DYNAMIC GEOMETRY AND THE THEORY OF VARIATION

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In this paper, the theory of variation in the tradition of phenomenographic research approach is placed in the context of Dynamic Geometry Environment (DGE). Central concepts of discernment, variation, simultaneity and space of learning in the theory of variation are discussed for simple dragging episodes in DGE to illustrate the potential partnership between the two enterprises. Implications for further research will be discussed at the end.

INTRODUCTION

What makes Dynamic Geometry Environment (DGE) a powerful mathematical knowledge acquisition microworld is its ability to visually make explicit the implicit dynamism of "thinking about" mathematical, in particular geometrical, concepts. By implicit dynamism I mean when engaging in mathematical activities or reasoning, one often tries to comprehend abstract concepts by some kind of "mental animation", i.e. mentally visualizing variations of conceptual objects in hope of "seeing" patterns of variation or invariant properties. The success of perceiving such patterns or properties usually helps to bring about understanding of the underlying formal abstract concept. However, this is usually a pure-thought process and often lacks corresponding representation in physical reality. In geometry, DGE provides a quasi-reality (Euclidean in nature) embedded in computational technology in which such a cognitive dynamism could be given a visual manifestation. In particular, one of DGE's power is to equip us with the ability to retain (keep fixed) a background geometrical configuration while we can selectively bring to the fore (via dragging) those parts of the whole configuration that interested us in a mathematical thinking episode. Research has been done in studying dragging strategies employed in DGE and the general conclusion is that dragging in DGE plays a key role in forming a mathematical conjecture (see for examples, Arzarello, F., 2000; Hölzl, R., 1996; Leung, A. & Lopez-Real, F., 2000). Recently, a dragging scheme was suggested by Leung and Lopez-Real (2002) to visualize a proof by contradiction for a geometrical theorem. The cognitive transition from experimental Mathematics (verification and conjecturing) to theoretical Mathematics (formal abstract concept and proof) is an elusive process that is yet to be fully understood. Dragging experiences in DGE open up a "space" in which the reification (Sfard, 1991) of mathematical processes into mathematical concepts might be able to take place. In a dragging episode, what can be experienced is a simultaneous variation of different aspects of an evocative computational object (Hoyle, 1991) and a temporal integration (a simultaneity of the past and the present) of accumulated mathematical knowledge. This confluence of simultaneities may hold the key to concept formation.

Discernment, variation and simultaneity are the central concepts in the phenomenographic research approach in which learning and awareness are interpreted under a theoretical framework of variation (see for example, Marton & Booth, 1997). In short, phenomenography is about categorizing the limited number of qualitatively different ways of seeing, or experiencing, a phenomenon in a hierarchical fashion. In particular,

> To discern an aspect is to differentiate among the various aspects and focus on the one most relevant to the situation. Without variation there is no discernment….Learning in terms of changes in or widening in our ways of seeing the world can be understood in terms of discernment, simultaneity and variation. (Bowden and Marton, 1998, p.7)

Marton, Runesson and Tsui further asserted

> that various degrees of expertise, that is, the capability of acting in powerful ways within a certain domain, is reflected in the various ways of seeing, i.e., in the various meanings seen in a particular scenario or problem. (Marton, Runesson and Tsui).

DGE is rooted in variation in its design. It is a milieu where mathematical concepts can be given visual dynamic forms subject to our actions, powerful or not. DGE is a natural experimental ground to experience the theory of variation since it has the built-in mechanism that enables the generation (via intelligent construction and dragging by us) of various qualitatively different ways of literally seeing a geometrical phenomenon in action. In the following discussion, I will attempt to describe, using simple examples in DGE, how the rudiments of the theory of variation can be perceived under DGE, and how the theory of variation can shed light on the process of concept formation in DGE.

DISCERNMENT

> In order to see something in a certain way a person must discern certain features of that thing. (Marton, Runesson and Tsui)

Discernment comes about when parts (features) are being focused and temporarily demarcated from the whole (background). In DGE, it is possible to define a way of seeing (discernment) in terms of actually seeing invariant critical features (a visual demarcation or focusing) under a continuous variation of certain components of a configuration. For example, a triangle ABC is constructed in DGE with the values of its interior angles and their sum measured and calculated respectively by prescribed functions in the particular chosen DGE (see Figure 1 for a Sketchpad version). The vertex C is then being dragged (a continuous action on the triangle ABC), hence varying the shape and the values of the interior angles of the triangle continuously (a continuous feedback of the dragging action). In such a dynamic episode of dragging resulting in a simultaneous twofold interdependent variation, what remains invariant is the sum of the interior angles, that is, it is always equal to $180°$. The visual constancy of $m\angle ABC + m\angle ABC + m\angle ABC = 180.00^{o}$ in midst of variations brings to the fore, hence the discernment of, a critical feature of a "generic" triangle, i.e. the sum of the interior angles of any triangle is equal to 180^{o}.

Variation

> …variation enables learners to experience the features that are critical for a particular

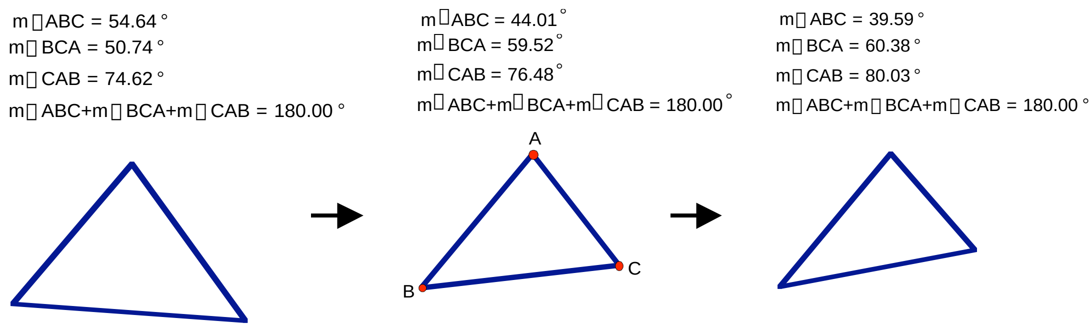

Figure 1.

> learning as well as for the development of certain capabilities. In other words, these features must be experienced as dimensions of variation. (Marton, Runesson & Tsui)

A dimension of variation is an aspect of the whole that can be subjected to vary. Dragging in DGE brings about a visual experience for different dimensions of variation in a geometrical situation. I will illustrate this with a simple construction problem in geometry:

> (P) Given two arbitrary points A and B, construct a circle that passes through A and B.

The key to this construction is to locate the centre of the circle. How does one see A and B as being points on the same circle? If A and B are regarded as the endpoints of a diameter of a circle, then the center C of the circle can easily be located as the midpoint of the line segment joining A and B. Points A and B can be dragged to different positions (see Figure 2) resulting in circles of different sizes with <u>C = the midpoint between A and B</u> as the centre. Hence we can think of C as a 'circle-valued' function with independent

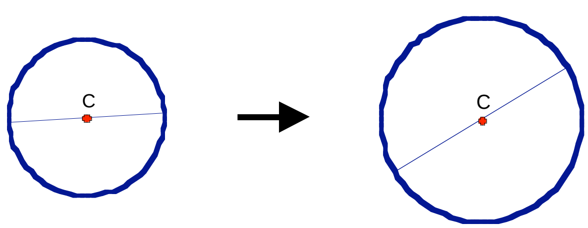

Figure 2

‘variables’ A and B. As A and B vary, C varies (hence the circle containing A and B) accordingly by keeping its relation to A and B invariant.

This provides a simple solution to the construction problem. However, instead of regarding AB as a diameter, A and B can be arbitrary points on a circle. In this case, to locate the centre C of a desired circle, one has to think about radius instead of diameter. This variation in the perception of the problem instantly opens up new dimensions of variation for the geometrical situation that are conducive to deeper understanding of the problem. Construct an arbitrary point C. Join A, C and B, C and measure their lengths. C can then be dragged to a position at which the numerical values of the lengths of AC and BC appear to be the same (see Figure 3).

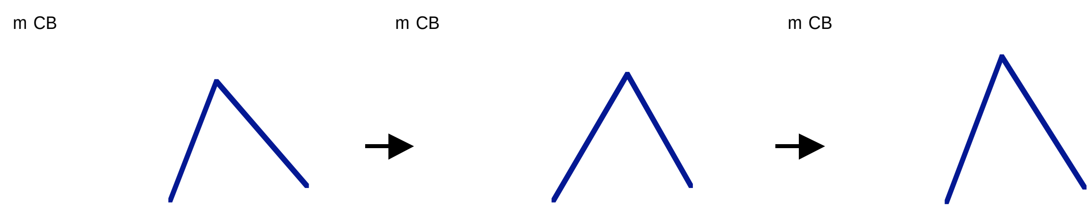

Figure 3

This position is then the centre of a circle that passes through points A and B. However, this is not only position such that this drag-to-fit strategy works (e.g. see Figure 4 for another such position with a different value for the radius). Consequently, one can construct more than one circle that passes through points A and B. By dragging (hence varying) C, we are literally looking for those positions on the computer screen at which the relation BC = AC holds.

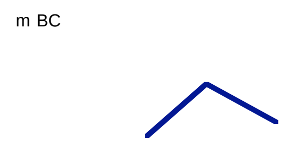

Figure 4

Contrast

> … in order to experience something, a person must experience something else to compare it with. (Marton, Runesson & Tsui)

The positions of C that are either circle producing or not circle producing bring about such a contrasting experience.

Generalization

“…in order to fully understand what ‘three’ is, we must also experience varying appearances of ‘three’...” (Marton, Runesson & Tsui)

Is it possible to find all circle producing positions for C? Can the ‘appearance’ of the invariance length BC = length AC be visualized?

In most of the DGE, it is possible to visually trace the movement of a point when the point is being dragged. In Figure 5, the path (or locus) of point C is being traced while it is dragged in a way that would keep the lengths of BC and AC as equal as possible. This kind of dragging strategy is called lieu muet dragging (Arzarello, Micheletti, Olivero and Robutti, 1998). The traced path (locus) traced is called a locus of validity by Leung and Lopez-Real (2002). In fact, the locus of validity for this problem is the perpendicular bisector of the line segment AB. Hence, in DGE, it is possible to have an objective visualization of 'generalization' in variation.

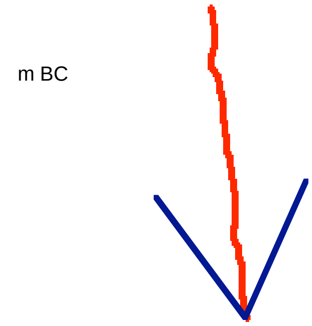

Figure 5

Separation

> In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant. (Marton, Runession & Tsui)

Throughout this dragging episode, the positions of A and B remain fixed while C is varying. Even though the invariance of the positions of A and B is not a given condition to the problem, it is a necessary condition to separate out the locus of validity when C varies in a specific way. The locus of validity is the critical feature that emerges under the two constraints:

C1: A and B are fixed,

C2: length BC = length AC.

Fusion

> If there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously. (Marton, Runesson & Tsui)

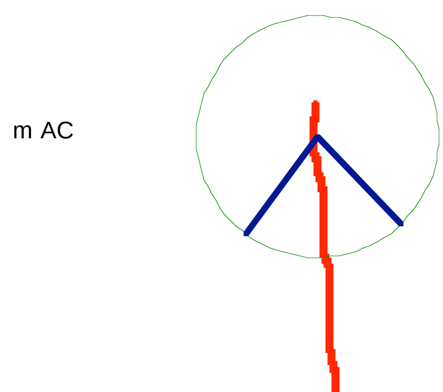

Figure 6

Constraint C2, the locus of validity and the circle that passes through points A and B are three critical aspects that can be manifested visually and simultaneously in a continuous manner. Figure 6 is a snapshot of such a fusing experience as C is dragged along the locus of validity. It is interesting to ponder on the idea that simultaneity seems to suggest equivalence, since in this case, the three critical aspects can be regarded as equivalent to each other.

Simultaneity

> …to experience variation amounts to experiencing different instances at the same time. (Marton, Runesson & Tsui)

Experiencing variations simultaneously is a unique feature of DGE. This is what makes DGE a powerful knowledge acquisition medium. Different parts of an evocative computational object can be varied continuously via different dragging strategies in real time. This brings about a simultaneous awareness of different critical aspects of a geometrical situation, hence creating a potential space for seeing mathematical structure and meaning in different qualitative ways. This kind of simultaneity is likely to constitute the crucial moment when mathematical conjecturing, and even formulating of mathematical proof, occurs. In expounding the construction problem (P) in DGE, the emphasis on seeing at the same time different parts of the whole, or variation under a fixed constraint, has been the underlying theme that carries the discussion. Two types of simultaneity can be distinguished, one is on space and the other one is on time.

Synchronic Simultaneity

> …a way of seeing something as the discernment of various critical features of an instance simultaneously.. .. (This) is the experience of different co-existing aspects of the same thing at the same time. (Marton, Runesson & Tsui)

This is a spatial type of simultaneity. Figure 6 is such an example of a synchronic simultaneity in DGE. Different critical features (the numerical values of the lengths of CB and AB, an approximate trace of the locus of validity, the circle that passes through A and B) 'co-exist' visually in the same picture at a point in time all being parts of one spatial configuration, hence producing a simultaneous experience of the whole-parts relations.

Diachronic Simultaneity

> In order to experience variation in certain respect, we have to experience the different instances that vary in that respect simultaneously, i.e., we have to experience instances that we have encountered at different points in time, <u>at the same time</u>. (Marton, Runesson & Tsui)

This is a temporal type of simultaneity. In DGE, diachronic simultaneity can assume a physical appearance. Dragging in real time composes a mini-movie episode (in some DGE, this can even be recorded and replayed), a weaving together of continuous temporal instances of variation of critical features in motion. In particular, in the case of the construction problem (P), diachronic simultaneity is visually manifested as the trace of a possible locus of validity that can bring about structure and meaning to the problem. Each 'previous' location of C on the locus of validity is a past experience (knowledge) and is visually in co-existence with the 'present' location of C.

Furthermore, diachronic simultaneity in DGE can be thought of as a temporal (in the sense of dragging in real time) integration (continuous summing up) of synchronic simultaneity. The integrated object, e.g. the locus of validity, can be thought of as a generalized and ever changing figure-ground structure that encompasses a totality of experiences of a dragging episode, bringing about an awareness of the mathematical concepts involved.

Space of Learning

> Creating a space means opening up a dimension of variation. (Marton, Runesson & Tsui)

The metaphor of a space of learning in variation theory is particularly apt when perceived in DGE. On an ontic level, DGE is a quasi-real virtual space in which geometrical objects and concepts can be visually constructed and manipulated. On an epistemic level, DGE is equipped with built-in devices (e.g. dragging, animation) to open up multiple dimensions of variation in any given geometrical situation, thus creating a space of learning in the sense of variation theory. A dimension of variation can be a measurement of certain geometrical quantity like length, a geometrical object like point, or any "drag-sensitive" (i.e., vary under dragging) part of a geometrical configuration. These dimensions of variation are "aspects of a situation, or the phenomena embedded in that situation, that can be discerned due to the variation present in the situation." (Marton, Runesson and Tsui) The choice of these dimensions makes up an *observable space* that has the potential (via dragging) to bring to awareness of certain pattern of variations (or invariance), e.g. the existence of a locus of validity. This in turn contributes to the understanding of a certain mathematical concept. For the construction problem (P), the locus of validity is the perpendicular bisector of AB and the 'answer' to, or the mathematical concept behind, (P) is:

> There are infinitely many circles that can pass through any two arbitrary given points.

Furthermore, the dragging episode opens up a space of learning (via exploration) in which variation leads to an experience of the equivalence between the problem (P) and its locus of validity. Equivalence is a deep mathematical concept.

IMPLICATIONS

I hope the above discussion carried enough conviction to suggest that there is a natural partnership between two well-established educational enterprises: DGE and the theory of variation. It would be worthwhile research to investigate how the framework of variation theory can enlighten our understanding on how students learn Mathematics in DGE and consequently, how to make DGE a pedagogically powerful environment to acquire mathematical knowledge. The idea of simultaneity seems to be a promising agent to help to bridge the gap between experimental Mathematics and theoretical Mathematics, or the transition between the processes of conjecturing and formalizing. This gap has been a cognitive black box that has yet to be totally opened. I hope the introduction of the theory of variation into DGE research can stimulate further insights, discussions and research agenda, and may even catalyze the process of opening this black box.

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