EXPERIENCING RESEARCH PRACTICE IN PURE MATHEMATICS IN A TEACHER TRAINING CONTEXT

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This paper presents the early results of an experiment involving a class of elementary student teachers within the context of their mathematics preparation. The motivation of the exercise centred on giving them an experience with mathematical research at their own level and ascertaining its impact on their attitudes and beliefs. The students spent the first month working on open-ended geometrical topics. In the second month, working alone or in groups of up to four, they chose one or more of these topics then worked on a problem of their own design. The students spent the class time developing their ideas using strategies such as generating examples and non-examples, generalising, etc. Reference to books was not accepted as a research tool, but the instruction team monitored student progress and was available for questions.

INTRODUCTION

The paper examines the experience of mathematics research at their own level to "bring [elementary students teachers] to a point where they know the mathematician in themselves" (Gattegno, 1974, p. 79). Much work has already been done in this direction, under such topics as 'doing mathematics' (ATCDE, 1967; Banwell et Al, 1972; Wells, 1986; Brown & Walter, 1990; Schoenfeld, 1994), or 'investigations' (Banwell, Saunders & Tahta, 1972; Brown & Walter, 1990; Pòlya 1957). A review of writings about mathematics research practice (Hardy, 1940, Woodrow, 1985, Rota, 1981, Restivo et Al., 1993, Burton, 1999), however, shows that investigation and similar class work still contrasts with 'real' mathematics research. In other words, although investigative work in the classroom is a step towards modelling students' classroom experience on the experience of mathematicians, more work is required.

So "what does it mean to *do* mathematics, or to act mathematically?" (Schoenfeld, 1994, p. 55). Though it is often perceived that way by students, "Mathematics is not a contemplative, but a creative subject" (Hardy, 1940, p. 143). This sharp contrast between practice in pure mathematics and the sorts of activities one thinks of in relation to the traditional, or even the reform, mathematics classroom raises the question: How can classroom practice be modified to reconnect with practice in pure mathematics and what effect would that have on the students?

The topic of this paper is an attempt to answer just this question in the context of an elementary education programme. The experiment took place in a service course given by a mathematics department with a strong background in pure research. The course module is one of four mathematics courses required for an elementary education qualification, apart from the teaching methods component. It took place bi-weekly, and was given to 37 students, 33 of which were female. The majority of the students were Juniors (3^{rd} year) or Seniors (4^{th} year), with at least 4 graduate students completing prerequisites for a Masters' programme. Of the 35 students who completed the pre-course survey, 3 assessed themselves as having excellent mathematical ability, 15 as being competent, 14 as average, and the remaining 3 as either weak/poor. 25 of the same 35 acknowledged preferring algebra, 4 geometry, and 6 arithmetic. The mixture of self-determined ability and attitude in the class added another dimension to the experiment, particularly pertaining to the students' choice of topics for their projects. This resulted in a wide range of levels of mathematical thinking and creativity, analytical rigor, and others.

PEDAGOGICAL COURSE OUTLINE

The course was carefully designed around several didactic methods in order to optimise the classroom environment for supporting the anticipated learning approach. First, a significant part of the course curriculum, which is normally dealt with using the assigned textbook, was left to the end of the term in order to establish an unthreatening power relationship between students and instructors. The intention of this move was to give the students control of the knowledge, making their own experience the key, emphasising that the instructors *did not* have the answers, nor was there a textbook which does. Second, the project work was made the core of the course, not an extra, a 'Friday afternoon activity', by making it the main part of the grade. Third, the students were asked to collect all their work into a final portfolio and were allowed to resubmit previous assignments for additional points. Finally, the teaching style of the instructors and the contribution of the teaching assistant were carefully monitored and directed in order to restrict the possibilities of the instructors 'taking over' the student's project, or leading them to the answers.

September

The term was subdivided into three main parts. In September, the students were introduced to some mathematical topics and encouraged to develop ways of thinking mathematically emulating 'real' mathematics research as discussed in the introduction. In order to facilitate this development, a portion of the first class was spent in a whole class discussion of a mathematician's role and job. In addition, the course guideline placed special attention on the process, communication and outcomes of the explorations. Finally, the 'mini-projects' developed for September were designed as templates for investigations, in addition to providing the topics, mainly by keeping them as open-ended as possible. The students were given a chance to practice using mathematical research strategies. They were taught to think in terms of examples and counter examples, cases and constraints, patterns and systems of rules, conjectures, justifications and proofs. They were asked why rules they conjectured might always be true, or in which cases, and they learned to frame problems. The intent of this part was for the students to "do a non-mathematical activity and then reflect on it mathematically" (Morgan, 2003). In order for the students to focus on this mathematical thinking, the mathematical topics were approached in a very accessible manner. The topics included geometrical subject matter involving proper colourings[1] of regular triangular grids and simple polyhedra, symmetries of the same, the Euler characteristic, and others.

The approach entailed visual or manipulative activities using paper and colouring tools or modular connector sets such as Polydrons. The activities began with low levels of symbolic notation and terminology and ramped up to more sophisticated modes of communication, including use of graph theory to represent polyhedra, use of higher level counting techniques to verify colourings and symmetries, and use of precise language such as duality of configurations. Throughout this part of the course, the students were asked, as part of their homework assignments, to think of an 'interesting thing to look at next' in the context of the class work.

October

Following the training stage, the students were asked to pick topics for their projects, based on their ideas for 'interesting things to look at next'. They were expected to pose, and attempt to solve, their own mathematical problem, taken from one or more of the topics investigated earlier. It was made clear that their work should be "a step into the unknown. [...] The principal hope for an investigation [being] that totally unexpected things turn up, that different kinds of approaches to problems should appear as different pupils tackle different aspects of the problem in different ways." (Driver, 1988). Library research was therefore deemed unacceptable as mathematics research does not consist of looking up results in books and then presenting them. The students broke up into 19 groups, one of 4, three of 3, nine of 2 and six single people. Interestingly, every group of students came up with a topic they wanted to explore further. Though the instructors had prepared a list of 'fall back projects' for students who were not finding a starting point, these were unnecessary.

In the first week, the students submitted a proposal for which they received very restrained feedback, typically "would be very interesting mathematically", or "there is lots to explore here". Though in many cases, the proposed problem had to be reframed during the evolution of the project, this was declared acceptable, even normal in this context, as in 'real mathematical research'. This is typified in Schoenfeld's (1994, p. 59) example of the reformulation of the 'natural' definition of polyhedra: "New formulations replace old ones, with base assumptions (definitions and axioms) evolving as the data comes in." (Schoenfeld, 1994)

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[1] A proper colouring is a way of colouring a tiling or polyhedron so that no two tiles or faces sharing an edge or side have the same colour.

In the following weeks, the students spent class time working on their projects, and the teaching team approached each group or individual in turn. This interaction was kept deliberately very hands-off, preserving the students' ownership of their projects. The guidance provided by the instructors was restricted to "validating [the students'] efforts" (Drake, 2003) and suggesting general strategies, preventing the closing off of any avenues of investigation. For example, Patrick (pseudonym), a student who was interested in examining the relationship between the number of faces of regular pyramids and their symmetrical proper colourings, was only considering pyramids made of regular polygons. This meant that he only had three cases to work with (the triangular, square and pentagonal pyramids, as the hexagonal case is degenerate). When it was suggested that he eliminate the regular polygon constraint, he was suddenly faced with an infinite class of polyhedra, giving him room to develop a conjecture involving the connection of primality and symmetry.

During October, the students were given the opportunity to work on mathematical problems of their own devising, at their own pace and at their own level, using methods of their choice. Though the pace, level and methods varied greatly, the class work in October was productive in most cases, and there were very few students who missed class, and in most cases, the students met and worked outside of class time. At the end of this part, the students were asked to hand in a 'first draft' of their project report. This document, which each student had to compose individually, including group members, contained three parts: First, the student described the initial motivation and topic chosen in September. In the second part, the student described "the project as it evolved, showing [their] work and thinking in full detail for each stage" (from guidelines). This part was expected to be the most extensive and was requested in order to emphasise to the student the importance given to the *process*, including direction changes and blind alleys. In order to help the students write up their work, they were given the following quote from the Gian-Carlo Rota's introduction to *The Mathematical Experience*: "A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure our minds are not playing tricks." (Rota, 1981)

Finally, the write up was to contain "a summary of [their] findings from section 2, referring to key examples and/or counter-examples. A precise statement of your claims and reasons why" (from guidelines). This section was considered to contain the results of their work, in contrast to the process of the previous section.

November and December

In November and December, regular classroom activities resumed, mainly for the purpose of course completion. The students were also given the opportunity to revise their project reports and everything, including homework re-submissions, was handed in at the end of term. Additionally, in the last few classes, the groups all gave an oral presentations of their projects.

STUDENT WORK

Many things can be said about the various projects. The level of enthusiasm and creativity achieved by the students, in particular, was unexpected. For example, one of the students began working on his project in the middle of September, not even waiting to hear the rest of the topics. Another group went way beyond the anticipated results, discovering a whole class of polyhedra, unknown to any of the instructors, with the property of self-duality. In order to accurately portray the work that was done in this experiment, both by the students and the instructors, this section will detail three projects. The description will place particular emphasis on the quality of the interaction between the students and the instructors. This is particularly important to illustrate the method of instruction, which is very difficult to pin point and relies heavily on teacher self-confidence and experience in mathematics research.

Project 1: relationships of properties

Carol and Samantha (pseudonyms) investigated proper colouring of polyhedra. They attempted to relate the number of edges of a polyhedron to the minimum number of colours required for a proper colouring. In order to investigate this problem, they began by generating 7 polyhedra with triangular, square and pentagonal faces, including a pentagonal dipyramid, a triangular prism, a cube with a pyramid on top, and a pentagonal anti-prism. They then generated tables of data, including descriptive tables containing the number of faces of each type for each polyhedron, and the number of faces, edges and vertices versus the minimum number of colours required.

They found that all their polyhedra could be coloured properly using either 3 or 4 colours. They also noticed that the pattern seemed to obey the following rules: “if the number of edges is divisible by 3, then the polyhedron needs 4 colors for proper coloring to occur” and vice versa. These rules are not mutually exclusive, and therefore their conjecture contains an internal flaw. This was not as dire as it could have been, however, as they disproved the validity of their conjecture using one of their existing shapes. The next conjecture they made concerned the number of faces at each vertex in relation to the number of colours necessary. Again, they found a counter-example in their collection. Though the students in this group did not arrive at a clear mathematical result, they made strong use of the general strategies that were developed in September. Interaction with the instructors was limited, for two main reasons. First, because of an “Artefact of the geography” (Drake, 2003): though the class was large, it was held in a very narrow room, and not all students were easily accessible. Second, they mostly kept to themselves and showed little interest in discussing their project during class. Suggestions were generally not followed. For example, the researcher suggested that the students concentrate on what forces the jump from 3 to 4 colours when constructing the polyhedron: “watch yourself colour it and focus on when it happens”. Interestingly, in the various write-ups and presentation, this group repeatedly emphasised their intention of working on a project that was relevant, in their view, to their future work in elementary education.

Project 2: mathematical creativity

Rob, Sandra and Jo's project was the one least connected to the September topics. They developed a mathematical system all of their own, involving a class of transformations, which they called manipulations, then explored the rules and constraints of the system. According to their definition, a manipulation is "either a change in length of a side or a change in angle measurement from a base shape." As their base shape of choice was the square, their manipulations netted them rectangles, parallelograms, etc. The subsequent exploration involved the determination of interdependence between the manipulations. In effect, can manipulation (a) occur without anything else changing, and if not, what does or can change. The types of problems they encountered early on concerned their counting and classifying system. If two opposite sides change length, does it count as one or two manipulations? They were also looking to find all possible resulting shapes, and finding their properties, including parallelism, symmetries, equality of length or angle. Though all this could easily be expressed in regular elementary school geometry terms, the significance of the project is in the students' 'stepping into the unknown' and in their attempt at formalising a system of rules they themselves had created and finding its structure.

Suggestions were made by the instructors throughout. An early example proposed that the students look at the 'manipulations' another way, for example as a displacement of a component, an edge or vertex, of the original square. Later, the course instructor suggested the students look for manipulations that can occur without others following, or which force others. Another suggestion involved the use of combinatorics to find all the possible combinations of manipulations, then eliminating the impossible ones. Two of these suggestions mainly centred on the *approach* to the question, the specific techniques used, rather than giving a redirection that would bring the question back to more conventional mathematics, or promote a solvable question. As for the one concerning interdependence of manipulations, it was made at a point when the group was looking for a direction to take, and therefore was timely and useful, and was in fact followed right up.

Project 3: analytical rigor

Although Patrick began his project with a very open field of study, things soon narrowed down to the relationship of symmetries of pyramids with regular polygonal bases to their proper colourings. He wanted to find out how many colours were necessary to colour a given pyramid, and how a resulting colouring impacted its symmetries. His first result showed that the base will always need to have a separate colour to all the other faces, since it touches them all. Following this, he found that pyramids with a base polygon having an even-number of edges will only need 3 colours, and that odd-numbered bases force a fourth colour. Examining the symmetries of the resulting pyramids, he found that reflection symmetries are only possible on even-numbered pyramids, as the odd ones have reflection planes passing through the edge between two sloping faces, which contradicts the proper colouring rule. He then continued his investigation by looking for symmetries in higher order pyramids. The results of this investigation showed that the factorisation of the number of sides of the pyramid plays an essential role in determining the symmetries of possible proper-coloured pyramids.

Though this project exemplifies a high level of mathematical rigor in the range of this class, it did not occur without consistent feedback on the part of the teaching team. The problem was narrowed down early on when the researcher suggested the student look at a class of simple cases, before the result was generalised to the student's initial choice. As mentioned in the course outline section, the field was opened up to include non-equilateral pyramids. Further along, a simple remark sent the student back to the work table: "Ah, but is this the only way to 4-colour this pyramid, or can you do it so some symmetry is conserved?" Though again the suggestions and feedback were designed not to close the question, but rather to facilitate a clearer view, Patrick made very intense use of them.

EARLY RESULTS AND CONCLUSION

The purpose of this experiment, from the research standpoint, concerns the students' potential change in view and attitude towards mathematics as well as the acquisition and development of general mathematical strategies. In order to evaluate this change, data were collected from all the participants. The students filled out a survey before and after the course, the discussion of the first class was recorded, the researcher observed and took notes in each class period, extensive interviews of the main instructor and teaching assistant were recorded, and everyone kept a journal.

Though the data have only just finished coming in, early observations can be made. In the journals, students expressed frustration in September and sometimes in October, though this was often temporary, and the following entry would express enthusiasm. In any case, this kind of frustration is quite common in mathematics research, and can even be a good sign. After all, as John Mason says in Thinking Mathematically, "Being stuck is an honourable state" (p. ix). In the after-course survey, the students were asked how they felt their project fit into their view of mathematics. 20 of the replies contained the word exploration or process. Terms used in other responses included creativity, innovative, and possibilities. When asked to rate which stage of the course they felt most useful for specific topics, 20 chose October as the most interesting. 20 chose October as the most instructive about teaching, while 11 found September to be most instructive. Though 23 students found that they learned most about mathematics in November (during regular book work), 15 found October the most instructive about mathematics. Finally, according to 34 students, their work in October emulated most closely the work of professional mathematician. In both pre- and post-course surveys, the students were asked to select three nouns that best describe mathematics as they view it. The following table shows the top three choices before and after the course, along with their score.

An exploration	Score before: 0	**Score after: 24**
Formulas	**Score before: 14**	Score after: 8
Numbers and Operations	**Score before: 20**	Score after: 5
Patterns and relations	Score before: 4	**Score after: 23**
Problem-solving	**Score before: 29**	**Score after: 23**

Table 1: Nouns most selected before and after, and their respective scores

Though these results only present a tentative view at best, the analysis promises to be interesting. In particular, the process section of the project hand-in, together with the journal entries, should prove enlightening.

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