A Modified Moore Approach to Teaching Mathematical Statistics:

An Inquiry Based Learning Technique to Teaching Mathematical Statistics.

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Abstract

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The author of this paper submits the thesis that learning requires doing; only through inquiry is learning achieved, and hence this paper proposes a programme of use of a modified Moore method in a Probability and Mathematical Statistics (PAMS) course sequence to teach students PAMS. Furthermore, the author of this paper opines that set theory should be the core of the course's pre-requisite with logic and calculus as antecedents to the set theory, an introduction to the theory of functions as subsets of $\mathbb{R} \times \mathbb{R}$ as consequents of set theory. The connections between logic, set theory, and proofs about probability, random variables and processes, & inferential mathematical statistics cannot be understated– – the better the student's pre-requisite knowledge the easier it is for the student to understand probability theory and flourish in a Probability & Statistics course sequence.

The author of this paper has experienced teaching such a course sequence for approximately fifteen years; mostly teaching the course at a historically black college. The paper is organised such that in the first part of the paper an explanation as to why Logic, Set Theory, and Calculus are proper pre-requisites to a Probability & Statistics course sequence and a brief overview is presented of the Moore method. The second part of the paper, presents justification for use of a modified Moore approach in teaching probability & statistics (or what is termed mathematical statistics often); both pedagogical and practical justification is submitted. In the third part of the paper, the author submits the model for the Probability & Statistics courses and focuses on what is effective for the students, what seems not useful to the students, and why. Also, explanation is presented as to why the courses were designed the way they were (content), how the courses were revised or altered over the years; hence, explaining what practices were refined, retained, modified, or deleted and how such was helpful or not for the faculty and students. The final part of the paper discusses the successes and lack thereof how the methods and materials in the PAMS courses established an atmosphere that created for some students an easier transition to graduate school, preparation for actuarial tests, to the work force in applied statistics, assisted in forging a long-term undergraduate research component in the major, and encouraged some faculty to direct undergraduates in meaningful research.

So, this paper proposes a pedagogical approach to mathematical statistics education that centres on exploration, discovery, conjecture, hypothesis, thesis, and synthesis such that the experience of doing a mathematical argument, creating a statistical model, or synthesising ideas is reason enough for the exercise - - and the joy of mathematics and statistics is something that needs to be instilled and encouraged in students by having them *do* proofs, counterexamples, examples, and counter-arguments in a Probability and Mathematical Statistics course (indeed in

any course).

I. Introduction, Background, and a Brief Overview of the Moore Method

Mathematics is formally a branch of Philosophy under the Epistemology subheading. As such, a principle role in any mathematical education focuses the student on the attempt to deduce that which is conditionally true based on certain assumptions (axioms). Whilst some mathematicians are concerned with Ontology or Axiology (namely is there an ultimate "Truth" to mathematics and can we discover it, what is the essence of the beauty of a proof (its elegance, brevity, etc.) or what is the value of the idea?) this paper presents the position that restricts the discussion to the epistemological level as far as to deduce that which is conditionally true, present evidence to suggest the possibility that a particular pedagogy has merit, and argues that Probability and Mathematical Statistics should be studied in a pseudo-Socratic manner rather than studied as a Sophist would study a subject.

Mathematics, and in particular Probability and Mathematical Statistics (PAMS), is built upon a foundation which includes axiomatics, intuitionism, formalism, logic, application, and principles. Proof is pivotal to mathematics as reasoning whether it be applied, computational, statistical, or theoretical mathematics. The many branches of mathematics are not mutually exclusive. Oft times applied projects raise questions that form the basis for theory and result in a need for proof. Other times theory develops and later applications are formed or discovered for the theory.

Probability and Statistics (mathematical statistics) being a branch of mathematics, therefore, required an axiom system and thanks to Kolmogorov such was produced. From the point of the establishment of the basic axioms of probability (along with fundamental Aristotelian logic, the Zermelo-Frankel-Cantor axioms of Set Theory, and the axioms of Analysis) the theory of mathematical statistics became progressively more rigourous such that when we consider probability theory we are able to say, for example, given S is a well-defined sample space and E an event it must necessarily be the case that the probability of the complement of E is one minus the probability of E to use a facile corollary to the axioms of probability theory as an example.[1]

Mathematical statistics education should be centred on encouraging a student to think for one's self: to conjecture, to analyze, to argue, to critique, to prove or disprove, and to know when an argument is valid or invalid. Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is proof - - the demand for succinct argument from a logical foundation for the veracity of a claim. Mathematical statistics is deductive science as is mathematics in general whereas it is the case that Applied Statistics, like its sister sciences Physics, Chemistry, Biology, Economics, Psychology, etc., is fundamentally an experimental science. But, Applied Statistics is founded upon the establishment of mathematical truth– – specifically, probability theory.

The author of this paper submits that in order for students to learn, students must be *active* in learning. Thus, the student must learn to conjecture and prove or disprove said conjecture. One cannot learn to conjecture from a book, we learn

[1] S is a well-defined sample space $\wedge$ E an event $\Rightarrow Pr(E^C) = 1 - Pr(E)$.

to conjecture by conjecturing.[2] Ergo, the author of this paper submits the thesis that learning requires *doing*; only through inquiry is learning achieved; and, hence this paper proposes a philosophy such that the experience of creating an idea and a mathematical argument to support or deny the idea is a core reason for an exercise and should be advanced above the goal of generating a polished result.

This paper outlines a programme of use of a modified Moore method (MMM) in a naïve Probability & Statistics (P & S) course sequence[3],[4] to teach students about axiomatic probability, conditional and marginal probability, mutual exclusivity, statistical independence, random variables (discrete and continuous), multivariate distributions, moments, probability mass functions (PMFs), probability density functions (PDFs), moment generating functions (MGFs), cumulative distribution functions (CDFs), conditional and marginal PMFs or PDFs, etc. by teaching them to do, critique, or analyse proofs, counterexamples, examples, or counter-arguments. Furthermore, the author of this paper opines that Set Theory should be the core antecedent to the course with logic and Calculus as the other antecedents to the P & S courses and ideally a basic understanding of functions which are a subset of $\mathbb{R} \times \mathbb{R}$ to compleat a foundation on which rises the Probability & Statistics.

R. L. Moore created or adapted a pseudo-Socratic method which bears his name [13], [18], [19], [21], [31], [60], [61], and [62][5]. He said, "that student is taught the best who is told the least."[6] It is the foundation of his philosophy and it sums up his philosophy of education simply, tersely, and succinctly. Moore believed that the individual teaches himself and the teacher is merely an informed guide who must not trample on the individual's natural curiosity and abilities.[7] The Moore method accentuates the individual and focuses on competition between students. Moore, himself, was highly competitive and felt that the competition among the students was a healthy motivator; the competition amongst students rarely depreciated into a negative motivator; and, most often it formed an *esprit d' corps* where the students vie for primacy in the class.[8]

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[2]This statement is not meant to be sarcastic but to illustrate how fundamental to the argument forwarded in this paper that the act of conjecturing is central to inquiry-based learning (IBL).

[3]Also oft titled: Mathematical Statistics I and II, Theory of Probability, Statistical Theory, or some other title. We assume it is a junior, senior, or first-year graduate course sequence that we are discussing herein that is designed to delve deeply into the underpinnings of Applied Statistics.

[4]We designate a naïve Probability & Statistics course or courses to mean that Lesbegue measure theory and probability measures are not pre-requisite or discussed within the courses.

[5]Some have argued that his advisor E. H. Moore (no relation) might be rightly credited with inventing the Moore method. This is of no interest to the author, it suffices that such a method exists for the sake of this paper.

[6] R. L. Moore, *Challenge in the Classroom* (Providence, RI: American Mathematical Society, 1966), videocassette. See, also, Miriam S. Davis, *Creative Mathematics Instruction* Ed.D. dissertation (Auburn, AL: Auburn University, 1970), 25; Benjamin Fitzpatrick, Jr., "The Teaching Methods of R. L. Moore." *Higher Mathematics* 1 (1985): 45; and, Lucille S. Whyburn, "Student Oriented Teaching - The Moore Method." *American Mathematical Monthly* 77, 4 (1970): 354.

[7] See Davis, *Creative Mathematics Instruction* and Paul R Halmos, How To Teach. In *I Want To Be A Mathematician* (New York: Springer-Verlag, 1985) for a more detailed discussion of Moore's tenets.

[8]See Davis, pages 21 and 119 and D. R. Forbes, *The Texas System: R. L. Moore's Original Edition*, Ph.D. dissertation (Madison, WI: University of Wisconsin, 1971), pages 168, 169, 172, and 188 for a detailed psychological analysis of the formation of *esprit d' corps* from competition.

Many educators opine that the Moore method is, perhaps, best suited for graduate-level work where there is a rather homogenous set of students who are mature. Moore's philosophy of education is too often considered a method of teaching and, as such, can be adopted and practiced. The author opines that this is an error and it is a *philosophy of education.* Therefore, adoption of the methods that Moore created and practiced would be meaningless and could lead to harm for the students *if the practitioner did not subscribe to Moore's philosophy.* Whyburn, rather poetically, notes that Moore's beliefs "gives one the feeling that mathematics is more than just a way to make a living; it is a way of life, an orderly fashion in which you want to consider all things."[9]

Given the 'harshness' of the Moore method or due to the change in American education since Moore's death (1974), there have been several proposed modified Moore methods (MMM) which are similar to but not identical to the one proposed herein for teaching mathematical sciences courses [1], [2], [17], and [20] which could be used in a mathematics, computer science, physics, or statistics course. If one agrees with the philosophical position conditional to the modified Moore method, then it is an entirely acceptable teaching methodology. However, it still requires that the individual learn without the aid of books, collaboration, subject lectures, and demands (uncompromisingly) talent from the individual.[10]

The use of books would cause the student to be a witness to mathematics, rather than a participant.[11] Since the Moore method is based on the assumption that this talent is dormant or latent within the student,[12] the student is expected to do all that is necessary to tap into this dormant talent. There is not an expectation on the part of Moore's philosophy that this dormant talent will awaken easily or quickly. Thus, the pace of learning is set by the student or students.[13]

Several authors opine that a constructivist approach to teaching mathematics or statistics courses [24], [47], [53], [54], and [56] is the proper method. The constructivist accentuates the community and focuses on cooperation amongst students. The constructivist approach includes alternate assessment, group projects, service learning, etc. and closely resembles pedagogically the National Council of Teachers of Mathematics [44] standards and Dewey's position [14], [15], and [16]. If one agrees with the philosophical position conditional to the constructivist method, then it may be an entirely acceptable teaching methodology. It seems that the constructivist method is best suited for elementary or secondary education where students have not completely matured and where the material is less sophisticated. The constructivist method is based on a philosophy that the individual learn with

[9] Whyburn, page 354.

[10] D. Reginald Traylor, *Creative Teaching: Heritage of R. L. Moore* (Houston, TX: University of Houston Press, 1972): page 171.

[11] Indeed this was the case in all the classes that I took where a Moore adherent was instructing. Interestingly, to this day I can not quote well known theorem like the 'Dedekind Cut Theorem,' but can sit down with a paper and pencil and reading the statement of the proof, prove it. Indeed, if memory serves me correctly, my proof of it is contained in my master's thesis.

[12] Davis, pages 74, 75, 79, 81, and 185. Also, Traylor, page 169.

[13] See Davis, page 94; Forbes, page 156; F. Burton Jones, "The Moore Method." *American Mathematical Monthly* 84, 4 (1977): page 275; Traylor, page 131; and, R. L. Wilder, Axiomatics and the Development of Creative Talents. In *The Axiomatic Method with Special Reference to Geometry and Physics* (Amsterdam: North - Holland, 1959), page 479.

others and that reality is constructed. In its radical form it maintains "individuals construct their own reality through actions and reflections of actions."[14] So, under such a philosophy a compleat relativism antecedes such that objectivism is relegated to oblivion.

Indeed an concrete constructivist might argue for a pedagogy that centres on Monte Carlo simulations, examples, exemplars, demonstration, illustration, machine produced data sets, etc. and does not expect any proof or argument forms to be a part of a mathematics course and most particularly would not or should not be a part of a Probability and Mathematical Statistics (PAMS) course. Such a position is very similar to the qualitative research theory of credibility and believability (see [26] or [32]). Such fails to take into consideration that a PAMS course is a bridge between and betwixt mathematics and statistics that creates for a mathematics student a wonderful contrast to the deterministic mathematics that one had studied previous to a PAMS course– – the student in the PAMS course enters the realm of stochasticism rather than simple determinism. Indeed, concrete constructivism seems to suggest that there is not a clear connection between and betwixt mathematics and statistics; and, it seems to relegate the teaching of PAMS to an applied level in Bloom's taxonomy rather than on a higher cognitive domain (see [51]).

If there was *a* way to teach mathematical statistics, then it might be the case that this paper would not exist. However, it is commonly accepted that (a) different individuals learn in different ways and (b) there is a basic knowledge base that is necessary for the average student to obtain so that he has a higher likelihood to succeed in subsequent work after a course (subsequent course, the work force, graduate school, etc.). It is not as commonly accepted, perhaps, but it *is* argued in this paper that (c) conjecturing, hypothesising, and proving claims true or false is a skill that can be mastered through limited exposition, much practice, and individual inquiry.

Much general educational research centred on point (a), thus we shall not bother wasting paper addressing in detail this point. Much work of professional associations (in particular the Mathematical Association of America (MAA)) recent research and policy statements centred on defining point (b) and revising, enhancing, and reviewing point (b) [5], [6], [7], [8], [9], [10], [11], and [12]. Regarding point (b) as it applies to a P & S course sequence, the author submits that the course should teach students how to do, critique, or analyse proofs, counterexamples, examples, or counter-arguments in the theory of probability and mathematical statistics.

As to point (c), proving claims true or false the author opines is a skill is grounded in the philosophy of William James and the practice of George Pólya [48]. Just as art schools teach composition techniques, architecture schools teach drafting, etc. schools of mathematics teach theorem proving as a skill that is grounded in logic. There are a finite number of techniques and students are encouraged to learn each one so as to have a basic competency when approaching mathematical claims.

So, herein is proposed a methodology which seeks not to dogmatise teaching. It is proposed that if (a) is true, then strict, uncompromising, and rigid adherence to a traditional lecture, recitation, or German seminar pedagogical method should *not* be employed (such as is found in [52] or [55]). This is because it is highly

[14] Steffe and Kieren, "Radical Constructivism and Mathematics Education," *Journal for Research in Mathematics Education* 25, no. 6 (1994): 721.

unlikely that a class would be composed so homogeneously as to allow for that inflexible method of teaching to be employed and the goal of any course should include maximisation of the likelihood of success for students in the course *and beyond the courses subsequent to the course in which the students are enrolled.*

Further, it is proposed that if (b) is true, then the constructivist method being employed by an instructor might cause the pace of the course to be slow (perhaps too slow) and cause a likelihood that a student might not understand the material but only parts, elements, or pieces of the material. This is because group work does not necessarily imply that all in the group equally worked on a project, that all in a group learn each and every part of the project, and that much difficulty arises between the work in a group translating to an individual being able to do the work without help. In some areas of academia this may be an acceptable outcome, but in mathematics − − especially in theoretical mathematics − − it can be lethal to a student's mastery of material in subsequent courses if mastery of present material is not obtained (or at least there is a maximisation of the possibility that the student mastered the material).

It is proposed that if (b) is true, then a traditional German seminar method being employed by an instructor maximises the amount of material that can be 'covered;' but, 'coverage' does not necessarily imply mastery. Indeed, it can be argued that the traditional German seminar method (recitation) creates a likelihood that the pace of the course will be fast (perhaps too fast). Maximisation of expository material does not imply maximisation of the probability that the student mastered the material. In some areas of academia it may be an acceptable outcome that the student is *aware of much of the material but has not mastered the material*, but in mathematics - - especially in theoretical mathematics - - it can compromise or retard a student's mathematical progress.

Thus, use of a modified Moore method insofar as it employs the classic Moore method (students *doing* the proofs, counter-examples, etc.) allows for pace of the course to not be too fast; but, use of the *modified* Moore method with a book as a guide or reference for fundamental points which would probably be best learnt through discovery (but by using discovery would cause the pace to be too slow) allows for the pace of the course to not be too slow. Expository material (especially definitions) are contained within a book or instructor notes such that they are available to the student *before* material is discussed in the classroom but the exploratory material is not assumed to have been accessed *before* discussion of said within the classroom. Therefore, the pace of the class is largely determined by the students' abilities, schedules, interest, and needs; but, somewhat determined by a traditional idea of a syllabus and basic knowledge base that is necessary for the average student to obtain so that he has a higher likelihood to succeed in upper division courses. The use of a modified Moore method for instruction allows for the potential that material scheduled for the end of a course can be discussed or studied; but, does not guarantee it will be discussed or studied. We shall take point (c) for granted (perhaps errantly, but we shall assume it). Thus, we shall assume (a), (b), and (c) are true for the subsequent discussion.

II. The Author's Modified Moore Method and its use in Probability & Statistics Courses

A basic tenet of the modified Moore method (MMM) employed by the author is 'if it works, then use it,' to paraphrase William James. The instructor must enter into the classroom without much 'baggage' - - that is to say he should be pragmatic, realistic, open to changes, revisions, and constantly assess whether or not the students are learning.

The MMM employed by the author is fundamentally derived from the Moore method: the author's position is based on Moore's philosophy of education but relaxes several aspects of the Moore method. Moore's philosophy of education stated that a person learns best *alone* - without help or interference from others. The author's modified Moore philosophy of education states that a person learns best and most completely alone; *but*, sometimes needs a bit of help, encouragement, or reinforcement.

The Moore method assumes the student has a natural inquisitiveness, he must be active in learning, and as a consequent self-confidence and self-directedness is established and builds within the individual.[15] However, the student is not always going to perform at peak efficiency given the constraints of human nature and the diversions of modern society.

Therefore, the MMM employed by the author assumes there is a natural inquisitiveness in all humans; but, it ebbs and flows or intermittently turns on or off much as a distributor cap distributes a charge in an engine. Therefore, a student sometimes needs a bit of help, encouragement, or reinforcement. The help, encouragement, or reinforcement **should not** be actualised by giving solutions to a student; but, by asking a sequence of directed questions that the instructor 'knows' is perhaps one path toward an argument for or against a proposition. It is best if the instructor tries to put himself in the place of the student and imagine that he does not know the solution. [16]

The Moore method demands that the student *not* reference any texts, articles, or other materials pertaining to the course save the notes distributed by the instructor and the notes the individual takes during class. Not every student is as mature and dedicated as to be able to follow such a regulation especially in a university setting and most especially in an undergraduate course. Thus, books are not banished in the classroom of our MMM. The class has a 'required' text that the author opines is fine for definitions and trite examples but is less than compleat or rigourous in its exposition or examples. The author opines that such a text is best so that it does not give to or impose upon the student too much.[17] This philosophy of education does not seek maximal coverage of a set amount of material, but standard competency with some depth and some breathe of understanding of material under

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[15] See Davis, pages 17, 78, and 173; D. R. Forbes*, The Texas System: R. L. Moore's Original Edition* Ph.D. dissertation (Madison, WI: University of Wisconsin, 1971), page 181; Traylor, page 13; and, Whyburn, page 354.

[16] This is easy for a person such as the author who readily forgets much and oft remembers little.

[17] Indeed, the student is allowed to use as many books as he opines is necessary to understand the material.

consideration.[18] This requires time, flexibility, and precise use of language.

The Moore method *demands* that the students not collaborate. Moore stated this position clearly:

> I don't want any teamwork. Suppose some student goes to the board. Some other student starts to make suggestions. Suppose some how or another a discussion begins to start. One person suggests something, then another suggests something else... after all this discussion suppose somebody finally gets a theorem... who's is it? He'd [the presenter] want a theorem to be his - he'd want a theorem, not a joint product![19]

The modified Moore method employed by the author tempers the position Moore proposed and demands *no collaboration on material before* student presentations and *no collaboration on any graded assignment* and requests minimal collaboration on material after student presentations. After student presentations, if a student does not understand a part of an argument or nuance of said argument, the students are permitted to discuss the argument as well as devise other arguments amongst themselves or with the instructor.

The Moore method does not include subject lectures. The MMM employed by the author includes minimal lectures before student presentations over definitions, notation, and terminology, an occasional exemplar argument, counter-argument, example, or counterexample (especially early in the course), as well as subsequent discussions (facilitated, directed, or led by the instructor) after the students discuss the work(s) presented when the instructor finds there is confusion or misunderstanding about the material amongst the students. However, the MMM employed by the author is not as 'lecture heavy' as a traditional class – – the instructor does not enter the class begin lecturing and only end recitation at the end of the period.

In the P & S courses where the author's modified Moore method is employed, everything *should* be defined, axiomatised, or proven based on the definitions and axioms whether in class or referenced. In this regard the MMM employed by the author is reminiscent of Wilder's axiomatic methods [57], [58], and [59]. Everything cannot be defined, discussed, etc. within class; hence, the allowance for reference material. Indeed, the MMM employed by the author avails itself of technology; thus, additional class materials are available for students to download from an instructor created web-site. The materials on the web-site have several purposes including delving deeper into a subject; clarifying material in a text; correcting a text used in the class; reaction papers to student work; alternate solution(s) by student(s) other than the student who presented a solution to a claim in class, an alternate solution by the instructor to a claim which was presented in class, or posing several additional problems and question in the form of additional exercises.

Moreover, instructor created handouts on the web-site present students with material previously discussed, claims which were made during the class (by students or the instructor), exercises beyond the scope of exercises in the text, and conjectures

[18] Ideally– – overall – – if depth and breathe are in competition, then depth should be considered more essential than breathe.

[19] Moore, *Challenge in the Classroom.*

that were not presented by students in the class along with proposed arguments as to the veracity of the claims. The students critically read the proposed arguments and note whether or not the proposed solution is correct. Thus, the modified Moore method employed by the author includes more reading of mathematics materials than the Moore method, though less than a traditional or constructivist method.

A superficial understanding of many subjects is an anathema to a Moore adherent; a Moore adherent craves a deep, full, and compleat (as compleat as possible) understanding of a subject (or subjects)[20] so, the pace in the author's P & S classes is set by the instructor tempered by the instructor's understanding of what the students grasp. 'Coverage' of material is not a hallmark of the Moore method. Traditional methodology includes the pace of the class set by the instructor (usually prior to the semester). 'Coverage' of material is a trademark of traditional methods. Maximal treatment of material is typical in a traditional classroom. However, the undergraduate experience is repleat with time constraints. Thus, pace is not determined by the students but is regulated and adjusted by the instructor. Hence, the P & S classes using the MMM employed by the author attempts to balance the student-set pace (Moore method or constructivism) with the instructor-set pace (traditionalism). The author's MMM acknowledges that not all questions can be answered and that each time a question is answered a plethora of new questions arise that may not be not answerable at the moment. Therefore, the MMM employed by the author is designed to balance the question of 'how to' with the question of 'why.' The author opines that a subject that is founded upon axioms and is developed from those axioms concurrently can be studied through answering (or at least attempting to answer) the questions 'why' and 'how to.'

The author's modified Moore method includes the concept of minimal competency, that a student needs some skills before attempting more complex material. So, aspects of 'coverage' are included in the author's P & S classes; that is to say that there is a set of objectives that the instructor attempts to meet when administering a class, that he is duty-bound to try to meet said objectives. However, the author's modified Moore method does not attempt to maximise 'coverage' of a syllabus. A syllabus designed by an instructor who adheres to the author's MMM would include 'optional' material and would have a built-in 'cushion' so that the set of objectives can be discussed (more than just mentioned), the students have a reasonable amount of time to work with the material, and more than that set of objectives is met each semester. The goal of education is not, under this methodology, 'vertical' knowledge (knowing one subject extremely well) nor 'horizontal' knowledge (knowing many subjects superfluously), but this philosophy attempts to strike a balance between the two.

Traditional methods include regularly administered quizzes, tests, and a final. The author's MMM also includes said assessments. However, a part of each quiz or test (for a test no less than ten percent nor more than thirty-five percent) is assigned as 'take home' so that the student may autonomously compleat the 'take

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[20] See Davis, page 70; Fitzpatrick, "The Teaching Methods of R. L. Moore." *Higher Mathematics* 1 (1985): 44; Fitzpatrick, Some Aspects of the Work and Influence of R. L. Moore, *A Handbook of the History of Topology* 1996), page 9; Forbes, page 194; Paul R. Halmos, How To Teach. In *I Want To Be A Mathematician* (New York: Springer-Verlag, 1985), page 262; and, Edwin E. Moise, "Activity and Motivation in Mathematics." *American Mathematical Monthly* 72, 4 (1965): page 409.

home' portion with notes, ancillary materials, etc. 'Take home' work is more natural (reflecting the work mathematicians and statisticians *do* after college); allows students to follow an honour code;[21] allows students time to work on their arguments or examples; and, allows students to tackle more challenging problems than could be included on an 'in class' quiz or test.

The author has found that what fundamentally drove him toward use of his MMM was that he learnt best under the Moore method (out of all the methods he was privy to be exposed to whilst an undergraduate) and that he could learn under other methods but with diminished results. Two of the primary reasons for the deminishment of results were his laziness and ability to memorise. The Moore method or modified Moore method does not seemingly reward superficiality or non-contextual rote memorisation.

It is the author's understanding that constructivist or reform methods include class discussions, use of technology, applications, modelling, and group assignments. P & S courses directed under the author's MMM includes class discussion and allows for the discussion to flow from the students (but be directed by the instructor). It should be expected that, on average, at least one-half of each class period be dedicated presentation of work, at least one-fourth of each class period be dedicated discussion of work presented or ideas about the definitions, axioms, or arguments. The author's MMM class allows students to use machines[22] for applications (minimal discussion of applications exists in the P & S courses since the emphasis is on the foundations of the theory of probability and mathematical statistics) and modelling (with regard to the fact that students present their arguments before the class and there exist exemplars for the students as well as critical reading exercises). The author's modified Moore method does *not include any kind of group assignments nor any kind of group work.* In this regard it is much more similar to the Moore or traditional methods than constructivism.

At Emory University, his Algebra instructor used the Moore method and his Analysis instructor a traditional method. In the Algebra class the author was enraptured by the material and found himself driven to try to do *every* exercise, example, proof, counter-argument, or counterexample. The author memorised proofs in Analysis and regurgitated them back on tests (one recalled with great ease is the proof that $\sqrt{2}$ is irrational). At Auburn University, the exact opposite was true:

[21]An honour code is *essential* in a Moore or modified Moore philosophy of education. The honour code is a positive affirmation of that which is most important in academia: the honest, open, and unbiased search for truth and the celebration of the hunt!

[22]Maple, Mathematica, Derive, SAS, SPSS, R, Bilog, etc. are acceptable (with restrictions - - they may be used outside of class, they may be used to explore, but they can not be used for graded assignments and the use of said is not encouraged). The use of Maple, for example, for empirical results, simulation, or exploration is fine (in small doses) and can be used to encourage and assist in providing ideas. However, overdependence on machines and on the external is not advisable nor is it a part of a Moore or modified Moore philosophy of education. Much as videos ruined many a song post 1980 (the introduction of MTV and its clones)– – imagination can be ruined because of the imposition of images from the videos, for example. Consider that a song is meaningful in different ways for different people and diverse images are created in individuals minds from the words of a song. Post-1980, many people 'see' only the images from the videos (especially through repeated incessant exposure); the words no longer excite the imagination– – the images of the video are imposed on the watcher or listener. Perhaps I have belaboured this point but when one views clouds and sees images and imagines shapes is 'better' than someone who just sees a cloud (and doesn't even care it is a cloud, water vapour, etc. the person just witnesses everything and acts on nothing).

his Analysis instructor used the Moore method and his Algebra instructor a traditional method. The author memorised proofs in Algebra and regurgitated them back on tests; whilst thriving in the Analysis class. It is not a contention forwarded by the author that traditionalism caused his lack of understanding of material but that because it was easy *for the author* to memorise arguments presented by the instructor, cram before tests, and take 'short cuts;' but, that the Moore method does not reward such study habits (e.g.: it is harder to fall into the 'the long memory & short on understanding trap' in a class organised under the aegis of the Moore method or a modified Moore method).

At Georgia State University, his Meta Analysis and Evaluation Theory, and Sampling Theory instructors used constructivist methods and his Mathematical Statistics and Linear Statistical Analysis instructors traditional methods, and his Edumetrics and Multivariate Statistics instructor used modified Moore methods. Frankly, it is surprising the amount of material that is recalled and not all together shocking that much (if not most) is forgotten from the courses. However, given that it was graduate school (and the second Ph.D. programme for the author who was in his 30s when at Georgia State) it is not noteworthy that the author did learn some things. However, he found that many of the constructivist 'activities' were mind-numbing or less than instructive (in his humble opinion) and it is entirely possible that the methods were not executed well rather than the fault of the method, per se.

A strength of the modified Moore method is that the focus of a MMM class is on the construction of arguments, examples, counter-arguments, or counterexamples. It is not the case that a traditionalist or constructivist teaching mathematics would do any less; but the MMM is very much suited for construction of arguments, examples, counter-arguments, or counterexamples and less suited for more applied pursuits or material.

Another strength of the modified Moore method is that there can be and oft is a detailed discussion and the instructor can focus student attention on the difference between and betwixt contradiction, contraposition, and contrarianism. The author has found that many student have great difficulty discerning the difference between and betwixt the three and oft confuse them. The author opines that contradiction, contraposition, and contrarianism are not usually properly contrasted and compared in a traditional class because the the class is often instructor-focused rather than student-focused. The author opines that contradiction, contraposition, and contrarianism are not usually properly contrasted and compared in a constructivist class because the the class often does not delve into material as deeply as with a MMM nor does it seem that contrasts and comparisons are done as much in a constructivist setting. [23]

An additional strength of the modified Moore method is that many ideas naturally percolate from members of the class and that many ideas can be provoked by the instructor by asking a sequence of question so that the end result is ideas arise from individuals in the class. Such percolation of ideas is a hallmark of any Moore class; and such is a primary objective for the instructor in the author's MMM schemata. It very much seems to be the case that in a traditional scheme there is

[23]It is all together possible that I could be wrong on this point; but, I have not been privy to much depth in courses instructed by a constructivist where I was a student nor have I been privy to much depth in courses instructed by a constructivist where I was a peer witness.

not such a movement of ideas from individuals to the group (the class) but from one individual (the instructor) to the group (the class) or from an outside source (the book) to the group (the class). Much of the literature about or advocating for constructivistic instruction in mathematics contains a similar focus on natural percolation of ideas from members of the class and as such share a commonality to a MMM class (they are 'student-centred' approaches as opposed to 'instructor-centred.'

A fourth (and possibly most important) strength of the modified Moore method as practiced by the author is that it encourages students to opine, conjecture and hypothesise by naming principles, lemmas, theorems, corollaries, etc. for individuals who proved the result or proposed the result. Such a technique, the author has found, advances the proposition of trying to opine and think about the ideas discussed in class - - hopefully giving a modest 'push' to the students to try to stretch beyond that which is before them and try to induce ideas new to them (but not necessarily or most often new). The author's MMM is predicated on the proposition that we do not really care who first developed an idea– – we are interested in the idea itself and whomever it was that thought of it or did it first does not matter for it is in the *doing of mathematics* we learn not through a discussion of history. But, does the history of a problem really matter in the greater scheme of things? Perhaps it did for such individual, but most often the ideas that the students propose are ideas first proposed and most often were solved by people who are dead and therefore are not complaining or seeking credit. [24]

A fifth strength of the modified Moore method as practiced by the author is that in such a scheme there is a pronounced, overt, and clear **celebration of effort** and an attempt at trying to solve a problem, create a proof, argue a point, forge an example, produce a counter-arguments, or construct a counterexample. Such was not the case in many classes in which the author was a student. In a class or two instructed by a person using a Moore method or modified Moore method there was not such a celebration for an attempt and in some cases there was revulsion for or a taunting of an individual who tried but failed to produce a valid result.[25] It seems to be quite the case that in a traditional scheme there is not such a celebration for the vast majority of the work is refined through recitation by the instructor or exposition in the text. On this point from his review of the literature about or advocating for constructivistic instruction, the author's MMM shares in a similarity with constructivism. A fortunate or fortuitous example exists in recent popular culture, the Disney film *Meet the Robinsons*, captured the sense of excitement the author attempts to create in his class and amongst his students for trying, trying again, celebrating the attempt, accepting that we are not always correct, and realising that we learn from mistakes (if we pay attention to the mistakes and analyse

[24]Let it not be misunderstood that in the author's MMM class there is some sort of nihilistic or narcissistic atmosphere. The credit for principles, lemmas, theorems, corollaries, etc. and study of the history of mathematics is all well and good but is not a part of a Probability and Statistics sequence nor is a focus of any course taught by the author.

[25]It should be noted that at least four individual instructors who used Moore methods or modified Moore methods were definitely NOT in this category: David Doyle of Emory University, Michel Smith of Auburn University, Coke Reed of Auburn University, and John Neel of Georgia State University. Those individual instructors were *very* encouraging, inspirational, and their methods formed the basis for the author's teaching style.

them), and always trying to "keep moving forward." [26] Therefore, though it may seem trite, every student in an author's MMM class receives credit in the form earning a 'board point' for attempting to solve a problem, present a solution, do a proof, etc. Said points add into the student's total points at the end of the semester and there are a plethora of example where said points produced the 'rewarding' result of a student's grade being positively impacted.

Much of the points that highlight the strengths of the modified Moore method may be summed up as the MMM *accents, celebrates, encourages, and attempts to hone an internal locus of control.* The traditional scheme there does not appear to be as prevalent a focus on the internal; indeed, rather there is a clear focus on ideas from the external (the instructor, a calculator, a computer, or the book). The constructivist scheme there does not appear to be as prevalent a focus on the internal; indeed, rather there seems to be a focus on ideas from the external (the group, a calculator, a computer, or the book) and the internal and individual are not primary.

Nonetheless, there are weaknesses to the MMM as practiced by the author, not the least of which the pace of the course is slow and almost always when a section of a course is taught by the author and someone else using traditionalistic methods, the author has found 'coverage' lacks in his section.[27] Sometimes it is the case that the an outside observer might opine that there is no pace seemingly at all in the course or that there is 'backward progression.' It is safe to say that sometime there is indeed 'backward progression' in a course taught by the author for if the author finds there seems to be a prevalent misunderstanding, confusion, or downright erroneous concept being embraced by members of the class; such is usually discussed, confronted, or debated.

One example stands out in the author's mind. He had a fellow faculty member visit his class and there was a student who volunteered to present proof to a rather difficult theorem about the mean of a particular random variable that day. The student did a wonderful job and she laid out the argument beautifully. Well, the claim was proven and there were not other volunteers that day. But, there was another claim that was true that the instructor thought of and wished for the class to consider. He called on another student in the class and had him go to the board. He asked the student about the claim (whether he thought it true or not) and the instructor quizzed the student on some material to a point at which the student felt he had an idea how to prove the result. He preceded to do so, 'winging it,' and not producing by any means a polished result, but the essence of the argument was there, he had presented the class with the rough sketch of a fine argument, and it seemed to be quite a productive class. However, the author found that his colleague was not impressed and bemoaned, "since Dr. McLoughlin knew that he will be observed, I wished he planned to present some new material to demonstrate his teaching."

The aforementioned incident is an exemplar of what seems to be the case: that is, the Moore method or a MMM such as the author employs is so different from

[26]A very important point to the author's MMM is to celebrate failures as well successes (in fact the failures oft lead to some great ideas and as my sainted mother, may she rest in peace, said we do not learn if we do not fail). It is from our failures whence we learn the most.

[27]However, it is definitely NOT the case that most often a standardised syllabus in a course has not been 'covered' in a section the author has taught.

a traditional classroom that a traditionalist can misunderstand the method and opine that nothing is being accomplished. It is the case that in an inquiry-based learning classroom nothing might be accomplished on a given day, but that was not the case in the example previously mentioned. Hence, one can easily misconstrue that which occurs in an inquiry-based learning (IBL) environment which could lead to professional difficulties (lack of an award of tenure, poor class assignments, etc.). There are many examples of problems between mathematicians who use the Moore method and those who don't from the twentieth century to fill several volumes. Thus, for practical reasons an instructor who creates an inquiry-based learning (IBL) class needs to have the support of his colleagues or at least the support of those in charge of the department.

Another 'thorn' of the MMM as practiced by the author (that it shares with the Moore method) is there is a heavy burden placed upon the student. Quite frankly, it seems there is a much greater expectation placed upon the student in the MMM class as described herein than under a traditional or a constructivist rubric.[28] The expectation is that the students are adults, they are responsible for their education, they are *not required* to attend class, they are responsible to do the work, they are not forced to do any work or hand in any work (other than quizzes or tests), they are expected to 'try & try again,' they are placed in a position in the class to usually experience a barrage of questions from the author and experience being interrupted often whilst presenting, they are oft questioned whilst someone else is presenting (meaning that during one student's presentation the other students have to 'be on their toes' to expect that they might be asked why something is so or whether or not it is or is not [which is a back-handed way to get students to argue with another student's work on the board in a kind way)], and they are asked to do all of this whilst attempting to take notes, etc. (which most do)).

Judging from some of the comments made by students in class, to the chairman of the department or colleagues, or on the Student Ratings on Instruction (SRI) at Kutztown University, the added expectation is not popular nor seemingly appreciated.[29] Furthermore, the author's MMM seems to be in direct conflict with

[28]Such was made very clear when the author moved to a new institution where there is a mathematics education programme, there seem to be as many traditionalists as at his previous institution, there are constructivists, and where there are no other Moore method faculty in the department.

[29]Such was the case at Morehouse College in some classes but the author received more positive feed-back there. The amount of positive feed-back was higher for higher-level courses at Morehouse. The exact opposite is the case at Kutztown University. Also, he received much positive feed-back from students after they graduated which included comments such as,"... at the time I did not care for it, but now I appreciate ..." The author has been in his present situation for three years, so, it may be such in the future with Kutztown University alumni. However, it may be a case of the cart before the horse since the author was at Morehouse College for 17 years so that students may have acclimated to the MMM used by the author and students not inclined to such avoided his section of a class. There is a tad of anecdotal evidence to suggest that may have been the case due to the following: During first semester of the 2007-2008 year, enrolment in the author's Probability & Statistics I course at Kutztown was lower than another section taught by another instructor; but, the previous 2 years the author was the only person who taught the Probability & Statistics I course. The author heard that there was much jockeying by several students to get into the other section and not be in his section. Moreover, the author heard from more than one student that one or two mathematics education majors in particular were "desperate" to get into the other section and celebrated when they achieved their objective. The other instructor taught in a traditional German seminar recitation style and expected no proofs to be

the prevailing approaches of instruction at Kutztown University of Pennsylvania (KUP) and how most classes at KUP are taught. There seems to be a two dominant formats to instruction at KUP: recitation and group work. Ergo, many classes are taught in the traditional German seminar manner and many are taught in the radical constructivist manner with groups, groups work, group projects, portfolios, etc.

The aforementioned 'problems' for or with students also could lead to professional difficulties (lack of an award of tenure, poor class assignments, etc.). At the university where the author teaches, there seems to be a heavy accent on student evaluations and some colleagues have advised him that faculty need to award many A's and few F's whilst maintaining a large enrolment (few withdrawals) to avoid employment problems. It would be interesting to note if such occurs at other universities or if such has any correlation to the existence of a College of Education at the university where the author teaches for there was not a College of Education at the college where the author taught and there seemed far less antipathy toward the Moore method or a modified Moore method and no pressure to inflate grades. The author could be wrong and there may not be antipathy amongst the faculty at the university, but there is no doubt there is antipathy amongst the students at the university toward the MMM and it exists most often amongst the student pursuing a Bachelor of Science in Mathematics Education.

So, under a modified Moore method (MMM) rubric for creating an inquiry-based learning (IBL) atmosphere in a Probability and Mathematical Statistics (PAMS) course sequence, the courses should be designed such that the instructor *guides* students through a carefully crafted set of notes that is forms a firm foundation in a naïve introduction to the theory of probability and mathematical statistics which then builds introducing more and more complex ideas. Further, the instructor *ought* constantly monitor the progress of individual students and adjust the notes or offer "hints," where appropriate so as to encourage inquiry and further study.

Use of the Moore method or a modified Moore method (MMM) cannot be undertaken or adopted as one changes shirts or ties dependent upon a whim, a mood of the day, or social convention - - one must 'buy into' a philosophical position that humans have a natural inquisitiveness - - we must be *active* in order to learn and we must be *engaged* when learning. Adoption of said philosophy is not enough - - it must be practiced - - hence, the author submits that the method of teaching that he suggests is a useful inquiry-based learning (IBL) model (which has had some success and has produced seemingly successful students) in teaching mathematics from the freshman to graduate level is a MMM which focuses on each of the individuals in a class **as individuals** and assists students' development of an understanding of the theory of probability and mathematical statistics.

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done in or out of class opining that the course was an 'applied' course and mentioning the author was a 'pure' person whilst he (the other instructor) was an 'applied' person. The same held for the second semester of the 2007-2008 year when comparing enrolment in the author's Foundation of Higher Mathematics course versus another section taught by another instructor.

III. Organisation of the Probability and Statistics Sequence When Using a Modified Moore Method

In this part of the paper the author submits the content he opines should be in the Probability and Mathematical Statistics sequence, an explanation is presented as to why the courses were designed the way they were (content), how the courses were revised or altered over the years, and how such worked or did not for the faculty and students.

The author has studied under professors who have taught in each of the three ways that have been noted: the Moore, traditional, and constructivist methods during his formal educational experience which spans from the 1960s to the 1990s.[30] The author's MMM has been developed over the years of his college-level teaching experience (1982– –present). It is constantly being analysed, refined, and evaluated so it is a dynamic rather than static programme of thinking about mathematics and teaching mathematics. As such the development, revision, and evaluation of the MMM used by the author is an example of an action research model forged empirically.

The author taught the P & S sequence course whilst at Morehouse College (MC) and at Kutztown University of Pennsylvania (KUP). At each institution P & S I is a required course in the Bachelor of Science (BS) in Mathematics programme. At MC, P & S I is an elective course in the Bachelor of Arts (BA) in Mathematics programme. At KUP, P & S I is a required course in the Bachelor of Science (BSE) in Mathematics Education programme. At each institution P & S II is an elective course in any of the versions of a mathematics major. The course outlines are similar at each institution so we shall focus on a discussion of the programme at KUP unless otherwise noted.

The first course in the sequence was designed to be taken by the students after Calculus II (in a three four-credit hour Calculus sequence) and Foundations of Advanced Mathematics (also called Bridge to Higher Mathematics (BHM), Foundations of Mathematics (FOM), Introduction to Advanced Mathematics (IAM), or Introduction to Set Theory (IST)). At KUP, in the model Bachelor of Science (BS) programme, the course is placed as a first semester junior course, and "is designed to provide the student with an intense foundation in fundamental concepts of stochastic mathematics used in advanced mathematics."[31]
The content is typical: [32]

- Preliminaries, basic probability, deterministic versus stochastic functions, & the axioms of probability.
- Theory of probability, claims about probability, and combinatorial Methods.

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[30] See http://faculty.kutztown.edu/mcloughl/curriculumvitae.html for a compleat curriculum vitae.

[31] Math 301, Probability and Statistics I, Course Objectives, http://math.kutztown.edu/ or http://faculty.kutztown.edu/mcloughl/Math301.asp

[32] See Committee on the Undergraduate Program in Mathematics (CUPM) *Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide 2004*, *A General Curriculum in Mathematics for College, 1965*, or *Pre-graduate Preparation of Research Mathematicians, 1963* for exact specifications.

- Conditional probability, independence versus non-independence, Bayes' Theorem, and claims about said material.
- Discrete random variables: Probability Mass Functions (PMF), Cumulative Distribution Functions (CDF), Moments, Specific PMFs, CDFs, and claims about PMFs or CDFs.
- Continuous random variables: Probability Density Functions (PDF), Cumulative Distribution Functions (CDF), The Gamma Function, Moments and Moment Generating Functions, Specific PDFs, CDFs, and claims about PDFs or CDFs.
- Joint Distributed Random Variables: Conditional probability revisited, marginal probability, statistical independence versus non-independence, Joint Probability Mass or Density Functions, Cumulative Distribution Functions, Marginal Probability Mass or Density Functions, Conditional Probability Mass or Density Functions, Moments, Specific JPDFs, JCDFs, and Claims about JPDFs or JCDFs, Covariance, Correlation, and Transformations of Random variables

Specific PMFs or PDFs considered in detail in P & S I usually include most of the following: Uniform, Bernoulli, Binomial, Gaussian (Normal), Geometric, Hypergeometric, Chi- Squared (χ^2), Erlang (Gamma), Exponential, Weibull, Cauchy, Beta, LaPlace, Poisson, Rayleigh, and Student (t).

The course has always ended somewhere in the Joint Distributed Random Variables part of the content objectives. The second course in the sequence begins with jointly distributed random variables (where the previous left off with a tad of review the first day to set the tone of the course). P & S II was designed to be taken by the students after Calculus III (in a three four-credit hour Calculus sequence) and Linear Algebra, whilst the content of P & S II is typical as was the content of the first course:

- Joint PDFs and PMFs, covariance, correlation, independence, marginal distributions, conditional distributions, applications.
- Properties of expectation: Expectation of a sum of random variables, covariance, moment generating functions, conditioning, and applications.
- Limit Theorems: Tchebyshev's inequality, weak law of large numbers, the Central Limit Theorem, the strong law of large numbers, bounding, and applications.
- Estimation Theory: Sufficiency, bias, relative efficiency, consistency, robustness, the method of maximum likelihood, the method of moments, and applications.
- The Theory of Hypothesis Testing: Statistical hypotheses, the Neyman-Pearson lemma, point estimates, confidence intervals, inferences about μ, inferences about σ^2, inferences about comparisons of parameters, the F and t distributions, and applications.
- The Theory of Linear Regression: Linear regression, the method of least squares, parametric inferences, and applications.

Specific PMFs or PDFs considered in detail in P & S II usually include most of the following: Multinomial, Gumbel, Fischer- Snedecor (F), Gaussian (Normal) bivariate, Chi- Squared, Student (Gossett) t, and generalisations of other distributions (indeed students oft conjecture that certain functions are well-defined PMFs or PDFs and try to prove their conjecture).

The author has never taught a Probability & Statistics II class that reached regression and at KUP the author's classes have never reached the theory of hypothesis testing.

The author recommends that the material in the course include and accent material that from [5] and [6] more than [7], [8], [9], [10], [11], and [12] for accomplishing the recommendation in each and the recommendations of [4] and [45]. The author opines that the 1963 - 1965 Committee on the Undergraduate Program in Mathematics (CUPM) materials delineated that which is fundamental to a strong undergraduate programme and preparation for graduate school and that much of the work produced by CUPM post-1985 centres on 'service' courses, 'mathematics appreciation,' and computational mathematics applications with computers.[33]

When one analyses the parts of the course in more detail one can see that there is much that is of import as introductory material that does not need to be delved in deeply if one were simply concerned with Applied Statistics. However, the Theory of Probability is the underpinning for Mathematical Statistics which is the foundation for Applied Statistics. Hence, a student might have a Sophistic understanding of Statistics without the Probability and Mathematical Statistics (PAMS) sequence as outlined herein; but, shan't have a Socratic understanding of Statistics. If one accepts the proposition that a Socratic understanding is superior to a Sophistic understanding of a subject, then the usefulness and credibility of an inquiry-based learning (IBL) scheme such as a MMM for teaching PAMS becomes, it seems, clear.

In the author's classes at Kutztown University of Pennsylvania (KUP), much time must be devoted to reminding students of that which they learnt (or should have) before the PAMS sequence. The background material which the author has found needs accenting, emphasis, bears repeating, should be stressed, or even (perhaps) drilled include: argument forms and logic (such as *modus ponens, modus tollens, reducto ad absurdum*, hypothetical syllogism, the law of the excluded middle, cases, contrarianism, the fallacy of assuming the conclusion, the fallacy of denying the hypothesis, and the fallacy of appealing to authority), the basic elementary properties of sets, the algebra of sets, and basic topological properties of $\mathbb{R}$, all aspects of functions (domain, codomain, range, corange of a function, image sets, inverse image set, the union or composition of functions, the creation of proofs or refutations about claims on well-defined spaces), as well as basic principles of Calculus (limits, continuity, derivative, Riemann-Darboux integrals, etc.). [34]

However, it is not the intent of this discussion to infer that at KUP the students are lacking or that the PAMS sequence is dominate by review; it is to simply posit that students forget or need refreshing on much pre-requisite material and that said is more needed now than twenty or so years ago (counter to [46]). The reasons for that probably would fill several volumes but it suffices to say that obviously this author is intimating that much of the problem of student retention is due to poor

[33]Post-1985 CUPM guidelines are not 'bad' or 'wrong' but do not seem to accentuate the kind of or strength of preparation for advanced work in mathematics that is the focus of this paper.

[34]My experience is contrary to the concept of 'pre-requisite free' mathematics (see O'Shea & Pollatsek [46])

pedagogy in the educational system before the PAMS sequence and might be rectified by more IBL methods in all mathematics classes from kindergarten forward.

Prototypical claims that are a part of the PAMS sequence include (most of these we shall discuss are from the Theory of Probability part of P & S I since it is rife with fun ideas and conjectures which *seem* to the student to be true, but which are, *of course, in some instances true (first item) and other not true*) include:

- when S is a well-defined sample space and $E \wedge F$ events claiming that $E \subseteq F \Rightarrow Pr\,(E) \leq Pr\,(F)$
- my favourite basic claim that is usually proposed is when S is a well-defined sample space and $E \wedge F$ events claiming that $E \subset F \Rightarrow Pr\,(E) < Pr\,(F)$.[35]
- what if for the axioms of probability, one were to not claim when S is a well-defined sample space $Pr(S) = 1$?
- why not have S is a well-defined sample space and an event E where $Pr(E) > 1 \quad \vee \quad Pr(E) < 0$?
- some student usually claims S is a well-defined sample space and E is an event $(Pr(E) = 0) \Rightarrow (E = \emptyset)$.
- moments of PMFs
- moments of PDFs
- in P & S II investigating S^2 versus S^2_*).[36]
- arguments about the weak law of large numbers
- must every branch of mathematics have an axiom system?
- and, how do I drop this major (a small joke)?

Many a healthy discussion develops about the concept of a well-defined sample space (the universe) S and an event (set) E where S is of cardinality greater than $\aleph_0$ and E is or is not. The author has found by using a sequence of directed questions about basic geometry, students can determine how to conceptualise probability theory with sample spaces and events that are not only infinite but uncountable; however, a caveat must be mentioned about the technique. Such an assumption as college students are familiar with basic geometry is becoming problematic– – KUP students are not required to have had high-school level Euclidean Geometry! It seems that (at least in Pennsylvania) the idea of a standard set of pre-college mathematics courses is not accepted or required (which seems in agreement with [46] and which it seems causes more than a few problems).

Some ideas, theorems, etc. have not (to this point) been created by the students spontaneously or without suggestion; for example, Tchebyshev's inequality, so those are given as ideas presented to the students for consideration and as fodder to prove or disprove (obviously proving it by the nature of the presentation that it is a theorem). This is not a problem since in all areas of mathematics the author has studied when under a Moore method or MMM so principles, concepts, theorems, etc. were given to the students by the instructor; for example, in Analysis and the Topology of $\mathbb{R}$ the classic Cantor middle 3^{rd} set was presented to the class in an inquiry-based learning manner such that the instructor (Dr. Coke Reed) presented the set and asked are there any interesting things that could be said about the set,

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[35]I could fill a book with all of the 'proofs' of this claim that have been produced by many a student. By and large, for this as well as many other false 'proofs' the problem for the student is not with the Theory of Probability but with pre-requisite knowledge (hence, the need for some much clarification of pre-requisite material in logic, set theory, and calculus)).

[36]S^2_* is the same as S^2 *except* the sum of squares is divided by n rather than $(n-1)$.

etc.

Such a presentation (in an IBL manner) is a part of the P & S sequence when taught with the MMM. To wit, each year or so the author encounters a particular PMF or PDF (one such example was the Maxwell PDF). The author presents the PDF as with any of the standard PDFs or PMFs in the sequence and asks the students, "what do y'all think about this? Is it a well-defined PDF; if so, what properties does it have, what is its mean, etc." By approaching the class in such a Socratic manner rather than Sophistic manner, the students are treated as adults, are challenged, and to whom are not condescended.

The author has used Wackerly, Mendenhall, & Scheaffer's *Mathematical Statistics with Applications* , Freund's *Mathematical Statistics*, Rice's *Mathematical Statistics*, Ross's *A First Course in Probability*, and Gharamani's *Fundamentals of Probability* . Currently, the author uses the 3^{rd}edition of the Gharamani text for P & S I and the Gharamani text along with Freund's text for P & S II. In addition his sequence of notes and hand-outs are liberally used. [37] The author has found that most students do not access the books except for problem sets. The expectation of rigour and proper construction of arguments, counter-arguments, examples, and counter-examples in the PAMS sequence makes the use of the texts rather superfluous. Some students attempt to use one or more books for exemplars and illumination but quickly find that due to the MMM being employed, such does not help as the student might think it would.[38]

The first meeting day of the P & S I class, students are given a syllabus, given a grading policy, told of the expectation of student responsibility, told of the web-site, etc. The author begins the process of memorising the students' names (last names are used – – never first names and students are addressed as Mr. X or Ms. Y, for example, just as is done in most Moore method or modified Moore method classes). Definitions are presented for a naïve introduction to the axioms of probability and the naïve idea of probability for finite sample spaces, and some of the concepts of logic, set theory, and calculus are mentioned. The second meeting day student presentations begin (but do not dominate the class as they do once the groundwork is set on the axioms of probability and getting students to conjecture). Rather than calling on students like the Moore method [62], volunteers are requested like Cohen's modified Moore method [2]. Throughout the first few weeks the class proceeds in this fashion with short talks about new definitions, methods to prove or disprove claims, and introduction to new terminology, notation, etc. By the end of the first third of the semestre, the amount of time the instructor talks decreases from perhaps half of the class period (at the end of the class session) to perhaps a fourth or not at all. In this manner, the students are encouraged to take more responsibility for their education and regard the instructor less as a teacher and more as a conductor. Nonetheless, it must be noted that some days there are no student presentations; so, the instructor must be prepared to lead a class in a discussion over some aspects of the material or be prepared to ask a series of questions that motivates the students to conjecture, hypothesise, and outline arguments that can later be rendered rigorous. If presentations are not forthcoming or time is not exhausted

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[37]Handouts, worksheets, ancillary materials, etc. are available at the author's web site: http://faculty.kutztown.edu/mcloughl/

[38]The books are a part of the course due to department policy (by and large) and it is hoped that soon they can be eliminated entirely.

before presentations are, then sometimes students are presented with claims and proposed proofs and counterexamples which they critique (faulty 'proofs,' correct proofs, faulty 'counterexamples,' correct counterexamples, etc.) The author tries to keep in the back of his mind at least a few such claims to 'run up the flag-pole and see who salutes it.' Such was how Doyle, Smith, Reed, and Neel taught and it is opined by the author that such creates an environment in class where students are treated with regard, there is a convivial relaxedness to discussions, but a serious formality to the treatment of the material and to students' work.

Moreover, by the end of the first third of the semestre, the discussion of the class focuses on basic matters past axiomatic probability and combinatorics and into more detail about independence versus non-independence, Bayes' Theorem, and what constitutes a random variable. Throughout these weeks the class proceeds in this fashion with presentations for most of at least the first half of the class, discussions on methods to prove or disprove claims (maybe another way to do the same claim), and late in the period some discussion over new definitions, terminology, notation, etc. In this manner, the students are continually encouraged to take even more responsibility for their education and the instructor reminds the students that he is not a teacher but a guide. Also still if presentations are not forthcoming or time is not exhausted before presentations are, then the author has material to discuss with the class, questions to ask the class, or sometimes a story or two and some encouraging words (if it seems members of the class are suffering from 'burn out').

The middle third part of the course focuses almost entirely on sequences, series, discrete random variables, PMFs, CDFs, specific ones and claims about the PMFs or CDFs. The last third of the course focuses almost entirely on functions whose domain has cardinality greater than $\aleph_0$ (e.g.: continuous random variables), the Gamma function, moment generating functions, specific PDFs, and claims about PDFs or CDFs. It is the case that a discussion of joint distributed random variables is actualised before the end of the semestre (usually to the point of statistical independence versus non-independence. The remainder of discussion and consideration of joint distributed random variables (as mentioned previously) is relegated to P & S II.

So, when the last third of the semestre is upon the class, typically discussions of joint distributed random variables and their aspects create reasons for the class to return to univariate discrete or continuous random variables for contrast and more illumination. Throughout these weeks the class usually is filled with presentations and there is little else done (perhaps a bit of talk still conversing about methods to prove or disprove claims, and perhaps some discussion over new definitions, terminology, notation, etc.). Toward the closing of a typical semester it is characteristic that many more claims are considered than are proven or disproven; but, oft it is the most satisfying part of the course for all since many students find they have a 'handle' on the material and the instructor has a wonderful time watching the students struggle, opine, revise, and often do very well with the 'competing' concepts of discreteness versus non-discreteness, the concept of dimension, and even what is mathematics, the world, our existence really all about?

Though the P & S I course contains a hodgepodge of introductory material the students will use or study in upper division courses, *all of it* truly centres on aspects of logic, sets, and calculus. For each subject, definitions, terminology, and notation are established and a series of facile claims are proposed for the students to prove or disprove. For example, even for something as 'small' as a combinatorial claim, students are encouraged to note the Peano axioms, the field axioms of the reals, the order axioms of the reals, and then given parochial claims on factorals, permutations, combinations, etc. Such matters arise again in Numerical Analysis, Number Theory, Abstract Algebra, as well as P & S II. It is therefore the case that it is noteworthy that P & S I being a required course in a BS major has less flexibility to deviate and meander (per the material) than P & S II.

For the PAMS sequence, (as with any sequence the author teaches since it is approached from an IBL perspective as far as the pedagogy is concerned) the intensity of the discussion deepens as the semester proceeds, the complexity (as opposed to sophistication which can easily fall into Sophistry) of the discussion increases, the claims included in the course are of a more challenging form (for the most part) than earlier in the course.[39] Oft times, students propose claims in the first class which are quite challenging. Indeed, some students offer claims in the P & S I class which are not answered during the semester (nor during the P & S II course). Such open-endedness of the ideas creates the opportunity for the author to then invite students to consider doing some studies independently, some undergraduate research, a thesis, or a Senior Seminar project over some of the open questions. In this manner, the PAMS sequence taught with the MMM fulfils the promise of academe - - opening new areas of inquiry for the student and leaving him with wanting more. Indeed, by the very nature of the manner in which the elucidation of the conjectures occurred: percolating up from the students causes more than a few to act upon their curiosity and study the conjectures in a directed reading course, independent study programme, or when they take Senior Seminar.[40]

It is in the discussions amongst the students and between the students and the instructor that the best elements of the Moore method and make for a wonderful educational experience (hopefully) for the students and a meaningful experience for the instructor. Management of the discussion centres on the instructor, but control of the discussion is left to the students. Students are free to debate the subject, discuss the subject (in the class - - not outside the class), opine, hypothesise, conjecture, and attempt to resolve the seemingly contradictory evidence before them. The instructor is responsible to explain the significance of the axioms and expose the students to the beauty of mathematics and proof. That is to say, that the axioms provide a framework or set of rules of a game or a puzzle, that logic provides the structure for deducing answers to questions or ways to solve the game or puzzle, that the students have the ability to solve the game or puzzle, and then encourage them to do so. In this manner, the MMM creates a student-centred experience as

[39]The claims about probability, random variables, or random processes are not *necessarily* more challenging.

[40] Senior Seminar is the 'capstone' course in the mathematics programme at Morehouse College. Students (in different traditions depending on instructor) choose an advisor and research a problem set; then do a formal paper (AMS style, research paper) and presentation at the end of the semestre. Senior Seminar is also the 'capstone' course in the mathematics programme at Kutztown University. The course seems to be organised in a combination of constructivist and German seminar traditions: Students choose a topic, research a problem set, then do a formal paper (not AMS style, more of a report or synthesis paper), and presentation at the end of the semestre. The KUP Senior Seminar is not as 'research intensive' (in a mathematical sense) as the MC Senior Seminar.

does the Moore or constructivist methods.

It must be noted that a focal point of the discussion of the methods of argument or counter-argument under the MMM is the uncompromising demand for justification. An instructor who employs the author's MMM must insist that his students (and he himself) justify every claim, every step of a proof (at least during the first third of the P & S I course), and explain to the students the rationalé for such a policy.

Consider, if one happens upon a fact but really does not know why the fact is indeed so, does he really know the thing he claims to know? Recall that in classical philosophy (epistemology) in order for person A to know X: (a) X must exist; (b) A must believe X; and, (c) A must justify why X is. An instructor who employs the author's MMM allows for (a), does not request the students adopt (b), but must insist on (c). This is because there are enough examples of truths in mathematical systems such that (a) and (c) are the case but (b) certainly is not for the majority. One can over time come to accept (b) because of the irrefutability of the argument that establishes the certainty of the claim.

The author's MMM requires the instructor adopt an approach such that inquiry is ongoing. A demand for understanding what is and why it is, what is not know and an understanding of why it is not known, the difference between the two, and a confidence that if enough effort is exerted, then a solution can be reasoned. In this way, the MMM is simply a derivative of the Moore method; it is perhaps a 'kinder, gentler' Moore method than the original. Consider:

> Suppose someone were in a forest and he noticed some interesting things in that forest. In looking around, he sees some animals over here, some birds over there, and so forth. Suppose someone takes his hand and says, 'Let me show you the way,' and leads him through the forest. Don't you think he has the feeling that someone took his hand and led him through there? I would rather take my time and find my own way.[41]

However, the confidence must be tempered with humility and realism. Not everything can be known. Hence, one must be selective. The instructor and students must realise that they are not the most intelligent creatures in the universe. Hence, one must accept his limitations.[42]

At least one quiz is administered approximately each week or week-and-a-half, part in class part take home, or all take home (on quiz work a majority of the work is 'take home') in which the students are asked to prove or disprove conjectures. They are required (of course) to work alone. The quizzes are graded and commentary included so that feedback is more than just a grade. Also, there are usually three major tests during the semestre for each class in the PAMS sequence

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[41] Moore, *Challenge in the Classroom.*

[42] My sister has termed this approach, 'quiet arrogance.' It is an approach to life and deeds our late father taught us– – one should not brag and 'stuff it down' other's throats that we can do something well or very well. We should let our work speak for itself and not 'toot our own horn.' By doing so the actions speak for themselves and we are not trapped in a Sophistic position of sounding as if we are conversant with something; we **are** but do not need to run about announcing it to the world. I added to our family crest, "Esse quam videri," so that such was made clear that it is a family tradition to be rather than to seem.

(since each carries 3 semester credit hours) and a comprehensive final; thus, the MMM is grading intensive for the instructor. The frequency of the quizzes creates a benchmark for the students so the students do not fall behind and so that some provocative claims can be made to the students by the instructor as well as some fundamental principles presented in an IBL manner. The presentations on the board, the quizzes, the tests, and the final give the students the ability to demonstrate competency over the course and an opportunity and responsibility to digest and synthesise the material. The testing schedule differs from the Moore and reform method and shares a commonality with the traditional method. It may be a tad more 'quiz intensive' than traditional methods, but the author has found that many of his colleagues who employ traditional methods grade homework (which is not a part of his MMM) so it might be similar to the traditional methods in that regard.

Experience with many different course sizes over the past twenty-five years has led the author to conclude that optimal course size is between approximately twelve and twenty. When there are less than about twelve students, then the class discussions often suffer for a lack of interaction. When the class size is more than about twenty students, then class discussions are often difficult to facilitate and can be problematic because so many students wish to be heard simultaneously. Also, if the class size exceeds approximately twenty, then the burden of grading so many papers becomes quite heavy and the turn around time lengthens which is detrimental. It seems that it is best to provide feedback in a timely manner so that the students have time to reflect on their work and discuss the work in follow-up session during office hours. If too much time has elapsed between the times students hand the papers in and they get the papers back, their memory of *why* they thought what they thought dwindles and the educational experience for the student suffers.[43]

Similar to the P & S I course, the P & S II course proceeds at KUP much as it did at MC. However, curiously, at MC the P & S II course would have an enrolment of 4 or so students and at KUP it is more typically 10 or more! KUP is a larger institution, but given the nature of the schools, one would possibly believe the enrolment figures would be reversed.

Insofar as use of the MMM in P & S II, it is as was the case in P & S I with the onus of responsibility on the individual students, more graded material is administered such that it is take-home (hence, more trust in students under the honour system), and most of the interesting claims made by students are typically about covariance or correlation and their relation to statistical independence early in the course. Later in the course, a *quite fertile* area of discussion that leads to much hypothesising is contained within the ideas of properties of expectation, estimation theory, and the method of maximum likelihood & the method of moments. Many open-ended discussions and sequences of claims that were a part of the P & S II

[43]The first semestre at KUP the P & S I class had 38 students in it and it was the worst experience I ever had as an instructor. Only one section per year of the course was offered at KUP when I joined it's faculty; hence, the overcrowding. Further, the policy was upper-division courses had typically 35+ students whereas lower-division courses had typically 25- students! Such was changed almost immediately after I joined the KUP faculty because the new Chair of the department, Paul Ache, opined such was not conducive to a healthy learning environment for students (I obviously agree with him totally and could not have stayed at KUP were it not for him).

course created the basis for students to do independent studies, undergraduate research, a thesis, or a Senior Seminar project.

IV. Summary and Conclusion

The author opines that a primary goal of *any course* is to establish an atmosphere that creates for students an interesting and challenging intellectual environment which ideally encourages students to further their study of mathematics. No matter the course the author attempts to use inquiry to create a basis for the encouragement of further study in mathematics by pointing out the student's ability to grasp the material and produce ideas rather than just read about ideas; this paper is one of a sequence of papers describing the author's MMM and its use across the mathematics canon (see [34], [35], [36], [37], [38], and [39] for other papers). It seems to me that such is easy when the class is taught within the context of the Moore method or a modified Moore method since the class is 'student-centred.' The encouragement of further study in mathematics is actualised by offering a suggestion of a course or course a student could take, perhaps suggesting a change of major (to mathematics or a related subject), or perhaps minoring in mathematics. Some of the most seemingly peripheral topics in the PAMS sequence have led to some wonderful research topics that the author has directed (for example, Aspects of the Gamma Function, Possibility Theory (Fuzzy Logic and Sets) versus Probability Theory, and Number Theory (Fibonacci and Fibonacciesque numbers) to reference but a few) as well as more obvious topics that arose from the PAMS sequence (for example, A Study of the Correlation Coefficient, Aspects of Moment Generating Functions, A Rigourous Study of the Chi-Squared Family of Functions, A comparative study of the Pearson Chi-Square and Kolmogorov-Smirnov Goodness of Fit Tests, An Introductory study of Item Response Theory, and On Bivariate Dirichlet Probability Density Function: An Analytical and Empirical Study of Aspects of Joint Dirichlet Random Variables to reference but a few).

From the Foundations of Mathematics (FOM), through the PAMS sequence, and beyond, the opportunity to encourage further study and in more depth arises within every course the author has taught and has assisted in forging a long-term undergraduate research component for some students (by identifying 'promising' students typically). The existence of Senior Seminar at Morehouse College allowed for such since students were aware they were required to do a Senior Seminar thesis which is not the case at Kutztown University. However, the author has been successful in finding students who are simply curious, who are in the Honours Programme at Kutztown, who are interested in a career in a field where mathematics is used, or who are interested in graduate school (so far).[44], [45]

Many students who were in the author's Calculus, FOM, Real Analysis, Probability & Statistics, or Senior Seminar courses went on to graduate school. In the period of 1999 through 2005 (beginning with the entering class of 1995), 17 students who were in the author's classes pursued post-baccalaureate work in the mathematical sciences. The author does not claim it was him but that by teaching in a manner that inquiry-based learning (IBL) can be achieved by the students–

[44] I opine that there are *many* students who probably would do undergraduate mathematics research if the opportunity arose and a faculty member simply offered such.

[45] I opine that undergraduate research is a *great* experience for students – – most of all because it helps the student focus on doing 'real' math slowly, purposefully. Such can assist the student in illuminating whether a love of math exists within him, in clarifying his objectives and goals (graduate school, etc.). IBL, the Moore method, or the MMM relay helps this point, I believe.

– *the modified Moore method* was key in encouraging or directing the students to pursue post-baccalaureate work. In fact, there is a possible explanation for the number of students who were in the author's classes who pursued post-baccalaureate work– – it may have been due to student self-selection. If the student in the back of his mind thought of the possibility of graduate school or subliminally had the self-confidence necessary to do such work, then he may have selected the author's classes because they were reputed to be 'hard' but 'fair,' and 'challenging.'

Indeed, there may be another a possible explanation for the number of students who were in the author's classes who pursued post-baccalaureate work - - the author's own bias toward 'smart' students![46] Again, there may be *yet* another possible explanation for the number of students who were in the author's classes who pursued post-baccalaureate work - - the programme at Morehouse College was redesign between 1998 and 2002 and was revised beginning in 2003. Such 'success' as the author had in teaching students who ultimately went to graduate school might not be as great in a department that is more focused on the 'applied' or mathematics education. Hence, there is a strong caveat in inducing any 'success' at all the author seems to have had from the number of students who pursued further study in mathematics. If such trends are found after 10 or more years at Kutztown University, then perhaps, a more credible case could be made for 'success' for the author.

It must be noted that descriptive papers about the teaching and learning of mathematics, such as this and [1], [2], [17], [20], [24], [29], [30], [42], [43], [47], [54], and [62], assist in creating anecdotal evidence to suggest a teaching method derived from the Moore method does seem to be successful. Furthermore, common sense seems to suggest that an inquiry-based learning (IBL) environment would seem likely to result in more students pursuing advanced degrees or more students having success in subsequent course-work in a mathematics programme by the very nature or IBL and human curiosity.[47] *It seems rather clear that a strong case for and a need for a dispassionate, objective, and quantitative study to be designed and executed that could delve into the question of whether or not a particular teaching method results in more students pursuing advanced degrees or more students having success in subsequent course-work in a mathematics programme. Such a study might prove impossible to create and might be controversial; but, the author opines it would be very interesting to do such and the results would be fascinating (no matter which teaching method showed promise or even if no difference (resulted in more students pursuing advanced degrees or more students having success in subsequent course-work) existed between and betwixt methods).*

In sum, the author described using a modified Moore method (MMM) to teach the courses in a Probability and Mathematical Statistics (PAMS) sequence and described the material in the, outlined some of the strategies employed, discussed the syllabi, policies, and ancillary materials used. Perhaps the most important part of this modified Moore method is the caution that one should remain flexible, attempt to be moderate in tone and attitude, be willing to adjust dependent upon

[46] There has always been claims by some that the Moore method favours the 'already mathematically inclined.' Such a view seems to assume there is a latent mathematical ability, not everyone possesses it or possesses as strong an ability, and that adherents to the Moore method subliminally favour 'better' students.

[47] However, such may be a circular argument.

the conditions of the class, and not be doctrinaire about methods of teaching.

It is the belief of the author that this method maximises educational opportunity[48] for most students by attempting to teach to as heterogeneous a group as possible. For each individual instructor, the teaching method employed *should be* that which is most comfortable for him and connects with the students. Nonetheless, this author opines that this pseudo-Socratic method, the modified Moore method, should be considered by more instructors who teach a Probability and Mathematical Statistics (PAMS) sequence. It is deemed so because many of the students taught in this method have gone on to graduate school or entered the work-force and have communicated with the author that they felt that the course taught in this manner (or other courses taught in the manner) was the most educationally meaningful for them. Whilst a student himself over the course of many years, the author was exposed to each of the methods discussed in this paper (traditional, German seminar, Moore, and constructivist) and aspects of those methods are a part of his modified Moore method because he found that each had its strengths and weaknesses. Thus, the author attempted to create a method that, hopefully, included the best of each and discarded to worst of each. It can be said honestly that he had moderate success in almost every class taught with the Moore, traditional, German seminar, or constructivist methods.

To put it succinctly and personally, that which I learnt the best was that which I *did myself*, rather than be told about, lectured to, or even read about. I must *do* in order to *understand*. That I can not explain something does not mean it does not exist, it simply means that I do not know it (at this point or perhaps it is never knowable). The MMM seeks to minimise the amount of lectures, but allows for students to read (a tad) from multiple sources and converse (after presentations). It acknowledges that learning is a never-ending process rather than a commodity or entity that can be given like the metaphor of an instructor cracking open the head of a student then pouring the knowledge into said head. In that regard it is very much reminiscent of reform methods and the philosophy of John Dewey. Dewey stated, "the traditional scheme is, in essence, one of imposition from above and from outside,"[49] and "understanding, like apprehension, is never final."[50]

The queries contain open questions from the perspective of the students (and perhaps the instructor) without indication as to whether they are true or false under the axioms assumed. But, unlike the Moore method, necessary lemmas or sufficient corollaries *are* oft included; thus, affording the students a path to construct their arguments.

In every paper I ever write that concerns inquiry-based learning (IBL), I mention the following because it is such a powerful and brief encapsulation of a Moore philosophy of education: P. J. Halmos recalled a conversation with R. L. Moore where Moore quoted a Chinese proverb. That proverb provides a summation of the justification of the MMM employed in teaching the transition sequence. It states, "I see, I forget; I hear, I remember; I *do*, I *understand*."

It is in that spirit that a core point of the argument presented in the paper is

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[48] *Not* educational outcomes.

[49] John Dewey, *Experience and Education* (New York: Macmillan, 1938), page 18.

[50] John Dewey, *Logic: The Theory of Inquiry* (New York: Holt and Company, 1938), page 154.

that the method of teaching the PAMS sequence should be carefully considered (I recommend a modified Moore method, of course) for the nature of the courses and material in the course seems to lend itself to said MMM. Also, the method of teaching the course should be carefully considered because in order to have an educationally meaningful experience for the students and in order to properly transition the student from an elementary understanding to a more refined understanding of any area of mathematics (not just probability or mathematical statistics), every effort should be made to see that the students reach beyond a mundane, pedestrian understanding (in every course, not just these courses). An innovation in the pedagogy proposed is that not all questions posed in the courses are answered. Many of the questions posed in the courses are left for the student to ponder during his matriculation and answer at a later date. Examples of proofs, counterexamples, etc. are given but *most* of the actual work is done by the students.

There has not been an inferential statistical study in the history of the world that has ever proven a thing; there has not been a Monte Carlo simulation in the history of the world that has ever proven a thing; but, under the aegis of Aristotelian logic, the axioms of set theory, axioms of the reals, and Kolmogorov axioms of probability theory much has been proven and the conditional truth of the lemmas, theorems, and corollaries that are a part of a PAMS sequence exist and are discoverable for students if the students are given the chance to explore and discover these truths. One can believe what one wants without regard to the evidence or facts. One can opine only with evidence and justification. One **knows** only through deduction and knows conditionally (see [37] for a discussion of why such is forwarded).

So, this paper proposes a philosophy such that the experience of doing a mathematical argument is reason enough for the exercise; but, the author recognises the practical need for task completion so student completed proofs, counterexamples, examples, counter-arguments, etc. form the framework type of mathematical education proposed herein.

Hence, this paper proposes a pedagogical approach to mathematics education that centres on exploration, discovery, conjecture, hypothesis, thesis, and synthesis such that the experience of doing a mathematical argument, creating a mathematical model, or synthesising ideas is reason enough for the exercise– – and the joy of mathematics is something that needs to be instilled and encouraged in students by having them *do* it rather than *see* it, *hear* it, or *read* it.

Nonetheless, it is not argued that this is <u>the</u> way to teach, for as Halmos asked in [29], "what is teaching?" Beats me! I do know; yet, I try to do it!

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