THE NATURE OF SCAFFOLDING IN UNDERGRADUATE STUDENTS' TRANSITION TO MATHEMATICAL PROOF

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This paper explores the role of instructional scaffolding in the development of undergraduate students' understanding of mathematical proof during a one-year discrete mathematics course. We describe here the framework adapted for the analysis of whole-class discussion and examine how the teacher scaffolded students' thinking. Results suggest that students who engage in whole-class discussions that include metacognitive acts as well as transactive discussions about metacognitive acts make gains in their ability to construct proof. Moreover, students' capacity to engage in these types of discussions is a habit of mind that can be scaffolded through the teacher's transactive prompts and facilitative utterances.

BACKGROUND FOR THE STUDY

Exploring Sociocultural Aspects of Students' Understanding of Proof

> "The concept of proof is one which not only pervades work in mathematics but is also involved in all situations where conclusions are to be reached and decisions to be made. Mathematics has a unique contribution to make in the development of this concept, and […] this concept may well serve to unify the mathematical experiences of the pupil" Harold P. Fawcett (1938)

Since the statement above was written, the assumptions about proof as a logical argument that one makes to justify a claim and to convince oneself and others, and its role in mathematics, have not changed. Mathematicians and mathematics educators unanimously agree on the importance of proof in mathematics and the necessity for students to develop both the understanding of concepts related to proof and the skills to read and write proofs. However, the ability to read and do proofs in mathematics is a complex one that depends on a wide expanse of beliefs, knowledge, and cognitive skills and that is uniquely shaped by the social context in which learning occurs.

Research on students' understanding of mathematical proof has focused on cognitive issues, including the development of students' proof schemes (Harel and Sowder, 1998) and students' misconceptions and difficulties with proof (e.g., Balacheff, 1988; Chazan, 1993; Porteous, 1990; Senk, 1985). However, the effect of sociocultural factors on students' transition to mathematical proof, particularly in undergraduate settings, remains a virtually unexplored domain. Thus, we are engaged in a study of the role of the social in how students in a one-year, undergraduate mathematics course come to understand proof.

We see an emphasis on the social character of proof as situated within the broader theoretical perspective that development cannot be understood apart form the *social* context in which it occurs (Vygotsky, 1962/1934). In particular, Vygotsky maintained that "higher voluntary forms of human behavior have their roots in social interaction, in the individual's participation in social behaviors that are mediated by speech" (Minick, 1996, p. 33), and that students' development of self-regulatory thinking occurs through a process of internalizing events that originate on the social plane. As part of this, he postulated the notion of a zone of proximal development (ZPD) as a way to conceptualize learning.

Scaffolding and the Zone of Proximal Development

The ZPD is defined as the space characterizing one's potential for development through the assistance of a more knowing other (Vygotsky, 1962/1934; Litowitz, 1993). As a diagnostic, the ZPD intends to assess not only those cognitive functions that one possesses, but also those that are in the process of development by virtue of the learner's *interaction* with more knowing others, cultural tools, and so forth (Kozulin, 1998). Since learning is viewed as a product of interaction, it follows that one's development within the ZPD is affected by the intellectual quality and developmental appropriateness of these interactions (Diaz, Neal, & Amaya-Williams, 1999). In other words, the extent of one's development within the ZPD is predicated in part upon how the more knowing other organizes, or *scaffolds*, the task at hand. Thus, if we intend to understand development within the ZPD, we must think about if and how tasks can be scaffolded to extend one's learning.

As a construct inseparable from the ZPD, *instructional scaffolding* is a mechanism for observing the process by which the learner is helped to effect his or her potential learning (Stone, 1993). Practically speaking, it refers to the "provision of guidance and support which is increased or withdrawn in response to the developing competence of the learner" (Mercer, 1995, p. 75), and it is based on the appropriation, not simple transfer, of ideas between teacher and student. However, understanding the subtleties by which this occurs is a complex process that requires sensitivity to the learner's goals as these goals emerge in the course of activity (Wells, 1999). Thus, within the classroom, scaffolding presupposes that the teacher is continuously attending to students' thinking in order to access their individual (and communal) ZPD. For example, knowing how to give hints that focus and challenge a student's thinking requires a deep knowledge of students' individual learning capacities with respect to the task at hand. The complexity increases for the teacher because hints are often given in large group settings that necessarily conceal individual differences and thus diminish the teacher's capacity to attend to them.

From this perspective, we came to view the nature of scaffolding and when and how one's learning is scaffolded as a critical part of understanding how students learn to construct mathematical proofs. Thus, within the broader purpose of exploring sociocultural factors in undergraduate students' transition to mathematical proof, we focus here on instructional scaffolding and how it supported the development of students' capacity to write and express rigorous mathematical proofs. In particular, we share our findings on the following specific questions:

h. What is the nature and meaning of instructional scaffolding in the classroom in the development of students' proof ability?

i. How do different types of scaffolding prompts from the teacher affect students' self-regulatory thinking about proofs?

METHODOLOGY

Participants, Data, and Setting

Participants for the study were two cohorts of undergraduate mathematics students, with 50 students per cohort, enrolled in a one-year discrete mathematics course that emphasized mathematical argumentation and proof. Classroom instruction was videotaped and selected small group discussions were audiotaped. Whole class and small-group episodes were selected for transcription and analysis. Additionally, students were given pre- and post-assessments which were analyzed to identify the generality, form and competency of students' arguments and which we took as evidence for shifts in students' capacity for self-regulatory thinking (see Blanton & Stylianou, 2002). Finally, students' individual written proof constructions were collected biweekly. The study reported here focuses on data collected during whole-class and small-group discussions that occurred in the first semester of the course.

The instructor (the same for both cohorts) worked to establish expectations that students explain their reasoning and make sense of and challenge each other's explanations and justifications. Students submitted regular assignments in which they wrote proofs and reflected about their thinking. Classroom activity focused on group problem solving and included alternative forms of assessment (e.g., group exams, reflective writings).

RESULTS

A Framework for Analyzing Instructional Scaffolding

We begin here by describing how our focus narrowed to instructional scaffolding and the subsequent framework we adapted for its analysis. Our previous work provided both a general description of how students evolved in their capacity for argumentation and written proof and quantitative results that students were learning to construct increasingly rigorous proofs (see Blanton & Stylianou, 2002). However, we wanted to more carefully detail the mechanisms of classroom interaction that mediated the collaborative, or public, development of students' proof ability. Consequently, our focus shifted to analyzing the discourse structure in whole class discussions, using each speaker's turn as the unit of analysis. For purposes of analysis, we found it useful to distinguish between *public* and *private* cognition, where we take public cognition to mean mathematical knowledge that is publicly owned and constructed. As we analyzed discourse data, it became apparent that the teacher's utterances, because of their intent coupled with her function as a more knowing other, were fundamentally different than those of students. Thus, we could not analyze these data as a group discussion such as that among peers, but had to attend to the dynamic created by the different purposes of the speakers. This redirected our attention to the teacher's utterances in order to detail the nature of instructional scaffolding and how it extended students' development within the ZPD.

We based our framework for analysis on the work of Kruger (1993) and Goos, Galbraith, and Renshaw (2002). In particular, we found Kruger's (1993) framework for the analysis of transactive discussion helpful in identifying each person's contribution to the collaborative structure of the whole class interaction. Transactive discussion is characterized by clarification, elaboration, justification, and critique (of one's on or one's partner's reasoning). Moreover, transactive discussion refers to the ways that people publicly engage with metacognitive utterances. Thus, we drew from the work of Goos *et al*, itself an extension of Kruger's framework, to analyze metacognitive utterances that functioned as "New Idea" or "Assessment". However, since the work of Kruger (1993) and Goos, *et al* (2002) is based on peer group analysis, we needed to extend their frameworks by analyzing the intent of the teacher's utterances as well. Thus, in our framework for analysis of whole-class discussion, utterances were cross-coded in terms of metacognitive acts (New Idea; Assessment), transactive utterances, and the nature of scaffolding in the teacher's utterances.

The Nature of Instructional Scaffolding in Students' Proof Construction

From our analysis, we found that the teacher's utterances consisted of *transactive prompts* and *facilitative utterances*. By facilitative utterances, we mean instances of revoicing or confirmation. We define transactive prompts to be a form of scaffolding in which the teacher's questions promote transactive discussion among students. In particular, the teacher's utterances consisted primarily of *requests for* clarification, elaboration, justification, and critique, all of which formed the basis for a complex, interconnected dialogue by which students engaged in metacognitive acts, transactive discussion, and transactive discussion about metacognitive acts. Goos, *et al*, (2002) found transactive discussion of metacognitive acts to be a significant factor in successful (small group) collaborative problem solving. We conjectured a similar effect on whole-class discussion and we argue that, to the extent that the teacher's transactive prompts were able to facilitate transactive discussion in whole-class dialogue, she was able to scaffold students' thinking in publicly constructing mathematical proofs.

To support this claim, we share here the coding and analysis of an excerpt from a 60-minute classroom episode that occurred during Week 4, where the task was to construct a proof that $\sqrt{2}$ is irrational. Codes of utterances are italicized in the protocol. The analysis focused on characterizing the structure of dialogue surrounding transactive prompts and facilitative utterances in the whole class discussion in order to understand how the teacher was able to scaffold student thinking and what this suggested about student development within the ZPD. Student names are pseudonyms.

Teacher: Why is that true ('$2q^2 = p^2$' fails for odd numbers)? (*request for justification)*

Anthony: We could prove that an odd times an odd is an odd. (*new idea)*

Teacher: Yeah. We could do something like that. That would certainly work. That would be a more general case in fact, instead of a particular case. *(revoice and confirm)*

Degan: We already know that 2 times any integer is going to be an even number anyway. *(new idea)*

Jarrod: That's what I was going to say. The left side (2 p^2) is always even. *(elaboration)*

Teacher: OK, So here's something ($2p^2$) that's always going to be even, so you're saying that if p is odd, [then] p^2 is odd, so you'd have an odd number equal to an even number? *(clarification)*

Jarrod: Yeah.

Teacher: True. So if p is odd, it fails. (*revoice and confirm)* Are we done? *(request assessment of proof status)*

In the above episode the teacher aims to scaffold the students towards the construction of a particular proof. The classroom had agreed the previous day that a proof by contradiction would be an appropriate strategy to use, and the teacher initiated the discussion by re-stating the agreed upon plan. The teacher restrains her comments in only three types: (a) requesting clarification, elaboration, justification, or assessment, (b) revoicing and/or confirming a student statement, and (c) elaborating on a student-originated idea. While the teacher herself avoids engaging in transactive discussion (except in the one case where she elaborates), her goal is to encourage her students to do so as they gradually progress in their proof construction.

The scaffolding here takes two forms. The obvious form of scaffolding is the teacher's confirmation of students' ideas. By revoicing and confirming student-originated ideas, the teacher lends authority and confidence to students, as the "more knowing other", to proceed along the student-suggested path. The second form of scaffolding is the teacher's repeated requests for students to engage in transactive discussions. And while by the first form of scaffolding the teacher shares responsibility for the proposed action (through a tacit approval), the second is a transfer of responsibility for a construction of a proof from the teacher to the students (through her requests for assessment and critique). A second difference is with respect to the overall goal of each form of scaffolding. The former involves utterances specific to a given mathematical problem. The latter is a theme that permeates the entire semester; it is about the development of the habit of mind of being inquisitive and engaging in metacognitive acts.

The question that arises is whether the teacher, through the two forms of scaffolding, accesses students' ZPD. Our coding and analysis suggest a tentative hypothesis: Students' proposal of "new ideas" and their subsequent elaboration and justification of these ideas in a way that furthered the construction of a proof indicates their development within the ZPD. Pre-test results (Blanton & Stylianou, 2002) suggested that prior to instruction students were not able to construct this proof. However, the teacher's transfer of the proof responsibility through transactive prompts supported students in making significant contributions to the proof. With respect to our first research question, we claim that while both types of scaffolding prompts impact student proof ability, it is likely that prompts that encourage transactive discussion are the most crucial in the development of students' proof ability. Further study is needed to understand whether the two types of scaffolding prompts impact students' reasoning differently at different stages of the proof construction.

We were further interested in examining possible patterns of transfer of the teacher's scaffolding prompts in students' small group discussions. We conjectured that transactive patterns in whole class dialogue, led initially by the teacher, eventually would be internalized by students in their acquisition of self-regulatory thinking. Our subsequent coding of students' small group discussions provided evidence that students assumed the role of scaffolding each other with the same transactive prompts their instructor earlier urged them to use to scaffold their own thinking about proof. In this sense, we argue that the forms of argumentation essential for proof-building were becoming a habit of mind for students *independent of the teacher's participation in the dialogue*. The following excerpt, which occurred during Week 5, is a small-group discussion for which the task is to prove that for any even integer, n, n^2+1 is odd.

Mike: Does this show this… this is true? *(request for clarification)*

Justin: Say, assume n^2+1 is even so then you can throw out… (*clarification)*

Mike: Right…. *(confirm)*

Justin: [voicing his algebraic work] n^2+4l^2+1 … n^2+4l^2…. *(elaboration)*

Mike: So that's where I show n is odd down here. *(confirm)*

Steve: Ummm….yeah *(confirm)*

Mike: But aren't you trying to show n is odd? *(request for clarification)*

Justin: I did. *(clarification)*

Mike: I don't know…I don't really think you… I don't think you proved it yet, but that could be close. Because you're trying to show its odd and all you proved is n^2+1 is even. *(critique)*

Steve: Alright, what are we trying to show? *(request for assessment of proof status)*

Justin: I know but I showed its even when I say $n=2l$. *(clarification)*

Steve: We showed its odd… n^2 we showed is…. *(clarification)*

The discussion in the small group is fundamentally different than the whole-class discussions. While in both episodes the main objective is to produce a correct proof to a given problem, in the small group discussion there is no instructional intent to scaffold student thinking. The "more knowing other" becomes the "more capable peer" in a discussion among equal partners. This difference is reflected in the type of utterances in the small group discussion that fall into two categories: (a) requests for clarification/elaboration/justification (transactive), and (b) responses to these requests. Justin appears to be the more inquisitive partner, but his requests do not imply that he assumes the role of the scaffolding instructor. His requests are the expression of his attempt to follow his partners' reasoning and negotiate meaning with them, not to get his partners to engage in transactive discussion for its own sake. However, *indirectly* and unintentionally, his requests have an impact similar to the teacher's earlier requests: The other two students are forced to clarify their reasoning and, subsequently, advance their own understanding and thinking. With respect to our second research question, we make an initial claim that students appropriated the structure of whole-class dialogue, scaffolded by the teacher's transactive prompts and facilitative utterances, and used this to advance their own and their classmates' proof construction.

DISCUSSION

Concerning the notion of scaffolding, Stone (1993) notes that little attention has been paid to the mechanism by which the transfer from mentor to student is accomplished. Indeed, little, if any, research has focused on instructional scaffolding in tertiary mathematics settings. As such, this study was intended to provide insights into how students appropriate strategies for advanced mathematical reasoning and how instructional scaffolding supports this. Our results suggest that students who engage in whole-class discussions that include metacognitive acts as well as transactive discussions about metacognitive acts make gains in their ability to construct mathematical proofs. Moreover, students' capacity to engage in these types of discussions is a habit of mind that can be scaffolded through the teacher's transactive prompts and facilitative utterances. This has serious implications for the nature of whole-class discourse that occurs in advanced mathematical settings, at least for those that deal conceptually with mathematical proof. In effect, it suggests that students can internalize public argumentation in ways that facilitate private proof construction if instructional scaffolding is appropriately designed to support this.

More work is needed, however, to further detail the nature of instructional scaffolding and its longitudinal effect on students' capacity for small-group and individual proof construction. For example, analyses that would establish the increasing use of transactive prompts in students' private proof construction subsequent to whole-class discussion would further our understanding of the development of students' proof conception and would provide a critical link between instructional acts and the development of one's transition to mathematical proof.

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