USING STUDENTS' WAYS OF THINKING TO RE-CAST THE TASKS OF TEACHING ABOUT FUNCTIONS[1]

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Recent research suggests that the examination of students' work may lead to changes in teaching practice that are more effective in terms of students' mathematical learning. However, the link between the examination of students' work and the teachers' actions in the classroom is largely unexamined, particularly at the secondary level. In this paper, I present the results of a study in which teachers had extensive opportunities to examine students' ways of thinking as the students developed models for exponential growth and decay. I describe two related aspects of the practice of one teacher: (a) how she listened to students' alternative solution strategies and (b) how she responded to these strategies in her practice. The actions of the teacher supported extensive student engagement with the task and the students' revising and refining their own mathematical thinking.

INTRODUCTION

The knowledge of subject matter alone is insufficient for effective teaching; subject matter knowledge is just one descriptor among many that attempt to capture the complexity of the nature of the knowledge base that is needed for teaching (Hiebert, Gallimore & Stigler, 2002; Shulman, 1986). Recent research on teachers' professional development would suggest that, among other things, effective teachers need to attend to students' ways of thinking about mathematical tasks (Schifter & Fosnot, 1993; Simon & Schifter, 1991). When teachers understand how students might approach a mathematical task and how their ideas might develop, this would seem to provide the basis for the teacher to support the student in ways that will promote student learning. However, most of the research on both students' ways of thinking and on teachers' understandings of student methods have focused on tasks in elementary mathematics, including important ideas in numeracy, rational numbers, and geometry (Ball, 1993; Jacobsen & Lehrer, 2000; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). Little research has been done on teachers' understanding of students' ways of thinking about tasks in secondary mathematics, such as functions, algebraic equations, Euclidean geometry, and data analysis.

THEORETICAL FRAMEWORK

In earlier work, I have argued that the focus of research on teacher knowledge needs to shift from examining what it is that good teachers *do* in particular situations to investigating how it is that good teachers *think* about particular situations (Doerr & Lesh, 2002). In other words, teaching is much more about seeing and interpreting the tasks of

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teaching than it is about doing them. A distinguishing characteristic of excellent teaching is reflected in the richness of the ways in which the teacher sees and interprets her practice not just in the actions that she takes. It is precisely a teacher's interpretations of a situation that influence *when* and *why* as well as *what* it is that the teacher does. Understanding teaching means knowing how teachers interpret the complexity and the situated variability of the practical problems of the classroom, how those interpretations evolve over time and across settings, and how and when those interpretations influence decisions and actions in the classroom. It is not enough to see what a teacher does, we need to understand how and why the teacher was thinking in a given situation, that is, interpreting the salient features of the event, integrating them with past experiences, and anticipating actions, consequences, and subsequent interpretations.

Increasingly, researchers on the development of teachers' knowledge have come to recognize two important perspectives on that knowledge: (1) that the knowledge for teaching is a complex and ill-structured knowledge domain (Feltovich, Spiro, & Coulson, 1997; Lampert, 2001) and (2) that to a large extent such knowledge is situated and grounded in the particularities of the contexts and constraints of practice (Borko, Mayfield, Marion, Flexer, & Cumbo, 1997; Lave & Wenger, 1991). These two perspectives on teachers' knowledge suggest that expertise in teaching is not a single, uniform image of a "good" teacher. Rather, expertise in teaching is highly variable, both across and within individuals and across multiple settings. This means that expertise can vary from teacher to teacher and that a given teacher's expertise can vary with different groups of students and with the same group of students in different situations. This is not to suggest a rejection of notions of accountability or assessment of good teaching. On the contrary, it is suggesting that teaching needs to be viewed as evolving expertise that will grow and develop along multiple dimensions in varying contexts for particular purposes.

One aspect of teachers' knowledge that has been the subject of much research (particularly, as I noted earlier, at the elementary level) is teachers' understanding of the landscape of children's conceptual development. In this paper, I emphasize that knowing this landscape is not the same as understanding one way of thinking or one way of developing or a particular learning trajectory. Rather, teachers need to recognize that within a given classroom children can be engaged in multiple ways of interpreting a problem situation and have multiple paths for refining and revising their ideas. The task for the teacher, then, includes seeing the multiple ways that children might interpret a situation, understanding that their ideas might be revised along various dimensions (while not being tested or refined along other dimensions), and acting in ways that will support the children's development towards more refined, more generalized, more flexible, and more integrated ways of thinking. The knowledge that teachers need consists of at least the mathematical understanding of the idea, an understanding of multiple ways that children's thinking might develop, the knowledge of typical mis-conceptions that students might have, an understanding of appropriate representations and the connections among those representations, and a knowledge of pedagogical strategies that will support that development.

In this research project, I examined teachers' knowledge of children's conceptual development by focusing on how the teachers saw and interpreted the events in their mathematics classrooms. I wish to recognize the complexity of those interpretations and the extent to which the teachers' interpretations are grounded in the context and constraints of practice. At the same time, I also wish to conceptualize the teachers' knowledge as that of evolving or emerging expertise along the multiple dimensions of practice. The central questions for this study are:

- How do teachers think about students' ways of thinking about exponential functions?
- How do teachers' interpretations of students' thinking influence their actions in the classroom?

By examining the particular characteristics of the teaching of an experienced teacher, I intend to generate an account of practice that moves beyond a description of what a "good" teacher does to an understanding of how teachers are seeing, interpreting and thinking about the tasks of teaching and evolving in their expertise.

DESCRIPTION OF THE STUDY

This particular study is part of a larger research project on the development of effective pedagogical strategies for teaching modeling tasks in technology-enhanced environments. The overall project includes modeling tasks that are intended to elicit the development of students' models (or conceptual systems) of linear change, exponential growth and decay, and periodic functions. The overall research design is that of the multi-tiered teaching experiment (Lesh & Kelly, 1999). This design enables researchers to examine the interpretations of teachers as they engage in understanding their students' interpretations of particular mathematical tasks.

The mathematical task that the students worked on is a well-known problem in exponential growth and served as the introductory lesson for the larger unit on exponential growth and decay. The task posed to the students was to investigate the pattern of pennies on a checkerboard when one penny is placed on the first square, two pennies on the second square, four on the third square, and so on. This simple recursive doubling pattern is very easy for the students to see. What is considerably more difficult for the students, despite their familiarity with the algebra of exponents and exponential functions from previous coursework, is to move from a recursive view of the function to an explicit form of a function that expresses the number of pennies on each square as a function of the number of the square.

Participants. The participant in this study was an experienced secondary teacher, who was teaching this particular lesson for the second time as part of a two-year project. This teacher had 30 plus years of experience and had strong content knowledge about exponential functions. The teacher, along with 11 others, had participated in two summer workshops where they explored the mathematics of exponential growth and decay and discussed various strategies that students might take in approaching this and other tasks in the sequence. The teacher also participated in monthly meetings during the school year in which student thinking and teaching strategies were discussed with colleagues and the researcher. Particular attention was paid to (1) listening to and identifying the different ways students might think about a problem and (2) supporting students so that they develop and revise their own solution strategies.

The teacher worked in a sub-urban public school with middle class students, with block scheduling, where the classes met for 75 minute periods for five times over a ten day period. There were 19 students, aged 16-18, with most taking this course as their final high school mathematics course. All of the students had graphing calculators and were generally familiar with their use from previous courses in mathematics.

Data Sources and Analysis. The teacher was video-taped during the lesson in which the Pennies Task was taught. The video-taping focused on the teacher and her interactions and exchanges with the students in her class. Field notes recording the researchers' observations were taken during the lesson. The video tape of the lesson, the transcript of the video tape, and the fieldnotes comprised the primary data sources for the analysis.

The data analysis was completed in two stages. The first stage of analysis involved open-ended coding (Strauss & Corbin, 1998) of the transcripts and field notes for the lesson. This coding was revised and refined through comparing the meaning of codes across similar lessons taught by other teachers in the study. This was followed by viewing the video tapes for the lesson, and adding annotations and clarifications to the transcript that were visible from the video tape. The coding of the transcript was then revised and refined in light of these annotations and clarifications.

The second stage of the analysis consisted of finding clusters of codes that defined the critical features the lesson. These features describe the dominant events that governed the lessons. In clustering these codes, I re-examined the data to find and interpret all those instances when the teachers were listening to and interacting with students for the purpose of understanding the students' thinking. This led to detailed descriptions of the teacher that were grounded in the data sources described above.

RESULTS

In this section, I briefly describe the implementation of the lesson by the teacher followed by the critical features of the lesson. The implementation began with Mrs. C asking the students to read the task and then to think about it independently. After a few minutes, she encouraged them to organize themselves into groups to work on the problem. From the outset, Mrs. C appeared to have a clear understanding that the central difficulty for the students will be in finding the equation that describes the number of pennies on each square as a function of the number of that square. As the lesson progressed, she repeatedly urged them to "think hard" and that what she wanted them to do was to "find the equation." Mrs. C allowed the students to engage in finding the equation for an extended amount of time. When the students had developed two different solutions to the problem, she asked both of those students to put their solutions up on the board and to explain how they arrived at their solution. She then posed that the differences in the form of the equation is a difficulty that the students must resolve.

The analysis revealed six critical features of Mrs. C's lesson:

(1) Setting an expectation for student thinking. Throughout the lesson, Mrs. C explicitly indicated that the problem did not have an immediate and obvious solution and that the real task for the students was to work hard and think about the task. Since this particular

lesson was the first task in an extended sequence of modeling tasks, Mrs. C. wanted to make clear that the rules of the game were changing; she would not provide them with easy answers, but would give them challenging tasks that they could productively think about. Throughout the lesson, she explicitly encouraged them to “work on the hard part” and "you've got to think now. That's the hard part."

(2) Focusing the task. From the teacher’s perspective, the central mathematical difficulty for the students would be to find the closed form equation to describe the pattern of the pennies. As the students began work on the task, they generated tables and graphs on both paper and in their graphing calculator. The teacher encouraged this work, but repeatedly focused their attention on “finding the equation.”

(3) Understanding students' ways of thinking about the task. Throughout the lesson, various groups of students attempted to model this exponential growth situation by using linear functions, investigating the rate of change using slope, finding quadratic functions, examining the behavior of the data at the origin, and exploring the patterns of perfect squares. In all of these cases, the teacher listened to how the students' were thinking about the task. The data revealed that the teacher was confident in her own ability to understand the diversity of students' thinking. She had anticipated the linear and quadratic approaches that students might take, but she also listened to and responded to new approaches:

Mrs. C	Okay. And that's what I want you to do is come up with an equation. [to Sara] Are you getting anywhere with it?
Sara	Does it have anything to do with perfect squares?
Mrs. C	[repeats to self] Anything to do with perfect squares...<pause> Like ... what are you looking at?
Sara	Like all the odd ones
Mrs. C	Okay <pause>
Sara	[cannot hear]
Mrs. C	Well, that's something that showed you something. Well, the even ones aren't perfect squares, though.
Mark	Yeah
Mrs. C	Well, so figure out why that might be.

The teacher was surprised at Sara’s observation of the pattern of perfect squares, as can be seen in how she repeated the observation to herself. She probed to try to understand the student's thinking by asking "what are you looking at?" Mrs. C quickly realized that every other entry in the table was a perfect square, something she had not thought about before. She then reflected back to the students that not all the entries were perfect squares and encouraged them to continue thinking about why that might be so. Significantly, the students' investigation of the pattern of perfect squares and another group’s investigation of the behavior of the data at the origin were novel ways of student thinking from the teacher's perspective. The teacher assimilated these approaches into her overall schema for students' ways of thinking about the task.

(4) Asking for student descriptions, explanations, and justifications. The teacher's response to the various ways of students' thinking was to ask the students to describe, explain and, on occasion, to justify their interpretations and reasoning. As we saw in the case with Sara (above), Mrs. C asked the student to explain her thinking so that the teacher could understand it. Mrs. C further encouraged the students to pursue a reason why their observation might be true. In the next example, when a student investigated slope, Mrs. C asked him to describe how he found the slope and to explain how he was using that to find an equation to fit the data. In the course of the student's explanation, he recognized that "the slope is always changing." This led him to consider a quadratic function rather than a linear one:

Mrs. C	Yes. It is the slope. Didn't you do a change in y over a change in x? [Bill looks on at Jerry’s work]
Mrs. C (to Bill)	No, he [Jerry] did the slope correctly.
Jerry	Well is there something wrong here?
Mrs. C	No, it is the slope, right?
Bill	Okay, okay, okay, I understand what you [Jerry] did. All right.
Mrs. C	It is the slope of the line connecting this [point] and this [point]. But [to Jerry] what did you just say?
Jerry	the slope changes, well, yeah
Mrs. C	So in other words, if you used different points than he did….
Jerry	you're going to get a different slope each time
Mrs. C	Which means?
Jerry	then, 'cause it's not a straight line. [Mrs. C shrugs affirmatively] So this will not work.
Mrs. C	So if you used this point and this point, you won’t get the same slope.
Jerry	So we have to figure out a parabola like
Mrs. C	[nodding] Right. You could figure out an equation, but what did you just say it wasn’t?
Jerry	a line
Mrs. C	Okay, so you've ruled that out anyway. Right?
Bill	It's challenging stuff
Mrs. C	Yes, I know. [Mrs. C moves to another group of students]

In asking for descriptions and explanations, the teacher created a situation wherein the student himself could refine his thinking and shift to a new way of thinking about the problem. In this instance, Jerry's initial investigation of slope led him to conclude that since the slope changes, the equation he is seeking cannot be that of a straight line. Jerry shifts his thinking to examining the possibility of a parabola. Mrs. C encourages him to pursue this new line of thinking! This gives the responsibility to the student to continue to seek explanations and justifications.

(5) Sharing and comparing solutions. After extended time on this task, the students had arrived at two different, but equivalent, solutions to the problem of finding an equation. One student wrote y=2^x/2 and the other student wrote y=2^(x-1). The teacher made these two solutions public on the board in the front of the classroom and asked two students to describe in considerable detail how they thought about the data in order to

arrive at their solutions. She intended to bring the students' description of their own reasoning as a part of the shared thinking of the class. She engaged the whole class in an extended discussion of why these solutions were the same and how to justify that claim.

(6) Anticipating development. Throughout the lesson, the teacher saw many ways that the students' ideas about exponential growth could develop. In her discussion with one student about the changing slope, the teacher encouraged him to explore this line of reasoning. Although it was not an immediate solution to the problem, the student was able to revise his own way of thinking himself, rather than simply being told that the function wasn't linear. The teacher recognized that the changing slope of an exponential function was an important characteristic of that function that would continue to be developed throughout the unit. This suggests that the teacher was anticipating the overall development of student ideas across the unit and within the lesson.

DISCUSSION AND CONCLUSION

This analysis reveals two ways that the teacher thought about students' ways of thinking about exponential function. The first way is about expectations: this teacher expected the students to think about the mathematics of the task. She clearly saw and explicitly communicated an expectation that the students should think hard about the task and that she expected them to spend time and to keep thinking even when they encountered difficulties. The second way of thinking about students' thinking is through having a flexible schema for how student's might approach the task. In the case of Mrs. C, she had a clearly developed set of schema that students might use that included using linear functions, investigating the rate of change using slope and finding quadratic functions. As the lesson progressed, she added two new ways of thinking about the problem through listening to unexpected student approaches: examining the behavior of the data at the origin and exploring the patterns of perfect squares.

The teacher's interpretations of students' thinking influenced her actions in the classrooms in two ways. First, having a well-developed schema for how students might approach the task enabled the teacher to press the students to explore and express their own ideas about the task. Even when students offered unfamiliar solutions (such as the pattern of perfect squares), she continued to ask the students for their descriptions and explanations. In asking for descriptions and explanations, the teacher created a situation where the student himself could refine his thinking and shift to a new way of thinking about the problem. Second, Mrs. C gave the students an extended amount of time to investigate their ideas about the function that could be used to describe the pennies data. She asked them to explain their ideas to her and, later in the lesson, to the whole class. She encouraged them to test their ideas by comparing the graph of their equations to the graph of the data. The teachers' understanding led her to ask for the comparison of solutions as a way to make public and shareable the students' ways of thinking about the task.

References:

Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373-97.

Borko, H., Mayfield, V., Marion, S., Flexer, R., & Cumbo, K. (1997). Teachers' developing ideas and practices about mathematics performance assessment: Successes, stumbling blocks, and implications for professional development. *Teaching and Teacher Education, 13*, 259-278.

Doerr, H. M. & Lesh, R. (2002). A modeling perspective on teacher development. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective on mathematics teaching, learning and problem solving*. Mahwah, NJ: Lawrence Erlbaum Associates.

Feltovich, P. J., Spiro, R. J., & Coulson, R. L. (1997). Issues of expert flexibility in contexts characterized by complexity and change. In P. J. Feltovich, K. M. Ford, & R. R. Hoffman (Eds.), *Expertise in context: Human and machine* (pp. 125-146). Cambridge, MA: AAAI/MIT Press.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*, 403-434.

Hiebert, J., Gallimore, R. & Stigler, J. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researcher, 31*(5), 3-15.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge, England: Cambridge University Press.

Lesh, R. A. & Kelly, A. E. (1999). Multi-tiered teaching experiments. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 197-230). Mahwah, NJ: Lawrence Erlbaum Associates.

Jacobsen, C. & Lehrer, R. (2000). Teacher appropriation and student learning of geometry through design. *Journal for Research in Mathematics Education, 31*(1), 71-88.

Schifter, D., & Fosnot, C. T. (1993). *Reconstructing mathematics education: Stories of teachers meeting the challenges of reform*. New York: Teachers College Press.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Simon, M. A., & Schifter, D. (1991). Towards a constructivist perspective: An intervention study of mathematics teacher development. *Educational Studies in Mathematics, 22*, 309-331.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage Publications.