USING INSTRUCTIONAL REPRESENTATIONS OF RATIO AS AN ASSESSMENT TOOL OF SUBJECT MATTER KNOWLEDGE[1]

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This study posits that instructional representations are viable assessment tools of subject matter knowledge within the context of lesson study. Evidence is provided and validated, suggesting that representations provide valuable insights into the depth and accuracy of the knowledge prospective teachers bring to instructional settings. It is conjectured that those with weak subject matter knowledge may ask too many open-ended questions, and over rely on their students to "remember." Those growing in their understanding may be reluctant to cede ownership to their students while those with strong understanding include many representations in their lessons.

PROBLEM, FOCUS, AND FRAMEWORKS

The development and growth of teachers' subject matter knowledge of ratio is the research problem addressed in this study. The link between teachers' knowledge of mathematics and the quality of classroom instruction was established in a number of studies (Ball, Lubienski, & Mewborn, 2002). Without a deep understanding of mathematics (Baturo & Nason, 1996), it is difficult for teachers to change their instructional practices to incorporate curricular reforms. While much has been learned about proportional reasoning over the last 20 years (Lamon, 1995) assessing teachers' content knowledge of ratio and proportion is difficult from an educative perspective. Teachers' knowledge, in the traditional sense of paper/pencil testing, does not correlate with traditional measures of student achievement. Wilson, Schulman, and Richert (1987) suggest that this failure is attributable to the narrow definitions given to teacher knowledge. They posit that an assessment include what is known about the subject matter <u>and</u> how to present that subject matter to others. We aim to study what prospective teachers know about ratio from the instructional representations they select in a lesson plan. Instructional representations are viewed as a link between content and pedagogy by a number of researchers (Wilson, Shulman, & Richert, 1987; Ball, Lubienski, & Mewborn, 2002). Words, pictures, graphs, objects, numbers, symbols, and contexts (including examples, metaphors, analogies) constitute the instructional representations that convey mathematical ideas. They serve as powerful connections between what teachers know about mathematics and what they know about teaching mathematics. The focus of this study is to determine if the initial instructional representations of prospective teachers are viable assessment tools of their subject matter knowledge. The study reported here provides base line information to inform a larger study on the growth of understanding of ratio among prospective teachers.

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Two theories conceptually guide this research: a) a model of the knowledge base of teaching proposed by Schulman (1986), and b) the theories of Pirie and Kieren (1994) that describe the growth of mathematical understanding. Of interest are two components of the knowledge base, subject matter knowledge and pedagogical content knowledge. Subject matter knowledge is defined as the procedural and conceptual knowledge of mathematics, as well as, the connections and relationships within ideas. Shulman defines pedagogical content knowledge as the subject-specific instructional strategies, instructional representations, and teachers' knowledge of students' understanding. The model of the growth of mathematical understanding, developed by Pirie and Kieren (1994), gives perspective to the emerging ideas expressed by the prospective teachers' selections of instructional representations of ratio. The Pirie-Kieren theoretical model is one of actions and interactions, tracing the back and forth movement of the learner's ideas between and among eight levels of understanding activities where the learner builds, searches, and/or collects ideas (Pirie & Martin, 2000). The movement toward inner layers is termed *folding back* and serves an important role in the growth of understanding as it may signal a fundamental shift in the learner's understanding. For this study we focus on the first five activities of understanding. The innermost level is that of *primitive knowing* consisting of all of one's previous knowledge and serves as the reservoir from which to build subsequent understanding. Moving outward within the model, *image making* and *image having* are learner activities that involve making a new image or revising an existing image, and then abstractly manipulating that image. It is these two levels of activities that play a prominent role in our analysis. Other levels of activities used here are *property noticing*, and *formalising*. Identifying properties of the constructed image defines property noticing and perhaps the images are formalised when a method, rule, or property is generalised (Pirie & Kieren, 1994). To begin the process, the researcher presents a problem or task to the learner so as to observe, record, and interpret the growth of understanding within a mathematical context.

OBTAINING AND ANALYSING THE EVIDENCE

Prospective elementary teachers at a large, Australian university volunteered to participate in a semester-long teaching experiment. It is noted here that instruction and terms associated with teaching ratio and proportion differ across countries. US students are introduced to fractional notation of ratios very early in instruction while most Australian students never use fractional notation. Here we report on the cases of three undergraduates beginning their third year of study. Stephanie and Maria were in their mid-30s, returning to the university after previous career and family experiences. Abbie was in her early 20s and had gone directly from high school into college. Lesson study was chosen for the teaching experiment of the larger study and here we focus on the first three tasks of lesson study. Described by Lewis (2002), lesson study is a model of Japanese professional development where communities of teachers come together as researchers to recursively develop, discuss, and teach a single lesson over an extended period of time. Teachers are cast in the role of researchers as they examine, test, and modify the lesson. Several researchers adapted these techniques to study the growth of understanding of slope, similarity, and right triangle trigonometry among prospective US high school mathematics teachers (Berenson, 2003). Findings suggest that prospective teachers come to the lesson study tasks with a variety of mathematical understanding, and as they engage in individual and group lesson study activities, grow in their understanding. Evidence was collected individually in the two videotaped interviews, and included the artifacts developed during the three activities of lesson study. In the initial interviews the prospective elementary teachers were asked to recall what they learned about ratio and proportion in school, what their teachers did, and what their textbooks showed. The interviewer assessed the undergraduates' ideas during this initial interview, and if necessary taught the missing subject matter. Then the subjects were asked to plan a lesson to introduce the concept of ratio to an average class of seventh graders. The initial, 15-minute interview was followed by 45 minutes of planning time where the undergraduates had access to textbooks, materials, and manipulatives. No particular format was requested for the plan although subjects were encouraged to write down their ideas to present in the 30-minute follow-up interview. Data were analyzed using categorical aggregation to find patterns. The conjectures or assertions drawn from the patterns of the data summarize the findings.

DIVERSITY OF REPRESENTATIONS AND UNDERSTANDING

Part – Part Comparisons	Simplifying	**Missing Value**	**Notation**	**Applications**
Table to show number of boys to girls* (T) G | B 15 | 10	How can we group girls and boys so that each group looks the same? (T)	If 15 G to 10 B is equal to 5G* to 2B, then how would we work out the number of girls to boys if there are 200 boys? (T)	3 : 2 = 15 :10 Or 3 girls = 2 boys 15 girls 10 boys (T)	If we know proportion of one thing to another we can predict for larger pop. (T)
Use counters or bears to represent class (S)	Use bear counters to find equal groups. (S) Make a table of trials: G., B., & Leftover. (S)	Ratio Table. G. 5 10 15 _ _ 125 500* B. 2 4 10 _ _ 50 200 (T)		Ask where seen e.g. fuel consumption, recipes (T & S)
	Use student models to show not equivalent e.g. 6 g & 4b leaves 3 g & 2b left over. Draw Table: G B Leftover 15 10 0 6 4 5 7 5 1 3 2 0 Arrows relate equivalence* (T)			Give real world problem to groups. If space shuttle uses 1000 litres fuel to travel to Mars which is 10,000 km from earth, how many litres?* (T & S)

S = student task; T = teacher task; * = accuracy/appropriateness questioned

Table 1: Maria's instructional representations

Tables 1-3 characterise the subjects' instructional representations from which are drawn the individual assessment of each teacher's subject matter knowledge of ratio. Images of the instructional representations are categorized according to the subject matter included in each individual plan. Maria's plan includes part – part comparisons, simplifying ratios, missing value problems, ratio notation, and applications (See Table 1 above). Data from Stephanie's plan is sorted into part – part comparisons, simplifying ratios, building up strategies, and notation (See Table 2). Images in Abbie's plan are rates, missing value problems, notation, and applications. Under each of these subject matter categories are the instructional representations included in each plan. For example, Maria selects an instructional representation of a data table of the number of girls and boys in the class and then moves to using teddy bear counters to represent part – part comparisons. Within each category, an order is implied from top to bottom but not across subject matter categories. Areas of understanding are identified in the narrative, and incomplete, questionable, or missing ideas are highlighted with an asterisk in the tables. The initial "S" or "T" in each cell denotes who will use the representation, students or teachers.

Maria appears to have a very good understanding of part – part ratios, using tables and manipulatives to represent the boy/girl comparisons of the classroom. Her initial choice of words, comparing boys to girls, and then the notation (15 g to 10 b) calls for monitoring a potential problem of order in writing ratios. She moves beyond an algorithmic understanding of simplifying ratios toward a conceptual understanding with the use of manipulatives to model possible equivalent ratios of 15 : 10. This understanding is incomplete as she was not aware that the bear representation of *6 green – 4 yellow – 5 left over* combination was equivalent to 15 : 10 and 3 : 2. It suggests that simplifying ratios means "lowest terms" rather than equivalence.

Her second use of ratio tables to find missing values provides information that Maria was able to construct multiple sets of equivalent ratios. Additional assessment is needed to determine if the misrepresentation of 15 : 10 = 5 : 2 and subsequently in the table of missing values is a computational problem or an indication of lack of planning time. The notation that Maria chooses to present to her students includes the colon and fractional notation. Her use of labels in the fractional notation provides confirming evidence that she understands the importance of order in writing ratios. In the representation of *the real world* space problem, we question the accuracy of the fuel and distance quantities. Overall, Maria's plan has more instructional representations than the other plans, and we assess her understanding of ratio subject matter to be deeper than the other prospective teachers. Maria will grow in her understanding of ratio and proportion as she collects more instructional representations, noticing the properties of the ideas embedded within the representation to formalize deeper understanding of the subject matter.

Two essential differences between Stephanie (see Table 2 below) and Maria's plans are the number of representations and who plans to use the representations. Maria has more instructional representations, yet gives students some of these representations to build understanding of difficult concepts. Stephanie uses fewer representations in her plan with little student ownership of the representations. Her representation of part – part summing to the whole is an example of property noticing of this type of ratio, and indicates a growth in her understanding of part – part ratios.

Part – Part Comparisons	Simplifying	Adding Up Strategy	Notation
Student taste test of 2 differ items intended to bake for par tea (S)	Reduce to low number. Hopefu (they) will divide 5.* (S)	How many of each it do we need to cook 100 people? (T&S)	Ratio – explain ter comparing two quantiti 10 : 15 (T)
Make a chart of preferences (T Item 1 Item 2 10 15	So for every 2 peo who like item 1, 3 l item 2.	2/3 adds up to 5 10/15 adds up to 25 So you need someth that adds up to 1((T&S)	
Explain: Out of 25 people in class 10 prefer item 1, 15 pre item 2. Item 2 is the m popular. (T)	Use colored counte Shows 10 yellow a 15 red. Then shows yellow and 3 red. *(T	Finding equivalent ra means same equivalent fractio Use fraction sticks remind them. (T)	
Both together add up to 25 total or whole. (T)			

S = student task; T = teacher task; * = accuracy/appropriateness questioned

Table 2. Stephanie's instructional representations

She chooses to teach simplifying ratios but "hopefully they will know to divide by 5" reflects her lack of knowledge of simplifying ratios. Representations of equivalent ratios using fraction bars may flag a weak understanding of proportion. Her building up strategy appears to indicate an image that is non-traditional in terms of introducing ratio problems but rooted in her own experience of cooking for 100 people. Overall, Stephanie's understanding of the subject matter is emerging as noted by the number of representations in her plan. Stephanie can grow in her understanding by collecting more instructional representations of ratios, simplifying ratios, equivalent ratios, and different types of problems and strategies.

Rate Comparisons	Missing Value	Notation	Applications
Introduce water bill classroom tap drippi The tap drips 5 tin every 2 minutes.* (T)	I left my office at 5pm yesterday an was going to measure the water loss putting a bucket under the leaking t but I forgot. How could I work out h much water was lost between the ho of …….? * (S)	What would be an ea way of presenting t data? Teacher introdu of concept of rate ra and how it is written. (	Can you think of other ways that we could use this idea of ratio? (T&S)*
Show chart: Drips Time ooooo 2,00 5 2 (T)	Student groups share their finding w class. Class questions findings. (S)		

S = student task; T = teacher task; * =

Table 3: Abbie's instructional representations

At first glance Abbie's plan (Table 3) appears to be exactly what we want beginning teachers to adopt, open-ended and student-centered instructional approaches. While the pedagogy may be sound, there are very few instructional representations of the subject matter. Abbie poses a problem that is computationally very challenging (5 drops : 2 min), and requires unit conversions of minutes to hours. The students are asked to solve the problem, explain and question the solution strategies, and come up with applications of ratios. The assessment that Abbie lacks the subject matter knowledge of ratio and proportion is based on what representations are missing rather than what she included. To grow in her understanding, Abbie needs to collect many more images of ratio, noticing properties to formalise her understanding.

VALIDATING THE ASSESSMENT

We conclude from the analysis of their instructional representations that Maria understands most of the ideas of her plan deeply, Stephanie understands some initial ideas, and Abbie may have very little idea understanding of the subject matter. We looked back to our initial interviews with each undergraduate and the extent to which they remembered or were able to communicate their ideas in the first interview. Maria's transcript speaks to very good understanding of ratio, proportion, and missing value problems. Her initial image of ratio was *how two different factors relate to each other* and her example was to use a rate comparison of dollars to lollies. When asked how she would explain ratios to students she replied, *You have to know the context. Suppose you have $2, then you can buy 3 lollies. If you have $4, how many can you buy?* To explain her definition of proportion to the interviewer, Maria wrote 2 : 3 = 4 : 6. Though unsure of her ideas, Stephanie was able to create images of ratios relating to her own experiences in cooking and entertaining, growing in her understanding by folding back to her primitive knowledge. In response to the interviewer's request to define "ratio," Stephanie said it was a *proportion like boys to girls. If there were 119 boys in a football club and 1 girls then we write this as 119:1.* When asked later in the interview what the term proportion meant to her, Stephanie said, *My expenditures are not in proportion to my income*. Stephanie was able to solve missing value problems without any additional instruction. While highly motivated to be an excellent eacher, Abbie's primitive knowledge of ratios and fractions was very thin. She recalled that math was her worst subject in school and that she did not enjoy math class. Her teachers were very strict and punished frequently. The only thing she remembered about ratios was how to represent them with a colon. It was necessary for the interviewer to teach Abbie some of the comparison ideas of ratios and fractions so that she was able to continue with the next lesson study activity, the lesson plan. During this instruction Abbie was able to grow in her understanding of ratio to solve several missing value problems. Triangulating the results with the initial interviews, we conclude that the assessments emanating from the instructional representations are valid.

IMPLICATIONS

<u>Conjecture 1</u>. *If lessons do not balance the number of open-ended questions with other instructional representations it may indicate weak subject matter knowledge*. Among some beginning teachers there is the belief that all teachers need to do is to ask students an open-ended question to "inform" the other students. What they fail to realize is that students are not capable of "spontaneously generating" knowledge without instructional

assistance. An open-ended approach requires deep subject matter knowledge in order to design a variety of instructional representations that lead to student understanding.

Conjecture 2. *If the teacher "owns" all of the instructional representations then it may indicate a level of uncertainty with the subject matter*. Teachers own almost none of the representations in the first conjecture, but it is more common to find teachers who own most of the representations. Teachers who are the primary users of the representation in the lesson appear to be collecting images of the subject matter before releasing them to the students. Teachers may be unsure of the subject matter and feels the need to "try out" these new images. They may even fold back to primitive knowledge to make new images during the process of teaching.

Conjecture 3. *If the teacher expects students to "remember" an algorithm or "guess" an answer correctly for a major concept in the plan, it may indicate the teacher's lack of deep understanding*. The primitive knowledge of many prospective teachers includes an algorithmic understanding of the subject matter. They are able to use procedures to solve problems and calculate answers. It is only when they begin to plan their lessons that they realize a deeper understanding of concepts, rules and definitions is necessary.

Conjecture 4. *If a teacher uses many accurate instructional representations, then it indicates a deep understanding of the subject matter*. It requires deep mathematical understanding for teachers to generate multiple representations of the subject matter. When teachers employ multiple representations of ideas and relationships that are accurate, the assessment of strong subject matter knowledge is made. The converse of this conjecture may also be true. If there are few accurate representations of the subject matter, then additional assessment and instruction are warranted.

Many teacher educators assign lesson plans to assess prospective teachers' use of pedagogy. A number of researchers advocate for preparation that moves beyond pedagogy as a primary focus. Programs are advised to incorporate subject matter knowledge along with pedagogy so as to provide prospective teachers with experiences in what and how to teach (Ball, Lubienski, & Mewborn, 2002; Sowder et al. 1998). Here we propose an assessment alternative of subject matter knowledge that moves beyond paper and pencil tests and finding the "correct" answer. These assessments are embedded within teaching tasks of lesson planning that are common to most preparation programs.

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