STUDENTS' UNDERSTANDING OF Z_N

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In this study, we explore six students' conceptions of $\mathbf{Z}_n$ *in an effort to understand students' conceptions of quotient groups in general. We discovered that there were three different ways our students thought about* $\mathbf{Z}_n$*, namely as infinite sets, element-set combinations, and representative elements. We explore how* $\mathbf{Z}_n$ *might be conceived of in terms of these three cognitively different interpretations, as well as our students' difficulties in working with* $\mathbf{Z}_n$ *using each of these interpretations. We conclude that* $\mathbf{Z}_n$ *is not a trivial example of a quotient group, and provide recommendations for teaching* $\mathbf{Z}_n$ *and other quotient groups.*

INTRODUCTION

The first course in abstract algebra is considered by mathematics students and teachers alike to be a troublesome and often disappointing experience. There may be many reasons why students struggle in abstract algebra; the course requires skills in proof, conceptual understanding of abstract structures, facility with complex notation, and mathematical breadth sufficient to understand examples developed using functions, matrices, complex numbers, and permutations. The course usually moves quickly and presupposes a good deal of mathematical maturity. Because of the difficulty of the course, students often leave with negative feelings toward proof, abstraction and formal mathematics

In our own experience with teaching abstract algebra, we have found that student attrition rises steeply when students encounter the concept of quotient or factor group. The relatively little research that has been done on students' understanding of abstract algebra seems to confirm our experience, but does not help us understand why students struggle with quotient groups. Dubinsky, Dautermann, Leron and Zazkis (1994) suggest that students' struggles are due to the complexity of the quotient group concept, building as it does on notions of subgroup, normality, and group operations. However, they provide little insight into exactly how students are thinking of these underlying constructs, and how their understanding contributes to or undermines their attempts to make sense of quotient groups. Asiala, Dubinsky, Mathews, Morics and Oktaç (1997) observe that students who were successful in working with quotient groups often computed coset products using the representative method (i.e., writing the coset aH as a). However, they note that students' use of this type of notation gave little information about how the students were thinking about cosets and quotient groups.

To better understand why students struggle with quotient groups, we decided to investigate students' conceptions of $\mathbf{Z}_n$, the cyclic subgroups of order n. We chose this collection of quotient groups because they are typically used in abstract algebra texts as straightforward, unproblematic first examples of quotient groups, in hopes that their mathematical familiarity will help students understand the more abstract case (cf Herstein, 1999). We explore in this paper why $\mathbf{Z}_n$ as a prototype of a quotient group is nevertheless difficult to understand, and what this might imply for understanding quotient groups in general. In particular, we address the following research questions:

1. What are students' conceptions of $\mathbf{Z}_n$ as a quotient group?
2. What difficulties do students have in understanding $\mathbf{Z}_n$ as a quotient group?

METHODOLOGY

The data we report here were collected as part of a larger study designed to investigate students' conceptions of quotient groups. The study consisted of two parts: the compilation of case studies (Stake, 1998) of six undergraduate students enrolled in an undergraduate introductory course in abstract algebra taught by Williams; and an analysis using grounded theory (Strauss & Corbin, 1998) of the cases and other class data to identify conceptions, themes and problems common across students in the class. The six students who participated in the case studies were recruited based on their willingness to participate and selected so that there were an equal number of men and women. Siebert conducted six 45-minute, semi-structured interviews with each student during the semester. Tasks in these interviews focused on students' understanding of groups and quotient groups, including $\mathbf{Z}_n$ for different values of n. These interviews were video-taped, and careful fieldnotes were created for each interview. Additional data were collected from the class as a whole, including videotapes of class instruction, detailed fieldnotes of class, and copies of the written work of all students in the class.

Our attempts to understand students' conceptions of $\mathbf{Z}_n$ represent a dialectic between a "top down" and "bottom up" approach. Our study was "top down" in the sense that we used our own conceptions of what is important to know about quotient groups to select tasks that would reveal whether or not students possessed the understanding we valued. At the same time, however, we were sensitive to the way students made sense of and approached tasks. In this way, our study was also "bottom up," in that students' solutions often caused us to rethink what ideas and images were important to understanding quotient groups. Thus, students solutions not only informed us about how they thought about $\mathbf{Z}_n$, but also led us to change our own understanding of $\mathbf{Z}_n$. Our newfound understanding often led to the development of new tasks leading to new data, and further revisions of our model of understanding $\mathbf{Z}_n$. We refer to this cyclic process as *grounded content analysis* (Lobato, personal communication, 1999), and recognize that content analysis cannot be conducted in the absence of the students to whom we wish to teach the content.

The second part of our study—the analysis using grounded theory of the cases and classroom data for common themes, conceptions, and problems—began during data collection. Important themes and problems were identified from class fieldnotes and from the journal entries Siebert wrote after each student interview. Subsequent interview questions pursued these themes and problems. Once all of the data were collected, we compiled descriptions of each target student's understanding of $\mathbf{Z}_n$ and quotient groups in general. We also identified and transcribed relevant segments from class instruction on $\mathbf{Z}_n$, and reviewed all the students' homework and test responses related to $\mathbf{Z}_n$. What emerged from these comparisons of correct and incorrect solutions was a framework of concepts that were foundational for our students to understand $\mathbf{Z}_n$, as well as insights into the difficulties students had with these concepts. Due to the space constraints of this paper, we only present our findings concerning three interpretations of $\mathbf{Z}_n$ and students' difficulties in perceiving $\mathbf{Z}_n$ in these three different ways.

THREE INTERPRETATIONS OF $\mathbf{Z}_n$

Based upon our analysis of students' thinking, we propose that there are three different ways that students might reason about $\mathbf{Z}_n$: as infinite sets, as element-set combinations, and as representative elements. We illustrate these three interpretations in Table 1 using $\mathbf{Z}_4$ as an example. Note that each of these three conceptions involves a unique representation of cosets and coset operations. However, the differences between these three interpretations of $\mathbf{Z}_n$ is more than mere notational variation. Each of these three conceptions involves an algebraic group composed of cognitively and experientially different, albeit mathematically equivalent, objects. In other words, while one may assert that there is no real mathematical difference between the structures of the three algebraic groups, we suggest that there is a vast difference in how students think about and operate on the objects that comprise each group. We briefly explain each one of these interpretations below.

Table 1: Three Interpretations of Z_n illustrated with Z_4.

	Infinite Sets	Element-Set Combinations	Representative Elements
Elements	{…, -8, -4, 0, 4, 8,…}	$0 + 4\mathbf{Z}$	0
	{…, -7, -3, 1, 5, 9,…}	$1 + 4\mathbf{Z}$	1
	{…, -6, -2, 2, 6, 10,…}	$2 + 4\mathbf{Z}$	2
	{…, -5, -1, 3, 7,11, …}	$3 + 4\mathbf{Z}$	3
Operation	Add all elements of one set to all elements of another	$(a + 4\mathbf{Z}) + (b + 4\mathbf{Z})$ $= (a + b) + 4\mathbf{Z}$	$a + b = (a + b)\text{mod}4$

Infinite Sets

Under this interpretation, the group $\mathbf{Z}_n$ is comprised of a collection of infinite sets. The group operation is set addition, defined to be the collection of all possible sums created by adding single elements from one set to single elements from the other. To form the sum, students need to add several single elements from one set to several single elements from the other set until they see a pattern and are able to write the infinite set that contains all possible sums.

Element-Set Combinations

In this second interpretation, group objects are comprised of an element and a set. The element serves as an operator on the set, because it shifts all of the elements of the set along the number line the same number of spaces as its value. For example, $1 + 4\mathbf{Z}$ is all the multiples of 4 shifted right one position. The group operation for these objects is grounded in the infinite set interpretation of cosets. To add shifted sets, we use the same type of addition as for infinite sets. However, this is equivalent to shifting the original set by the sum of the two element operators. For example, $(1 + 4\mathbf{Z}) + (2 + 4\mathbf{Z})$ is the same as

$(1+2)+4\mathbf{Z}$, because when we add any multiple of 4 shifted 1 to the right to any multiple of 4 shifted 2 to the right, we get a multiple of 4 shifted 3 to the right.

Representative Elements

In this third interpretation, group objects are single elements. The group operation consists of adding elements modulo *n*, also referred to as clock arithmetic. In this interpretation, the set aspect of the group elements is hidden by the operation. Thus, students are able to do calculations without ever invoking the image of a set.

Relative Strengths of the Three Interpretations

Each interpretation provides our students with unique conceptual insights into $\mathbf{Z}_n$. The infinite sets interpretation brings to the foreground the set nature of cosets. Students cannot avoid thinking about the group operation in $\mathbf{Z}_n$ as operating on an object comprised of a collection of elements. Furthermore, the infinite set interpretation is needed to provide a satisfactory conceptual justification for why the group operation for the element-set interpretation is defined the way it is, and why it is not defined as $(a+n\mathbf{Z})+(b+n\mathbf{Z})=(a+b)+2n\mathbf{Z}$. On the other hand, the representative interpretation of $\mathbf{Z}_n$ brings to the foreground the element operators on the original subgroup, and drastically reduce the cognitive load required for computation. Finally, the element-set interpretation balances both a set interpretation and an element operation interpretation. As such, it plays a critical role in proofs and conceptualizing the structure of $\mathbf{Z}_n$.

STUDENTS' UNDERSTANDING OF THE THREE INTERPRETATIONS OF $\mathbf{Z}_n$

Although students varied in their ability to work successfully with the three different interpretations of $\mathbf{Z}_n$, we were nonetheless able to identify common trends across the six target students in our study. We present these trends below for each of the three interpretations of $\mathbf{Z}_n$.

$\mathbf{Z}_n$ as a Group of Infinite Sets

We found that all six students were able to write examples of $\mathbf{Z}_n$ as a collection of infinite sets and perform the group operation on those sets. However, students universally had a different understanding of the group operation than we had intended. To add infinite sets, the students took one element from one set, added it to one element in the other set, and then identified the infinite set that contained the sum of the two elements as the sum of the two infinite sets. While this method of operating always leads to correct results, it is different from thinking about operating on infinite sets as whole objects. This led to problems when students were asked to explain why they could add whole sets by just adding two elements. For example, David tried to justify this method for adding two cosets in $\mathbf{Z}_{10}$ by listing the two sets and adding elements that were vertically lined up, as shown in Figure 1. David was unable able to recover once he recognized that his result was actually an infinite set from $\mathbf{Z}_{20}$, not $\mathbf{Z}_{10}$. In fact, we found that students' most common methods for trying to add two infinite sets were either vertical addition of elements, as David did, or set union, despite having seen the instructor demonstrate the correct method for adding infinite sets in class. In the interviews, only two students were able to produce the valid group operation for adding whole infinite sets, and then only after a great deal of experimentation.

$$
\begin{aligned}
0+10\mathbf{Z} &= \{\ldots,-20,-10,0,10,20,\ldots\}\\
6+10\mathbf{Z} &= \{\ldots,-14,-4,6,16,26,\ldots\}\\
\hline
&\{\ldots,-34,-14,6,26,46,\ldots\} = 6+20\mathbf{Z}
\end{aligned}
$$

Figure 1: David's incorrect addition of infinite sets.

$\mathbf{Z}_n$ as a Group of Element-Set Combinations

We found that all six of our students could write examples of $\mathbf{Z}_n$ in the form of element-set combinations and perform the group operation correctly. Furthermore, students were generally more successful in justifying the group operation for element-set combinations than they were for infinite sets. Four of the six students noted that subgroups of **Z** are normal, and thus by a theorem they had investigated in class, the coset operation was well-defined. However, despite these successes, there were occasional lapses in students' attention to and understanding of the set part of the element-set combination notation. These lapses showed up in students' notational mistakes. For example, when given the problem of determining what $4\mathbf{Z}+6\mathbf{Z}$ produced, several students interpreted $4\mathbf{Z}$ and $6\mathbf{Z}$ as the cosets $4+\mathbf{Z}$ and $6+\mathbf{Z}$, and then added the element parts of these cosets to get $10+\mathbf{Z}$, which they wrote as $10\mathbf{Z}$. Occasionally students inserted elements into the set part of the coset notation, as Mandy did when she wrote cosets in $\mathbf{Z}_{12}/(4_{12})$ in the form of $n+4_{12}$ instead of $n+(4_{12})$. These mistakes suggest that students were often thinking of cosets in terms of elements and not as sets.

$\mathbf{Z}_n$ as a Group of Representative Elements

In general, our students were most successful with the representative interpretation of $\mathbf{Z}_n$. Students were adept at working with the clock arithmetic group operation for representative elements. We looked for evidence of misconceptions concerning students' understanding of this interpretation of $\mathbf{Z}_n$ and were unable to find any. Students were able to correctly solve all problems involving the representative element interpretation of $\mathbf{Z}_n$. However, as demonstrated above, their success with this interpretation cannot be interpreted as an indication of solid understanding of $\mathbf{Z}_n$. In particular, by operating with representative elements, students are not required to address the complexity of thinking of $\mathbf{Z}_n$ as a collection of sets.

Students' Flexibility with the Three Interpretations

For the purposes of proof and conceptual understanding, the element-set interpretation is likely to be the most powerful. However, it derives its power from coordinating the infinite set interpretation with the representative element as an operator. In other words, unless students are able to maintain meaning for both the element and the set aspects of the element-set objects, they become little more than formal symbols. We found that in many instances, the set component of the element-set objects became merely a notational baggage with little meaning, as demonstrated above with students' syntactical errors in writing cosets in element-set notation.

We hypothesize that students often lacked meaning for the set component of the element-set interpretation because they did not understand how to operate on infinite sets. In other words, while students knew the definition of the operation on element-set objects and generally recognized that the element-set operation was well-defined, this did not help them understand why $(a + n\mathbf{Z}) + (b + n\mathbf{Z}) \neq (a + b) + 2n\mathbf{Z}$. Naturally, an instructor might provide mathematical arguments justifying the element-set interpretation by appealing to group closure or the operation being well-defined, but these are not cognitively satisfying. Students' experience with adding algebraic expressions suggests that they should add like terms, so that $(a + n\mathbf{Z}) + (b + n\mathbf{Z})$ should yield $(a + b) + 2n\mathbf{Z}$. When students discover that this is not how element-set objects are added, then it is difficult for them to maintain meaning for the set component, because it does not receive the same status in computations as does the element. In other words, the element-set objects become essentially representative elements with the set appended at the end. Thus, the operation for element-sets does not contain any more explanatory power than the operation for representative elements.

CONCLUSIONS AND RECOMMENDATIONS

The cyclic groups $\mathbf{Z}_n$ are in many ways the prototypical examples of quotient groups. It is often assumed that because they are formed from a familiar set by a simple relationship, they are easily accessible to students. However, our analysis of students' thinking about $\mathbf{Z}_n$ suggests that $\mathbf{Z}_n$ is a cognitively complex algebraic structure that involves three different cognitive interpretations. Because of this complexity, abstract algebra instructors cannot assume that their students will naturally and easily grasp the complexities of $\mathbf{Z}_n$ as an example of a quotient group.

We found that students were able to do computational problems within $\mathbf{Z}_n$ without difficulty by thinking of it as a collection of representative elements under clock arithmetic. Indeed, our students tended to reduce the objects in $\mathbf{Z}_n$ into single elements whenever possible. While it is often useful to think about group objects as single elements when working with quotient groups, it is also important to be able to flexibly return to thinking of the group objects as sets when necessary. To move beyond computational facility, our students needed to think of $\mathbf{Z}_n$ in terms of element-set combinations. Such an understanding is built upon an understanding of elements of $\mathbf{Z}_n$ as infinite sets – specifically, as sets created by "shifting" the subgroup n**Z** by adding an integer to each element. Thus, full understanding of $\mathbf{Z}_n$ as element-set combinations must take into account both the infinite set n**Z** *and* the element that shifts it. Given this understanding, the definition of addition in $\mathbf{Z}_n$ becomes natural, and students' ability to flexibly deal with problems and proofs is greatly enhanced. Our data suggest that a full understanding of $\mathbf{Z}_n$ as element-set combinations that allowed for such flexibility was not common among our students.

Recommendations for Teaching $\mathbf{Z}_n$

Students will need substantial help in understanding $\mathbf{Z}_n$. In particular, instructors will need to help their students understand the three cognitively different interpretations of $\mathbf{Z}_n$, because the ability to work with $\mathbf{Z}_n$ as a quotient group often requires students to work flexibility within and between these three different interpretations of $\mathbf{Z}_n$. Furthermore, our research suggests that the most difficult part of understanding and coordinating the three interpretations of $\mathbf{Z}_n$ is helping students to focus on and understand the key role that sets play in the construction of $\mathbf{Z}_n$. Our students tended to lack this flexibility in moving from thinking of group objects as elements to thinking of them as sets again. We suggest that students need specific experiences in working with $\mathbf{Z}_n$ as a collection of infinite sets and exploring all the different possible group operations to identify an operation that yields group structure. Students also need opportunities to move back and forth from element-set combination and representative element interpretations to the infinite set interpretation.

Recommendations for Teaching Other Quotient Groups

Upon reflection, we feel that understanding a quotient group in terms of sets, element-sets, and representative elements would be helpful when studying any quotient group, not just $\mathbf{Z}_n$. Each of these interpretations is not only applicable to other quotient groups, but also emphasizes different aspects of a quotient group, and thus enhances understanding. As with $\mathbf{Z}_n$, the set interpretation of the quotient group brings to the foreground the set nature of cosets, and also motivates and justifies the operation on element-set objects. In contrast, the representative element interpretation reduces the cognitive complexity of the group structure by associating them with their element operators, perhaps allowing students to more easily perceive and recognize emergent properties of the resultant quotient group. Finally, an understanding of the element-set interpretation supports and in turn is supported by an understanding of the other two interpretations. The element-set interpretation is particularly crucial because it is the representation that coordinates both the set and element operator aspects of cosets. For these reasons, we recommend that instructors of abstract algebra consider addressing these three interpretations of quotient groups for any of the quotient groups their students study.

References

Asiala, M., Dubinsky, E., Mathews, D. M., Morics, S., & Oktaç, A. (1997). Development of students' understanding of cosets, normality, and quotient groups. *Journal of Mathematical Behavior, 16*(3), 241-309.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*(3), 267-305.

Herstein, I. N. (1999). *Abstract Algebra* (3rd ed.). New York: John Wiley & Sons.

Stake, R. E. (1998). Case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *Strategies of qualitative inquiry* (pp. 86-109). Thousand Oaks, CA: Sage Publications.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage.

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