

YOUNG CHILDREN'S ABILITY TO GENERALISE THE PATTERN RULE FOR GROWING PATTERNS

Elizabeth Warren

Australian Catholic University

A common approach used for introducing algebra to young adolescents is an exploration of visual patterns and expressing these patterns as functions and algebraic expressions. Past research has indicated that many adolescents experience difficulties with this approach. This paper explores teaching actions and thinking that begins to bridge many of these difficulties at an early age. A teaching experiment was conducted with two classes of students with an average age of nine years and six months. From the results it appears that young children are capable of not only thinking about the relationship between two data sets, but also of expressing this relationship in a very abstract form.

INTRODUCTION

Mathematics activity is seen as the domain of reasoning about objects and their relations, and involves examining and investigating the truth of claims about those objects and relations (Carpenter, Franke & Levi, 2003). The power of mathematics lies in relations and transformations which give rise to patterns and generalisations. Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning in the research literature (Jonassen, Beissner & Yacci, 1993; Sfard, 1991). Thus mathematics teaching should focus on fostering fundamental skills in generalising, and expressing and systematically justifying generalisations (Kaput & Blanton, 2001). Such experiences give rise to understandings that are independent of the numbers or objects being operated on (e.g., $a+b = b+a$ regardless of whether a and b are whole numbers, decimals, or variables). Ohlsson (1993) names such understanding *abstract schema* and argues they are more likely to promote transfer to other mathematical notions than a schema based on particular numbers or content.

Traditionally, elementary schools give little emphasis to relations and transformations as objects of study. It appears that, as Malara and Navarra (2003) argued, classroom activities in the early years focus on mathematical products rather than on mathematical processes. Strings of numbers and operations in arithmetic are not considered as mathematical objects but as procedures for arriving at answers (Kieran, 1990). Fundamental to relations and transformations is the concept of the function, a schema about how the value of certain quantities relate to the value of other quantities (Chazan, 1996) or how values are changed or mapped to other quantities, referred to in the literature as co-variational thinking.

A common activity that occurs in many early years' classrooms in the Australian context is the exploration of simple repeating and growing patterns using shapes, colours, movement, feel and sound. Typically young children are asked to copy and continue these patterns, identify the repeating or growing part, and find missing

elements; a focus on single variational thinking where the variation occurs within the pattern itself. Approaches for introducing algebra to young adolescents (12-13 years) build on early explorations of visual patterns, using the patterns to generate algebraic expressions (Bennett, 1988). Such patterns are predominantly growing patterns. Students are asked to form the functional relationship between growing patterns and their position, and use this generalisation to generate other visual patterns for other positions, that is, they are asked to reconsider growing patterns as functions (i.e., as a relationship between the pattern and its position) rather than as a variation of one data set (i.e., as relationship between successive terms within the pattern itself). This often involves generating the visual representation, recording data in a table (the position and number of elements at that position), and from the table identifying the relationship between the two data sets. Past research has indicated that many young adolescents experience difficulties with the transition to patterns as functions (Redden, 1996; Stacey & MacGregor, 1995; Warren, 1996, 2000). These difficulties include the lack of appropriate language needed to describe this relationship, the propensity to use an additive strategy for describing generalisations (i.e., a focus on a single data set), and an inability to visualise spatially or complete patterns (Warren, 2000). However young children are believed to be capable of thinking functionally at an early age (Blanton & Kaput, 2004). This research investigates teacher actions that begin to assist young children to view and describe growing patterns in terms of their positional relationships, that is, to begin to bridge the gap between single variational thinking and functional thinking before they commence formal algebraic thinking. The specific aims of this research were to: (i) investigate models and instruction that help young students to create unknown steps/positions in growing patterns, and (ii) articulate the generality of the growing pattern in terms of its position in the pattern.

METHOD

Two lessons were conducted in two Year 4 classrooms from two middle socio-economic elementary schools from an inner city suburb of a major city. The sample, therefore, comprised 45 students (average age of 9 years and 6 months), two classroom teachers and 2 researchers. The lessons reported in this paper were those conducted by one of the researchers (teacher/researcher). The lessons were of approximately one hour's duration. The first lesson focused on copying and continuing simple growing patterns, describing the patterns in terms of positional language, and using this relationship to predict and create the pattern for other positions, for example, the 10th position. In this instance the patterns chosen were those where the links between the pattern and its position were visually explicit. (e.g., a pattern where its width is its position and its height is always 2). The second lesson entailed re-examining some of these patterns, extending young children's language and thinking to describe and predict the patterns for any position, and reversing the thinking (i.e., identifying the position when given the pattern). It was decided not to record the data in a table but to ascertain if children could link the generalisation with the construction of the pattern itself. Using Halford's structural mapping theory,

Warren (1996) found that converting a visual pattern to a table of values increased the processing load, making the task more difficult. In fact, recording data sequentially in a table appeared to encourage single variational thinking, that is finding relationships along the sequence of numbers instead of finding the relationship between the pairs, hence the omission of this step in this research.

Data gathering techniques and procedures

During the teaching phases, the other researcher and classroom teacher acted as participant observers. The lessons occurred sequentially. In each instance the other researcher and classroom teacher recorded field notes of significant events including student-teacher/researcher interactions. Both lessons were videotaped using two video cameras, one on the teacher and one on the students, particularly focussing on the students that actively participated in the discussion. At the completion of the teaching phase, the researcher and teacher reflected on their field notes, endeavouring to minimise the distortions inherent in this form of data collection, and come to some common perspective of the instruction that occurred and the thinking exhibited by the children participating in the classroom discussions. The video-tapes were transcribed and worksheets collected. A pre and post-test were administered before the first lesson and two weeks after the completion of the second lesson. The two tests comprised three questions as shown in Figure 1.

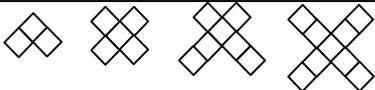
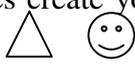
Draw the next step in these growing patterns	
1(a) 	1(b) 
1(c) 	2. Using these two shapes create your own growing pattern. 
3.  1 st 2 nd 3 rd 4 th 5 th 10 th	(i) Fill in the missing steps (ii) Write the general rule for this pattern. _____

Figure 1. Growing pattern questions on the Pre & Post test.

Questions 1 and 2 were included to ascertain children's understanding of growing patterns while Question 3 probed their ability to predict further positions in the pattern and describe, in general terms, the relationship between the pattern and its position. These tests mirrored the types of activities and discussions that occurred during the teaching phase. The delayed post test served to ensure that the responses reflected children's own thinking, rather than simply recalling the discussions that ensued during the teaching phase.

RESULTS

The results of the pre and post-tests (see Table 1) indicated that there was growth in children's understanding of growing patterns and in their ability to describe in general terms the relationship between the pattern and its position.

Table 1. Frequency of response to growing pattern questions on the Pre & Post test

	Pre test					Post test				
	1a	1b	1c	2	3(i)	1a	1b	1c	2	3(ii)
Incorrect	16	21	21	24	5	8	16	13	20	1
Correct	29	22	21	19	31	36	28	31	24	44
No answer		2	3	2	9	1	1	1	1	0

As these results indicated, at the beginning of the teacher phase many children experienced difficulties in simply continuing and creating growing patterns. Even though after the two lessons, many more children were successful in these activities, there were still many who exhibited some difficulties with these tasks. Responses to Question 3(i) indicate that by the completion of the two lessons more children were able to correctly draw the pattern when given differing positions.

The responses to Question 3(ii), the question relating to writing the general rule for a simple growing pattern, were categorised. The responses fell into 7 broad categories ranging from descriptions that gave no indication of the relationship between the pattern and its position to responses that specifically related the pattern to its position. The next section describes each category with a typical response for each.

Category 1. No response.

Category 2. No direct relationship to the question asked (No relationship)

Typical response: You do your original number.

Category 3. Stating that the pattern is simply growing (Single variation)

Typical response: *The patterns keep on growing and growing.*

Category 4. Describing a relationship within the pattern itself

Typical response: *Always the same as the tops as the bottom.*

Category 5. Stating that the pattern is growing in 2 (Quantifying the single variation)

Typical response: *Goes up by two. One more on each end.*

Category 6. Relating the position to the total number of tiles required for that position

Typical responses: Each step number $\times 2 =$ number of *'s or It's double the step number.

Category 7. Relating the position to a description of the pattern

Typical response: *The top and bottom row of the stars is the same number as the step*

Table 2 summarises the frequency of responses for each category for the pre-test and the post-test.

Table 2. Responses to Question 4(ii): Write the general rule for this pattern

Category	Pretest	Posttest
1. No response	12	4
2. No relationship	2	4
3. It grows – Single variation	6	3
4. Relationship within the pattern itself	0	3
5. It grows in twos – Quantifying single variation	22	12
6. Relationship between position and pattern – total number of tiles	2	7
7. Relationship between position and pattern – visual description	1	12

The results indicated that there was significant growth in these children's ability to describe the pattern in general terms, that is, specifically relate the pattern to its position (co-variational thinking).

Examination of the lessons gave some insights into the teaching actions that assisted this growth in understanding. The next section describes some teaching actions that supported this growth and thinking that hindered the generalisation process.

Supporting processes

The use of concrete materials The use of concrete materials appeared to assist many children ascertain the missing steps in the pattern. A number, when completing the accompanying pen and paper worksheet, recreated the pictorial pattern with the tiles and then used the tiles to create the 5th and 10th step. They then drew a picture of their solution on the worksheet.

Patterns where the relationship between the pattern and position were explicit These types of patterns appeared to assist children to verbally describe the relationship between the pattern and the position, for example, it is twice the step number, it is the same as the step number, it is one more than the step number.

Explicit questioning to link the position to the pattern For the pattern presented in Question 3, when asked to describe the 4th step, one child responded that it was 8 tiles. Explicit questions needed to be asked to ensure that the children connected the pattern's shape to its position. These questions were of the form – What does the pattern look like? How many rows? How many in each row? For the 3rd step, how many on the bottom, how many on the top? The questions explicitly related the position to the pattern's visual components.

Generalising from the pattern in small position numbers, to large position numbers. It was found that to articulate the relationship between position number and the visual pattern in general terms, children needed to discuss the relationship for increasingly larger positions, for example, describe the 10th step and the 20th step. Most children successfully completed this task. To ensure that children were linking the pattern's position to the pattern itself, several more discussion ensued, with each time the step number increasing, for example, what would the 100th step look like? 1000th step, 3000th step? While most children appeared to successfully complete this task, on the post-test over half the sample reverted to a single variation description of the pattern.

Hindering processes

Language used to describe the generalisation

Most children experienced difficulty in precisely describing a visual pattern. For example, when they created a 20 by 3 array, most described this as 60. With probing, some indicated it was 3 across, 3 rows of 20 and eventually 3 columns of 20.

Writing the generalisation as compared with saying it orally

The classroom discussions indicated that these children found it much easier to verbalise the generalisation than to provide a formal written response. When asked to share their written responses for the pattern delineated in Question 3, one child shared 'it increases by 2 every time' another 'always the same number on the bottom and the top' and two more said 'Each step has step number on the bottom and top' and 'Just put them in groups of 2 one on each other'. The range of responses indicated that even though many could verbally say the generalisation, when it came to writing it

many experienced difficulties, with the tendency to give responses that focused on the single variation of the pattern.

Completing patterns – single variation

Their propensity to think of growing patterns as adding on the growing part to the preceding step affected their ability to create missing steps within the pattern and to create the step number when given the total number of tiles. For example, we presented visual patterns giving only 1st, 2nd and 5th steps; children were asked to complete the missing steps. The most common strategy was to simply compare the 1st and 2nd step and continue adding on tiles to reach the 5th step. This single variation thinking (Additive strategy) was best exhibited in the following example where they were given the 1st and 3rd step (■&■■■■■). Nearly all of the children gave ■■ as the 2nd step. When challenged they simply recreated the 3rd step to fit their pattern.

Reversing the thinking We also presented the total number of tiles and asked which step this represented. Most children found this very difficult, perhaps for two reasons. First, it relied on linking the position to the step number, which many struggled with, and second in some instances it required a good understanding of number patterns.

Expressing the generalisation in language Many children could not express the pattern in general language, and when using the language there was confusion between the ordinal language and number of tiles.

- T. What if I had the n th position? What would the pattern look like?
 C1 n th on the top and n th on the bottom.
 T. Describe the pattern in terms of the number of tiles.
 C2 n tiles on the top and n tiles on the bottom.

On the more positive side, there were at least five children in each class that could not only describe the generalities in correct mathematical language but also write these generalities using abstract notation systems (e.g., for the n th step there are n blue tiles and $n + 1$ yellow tiles).

DISCUSSION AND CONCLUSION

As indicated by the results of the pre-test nearly half of the children could not complete the next step in simple growing patterns nor create their own growing pattern. This could be for two reasons. First, they had had limited experiences with growing patterns in the early years, or second, growing patterns are not as easy as they first appear. An examination of curriculum documents and commonly used classroom texts would suggest that the predominant focus in the early years is on repeating patterns. These children certainly did not experience the same difficulties with repeating patterns (due to space restrictions this data cannot be reported in this paper). By the completion of the teaching phase there had been some improvement in their ability to complete and create growing patterns, indicating that perhaps the difficulties did indeed stem from a lack of experience in this area. As indicated by past research, many young adolescents experience difficulties with the transition to patterns as functions. The inability to visualise spatially or complete patterns (Warren, 2000) is a key impediment to this. The impact that earlier classroom

experiences have on this thinking requires further investigation. This research appears to indicate that such experiences purposely built into elementary classroom experiences may indeed commence to address this impediment.

The results confirm the conjecture of Blanton and Kaput (2004) that young children are capable of thinking functionally. They also suggest that there are a variety of teaching actions that support this thinking, namely, using concrete materials to create patterns, specific questioning to make explicit the relationship between the pattern and its position, and specific questioning that assist children to reach generalization with regard to unknown positions. Young children are not only capable of thinking about the relationship between two data sets but also of expressing this relationship in a very abstract form.

While young children are capable of thinking functionally, it appears from this research that single variational thinking is perhaps cognitively easier or so entrenched in early experiences that a propensity to revert to this thinking is understandable. It was conjectured that not recording the data in a table would reduce the probability of this occurring. However, instead of looking for patterns in sequences of numbers, they appeared to look for patterns in the sequence of tiles, that is, instead of saying we keep adding on 2 for the sequence of numbers in the table, they said “we add on two tiles as we proceed along the steps”. This thinking was so entrenched that some children were even willing to change the examples given to make them fit their sequential thinking pattern.

The interaction between oral description of patterns and putting this description in written form also requires further investigation. Many children exhibited an ability to express the generalisations orally, but such descriptions often lacked precision. While their oral responses appeared ‘correct’, one wonders how much ‘filling in’ the listener does when hearing the responses to questions asked. A review of the video-tapes indicated that this was indeed the case, suggesting that the precision needed for correct written responses can be missing from classroom conversations. In this instance, gestures and manipulation of materials add to the conversations, elements that are missing from written responses. These children also appeared to lack some of the mathematical vocabulary needed to give precise responses, words such as row and column and describing an array as 2 rows by 4 columns. Thus on many occasions they could model the functional relationship with concrete materials and could attempt to describe this relationship using imprecise language embellished with gestures, they often reverted to ‘lower level’ responses when asked to write their generalization in a written form (e.g., “add on 2” instead of “the number of tiles is double the step number”). This could begin to explain the large variations in responses on Question 3(ii) on the post test, a problem that nearly all could complete and describe orally within the context of the classroom discourse.

This research commences to not only identify teacher actions that support examining growing patterns as functional relationships between the pattern and its position, but also delineate thinking that impacts on this process. Many of the difficulties these

children experienced mirror those found in past research with young adolescents. This suggests that perhaps these difficulties are not so much developmental but experiential, as these early classroom experiences began to bridge many of the gaps.

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