulty with perimeter and area concepts, especially situations in which they had to explain or justify answers" (Silver & Kenney, 2000, p. 194). With regard to area in particular, NAEP stated, " ... the concept of area is often difficult for students to understand ... " (Silver & Kenney, 2000, p. 223).

In teaching the areas of the rectangle, square, parallelogram, rhombus, triangle and trapezoid, it is important to consider the logical sequence in which to introduce the various shapes so that concepts can build on one another. In this article, the merits and demerits of two sequences are examined and discussed.

2. Conventional sequence

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When using this sequence, the teacher typically starts with the rectangle and establishes the formula $A = L \times W$ by using square tiles or grid paper and inductive reasoning. Some teachers tell students the formula without having students do any exploratory or investigative work. On no account would the author recommend or support this practice. Having established the formula for the area of a rectangle, the teacher can help students to deduce the formula $A = S^2$ for a square, because the square is a special rectangle ($L = W = S$).

In the next step of the sequence, the parallelogram is addressed (From the rectangle it can be moved to the triangle. The author will return to this later). The formula $A = b \times h$ for the area of a parallelogram is usually derived by transforming the parallelogram into a rectangle by cutting off the right triangle from one end and moving it to the other end (Usnick, Lamphere & Bright, 1992; Peterson, 1990). This can be accomplished by getting students to actually cut out the triangle and move it. The base and height of the parallelogram become the length and width respectively of the rectangle. Graph or grid paper works well for this activity. Thus the area of the parallelogram is the same as the area of the rectangle; therefore, the formula for the area of a parallelogram is $A = b \times h$. It can then be argued that because the rhombus is a special parallelogram, the formula for its area is the same ($A = b \times h$).

The formula, $A = ½b \times h$, for the area of a triangle can be derived in at least two ways. A parallelogram can be cut along one of the diagonals thus obtaining two congruent triangles, each of which has an area that is one-half the area of the parallelogram (Usnick, Lamphere & Bright, 1992); the congruence of the triangles can be confirmed by superimposing one on the other; so the area of a triangle is $½b \times h$. Alternatively, students can arrange two congruent triangles to form a parallelogram; so the area of each triangle is one-half the area of the parallelogram. In each case, the base and height of the parallelogram form one of the base-height pairs of the triangle. Students can also cut a rectangle along one of the diagonals to obtain two congruent right triangles, each of which is half the rectangle. This idea can be applied to obtain the area of any triangle (Usnick, Lamphere & Bright, 1992). If the perpendicular, h, is drawn from one vertex of the triangle to the opposite side, b, the triangle is divided into two right triangles, each of which is half of a rectangle. The area of one right triangle is $½b_1h$ (because it is half of the rectangle with base b_1 and height h); and the area of the other right triangle is $½b_2h$ (because it is half of the rectangle with base b_2 and height h) where $b = b_1 + b_2$. Thus the area of the large triangle is $½b_1h + ½b_2h = ½(b_1 + b_2)h = ½bh$.

The formula, $A = ½(a + b)h$ (where a and b are the lengths of the parallel sides and h is the height—distance between the parallel sides), for the area of a trapezoid can be derived in several ways. In some of these ways, the trapezoid is divided into shapes, the areas of which can be found by using previously derived formulas (Usnick, Lamphere & Bright, 1992; Peterson, 1990). We will look at four ways to dissect the trapezoid:

(1) Divide the trapezoid into two triangles by drawing a diagonal;

(2) Divide the trapezoid into a parallelogram and a triangle;

(3) Divide the trapezoid into two triangles and a rectangle;

These three approaches involve deductive reasoning and some algebra. They may be appropriate for high school students.

(4) Students can cut out two identical trapezoids (with the lengths of the parallel sides being a and b and the distance between the parallel sides being h) and arrange them to form a parallelogram. The area of each trapezoid equals to ½ of the area of the parallelogram.

Some of the merits of the approaches within this sequence are use of multiple representations, use of hands-on activities, and use of inductive reasoning and deductive reasoning. Two demerits are firstly, some require much use of algebra so they may not be appropriate for middle school students. Secondly, and perhaps more important, is that the relationships among the formulas may not be easily recognized and students end up with having to memorize several formulas. Is there a sequence which will enable students to recognize the connections among the formulas? The author thinks there is. This brings us to an alternative sequence which, hopefully, will eliminate the demerits of the conventional sequence.

3. Alternative sequence

Suppose the formula, $A = \frac{1}{2}(a + b)h$, for the area of the trapezoid, where a and b are the lengths of the parallel sides and h is the height, can be established without telling (One suggestion on how to establish this formula is outlined later). It can be argued that the rectangle, parallelogram and triangle can be obtained from simple transformations of the trapezoid (Craine & Rubenstein, 1993; Usnick, Lamphere & Bright, 1992). For example, the trapezoid becomes a parallelogram when the parallel sides are congruent; that is, $a = b$. When $a = b$, the formula, $A = \frac{1}{2}(a + b)h$, becomes $A = \frac{1}{2}(b + b)h = b \times h$.

The trapezoid becomes a rectangle when the parallel sides are congruent and perpendicular to the other two sides. The triangle can be considered as a “degenerate” trapezoid (Usnick, et al., 1992) with the length of one of the parallel sides being zero.

When $a = 0$, for example, the formula, $A = \frac{1}{2}(a + b)h$, becomes $A = \frac{1}{2}(0 + b)h = \frac{1}{2}b \times h$.

In this sequence, the formulas for the rectangle, parallelogram and triangle are derived from one formula, that of the trapezoid; so the students can recognize the connections among the formulas. The students then do not have to memorize several formulas; only one is needed because all the other formulas are derived from this one. There is extensive use of deductive reasoning which may be a challenge for middle school students. On the other hand, don’t we want students to develop reasoning skills? What better way for students to develop these skills than to provide tasks that will induce them to use these skills?

If this alternative sequence is to be used, the formula for the area of the trapezoid has to be established first. How can this be done without telling the students? We can establish Pick’s rule which states that the area of any simple polygon drawn on a lattice so that each vertex of the polygon is on a lattice point is given by $A = \frac{1}{2}b + I - 1$, where A is the area of the polygon, b is the number of dots on the boundary of the polygon, and I is the number of dots inside the polygon (Hirsch, 1974; Naraine & Hoosain, 1998; Russell, 2004). This rule can be established using square dot paper and inductive reasoning. Pick’s rule can then be used with square dot paper and inductive reasoning to establish $A = \frac{1}{2}(a + b)h$ for the trapezoid. For example, each student can draw a trapezoid on square dot paper so that Pick’s rule can be applied to find the area. The lengths of the parallel sides (a and b), height (h), and area (A) for each trapezoid can be obtained from the drawing and inserted in a table. The relationship among a, b, h and A can be investigated. The identification of the relationship may require hints and skillful questioning by the teacher.

4. Conclusion

Both sequences have strengths and weaknesses. Both emphasize reasoning (inductive and deductive), multiple representation and connections among mathematical concepts. The sequence to use depends on the grade level, objective of the lesson, available resources and the teacher’s beliefs about mathematics and the teaching of mathematics. The alternative sequence has one major advantage over the conventional sequence. With the conventional sequence, the learner has to memorize several formulas; whereas with the alternative sequence, only one formula (that of the trapezoid) has to be memorized. All other formulas are derived from this one. Therefore, the use of the alternative sequence is recommended.

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