Understanding Division of Fractions: An Alternative View

By E. Fredua-Kwarteng, OISE/University of Toronto, Ontario, Canada.
Francis Ahia, TYP/University of Toronto, Ontario, Canada

***Abstract**: The purpose of this paper is to offer three alternatives to patterns or visualization used to justify division of fraction' algorithms" invert and multiply". The three main approaches are historical, similar denominators, and algebraic that teachers could use to justify the standard algorithm of division of fraction. The historical approach uses the Babylonia definition of division as the dividend multiplied by the reciprocal of the divisor. The similar denominators approach converts dissimilar denominators to similar ones and proceeds as division of whole numbers. The algebraic approach uses the property of division of numbers, along with a little algebraic manipulation to show how the "invert and multiply" algorithm can be justified. The pedagogical merits of each approach are discussed. The paper concludes that difficulties with fractions, especially, division of fractions can be traced to students' and teachers' personal experiences or image. In particular, there is confusion between fractions as mathematical objects and the mathematical operations of fractions.*

Key words: algorithm, denominator, justification, mathematical objects, mathematical operations, pattern, property of division, reciprocal, visualize, whole numbers.

Our observation is that the teaching of division of fractions presents considerable difficulties for elementary, middle, and junior high school teachers. This is because most are unable to provide any justifications for the standard algorithm "invert and multiply." Consequently, elementary school students learn that algorithm and apply it mechanically without understanding the rich mathematical ideas undergirding the algorithm (Sharp & Adams, 2002). However, elementary school teachers who are able to offer any explanations for the standard algorithm usually use logic, creativity and visualization.

For example, let us have a look at this question. Mrs. Smith wants to serve tuna sandwich using bagel bread in her birthday. She plans to cut each bagel into three pieces for the sandwich. (A) How many sandwiches can she make if she has 2 dozens of bagels? (B) While Mrs. Smith was trying to figure out that problem, her daughter rushed to the kitchen saying, "Mom, I have estimated that 48 guests will be attending your birthday party". Based on her daughter's estimate, how many bagels will Mrs. Smith need? For the first question, the teacher may ask students to visualize how many ⅓s are in 24 (12 x 2), given that I bagel is partitioned into three parts. That is, 24 ÷ ⅓ is equivalent to 24 x 3/1 or the inverse of ⅓, which is 72 sandwiches. This pattern is used to justify the standard procedure "invert and multiply".

Nevertheless, Wu (1999) has criticized this approach. First, he states that such an approach gives students the false impression that the only problems they can do are those they can visualize and relate to practical phenomena. He asks, how could students cope with 2/97 ÷ 31/17? Second, he questions how such an approach could help to prepare students to study algebra if abstractions are eliminated in the learning of division of fractions. Third, he argues that teaching students only common fractions – those they can visualize- and nothing else would eventually result in deficient understanding of the division of fractions. Finally, he contends that students tend to develop a sense of insecurity and inadequacy when they are confronted with division of fractions other than common fractions.

To help teachers learn how to justify the standard algorithm, “invert and multiply” Wu (1999) uses the following algebraic approach. According to him, if m, n and k are natural whole numbers, $m \div n = k$, then $nk=m$. In terms of the division of fractions, $a/b \div c/d = x/y$ is similar to $a/b = x/y \times c/d$. To find x/y, multiply both sides of the equation by d/c which is the inverse of c/d. Wu (1999) concludes that by using such approach” The invert and multiply is the result of deeper understanding of fractions than that embodied in the naïve logical and visual thinking skills” (p.3).

Nonetheless, the problem with Wu’s algebraic approach for justifying the division of fractions standard algorithm is that he does not discuss at what grade level teachers could use his approach. We assume that if at the seventh, eighth, or ninth grade level students have a grasp of simple linear algebra, then Wu’s approach is very appropriate. Having said that, we agree with Wu that students should not only be taught things they can visualize, but also things they could imagine. In fact, basing instruction of the division of fractions solely on what students can visualize puts unnecessary restrictions on students' learning and curriculum possibilities (Egan, 2003). Rather, teachers should teach students things they can imagine as well as things they can visualize. This way, students are encouraged to think in both abstract and concrete terms.

The primary purpose of this paper is not to engage in a critique of Wu's view of the teaching and learning of division of fractions. Instead, we want to offer an alternative view to justify the standard algorithm: To divide two fractions, invert the second fraction, change the division sign to multiplication, and then proceed like multiplication of fractions. Radu (2002) has stated that Wu's mathematical justification for the standard algorithm for the division of fractions is just one method among many others. Accordingly, we offer three approaches to justify the mathematical truth of the standard algorithm and discuss the pedagogical advantages of each. Teachers may use these approaches when students demand a justification of the standard algorithm for division of fractions. In the conclusion, we briefly discuss why fractions as a mathematical concept, in general, are so difficult to teach and learn by students(see Mack, 1990,1995).

Historical Approach

We provide a brief historical background of the division of fractions, since this, we strongly hope, would give us an insight into the standard algorithm" invert and multiply". The Babylonians defined division of numbers in terms of multiplication of the dividend and reciprocal of the divisor (Aaboe 1964; Neugebauer and Sachs 1986). Incidentally, nonetheless, this definition is in accord with operations on groups in abstract algebra, where there are exclusively two defined operations of addition and multiplication (Artin, 1998; Stewart, 1995). So, for the Babylonians division of fractions did not present any problems as the division of whole numbers -- since for the Babylonians division of fractions essentially becomes multiplication of fractions.

We illustrate the Babylonian definition of division of fractions with two numerical examples below:

1) 6÷ 1/2= 6x 2/1, because 2/1is the reciprocal of 1/2.
2) 240 ÷ 12 = 240 x1/12, where 1/12 is the reciprocal of 12/1. Consequently, to the Babylonians every natural number is divisible by 1, or every natural number can be written as a fraction with 1 as its denominator.

Algebraically the Babylonian definition can be illustrated as follows: a/b ÷c/d = a/bx d/c, since the reciprocal of c/d is d/c because c/d x d/c = 1. From our observations, seventh grade teachers who present division of two whole numbers as multiplication of the dividend by the reciprocal of the divisor will subsequently experience little difficulties when teaching division of fractions to the students. This is because the underlying mathematical justification for the “invert and multiply” procedure becomes much clear to the students. As well, since fifth graders have no difficulties understanding that any natural number multiplied by 1 does not change that natural number, they should have no problems conceptualizing that any number divided by 1 does change the result. In addition, writing the reciprocal of a number should not present any problems for fifth, sixth, seventh or eighth graders. Thus, teachers could use the Babylonian definition when students at these grade levels demand a justification for the standard algorithm for division of fractions.

Similar Denominator Approach

In this approach, different denominators of fractions are converted into similar denominators. An example will help to illustrate the novelty of this approach. With $2/3 \div 1/4$, the denominators are not the same. So we convert them into the same denominators by multiplying the numerators and denominators of the first fraction by 4 and the second fraction by 3. Thus we have:

{(2/3) x4}÷ {(1/4) x3} = 8/12 ÷ 3/12 = 8(1/12) ÷ 3 (1/12). Now we proceed like division of whole numbers by dividing across: 8÷ 3 = 2 2/3.

This approach has four main pedagogical merits. First, it results in ordinary division of whole numbers which students can handle easily. Second, since addition and subtraction of fractions of dissimilar denominators involve converting them to similar denominators, the same skill can be applied in division of fractions. This ensures a consistent application of that skill. Third, there is no need for students to memorize the standard algorithm “invert and multiply”. All that they need to understand is how to divide whole numbers. Fourth, the relationship between division as an operation and fractions as quotients becomes clear to students (Toluk& Middleton, 2004).

Property of Division Approach

In this approach, we use the following property of division of whole numbers to prove the mathematical logic behind the standard algorithm, “invert and multiply.”

1. Property of Division of Numbers:
   No matter the definition or the method of computation that is used for division of whole Numbers, the following relationships hold:

   i) $m \div n = (m \times k) \div (n \times k)$ where k is not zero.

   ii) $m \div n = \left(\dfrac{m}{k}\right) \div \left(\dfrac{n}{k}\right)$ where k is not zero.

Using these two properties of division in sequence

$$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \times d\right) \div \left(\frac{c}{d} \times d\right)$$
$$= \left(\frac{a}{b} \times d\right) \div c$$
$$= \left(\frac{a \times d}{b} \times \frac{1}{c}\right) \div \left(c \times \frac{1}{c}\right)$$
$$= \left(\frac{a \times d}{b \times c}\right) \div 1$$
$$= \left(\frac{a \times d}{b \times c}\right) = \frac{a}{b} \times \frac{d}{c}$$

There are so many ways this can be established and since we use whole numbers it will fit in with the many approaches to fractions we use. We can also view the above as:

$$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \times \frac{d}{c}\right) \div \left(\frac{c}{d} \times \frac{d}{c}\right)$$
$$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \times \frac{d}{c}\right) \div 1 \quad \text{which gives:}$$
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

2. Relationship between Multiplication and Division

$$m \div n = k \Leftrightarrow m = n \times k$$

And so if :

$$\frac{\mathrm{a}}{\mathrm{b}} \div \frac{c}{d} = q \qquad then$$

$$q \times \frac{c}{d} = \frac{a}{b}$$

But

$$\left(q \times \frac{c}{d}\right) \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}$$

$$q \times \left(\frac{c}{d} \times \frac{d}{c}\right) = \frac{a}{b} \times \frac{d}{c}$$

$$q \times (1) = \frac{a}{b} \times \frac{d}{c}$$

$$q = \frac{a}{b} \times \frac{d}{c} \qquad \text{That is}$$

$$\frac{\mathrm{a}}{\mathrm{b}} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

We can also use the above in the following way:

$$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \times \frac{d}{d}\right) \div \left(\frac{c}{d} \times \frac{b}{b}\right)$$

$$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a \times d}{b \times d}\right) \div \left(\frac{c \times b}{d \times b}\right) \quad \text{The equivalent fractions on right have the same denominator, hence :}$$

$$\frac{a}{b} \div \frac{c}{d} = (a \times d) \div (c \times b) \qquad \text{which leads to :}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{c \times b} = \frac{a}{b} \times \frac{d}{c}$$

Using a numerical example 3/7 ÷ 2/5, we have (3/7×5/1) ÷ (2/5 ×5/1) = (3/7×5/1) ÷ 2/1 = (15/7 ×1/2) ÷ (2×1/2) = (15/14) ÷ 1 = 1 1/14.

There are a variety of ways in which the above algebraic approach can be established, since division of whole numbers fits perfectly with division of fractions. However, this approach cannot be used to teach either elementary or middle school students the division of fractions because of the algebraic manipulations involved. But the approach could be used to teach rational numbers to grade 10 or 11 students, in that in those grades students might have acquired algebraic understanding to appreciate the underlining mathematical ideas and beauty.

Concluding Remarks

Vinner (1991) regards concept definition and concept image (experiences, picture, etc) as two distinct elements in students' cognitive structure and suggested three possible situations in which a definition of a concept could occur:

i) The concept image changes to accommodate the definition;

ii) The concept remains as it is, the definition is forgotten or distorted;

iii) The concept image and definition are both present but not linked together (p.69).

Tall (1991) also states that in formal mathematics students invariably use definitions to identify the properties of a mathematical concept, not the concept itself. Tall (1991) and Sierpinsha (1992) further argue that using a concept to identify its properties may cause epistemological problems for students. From these theoretical perspectives, we can argue that many of the difficulties students have with fractions, in particular division of fractions, can be attributed to their misunderstanding of fractions from their own personal experience. Students and teachers alike use their own experiences or mental picture to formulate the definition of division of fractions which may be wrong; hence, their difficulties with division of fractions.

Fractions are embedded in our daily activities. We encounter them in different settings – shopping, cooking, sewing, money market, assembly plant and building industry. In fact, fractions are mathematical objects that find applications in real life situations. And if the application is taken to be the object, then there is bound to be difficulties in its mathematical operations. For example, if one quarter is one part of a pizza cut into 4 equal parts, how does one make sense of one quarter multiplied by another one quarter? What about the fact that 6÷ 3= 2, yet 1/2 ÷ 1/4= 2? Unless teachers understand these questions, they may not able to explain the answers.

References

Aaboe, A. (1964). *Episodes From Early History of Mathematics*. USA: Random House.

Artin, E. (1998) *Galois Theory*. New York: Dover Publications Inc.

Egan, K. (2003). Start with what the student Knows or with what the Student can imagine? *Phi Delta Kappa*. 84(60), 443-445.

Mack.N, (1990).Learning fractions with understanding: The Case of informal knowledge. *Journal for Research in Mathematics Education*, 27(2), 170-193.

Mack, N., (1995).Confounding whole-number and fraction

Concept when building on informal knowledge. ***Journal for Research in Mathematics Education***, 26(5), 422-441.
Neugebauer, O. and Sachs, A. (edited) (1986).***Mathematical Cuneiform*** Text. New Haven, Connecticut: American Oriental Society.
Radu, M. (2002). Basic skills versus conceptual understanding in Mathematics Education: The case of Fraction Division. ***ZDM,*** 34(3),1-3
Sharp, J. & Adams, B. (2002). Children's constructions of Knowledge for fraction division after solving realistic Problems. The Journal of Educational Research, 95(6),33-380.
Sierpinsha, A. (1992). On Understanding the Notion of Function. In G.Harel and E. Dubinsky (eds***).The Concept of Function***, MAA, Notes 25, 25-58.
Stewart, I. (1995). ***Concepts of Modern Mathematics***. New York: Dover Publication Inc.
Tall, D. (1996). Student Teachers' Conceptions of rational numbers. ***Proceedings of PME20***, Valencia, Vol.4 pp.139-146.
Toluk, Z. & Middleton, J. A. (2004) The development of children's understanding of the quotient: A teaching experiment. ***International Journal of Mathematics Teaching and Learning***. Retrieved September 13, 2006 from http://www.cimt.plymouth.ac.uk/journal/middleton.pdf

Vinner, S. (1991). The Role of Definition in Teaching and Learning Mathematics. ***In D.Tall (Ed.) Advanced Mathematical Thinking (***pp.65-81).Dordrecht: Kluwer
Wu, H (1999). Basic skills versus conceptual understanding. A bogus dichotomy in mathematics education***. American Educator/American Federation of Teachers***. Fall 1999.