

UNJUSTIFIED ASSUMPTIONS BASED ON DIAGRAMS IN GEOMETRY

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We investigated unjustified assumptions made by students when proving geometric statements. Geometric statements can be presented with a diagram or without. Diagrams can be accurate or sketchy. Unjustified assumptions may originate in an accompanying diagram. We thus asked whether the way in which a statement is presented has an effect on unjustified assumptions. We also attempted to find out what motivates students to make unjustified assumptions. Data were collected by means of written questionnaires and individual interviews. The main findings were that among all incorrect answers, 72% were based on unjustified assumptions, and that students make unjustified assumptions with good reasons such as in order to remove obstacles.

THEORETICAL BACKGROUND

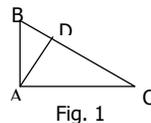
Fischbein (1993) introduced the term "figural concepts" to stress the double nature of geometric figures: conceptual and figural. The conceptual nature includes characteristics such as completeness, abstraction and generalization while the figural nature includes characteristics such as color, size and shape. The conceptual and figural characteristics used when proving depend both, on the conceptual system that includes abstract ideas and concepts and on the figural system that includes mental representations and images. For example, when attempting to make two triangles overlap, concepts like angle, side and triangle are needed as well as figural information like suitable angles and sides (Tall & Vinner, 1981). In every process like this, there is a tension between the conceptual system and the figural system and many of the difficulties in geometry can be interpreted due to this tension. Definitions, concepts and theorems impose characteristics on geometrical objects. However, these definitions and concepts are not always clear to students and often they are forgotten. As a result, the figural component tends to free itself from the formal control and to act independently (Fischbein, 1993; Mariotti, 1997). As a result, diagrams in geometry can be obstacles when proving geometrical statements. These obstacles are divided into three types:

Particularity of Diagrams: Most diagrams in high school geometry are intended as models. They are meant to be understood as representing a class of objects and contain the essence of the situation. Nevertheless, every diagram has characteristics that are individual and not representative of the class. For example, a specific acute triangle ABC which is meant to represent all triangles is by no means a universally valid representation since it does not depict obtuse angles. This obstacle causes students to be trapped by the one case concreteness of an image or diagram which may contain irrelevant details or may even introduce false data (Yerushalmy & Chazan, 1990).

Prototypical Diagrams as Models: If students link a definition to a standard, prototypical diagram, the particularity of the diagram can lead to another obstacle: a prototypical image may induce inflexible thinking thus preventing the recognition of a concept in a non-standard diagram. Students' definitions may include irrelevant characteristics of the standard diagram, causing difficulties in creating or interpreting diagrams (Yerushalmy & Chazan, 1990). For example, among rectangular triangles, the one with the perpendicular sides in vertical and horizontal position is prototypical and students as well as teachers have great difficulty in identifying other positive examples (Hershkowitz, 1989).

Inability to "See" a Diagram in Different Ways: Psychologists often test spatial ability by means of embedded figures tasks, in which a simple figure must be identified in a more complex figure. Yerushalmy & Chazan (1990) consider this sort of reorganization to be a central aspect of mathematical creativity. In geometry, there are situations, where students are asked to do this sort of reorganization. However, the ability to attend selectively to parts and whole does not come easily for many students. According to Hoffer's (1981) formulation, the van Hiele stages suggest that at level 1 (recognition) the student recognizes a shape as a whole. It is only at level 2 (analysis) that the student can focus on parts of a diagram and analyze properties of figures. For example, students may not be able to see AD as a side of triangles ABD and ACD because it is seen only as the altitude of triangle ABC (Fig.1) (Hoz, 1981).

These three obstacles may lead students to make unjustified assumptions, i.e. to assume properties that are not given and are not essential for proving the statement at hand. Geometric statements can be presented with a diagram or without. Diagrams can be accurate or sketchy. Unjustified assumptions may originate in an accompanying diagram.



This led to the following research questions:

- 1a) Do students make unjustified assumptions, when proving geometric claims?
- 1b) Do students make more unjustified assumptions when the statement is given with an accompanying diagram or when it is given without diagram?
- 1c) Do students make more unjustified assumptions when the diagram accompanying the statement is accurate or when it is sketchy?
- 2) What motivates students to make unjustified assumptions?

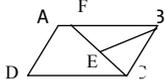
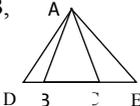
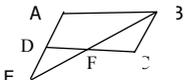
METHOD

In order to investigate these research questions, data were collected by means of written questionnaires and by means of individual interviews. Questionnaires were administered to 92 students from four classes in three different schools in Israel who were enrolled in a full-year 10th grade geometry course. Seventeen of these students were subsequently interviewed. Interviews were audio-recorded and transcribed.

Research Instruments

Questionnaire: Three questionnaire versions were used. All versions included the same three statements and proof tasks: two about parallelograms and one about an isosceles triangle. However, the statements were presented differently in the different versions. The tasks were all within the field of experience of the students, and of a level they could be expected to prove in class or in an examination (Table 1).

Table 1: The questionnaire tasks with the sketchy diagram

<i>Statement</i>	<i>Task</i>	<i>Goal</i>
Given: ABCD parallelogram. BE, CF bisect angles B, C. 	Prove: $FE=EC$	To investigate whether students make unjustified assumptions, such as $FB=CB$ or $BE \perp FC$ with an accurate and a sketchy diagram and without diagram.
Given: $AC=AB$, $\angle DAB = \angle CAE$ 	Prove: $DB=CE$	To investigate whether students make unjustified assumptions, such as $AD=AE$, $\angle D = \angle E$ with an accurate and a sketchy diagram and without diagram
Given: ABCD parallelogram. Point F is the middle of DC 	Prove: $AD=DE$	To investigate whether students make unjustified assumptions, such as: $BF=FE$ with an accurate and a sketchy diagram and without diagram

Each statement was presented with an accurate diagram in one version, with a sketchy diagram in another version and without diagram in the third version. Each version included one statement with an accurate diagram, one statement with a sketchy diagram and one statement without diagram. The order of presentation of the statements was the same in each version (Table 2).

Table 2: The three versions of the questionnaire

<i>Task</i>	<i>Version A</i>	<i>Version B</i>	<i>Version C</i>
1	accurate diagram	sketchy diagram	without diagram
2	without diagram	accurate diagram	sketchy diagram
3	sketchy diagram	without diagram	accurate diagram

Each student was given one version of the questionnaire at random. Writing the questionnaire in three versions was intended to

- a) eliminate the influence of any particular task or of any particular representation,
- b) separate between the three ways of presentation (accurate diagram, sketchy diagram and without diagram), in order to get a broad picture about the influence of each way on students' proving and making unjustified assumptions.

Interviews: Seventeen individual interviews were conducted in order to explore what motivates students to make unjustified assumptions in tasks with an accurate diagram, in tasks with a sketchy diagram and in tasks without diagram. The interviewees were selected according to three criteria:

- between three to five students were selected from each class, in order to represent
- each one of the classes appropriately, five or six students were selected for each version of the questionnaire, in order to get an appropriate picture of each one of the three presentations,
- the students were selected according to their performance in the questionnaire.

FINDINGS AND DISCUSSION

Table 3 presents the results of the three tasks according to the following categories: correct answers, incorrect answers and no answer. The category "correct answers" includes only answers with satisfactory proofs of the statement; all answers with unjustified assumptions were counted incorrect. The numbers in brackets in the cells of the incorrect answers refer to unjustified assumptions.

Table 3: Distribution of correct and incorrect answers (N=92)

<i>Task</i>	<i>Correct answers</i>	<i>Incorrect answers (unjustified assumptions)</i>	<i>No answer</i>	<i>Total</i>
1	60	27 (22)	5	92
2	75	9 (5)	8	92
3	73	14 (9)	5	92
Total	208	50 (36)	18	276

These results show that the majority of students responded correctly to the tasks, while some did not respond at all. But our focus is on the incorrect responses. The important finding is that among the 50 incorrect responses, 36 (72%) were based on unjustified assumptions. Furthermore, in each task, more than 50% of the incorrect responses were based on unjustified assumptions. These unjustified assumptions appeared in 36 responses and were due to 33 students (36% of the population). Thirty students made unjustified assumptions in one task and three in two tasks. This answers research question 1a.

In order to answer research question 1b, we compare the influence of the two presentations, with diagram or without diagram, on unjustified assumptions. Similarly, in order to answer research question 1c, we compare the influence of the two kinds of diagram, accurate or sketchy, on unjustified assumptions. Table 4 presents the results according to the three kinds of presentation.

Table 4: Unjustified assumptions according to presentation (N=36)

<i>Task</i>	<i>Accurate diagram</i>	<i>Sketchy diagram</i>	<i>Without diagram</i>	<i>Unjustified assumptions (Total)</i>
1	6	5	11	22
2	0	2	3	5
3	3	2	4	9
Total	9	9	18	36

Table 4 shows that the number of unjustified assumptions in tasks without diagrams was twice as large as in tasks with diagrams and that no difference was found between the number of unjustified assumptions in tasks with accurate diagrams and in tasks with sketchy diagrams. This answers research questions 1b and 1c.

Table 5: Number of students making main unjustified assumptions

<i>Task</i>	<i>assumption</i>	<i>This assumption</i>	<i>Unjustified assumptions (Total)</i>
1	Drawing the line FG, where G is the intersection point between BE and CD, and assuming that it parallels to BC	11	22
2	AD=AE	2	5
	$\angle ADE = \angle AED$	2	
3	EF=FB	3	9

Table 5 presents in the first column the most frequent unjustified assumptions, in the second column the number of students who made these assumptions and in the third column the total number of students who made unjustified assumptions. For each unjustified assumption, a detailed analysis was carried out according to the following aspects:

- Motives for making the unjustified assumption,
- Using the unjustified assumption for proving the relevant statement,
- Using the unjustified assumption with a backward or forward view,
- Awareness of using the unjustified assumption,
- The effect of the presentation on unjustified assumptions.

In this paper, we present this analysis for only one unjustified assumption, namely the one in Table 5, which was made in task 1. This assumption was the most frequent unjustified assumption made in this study. However, we then present an overview of the results of the analysis of all unjustified assumptions.

As mentioned in Table 5, the main unjustified assumption in task 1 was drawing the line FG, where G is the intersection point between BE and CD, and assuming that this line was parallel to BC. The most common motive for this assumption was to create a parallelogram FBCG (aspect a). Since FBCG is a parallelogram, its diagonals bisect each other and it was therefore easy to complete the proof (aspect b). Almost half of the students who made this assumption, did it with a backward view: looking at the givens, they already had a plan in their mind leading them to the end of the proof. The remaining students did it with a forward view: they considered how to use the givens in order to solve the task (aspect c). Most of the students who made this unjustified assumption were not aware of their mistake.

During the interviews, they were very sure about their answers and did not hesitate at all (aspect d). This assumption was made about equally frequently in all three presentations. Thus, in this case, the presentation had no effect on the unjustified assumption (aspect e).

Discussion according to aspects a-e

a) Motives for unjustified assumption: Two main motives were found: a mathematical motive and a purely visual motive. According to the mathematical motive, students made unjustified assumptions in order to reach a specific stage in the proof, which led easily to the given statement. According to the purely visual motive, the way the diagram looked strengthened the students' feeling and intuition that the stage they wanted to reach (according to the mathematical motive) was correct, because they could see it in the diagram.

b) Using the unjustified assumption for proving the given statement: Students made unjustified assumptions instead of propositions they wanted to reach but didn't know how, or instead of propositions that they believed were correct.

c) Using the unjustified assumption in a backward or forward direction: The students who made unjustified assumptions, did it with one of two views: backward or forward. Those who did it with a backward view, thought of the stages they had to reach, in order to prove the given statement (from end to beginning). Those who did it with a forward view, thought how to use the givens and the relevant theorems and propositions in order to reach the end of the proof (from beginning to end).

d) Awareness of using the unjustified assumption: Fourteen out of the seventeen interviewees were not aware at all of their unjustified assumptions, neither when answering the questionnaire, nor during the interview. Two others became aware during the interview. Only one student mentioned that she was aware of the unjustified assumption while solving the task. She explained that she felt the assumption was correct, but she did not know how to reach it correctly.

e) The effect of the presentation on unjustified assumptions: In almost each task, there was a purely visual motive in aspect a. This motive demonstrates that the diagram affected the students' way of thinking and making unjustified assumptions. There were also cases where students, themselves, built diagrams, in which the unjustified assumptions were shown. Therefore, they used their diagrams as evidence why these assumptions were correct.

CONCLUSIONS

- About four fifths of all answers were correct and only one fifth was incorrect. However, among the incorrect answers, 72% were based on unjustified assumptions.
- The number of unjustified assumptions in tasks without diagrams was twice as large as the number of unjustified assumptions in tasks with diagrams.
- No difference was found between the number of unjustified assumptions in tasks with accurate diagrams and the number of unjustified assumptions in tasks with sketchy diagrams.
- Unjustified assumptions were made with the purpose of reaching a critical step in the proof.
- Unjustified assumptions facilitated dealing with the tasks, removed obstacles and led immediately to the goal.
- Unjustified assumptions were made when students believed they were correct.
- Unjustified assumptions were made when students were stuck.
- In most cases, students made unjustified assumptions without being aware.

In order to prevent the development of misconceptions regarding this phenomenon, teachers should be equipped with appropriate tools for working with their students and have to suggest a variety of problems given in the three ways of presentation.

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