EMERGENCE OF MATHEMATICAL KNOWLEDGE STRUCTURES. INTROSPECTION[1]

Nada Stehlikova

Charles University in Prague, Faculty of Education

In the study, a part of longitudinal research focused on the emergence of mathematical knowledge structures in a learner's mind is presented. It concentrates on the analysis of introspective data gained from the author's study of a non-standard arithmetic structure in terms of the model of abstraction in context (Hershkowitz, Schwarz and Dreyfus). Some episodes are described by factual and interpretative accounts. It is shown that the above model can be applied to introspective data and used for their interpretation. The parallel between the model and Duffin and Simpson's idea of understanding is discussed.

This study is a part of longitudinal research aimed at describing the emergence of mathematical knowledge structures in a learner's mind. It is a very broad field of research and we concentrate on the construction of a new structure as an analogy to an existing structure. The process of constructing an internal mathematical structure is a mental activity, i.e. it is not directly observable. The methodology we used consists of think-aloud interviews with university students and the author's introspection. Introspection has been chosen because we believe that by studying ourselves from the inside we can infer about mental processes of other people, we "develop sensitivity" (Mason, 1998). "By introspection we mean constantly seeking to discern our individual perceptions of experiences, both past and present, and our reactions to them" (Duffin & Simpson, 2000a). Some of the research results have been reported elsewhere (Stehlikova & Jirotkova, 2002; Stehlikova, 2002). Here we will concentrate on the introspective part.

THEORETICAL FRAMEWORK

One of the central aspects of learning mathematics are the processes of the emergence of mathematical knowledge structures. Several theories are available which have a common goal: "They aim to provide a means for the description of processes during which new mathematical knowledge structures emerge" (Dreyfus & Gray, 2002). For our analysis, we have chosen the model of abstraction in context.

Hershkowitz, Schwarz and Dreyfus (Dreyfus, Hershkowitz & Schwarz, 2001a; Hershkowitz, Schwarz & Dreyfus, 2001; Schwarz, Hershkowitz & Dreyfus, 2002) have recently proposed a model of dynamically nested epistemic actions for processes of abstraction in context which has since been elaborated (e.g. Dreyfus, Hershkowitz & Schwarz, 2001b; Tabach, Hershkowitz & Schwarz, 2001; Tabach & Hershkowitz, 2002; Tsamir & Dreyfus, 2002). The proposers of the theory characterise abstraction as "an activity of vertically reorganising previously constructed mathematics into a new mathematical structure". By reorganising into a new structure, they mean the establishment of mathematical connections (making a new hypothesis, inventing or reinventing a mathematical generalisation, a proof, or a new strategy for solving a

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problem). On the other hand, neither learning to mechanically perform a mathematical algorithm nor rote learning qualify as abstractions.

The authors of the theory also claim that abstraction strongly depends on context, on the history of the learner and on artefacts available to them and in this sense structure is internal, "personalised" (Schwarz, Hershkowitz & Dreyfus, 2002). Thus hereinafter by a structure, we will mean a mental image which a person holds in his/her mind about a mathematical structure.

The authors of the theory call mathematical methods, strategies, concepts, etc. structures. We would have preferred to reserve the word 'structure' for more complex knowledge and simply call what is being built 'mathematical knowledge'. Similarly, the term abstraction has a more specific meaning in mathematics for us, thus instead of 'processes of abstraction', the term 'processes of construction of knowledge' seems to us to be more appropriate.

The genesis of abstraction is seen as consisting of three stages (Hershkowitz, Schwarz & Dreyfus, 2001):

1. A need for a new structure,
2. The constructing of a new abstract entity in which recognizing and building-with already existing structures are nested dialectically, and
3. The Consolidation of the abstract entity facilitating one's recognizing it with increased ease and building-with it in further activities.

Three epistemic actions which are constituent of abstraction are (Schwarz, Herhskowitz & Dreyfus, 2002):

> *Constructing* is the central action of abstraction. It consists of assembling knowledge artefacts to produce a new knowledge structure to which the participants become acquainted. *Recognizing* a familiar mathematical structure occurs when a student realizes that the structure is inherent in a given mathematical situation. *Building-With* consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement.

STUDY

The tool of our investigation of an internal mathematical structure is an arithmetic structure $A_2 = (\underline{A}_2, \oplus, \ )$ which we call *restricted arithmetic* (hereinafter RA) [2] where $\underline{A}_2 = \{1,2,3,...,99\}$ is the set of *z*-numbers. The gate to RA is the mapping $r: \underline{N} \to \underline{N}$, $n \mapsto n - 99 \cdot [n/99]$, which we call *reducing mapping*; here $[y]$ is the integer part of $y \in \underline{R}$. Reduction can also be introduced as an instruction illustrated by several concrete examples[3]: *Perform a 'double-digit sum' operation on a natural number until you get a one or two digit number. A double-digit sum operation is similar to a digit sum operation but instead of adding digits, we add two digits at a time.* For example, $r(682) = 82 + 6 = 88$, $r(7\,945) = r(45 + 79) = r(124) = 24 + 1 = 25$.

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[2] It was elaborated by Milan Hejny.

[3] The isolated models of reduction, e.g. the numerical examples, are introduced to the student at the same time as its verbalised universal model (for the theory of isolated and universal model see Hejny, in press).

The reducing mapping r is used to introduce binary operations of z-addition $\oplus$ and z-multiplication $\odot$ in $\underline{A}_2$ as follows: $\forall x, y \in \underline{A}_2$, $x \oplus y = r(x + y)$ and $x \odot y = r(x \cdot y)$. For instance, $72 \oplus 95 = 68$, $72 \odot 95 = r(6\,840) = 9$.

Note: This context has been chosen as a tool of our research and not a different part of mathematics because it presents a fresh part of mathematics, not elaborated elsewhere, it is suitable with respect to the author's mathematical knowledge and abilities and the analogy with ordinary arithmetic allows her to pose questions and develop solving methods herself.

The only **participant** of this part of our research is the author, a 30-year-old researcher. The **data** available for analysis consists of the author's detailed notes of her solutions to problems. The notes are dated and besides the solutions themselves also contain her comments on them as they occurred to her at the time of writing. In addition, the author used, if possible, different pens at different times for writing her notes. When the problems were considered to be solved, a descriptive table was made which consisted of: the task, its solution, its interpretation by the author. The table together with all the notes was subject to analysis.

The **problems** solved by the author which we will describe here consist of the study of squares, of general powers in RA and of looking for multiplicative groups in RA. The study spanned four months and used some results which the author found out previously. The author's investigations will be divided into several episodes which will be described by factual accounts (written in first person singular in italics) which are shorter versions of our comments in the descriptive table and interpretative accounts. The interpretation will be mainly done in terms of abstraction in context.

FACTUAL AND INTERPRETATIVE ACCOUNTS OF INVESTIGATIONS

Study of Squares

x^2	x
1	1, 10, 98, 89
4	2, 20, 79, 97
9	3, 30, 36, 63, 69, 96
16	4, 59, 95, 40

The study of squares was motivated by my effort to get an insight into the solutions of quadratic equations for which I needed to know which z-numbers were squares. Using the notion of squares from ordinary arithmetic and the fact that additive inverses have the same square, I made a list of squares (a part of it is in fig. 1). I noticed anomalies and regularities (e.g. $45^2 = 45$, $55^2 = 55$, $22^2 = 88$ and $88^2 = 22$; $(ab)^2 = (ba)^2$, $a^2 = a0^2$, where a, b are non-zero digits) and gradually 'chains' and 'cycles' emerged. For instance, $4^2 = 16$, $16^2 = 58$, $58^2 = 97$, $97^2 = 4$, etc. The list was not illustrative enough and I tried to represent it in a graphical form. I decided to use arrows for the relationship 'being a square of' and redrew the list several times. Finally, I got a visual diagram consisting of nine clusters (one of them is in fig. 2). I had a strong sense of satisfaction because the diagram was pleasing to the eye and it consisted of clusters of certain shapes and I felt confident that I would be able to get new information from it.

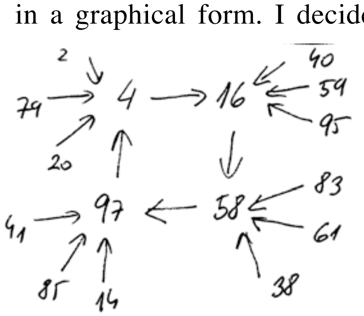

Figure 2.

Next, I felt prepared to prove some of the found regularities. While doing so, my attention was attracted by the fact that the numbers in the diagram made various sets. It seemed an obvious choice to check them for group properties with respect to z-multiplication. I checked the following sets: numbers from the inner 'rectangle' (e.g. numbers 4, 16, 97, 58 from fig. 2); all numbers from the cluster; numbers which have the same square.

While checking the last sets, I became interested in the questions how the numbers with the same square are connected. It was very easy for NN numbers[4]. To solve the problems for zero-divisors was more difficult but I managed to find a rule which worked.

What has been constructed here, is <u>the structure</u> (in the author's head) <u>of squares, their mutual relationship and of regularities</u> (we will call it S1) which has quite a simple visual representation (its part is in fig. 2). The need for the new structure was given by our study of quadratic equations. Using the notion of squares from ordinary arithmetic and the fact that additive inverses have the same square (recognising previously constructed knowledge and building-with it something new), a list of squares was constructed. By studying the list and noticing anomalies and regularities, 'chains' and 'cycles' were constructed (recognising & building-with the knowledge of the relationship 'being square of'). By further recognising and building with the relationship 'being square', with observed regularities and with the idea of using nodes and arrows for the visual representation, the visual diagram was constructed in a rather raw form and by several redrawings, the diagram consisting of nine clusters was constructed and S1 was consolidated[5]. We think that the structure S1 was also consolidated when the author proved regularities because she had to reflect on the properties of squares. The suggestive clusters of the diagram led naturally to distinguishing some subsets and investigating them for group properties (recognising & building with sets of numbers which seemed to be mutually connected and with the knowledge of group structure).

The last part of the study was motivated by the author's natural curiosity leading her to the question if there was a simple rule connecting all the numbers in a cluster. It contributes to the understanding of the structure S1 and enriches it. This raises the question of what the construction is. Shall we say that a structure has been constructed if after some time we will find out that we have missed some of its important characteristics? Or shall we say that the structure is being constructed until all the characteristics have been found? It would then require an external authority which would judge that the learner has discovered everything about the structure and it has thus been constructed. As we deal with introspection, it seems natural to speak about construction when the solver feels that he/she constructed something which he/she did not know previously and that he/she understands it in terms of the definition of understanding given by Duffin and Simpson[6]: "When I understand, I feel comfortable ; I feel confident; I feel able to forget the detail, confident that I can reconstruct it whenever I need it; I feel that the thing belongs to me; I can explain it to others" (Duffin & Simpson, 2000b). The

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[4] By NN numbers we will mean non-zero *z*-numbers which are not zero divisors.

[5] An indication that it was consolidated is that when the author had to draw the diagram again after a long time, she was able to reconstruct it without having to go through the same process again.

[6] As we only deal here with introspection, we will not consider the outer demonstrations of understanding the authors provide.

process of construction is thus an iterative process and it belongs to a person, is 'personalised'.

Study of Sequences

k	a
30	2, 50, 68, 83, 5, 20...
15	4, 25, 16, 31...
10	8, 62, 17, 35,...

The discovery of some groups in the study of squares motivated me to go back to my study of sequences {a^k,   a $\underline{A}_2$, k   N}. I already knew that structures ({a^k, a is NN, k   N},   ) are groups. However, I was not satisfied with this result and I wanted to explore the structures more and to see what the situation with zero divisors was. I made a table for NN (its part is in fig. 3): k is the number for which $a^k = 1$ (I knew from Euler's theorem that k must be a divisor of 60). It was not difficult to fill in the table because I used the knowledge that (a) inverse numbers generate the same set (e.g. 2 and 50) and (b) if I know that e.g. $2^{30}=1$, then $4^{15}=1$ and similarly $8^{10}=1$. A similar table was made for zero divisors where also the length of the pre-period (if any) was given. I investigated sets from the table for group properties and discovered some groups. I made an important 'discovery' of a different multiplicative neutral element other than 1 – number 45.

What is being constructed (or rather what has started to be constructed) here is <u>the structure of multiplicative subgroups</u> (we will call it S2). The need for it stemmed from our feeling that there had to be an inner organisation of subgroups which had been so far found in a rather haphazard way. Within it another construction was made – the construction of <u>the structure of sequences</u> (we will call it S3). The structure S3 was constructed via recognising and building with some knowledge taken from ordinary arithmetic and adapted for RA (see (a) and (b) above).

As for the neutral element, it was not a discovery as such. During her university studies, the author met many examples of different neutral elements other than those found in ordinary arithmetic. However, the time showed that this knowledge was not immediately available when needed (as the knowledge of e.g. group properties was). Some previous knowledge gained at university was consolidated through a rediscovery in a different context.

Study of Third Powers

I wanted to find more subgroups and realised that if I studied third powers similarly to squares, I might find some new ones. I made a list and a diagram of third powers. I was surprised to see that unlike squares, third powers 'repeated' ($x^3 = (x + 33)^3 = (x + 66)^3$), and that they occurred 'in threes' (8, 9, 10; 17, 18, 19, etc.). From the visual diagram I was able to discover a set {1, 10, 89, 98} (NN for which $a = a^3$) which together with z-multiplication formed group. By further investigating sets of numbers with the same third power, two more groups were found of order 3.

The process of the construction of S2 continued, new groups were identified. The structure of squares was further consolidated as it was used as a basis for the construction of the structure of third powers

Further Study towards the Construction of S2

The lack of space does not allow for the detailed description of the rest of the process. Let us only say that other subsets of *z*-numbers were investigated for group properties and subgroups were organised in a table according to their order.

> I felt the need to summarise all the found subgroups to see if the list was complete. I noticed the relationship between the order of subgroup and the order of group and I remembered Lagrange's theorem. I listed all possible orders of subgroups and filled in a table. Two types of subgroups were missing – of order 12 and 20. I went over all my notes trying to find a clue even though I did not know what I am really looking for. Suddenly I saw that the set $\{4^k$, k N) equals the set of all squares which are NN. It occurred to me that something similar could be true for third powers. When I checked the set of all third powers which were NN, I got a multiplicative group of order 20. Then I investigated the fourth powers but it did not bring anything new. (I started to use Maple at this stage.) When I investigated the fifth powers, I discovered the last subgroup – of order 12. I really felt a sense of accomplishment.

The solver felt that the construction of S2 had finished (even though later, she went on with the study and developed S2 further). This time the author built with the theory (Lagrange's theorem) and with all the knowledge she had constructed so far in her investigations. An important part of the whole process was the chance recognition of the equality of the two sets. From then on, she could build with the knowledge that if we investigate the third, fourth, fifth, etc. powers, other subgroups might be found. Lagrange's Theorem was re-constructed similarly to the neutral element above.

The structure of subgroups was later consolidated when the author had to present it to others and describe it verbally. Moreover, this consolidation also went on when the author carried out the presented analysis of her own work! She had to reflect on S2 even more deeply than when investigating it earlier.

DISCUSSIONS AND CONCLUSIONS

Visual representations: The structures S1, S2 and to a certain extent S3 too are specific in that there is a visual representation available for them (the visual scheme and tables) and thus it is possible to analyse them more easily than structures with no visual representation.

Introspection: We accept that introspection as a research method is rather controversial. In agreement with Duffin and Simpson (2000a), we take into account that introspection should be complemented by other techniques in order for us to get more creditable results, and thus we made an attempt to accompany it by co-spection (Duffin and Simpson characterise it as "sharing of our own personal reactions to experience"). In our case, it is the sharing between the author and a colleague of hers to whom she presented her account of her work and who tried to find flaws and inconsistencies in it.

One of the dangers when using introspection is that the researcher may reinterpret (albeit unconsciously) his/her former reasoning in the light of what he/she knows at the time of analysis. On the other hand, when analysing other people's solutions, we naturally use our own experiences and interpret them accordingly. We believe, that no two researchers will analyse a solution in the same way. Thus, the findings from the introspection will be complemented by results from interviews with university students.

Abstraction in context: The model of abstraction in context was used for different kinds of data than previously. As far as we know, the model has been used for (a) an interview with a single student, (b) an interview with a pair of students, (c) a series of interviews with a single student, (d) a series of interviews with a pair of students. Here we applied the model to introspective data. Moreover, we showed that the consolidation of new knowledge can also be done when one is analysing one's own work, not merely reflecting on it. An illustration was given as to the consolidation of the knowledge gained some time ago. The question was raised if the consolidation can come about when proving.

The model of abstraction proposed by the theory of abstraction in context seems to be able to account for the part of data of our research on structuring mathematical knowledge presented above. It remains to be seen how the theory of abstraction in context can be used for other data from our research and for results which have already been found in terms of the grounded theory approach, procept theory or the theory of isolated and universal models. Besides, our study brought to light some problems with terminology which we had when using abstraction in context.

Understanding: When analysing the data, we could see a parallel between the processes of abstraction and the processes of building understanding. If we have constructed something, we understand it, understanding is in the very heart of constructing. Duffin and Simpson (2000b) define understanding as "an ongoing process of the development of connections (building), a state of the available connections at a given time (having), and the act of using the connections in response to a problem (enacting)". They characterise the act of knowledge construction as follows: "Indeed, in solving a substantial problem, a learner may use some recalled facts, enact some of their understanding, get stuck, find, and resolve conflicts by building new connections, enact the understanding inherent in those new connections, bring in more recalled ideas, and so on." This characterisation resembles the characterisation of the knowledge construction given by the model of abstraction in context. The connection between these two theories will be further studied.

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