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ABSTRACT

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ABSTRACT.

A Methodology in the Teaching Process of the Derivative and its Motivation.

The development of the derivative because of being part of calculus in permanent dialectic, demands on one part an analytical, deductive study and on another an application of rochrematic methods, sources of resources, within calculus of derivative which allows to dialectically confront knowledge in its different phases and to test the results. For the purposes of this study, the motivation of the derivative in calculus, the characteristics of motivation, correlation between theory and practice, the concrete and abstract, creativity, and the systematization in the teaching of derivative in calculus, are presented.

Title: A Methodology in the Teaching Process of Calculus and its Motivation.

To form a first idea of what motivation represents, and at the same time detect the different parameters to which Motivation can respond, one thinks on the following question:

Why has Calculus been chosen and not another science?

Each student must think about his experiences and express his ideas.

The reasons which have been presented most frequently are these:

- One has EASE for this discipline.
- GOOD NOTES have been obtained.
- There is GOOD DEMAND for Calculus teachers.
- By COINCIDENCE (there was no professor who described my abilities).
- One wants to prove to the previous teachers that one IS CAPABLE.
- By the POWER OF PROVING THAT CALCULUS HAS.
- Because Calculus is PLEASANT.

1.1 WHAT IS MOTIVATION.

Motivation can be understood as a psychological impulse in a person toward the achievement of some goals, to bear in mind that man by nature needs to be motivated to reach the full achievement of his abilities and his will. On the other hand, one must be aware that each person acts according to a motive, to a need and to a personal expectation; so one should not be surprised that a certain activity stimulates some and does not affect others. One must be also aware that motivation cannot be detached from the person's reality, and still less insist on forced motivations.

> "To think that the pleasure of contemplation and the search can be favored by means of coercion and the sense of duty is nothing but a large-bore mistake." EINSTEIN.

> "With such procedure, taking place without punishment nor rigor, light and softly, without any coercion and as in a natural manner..."
> COMENIUS

1.2 CHARACTERISTICS OF MOTIVATION

The following are the main characteristics of motivation:

1.2.1. The motivated behavior is cyclical. First appears a desire or impulse, then a means of satisfying or reducing such impulse, later comes the achievement of objectives and finally one returns to the initial state.

1.2.2. Motivation is selective. The person tends to satisfy in first place the needs which respond to a specific motive or to the strongest stimulus, to later satisfy others.

1.2.3. The motivated behavior is active and persistent. To more motivation always corresponds more activity and the person insists in the achievement of the inherent objectives to the type of motivation received.

1.2.4. Motivation is homeostatic, that is, conscious or unconsciously it is constantly bolstered up.

1.3 MOTIVATION IN TEACHING.

It is thought: How to motivate teaching?

That is in accordance with the studies and experiences, what methods would be used to motivate teaching? After listening and discussing the opinions of the students it is possible to conclude that motivation in teaching can be done by the application of the following didactic principles:

- Activity in teaching.
- Correlation between theory and practice.
- Unity between the concrete and the abstract.
- Systematization.
- Correlation with other sciences.
- Applicability.

1.3.1. These didactic principles do not function ones independently from the others, but they are intimately related; because of that, when speaking of one of them, one speaks direct or indirectly of the rest.

1.3.1.1. Activity in the teaching.

The infant is active and restless by nature, his organs develop and strengthen through physical and mental activity, but he must be concerned that such development is harmonic providing adequate activities, pleasurable for him and the development of his will and his creativity.

It is also said that activity in the teaching eliminates or minimizes the magisterial lectures which produce so much tedium in the students.

> "The magisterial lecture can be useful if the students solicit it as a means of elucidating issues already debated." The new pedagogy. Salvat Library, G.T.

> "The laws of Physics are not enunciated; one trusts that they are discovered by the students." GOODMAN

1.3.1.2. Correlation between theory and practice.

In the teaching there must be a biunivocal relationship between theory and practice. One cannot be without the other not risking falling into an incomplete and partialized teaching. Practice needs the enlightenment of theory and in turn, theory requires the sense that practice provides.

The importance of practice in the fixation of knowledge has been experimentally proven in the following way: A group of students is explained how to do a certain experiment; another group watches the teacher while he carries it out; a third group does the experiment by its own means. After six months the fixation level of the first groups is similarly low while the third group shows a high level of fixation.

1.3.1.3. Unity between the concrete and the abstract.

Following the didactic suggestion that in the achievement of knowledge one must go from THE NEAR TO THE REMOTE, FROM THE EASY TO THE DIFFICULT, FROM THE KNOWN TO THE UNKNOWN; the following steps are recommendable.

Contact with the	→	Graphic representation of	→	Deduction and analysis
objective reality.		the objects and the		of properties.
-		relationships between		Formulation of
-		them.		hypothesis, postulates,
-		-		theories...
-		-		-
-		-		-
CONCRETE	→	DIDACTIC	→	ABSTRACT

The recommendation of beginning the teaching in the concrete is based on the proven fact that the more SENSES are used in the acquisition of knowledge, the more solid the learning is.

> "A good characteristic is worth more than entire volumes of political oratory." LANCELOT

> "There are things which having been presented clearly by Mechanics, have been later proven by Geometry, because the first method lacks the proving force which is inherent to the second. It is frequently easier, after having, thanks to the mechanical method, a first idea on the issue, to elaborate the demonstration than without having the preliminary mechanical experience." ARCHIMEDES...

One is aware, none the less, that a good learning does not remain in the concrete, but it moves forward toward abstraction in an organized and conscious manner on the individual's part.

1.3.1.4. Systematization in the Teaching.

The systematization is of vital importance in all human activities, especially in those of the educational kind. Calculus could be named the Science of Order and an exquisite organization of the contents is therefore essential to achieve greater clarity in the teaching as well as in research.

Some of the aspects that must be taken into account for a good systematization of the teaching of Calculus are the following:

Organize the global content into units, each unit divided into parts and each part with one or more gravity centers.

Link the known subject with the new one.

Constantly strengthen (revise, insist) the contents already seen.

Regularly control and evaluate the subject, the method, the teacher...

Have a sequential order.

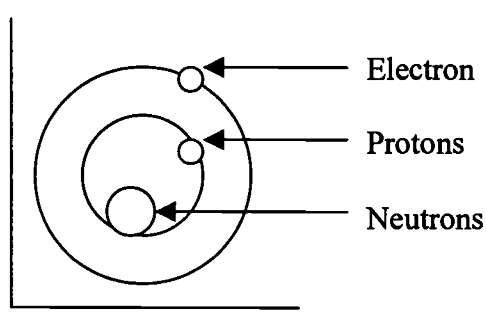
1.3.1.4. Correlation with other Sciences.

Every discipline that is a study object must be related with other areas of knowledge so it does not lose sense, when appearing isolated from the curricular context and so it awakens the most diverse interests in the students. A way to achieve correlation or integration of different subjects is through applications.

Example: Calculus and Archaeology.

Who knows the C^{14} method to determine the age of materials of organic origin?

Discuss for about 15 minutes and afterwards present before the class.



When adding neutrons to the nucleus, the latter becomes unstable and emits energy in the form of radiations.

C_{14} ATOM

Fig. 1-1

In living beings the C^{14} percentage is of 0.002%. Upon death, the percentage of C^{14} decreases when emitting energy in the form of radiations, then the age is determined according to the amount of C^{14} in the remains.

Be:

m: mass of one radioactive atom in the moment "t"

"m" is a decreasing function of "t" since "m" decreases by the emission of radiations.

The variation in the amount of matter is proportional to the mass in a given moment:

$$\frac{-dm}{dt} = km$$

$$\frac{-dm}{m} = kdt$$

$$-Ln(m) = kt + c$$

$$Ln(m) = -(kt + c)$$

$$m = e^{-kt} - c$$

$$m = e^{-kt} \cdot e^{-c} \quad (1)$$

For $t = 0$. $m_0 - e$

Questions to pose:

1) How is e^X defined?

A: $e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots + \frac{X^n}{n!}$

There is a reason for it being called exponential.

$$e = \underset{x\to\infty}{Lím}\left|x + \frac{1}{x}\right|^x; \quad e = \underset{h\to 0}{Lím}(1+h)^h$$

2) Why that function is important?

A: Because the change or variation of the function is proportional to the original:

$$\frac{de^x}{dx} = e^x$$

For what time "t", the mass will be reduced to half?

In (1)

$$\frac{m_0}{2} = e^{-kt_1} \cdot m_0$$

$$Ln\left(\tfrac{1}{2}\right) = -kt_1$$
$$kt_1 = -Ln\left(\tfrac{1}{2}\right) \qquad t_1 = \frac{Ln(2)}{k} = \frac{0.693}{k} : \qquad \text{Half-life period.}$$
$$kt_1 = Ln(2)$$

In the case of Radium, k = 0.00064/year.

$$t_1 = \frac{0.693}{0.00064} = 1.575 \ years$$

1.2.1.6. <u>Principle of Rochrematic Applicability</u>.

The rochrematic applications play a very important role in the teaching of science and very specially in the teaching of Calculus.

"Of everything perceived, consider at once what use it may have so nothing is learned in vain" COMENIUS.

Frequently one finds students who say: THAT, WHAT FOR? Attitudes like this show lack of interest or motivation. The student is not able to see neither the importance nor the usefulness of the subject. To avoid these situations it is CONVENIENT to propose certain typical problems to the class and when the student finds out that he cannot solve them, and then explain the necessary theory for it.

Some examples of how the above can be related:

SOME PROBLEMS THAT DRIVE YOU TO THE DERIVATIVES.

1°) The slope of the tangent to a curve in one of its points is the limit of the incremental crescent $\frac{\Delta y}{\Delta x}$, when the $\Delta x \to 0$. If this limit exists it will exist also tangent to the curve in the point.

Abstract: Slope of tangent = $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, (1)

2°) The velocity of a varied movement in an instant, is the limit of the incremental quotient $\frac{\Delta e}{\Delta t}$, when the $\Delta t \to 0$.

Abstract: Velocity = $\lim\limits_{\Delta t \to 0} \frac{\Delta e}{\Delta t}$, (2)

3°) The density of a heterogeneous substance, in a x height from the bottom of a recipient, supposing that the quantity of substance is a function, C = C(x), of the height, it comes by the limit of the incremental quotient, $\frac{\Delta C}{\Delta x}$, when the $\Delta x \to 0$.

Abstract: Density in a height x=lim $\frac{\Delta C}{\Delta x}$, (3)
$\Delta x \to 0$

If we notice the expressions (1), (2) and (3) we observe that the three of them have the same mathematical structure: They are limits of increasable quotients, that is to say, limits of the increment from a function divided by the correspondent increment of the variable, when the increment of the variable tends to zero.

Let's see new examples.

4°) Instant Intensity of an Electric Current.

The intensity of a current it is defined as the reason of the quantity of electricity that has passed from a section of the conductor, at the inverted time in the crossing.

Now two cases can be presented: a) That the flux of current is uniform, this is, that any time that is considered, the reason of quantity of electricity by the time it is constant, this is that the intensity is constant. B) That the intensity is variable, they are moments when it passes more electricity and moments when it passes less. In this case it can interest us how much the intensity of the current in a certain instant, *to*, is. For that it is proceeded in this way: It is considered an interval of time that includes the instant *to*, *Δto* and the quantity of electricity that passed at that interval of time *Δco* is by the same time: $\frac{\Delta C_o}{\Delta t_o}$, (4); this reason will give us an approximate value of the intensity of the current in *to*, as approximate as small is the Δto of time that surrounds the instant *to*; if we want the exact value of the current intensity in the instant *to*, we will find the limit of the incremental quotient (4), when $\Delta to \to 0$, getting then to the next conclusion: Current intensity in an instant $to = lim \frac{\Delta C_o}{\Delta t_o}$, (5)

$\Delta to \to 0$

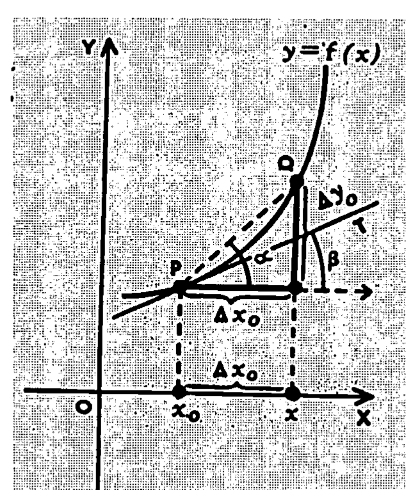
5°) Velocity of a chemical reaction.

When by means of a chemical reaction a certain substance is forming, the rate obtained when a quantity of substance formed in a certain time is divided by such time, should be considered; if this rate is constant, whatever the considered time is, receives the name of velocity of the chemical reaction; but the most common case is when the velocity of the reaction is not constant, but that in some instants more substance is formed than in others. In this case the velocity of reaction in the instant *to,* it can be interesting for us. To find it, it is proceeded this way: An interval of time, Δto, is considered, that includes *to*, as center, and the quantity of substance, A Co, that is formed in the interval of such time, is found. The incremental quotient, $\frac{\Delta C_o}{\Delta t_o}$, (6), will be an approximation of the reaction's velocity, as fine, as small Δto, will be; if we want the exact value of the reaction's velocity in the instant *to*, we will find the limit of the incremental quotient (6), when $\Delta to \to 0$, getting then to the next conclusion.

Velocity of reaction in $to = lim \frac{\Delta C_o}{\Delta t_o}$, (7).

$\Delta to \to 0$

Synthesis: They exists so many problems, of Mathematics, Physics, and Chemistry, which requires a correspondent increment to its variable, when the increment of such variable tends to zero. By this reason, the Pure Mathematic has studied the methods for calculate this limits of incremental quotients, the ones that are called derivatives. To the different kind of functions, and this methods that later are exposed, they constituted the Differential Calculus.



MATHEMATICAL CONCEPT OF DERIVATIVE.

Explanation:

Being $y=f(x)$ a uniform and continuous function in the (a,b), y x_0 interval, a relevant point to such interval $(a< x_0<b)$. If that point x_0 we increment it in Δx_o, it will pass to another point $x= x_0 + \Delta x_0$; we will suppose that the point x it also belongs to the interval of definition (a,b) of the function. The increment that the function experiments when it passes throw the point x_0 to $x= x_0 + \Delta x_0$ we represent it by Δy_o, being therefore $\Delta y_o, = f(x)-f(x_0)$. (See the figure 1-2). Finally we will form the incremental quotient.

(Fig. 1-2)

$$\frac{\Delta y_o}{\Delta x_o} = \frac{f(x_o+\Delta x_o)-f(x_o)}{\Delta x_o} = \frac{f(x)-f(x_o)}{\Delta x_o} = \frac{f(x)-f(x_o)}{x-x_o}, \quad (1)$$

Definition: It is called derivative of the function $y=f(x)$ in the point x_0, to the limit (if it exists) of the incremental quotient (1) when $\Delta x_o \rightarrow 0$.

It is represented by these notations:

$$y'_0=f'x_0=Df(x_0)=\lim_{\Delta x_0\rightarrow 0} \frac{\Delta y_o}{\Delta x_o} = \lim_{x\rightarrow x_0} \frac{f(x_o+\Delta x_o)-f(x_o)}{\Delta x_o} = \lim_{x\rightarrow x_0} \frac{f(x)-f(x_o)}{x-x_o}, \quad (2)$$

If we observe the last term of (2) we see that the derivative is a functional limit, because the expression $\frac{f(x)-f(x_o)}{x-x_o}$, by x_0 it is considered a firm point, is a function of x.

Geometric Cartesian Interpretation. Even when it has been treated like one of the problems of the introduction, here we precise it even more.

The (fig. 1-2) shows that $\frac{\Delta y_o}{\Delta x_o} = \text{tg}\propto$, (3) . ($\propto$ is the angle of the chord PQ with the half-axis positive of the x-s). When taking limit in (3), when $\Box x_0\rightarrow 0$, the point Q will tend to be confused with the P, the chord PQ will tend to be confused with the tangent PT and the angle $\propto$ of the chord, with the β of the tangent, with the half-axis positive of the x-s.

Therefore: $\lim\limits_{\Delta x_0 \to 0} \frac{\Delta y_0}{\Delta x_0} = tg\ \beta$, (4); but because the tangent β is the slope of the line PT tangent to the curve in P, it is conclude that:

The derivative in the point x_0 is the slope to the curve in the point P of the abscissa x_0.

With this, we can obtain immediately the equation of the tangent to a curve, representative of the function y=f(x), in the point of the abscissa x_0. The coordinates of this point are $(x_0, f(x_0))$ and by passing the tangent throw it, it will be $y-f(x_0)=m\ (x-x_0)$, (5), and because the slope of the tangent is $f'(x_0)$, substituting this value in the (5) instead of m, we have:

$$\boxed{y-f(x_0) = f'(x_0)\ .\ (x-x_0)} \quad (6). \quad \text{(Equation of the tangent)}$$

THE DERIVATIVE FUNCTION.

When a function, y=f(x), defined in the interval (a,b), it has derivative in all the points of (a,b), that is, that for all point, x_0 just as ($a< x_0<b$) exist limit of the incremental quotient (2), the correspondence that exists between the points of (a,b) and the values of the derivative y=f(x) in each one of them receives the name of derivative function; known this, it is enough to substitute x by x_0 to have the derivative in the point x_0.

GENERAL RULE FOR FIND A DERIVATIVE OF A FUNCTION.

To find the derivative of a function, y=f(x), the incremental quotient is formed $\frac{f(x+\Delta x)-f(x)}{\Delta x}$, (7), and the limit is found when $\Delta x \to 0$.

EXAMPLE: Find the derivative of $y=x^2$. In this case, $f(x)=x^2$, $f(x+\Delta x)= (x+\Delta x)^2$. Substituting in (7) we have:

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{(x+\Delta x)^2-x^2}{\Delta x} = \frac{x^2+2x.\Delta x+\Delta^2 x-x^2}{\Delta x} = 2x+\Delta x, \quad (8).$$

Taking limits in the equality (8), when $\Delta x \to 0$, we have:

$$\lim \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\ (2x+\Delta x) = 2x, \quad (9).$$

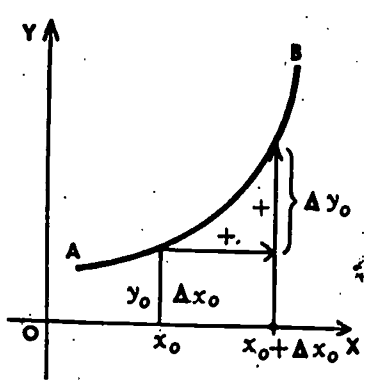
$\Delta x \to 0$ $\Delta x \to 0$

(Think in the theorems about limits that have been applied for calculating the limit from above).

The formula (9) it says that, if $y=x^2$, $y'=2x$, that is, the derivative of x^2 is $2x$.

EXAMPLE: Let us suppose that the space travelled by a particle that moves over a line, in function of the time, it comes by the formula $e=t^2$.
Which will be the velocity of the particle in the fifth second of movement?
SOLUTION: Taking into account that the velocity is the derivative of the space, and that the derivative of t^2 is $2t$, we have: $V=2t$ and substituting t by 5 we have: $V(5)=2x5=10$. If we are using the c.g.s, system, the result will be 10cm/ sec.

VARIATION OF THE FUNCTIONS



a) INCREASE AND DECREASE.

Definition: A function is increasing in a point, x_0, (figure 1-3), when to a positive or negative increment of x_0, corresponds a positive or negative increment of the y_0, that is to say, the increments of a variable and of a function are of the same sign.

(Fig. 1- 3)

In the branch of curve AB in the figure 1-3, it has been indicated with arrows that to a positive increment of x_0, Δx_0, corresponds an increment of y_0 also positive; the branch of curve AB is increasing. In a more concise way, but not to adequate for establishing the analytic condition of increasing, that we later will demonstrate, is usually said that: A function is increasing in a point x_0, when x_0 increases, y_0 will increase as well .

EXAMPLES: The next functions are increasing: y=sen x, in all interval points $\left(0, \frac{\pi}{2}\right)$ the function $y=x^2$ in all the points $x>0$.

When a function is increasing in all the points of an interval, it is said that is increasing in an interval, which case, the branch of curve correspondent to the interval, adopts the ascending form of the figure 1-3.

ANALITYC CONDITION OF INCREASING.

If a function is increasing in one point, being the increments of the same sign, the incremental quotient will be positive $\frac{\Delta y}{\Delta x}$ =+

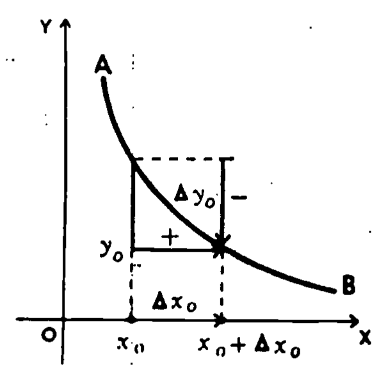
(1), and considering the property of the functional limits, if $\lim_{x \to a} f(x) = \beta$, and $M > \beta$,

a reduced environment from the point *a* exist, in which all the values that the function takes are lesser than M; symmetrically, if $N < \beta$, all the values that the function takes in a certain reduced environment from *a* are greater than N. In particular, if β is positive, all the values of the function in a certain reduced environment of *a,* are positive, and if β is negative, they will be negative; it is said, by this, that, from one value, the function has the same sign than its limit, with the possible exception of f (a); its limit, that is, the derivative, it will also be positive, getting then to the analytic condition of increasing: The condition for that a function is increasing in a point, is that the derivative is positive in such point.

In the examples shown above, if y = sen x, y' = cos x, that it is positive in all the points of the interval $\left(0, \frac{\pi}{2}\right)$; the function $y = x^2$ has its derivative y' = 2x positive, always that x>0. The function $y = x^3$ has by derivative $y' = 3x^2$, that it is positive in all the real campus, therefore, the function is increasing in all x.

Abstract: The points of an increasing branch are characterized because in all of them the derivative is positive.

MEANING OF INCREASE. A function is decreasing in a point x_0, fig. 1-4 when a positive (negative) increment of x_0, corresponds an negative (positive) increment of y_0, that is, the correspondent increments are of opposite signs.



(fig. 1-4)

In the branch AB of the figure 1-4 it has been indicated with arrows that to a positive increment x_0, Δx_0, it corresponds a negative increment of y_0; the branch of curve AB of such figure is decreasing. In a more concise way but not to adequate for demonstrating the analytic condition of decreasing usually it is said that: A function is decreasing in a point x_0, when the x_0 increase, the y_0 decrease. When a function is decreasing in the point of an interval it is said that is decreasing in the interval.

EXAMPLES: The next functions are decreasing: y = cosx in the interval $\left(0, \frac{\pi}{2}\right)$, $y = x^2$ for x<0, the function y =1/x in all the real campus (the zero excluded, in the one that is not defined). All the curves correspondents to the examples, that are familiar to the reader, are "descending" in the intervals that are indicated.

ANALYTIC CONDITION OF INCREASING

If a function decrease in one point, because the increasing are of different signs, the incremental quotient, $\frac{\Delta y}{\Delta x}$, will be negative, and its limit, that is, the derivative in the point, it will also be negative, getting then to the decreasing condition:

The condition for that a function could be decreasing in one point is that the derivative has to be negative in such point.

When all the points of a branch AB (fig.1-4) comply with the condition, it is said that the branch is decreasing.

Let us prove this condition in the examples from above. If y = cos x, y'= - sen x, and as - -sen x>0 in (0,π), - sen x, that is the derivative, will be negative and the function decreasing in (0,π); if y = x^2, y'=2x, that is negative for $x < 0$; si $y = \frac{1}{x} = x^{-1}$, $y' = -1 . x^{-2} = \frac{-1}{x^2}$, that is negative for all x (with the exception for x=0) for being x^2essentially positive.

b) MAXIMUMS AND MINIMUMS

Definition 1. It is said that f (x) gets a maximum relative (*) in x_0, when the ordinate x_0, $f(x_0)$ is greater in all the ordinates in a small enough environment of x_0.

In symbols: x_0 corresponds to a maximum relative if exists a $\varepsilon > 0$, small enough such that:

$f(x_0) > f(x)$ always that $(x_0 - \varepsilon < x < x_0 + \varepsilon)$.

In the (fig. 1-5) you can see a point x_0 in the one that the function reaches a maximum relative.

Definition 2: It is said that f (x) reaches a minimum relative in x_0 when the ordinate in x_0, $f(x_0)$ is lesser than all the ordinates in a sufficiently small environment of x_0.

[(*) Let us distinguish the maximums and minimums relatives of the absolutes; the first ones refer to a sufficiently small environment of a point and the seconds to all the campus where the function is defined.]

In symbols: x_0 corresponds to a minimum relative if it exists a $\varepsilon > 0$, sufficiently small such that $f(x_0) < f(x)$ always that $(x_0 - \varepsilon < x < x_0 + \varepsilon)$.

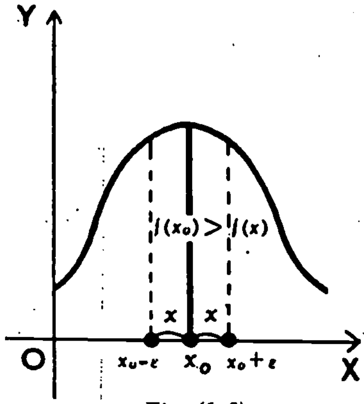


Fig. (1-5)

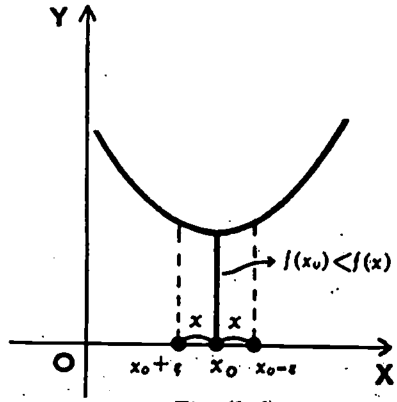
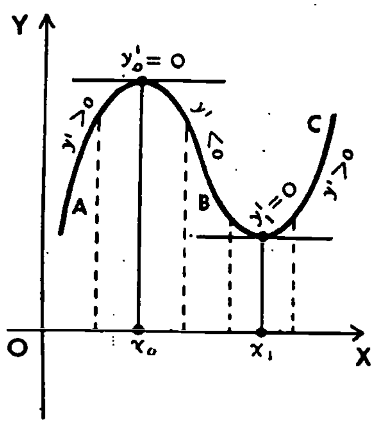


Fig. (1-6)

In the figure 1-6 you can see a point x_0, in which the function reaches a minimums relative.

NECESSARY CONDITION OF MAXIMUMS OR MINIMUMS (NOT SUFFICIENT).



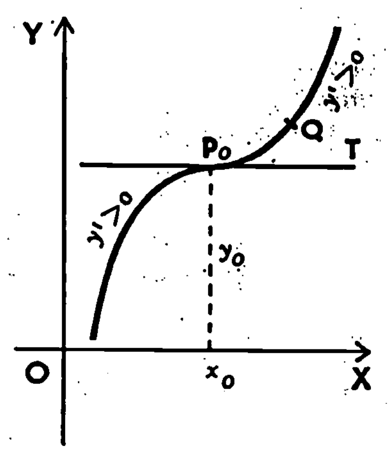
If we remember that in an increasing branch the derivative is positive and that in a decreasing is negative and we look at figure 1-7, we see that in a maximums the derivative goes from positive to negative, and therefore, if it is supposed that the derivative continues, it will be annulated in such point; in an analogous way, in a minimums the derivative passes from negative to positive and therefore it will also be annulated in such point.

Fig. (1-7)

In a symbolic way: If x_0 corresponds to a maximums $f'(x_0)=0$. If x_1 corresponds to a minimums, $f'(x_1)=0$. As in x_0 like in x_1, the tangents to the curve are parallel to 0x($y'=0$, null slope). That is how we get to the necessary condition of maximums or minimums.

To have a maximum or a minimum of f (x) correspond to a value of x, the derivative f'(x) should be annulated in such value.

This condition is not enough, what means that they can exist points, x, in which f'(x)=0, and for the ones the function doesn't reach neither one maximum value nor a minimum value. In the fig. 1-8 one of those points, called points of inflection, is shown. The tangent to the curve in Po is parallel to the axis OX and therefore $f'(x_0)=0$; however, in accordance with the given definitions from this points, Po doesn't correspond neither to a maximum nor to a minimum, because the ordinates from the left of x_0, are lesser than y_0 and the ones from the right are greater than y_0.



(Fig. 1-8)

The tangent in the inflexion point, Po, goes throw the curve (Remember that a tangent from a curve in one of its points, Po, is the limit position of the secants Po Q, when Q $\rightarrow$ Po).

DISCUSIONS FROM THE MAXIMUMS, MINIMUMS AND INFLECTIONS.

According to what we just see, a point, x_0 , to the one $f'(x_0)=0$, it can correspond to a maximum, a minimum or an inflexion point; therefore, the necessity of giving supplementary criterions, (besides of being $f'(x_0)=0$) is imposed for distinguish, between the points that annul the derivative, which corresponds to a maximum, which to a minimum and which ones to an inflection point.

First criterion; Variation of the function

Being x_0 the point that we tried to classified (it is supposed that x_0 is a solution to the equation $f'(x_0)=0$).

A slightly smaller $x_0 - \varepsilon$ value and another slightly greater $x_0 + \varepsilon$ value, are taken.

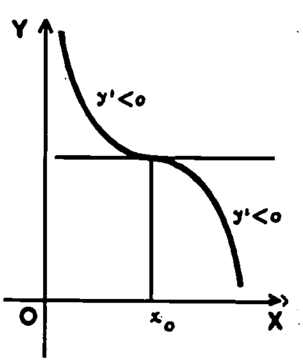
$$\text{If} \quad \begin{cases} f(x_0-\varepsilon) < f(x_0) \text{ and } f(x_0+\varepsilon) < f(x_0) \text{ there is maximum in } x_0. \\ f(x_0-\varepsilon) > f(x_0) \text{ and } f(x_0+\varepsilon) > f(x_0) \text{ there is minimum in } x_0. \\ f(x_0-\varepsilon) >< f(x_0) \text{ and } f(x_0+\varepsilon) >< f(x_0) \text{ there is an inflection point in } x_0. \end{cases}$$

Second criterion; Variation of the derivative.

Observing the figure 1-7, the next criterion is obtained:

$$\text{If} \quad \begin{cases} f'(x_0-\varepsilon) > 0 \text{ and } f'(x_0+\varepsilon) < 0 \text{ there is maximum in } x_0. \\ f'(x_0-\varepsilon) < 0 \text{ and } f'(x_0+\varepsilon) > 0 \text{ there is minimum in } x_0. \\ f'(x_0-\varepsilon) >< 0 \text{ and } f'(x_0+\varepsilon) >< 0 \text{ there is an inflection point in } x_0. \end{cases}$$

(fig. 1-9)



In the figure 1-8 you can see that, as $f'(x_0 - \varepsilon)$ as $f'(x_0 + \varepsilon)$ is positive, because the function is increasing; they are inflection points, like the one in the figure 1-9, in which the derivative maintains negative as in the left as in the right of the inflection point, for being the branch of decreasing curve in the point environment.

Third criterion; Variation of the second derivative.

Before annunciating this criterion we have to define the successive derivatives. If the function y =f(x) is derivable, being their derivative $y'=f'(x)$, and if we apply again the derivative operation to the function $f'(x)$, we obtain another function that is called the second derivative of y=f (x) and denotes by $y'''= f''(x)$, and if we derivate again the $f''(x)$ we will obtain the third derivative that is represented by $y''=f'''(x)$, and so on the forth, fifth, ..., umpteenths derivatives are defined .

EXAMPLES: 1 The second derivative of $y=3x^3 -2x^2+5x-4$, will be successively $y'= 9x^2 -4x + 5$,$y''=18x-4$.

2. The fifth derivative of y= sen x is obtained by this way: $y'=\cos x$, $y''=-\text{sen } x$, $y'''=-\cos x$, $y^{IV} =\text{sen } x$, $y^{V} =\cos x$.

We go now to explain our third criterion. Again, we observe the fig. 1-7; a branch of curve just like AB we will say that is negative concave, or that presents its concavity toward the axis –OY, or that it is convex.

A branch of curve just like BC we will say that is positive concave, or that presents its concavity towards +OY, or that it is just concave. In the branch AB (negative concave), the derivative is decreasing (because it pass from positive to zero and to negative); then it derivative, that is, the second derivative, will be negative.

In the branch BC (positive concave), the derivative is increasing (because it passes from negative to zero and to positive); then its derivative, that is, the second derivative, will be positive.

So then we have the next criterion to decide the concavity o convexity of a branch of curve: The sign of concavity matches with the sign of the second derivative.

A branch of curve that in all its points has a positive second derivative, it will be positive concave, or just concave.

A branch of curve that in all its points has a negative second derivative, it will be negative concave, or just convex.

The maximums belongs to a negative concave branch (see fig. 1-7), then in it $f''(xo) <0$.

The minimums belongs to a positive concave branch (see fig. 1-7) then in it $f''(x_1)>0$.

Finally, in the inflection point, (see figures 1-8 and fig.1-9), in the case of the fig. 1-8, the concavity passes from negative to positive, and therefore the second derivative, from negative to positive, and if we suppose that the second derivative is continue, when passing from negative to positive from left to right of Po, in Po it will be annulated, that is $f''(x_0) = 0$. The reader should argue in an analogist way that in the case of the inflection point in the figure 1-9, it will also be $f''(x_0) = 0$.

As an abstract, we could enunciate the third criterion to distinguish between the points that annul the derivative, that is to say, between the solutions of the equation f′(x)=0, which ones correspond to maximums, which to minimums and which to the inflection points:

$$\begin{cases} x_0 \text{ will correspond to a maximum if } f''(x_0) < 0. \\ x_0 \text{ will correspond to a minimum if } f''(x_0) > 0. \\ x_0 \text{ will correspond to an inflection point if } f''(x_0)=0 \text{ y } f'''(x_0) \neq 0. \end{cases}$$

Very important observation: 1. If $f''(x_0)=0$, for secure that x_0 corresponds to an inflexion point, let us add that $f'''(x_0) \neq 0$. When $f'''(x_0)=0$, the point x_0 can correspond to a maximum, to a minimum, or to an inflection point depending in the order and the sign of the first derivative not null in x_0. The complete demonstration of these affirmations, that is based in the development of the function y =f(x) in the Taylor's series in the environment of x_0, it goes out of reach of this introduction text, but however we are going to enunciate, with out demonstrating it, the general rule of discussion of maximums and minimums, and inflection points, for that the reader can form a clear idea of this important question of the Differential Calculus.

General Rule: If in the point x_0 complies with:

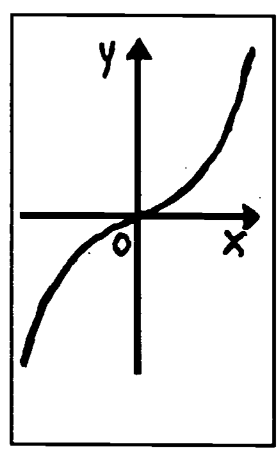
$$f'(x_0)=0,\ f''(x_0),\ldots,f^{(n-1)}(x_0)=0,\ f^{(n)}(x_0) \neq 0.$$

We will have:

$$\begin{cases} 1°) \text{ If } n \text{ is odd, } x_0 \text{ corresponds to an inflexion point.} \\ 2°) \text{ If } n \text{ is even and } f^{(n)}(x_0) > 0, \text{ corresponds to a minimum.} \\ 3°) \text{ If } n \text{ is even and } f^{(n)}(x_0) < 0,\ x_0 \text{ corresponds to a maximum.} \end{cases}$$

Examples: 1. The function $y = x^3$ complies: $y' = 3x^2$, $y'(0) = 0$; $y''(0) = 0$; $y''' = 6$, $y'''(0) = 6 \neq 0$. In accordance to the previous the first derivative not null in x=0 it has been the third, of odd point x=0 corresponds to an inflection point of the curve (see 10).

2. The function $y = x^4$ complies: $y' = 4x^3$, $y'(0) = 0$; $y'' = 12x^2$, (0)=0; $y''' = 24x$, $y'''(0) = 0$; $y^{(IV)} = 24 > 0$. Like the first derivative not null in x=0 is of even order and is positive, the presents a minimum in the origin.



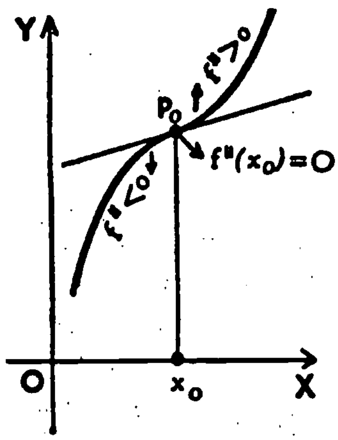
y''= 6x,
rule, like
order, the
fig. 1-

y''

curve

(fig. 1-11)

(fig. 1-10)



Another observation: 2.If in one point, x_0, it complies $f''(x_0) = 0$, being $f'(x_0) \neq 0$, it is secure that is an inflection point, because being $f'(x_0) \neq 0$ it can't be a maximum nor a minimum. When is not $f''(x_0)$ neither positive nor negative, the point can't belong to positive concave branch nor to negative concave; it is the union point of two branches of curves with different concavity. When there is a change of sign of the concavity, the proper occurs in the second derivative, that if is continues, when passing from left to right of x_0, the sign is changed, it should be annulated in x_0. The fact of being $f'(x_0) \neq 0$ indicates that the tangent in the inflection point, is not parallel to the axis OX. It has a point as the P_0 of the fig. 1-11.

The theory about maximums and minimums, that we just exposed, has multiples and interesting applications to the Geometry, Physics, Chemistry, and Biology's problems. Next, we will give examples completely solved, with the purpose that help in the problems. We resume all the answers obtained in one practical rule of action:

Practical rule: To solve a maximums and minimums problem by the methods of the Differential Calculus, the following stages are executed: 1^{th}) The function that has to be maximum or minimum, is obtained in the supposed that they don't give it directly.

If a function becomes with more than one variable, some relations that should comply the variables and let them all be eliminated but one, in the enunciate problem they should be looked for, in the way that they will be in function in the form $y = f(x)$.

2^{nd}) The function $y = f(x)$ is derivated. 3^{rd}) The equation that results from annualizing the derivative $f'(x) = 0$ is solved. 4^{th}) To the roots of this equation, one of the three criterions studied to decide if they correspond to maximums, minimums or inflections, is applied; reverently, for this discussion, the third criterion called: of the second derivative, is used. 5^{th}) As the last stage, if is about a Geometric or Physic problem, the solution it should try to be interpreted geometrically or physically.

SOLVED EXAMPLES OF PROBLEMS ABOUT MAXIMUMS AND MINIMUS.

1. Find the maximums, minimums and inflexion of the curve $y = 3x^4 - 4x^3 + 1$, as well the intervals of positive and negative concavity.

Solution: In this case they give the function directly, in way that the first phase is all ready done.

2^{nd} phase: Derivative, $y' = 12x^3 - 12x^2 = 12x^2(x-1)$.

3^{th} phase: $12x^2(x-1)=0$, which roots are: $x_1=0$ (double), $x_2=1$.

4^{th} phase: $y''=24x(x-1)+12x^2=36x^2-24x$.

Let's analyze the root $x_1=0$ of the derivative. $y''(0)=0$ and like $y'''= 72x-24$, $y'''(0) \neq 0$, and therefore the point of the curve of null abscissa is an inflexion point.

Let's analyze the root $x_2 = 1$ of the derivative. $y''(1) = 12 > 0$, ad therefore the point of the abscissa 1 is a minimum. (Remember the discussion rule of the third criterion).

5 th phase: Doesn't have place in this problem.

Abstract: The curve $y=3x^4-4x^3+1$, has an inflexion point, for $x=0$, that is to say the point (0,1), and a minimum in the point of abscissa $x=1$, that is to say in the point (1,0); the axis OX is tangent in that point.

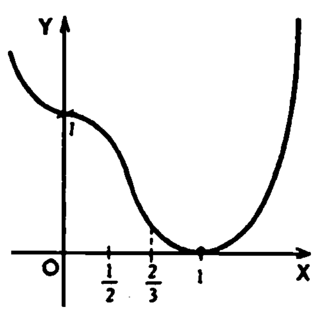
Let us see if another inflexion point from non parallel tangent to axis OX, exists. Annualizing the second derivative we have: $36x^2-24x=0$, which roots are: $x=0$ (all ready found) y - - $x = (2/3)$.

Study of the concavity: For $x<0$, $y''= 12x.(3x-2) >0$ (because both factors are negative and their product is positive).

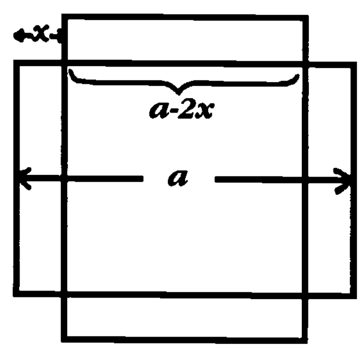
For $(0<x<(2/3))$ for example, for $x = (1/3)$, $y'' = 12.(1/3)\ (3.(1/3)-2) = -4<0$. Finally, for $x>2/3$, for example, for $x =(3/3)= 1$, $y''= 12.1=12>0$.

As consequence of the second derivative signs, we can conclude that:

- For $x<0$ the curve is positive concave.
- For $(0<x<\underline{2})$ the curve is negative concave.
- For $x>2/3$ the curve is positive concave.



(Fig. 1-12)



(Fig. 1-13)

Putting together all this information we can draw the curve, which form is the one from the figure 1-12.

2. We have a thin and squared cardboard and it is wanted to make a box cutting four squares, all the same, from the vertex and folding the lateral parts. (See figure 1-13) What dimension should be given to the cut of the corners for that the volume of the box can be the maximum?

Let us represent the side of the square by *a,* and the one of the little square that should be cut by the corners, by *x*.

1^{st} Phase: If we observe the figure 1-13 we can say that the volume of the box will be given by: $V= (a-2x)^2x$.

2^{nd} Phase: $V' = 2(a-2x) . (-2) .x + (a-2x)^2 = (a - 2x) \ [a - 2x - 4x] = (a - 2x) . (a - 6x)$.

3^{rd} Phase: $V' = (a - 2x) . (a- 6x) ==0$, which roots are: $x_1 = a/2$, $x_2 = a/6$.

4^{th} Phase: $V'' = -2. (a - 6x) + (a - 2x). (-6) = -2a + 12x - 6a + 12x = 24x-8a$.

For $x_1 = a/2$, $V'' (a/2) = 24.(a/2) - 8a = 12a - 8a = 4a>0 \longrightarrow$ Minimums.

For $x_2 = (a/6)$, $V'' (a/6) = 24.(a/6) - 8a = 4a - 8a = -4a<0 \longrightarrow$ Maximums.

5^{th} Phase: When $x = a/2$ the cardboard would be broken in four little squares exactly the same; the volume is null, because it will be a box which bottom would be reduced to one point. As you can see in this example, the solution of minimums has a really clear interpretation.

3. *Break down the number 4 in two addends such that the sum of the cube from the first one plus the triple of the quadratic from the second has a maximum value.*

1^{st} Phase: If we represent one of the addends by *x,* the other will be (4-x), by reason of the principle, the function that we want to make maximum is:

$$y = x^3 + 3 (4 -x)^2 = x^3 + 3x^2 -24x + 48.$$

2^{nd} Phase: $y' = 3x^2+6x - 24$.

3^{rd} Phase: $y' = 3x^2+6x -24 = 0$, which roots are: $x_1 =2$, $x_2=-4$.

4^{th} Phase: $y''=6x+6$. For $x_1 =2$, $y''(2)==18>0 \longrightarrow$Minimum.

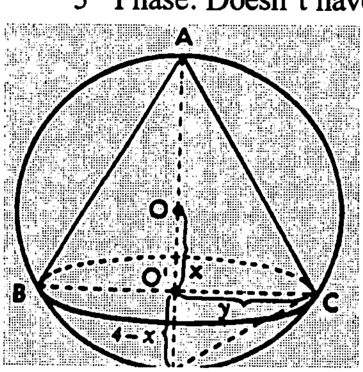
For $x_2 =-4$, $y''(-4)= -18<0 \longrightarrow$ Maximum.

The solution to the problem is therefore: $\begin{cases} -4 \text{ (first part)} \\ 4-(-4) =8 \text{ (second part).} \end{cases}$

The maximum ordinate of the function is:

-64+3.64=128.

5^{th} Phase: Doesn't have place in this problem.



4. Find the height of the revolution cone of maximum volume that can be record in a sphere of 4 cms by radius.

1^{st} Phase: Let us take for incognitos the distance between the sphere center and the center of base of the cone OO'=x, and also

the radius of base of the cone O'C=y (See fig.1-14). The volume of the cone is the third of the height times area of the base, that is:
$V=(1/3)\pi \cdot y^2 \cdot (4+x)$, (a).The function V depends of two variables *x*, *y* and therefore we have to find the relation that let us eliminate one of them. In the triangle ACA', rectangle in C, the height O'C=y is proportional media between the segments that the hypotenuse is divided by, that is to say:

fig. 1-14

$y^2=(4+x).(4-x)$; substituting this value in the formula (a), is:
$V= (1/3)\pi \cdot (4+x).(4-x).(4+x)=(1/3)\pi\ (4+x)^2.(4-x)$. This function is maximum or minimum for the same values of x than the auxiliary function $\Phi = (4+x)^2 \cdot (4-x)$, because the constant factor $(1/3)\pi$, doesn't affect the roots of the derivative equalized to zero; therefore we will find the maximums and minimums of the auxiliary function Φ, and this, we will do it in all problem were the function that has to be studied, to find the maximums and minimums, has a constant factor, that is to say, the constant factors are eliminated before the derivation. In definitive: The function that needs to be studied is the $\Phi = (4+x)^2.(4-x)$.

2nd Phase: $\Phi' = 2(4+x)(4-x) + (4+x)^2.(-1)=(4+x)[2(4-x)-(4+x)]=(4+x).(4-3x)$.

3rd Phase: $\Phi' = (4+x)(4-3x)=0$, which roots are: $x_1=-4$, $x_2 = (4/3)$

4th Phase: $\Phi'' = 4-3x+ (4+x).(-3) = 4-3x-12-3x = - 8-6x$.

For $x_1 = -4$, $\Phi''(-4) = -8-6.(-4) = 16>0 \longrightarrow$ Minimum.

For $x_2= (4/3)$, $\Phi''(4/3) =-8-6.(4/3) = -16<0 \longrightarrow$ Maximum.

Solution: The volume will be maximum when the height, $O'A = 4+(4/3) = 4(1+(1/3))$ that is, when the height is equal to the radius plus a third of the radius.

5th Phase: For $x_1= -4$, the height of the cone would be equal to 4-4 = 0 and therefore the volume would be zero (that is the minimum value), and this interpretation coincides with the result of the analysis.

5. Determine, between all the rectangles of 2p by perimeter, the one with maximum area.

1st Phase: the area comes by S=x.y. (fig.1-15). The function has two variables, but if the perimeter has to be equal to 2p, it will have x+y=p, from were y=p-x, so that the area is:
$S=x.(p-x) = xp - x^2$.

2nd Phase: $S'= p-2x$.

3rd Phase: $S'= p-2x = 0$, from were $x = \frac{p}{2}$ and therefore $y = \frac{p}{2}$.

4th Phase: $S'' = -2<0 \longrightarrow$ Maximum

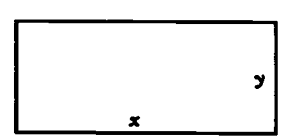


fig. 1-15

5th Phase: Geometric interpretation. We have found $x = y = \frac{p}{2}$, and therefore, given the perimeter, the rectangle of maximum area is the perfect square.

6. A harvester calculates that if the fruit recollection is made today it will obtain 120 hectoliter of it and he could sell it for 25 pesos per hectoliter, while if he waits some time, the harvest will increase 20 hectoliters per week and that will make the price decrease in

2.50 pesos per hectoliter and per week. In which time will be better to make the recollection to have the maximum benefit?

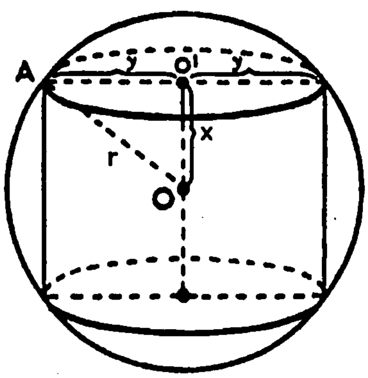
1st Phase: If the recollection is made in *x* weeks, the fruit quantity will be (120+20x) hectoliters, and the price will be (25-2.50x). The value of the sale will be:

V= (120+20x). (25-2.50x), that is the function that has to be maximum.

2nd Phase: $V' = 20.(25-2.5x) + (120 + 20x)(-2.5) = 500 - 50x - 300 - 50x = 200 - 100x$.

3rd Phase: $V' = 200 - 100x = 0$, where $x = 2$.

7: Which is the height of the cylinder of maximum volume that can be inscribed in a sphere with radius r? (Fig. 1-16)



Phase 1°: Observing the figure we can see that the volume from the cylinder is: $V = \Phi\ \pi\ .\ y^2.2x$. The function contains two variables, x, y, but applying the Pythagorean theorem to the triangle rectangle OO'A we have: $y^2 = r^2 - x^2$,which what the function V it is: $V = 2\ \pi\ .\ x\ \ (r^2 - x^2)$. We can consider the auxiliary function $\Phi\ = x.\ (r^2 - x^2)$, dispensing from the constant factor $2\ \pi$.

Phase 2°: $\Phi' = r^2 - x^2 + (-2x).x = r^2 - 3x^2$.
Phase 3°: $\Phi' = r^2 - 3x^2 = 0$ which roots are $x_1 = r(\sqrt{3})/3$, $x_2 =$
$= -r(\sqrt{3})/3$

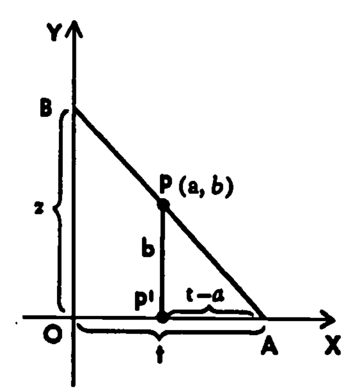
Fig. 1-16

We get rid of the negative root that doesn't have a real sense in the problem.

4th Phase: $\Phi'' = -6x, \Phi''(r(\sqrt{3})/3) = -2r(\sqrt{3}) < 0 \longrightarrow$ Maximum.

5th Phase: If the side of an equilateral triangle, in function of the radius is remembered that comes by $r\sqrt{3}$, we obtain the next interesting conclusion: The height of the cylinder of the maximum volume inscribed in a sphere is equal to 2/3 of the side from the equilateral triangle inscribed in a maximum circle.

-8. By one point P of coordinates (a,b) passes throw a line that cuts in the coordinate axis, segments OA and OB. Calculate the lengths from OA and OB when the triangle's area OAB is minimum.



1st Phase: We represent OA by t and OB by z. The triangle's area OAB is: $S = (1/2)\,t\,z$, (1) where it figures two variables; for eliminate one we observe that the triangles OAB and PP'A are similar, (fig. 1-17), and therefore: $(z/t) = (b/(t-a))$, from where $z = (bt)/(t-a)$ with what the (1) transforms in : $S = (1/2)(t).(bt/(t-a)) = (1/2).(bt^2/(t-a))$ with the only variable t.

2nd Phase: $S' = (1/2).(bt\,(t-2a))/(t-a)^2$ (after simplified).

3rd Phase: $S' = (1/2).(bt\,(t-2a))/(t-a)^2 = 0$. This expression is annulated when the numerator is also annulated, that is to say, when $bt.\,(t-2a) = 0$, from where: $t_1 = 0$, $t_2 = 2a$.

4th Phase: $S'' = (b(t-a)^2 - bt(t-2a))/(t-a)^3$ (after simplified).

For $t_1 = 0$ no triangle is formed. For $t_2 = 2a$,

$S''(2a) = \dfrac{ba^2 - 2ba.\,(2a-2a)}{a^3} = \dfrac{ba^2}{a^3} = \dfrac{b}{a} > 0 \longrightarrow$ Minimum. (We have supposed positives the coordinates of P in all reasoning).

5th Phase: When $t = 2a$, $z = [(bt)/(t-a)] = [\,b.2a/(2a-a)] = 2b$.

The triangle of minimum area is the one that cuts, throw the coordinate axis, segments OA and OB, equals to the double of the abscissa and of the ordinate of P, respectively. The area of this triangle OAB is four times the one from triangle PP'A.

9. Problem of the mirror.

A ray of light parts from a point A, getting another point B after been reflected in a flat mirror A'B' (see fig.1-18). Which relation will exist between the symbols α and β, (incidence angle and reflection angle), knowing that the trajectory that ray follows to get from A to B, AP+PB, is the minimum?

(Problem solved by Heron from Alexandria, years before Jesus Christ).

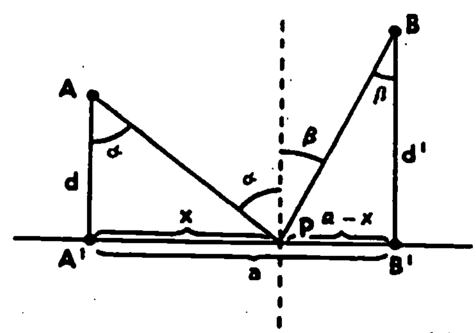


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Fig. 1-18

Solution: In the figure 1-18, the orthogonal projections of the points A and B are represented over the mirror, by A'B', and the distance A'B' by a. Finally the segment A'P it has been represented by x and the segments AA' and BB' by d' and d' respectively.

If the line APB is the one followed by the ray of light, the sum $l = AP + PB =$

$\sqrt{x^2 + d^2} + \sqrt{(a-x)^2 + d'^2}$ will be minimum and therefore will complied that:

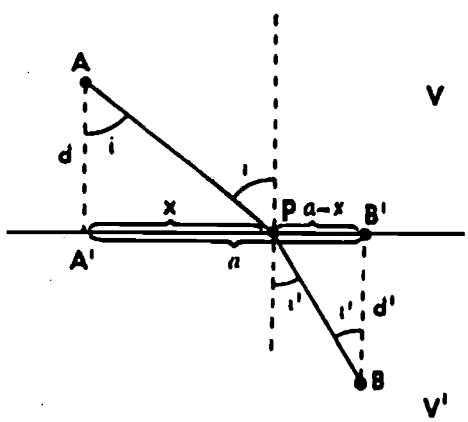
$$l' = \frac{x}{\sqrt{x^2+d^2}} - \frac{a-x}{\sqrt{(a-x)^2+d'^2}} = 0, \text{ that is :}$$

$$\frac{x}{\sqrt{x^2+d^2}} = \frac{a-x}{\sqrt{(a-x)^2+d'^2}}$$ (1), or what is the same:

sen α = sen β (See fig. 1-18 that the first term of (1) is sen α and the second sen β).

The incidence angle is equal to the reflection angle, Therefore: $\alpha = \beta$ that is the Physic's law already known.

10. A ray of light propagates from one point A to other, B, situated in two environments of different density, separated by a flat surface (for example: air and water). If we represent by v and v' the light velocities in two environments, by I the angle of the incident ray with the normal to the surface, and by i' the angle of the refracted ray with that normal; it is asked to determine the relation between the angles i, i' and the velocities v, v', supposing that the light propagates in the way that the inverted time, so that the ray gets from A to B, is minimum.



Solution: See figure 1-19. The inverted time in the trajectory AP will be equal to space divided by the velocity, and in an analogy way, the inverted in the trajectory PB. The sum of these times is minimum, that is:

$$T = [AP / v] + [PB / v'] = [\sqrt{x^2+d^2} / v] + [\sqrt{(a-x)^2+d'^2} / v'] \text{ is minimum.}$$

Fig. 1-19.

Therefore it will comply that: $T' = \{[1/v].[x/(\sqrt{x^2+d^2}] - [1/v'].(a-x)/\sqrt{(a-x)^2+d'^2})\} = 0$

From where is obtained: $(1/v).(x / \sqrt{x^2+d^2}) = (1/v').(a-x / \sqrt{(a-x)^2+d'^2})$ to what it is

The same $(v/v') = \frac{x}{\sqrt{x^2+d^2}} : \frac{a-x}{\sqrt{(a-x)^2+d'^2}}$ and having in minded that

$\frac{x}{\sqrt{x^2+d^2}}$ = seni, $\frac{a-x}{\sqrt{(a-x)^2+d'^2}}$ = seni′ in definitive has:

$$\boxed{\frac{v}{v'} = \frac{\text{seni}}{\text{seni}'}}$$ That is the request relation.

The reason of the velocities of the light in both environments is equal to the reason of the sins of the incidence and refraction angles. This reason is called "refraction index". We have demonstrated the refraction law of the light, from Snellius.

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THE FUNDAMENTAL LIMIT OF THE SEQUENCE

$\{(1+1/n)^n\}$: THE NUMBER e.

The sequence $\{(1+1/n)^n\}$ appears very frequently in the solution of pure and applied mathematics problems; in this part we will see some of these problems.

In this part we will demonstrate that $\{(1+1/n)^n\}$ is increasing but bounded, and by virtue of the limited growth principle it will have this limit which is represented by the letter *e* and is one of the fundamental limits of Mathematics. We will expound the demonstration in several stages with the intention of facilitating its comprehension.

1st) Applying Newton's formula to obtain the expansion of the power of binomial to the nth term, one has:

$$(1+1/n)^n = 1^n + n\cdot 1^{n-1}\cdot\frac{1}{n}+\frac{n(n-1)}{1\cdot 2}\cdot 1^{n-2}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot 1^{n-3}\cdot\frac{1}{n^3}+\cdots$$

$$\cdots+\frac{n(n-1)(n-2)\cdots[n-(n-2)]}{1\cdot 2\cdot 3\cdots(n-1)}\cdot 1\cdot\frac{1}{n^{n-1}}+\frac{n(n-1)(n-2)\cdots[n-(n-1)]}{1\cdot 2\cdot 3\cdots(n-1)\cdot n}\cdot\frac{1}{n^n}$$

2[nd]) Let us simplify the terms of the above formula, suppressing the 1 factors, which do not affect the result; also, the fractions which appear in the formula have in their numerators products of decreasing factors, starting from n, and in their denominators powers of n whose exponent *coincides* with the number of factors of the numerator. Dividing each factor of the numerator by a n factor, contained in the power of n of the denominator, one reaches:

$$\left(1+\tfrac{1}{n}\right)^n=1+1+\frac{1}{1\cdot 2}\cdot\left(1-\tfrac{1}{n}\right)+\frac{1}{1\cdot 2\cdot 3}\cdot\left(1-\tfrac{1}{n}\right)\cdot\left(1-\tfrac{2}{n}\right)+\cdots$$

$$\cdots+\frac{1}{1\cdot 2\cdots(n-1)}\cdot\left(1-\tfrac{1}{n}\right)\cdot\left(1-\tfrac{2}{n}\right)\cdots\left(1-\tfrac{n-2}{n}\right)+\frac{1}{1\cdot 2\cdots n}\cdot\left(1-\tfrac{1}{n}\right)\cdot\left(1-\tfrac{2}{n}\right)\cdots\left(1-\tfrac{n-1}{n}\right),\quad (1).$$

3[rd]) The factors which figure contained between parentheses in the above formula are different and when n increases the subtrahends decrease with which the value of such differences *increases*; therefore, when n increases, the value of $\left(1+\tfrac{1}{n}\right)^n$ *increases*, for two reasons: first, because the number of terms of it's expansion increases, and second, because the terms become greater.

SUMMARY: *The sequence* $\left\{\left(1+\tfrac{1}{n}\right)^n\right.$ *is increasing.*

4[th]) The parentheses of the second member of (1) are lesser than one and therefore are contraction factors (when a number is multiplied by a factor lesser than one the number is decreased, for example 8 · 0,5 = 4); therefore, if we suppress the parentheses in the second member of (1) we will obtain an expression of *higher value*, that is:

$$\left(1+\tfrac{1}{n}\right)^n<1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots+\frac{1}{1\cdot 2\cdot 3\cdots(n-1)}+\frac{1}{1\cdot 2\cdot 3\cdots n},\quad (2).$$

5^{th}) If in the second member of (2) we substitute the denominators of the fractions, by powers of base 2, which are *smaller* than such denominators, the fractions will increase; but we have that $1 \cdot 2 \cdot 3 > 2^2$, $1 \cdot 2 \cdot 3 \cdot 4 > 2^3$, ..., $1 \cdot 2 \cdot \ldots \cdot (n-1) > 2^{n-2}$, $1 \cdot 2 \cdot 3 \cdot \ldots \cdot n > 2^{n-1}$, with what one concludes that:

$$\left(1+\tfrac{1}{n}\right)^n < 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-2}}+\frac{1}{2^{n-1}}\ , \qquad (3).$$

6^{th}) The terms in the second member of (3), starting from the second, constitute a geometric progression of a ratio of ½: we know that the sum of infinite terms of one such progression comes given by the formula $\lim\limits_{n\to\infty} S_n = \frac{a}{1-r}$, which we will simply write like this: $S=\frac{a}{1-r}$. This formula is of great use in Mathematics and is expressed as: The sum of the infinite terms of an undefined geometric progression, of a ratio lesser than one, is equal to its prime term divided by one minus the ratio. Applying such formula to the second member of (3) we come to, however large *n* may be,

$$\left(1+\tfrac{1}{n}\right)^n < 1+1+\frac{1}{1-\tfrac{1}{2}} = 1+\frac{1}{\tfrac{1}{2}} = 1+2 = 3\,.$$

CONCLUSION: All the terms of $\left\{\left(1+\tfrac{1}{n}\right)^n\right\}$, however large *n* may be, are lesser than 3; therefore:

The sequence $\left\{\left(1+\tfrac{1}{n}\right)^n\right\}$ *is bounded.*

Having demonstrated that the sequence $\left\{\left(1+\tfrac{1}{n}\right)^n\right\}$ is increasing, but bounded, it will have a finite limit, Cantor's "limited growth principle". This limit is represented by the letter *e* (initial of it's discoverer's name, *Euler*).

The number e is an irrational number (moreover it is *transcendent*, that is that it is not the solution of an algebraic equation). The first ciphers of e are obtained by giving values to n, successive, in the expression $(1+1/n)^n$ whose limit is precisely the said number e.

The approximate value of e to the millionths, is:

$e = 2{,}718281\ldots$

OTHER SEQUENCES WHICH HAVE THE NUMBER e FOR LIMIT.

If we observe the *nth* term of the sequence that defines e, we see that it is composed of: "*One plus an infinitesimal raised to the reciprocal of the infinitesimal.*" The sequences whose *nth* term is constructed in accordance to this rule, have the number e for a limit. Let us see some examples.

The sequences $\left\{\frac{1}{2^{n-1}}\right\}$, $\left\{\frac{1}{n^2+1}\right\}$, $\{\sin a_n\}$, when $a_n \to 0$, $\left\{\frac{1}{n^3-9}\right\}$, are infinitesimal; in accordance with the previous rule we will have:

$$lím\left[1+\frac{1}{2^{n-1}}\right]^{2^{n-1}} = e, \quad lím\left[1+\frac{1}{n^2+1}\right]^{n^2+1} = e, \quad lím\left[1+\sin a_n\right]^{\frac{1}{\sin a_n}} = e, \quad \text{when} \quad a_n \to 0,$$

$$lím\left[1+\frac{1}{n^3-9}\right]^{n^3-9} = e.$$

Sometimes the structure "one plus an infinitesimal raised to the reciprocal of the infinitesimal" cannot be seen very clearly, but by simple algebraic transformations one can reach it.

EXAMPLE. If we are asked to find the limit for $\left\{\left[\frac{3n+4}{3n}\right]^{\frac{3n}{4}}\right\}$ (1), being that $\frac{3n+4}{3n}-1=\frac{3n+4-3n}{3n}=\frac{4}{3n}$, the expression (1) could be written like this: $\left[1+\frac{4}{3n}\right]^{\frac{3n}{4}}$ and in this form one sees that it is "one plus an infinitesimal raised to the reciprocal of the infinitesimal" and therefore: $lím\left[1+\frac{4}{3n}\right]^{\frac{3n}{4}}$ is *e*.

SOME PROBLEMS WHOSE SOLUTION DEPENDS ON THE NUMBER e

FORMULA OF CONTINUOUS UNIFORM GROWTH

Explanation of the problem:

Let us suppose that a substance grows continuously. It can be a living being, the wood of a tree, a substance being formed through a chemical reaction, a capital to which its interests were being continuously accumulated, a ball rolling on a snow-covered road with uniform velocity.

Furthermore let us suppose that the growth ratio is uniform, this is, constant along time, which means that, if a unit of substance turns into $1 + r$ in a certain time period, the same will occur along the entire time period through which the phenomenon is considered. This condition is only observed approximately in natural phenomena; for example, the growth of a tree is not uniform, for there are seasons in which the growth is faster than in others.

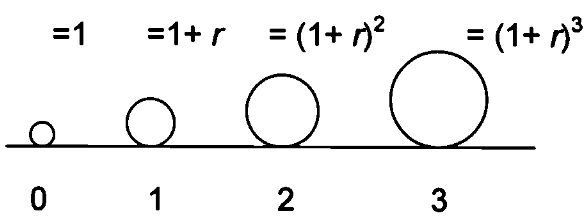
Now we will pose the following fundamental problem: Given an initial amount of substance, *c*, and a time period, *t*, (*t* time units), *what will be the final amount of substance,* C, *that will have been formed from the initial amount, c, through a continuous uniform growth, being **r** the rate per* one *of growth in the time unit, for **t** time units?*

To solve this problem, typical of Infinitesimal Calculus, we will consider three stages: In the first one we will assume that the growth of the substance occurs *every fixed time period*, that is, that the substance remains constant until the period expires, instant in which an instantaneous growth occurs, to remain constant once more until the period expires again, another instantaneous growth happening, at the end of the second period, and so on.

In the second stage we will divide the time period into n equal parts and we will assume that there is growth at the end of each *nth* part of the initial period, and finally, in the third stage, we will go to the limit when n tends to infinity, with which the *nth* part of the period considered initially will tend to zero and the accumulation of substance will happen by instants, infinitely small, with which we will have the continuous growth. This last stage, in which a step to the limit is taken, corresponds to a typical process of the Infinitesimal Calculus and it will allow us to obtain the formula of continuous uniform growth.

First Stage: Let us assume that the growth of the substance occurs at the end of each time period whose duration we will take as time unit. In the intermediate instants it is assumed that the amount of substance remains constant. Let us see into what does a unitary amount of substance transform itself after t time periods in which there is accumulation.

See the following Figure 1-2.



(Fig. 1-2)

Figure 1-2 shows us that when the process begins, in instant zero, the amount of substance is equal to 1; in the instant of expiring a period the substance unit grows transforming itself into $(1 + r)$, being r the "*rate per one of uniform growth*", in the considered period. In the instant of expiring the second period a new growth occurs, and it is clear that, if a substance unit transforms itself into $(1 + r)$, the units we had at the end of the previous period, that is $(1 + r)$, will transform themselves into what one unit transforms itself multiplied by the existing units, that is, into $(1+r) \cdot (1 + r) = (1 + r)^2$, because it is a growth proportional to the substance, that is to say, that double, triple, etc., amount of substance, grows double, triple, etc.; and generally, when multiplying the amount of existing substance by any real number the growth is multiplied by the same number. Similarly, upon ending the third period a new growth takes place; the substance there was at the end of the second period transforms into what one unit transforms itself, that is, $(1 +r)$, multiplied by the existing units, $(1 + r)^2$, reaching $(1 + r) \cdot (1 + r)^2 = (1 + r)^3$; so following to the end of the tth period, the initial substance unit will have become transformed into $(1 + r)^t$.

Now one easily understands, by the proportionality previously alluded, that if one substance unit transforms itself in t periods into $(1 + r)^t$, c units will transform themselves in the same time into $c \cdot (1 + r)^t$.

That way one comes to the famous formula for discrete or discontinuous growth (performed by jumping).

If we represent by C the final amount of substance, after t time periods, we will have:

$$C = c \cdot (1 + r)^t \qquad (1),$$

being r the rate per one of growth in the considered time unit.

When c is interpreted as the initial capital, r as the rate per one of annual interest (what one dollar produces in one year), and t as the number of years, formula (1) is the famous compound interest formula; in this case C represents the final capital which is attained placing a capital c for t years at a compound interest of r per one annual rate, with accumulation of interests every year. But, by the explanation we have given here of this formula, it is understood that it can be applied to other cases of growth.

Second Stage: In this second stage we divide the time unit from the previous stage into *n* equal parts, and assume that the growth occurs every nth part of the unitary period of the previous stage.

Since *r* represents the rate per one of growth in the time unit, and we assume that the growth is uniform, that is, proportional to time, in a period equal to one nth of the unit, the rate per one of growth will be $\frac{r}{n}$. Now, in *t* time units there are t · n nths of a unit, or, t · n accumulation periods. By a reasoning analogous to that of the first stage, except that here the accumulation periods (of substance growth) happen every nth of the period of the first stage and that the rate per one of growth in this new period is of $\frac{r}{n}$, and that instead of *t* accumulation periods there are $t \cdot n$ periods, one reaches the formula:

$$C = c\left(1 + r/n\right)^{nt} \quad (2)$$

Formula (2) corresponds also to a discontinuous growth, but the growths of the substance occur every nth part of the time in the first stage.

When *c* represents the initial capital and *t* the number of years, formula (2) is that of compound interest with interest accumulation every nth of years; *r* still represents the rate per one of annual interest. For example, if $n = 4$, one has: $C = c\left(1 + r/4\right)^{4t}$, which is the formula for the compound interest with quarterly accumulation of interests. It is easily understood that the final capital which is attained by this type of interest accumulation to the capital every fraction of a year, is greater than that which would be attained by annual accumulation.

Third Stage: If we assume now that $n \to \infty$ (reads: n tends to infinity), the growth will happen every instant, for $\frac{1}{n} \to 0$, and we will have the continuous growth.

Formula (2) can be written like this:

$$C = c\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{r \cdot t} \qquad (3).$$

If in formula (3) we reach the limit when $n \to \infty$, one notices that the expression contained between the brackets is composed of "one plus an infinitesimal raised to the reciprocal of the infinitesimal", and therefore, the said expression tends to the number *e* when $n \to \infty$, reaching in this way the formula for continuous uniform growth:

$$C = c \cdot e^{r \cdot t} \qquad (4).$$

To reach this formula we have used the theorem: "The limit of a power is equal to the limit of the base raised to the exponent of the power".

Formula (4) is broadly used, especially in Physics and Chemistry.

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