ON THE NATURE OF MATHEMATICAL THOUGHT AND INQUIRY: A PRELUSIVE SUGGESTION.

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ABSTRACT

ON THE NATURE OF MATHEMATICAL THOUGHT AND INQUIRY: A PRELUSIVE SUGGESTION.

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The author of this paper submits that humans have a natural inquisitiveness; hence, mathematicians (as well as other humans) must be active in learning. Thus, we must commit to conjecture and prove or disprove said conjecture. Ergo, the purpose of the paper is to submit the thesis that learning requires doing; only through inquiry is learning achieved, and hence this paper proposes an archetype of mathematical thought such that the experience of doing a mathematical argument is the reason for the exercise along with the finished product; and, that the nature of mathematical thought is one that can be characterised as thought through inquiry that relies on inquiry though constructive scepticism.

To opine mathematical thought is rooted in a disconnected incidental schema where no deductive conclusion exists or can be gleaned is to condemn the field to a chaotic tousle; whereas, to opine that it is firmly entrenched in a constricted schema which is stagnant, simple, and compleat is to deny its dynamic nature. So, mathematical thought must be focused on the process of deriving a proof, constructing an adequate model of some physical or latent occurrence, or providing connection between and betwixt the two. The two aforementioned ideas, the theoretical and practical are further convoluted by the seemingly axiological contrarians of experiential process and final product. The experiential process and final product cannot be disconnected. Thus, to paraphrase John Dewey, the ends and the means are the same.

The paper is organised in the following manner. In the first part of the paper the author gives a synopsis of the major philosophical influences of the thesis: Idealism, Realism, and Pragmatism. In the second part the author argues that the four basic ideas of mathematical thought, Platonism, Logicism, Formalism, and Intuitionism, all share the aspects of 'constructive scepticism' which forms the core of the author's argument regarding the nature of mathematical thought. In the third part of the paper the author submits that the single most important feature of mathematics that distinguishes it from other sciences is 'positive scepticism.' What binds and supports mathematics is a search for truth, a search for what works, and a search for what is applicable *within the constraints of the demand for justification*. It is not the ends, but the means which matter the most - - the process at deriving an answer, the progression to the application, and the method of generalisation. These procedures demand more than mere speculative ideas; they demand reasoned and sanguine justification. Furthermore, 'positive scepticism' (or the principle of *epoikodomitikos skeptikistisis*) is meant to mean demanding objectivity; viewing a topic with a healthy dose of doubt; remaining open to being wrong; and, not arguing from an *a priori* perception. Hence, the nature of the process of the inquiry that justification must be supplied, analysed, and critiqued is the essence of the nature of mathematical enterprise: knowledge and inquiry are inseparable and as such must be actively pursued, refined, and engaged. Finally, the author argues that not only is constructive scepticism an epistemological position as to the nature of mathematics, but it is also an axiological position for it is a value-judgement that inquiry into the nature of mathematics is positive.

P. J. Halmos recalled a conversation with R. L. Moore where Moore quoted a Chinese proverb. That proverb provides a summation and provides incite into the foundation of the philosophy of positive scepticism. It states, "I see, I forget; I hear, I remember; I *do*, I *understand*." It is in that spirit that a core point of the argument presented in the paper is that multiple methods are needed and must be employed in the execution of learning in order to have an educationally meaningful experience and in order for us to transcend from rudimentary to more refined epistemological and axiological understanding of mathematics.

So, this paper proposes a philosophical position that deviates from both a disconnected incidental schema (usually termed phenomenological, hermeneutical, or constructivistic schema) and a constricted schema (usually termed traditionalistic schema). We should acknowledge that conditional truth can be deduced, recognise the pragmatic need for models and approximation, and the author suggests that such is based on the experience of **doing** rather than witnessing.

INTRODUCTION

Mathematics is built on a foundation which includes axiomatics, intuitionism, formalism, logic, application, and principles. The act of inquiring is central to mathematics as a subject, as a form of reasoning, and as a discipline be it applied, computational, statistical, or theoretical. The many branches of mathematics are not mutually exclusive nor is there but one way of creating, discovering, or doing mathematics. Mathematicians conjecture, analyse, argue, critique, prove or disprove, and can determine when an argument is valid or invalid. Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is a need for justification that is open to criticism and can withstand scrutiny - - there is a stated or understood demand for succinct argument from a logical foundation for the veracity of an assertion.

The author of this paper submits that we humans have a natural inquisitiveness; hence, mathematicians (as well as other humans) must be active in learning. Thus, we must commit to conjecture and prove or disprove said conjecture. Ergo, the purpose of the paper is to submit the thesis that learning requires doing; only through inquiry is learning achieved, and hence this paper proposes an exemplar of mathematical thought such that the experience of doing a mathematical argument is the reason for the exercise - along with the finished product and that the nature of mathematical thought is one that is centred on constructive scepticism.

The paper is organised in the following manner. In the first part of the paper the author gives a synopsis of the major philosophical influences of the thesis: Idealism, Realism, and Pragmatism. In the second part the author argues that the four basic ideas of mathematical thought, Platonism, logicism, formalism, and intuitionism, all share the aspects of ‘constructive scepticism’ which forms the core of the author’s argument regarding the nature of mathematical thought. In the third part of the paper the author submits that the single most important feature of mathematics that distinguishes it from other sciences is ‘positive scepticism’ or ‘constructive scepticism.’ ‘Positive scepticism’ is meant to mean demanding objectivity; viewing a topic with a healthy dose of doubt; remaining open to being wrong; and, not arguing from an *a priori* perception. Hence, the nature of the process of the inquiry that justification must be supplied, analysed, and critiqued is the essence of the nature of mathematical enterprise: knowledge and inquiry are inseparable and as such must be actively pursued, refined, and engaged. Finally, the author argues that not only is constructive scepticism an epistemological position as to the nature of mathematics, but it is also an axiological position for it is a value-judgement that inquiry into the nature of mathematics is positive.

BACKGROUND

The first tradition of philosophical thought from whence positive scepticism is derived is Idealism. Idealism basically holds that knowledge and truth are obtainable and that there are absolutes that exist (which we shall capitalise). Idealism holds that the universe is fundamentally rational and orderly; hence, intelligible. Idealism holds that an objective body of Truth has existence and can be known (though perhaps not in its entirety) by the human mind. The act of knowing is in some form a reconstruction of the ideal into intelligible ideas and systems of ideas. The criterion for truth of an idea is believability, reliability, coherence, and consistency with the existing and accepted body of truth.

The second tradition of philosophical thought from whence positive scepticism is derived is Realism. Realism holds that there exists a world of things, events, and relations amongst these things and events and the world in which individuals live which is not dependent on the individuals. Hence, the reality is independent of the knower and that fact can exist apart from consciousness. Furthermore, the world, as it is, can be known, at least in part, as it is unto and in itself. Knowledge and truth are obtainable.

The third tradition of philosophical thought from whence positive scepticism is derived is Pragmatism or Pragmaticism.[1] Pragmatism is the idea that the meaning of a word is defined by its practical consequences. Words and ideas that cannot in any way be tested practically are inconsequential. However, it is not surprising that there is no one general definition of pragmatism that covers all the philosophical doctrines that have been given that name. H. S. Thayer defines pragmatism as 1) a procedural rule for explicating meanings of certain philosophical and scientific concepts; 2) a theory of knowledge, experience, and reality maintaining that thought and knowledge are evolved modes by means of adaptation and control; b) reality is transitional and thought is a guide to satisfying interests or realizing purposes; c) "all knowledge is a behavioral [sic] process evaluative of future experience" and thinking is experimentally aimed at organizing, [sic] planning, or controlling future experience; and 3) "a broad philosophic attitude toward our

[1] Pragmatism is what mort reference when speaking of the works of Charles Pierce and William James. Charles Pierce later referenced his ideas as pragmaticism to differentiate it from James's work.

conceptualization [sic] of experience."[2] Nonetheless, it suffices to say that Pragmatism fundamentally holds that the truth of any proposition is determined by the success or failure of action based on it. At one extreme William James seemed to regard personal experience as a sufficient source and test of truth; whilst Charles Pierce seems to have held that an ideal community of minds form opinions that in the long run are destined to converge on the one unalterable Platonic truth.

It has become rather accepted in the modern academy and throughout much of mathematics education to include in discourse a position statement or a statement from whence someone argues. Some educators argue that all knowledge is tinted by the background and perspective of the individual or more often from the group that person is a member of (be it religious, racial, etc.). This position is constructivism or radical constructivism (it has also been called phenomenology or hermeneutics). In such a schema, there is no global truth, all is relative, and the best one can hope for is a sharing of perspectives but no conclusion can necessarily be drawn. However, such a position belies the great work done by mathematicians throughout the centuries and negates the consequences of the discoveries, inventions, observations, and realisations that were created. There has to be some foundation of objectivism that underlies a proper philosophy of mathematical inquiry and thought.

Several authors submit a constructivist approach to the learning of, teaching about, or even doing mathematics [14, 22, 23, 28]. The constructivist accentuates the community and focuses on cooperation amongst learners. If one agrees with the philosophical position conditional to the constructivist method, then it may be an entirely acceptable learning or teaching methodology and might be a position grounded for a philosophy of mathematical inquiry but it seems to be highly suspect as a philosophy of mathematical thought. This is because it seems of little practical use in the *doing* of mathematics and is not a *foundation* upon which conclusions can be drawn; hence, how would one be able to convey results, argue veracity, or generalise with any reliability? It seems that the constructivist method is best suited for elementary problems where inquirers have not completely matured and where the material is less sophisticated. The constructivist method is based on a philosophy that the individual learn with others and that reality is constructed. In its radical form it maintains "individuals construct their own reality through actions and reflections of

[2] Thayer, *Meaning and Action: A Critical History of Pragmatism* (1968): 431.

actions."[3] So, under such a philosophy a compleat relativism antecedes such that objectivism is relegated to oblivion. As a matter of the opinion of the author, constructivism seems to be a quite nihilistic, solipsistic, and a hopelessly subjective philosophical position. A constructivist negates the transcendent, universal, and objective nature of mathematics.

The need for some objectivism predicates the position of positive scepticism and is based (at least in part) on several traditions of philosophical thought. In this paper we shall depart from the classical philosophical position, call it $\mathcal{C}$, that that person M knows that thing p is true if and only if 1) M believes p; 2) p is true; and, 3) M is justified in believing that p is true. We shall call the position, call it $\mathcal{M}$, that person M knows that thing p is true if and only if 1) p is true and 2) M is justified in opining that p is true. That p is true implies that there is something that can be known apart from the individual M. That M is justified in opining that p is true requires a method of argument from the justification, requires that the justification be understandable, and that there was an accepted schemata employed for providing said justification. The author holds that belief is not a necessary condition for obtaining mathematical truth for it seems that belief is a consequent rather than an antecedent for knowing something and might not be needed even after obtaining knowledge. For example, even a student of Calculus knows that the area of the region, R, bounded by y = 0, x = 1, $y = \frac{1}{x}$, to the right of x = 1 does not exist since $\int_1^\infty \frac{1}{x}\,dx$ does not exist. Yet, the volume of the resulting object, T, obtained by rotating R about the x-axis does exist since $\int_1^\infty \frac{\pi}{x^2}\,dx$ exists. Hence the region R has no area (i.e.: $\nexists\, a \in \mathbb{R} \ni$ R has a units^2 area) but the region T (based on R) has volume (π units^3). The student need *not* believe a result in order to *deduce* it or *know* it.

Hence, we shall adopt a modicum of objectivism (there are ideals, there is a real world, it has meaning, and we can know some of the things that exist) along with a position that knowledge is gained through $\mathcal{M}$.

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[3] Steffe and Kieren, "Radical Constructivism and Mathematics Education," *Journal for Research in Mathematics Education* 25, no. 6 (1994): 721.

FOUNDATIONS OF POSITIVE SCEPTICISM

It should not pass without comment that these three philosophical schools do not exhaust the foundations of constructive scepticism. Socraticism, empiricism, humanism, foundationalism, constructivism, utilitarianism, positivism, and others also form parts of the core to positive scepticism. That is because a fundamental tenet of positive scepticism is moderation. From each school one takes what is necessary and sufficient and meshes the parts into a coherent whole. Along with moderation it is important to acquire balance. A balanced approach to ideas allows for an open, liberal elucidation of the concepts and forges a firm path toward solving the problem, understanding the problem, and solving the problem. In this regard the position of moderation and balance is decidedly pragmatic.

The four basic ideas of mathematical thought, Platonism, logicism, formalism, and intuitionism, all share the aspects of ‘constructive scepticism’ which form the core of the argument regarding the nature of mathematical thought. Actually there are many variations and distinctions between and betwixt the schools of thought, but we will not complicate the discussion.

Nonetheless, one should begin with Platonism or neo-Platonism and objectivism when one is discussing the philosophy of mathematics. These form the roots of modern mathematical thought and are antecedents to logicism, formalism, and intuitionism insofar as ontology is antecedent to epistemology and axiology.

The major idea of Plato's was considering abstract forms as ontological entities more basic than material things. Plato asserted that mathematics represents a separate universe of abstract objects existing outside of space and time. Mathematical objects aren't created by humans. They always existed. Platonism asserts that a mathematician is an empirical scientist who can only discover what is already there. He can't invent new mathematics - - epistemologically true knowledge is conditioned on the ontological entities. Mathematical truth possesses absolute certainty. There is much to be said for this position insofar as the existence of real numbers. However, one does need the axioms of the reals in order to truly understand and experience reals. Nonetheless, π exists. It existed before man first walked the earth. It existed before the author was born. It will exist after he dies. It will exist after man no longer exists. The development of differential calculus is evidence for Platonism. Both Isaac Newton and Gottfried Wilhelm von Leibniz had the idea of Calculus at about the same time. That such fundamental and seminal work could be created by more than one person at approximately the same time indicates there is a mathematical reality.

Yet there is a problem with Platonism. It suffered from a ‘mysticism’ and pseudo-religious view of mathematics. Lobachevskian geometry, quaternions, anticommutative algebra, etc. seemingly contradicts the idea that there is one absolute truth. However, absolute truth need not be unitary and as such might exist; hence, allowing for the realistic underpinning of the Platonic ideals (a neo-Platonist approach) seems warranted.

Objectivism holds that a proper understanding of abstraction is a prerequisite for explaining mathematical concepts. Rooted in this is a position that is similar to Platonic realism and Kantian idealism. The identification of the nature of universals and the analysis of the process of abstraction seem to be central to the objectivist position. Objectivism recognises a deep connection between mathematics and philosophy than other mathematical philosophies posit. Objectivist theory concentrates on the process of concept-formation involves the grasp of quantitative relationships among units and the omission of their specific measurements. It thus places mathematics at the core of human knowledge as a crucial element of the process of abstraction. Thus, the ontological debate is lessened and the epistemological question is highlighted in this schemata.

Formalism contends that mathematics must be developed through axiomatic systems. Formalist and Platonist positions agree on the principles of mathematical proof, but formalists do not necessarily recognise an external world of mathematics. Formalism is most centred on consistency and completeness. For a system to be consistent, no contradictions may exist within the system. For a system to be complete, its axioms must be sufficient to prove any proposition either true or false within the system. Formalists argue that are no mathematical objects until one creates the mathematical object. Humans create the real number system by establishing axioms to describe it. All mathematics needs is inference rules to progress from one step to the next. A formalist proves a claim within the framework of established axioms, theorems, and definitions in a mathematical system that is consistent. Formalists sought to express mathematics as strictly formal logical systems, and to study them as such, without concern for their meaning. Their primary motivation was to justify Cantor’s mathematics of infinite sets. The formalists hoped to express the mathematics of infinite sets in such a system, and to establish the consistency of that system by finite methods. If they succeeded in this, they thought, they would have justified the use of infinite sets without having to address the thorny question of just what such sets are.

Yet there is a problem with formalism. It suffers because the consistency aspect cannot be established within a given system as a consequent of Gödel's Incompleteness Theorem. Gödel's work showed that the formalist programme was untenable. Furthermore, many results were created and used before an axiom system was devised to justify the result (for example Newton's and Leibnitz's differential calculus; Riemann's integral calculus; and, probability before Kolmogorov to note but a few). Much of the modern exploratory, applied, and computational mathematics is not in harmony with the formalist philosophy that one does not do mathematics unless a hypothesis is stated and a proof begun.

Logicism claims that mathematics is a vast tautology. All of mathematics is derivable from principles of logic. Many of the logistic ideas are similar to those of the formalists, but the latter group does not believe that mathematics can be deduced from logic alone (which contrasts with formalism which studies formal logical systems them as the systems themselves, without concern for their meaning whereas the logicists seek to establish the meaning of mathematical notions by defining them in terms of concepts of logic). Amongst other things, the logicists attempted a logical construction of the real number system, whereas the formalists constructed it axiomatically. Logicism also uses mathematical sets in its logical development. However, logicism could not adequately resolve the paradoxes that arise in set theory. Again Gödel's work caused a major problem for logicism's contention that mathematics is a tautology (as for formalism's contention of mathematical systems having internal consistency). Indeed, logicism seems to suffer as a major school of mathematical thought insofar as logicists reject the principle of mathematical induction. Logicism had as its purpose to "reduce mathematics to logic." Logicism's conception of logic is radically different from the objectivist, or more generally, the classic conception of logic; and it is a view of logic presupposed in most modern mathematical philosophy. Nonetheless, one can argue from a position such that logicism and objectivism are not contrarians - - that consciousness is intentional, that it is always of or about a world that exists and that the world has identity independently of consciousness.

Intuitionism claim that mathematics originates and thrives within the mind. Human minds intuitively possess the forms of time and space. The natural numbers are given intuitively, and they represent the fundamental foundation of mathematics from which springs all meaningful mathematics. Mathematical laws are not discovered by studying nature; rather, they are found in the recesses of the human mind. Intuitionism holds that only those mathematical concepts that can be demonstrated, or constructed, following a finite number of steps are legitimate. Yet (again) there is a problem - - this time with Intuitionism. It is the case that concepts differ from person to person so that the concept of number may be different depending on the perceiver. Hence, is it reasonable or indeed advised to assume that people have the same intuitive understanding or view of mathematics? If not, then what does this say for mathematics?

The intuitionists are rooted philosophically in Kantian philosophy. Their position on the law of excluded middle demands that a statement be established as meaningful *before* the laws of logic are applied to it, a demand that objectivism seemingly might endorse. Their insistence on constructive proofs may be seen as a means of specifying what is meant by the existence of a number. Intuitionism is indeed perhaps best understood by noting that intuitionism is quite conservative regarding infinity. Intuitionists are opposed to the application of the law of excluded middle to statements involving mathematical infinitudes, as in a proof that takes the following form: either there is a number with the property P or there is not; if not, a consequence follows that is known to be false; therefore there exists a number with the property P. Such proofs do not tell us what the number in question is, or why it has the property. Constructive proofs, by contrast, do provide this information, and intuitionists require constructive proofs of mathematical theorems. Indeed many of us have proven that $\sqrt{2}$ is irrational using proof by contradiction (*reducto ad absurdum*). In its radical form intuitionism would not allow for this since the object must be constructed. The insistence on the absence of the use of the law of the excluded middle in proof is a major fault for intuitionism (one may or may not indict them for the rejection of the axiom of choice as a major fault, but with regard to the law of the excluded middle there is much fodder). It is also akin to general constructivism in its relativistic tendencies, and constrains and constricts the possible outcomes that the investigator may deduce. Furthermore, intuitionism fails to address the question, “why teach mathematics if it is all simply intuitive?”

Whilst summarising the fundamental schools of mathematical thought, it should be noted there are actually many variations and distinctions between and betwixt the schools, but we will not elaborate beyond the simple exposition heretofore mentioned. It should not pass without comment that these philosophical schools of mathematics *also* do not exhaust the foundations of constructive scepticism. Empirical investigation, statistical thought, Pólya’s theory on problem-solving, foundationalism, and others also form parts of the core to positive scepticism. Moderation and a balanced approach to ideas form the structure that creates an open, liberal investigation of concepts and forges a firm path toward solutions to pure, applied, and mixed problems. Once again taking a pragmatic approach to mathematics requires the position of moderation and balance.

To opine mathematical thought is rooted in a disconnected incidental schema where no deductive conclusion exists (radical constructivism, phenomenology, or hermeneutics) can be gleaned is to condemn the field to a chaotic tousle; whereas, to opine that it is firmly entrenched in a constricted schema (idealism or realism) which is stagnant, simple, and compleat is to deny its dynamic nature. So, the nature of mathematical thought *must* be focused on the process of deriving a proof, constructing an adequate model of some physical or latent occurrence, or providing connection between and betwixt the two. It is important to use elements from objectivism, formalism, logicism, and intuitionism in approaching questions and not constrain oneself to an *a priori* position which clouds the question and predisposes the investigator into predetermined or predestined conclusions. The ideas, the theoretical and practical results, are further convoluted by the seemingly axiological contrarians of experiential process and final product. The experiential process and final product cannot be disconnected. Thus, to paraphrase John Dewey, the ends and the means are the same. This pragmatic imperative is most often illustrated by the many different proofs that mathematicians devise for a claim and the axiological (both the aesthetic and ethical) judgments that are consequent to the presentation of the proofs. One need only peruse the literature to note that there are many articles that present new, interesting, or varied methods that prove the veracity or lack thereof of claims that are already a part of the canon. So, the process of deriving a solution, a proof, an argument, or a model is focal to the exercise.

POSITIVE SCEPTICISM AS ONE DIFFERENTIATOR OF MATHEMATICS TO OTHER SCIENCES

One thing that most (indeed one opines most) mathematicians would agree upon is that mathematics is fundamentally different from other sciences. First and foremost, mathematics is abstract and that it consists of primarily of results of reasoning about or with regard to non-figurative, intangible, or ideal concepts. Indeed, the fundamental position in mathematics is deduction. The truth or lack thereof of a proposition in mathematics is resolved by a process of deductive reasoning predicated on the basis of assumed truths (axioms) and consequents to those axioms (lemma, corollaries, and theorems). Hence, the position taken in this paper is that mathematical truth is conditional.

The conditional truth derived from mathematical reasoning is essentially different from other sciences because there is less of an accent (indeed in some case a compleat lack of) on the physical manipulation of events in our world. The other sciences owe much more to empiricism than mathematics. In most sciences one test hypotheses and theories empirically and 'sees' if the hypothesis or theory reasonably holds (in 'hard' science with great reliability and in social science with lesser reliability). However, mathematical truth can be (and is oft) derived without any manipulation of the physical world and lacks the consequential 'seeing' if the hypothesis holds or not. Much of the reasoning that is employed in empirical sciences is inductive and disallows the firm conclusive results that mathematics produces. This reasoning pattern is a differentiator between mathematics and other science for they are in large inductive and mathematics in large deductive. The fact that mathematicians write and communicate with symbols and language is not enough to opine that the mathematician's thought and inquiry is akin *in summa* to the empirical scientist's thought and inquiry; however, for some applied and computation mathematical works there is enough similarity such that one should not completely disregard that there are mathematical activities that are parallel.

The author submits that perhaps the single most important feature of mathematics that distinguishes it from other sciences is 'positive scepticism' or the principle of '*epoikodomitikos skeptikistisis.*' Positive or constructive scepticism is meant to mean that the mathematician never loses sight that he may be wrong but *can* derive correct arguments, that each and every claim must be justified (belief does not suffice), and that one reasons in a mathematical system that is not necessarily unique so the truth derived is indeed conditional. This may be understood better by noting what it is not. It is not a rabid scepticism that is destructive, which would seek to tear down, detract, or offer no modification, enhancement, or solution. Positive scepticism is not is meant to mean anti-constructivism, anti-Platonism, etc. it is an active rather than a reactive philosophical position. Positive scepticism has its proximate roots in objectivism and optimism. Constructive scepticism is also an objective view that inquiry can (and oft) leads to better ideas, processes, or innovations.

Objectivism, formalism, logicism, and intuitionism all share in the fundamental aspect of mathematical thought and inquiry - - the desire to glean new information, unique insights, and drive toward a better understanding of mathematics (pure and applied). These schools of thought share features of constructive scepticism. One approaches a problem with an open mind and does not allow for belief to cloud judgement and force a conclusion. The investigator's belief in the veracity or lack thereof of a claim does not override the sincere inquiry. For each step of reasoning the investigator questions his deduction. When a finished argument, proof, or solution is arrived at the researcher opens the entire result up to scrutiny by his peers for thorough review. A classic elementary example of such an approach is in a Texas (Moore method) classroom where the student presents his work to his peers and the peers listen attentively and attempt to find 'holes' in the argument. Such 'attacks' by the 'tribunal' of peers is not meant to harm but to adequately gauge whether or not the presenter is correct and really knows what he is doing. The active participation of the peers after the presenter has derived his argument constitutes a sceptical approach to the presenter and his work. Criticism is tempered with suggestions for improvements, modifications, or apotheoses which demonstrate a lack of verisimilitude of the claimed work. The peer review creates an *a posteriori* component of a research experience in aiding in the creation of an argument's strength, weight, and credibility. So, what results is an *a forteriori* argument that should be closer to being veridical than if no such an experience occurred. Again the processes, the inquiry; demonstration; and, examination by peers, are situated in a constructive or positive context.[4]

What binds and supports mathematics is a search for truth, a search for what works, and a search for what is applicable within the constraints of the demand for justification. It is not the ends, but the means which matter the most - - the process at deriving an answer, the progression to the application, and the method of generalisation. These procedures demand more than mere speculative ideas; they demand reasoned and sanguine justification. Furthermore, 'positive scepticism' is meant to mean demanding objectivity from inquirer and peers; viewing a topic with a healthy dose of doubt; remaining open to being wrong; and, not arguing from an *a priori* perception. Hence, the nature of the process of the inquiry that justification must be supplied, analysed, and critiqued is the essence of the nature of mathematical enterprise: knowledge and inquiry are inseparable and as such must be actively pursued, refined, and engaged.

Mathematicians forge their understanding of the subject (and perhaps beyond a subject) through reason. Reinforcement of biases, prejudgements, etc. is not part of the job description of a mathematician. Such as mathematics was and mathematics is, a position of positive scepticism led to the creation of branches of mathematics that could not possibly have been created without being people being sceptical of beliefs, claims, and theories. Consider, if Euclidean geometry was a unary absolute and *all* mathematicians thought that the laws of Euclidean geometry *had* to be accepted without question as indubitable truths about the universe (as was thought); then, the work of Bolyai

[4] For a more refined and detailed discussion of the author's teaching philosophy see McLoughlin, 2002, 2003a, and 2003b.

and Lobachevsky would not nor could not have happened. If all mathematicians thought as Kronecker, then Dedekind's and Cantor's work would have been universally rejected (though it was by some like Kronecker). Throughout the history of mathematics, mathematicians have speculated, dreamt, and imagined beyond that which was obvious. Positive scepticism is in ways the manifestation of intuitionism. It seems rather clear that some form of constructive scepticism was operating by and for mathematicians in the past and it seems that it operates to this day.

Mathematical thought requires discipline and rigour. Hence, logicism and formalism are part of mathematical thought and inquiry. Nonetheless, constructive scepticism is manifest within these philosophical strains since with any formalist proposition is the inherent question, is that all there is or can it be reduced or expanded? Likewise, simply by noting the demand for succinct reasonable arguments at every step of a proposition's possible solution is constructively criticising the underlying structure of the theory or argument. Hence, the position of positive scepticism intertwines within the thought process of the person who proposes the solution and the person who critiques the proposed exposition.

When we reflect on Gödel's work, that internal consistency can never be established by methods of mathematical proof, the realisation that every logical system must contain statements that cannot be proven within the system. So, a formal mathematical system could never prove its own consistency. Some ideas must be accepted (axiomatised). Furthermore, an axiom such the Euclidean "for a plane, two non-parallel lines intersect at a unique point" is accepted as true without proof. However, there are simple non-Euclidean geometries where this axiom is not assumed. However, in a mathematical system there is not room for the concept of statistical robustness. One does not allow for a relaxing of the axioms and a pseudo-axiom to hold so that, for example, a real number usually follows the trichotomy law but sometimes might not. This differentiation between empirical and statistical science and mathematics is an important one.

When one constructs a study and hypotheses about some (constructs a null hypothesis, H_0), collects data, tests the hypothesis, and finds there is sufficient evidence at the 0.05 level to reject a null hypothesis and conclude there is evidence to suggest the null is false - - that does not prove that the alternate hypothesis is true. Likewise, suppose the researcher finds there is not sufficient evidence at the 0.05 level to reject a null hypothesis - - that does not prove H_0 (nor does one accept H_0, but that is the subject for another paper). Replication after replication study will not yield convergence to the truth. In fact, with empirical or statistical sciences we really can never really know the truth - -

empiricist inquiry doe not necessarily lead to the truth, but in mathematical sciences there is truth to be found! Hence, one can reasonably, facilely, and *correctly* conclude based on the Kolmogorov axioms that claim M: Given events E and F within a well defined sample space S that $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$ is true.[5] The argument can be 'attacked,' decomposed, debated, rehashed, etc. through scepticism and the sceptic may doubt the veracity; but the positive sceptic (the mathematician) is forced to conclude that claim M is true. Ergo, 1) M is true and 2) we are justified in opining that M is true (conditioned on the axioms). It is not necessary that we believe it for we are forced to conclude it. This is the fundamental difference between scepticism and positive scepticism and in a key differentiation between the general classical philosophical position, $\mathcal{C}$, that requires three conditions and the mathematical philosophical position, $\mathcal{M}$, that requires but two conditions.

The key distinguishing characteristics can be summarised as contrasts between the ideas of scepticism and positive scepticism and betwixt empiricism and constructivism and objectivism. The original Greek meaning of *skeptikos* was "an inquirer," someone who was unsatisfied and still looking for truth. That is quite different from the modern idea of a sceptic and indicates the nature of positive scepticism. Scepticism throughout history has played a dynamic role in forcing dogmatic philosophers to find better or stronger bases for their views and to find answers to a sceptics critique. It has forced a continued re-examination of previous knowledge claims and has stimulated creative ideas. Without a doubt (pun intended) practically everyone is sceptical about some knowledge claims; but a sceptic raising doubts about *any* knowledge beyond the contents of directly felt experience is too reactionary. Being sceptical of everything is more likened to a statistical position and being positively sceptical more a mathematical position.

Furthermore, empiricism and constructivism as previously noted are rather restrictive philosophical positions. It seems the empiricist is too dogmatic and extreme in that they seemingly dismiss out-of-hand anything that does not conform to 'matters of fact' based on the data of experience, and repudiate speculation regarding the nature of reality that goes beyond any possible evidence that could either support or refute 'transcendent' knowledge claims. It seems the constructivist is too solipsistic and extreme in that they seemingly dismiss out-of-hand anything that might be systematic, and repudiate speculation regarding the nature of reality that addresses possible evidence that could either support or refute 'standard' knowledge and in their rejection of universals.

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[5] Pr(•) indicates the probability of the event denoted in the parentheses.

Once again, rather than be fixated on an *a priori* position from which to argue, objectivism allows one to be moderate and balanced. To recognise there is an external world, there are things within experience that can be empirically investigated, just as there are things beyond experience that must be investigated with reason. Empiricism alone cannot solve every problem, constructivism creates (seemingly) infinite solutions (every belief is justified since it is held), whereas, objectivism admits rationality, reasonableness, deduction, and verisimilitude. It allows that some problems have solutions, other seem to not (though we might not have the tools or intelligence to discern the solution presently), and some humanity may not be able to solve. *In actu*, there is an emphasis on questioning authority, independent thinking, individual creativity, and the empowerment of the individual to reason through a problem and create a solution.

CONCLUSIONS

The philosophy of mathematics is the philosophical study of the concepts and methods of mathematics. Mathematical philosophers are concerned with the nature of numbers, geometric objects, and other mathematical concepts; they are concerned with the cognitive origins and with the application of the concepts to reality. The philosophy of mathematics addresses the validation of methods of mathematical inference, the basis of the thought, and with the logical problems associated with mathematical arguments. It goes without saying that amongst the sciences, mathematics has a unique relation to philosophy. Mathematics is an abstract, the model of logical perfection, with clarity of concepts and certitude in its conclusions, and we therefore must devote much effort to explaining the nature of mathematical thought and inquiry.

This paper simply offers a view as to an aspect of mathematical thought and inquiry which the author has come to opine might be part of the substance which makes mathematical inquiry different from inquiry in the other sciences and other areas of academia. The paper provides a cursive introduction to the major foundations of the philosophy of mathematics, and some of the historically important traditions on these issues.

We have mathematics which has universals; there exist principles to be discovered, created, or invented. It could be there are others in the universe; yet, they would have π (though perhaps of a different name but it would be π) and it exists as it exists here. This is quite a different situation than the social sciences, arts, humanities, etc. which hinge on a subjective slant and relative interpretation. We are not bound by the idea of *interpreting* the meaning of π, it simply *is*. This demonstrates that we can understand it but must also *get it right*. Hence, the philosophy of mathematics is inexorably bound to the notion of being correct, of bounding error (when error exists), and of being able to note when we are wrong. The root of objectivism is fundamental to mathematical thought and inquiry.

It would be advised if we concentrated on a kind of mathematical realism such that we could agree that mathematical entities exist independently of the human mind. Whether or not humans invent mathematics or discover it is of no import, what is important is that there are real things in mathematics that do exist and it would be logical to opine any other intelligent beings in the universe would presumably do the same mathematics no matter what they call it.

Formalism is utile insofar as mathematical statements may be thought of as statements about the consequences of certain string of rules to be manipulated, so, as with a "game" of mathematics (which can be considered of some strings called "axioms", and some "rules of inference" to generate new strings from given ones), one can prove that a claim holds. Any game is as good as another, and one should revel in the playing of the game (most oft when the game involves proving things as consequence of the initial rules). A better form of formalism (deductivism) the author opines is quite utile because we do not arrive at an absolute truth, but a conditional one. Hence, *if* one assigns meaning to the strings in such a way that the rules of the game become true, *then* one has to accept the theorem. The more games we play and study, the better.

We should acknowledge that logicism is clearly one of the cornerstones of mathematical thought, and that all mathematical statements are necessarily decomposable to logical truths. For instance, the statement "If Socrates is a human, and every human is mortal, then Socrates is mortal" is a necessary logical truth. We can deduce analytic truths.

We should also acknowledge that the intuitionist insistence on mathematical entities which can be explicitly constructed have a claim to existence; and, they should be the only ones admitted in mathematical discourse is too restrictive. However, it seems reasonable that an entity which can be constructed does seem to be ‘better’ than an entity that is not constructed. It could be looked upon as more meaningful, but to relegate all that is deduced by contradiction to oblivion is ill-advised.

Phenomenology, hermeneutics, radical constructivism, and their kin see mathematics primarily as a social construct, as a product of culture, subject to correction and change. Like the other sciences, mathematics is viewed as empirical endeavors whose results are constantly compared to reality and may be discarded when not in agreement with observation, seem pointless, or are ‘too abstract.' The belief that mathematics is hounded by the fashions of the social group performing it or by the needs of the society financing it is seemingly without merit given the permanence of mathematics.

Some practitioners of the philosophy of mathematics have attempted to relate mathematics to other aspects of philosophy: ontology, aesthetics, and ethics in particular. Those concerns are not discussed in this paper. Some philosophers of mathematics opine that an account of mathematics, mathematical practice, and the 'mathematical community' should be interpreted. Such a position is not held or supported in this paper.

What mathematicians seemingly do best is create, think, and critique. Hence, criticism is a more apt than interpretation for a discussion of mathematical thought and inquiry. Constructive criticism is quite useful in any endeavour and most especially for mathematical practice and claims for finished mathematics. Hence, a philosophical position that mathematics hinges upon the idea of positive scepticism seems warranted and could be of direct interest to working mathematicians, particularly in new fields where the process of inquiry is of interest. The ability to detect errors of reasoning, logic, or subject can thus only be reduced by knowing where they are likely to arise and how they happen. This, the author opines, is a prime concern of the philosophy of mathematics.

What binds and supports mathematics is a search for truth, a search for what works, and a search for what is applicable within the constraints of the demand for justification. It is not the ends, but the means which matter the most - - the process at deriving an answer, the progression to the application, and the method of generalisation. These procedures demand more than mere speculative ideas; they demand reasoned and sanguine justification. Furthermore, 'positive scepticism' is meant to mean demanding objectivity; viewing a topic with a healthy dose of doubt; remaining open to being wrong; and, not arguing from an *a priori* perception. Hence, the nature of the process of the inquiry that justification must be supplied, analysed, and critiqued is the essence of the nature of mathematical enterprise: knowledge and inquiry are inseparable and as such must be actively pursued, refined, and engaged. Finally, it is not only that constructive scepticism is an epistemological position as to the nature of mathematics, but it is also an axiological position for it is a value-judgement that inquiry into the nature of mathematics is positive.

P. J. Halmos recalled a conversation with R. L. Moore where Moore quoted a Chinese proverb. That proverb provides a summation of the justification of the methods employed in teaching students to do mathematics with the fusion method and provides incite into the foundation of the philosophy of positive scepticism. It states, “I see, I forget; I hear, I remember; I *do*, I *understand*.” It is in that spirit that a core point of the argument presented in the paper is that multiple methods are needed and must be employed in the execution of learning in order to have an educationally meaningful experience and in order for us to transcend from rudimentary to more refined epistemological and axiological understanding of mathematics.

So, this paper proposes a philosophical position that deviates from both the disconnected incidental schema (usually termed phenomenological, hermeneutical, or constructivist schema) and the constricted schema (usually termed traditionalistic schema) that the nature of mathematical thought is one that is centred on constructive scepticism which acknowledges conditional truth can be deduced, recognises the pragmatic need for models and approximation, and suggests that such is based on the experience of doing rather than witnessing.

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