WCER Working Paper No. 2009-3
April 2009

Middle School Students' Conceptual Understanding of Equations: Evidence From Writing Story Problems

Martha W. Alibali
Department of Psychology/
Wisconsin Center for Education Research
University of Wisconsin–Madison
mwalibali@wisc.edu

Alayna N. Brown
Department of Psychology
University of Wisconsin–Madison

Ana C. Stephens
Department of Curriculum and Instruction/
Wisconsin Center for Education Research
University of Wisconsin–Madison
acstephens@wisc.edu

Yvonne S. Kao
Department of Psychology
Carnegie Mellon University
ykao@andrew.cmu.edu

Mitchell J. Nathan
Department of Educational Psychology/
Wisconsin Center for Education Research
University of Wisconsin–Madison
mnathan@wisc.edu

WCER
Wisconsin Center for Education Research
School of Education • University of Wisconsin–Madison • http://www.wcer.wisc.edu/

Copyright © 2009 by Martha W. Alibali, Alayna N. Brown, Ana C. Stephens, Yvonne S. Kao, and Mitchell J. Nathan
All rights reserved.

Readers may make verbatim copies of this document for noncommercial purposes by any means, provided that the above copyright notice appears on all copies.

WCER working papers are available on the Internet at http://www.wcer.wisc.edu/publications/workingPapers/index.php. Recommended citation:

> Alibali, M. W., Brown, A. N., Stephens, A. C., Kao, Y. S., & Nathan, M. J. (2009). *Middle school students' conceptual understanding of equations: Evidence from writing story problems* (WCER Working Paper No. 2009-3). Madison: University of Wisconsin–Madison, Wisconsin Center for Education Research. Retrieved [e.g., April 15, 2009,] from http://www.wcer.wisc.edu/publications/workingPapers/papers.php

The research reported in this paper was supported by the McDonnell Foundation (Grant No. JSMF 97-56), the Interagency Education Research Initiative (Grant No. REC 0115661), and the Wisconsin Center for Education Research, School of Education, University of Wisconsin–Madison. Any opinions, findings, or conclusions expressed in this paper are those of the authors and do not necessarily reflect the views of the funding agencies, WCER, or cooperating institutions.

Middle School Students' Conceptual Understanding of Equations: Evidence From Writing Story Problems

Martha W. Alibali, Alayna N. Brown, Ana C. Stephens, Yvonne S. Kao, and Mitchell J. Nathan[1]

One goal of mathematics instruction in the middle school years is to help students become skilled at using the symbol system of algebra. Facility with the symbol system of algebra involves many component skills and concepts: knowledge of procedures for algebraic manipulation, the ability to use equations and inequalities to symbolize mathematical situations, and the ability to understand mathematical relations that are expressed symbolically.

These skills and concepts are important but difficult for students to learn. The literature is replete with reports of middle and high school students' difficulties in solving algebraic equations (e.g., Koedinger & Nathan, 2004), interpreting symbolic expressions (e.g., Stephens, 2003), and symbolizing mathematical situations (e.g., Clement, 1982; Heffernan & Koedinger, 1997; Kenney & Silver, 1997; MacGregor & Stacey, 1993). Students' difficulties are often construed as indicating gaps in their conceptual understanding of mathematical symbols. The claim is that because students' conceptual understanding is lacking, they sometimes misapply procedures learned by rote or generate symbolic expressions that are syntactically incorrect or do not appropriately capture the mathematical relations they wish to express.

How can students' conceptual understanding of symbolic expressions be assessed? In past work, researchers have asked students to solve algebraic equations (e.g., Herscovics & Linchevski, 1994) or to translate word problems into algebraic equations (e.g., Swafford & Langrall, 2000). For students who have had some exposure to instruction in the symbol system of algebra, however, such tasks may be viewed as routine. Students can succeed on routine tasks without conceptual understanding if they have learned procedures by rote. We therefore argue that students' performance on such tasks does not provide full information about their conceptual understanding. Instead, novel tasks are needed to provide a more accurate picture. Students given a novel task do not have readily available procedures for solution and must therefore rely on conceptual understanding to guide their approach to the task.

In the present study, we used the novel task of generating a story to correspond with a given equation as a means of investigating middle school students' conceptual understanding of algebraic equations. We also asked students to solve a set of symbolic equations so we could assess the relationship between their conceptual understanding and their equation-solving abilities.

To preview the findings, our results suggest that middle school students have substantial difficulty generating stories to correspond with algebraic equations. Not surprisingly, students who were more successful generating stories were also more successful solving such equations.

---

[1] We would like to thank Kate Masarik, Leslie Tengwall, and Paige Larson for assistance with data collection, and Carrie Pritchard for assistance with coding. Correspondence concerning this article should be addressed to Martha W. Alibali, Department of Psychology, University of Wisconsin, 1202 West Johnson Street, Madison, WI 53706. Email: mwalibali@wisc.edu

Students made many types of errors in generating stories. The nature of these errors revealed two broad areas of concern in students' conceptual understanding. First, students' errors indicated that their conceptual understanding of some arithmetic operations—in particular, multiplication—was weak or incomplete. This finding is consistent with research documenting middle school students' difficulties in identifying which operations need to be performed to *solve* story problems (Sowder, 1988) and reports that eighth-grade students' intuitive understandings of multiplication are weaker than their understandings of addition (Dixon, Deets, & Bangert, 2001). Second, students' errors indicated that they had difficulties combining multiple operations into coherent stories. This result is reminiscent of findings that students have difficulties solving and symbolizing story problems that involve multiple operations (Heffernan & Koedinger, 1997, 1998; Koedinger, Alibali, & Nathan, 2008).

Method

Participants

Participants in the study were 257 students (213 sixth-grade students and 44 seventh-grade students) from a middle school in Boulder, Colorado. Students in both grades utilized the *Connected Mathematics* curriculum (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). All students had experience solving equations, but none had been exposed to the novel task of writing a story that could be represented by a given equation. Due to absences, 13 students did not complete the equation-solving task, and three students did not complete the story-writing task.

Materials

For the equation-solving task, students were asked to solve for *n* in each of 12 equations. The equations varied systematically along three parameters: position of the unknown (start vs. result), number of operations (one vs. two), and operation type (addition, subtraction, or multiplication for one-operator equations and addition-subtraction, multiplication-addition, or multiplication-subtraction for two-operator equations). The problems used are presented in the appendix. Order was counterbalanced across two different test forms.

For the story-writing task, students were given a set of equations and asked to write corresponding stories. The equations were generated using the same three parameters used in the equation-solving task, resulting in a total of 12 types of equations. These equation types were divided into two sets, which we refer to as *versions*, each of which contained three result-unknown equations and three start-unknown equations. Version A included result-unknown addition, result-unknown multiplication, result-unknown multiplication-subtraction, start-unknown subtraction, start-unknown addition-subtraction, and start-unknown multiplication-addition problems; Version B included result-unknown subtraction, result-unknown addition-subtraction, result-unknown multiplication-addition, start-unknown addition, start-unknown multiplication, and start-unknown multiplication-subtraction problems. In addition, for each equation type, two different number sets were used. Finally, each set was presented in forward and reverse order. Thus, there were a total of eight different story-writing forms, each consisting of six equations (two versions × two number sets × forward/reverse order). The equations used in the story-writing task are presented in the appendix.

To minimize demands on their creativity, students were instructed to use one of eight scenarios as the basis for their stories. The following scenarios were printed at the top of each page of the story-writing booklet: (a) Kevin lives on a farm, (b) Nicole is going shopping, (c) Ian collects CDs, (d) Emily is playing basketball, (e) Tara is saving to buy a bicycle, (f) Mike is baking cookies, (g) Alayna has some M&M’s, and (h) Beth is having a birthday party. Students were told that they did not have to use all eight of the scenarios when writing their stories and that they could use the same scenario more than once. To clarify the task, students were given a sample equation, $22 - 8 = n$, accompanied by the following sample solution: “Kevin lives on a farm. He had 22 pigs, but he sold 8 of them. How many pigs does he have left?”

Procedure

The classroom teachers administered the paper-and-pencil assessments. Each student was randomly assigned to one of the two equation-solving forms and one of the eight story-writing forms. One of the two participating teachers administered both assessments on the same day; the other administered them on two consecutive days. All students completed the story-writing assessment before the equation-solving assessment.

Students were instructed to show all of their work, making no erasures, and to draw a circle around their final answers on the equation-solving form. Calculators were not allowed. The teacher collected the forms at the end of each testing session.

Coding

***Equation-solving task*.** Students’ solutions to the equation-solving task were given a score of 1 if they were correct or if they showed evidence of a correct procedure with a computational error. Solutions that were otherwise incorrect were given a score of 0.

Story-writing task. Students’ solutions to the story-writing task were given a score of 2 if they were well-formed story problems that corresponded with the numbers and operations in the given equation, a score of 1 if they were incorrect attempts, and a score of 0 if no attempt was made. Cases in which students solved a given equation for n and then integrated that solution into the story rather than posed a question were also treated as correct. For example, for the equation $19 + 33 = n$, one student wrote, “Ian has 19 CDs one month. The next month, he collected 33 more. Now he has 52 CDs.” This story was considered correct because it correctly corresponds with the given numbers and operations.

Each incorrect solution was assigned one or more codes describing the nature of the error(s). Error categories and accompanying examples are presented in Table 1. To assess the reliability of the coding procedures, a second trained coder recoded 10% of the story-writing data. Agreement was 84% for identifying errors and 83% for classifying errors into categories.

Table 1
Error Categories and Examples

Category of Error	Equation	Example	Percentage of problems with error
No response		(Student leaves problem blank)	3.1
Incomplete story	$6 \times n + 23 = 89$	Ian collects CDs. He was trying to figure out how many he has.	1.4
Wrong operation	$63 + n - 13 = 91$	Alayna has 63 M&M's and she gives some to a friend. Then another friend gives her 13 M&M's. Now she has 91 M&M's. How many did she give her friend?	5.3
Missing mathematical content	$45 - n = 21$	Kevin has some pigs. He gave away a certain amount. Now Kevin has 21 pigs. How many pigs did Kevin give away?	5.5
Added mathematical content	$6 \times 13 = n$	Alayna has some M&M's. She has 6 of them, but she buys 13 more bags that hold 6 each. How many does she have now?	5.6
No story action	$6 \times 13 = n$	Ian has 6 × 13 CDs. How many CDs is that?	8.5
Wrong question	$63 + 41 - 13 = n$	Ian had 63 CDs and got 41 new ones. 13 of the new CDs didn't work. How many new CDs did work?	3.0
No end statement	$6 \times 13 = n$	Tara is saving for a bicycle. She is making $13 an hour for watching her younger brother. She watches him for 6 hours.	3.5
Conversion of two-operator to one-operator equation	$21 \times 4 - 17 = n$	Mike is baking cookies. He has 84 cookies made. Then the dog eats 17. How many cookies does Mike have left?	1.7
Conversion of start-unknown to result-unknown	$45 - n = 21$	Sara has 45 pencils. She broke 21 pencils. How many are left?	3.4

Results

We focus first on how structural characteristics of the equations (position of unknown, number of operations, and operation type) influenced students' performance on the two tasks. We then examine the most common types of student errors on the story-writing task, with an eye toward investigating what such errors suggest about students' conceptual understanding of algebraic equations. Unless otherwise noted, all reported statistics are significant with alpha set at .05.

Equation-Solving Performance

On average, students succeeded on 9.8 out of 12 equation-solving items. Figure 1 presents the proportion correct for each problem type. It is apparent in the figure that both number of operations and unknown position influenced students' performance on equation solving. To statistically evaluate these patterns, we utilized 2 (number of operations: one vs. two) × 2 (unknown position: result vs. start) × 2 (grade level: six or seven) repeated measures analysis of variance (ANOVA), with repeated measures on the number of operations and unknown position factors. We utilized number correct as the dependent variable. Surprisingly, sixth-grade students performed slightly better than seventh-grade students, $F(1, 242) = 3.8$; this may have been due to the fact that the sixth-grade sample included students in accelerated classes, whereas the seventh-grade sample did not. Students performed better on one-operator equations than on two-operator equations, $F(1, 242) = 120.2$, and this effect was greater for sixth-grade students than for seventh-grade students, $F(1, 242) = 6.7$. Students also performed better on result-unknown equations than on start-unknown equations, $F(1, 242) = 95.8$. The combination of two-operator, start-unknown proved to be more difficult than would be expected based on the additive effects of the two factors, $F(1, 242) = 16.8$. The remaining interactions were nonsignificant.

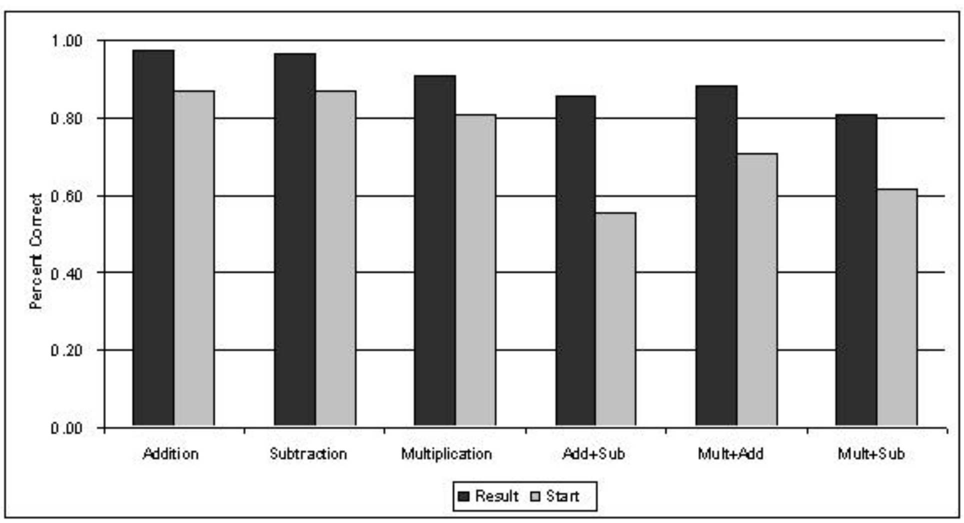

Figure 1. **Mean percentage correct on the equation-solving task for each operation or operation combination and each position of the unknown.**

Story-Writing Performance

On average, students generated correct stories for 3.8 out of 6 story-writing items. Figure 2 presents the proportion correct for each problem type. As was the case for equation solving, both number of operations and unknown position influenced students' performance on story writing. To statistically evaluate these patterns, we again utilized 2 (number of operations: one vs. two) × 2 (unknown position: result vs. start) × 2 (grade level: six or seven) repeated measures ANOVA, with repeated measures on the number of operations and unknown position factors. We utilized average score on problems of each type as the dependent variable. Note that because each participant solved six problems, some of the scores were based on two problems, and others were based on a single problem. Sixth-grade students outperformed seventh-grade students by a small margin (sixth $M = 1.59$ vs. seventh $M = 1.46$ out of 2), $F(1, 252) = 4.8$. There were significant main effects of both number of operations, $F(1, 252) = 169.9$, and unknown position, $F(1, 252) = 31.5$. Students were more successful writing stories for one-operator equations than for two-operator equations, and they were more successful writing stories for result-unknown equations than for start-unknown equations. The interaction of these two factors was also significant, $F(1, 252) = 4.7$, indicating that the combination of two-operator, start-unknown was more difficult than would be expected based on the additive effects of the two factors. All other interactions were nonsignificant.

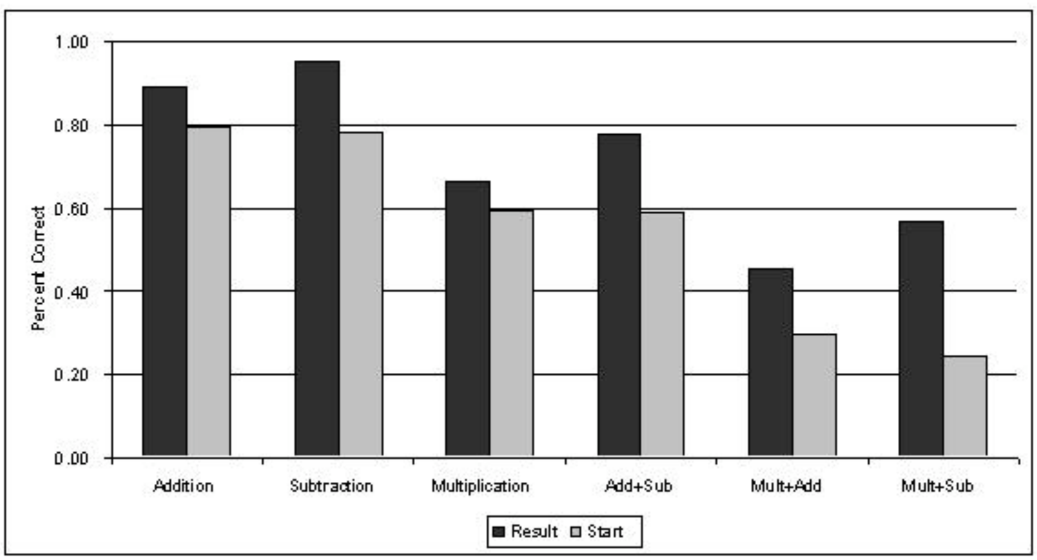

Figure 2. **Mean percentage correct on the story-writing task for each operation or operation combination and each position of the unknown.**

We next examined whether each type of two-operator problem was more difficult than would be expected based on the corresponding one-operator problems. Such a pattern has been called a *composition effect* in past research (Heffernan & Koedinger, 1997) because it implies that combining operations adds an additional source of difficulty. We estimated the probability of success on each of the six types of two-operator problems (i.e., addition-subtraction, multiplication-addition, and multiplication-subtraction for start- and result-unknown equations) by multiplying the rates of success on the relevant one-operator problems. We then compared these estimated probabilities of success with the actual probabilities of success in the data. In all six cases, the two-operator problems were more difficult than would be expected based on the corresponding one-operator problems, $t(5) = 4.28$. Thus, combining operations in stories presented a substantial challenge for students.

Performance on the equation-solving task and the story-writing task was significantly correlated, $r(240) = .44$. This finding is consistent with reports in the literature from other domains indicating that students' conceptual knowledge and procedural skill are positively associated (e.g., Baroody & Gannon, 1984; Dixon & Moore, 1996; Hiebert & Wearne, 1996; Rittle-Johnson & Alibali, 1999).

Analysis of Story-Writing Errors

We turn next to an analysis of the errors students produced in story writing. Here, we present a detailed analysis of those error categories that were assigned for more than 5% of all problems (with the exception of *other*, which was a heterogeneous category): (a) wrong operation, (b) no story action, (c) missing mathematical content, and (d) added mathematical content. The analyses of these errors converge to suggest that students lacked full-fledged conceptual understanding of the operation of multiplication.

Wrong-operation errors. Wrong-operation errors are errors in which some aspect of the student's story reflected an operation different from the one in the given equation. For example, given the equation $6 \times 13 = n$, one student wrote: "Kevin lives on a farm. He has 6 cows, and he buys 13. How many does he have?" In this story, the student used a story action that reflected addition rather than multiplication. Table 2 presents the distribution of different types of wrong-operation errors across problem categories for one-operator ($N = 31$) and two-operator ($N = 48$) problems. As seen in the table, the large majority of cases involved converting multiplication to addition.

Table 2
Distribution of Wrong-Operation Errors

Operation	**One-operator problems**	**Two-operator problems**
Addition		
To multiplication	0.06	0.00
To subtraction	0.10	0.06
Addition total	*0.16*	*0.06*
Subtraction		
To addition	0.03	0.10
Subtraction total	*0.03*	*0.10*
Multiplication		
To addition	0.68	0.63
To subtraction	0.03	0.00
To division	0.06	0.00
Multiplication total	*0.77*	*0.63*
N	31	48

Note. Totals do not sum to 1.0 because in some cases the operation could not be coded and because some two-operator problems included multiple wrong-operation errors.

No-story-action errors. No-story-action errors are errors in which the student did not provide a story context for some component of the given equation. For example, given the equation $4 \times 13 + 25 = n$, one student wrote: "Kevin lives on a farm. He has 4 × 13 pigs. The next day he gets 25 more. How many does he have now?" In this story, the student did not provide a story context for the multiplication operation. Table 3 presents the distribution of problem components that were not described in story form for one-operator ($N = 29$) and two-operator ($N = 101$) problems. As seen in the table, when students omitted a story component, it was most often the component that corresponded with multiplication in the given equation.

Table 3
Distribution of No-Story-Action Errors

Content	One-operator problems	Two-operator problems
Addition operation	0.00	0.36
Subtraction operation	0.00	0.27
Multiplication operation	1.00	0.80
Result quantity	0.00	0.02
N	29	101

Note. Total for two-operator problems does not sum to 1.0 because some problems included multiple no-story-action errors.

Missing-mathematical-content errors. Missing-mathematical-content errors are errors in which the student failed to include some of the mathematical content from the given equation in his or her story. For example, given the equation $6 \times n = 78$, one student wrote: "Alayna has some M&M's. A bag has 6 M&M's in a bag. How many more bags does she need?" In this story, the student described a multiplicative relationship involving 6 but did not include the result quantity, 78. Table 4 presents the distribution of problem components that were missing for one-operator ($N = 23$) and two-operator ($N = 61$) problems. As seen in the table, when a component was missing, it was most often either the start or result quantity. However, when a mathematical operation was missing, it was most often multiplication.

Table 4
Distribution of Missing-Mathematical-Content Errors

Content	One-operator problems	Two-operator problems
Addition operation	0.09	0.12
Subtraction operation	0.00	0.13
Multiplication operation	0.04	0.26
Start quantity	0.48	0.39
Result quantity	0.44	0.38
N	23	61

Note. Totals do not sum to 1.0 because some problems included multiple missing-mathematical-content errors.

Added-mathematical-content errors. Added-mathematical-content errors are errors in which students included mathematical content in their stories that was not present in the given equation. Such errors were coded only when the added content was integral to the solution of the story problem, not when it was simply "distractor" information that was not needed for solving the problem. In coding the data, it became apparent that students often made added-mathematical-content errors of a particular type when the given operation was multiplication. Specifically, given the expression $n \times m$, students would express the initial quantity on its own before describing the multiplication operation. Combining these statements, the mathematical relationship described was $n + n \times m$ rather than $n \times m$. For example, given the equation $4 \times 21 = n$, one student wrote: "Mike is making cookies for a school bake sale. He has made 21 but now needs to make 4 times that amount. How many cookies will he have made altogether?" Inspection of the added-mathematical-content errors indicated that fully 79% were of this type (74% of the added-mathematical-content errors made on one-operator problems, and 81% of such errors made on two-operator problems).

Distribution of Story-Writing Errors on One- and Two-Operator Problems

We next examined whether particular story-writing errors were especially likely to occur on two-operator problems. To address this issue, we examined whether particular types of errors were more likely to occur on two-operator problems than would be expected based on their frequency on the corresponding one-operator problems. We performed this analysis on all of the error categories that occurred on more than 5% of all problems: (a) wrong operation, (b) no story action, (c) missing mathematical content, (d) added mathematical content, and (e) other.

We estimated the probability of each type of error for each of the six two-operator problems (i.e., start- and result-unknown versions for addition-subtraction, multiplication-addition, and multiplication-subtraction) by adding the probabilities of that type of error for the relevant one-operator problems and then subtracting their joint probability. For example, to estimate the probability of a wrong-operation error on a start-unknown multiplication-addition problem, we added the probabilities of wrong-operation errors on start-unknown addition problems (3.1%) and start-unknown multiplication problems (8.6%) and then subtracted their joint probability (0.27%). We then compared these estimated probabilities with the actual probabilities for that error category.

The actual frequency of wrong-operation, missing-mathematical-content, and added-mathematical-content errors on two-operator problems did not differ from what would be expected based on their frequency on the corresponding one-operator problems. However, no-story-action errors occurred more frequently on two-operator problems than would be expected on the basis of their frequency on the corresponding one-operator problems, $t(5) = 4.84$, $p < .005$. This finding suggests that, for two-operator problems, students often avoided generating a story, rather than face the challenge of generating a coherent two-operator story.

Errors in the *other* category also occurred more frequently on two-operator problems than would be expected based on their frequency on the corresponding one-operator problems, $t(5) = 2.21$, $p < .05$, one-tailed. Because *other* is a heterogeneous category, it is not clear how this finding should be interpreted. Nevertheless, some of the errors observed in the *other* category are of interest because they illustrate the difficulties students experience integrating multiple operations into a coherent story. In some cases, students generated stories that were incoherent because different units applied to each operation. For example, given the equation $14 \times 7 - 23 =$

n, one student wrote: "Nicole wants to buy some necklaces for her[self] and her friends. They come in packs of 14 for $7. She wants to have a few leftovers for her[self], so if she has 23 friends, how many will she keep for herself?" In this example, the multiplication component of the story focuses on the *cost* of the necklaces, but the subtraction component of the story focuses on the *number* of necklaces. In other cases, students appeared to have difficulties assigning meaning to quantities that involved operations. For example, given the equation $63 + 41 - 13 = n$, one student wrote: "Kevin lives on a farm. He has 63 cows, 41 ducks, and 13 pigs. The pigs are on a sale, though. How [many] animals will he have after the pigs are sold?" In this example, the student incorporated story actions that reflect addition (finding the total number of animals) and subtraction (selling the pigs) but treated the value $63 + 41$ as indicating the number of animals *including* the pigs, rather than only the number of cows and ducks. In both of these examples, students displayed some understanding of the operations involved in the equations but had difficulty integrating multiple operations into coherent stories.

Discussion

Our primary aim in this study was to investigate middle school students' understanding of symbolic expressions. In past work, such understanding has often been assessed by asking students to solve equations. We, too, asked students to complete an equation-solving task; however, we also employed a novel task that we hoped would provide further insight into students' conceptual understanding by making it impossible for them to rely on memorized procedures. Our findings suggest that the story-writing task did indeed reveal much about students' thinking.

Although students in our study were fairly successful at solving algebraic equations, they experienced difficulties with equations that involved either two operations or unknown starting quantities. They experienced even greater difficulty when these two factors were combined in two-operator, start-unknown equations. Students' performance on the story-writing task showed a similar pattern, with two-operator equations being more difficult than one-operator equations, start-unknown equations being more difficult than result-unknown equations, and equations with these combined factors being the most difficult of all. These results are consistent with reports of middle and high school students' difficulties interpreting word problems (Kenney & Silver, 1997; Koedinger & Nathan, 2004; Sowder, 1988) and symbolic equations (Stephens, 2003).

The nature of students' story-writing errors suggests two main issues. First, students lacked a robust conceptual understanding of multiplication. Second, students demonstrated difficulty combining multiple mathematical relationships into coherent stories. We consider each of these issues in turn.

A closer analysis of student work falling into four common error categories indicated that students' conceptual understanding of multiplication was weak or incomplete. When students made wrong-operation errors, the operation that they represented incorrectly was multiplication in the overwhelming majority of cases. In most of these cases, students wrote stories reflecting the operation of addition instead of multiplication. When students made missing-mathematical-content errors, they often neglected the equation's starting or resulting quantity; however, in cases where the omitted portion of the equation was an operation, that operation was usually multiplication, particularly on two-operator equations. Students who made no-story-action errors were most likely to have had difficulty generating a story situation that could be represented by a given multiplication operation. Finally, students' added-mathematical-content errors again indicated difficulty generating a story that appropriately corresponded to a given multiplication operation. The vast majority of added-mathematical-content errors occurred when students composed a story reflective of the expression $n + n \times m$ rather than $n \times m$.

Carpenter, Fennema, Franke, Levi, and Empson (1999) noted that even very young children can solve multiplication word problems such as the following: "Megan has five bags of cookies. There are three cookies in each bag. How many cookies does Megan have all together?" (p. 34). Students' success on such problems indicates that they do have some understanding of the operation of multiplication. We suggest, however, that the link between such a story situation and its symbolic representation (5×3) may be tenuous for many students. Whereas students often successfully model and subsequently solve multiplication word problems using repeated addition of groups (Carpenter et al., 1999), students who are provided a multiplication operation in symbolic form do not necessarily connect these symbols to a repeated-addition scenario (Koehler, 2004). This points to the importance of helping students make stronger connections between verbal and symbolic representations.

A second area of concern raised by students' performance on the story-writing task has to do with their abilities to combine multiple operations into coherent stories. Our data point to the existence of a composition effect in story writing as well as in symbolization. Students often simply avoided generating story actions in two-operator problems—and did so much more frequently than would have been expected based on the frequency of such errors on one-operator problems. This finding suggests that students found it difficult to integrate multiple mathematical operations. Consistent with this view, when students did generate stories, they often included all the relevant numbers, but not in ways that fit together conceptually. For example, students sometimes generated stories in which different units applied to each operation, rendering the stories as a whole incoherent. The present findings are reminiscent of past research indicating that students have difficulties symbolizing story problems that involve multiple operations (Heffernan & Koedinger, 1997) as well as solving equations that involve multiple operations (Koedinger et al., 2008).

The story-writing task was designed to assess students' conceptual knowledge of symbolic expressions. We believe that it did in fact provide insight into such knowledge—particularly concerning multiplication and operation composition issues—that the equation-solving task on its own did not reveal. Although performing multiplication operations was not necessarily difficult for students (as was evident in their good performance on the equation-solving task), the story-writing task revealed difficulty with the underlying meaning of multiplication. Likewise, students' abilities to generate stories to correspond with two-operator equations were poorer than their abilities to solve comparable equations. The nature of students' errors suggests that integrating operations posed a special challenge.

Finally, our results are consistent with those of others who have documented a correlation between knowledge of concepts and knowledge of procedures. Although we do not wish to argue that the equation-solving task is purely procedural, we believe that students familiar with such tasks can be successful without deep conceptual knowledge of symbolic expressions. We believe that the novel nature of the story-writing task, on the other hand, encouraged students to rely more exclusively on their conceptual knowledge and thus provided greater insight into their understandings of symbolic expressions.

Our findings have implications for the mathematics instruction of students in the elementary and middle grades. First, our findings support Schifter's (1999) call for an increased focus on operation sense. We suggest that the meanings of the arithmetic operations should be an explicit focus of instruction in the early grades. In particular, students need opportunities to develop their understandings of multiplicative relationships. Second, our findings suggest that students could benefit from instructional activities that focus on combining multiple mathematical relationships. One such activity might involve interpreting various components of equations, including not only the numbers and operations, but also expressions such as 14×7 or $14 \times n$. More generally, our findings underscore the importance of activities that promote meaning-making, which is a key aspect of conceptual knowledge.

References

Baroody, A., & Gannon, K. E. (1984). The development of the commutativity principle and economical addition strategies. *Cognition and Instruction, 1*, 321–339.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction.* Portsmouth, NH: Heinemann.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13*, 16–30.

Dixon, J. A., Deets, J. K., & Bangert, A. (2001). The representations of the arithmetic operations include functional relationships. *Memory & Cognition, 29*, 462–477.

Dixon, J. A., & Moore, C. F. (1996). The developmental role of intuitive principles in choosing mathematical strategies. *Developmental Psychology, 32*, 241–253.

Heffernan, N., & Koedinger, K. R. (1997). The composition effect in symbolizing: The role of symbol production versus text comprehension. In M. G. Shafto & P. Langley (Eds.), *Proceedings of the 19th annual conference of the Cognitive Science Society*. Mahwah, NJ: Lawrence Erlbaum.

Heffernan, N., & Koedinger, K. R. (1998). A developmental model for algebra symbolization: The results of a difficulty factors assessment. In M. A. Gernsbacher & S. J. Derry (Eds.), *Proceedings of the 20th annual conference of the Cognitive Science Society* (pp. 484–489). Mahwah, NJ: Lawrence Erlbaum.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*, 59–78.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding and skill in multidigit addition and subtraction. *Cognition and Instruction, 14*, 251–283.

Kenney, P. A., & Silver, E. A. (1997). *Results from the sixth mathematics assessment of the National Assessment of Educational Progress*. Reston, VA: National Council of Teachers of Mathematics.

Koedinger, K. R., Alibali, M. W., & Nathan, M. J. (2008). Trade-offs between grounded and abstract representations: Evidence from algebra problem solving. *Cognitive Science, 32*, 366–397.

Koedinger, K. R., & Nathan, M. J. (2004). The real story behind story problems: Effects of representation on quantitative reasoning. *Journal of the Learning Sciences, 13*(2), 129–164.

Koehler, J. L. (2004). *Learning to think relationally: Thinking relationally to learn.* Unpublished doctoral dissertation, University of Wisconsin–Madison.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (1998). *Connected mathematics*. Palo Alto, CA: Dale Seymour.

MacGregor, M., & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for Research in Mathematics Education, 24*, 217–232.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*, 175–189.

Schifter, D. (1999). Reasoning about operations: Early algebraic thinking in grades K–6. In L. V. Stiff (Ed.), *Developing mathematical reasoning in grades K–12* (pp. 62–81). Reston, VA: National Council of Teachers of Mathematics.

Sowder, L. (1988). Children's solutions of story problems. *Journal of Mathematical Behavior, 7*, 227–238.

Stephens, A. C. (2003). Another look at word problems. *Mathematics Teacher, 96*, 63–66.

Swafford, J., & Langrall, C. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education, 31*, 89–112.

Appendix

Problems Used in the Equation-Solving Task

$17 + 54 = n$
$67 - 41 = n$
$5 \times 19 = n$
$28 + n = 74$
$84 - n = 53$
$7 \times n = 91$
$42 + 26 - 13 = n$
$4 \times 12 + 21 = n$
$16 \times 5 - 27 = n$
$35 + n - 18 = 46$
$5 \times n + 23 = 93$
$13 \times n - 22 = 56$

Problems Used in the Story-Writing Task

Version A

Number Set 1	Number Set 2
$19 + 33 = n$	$43 + 18 = n$
$63 + n - 13 = 91$	$37 + n - 15 = 46$
$45 - n = 21$	$93 - n = 61$
$21 \times 4 - 17 = n$	$14 \times 7 - 23 = n$
$6 \times 13 = n$	$4 \times 21 = n$
$6 \times n + 23 = 89$	$4 \times n + 25 = 77$

Version B

Number Set 1	Number Set 2
$93 - 32 = n$	$45 - 24 = n$
$37 + 24 - 15 = n$	$63 + 41 - 13 = n$
$43 + n = 61$	$19 + n = 52$
$4 \times 13 + 25 = n$	$6 \times 11 + 23 = n$
$4 \times n = 84$	$6 \times n = 78$
$14 \times n - 23 = 75$	$21 \times n - 17 = 67$