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ABSTRACT

Although proof and reasoning are seen as fundamental components of learning mathematics, research shows that many students continue to struggle with geometric proofs. In order to relate pedagogical methods to students' understanding of geometric proof, this 3-year project focuses on 2 components of student understanding of proof, namely, students' beliefs about what constitutes a proof and students' proof-construction ability. The classroom environments in the first year of the study were generally teacher-centered learning environments in which proof was logical exercise rather than a tool for establishing a convincing argument. Students harbored several ill-founded beliefs including: general claims may be established on the basis of checking critical examples, the form of an argument is more important than its chain of logical reasoning, and proofs are only valid for their associated diagrams, even if specific features of the diagram are not incorporated into the proof. In addition, students had great difficulty constructing proofs unless the key relationships necessary to establish the proof were outlined for them. (Author)

INVESTIGATING THE TEACHING AND LEARNING OF PROOF: FIRST YEAR RESULTS

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Abstract: Although proof and reasoning are seen as fundamental components of learning mathematics, research shows that many students continue to struggle with geometric proofs. In order to relate pedagogical methods to students' understanding of geometric proof, our three-year project focuses on two components of student understanding of proof, namely, students' beliefs about what constitutes a proof and students' proof-construction ability. The classroom environments in the first year of the study were generally teacher-centered learning environments in which proof was logical exercise rather than a tool for establishing a convincing argument. Students harbored several ill-founded beliefs including: general claims may be established on the basis of checking critical examples, the form of an argument is more important than its chain of logical reasoning, and proofs are only valid for their associated diagrams, even if specific features of the diagram are not incorporated into the proof. In addition, students had great difficulty constructing proofs unless the key relationships necessary to establish the proof were outlined for them.

Introduction

Proof is fundamental to the discipline of mathematics because it is the convention that mathematicians use to establish the validity of mathematical statements. In addition, the teaching of proof as a sense-making activity is fundamental to developing student understanding in geometry and other areas of mathematics. Despite the fact that student difficulty with proof has been well established in the literature, existing empirical research on pedagogical methods associated with the teaching and learning of geometric proof is insufficient (Chazan, 1993; Hart, 1994; Martin & Harel, 1989). Our work in this area has begun to address the need for research into the pedagogy of geometric proof instruction. We focus on geometric proof because geometry is traditionally the course in which students are first required to construct proofs. We have begun a three-year study to develop an empirically grounded theoretical model that relates pedagogy to student understanding of proof.

In order to assess the effectiveness of the pedagogical methods used by participating teachers, the project focuses on two components of student understanding of proof, namely, students' beliefs about what constitutes a proof and students' proof-construction ability. Specifically, the first year of the project has addressed three objectives:

1. To document student understanding of proof in order to update and expand existing research in this area;

2. To characterize evolving student beliefs about what constitutes a proof in proof-based geometry classes and to link these characterizations to aspects of the pedagogy including sociomathematical norms, nature of the activities, and teacher beliefs; and
3. To characterize students' evolving proof-construction ability in proof-based geometry classes and to link these characterizations to aspects of the pedagogy including sociomathematical norms, nature of the activities, and the teacher's instructional philosophy.

Perspectives

Existing research documents students' poor performance on proof items and identifies common, fundamental misunderstandings about the nature of proof and generalization in a number of mathematical content areas (Chazan, 1993; Harel & Sowder, 1998; Hart, 1994; Martin & Harel, 1989; Senk, 1985). In trying to make sense of students' difficulties with geometric proof, Dreyfus and Hadas (1987) articulate six principles which form a basis for understanding geometric proof. These principles address many of the student misunderstandings of proof cited in the literature.

The theories guiding our research come from proof-related research projects including the work of Harel and Sowder (1998), Hoyles (1997), and Simon and Blume (1996). Some researchers (Balacheff, 1991; Harel & Sowder, 1998; Knuth & Elliott, 1998) have proposed similar theories that describe increasingly sophisticated strategies used by students to construct proofs. At the least sophisticated level, students appeal to external forces for mathematical justification. At the next stage, students base their justifications on empirical evidence. Finally, students are able to use more abstract and mathematically appropriate techniques when proving statements. The findings from our three-year study will be used to make connections between pedagogy and various levels of student understanding.

Methods

During the first year of the study, we collected data in the classrooms of two teachers in a large high school in the mid-western United States, recording the beliefs and proof construction ability of the students as well as the beliefs held and the pedagogical methods used by their teachers. The teachers participated in a summer workshop, prior to the school year, during which they read and discussed existing research on geometric proof and experienced methods for investigating proof. Teachers also worked collaboratively with the researchers and graduate assistants to plan for and reflect on classroom events. In order to capture classroom events, we conducted daily observations and videotaping of classroom activities during the four months in which proof was a major focus of the curriculum.

In order to document beliefs, we revised and extended Dreyfus and Hadas' (1987) six principles, then constructed the Proof Beliefs Questionnaire that assessed students' agreement with these revised principles. It was necessary to add more detail to Dreyfus and Hadas' (1987) principles, in order to address a broader set of beliefs and to reliably map items to particular principles. Questionnaire items consisted of items modified from instruments used by Chazan (1993), Healy and Hoyles (1998), and Williams (1980), as well as some original items.

To assess students' ability to construct proofs, we developed a performance assessment instrument, the Proof Construction Assessment. The Proof Construction Assessment included items in which students must construct partial or entire proofs, as well as generate conditional statements and local deductions. In addition to some original items, the instrument includes items modified from Healy and Hoyles (1998), Senk (1985) and from the Third International Mathematics and Science Study (TIMSS) (1995).

The Proof Beliefs Questionnaire was given to all students during the first semester, about three weeks after proof had been introduced. We conducted follow up interviews with six focus students in each class to clarify their beliefs. These students were selected on the basis of their performance on the questionnaire and teacher recommendations. During the second semester, the Proof Construction Assessment was administered to all students in the two classes, and another set of interviews was conducted with the 10 of the 12 focus students.

The multiple sources of data helped us to learn about the context for the development of students' beliefs about what constitutes a proof and their ability to construct proofs in order to interpret this information and connect it to pedagogy. Other data sources included audiotaped planning meetings with researchers and teachers as well as interviews with the classroom teachers.

Results

In order to get a sense of the classroom environments in which the research took place, we first describe general features of the two classrooms as well as some of the typical classroom practices. One of the two participating teachers had been teaching for 5 years and the other for more than 20 years. Despite the difference in years of experience, there were several commonalities in the teachers' classroom practices. Both teachers followed the order and scope of the textbook quite closely. The typical daily routine involved discussing homework, introducing new material, and practicing new material. At the suggestion of researchers, student desks were arranged in pods of four to facilitate student dialogue.

In analyzing videotapes and field notes of classroom sessions, we have identified several features of the classroom environment, including social norms, sociomathematical norms, and other factors, that may have influenced students' learning.

The social norms, or standards of social behavior in the classroom, included:

- *The teacher was the mathematical authority in the classroom.* Teachers provided counterexamples to student conjectures, rather than remaining neutral or turning conjectures back to the class. They also posed rhetorical questions so that students were essentially asked to agree with the correct answer.

- *There was limited time for thinking and answering questions*. Teachers often asked and answered their own questions. Wait time was very short. In-class work time for groups was very limited. Teachers often interrupted this time with hints, advice, and examples.

The sociomathematical norms, or standards for mathematical behavior in the classroom, included:

- *There were few opportunities for sense-making*. Students appealed to facts such as "you can't divide by zero," but were not generally asked to explain or make sense out of these facts. Students were able to make claims about geometric relationships without justifying their claims.
- *Problems and proofs always worked out nicely*. Problems had solutions and proofs contained all the necessary information to prove the desired result. In instances where this was not the case, it was due to a "typo" either in the text or on teacher-made worksheets. Students were directed to fix the mistake and solve the problem or complete the proof.
- *It was not clear that there was a need for proof*. There was very little opportunity to make conjectures or prove conjectures. Proofs that students had to construct were generally proofs of given "facts."

An additional factor that may have influenced the students learning environment was:

- *Teachers' pedagogical practices were limited by their content knowledge*. The less experienced teacher rarely strayed from teacher-directed activities. When she did, errors in reasoning and in logical structure were documented. The more experienced teacher made fewer errors and was more willing to follow up on students' mathematical suggestions.

These general features established an environment in which the teacher had most of the responsibility for constructing convincing arguments and the students were left to mimic the expert practices of the teachers.

Beliefs About Proof

We have used our revisions of Dreyfus and Hadas' six principles (1987) as a framework for our analysis of student beliefs. The Proof Beliefs Questionnaire, which is aligned with these six principles, formed the basis of students' self-reported beliefs about proofs. By synthesizing Proof Beliefs questionnaire data with interview data and classroom observations, we have developed some preliminary findings. These findings are organized by principle.

Principle 1: A theorem has no exceptions. Some students claim to believe that this is true. During clinical interviews with students regarding their beliefs, students stated that theorems don't always have to be true and there can be some exceptions to the theorems. However, they sometimes allow for exceptions to the rule when faced with counterexamples. In the classroom, the teachers used counterexamples to respond to students when they made a false claim. (For example, one student asked if AAA was a congruence theorem. The teacher sketched a pair of similar triangles to show that it was not a theorem.) However, the teachers did not take the opportunity to emphasize that here was a counterexample being used to refute a statement.

Principle 2: The dual role of proof is to convince and to explain. Despite students' claim that proofs are required to establish validity, they are often unconvinced by general proofs. In fact, they often claim that examples are more convincing than proofs. It is not clear that the explanatory role of proof has hit home with these students. The statements that they were asked to prove in the classroom were generally statements they already believed to be true. In other words, the explanatory role of proof was not a critical role for students, because, in their own minds, they had already ascertained the validity of the statement. Opportunities for students to make conjectures then prove these conjectures were rare, and generally out of the comfort zone for the participating teachers.

Principle 3: A proof must be general. Students believe that empirical evidence constitutes a proof. They also believe that checking critical cases (e.g., an isosceles triangle, a right triangle, an obtuse triangle, etc.) satisfies the requirements for generality in an argument. They view a specific triangle as a reasonable representative for all triangles in the classification. In the classroom, teachers appealed to specific examples to help demonstrate the validity of a statement and the application of a statement (not necessarily clearly distinguished), possibly contributing students' belief that examples constitute a convincing argument.

Principle 4: The validity of a proof depends on its internal logic. Students claimed to prefer two-column proofs to any other style of formal proof (e.g., paragraph or flow chart). They believed that two-column proofs were more organized and easier to understand. In assessing the relative value of multiple proofs, they appealed to form over internal logic. In addition, when checking proofs, some students were not particularly attentive to issues of logical order, with the exception of the location of given statements in a proof. In the classroom, the teachers almost always used two-column proofs for direct reasoning and reserved the paragraph format for indirect proofs. Although the teachers experimented with flow chart proofs, they often aligned them as if they were two column proofs and sometimes misrepresented the logical connections.

Principle 5: Statements are logically equivalent to their contrapositives, but not necessarily to their converses or inverses. Students appealed to context to determine the validity of various forms of a statement and not to the logical equivalence of the form of the statement. If the context is a nonsense context, students will translate the context to a "real life" context in order to reason in context. In the classroom, converses, inverses, and contrapositives were treated as an independent section at the beginning of the school year. They were not connected to later treatments of proof, which were generally focused on proving positive statements. There was no link to these forms during the section on indirect proof either.

Principle 6: Diagrams that illustrate statements have benefits and limitations. Students believe that a diagram is valuable to forming a proof. However, many students are unclear about which aspects of a diagram are general (i.e., meant to represent a class of figures) and which are specific. Some students also believe that a proof is only valid for its accompanying figure, or at least its accompanying class of figure (e.g., obtuse triangles), even if the specific features of the figure (e.g., obtuse angle) are not incorporated into the proof. The role of diagrams as general representations was not explicitly discussed in class.

Proof Construction Ability

Student proof construction ability was determined using three types of data collected during the project year. First, the Proof Construction Assessment instrument was developed to measure students' varying levels of ability to engage in formal logical reasoning. Second, data was collected during classroom observations. Observers took field notes and video recorded classroom sessions of proof instruction as well as students working in groups or with technology to develop proofs. Third, a set of ten focus students participated in clinical interviews with researchers. The interviews focused on some aspects of the Proof Construction Assessment and required focus students to create at least one original proof during the session.

The Proof Construction Assessment included items with varying amounts of support in order to assess proof construction ability at four levels. Items at the first level, which offered students the greatest support, required students to fill in the blanks in a partially constructed two-column proof. Items at the second level of support addressed specific components of proof construction. The first type of item at this level addressed students' understanding of conditional statements. Students were asked to separate the *if* and the *then* components of a conditional statement in order to identify which component was associated with the relationships that were given and may be assumed to be true and which component required proof or justification. The local deductions items were also at this second level of support. These items assessed students' ability to draw one valid conclusion from a given statement and to justify the conclusion. This was less supported than the fill-in type item because the students were required to draw the conclusion themselves, without being told what they were to justify or what the justification was for a missing statement. This type of task is equivalent to producing and justifying only one step in a logical argument. Items at the third level of support required multi-step reasoning. These items required students to construct proofs for which hints were provided. These hints identified some of the key elements in the proof's logical chain of reasoning. At the fourth level of support, students were asked to generate a complete, multi-step formal proof, independently.

Results from the Proof Construction Assessment are found in Table 1. Student performance on the instrument suggests that the students in both classes seemed to have the greatest difficulty with items 2 and 4, which provided the least amount of scaffolding. These items required students to write original proofs of statements based on given conditions. Even though students also needed to write a proof for item 5, they were provided with ideas for outlining the proof. Strong student scores on item 5 also might be due to the fact that students were most familiar with the content (similar triangles) since they had just completed a unit on similar triangles in class. Student performance was best on items 1 and 3, which provided the most scaffolding. For item 1, students were asked to fill in the missing statements or reasons for a proof that had been developed for them. For item 3, students were required to write a conditional statement and then use this statement to determine what information was given and what was necessary to prove if asked to justify the conditional statement. By synthesizing Proof Construction Assessment data with interview data and classroom observations, we have developed a few preliminary findings.

Content knowledge is a major factor in student proof construction ability. Student performance on the Proof Construction Assessment was discussed during clinical interviews with students. During these interviews, several students claimed to have difficulty with those items whose content was unfamiliar or whose content was from lessons earlier in the school year. When the geometric content was somewhat familiar to the students, they were able to talk through aspects of the given diagram (or provide their own diagram) that eventually led them to at least an elementary understanding of what was needed to write a proof. When the content was unfamiliar, at

Table 1. Student Performance on the Proof Construction Assessment.

Item Number	**Average Score as a percent for Mrs. A's students**	**Average Score as a percent for Mrs. C's students**
1	61.6	69.5
2	22.3	27.4
3	66.5	76.7
4	33.1	30.5
5	52.3	70.5
6	42.3	39.0
Total	46.7	51.1

least one student was unable to provide even one valid conclusion from a given statement. Field notes of classroom observations indicate that the classroom teachers often "brainstormed" with students about a given situation and wrote an outline for a proof prior to requesting that students write a formal proof on their own. These "brainstorming" sessions helped all students recall the content needed to complete the proofs.

Format for writing formal proofs was over-emphasized in class (understanding the need for proof was under-emphasized). At the start of instruction on proof writing, both classroom teachers modeled a variety of proof writing techniques and allowed students to write proofs in flow chart form, paragraph form, or two-column form. However, after about two weeks of proof instruction, the teachers only showed proofs in two-column form. Thus, this became the accepted method for writing a formal proof. By this point students had also begun to "validate" their proofs by checking that their statements and reasons matched those demonstrated by the teacher in number and content. Students were often convinced that their proof was valid if the number of steps in the proof matched the number of steps in the proofs constructed by other students in the class. These ideas relate to the belief that the form of the proof is more important than the substance of the proof.

Students were not given many opportunities to explore mathematical ideas and proof writing on their own. One goal of proof writing is that the writer will come to a deeper understanding of the mathematical concepts involved. For this to happen, the writer must see proof development as a logical process that begins with exploration of mathematical ideas. Often, students in the research classroom were merely given new geometric ideas, such as the fact that parallel lines have the same slope, and expected to use these ideas to prove statements or situations that were provided for them. Moreover, the teachers frequently demonstrated how to complete a proof using the new ideas before allowing the students to explore the ideas or to write similar proofs on their own.

Conclusions

Some of the findings from the first year of the study echo the results of earlier studies such as the fact that many students believe that a set of examples constitutes a proof (Chazan, 1993; Harel & Sowder, 1998). In addition, students' poor performance on writing original proofs supports Senk's (1985) findings. It is our investigation of the classroom environment and its connection to students' understanding that sets our work apart from the existing literature.

The classrooms we studied were teacher-centered environments in which successful proof-writing consisted of using an acceptable format to link a collection of definitions, postulates, and theorems in a repeatable pattern of sequenced steps. In this environment, students developed some beliefs that were contrary to their teachers' expectations and to generally accepted principles of proof understanding. The classroom's social and sociomathematical norms gave rise to specific classroom practices that were generally detrimental to developing student understanding. For example, there were few opportunities for sense-making. Because students were not expected to reason about geometric relationships in an informal context in the classroom, they never practiced proof-writing as a form of making sense of geometric relationships. Although they claimed to believe that proofs were useful for explaining relationships, students judged the validity of arguments based only upon a proof's format or on whether they believed that the statement to be proved was true.

The teacher's role as the mathematical authority in the classroom also impacted students' beliefs. In particular, when the teacher led classroom discussions, students were easily distracted, because they had little responsibility for making mathematical decisions. They were also reluctant to investigate or make conjectures because there was usually not long to wait before the teacher would provide the correct answer or provide the next step in a proof.

An environment in which all problems can be solved and we only prove true facts also undermines the value of proof as a sense-making tool. It is not surprising that students do not see a need for proof in a situation in which everything we try to prove is true and if it cannot be proved, then we can safely assume that needed information was inadvertently omitted.

Our first-year results show that environmental aspects of the classrooms certainly have the potential to impact students. Social norms and sociomathematical norms can give rise to classroom practices in which students' main goal is to generate work that looks like the teacher's examples. The effect of this is that students may be less likely to do their own thinking about the given situation and more likely to simply follow the format provided, even if they experience little success in implementing the practice. If we take the position that students must construct their own knowledge by doing and experiencing, this model for teaching proof construction may have a detrimental effect on student learning.

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