DIALECTICS BETWEEN THEORY AND PRACTICE: THEORETICAL ISSUES AND ASPECTS OF PRACTICE FROM AN EARLY ALGEBRA PROJECT

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Our teaching conception acknowledges the teacher's central role as a decision maker, influenced by knowledge, beliefs, and emotions. We believe that teachers' education must be focused on teachers' awareness of the complexity of the teaching process, of the incidence of these factors in it, and of the importance of looking at theory as a strong component of their professional development. In this framework, we face the question of the relationship between theory and practice, taking into account some aspects of our Project on Early Algebra (ArAl), which is also an in-service education process. We present the main features of the Project, highlighting not only its influence on teachers' knowledge and beliefs – and, consequently, on their practice – but also the way in which an analysis of such practice has given us a greater awareness of teachers' difficulties in reshaping their teaching, as well as some indications for our future research.

INTRODUCTION

There are many ways of looking at the relationship between theory and practice, depending on the point of view from which you look at the two poles in question. There exists a researchers theory, a teachers theory and even a mathematicians theory, just as there is a researchers practice, a teachers practice and a mathematicians practice. Each of these different combinations provides a different reading key for this relationship. Here we shall concentrate on the most common combination, thus looking at theory, as a body of knowledge on Mathematics Education (ME) in the hands of researchers, and at practice, as the actual teaching carried out by teachers.

ME is also a multifaceted discipline, and various are the beliefs as to what it is or should be. These conceptions underlie the choices of the individual researcher, together with his or her own values, but are rarely made explicit. For this very reason, we prefer to clarify our own idea of ME and of its aims.

We conceive ME as a discipline essentially constituted by problem-driven research (Bishop 1992; Zan 1999; Arcavi 2000), and as a *Science of Practice* - which studies the concrete action of the teaching by carrying out a mediation among mathematics (with its history and epistemology), pedagogy and other disciplines (psychology, anthropology, sociology, etc.), from the integration of which it acquires its own peculiarity and authenticity (Wittmann, 1995; Hiwasaki 1997; Pellerey 1997; Speranza 1997). Using the Stokes-Shoenfeld metaphor (Shoenfeld 2002, p. 446)[1], we see ME in the "Pasteur's Quadrant".

[1] As to the theory and practice relationship, Schoenfeld applies to educational research the perspective elaborated by Stokes (1997) for describing the tension between theory and applications in science and technology. In this perspective, basic research and utility are separate dimensions of research. The various combinations of two dimensions are represented through a Carrol square. The Pasteur Quadrant concerns "use-inspired basic research".

This does not mean that we deny the value of theory. Starting from problems of practice, it is possible to identify conditions that promote (or limit) mathematics teaching/learning, or variables that influence didactical processes, or theoretical constructs that objectivate key elements of didactical processes (according to the way we look at them). Furthermore, it is also possible to generate teaching models, or design innovation plans. A certain type of research can also be developed without any immediate or direct relationship with practice, however in our own thinking the ultimate objective must be that of contributing to the creation of a body of knowledge to be invested in the improvement of the quality of teaching. This vision, of course, depends on our cultural background, and particularly on the historical path through which ME developed in our own country (Barra et Al. 1992, Arzarello & Bartolini Bussi, 1998; Malara 2000).

Internationally, in some cases ME tends to be accepted at the level of pure scientific speculation, with no connection to social reality and the most pressing needs of teachers. Already in the Eighties, some important scholars had pinpointed this separation (Kilpatrick 1981; Freudenthal 1983). Recently Wittmann (2001), also quoting others, has argued in favour of a re-orientation of research forwards practice; moreover, other scholars have underlined that communication and spreading of research results must be increased among teachers (Bishop 1998; Lester 1998; Lester & Wiliam 2002). In particular, Lester & Wiliam have written:

> We promote a renewal of a sense of purpose for our research activity that seems to be disappearing, namely, a concern for making real, positive, lasting changes in what goes on in classrooms. We suggest that such changes will occur only when we become more aware of and concerned with sharing of meaning across researchers and practitioners. (p. 496)

We agree with these scholars, and believe that research in ME, especially when theoretical, finds its natural validation in practice, and that teachers must have access to research results. This validation does not happen only in the daily managing of classroom activities, but also on all occasions when researchers and teachers come together to share ideas on teaching/learning issues (through meetings, discussions, reading journals, planning projects, e-mail dialogues, Web forums, etc). These occasions, when experienced by teachers, cause them to reflect on their knowledge and beliefs; so, in the time, they can refine or (re)construct their professional identity, and acquire a more adequate competency, to face their work according to new educational needs. Of course, for all this to happen, it is necessary for: a) researchers to feel the social purpose of their work; and b) researchers to consider it their social duty to create opportunities for sharing theory with teachers.

TEACHING AND TEACHERS

The socio-constructivist approach to the learning of mathematics has two important implications for teaching. The first is that the teacher figure becomes exalted as a person with an individual interpretation of reality, and in particular of his/her teaching discipline, and of the aims and tools of its teaching (Cooney 1994, p. 612; Arsac et Al. 1992 p.7; Carpenter 1988). The second implication is that mathematics teachers have the responsibility of creating an environment that allows pupils to build up a mathematical understanding, but they also have the responsibility to make hypotheses on the pupils' conceptual constructs and on possible didactical strategies, in order to possibly modify such constructs. This implies that teachers must not only acquire pedagogical content knowledge, in Shulman's sense (1986), but also knowledge of interactive and discursive patterns of teaching (Wood 1999).

The complexities of classroom and school life oblige teachers to continually make decisions. These decisions, even though they often are fruits of practical wisdom, do not only involve the solution of problems arising in the classroom, but also their identification (Thompson 1992; Cooney & Krainer 1996; Jaworski 1998). In this sense, teaching can indeed be seen as a problem-solving activity, but also a problem-posing one.

Lester & Wiliam (2002, p. 494) stressed that "*the speed with which decisions have to be made means that the knowledge brought into play by teachers in making decisions is largely implicit rather than explicit*". Thus, it is important that they are able to recognize and control it. This implies that they must be able to analyse their actions and reflect on the reasons that produced them.

Recent research in mathematics teaching points out the need for teachers to reflect on their own practices (Lerman 1990; Mason 1990, 1998; Jaworski 1994, 1998, 2003). Jaworski (1998, p. 7) uses the following words to define the kind of practice that results from this reflection, i.e. reflective practice: "*The essence of reflective practice in teaching might be seen as the making explicit of teaching approaches and processes, so that they can become the objects of critical scrutiny*." Through reflective practice, teachers become aware of what they are doing and why: awareness is therefore the product of the reflective process.

We consider awareness as an essential element in the construction of a teacher's qualified professional identity, and agree with Mason (1998), who emphasizes that what supports effective teaching is "awareness-in-counsel"[2].

In this framework, we cannot forget teachers' beliefs (i.e. their conceptions, convictions and epistemology about the discipline and its teaching), which always form a strong part of teachers' tacit knowledge and underlie their basic decisions. Thus, it is important to make teachers aware of their beliefs and, moreover, to take into account teachers' beliefs in creating experimental projects. Sometimes, even if teachers agree with the aims of a project and its features, it happens that a teacher's sudden choice can go against the very spirit of a project. But the mismatch between espoused beliefs and beliefs-in-practice can be minimized by making teachers reflect upon it.

Moreover, teachers' decisional processes are influenced not only by their beliefs, but also by their emotions. Context constraints, such as syllabus prescriptions and their interpretation according to their own values and beliefs3, or, more simply, students'

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[2] Mason argues that being a real teacher involves the refinement and development of complex awareness on three levels: i) awareness-in-action; ii) awareness of awareness-in-action, or awareness-in-discipline; iii) awareness of awareness-in-discipline, or awareness in counsel. Mason suggests that awareness-in-discipline is what constitutes the practice of an expert, but what supports effective teaching in that discipline is awareness in counsel.

[3] For instance, Berdot et al. (2001) speak of teachers' traumas due to the disappearance from French syllabuses for mathematical subjects of such elements as radicals and real numbers, deemed by them to be essential mathematical knowledge for the students.

numbers and level, time needed to explain a topic, etc., elicit emotions, which influence a teacher's decisional processes. Time is quite a good example in this respect, because it arouses anxiety. The role of emotions connected to the interaction between teachers and pupils is particularly important in the interactive phases of classroom work, in which there is no possibility of pondering before deciding. Here, too, awareness appears to be crucial, in order to minimize the consequences of this influence.

To summarise, our teaching conception acknowledges the teacher's central role as a decision maker, whose decisions are influenced by knowledge, beliefs, and emotions. We therefore stress the importance of teachers being aware of the incidence of these factors in their own teaching and, moreover, of their living their profession with the attitude of a research – hypothesizing situations and student behaviour, reflecting on what they are doing, and enquiring about the factors influencing their results.

TEACHERS AND RESEARCH

In the study of classroom situations or teaching experiments it is meaningful, but also unavoidable, to take into account the influence of teachers' decisions on their pupils' learning processes.

For a long time, teachers were treated as a "constant" in classroom studies. However, the failure of many innovative programs – even if extremely careful in foreseeing most of the important decisions for the teacher (for example regarding content, activities, and even assessment) – and the difficulties in reproducing experimental situations underline the dramatic importance of the teacher as a "variable" (Balacheff 1990; Artigue & Perrin-Glorian 1991; Arsac et Al. 1992). This research acknowledges the existence of obstacles created by teachers' unforeseen decisions in reproducing teaching experiments. Thus, in order to make research usable, it is extremely important that teachers undergo some preliminary training on aspects that influence decisional processes. Only through a carefully managed training programme can teachers avail themselves of theory, and become able to modify knowledge and conceptions, thus acquiring a new emotional involvement and a greater awareness of their role. This change, however, does not take place through a direct external intervention (where someone says to the teacher "do this, don't do that!"), but occurs as a progressive growth of the teacher's awareness, induced by theory and by reflection on it.

From this point of view, the model of the teacher as a decision maker bridges the gap between pragmatic and theoretically relevant research4. But, in order to make sure that theoretically relevant research has a direct influence on teachers, two preconditions are required:

- Teachers must be able to "absorb" such research; in particular, they must be aware of their role as "decision makers";

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[4] Sierpinska (1993) indicates the distinction between 'pragmatic relevance' and 'theoretical relevance': "*something is pragmatically relevant in the domain of mathematics education if it has some positive impact on the practice of teaching; it is cognitively relevant if it broadens and deepens our understanding of the teaching and learning phenomena*." She observes that, if we accept the idea that the ultimate goal of research is the improvement of the practice of teaching, each theoretically relevant research must be pragmatically relevant, too.

- The research itself must be made available in such forms as are also accessible to practitioners.

The second point is particularly important: if the presentation style of the research is too sophisticated and full of theoretical constructs, the research itself becomes meaningless, if the intended user cannot interpret its language expressions. It is the training process, however, which improves the legibility factor of research materials, as documented in many studies (see for instance Even, 1999; Jaworski, 1998; Malara & Iaderosa 1999). When teachers take part in training projects or in long-term teaching experiments, thanks to the mediation of educators, researchers or even more experienced colleagues, their approach to literature becomes slowly but increasingly friendlier.

Thus, the crucial aspect lies in getting teachers to embrace the idea that theory is indispensable to their professional growth and therefore also to their teaching. This is what we have aimed at in our own country, and has brought about the establishment of the so-called "Italian Model for Innovation Research" (Arzarello & Bartolini Bussi 1998, Malara & Zan 2002).

Following an old tradition, our research for innovation develops into a close collaboration between teachers and researchers. Researchers offer access to theory: they suggest what to read, highlight problems, propose research hypotheses, and in the end act as models in carrying out research[5]. Through the interaction with theory and thanks to the model researcher with whom teachers get in contact, the latter gradually achieve the professionalism of researchers[6]. In particular, teachers-researchers acquire a new awareness of the complexity of pupils' learning processes. This awareness gradually modifies their "practice": the role of researcher creates a new teacher model, which slowly replaces the old one[7]. This evolution is the result of a training process enacted alongside with the relationship with theory, which influences teachers-researchers' choices and decisions, by modifying their knowledge, beliefs, awareness, and emotions.

On the other hand, examining this process from the researcher's side, we can see that, as a result of his interaction with the teacher, the researcher has the opportunity of getting into the live reality of the school world, becoming aware of the conditions in which the teacher has to operate or to which he or she is subjected. This helps the researcher to set research topics into a wider perspective and to co-ordinate research aims with teaching objectives. Thus, this interaction affects not only the choice of research problems, but also the strategies with which to tackle them. In time, this collaborative effort gives the

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5 Borrowing Even's words (1999), committed teachers "build upon and interpret their experience-based knowledge using research-based knowledge and vice versa they examine theoretical knowledge acquired from reading and discuss research in the light of their practical knowledge".

6 Today some Italian teachers-researchers are well-known and independently publish their articles in periodicals and proceedings of international conferences, as well as writing books for teachers. For more details, see Malara & Zan (2002).

7 This is witnessed by this declaration of a teacher-researcher of our group: " *Meeting the world of research puts a teacher in a condition of tension towards a study that, beyond every deadline, never ends, because one sees that knowledge must be built up day by day, it is not a ready-made stock to be conveyed: this is very important and it belongs to the teaching profession as soon as it becomes an attitude to be conveyed, with one's experience, to other teachers too*" (R. Iaderosa in Garuti & Iaderosa 1999).

researcher an ever-increasing awareness of the variety of factors affecting some teaching problems, pushing him or her to tackle ever more complex research challenges. So, if contact with theory (slowly) changes the teacher's decisional processes, and therefore the practice, analogously contact with practice (slowly) changes the researcher's decisional processes, and therefore the theory. The two processes, which we have examined separately – starting either from practice or theory, related to the changes of teachers and researchers – have to be seen as connected components of a same "object", as in a Möbius Strip.

AN EXAMPLE OF A COMING TOGETHER OF THEORY AND PRACTICE: THE ARAL PROJECT

In order to show how reciprocal influences between theory and practice develop, let us examine our *ArAl Project: arithmetic pathways towards favouring pre-algebraic thinking* (Malara & Navarra 2003a), also showing the role acquired by the teachers-researchers, which has become more evolved compared with the past. We shall dwell on some aspects of the project implementation in the classroom, and examine a discussion extract from the point of view of the teacher's decisions-actions. Finally, we shall reflect on the impact these aspects have on our research.

The Aral Project

The ArAl Project was born in 1998[8], within the framework of our previous studies, carried out between 1992 and 1997 and devoted to the *renewal of the teaching of arithmetic and algebra* in middle schools (grades 6^{th}–8^{th}). Among the results of experimentations in middle schools, there became apparent – at the level of meaning for pupils – the strong potential of an approach to algebra as a language to be used in modelling, solving problems and proving (Malara & Iaderosa 1999); but we also found, as indicated in the literature (see, for instance, Kieran 1992), the negative influence of the type of teaching received in primary school, which is essentially procedural and concentrated on calculations results. This led us to consider a possible revision of the teaching of arithmetic in primary schools in a pre-algebraic sense (Linchevski 1995), which became a reality, thanks also to the various training requests from several institutions within the territory. The studies carried out so far within the ArAl Project have confirmed the richness and productivity of the approach implemented by us (Malara & Navarra 2001, 2003a, 2003b), and have also made us consider the possibility of a wider spread in schools.

The Hypothesis

The specific hypothesis on which the ArAl Project is based is that the mental framework of algebraic thought should be built right from the earliest years of primary school – when the child starts to approach arithmetic thought – by teaching him or her to think of arithmetic in algebraic terms. In other words, this means constructing algebraic thought in the pupil progressively and as a tool and object of thought, working in parallel with

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[8] In 2001, the Project won first place in the national S&T (*Science & Technology Education*) competition over close to 600 candidates. The Project is still in progress and currently (2002/03) involves 63 teachers and almost 2000 pupils, in two provinces of north-eastern Italy.

arithmetic. It means starting with its meanings, through the construction of an environment which might informally stimulate the autonomous processing of that we call algebraic babbling[9], and then the experimental and continuously redefined mastering of a new language, in which the rules may find their place just as gradually, within a teaching situation which is tolerant of initial, syntactically "shaky" moments, and which stimulates a sensitive awareness of these aspects of the mathematical language.

The perspective to start off the students with algebra as a language, continually thinking back and forth from algebra to arithmetic, is based on the negotiation and then on the rendering explicit of a didactical contract, in order to find the solution of problems based on the principle "first represent, then solve". This prospect seems very promising when facing one of the most important issues in the field of conceptual algebra: the transposition in terms of representation from the verbal language, in which problems are formulated or described, to the formal algebraic language, into which relationships are translated. In this way, the search for the solution is part of the subsequent phase. From this point of view, translating sentences from verbal (or iconic) language into mathematical language, and vice versa, represents one of the most fertile areas within which reflections on the language of mathematics may be developed, even for the deep differences between the morphologies of the two languages. "Translating" in this sense means interpreting and representing a problem situation through a formalised language or, conversely, recognising a situation described in symbolic form.

Such an innovative vision requires a process of authentic reconstruction of teachers' conceptions in the field of mathematics and methodology, which is also among the objectives of the Project itself.

The Methodology

The methodological structure of the ArAl Project constitutes an evolution, compared to that of our previous studies, which were conducted according to the Italian model for innovation research, and is certainly more complex, not least for the number of schools now involved. It can be seen as a result of the coming back of theory to practice, for the different role played by the teachers-researchers. It has various protagonists: the pupils (P), the teachers-experimenters (TE), the teachers-researchers (TR), the university researcher (UR), responsible for all scientific aspects; all of these variously interrelate with one other. There are two types of privilege relationships, that between UR and TR, and that between TR and TE, which are based on trust and dialogue. The teacher-researcher (TR) plays a strong mediating role between the university researcher (UR) and the teachers-experimenters (TE), both as regards theory (circulating summaries of articles and their comments) and practice. The initial experimental activities are conducted by the TR, assisted by the TE, who provides live models of behaviour for tackling problems, and for the activation and orchestration of discussions. This reduces the TE's fears and anxieties. As the activities carry on, class collaboration between the TE and TR

[9] We employ the " *babbling*" metaphor, because when a child learns a language, he or she masters the meanings of words and their supporting rules little by little, developing gradually by imitation and self-correction, right up to the study of the language at school age, when the child begins to learn to read and reflect on the grammatical and syntactic aspects of the language.

encourages a hot confrontation in the face of emerging habits, stereotypes, convictions, misconceptions, etc., and encourages the TE to express points of view, doubts, perplexities, important indicators of his or her conceptions. The joint analysis of pupil protocols and discussions reveals conceptual knots of the intertwining between arithmetic and algebra, and provides an opportunity to show up conceptualisation gaps in the mathematics education of TE's. These gaps can then be the object of a critical analysis. All these aspects fall back on the research side, favouring subsequent solutions, fine analyses, and in-depth examinations, developed within the TR-RU relationship. This methodology allows for the activity to be conducted on three distinct – yet concomitant – levels (research, experimentation, training), tackling issues that are strongly intertwined between conceptions and personal attitudes, and teaching methodologies. The latter point is the subject of the paragraphs that follow.

The "Units" of the ArAl Project

An important result of the ArAl Project is the creation of various "Teaching Sequences", roughly called "Units", to facilitate communication among teachers. These "Units" were conceived with the aim of producing a wide spreading of the Project itself. They are the result of the progressive refinement of numerous experimentations and are fine-tuned on the basis of cross-analyses of class diaries or records of class activities, and of comparisons of reflections between UR, TR and TE. Their fine-tuning process is very slow, lasting about three years[10].

The Units can be seen as models of teaching processes[11] of arithmetic in an algebraic perspective. They are structured in such a way as to make the teaching process transparent in relation to the problem situation being examined (methodological choices, activated class dynamics, key elements of the process, extensions, potential behaviour of pupils and difficulties they may encounter). The final goal is therefore to offer teachers the opportunity to reflect on their own knowledge and *modus operandi* in the classroom, before actually providing them with didactical pathways that they should follow. Thus, the Units are not tools for immediate use in the classroom, but require a theoretical study,

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[10] This process can be summarised as follows: Selection of Contents : During seminars taking place at the beginning of each school year, the TE's are presented with themes and work outlines, around which the experimental activities of joint classes will be developed. Joint Classes and Meeting Diaries: Each year, 120-140 joint classes (8-10 hours per class) take place, in which both TE's and TR's participate. These joint classes are recorded by the TE (mainly on audio equipment), who are sometimes helped by students from teachers training colleges. Class diaries are a key tool for analysing the teaching/learning process within the Project. From the Diaries to the Units: After being transferred to computers by the TE and reorganised by the TR, the class diaries are periodically discussed in workgroups (nodal points of the teaching-learning process, refinement of certain tasks, teachers or pupils behaviour in different classes, reflections of the teachers, etc., are considered). At the end of each school year, the diaries are reorganised jointly by UR and TR into Teaching Units, which will subsequently be tested on participating classes. The Units in their final version: After these new checks, the Units – consisting of some 25-30 pages – are re-processed and made available on the Net, together with other relevant materials for teachers (the theoretical framework of the Project and related papers, a glossary of clarification of used theoretical constructs, documentation of work of the various classes, etc.).

[11] Of course, these models are not theoretical tools for researchers (Schoenfeld, 2000), but tools for the renewal of classroom practice.

before being put into practice. To this end, the Project's two key tools were created: the Theoretical Reference Framework and the Glossary, which contains more than 70 terms. Through the combined use of these tools, teachers can attain a double goal: the first, immediate and local, concerns the guiding of pupils in the collective exploration of proposed problems; the second one, more general and attainable in the longer term, concerns the objectivation of "hypothetical learning trajectories"[12] (Simon, 1997) as to the subject in question, according to the spirit of the Project. But teachers who intend to embrace these innovative teaching approaches must be prepared to combine their existing knowledge, competences and beliefs with a mix of far-from-marginal methodological and organizational aspects – to stimulate activities with a high metacognitive content, to favour the reflection on language, to promote verbalization and argumentation, to reach a fine analysis of protocols. All these aspects operatively support an actual *culture of change*.

ASPECTS OF CLASSROOM IMPLEMENTATION OF THE ARAL PROJECT – THE TEACHER'S ROLE AND THE RESEARCHER'S POINT OF VIEW

We will now dwell on some aspects that emerged from monitoring an experimental activity carried out in 2002 for and with teachers at their first entry into the Project, but who had participated in a study phase of the Project's theoretical framework; on orchestration work of class discussions (Bartolini Bussi 1998, Yackel 2001); and on a critical analysis of some Project Units. This activity in 2002 concerns the implementation of the Project Unit "From the Scales to the Equations" (Grades 5^{th}–6^{th}). This Unit was conceived working from experience to theory, and uses the well-known scales scheme as an aid to a symbolic representation that can create a semantic basis for the introduction of algebraic formalism[13]. For reasons of space constraints, we shall here concentrate on a single class episode, though many would deserve being reported. It is an extract of a discussion, which is to be read from a viewpoint of the teacher's decisions-actions (see Table 1). At that point in time, the teacher had changed his conceptions of algebra and its

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12 According to Simon, "The hypothetical learning trajectory is made up of three components: the learning goal, the learning activities, and the hypothetical learning process – *a prediction of how the students' thinking and understanding will evolve in the context of the learning activities*" (1997, p. 78).

13 The Unit starts with the simulation of problem situations on the scales, which are then solved by subtractions or divisions of identical quantities from both balance plates. Reflecting collectively on the actions taken to find a solution, students discover 'the principle of equilibrium' and the two principles of equivalence. The problem then arises of how to represent the situations already examined. This phase involves the progressive simplification of the representation of the scales, slowly arriving at the equal sign and the choice of representation of unknown quantities, which leads to the 'discovery' of letters in mathematics and equations. Even the procedures for the solution of equations are progressively elaborated and refined through collective and individual activities, during which students elaborate and compare various representations, refine their competence of natural language translations and symbolic ones, and vice versa, and students, moreover, become accustomed to using letters as the unknown entity. A sequence of appropriately organized verbal problems of different levels of difficulty lead students to investigating how to solve problems using algebra.

teaching, learned to appreciate the value of theoretical study, and already started on Unit experimentation[14].

This discussion was inserted into the representation phase of the problem situations under examination, and concerns particularly the choice of the way in which unknown entities are to be represented. The class had already tackled the problem of representing the scales in equilibrium, which had been solved in a process of progressive simplification, which had brought to the choice of this symbol _!_, which in time was changed by pupils to the "=" symbol. The discussion deals with ways of representing the weight of a packet of salt, rice, etc., and widens to how best to represent the weight of several packets.

This discussion extract highlights how difficult it is for a teacher – however culturally and emotionally committed – to move to an innovative class practice.

As we can see when reading it, it is a problematic discussion, since the teacher, very probably suffering from latent anxiety, and affected by his usual way of being with the class, repeatedly intervenes, approves correct hypotheses at their first appearance, tends to interrupt those contributions he considers less than productive, anticipates the reasons why certain hypotheses must be discarded, does not ask pupils for justifications of their hypotheses, and decides conclusions *de facto*. The positive aspect is that, after a transcript analysis with the RU, the teacher writes in his reflection commentary:

> I tend to impose too strongly the path we must follow. … Perhaps I tackled the problem of the introduction of the letter too hurriedly; but it is important to be aware of this. It will come up again on other occasions, and then we can carry on the discussion.

This is a paradigmatic example, since those factors we had highlighted in the theoretical analysis as affecting the teacher's decisions (knowledge, beliefs, and emotions) appear transparently, and furthermore there is a highlighting – albeit a posteriori – of the importance of the role of awareness. More specifically, (new) knowledge and conceptions are at the basis of the decision to tackle experimentation according to socio-constructive modalities, whilst pre-existing conceptions about the best way to guide the students in the

Teacher: We are going to represent the starting situation, then we'll try to represent the actions we carried out and, finally, the result, that is the value of the unknown quantity… . We have three moments in the symbolic representation… . The first is to define the starting situation… . The second is the description of our actions, and the third is to reach a result, that is, to find the unknown quantity. We have to agree on symbols. Now I'll ask you: the famous 270 g and 50 g, when we come across them, how should we represent them symbolically?

Alex: We could use little drawings of weights.

Teacher: *Alessandro, it seems complicated to draw all the little drawings.*

Margy: We have to write 270 g. … My opinion is that it is essential to specify the unit, since 270 could also be kg, unless we always work only in g.

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[14] This transpires from the following extract of the teacher's reflection: " *It is important to think about paths for deepening the study via a specific bibliography, visits to exhibitions, participation in conventions and seminars. In our own small way, we have had significant experiences in this regard. The relationship with Nicolina has been a very special one: of dialogue, but with strong theoretical and methodological connotations, and based not only on experience, but also on a wide-angle intellectual opening and true personal commitment. This inevitably involves moments of crisis, disagreements, lively discussions - a SEISMIC TREMOR!"* (R.N.)

Teacher: It seems an interesting convention and I would like you to vote on it. … *Do you agree? All of you? If we reach an agreement, we can avoid using a* ***g*** *for grams, just as long as you are in agreement.*

Stefano: I would suggest we write g's on the balance plates… for me it's fine what she said.

Teacher: If you agree to avoid using g for grams, raise your hand… *15 out of 20… It looks like a good majority… Make a note of this criterion and start using it… In the various situations, we'll omit the measuring unit because…?*

Alex: We'll always use grams!

Teacher: I repeat… Good… We must represent some packets, and choose a criterion for their mathematical representation Remember that drawings vary from person to person.

Giulia: We could write the initial letters, only the first letter

Elisa: I have always written all words… but Giulia's criterion is ok too.

Alex: I would have a number in front, when there are 3. I'd put a number before the letter.

Stefano: I would put the unknown entity within a square… When there are several packets.

Marco: I'd like to do like Alessandro says… but writing "3 packets of salt" in full.

Teacher: *It becomes lengthy.*

Margy: I'd have the letter in capitals; in long hand everyone has his own handwriting.

Majid: I agree with Alessandro… but we'll have a "by" before the 3, otherwise we might confuse it with another number.

Alex: But what's the "by" for?

Teacher: Could we not insert it between the *3 and the P?*

Stefano: A dot, because you don't want the "by", sir!

Teacher: Yes, of course, the dot… . Raise your hand all those who want to use the initial letter… . Yes, an overwhelming majority… . Hands up now those who want it capitalised… . Yes, an overwhelming majority… . Now we must decide on the script: block caps or long hand?... Hands up those who want long hand. Nobody….. so it's block caps... Write in your exercise book that the overwhelming majority has decided to use the initial letter, in block caps… . Then there is what Alessandro was saying, with Majid's variant… . *Alex said we should write three packets as "3P"; Majid said we should write "3•P"*. (Here the different options are written up on the blackboard.)

Luca: Perhaps we could write "P 'by' 3".

Stefano: 'by' P 3.

Teacher: But if I need another operation symbol what do I do? I think I have to pull rank here and discard this one…… Or do you want it included? Hands up all those in favour of rejecting Stefano's suggestion… . Yes, an overwhelming majority.

Teacher: This time, each of you must vote for only one of these three: 3P; 3•P); P•3)….. The results, in order: 9, 2 and 7 votes. Let's write this down: "Every time we'll find a number followed by a letter, we'll always mean the internal multiplication… . We'll take for granted the 'by' between the number and the letter". Now we can get on with representing the situation.

Table 1. A discussion on the introduction of a letter to represent a quantity (6th grade class)

classroom – intertwined with tacit emotions relating to the novelty of the task in hand – are subordinate to the choices made by the teacher in conducting the discussion (in Table 1, the teacher's interventions that were anxious, lacking dialogue, or too decisional are highlighted in italics).

Let us now reflect on these experiences from the viewpoint of the impact for the researcher.

This and other episodes analysed by us – in which teachers show that they do not grasp a pupil's reasoning or fail to value and let drop significant contributions, or are conditioned by some pupils' invasiveness, or are even unable to use appropriate silent pauses – highlight how rich and at the same time also how dangerously delicate the situation is, precisely because in the midst of the overwhelming energy of a participating class, "traps" for the teacher lie everywhere (unforeseeable diverging solutions, potentially fruitful but perhaps not too clearly expressed; time that flies; the need to keep alive the pupils' general attention; the need to consolidate achievements, rather than disperse them, etc.).

All this shows us very clearly the importance of a fine teachers education on listening to their pupils, and poses us the hard challenge of how to best help them to "fine-tune their antennas" and acquire that "local flexibility" which enables them to adapt to the flux of thoughts which emerges from the class, to grasp the potentialities, to develop them and adequately insert them into the working context. The task is far from being easy, since it is not a matter of dialogue on a mathematical knowledge level, but on the more complex and delicate level of behaviour – mostly subconscious – that is rooted in the teacher's past life experiences. Furthermore, it is not a question of giving teachers an awareness of what is wrong with the way they operate (what they tend to anticipate or, on the contrary, even to omit in the midst of live classroom action), but rather more a question of heightening this awareness, in order to create a new, more adequate behaviour.

These experiences have made us aware of the fact that we have to implement even finer modalities, to encourage teachers to reflect upon their own actions, thus acquiring new abilities towards "knowing-to-act in the moment" (Mason & Spence, 1999). For example, we deem it indispensable to make use of tools such as video recordings of class interventions (up until now only marginally used in Italian research), to help teachers reflect on their micro-decisions and to analyse the use and incidence of non-verbal language. Needless to say, this "local flexibility" of teachers – deciding for innovation – represents the result of a process which can in the final analysis be defined as "joint (self)education", involving study, comparison and experience.

A further, completely different, and important ground for reflection is for us the incidence of the network of socio-emotional relationships within the classroom (leaderships, power groups, median roles, singles) in the development of discussions. In many cases, we observed rivalries between groups of different sexes[15], complicities between singles, or even a refusal on the part of pupils to involve themselves. In this respect, our teacher writes an emblematic commentary on his experimentation of the same Unit with two classes:

> We must underline the progressive emergence between the two classes of a strong differentiation, with regard to a fundamental theme: the ideological clash on the critical comparison of ideas. In the first class, where this clash was more apparent and wide-reaching, students displayed a positive attitude towards the clash itself. Contrasting other people's ideas was not seen as "humiliating" fellow students, but, on the contrary, as giving them an extra opportunity to show their individuality and personal convictions. In

[15] The social equality between sexes, which prevails in our country, is reflected in the way in which teachers – mainly women – regard their pupils. However, we have been able to ascertain that sexual differences affect aggregations and subsequent performance in the development of discussions.

> the second class, however, where contrast was more limited, there appeared an idea of confrontation as an "encroachment on individuality", thus as a negative event, which should preferably be avoided. What the students in the first class actually sought, was deliberately avoided in the second class, as a threat to established social roles. The second class proved therefore to be a conservative group on the social front. In this respect, we should remember the role played by "dominant girls", i.e. by the group of the "clever girls". (R.N.)

In spite of the unquestionable validity of class discussion as a tool to activate social construction processes of authentic knowledge, these experiences have forced us to address questions that we had hitherto underestimated in our research. All this exemplifies how contact with practice can influence and modify a researcher's conceptions.

6. A BRIEF CONCLUDING COMMENTARY

Until a few years ago, our research was characterised by a joint and peer conduction between university researchers and teachers-researchers. The latter used to actively participate in all research stages, sharing even tacit hypotheses (involving knowledge, beliefs, and emotions), but, above all, they used to carry out themselves their own observations of classroom processes, claiming this role as theirs (Arzarello 1997, Malara & Iaderosa 1999, Malara & Zan 2002). All this rendered our research necessarily *teacher-free*. Being mediated by the teachers themselves, results concerned solely the quality of the educational project as seen from the mathematical viewpoint, and assessed on the basis of the fineness of the student production.

During the last few years, research projects have become more complex, both because of our evolution and because they are intertwined with some major ministerial initiatives for the training both of future and in-service teachers. This on the one hand has allowed and still allows a certain general spreading of research results (not only Italian), but at the same time has also put before us some new scenarios. Nowadays, the focus of our research has of necessity been shifted to the variable "teacher". Our most recent experience makes us see in a new perspective the themes we have traditionally studied, more closely related to a knowledge of pedagogical content, binding with them aspects connected with the teacher's role, the impact of his/her personality, and also socio-emotive issues within the class group. Our shift in perspective necessarily forces a revision of our research methodology, and also shows us the limits and sometimes the naivety of our past research.

Here we conclude. For reasons of space constraints, we cannot go any further with our considerations. We would like to close reviving the idea of a "story", as expressed by John Mason (1994) and Erna Yackel (2001) in their PME plenaries. We too have told our story of the close interweaving between theory and practice. It is an account that we hope will prove helpful to those who in the future will work in our research field.

Acknowledgments

I wish to thank Paolo Boero, Jordi Deulofeu, Giancarlo Navarra and Rosetta Zan for their helpful comments on the draft version of this paper. In particular, I wish to thank Rosetta Zan for her contribution to our previous work on this topic; without our exchanges that took place in those days, this would have been a very different paper.

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