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ABSTRACT

In this paper, a central claim will be that one of the most important influences that technology should have on mathematics education is that many of the most important goals of mathematics instruction should consist of helping students develop powerful, sharable, and re-usable conceptual technologies for constructing (and making sense) of complex systems. A second claim will be that these new conceptual tools don't involve introducing completely new topics into the mathematics curriculum as much as they involve dealing with old topics in new ways that emphasize mathematics-as-communication (description, explanation) more than mathematics-as-rules-for-symbol-manipulation. A third claim will be that, even though technology-based tools create the need to teach these new levels and types of understandings and abilities, in many cases, technology-based tools are not needed to teach them effectively. So, it is not necessary to have lots of classrooms full of educational technologies in order to provide learning experiences for students that are wise to the needs of a technology-based society. (Author)

What Mathematical Abilities Are Most Needed for Success Beyond School in a Technology Based Age of Information?

by

Dr. Richard Lesh

WHAT MATHEMATICAL ABILITIES ARE MOST NEEDED FOR SUCCESS BEYOND SCHOOL IN A TECHNOLOGY BASED *AGE OF INFORMATION*?[1]

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In this paper, a central claim will be that one of the most important influences that technology should have on mathematics education is that many of the most important goals of mathematics instruction should consist of helping students develop powerful, sharable, and re-usable conceptual technologies for constructing (and making sense) of complex systems.[2] A second claim will be that these new conceptual tools don't involve introducing completely new topics into the mathematics curriculum as much as they involve dealing with old topics in new ways that emphasize mathematics-as-communication (description, explanation) more than mathematics-as-rules-for-symbol-manipulation. A third claim will be that, even though technology-based tools create the need to teach these new levels and types of understandings and abilities, in many cases, technology-based tools are not needed to teach them effectively. So, it is not necessary to have lots of classrooms full of educational technologies in order to provide learning experiences for students that are wise to the needs of a technology-base society.

How has technology influenced what's needed for success beyond schools in a technology-based *Age of Information*?

When people speak about appropriate roles for calculators, computers, and other technology-based tools in instruction, their comments often seem to be based on the implicit assumptions that: (i) the world outside of schools has remained unchanged – at least since the industrial revolution, and (ii) the main thing that new technologies do is provide "crutches" that allow students to avoid work (such as mental computation or pencil-&-paper computations) that they should be able to do without these artificial supports. But, technology-based tools do a great deal more than provide new ways to do old tasks. For example, they also create new kinds of problems solving situations in which mathematics is useful; and, they radically expand the kinds of mathematical understandings and abilities that contribute to success in these situations. In fact, one of the most essential characteristics of a technology-based *age of information* is that the constructs (and conceptual tools) that humans develop to make sense of their

[1] References for most of my comments come from two recent books that I co-edited: *Handbook of Research Design in Mathematics & Science Education* (Kelly & Lesh 2000) and *Beyond Constructivism: A Models & Modeling Perspective on Mathematics Problem Solving, Learning & Teaching* (Doerr & Lesh, in press). Each book contains chapters by more than thirty leading math/science educators. So, even though the views expressed here are my own, they were informed by a great many others.

[2] Here, I am using the term "complex system" is a somewhat more general sense than it is used in the newly emerging field of "complexity theory" in mathematics. Nonetheless, later in this paper, readers will see that the two uses of the term are closely related. Terms with fewer technical associations could have been used. For example, instead of calling these systems "complex", I could have called them "structurally interesting " or "mathematically significant". But, regardless what terminology is used, it should be understood that a system that is "complex" (or "structurally interesting" or "mathematically significant") is different for a child than for an adult.

experiences also mold and shape the world in which these experiences occur. Consequently, many of the most important mathematical "objects" that impact the everyday lives of ordinary people are complex, dynamic, interacting systems that are products of human constructions - and that range in size from large-scale communication and economic systems, to small-scale systems for scheduling, organizing, and accounting in everyday activities. Therefore, people who are able to create (and make sense of) these complex systems tend to enjoy many opportunities; whereas, those who don't risk being victimized by credit card plans or other systems created by humans.

To see evidence of the kind of changes that are being introduced into our lives by advanced technologies, look at a daily newspaper such as *USA Today*. In topic areas ranging from editorials, to sports, to business, to entertainment, to advertisements, to weather, the articles in these newspapers often look more like computer displays than like traditional pages of printed prose. They are filled with tables, graphs, formulas, and charts that are intended to describe, explain, or predict patterns or regularities associated with complex and dynamically changing systems; and, the kinds of quantities that they refer to go far beyond simple counts and measures to also involve sophisticated uses of mathematical "objects" ranging from rates, to ratios, to percentages, to proportions, to continuously changing quantities, to accumulating quantities, to vector valued quantities, to lists, to sequences, to arrays, or to coordinates. Furthermore, the graphic and dynamic displays of iteratively interacting functional relationships often cannot be described adequately using simple algebraic, statistical, or logical formulas.[3]

For a simple example to illustrate some of the impacts of the preceding new-uses-of-old-ideas on the everyday lives of ordinary people, consider the section of a local newspaper that gives advertisements for automobiles. Then, think about how these advertisements looked twenty years ago. They've changed dramatically! Today, it's often difficult to determine the actual price of cars that are shown. What's given instead of simple prices are mind boggling varieties of loans, leases, and buy-back plans that may include many options about down payments, monthly payments, and billing periods. Why have these changes occurred? One simple answer is: *graphing spreadsheets* (like the one shown in Figure 1).

Spreadsheets with graphs, like the one shown in Figure 1, provide dynamic and easily manipulable conceptual tools for describing and exploring relationships among time, interest rates, monthly payments, and the amount of money remaining to be paid (or that has been paid) at any given time. Therefore, such tools make it easy for car dealers to develop sophisticated buying, leasing and loan plans – based on a few "new ideas" dealing with iteraction, recursion, trends, and matrix-based organizations of information – but mostly based on new ways of using old basic ideas from elementary mathematics. Yet, these new ways of using old ideas emphasize mathematical under-

[3] It is now well known that several iteration of simple algebraic function can lead to a system that is essentially chaotic – with many characteristics that are unpredictable, with emergent characteristics that are not simply derived from characteristics of the interacting elements of the system, and often with feedback loops in which second-order effects often overwhelm the impact of first-order effects. Yet, new fields of mathematics, such as "complexity theory" or "discrete mathematics", don't call for new "foundation-level ideas and abilities" as much as they require much less narrow and shallow treatments of old topics in "elementary mathematics".

standings and abilities that are quite different than those that have been emphasized in traditional schooling. For example, instead of operating on pieces of information, operations often are carried out on whole lists of data. Instead of simple one-directional "input-output" rules, the kind of functions that are involved often are iterative and recursive (sometimes involving sophisticated feedback loops); and, the results that are produced often involve multi-media displays that include a variety of written, spoken, constructed, or drawn media.

Year	Amount Owed	Interest Rate	Payment/Month	Total Paid
0	$10,000	7.0%	$300	$0
1	$7,100	7.0%	$300	$3,600
2	$3,997	7.0%	$300	$7,200
3	$677	7.0%	$300	$10,800
4	($2,876)	7.0%	$300	$14,400
	($6,677)	7.0%	$300	$18,000

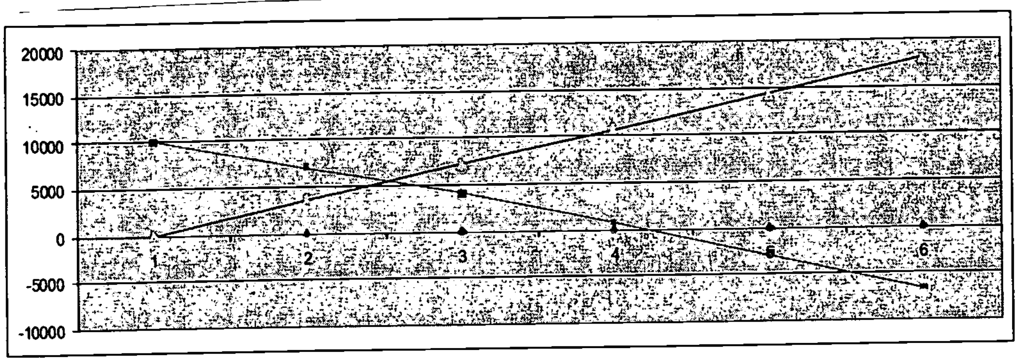

Figure 1. A Spreadsheet for Determining Interest Payments for Car Loans

Therefore, representational fluency is at the heart of what it means to "understand" many of the most important underlying mathematical constructs; and, some of the most important mathematical abilities that are needed emphasize: (i) mathematizing (quantifying, dimensionalizing, coordinatizing, organizing) information in forms so that "canned" routines and tools can be used, (ii) interpreting results that are produced by "canned" tools, and (iii) analyzing the assumptions that alternative tools presuppose – so that wise decisions will be made about which tools to use in different circumstances.

Figure 2 emphasizes another point about the kind of problem solving situations that occur with increasing frequency in everyday situations today. That is, unlike the kind of word problems that have been emphasized in traditional textbooks and tests, where the products that students produce are simply short answers to narrowly specified questions about specific situations, in more realistic situations where mathematics is useful, it's often the case that the construction of relevant conceptual tools is not simply a process on the way to producing "an answer". Instead, the conceptual tools ARE the products that are needed. --- For example, a textbook word problem might be about determining how much money to leave as a tip for the waiter at a restaurant if the bill is $23.52 and you want to give a 15% tip. Or, in a "real life" situation that's similar to the situation involving spreadsheets and automobile sales, the student might be asked to program a calculator so that, no matter what percent tip we want to give, and no matter how large the bill may be, the calculator will tell us how much money to give the waiter as a tip. --- In such as situation, the calculator routine or the spreadsheet provide conceptual tools that should be sharable, manipulable, modifiable and reuseable in a variety of situations. Furthermore, students need to go beyond thinking with these tools to also think about them - for example, by thinking about the assumptions that they implicitly presuppose.

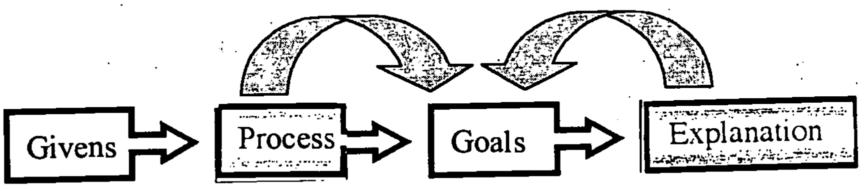

Figure 2. In real life situations where mathematics is useful, it's often the case that the process IS the product that's needed

An important fact to notice about the preceding kinds of conceptual tools is that, even when they reduce the burden of computing results, they often radically increase difficulties associated with describing situations in forms so that the conceptual tools can be used; and, they also my increase difficulties associated with interpreting the results that the tools produce. --- It's these facts that I've referred to when I say that "thinking mathematically" is about constructing, describing, and explaining at least as much as it is about computing.

Another example that will be given in the next section of this paper emphasizes the fact that describing situations mathematically may involve processes that range from quantifying qualitative information, to assigning "weights" to a variety of different kinds of qualitative and quantitative information, to operationally defining constructs (such as "productivity" for workers, or "cost-efficiency" for cars). For the purposes of this section, the main point that I want to emphasize is that the preceding kinds of mathematizing activities generally emphasize almost exactly the opposite kind of processes than those that have been emphasized in traditional word problems in textbook or tests. That is, in traditional word problems, what's problematic is (beyond the computational skills that such problems are intended to emphasize) that students must try to make meaning of symbolically described situations. But, in tasks that emphasize mathematizing activities, what's problematic is to make a symbolic description of meaningful situations.

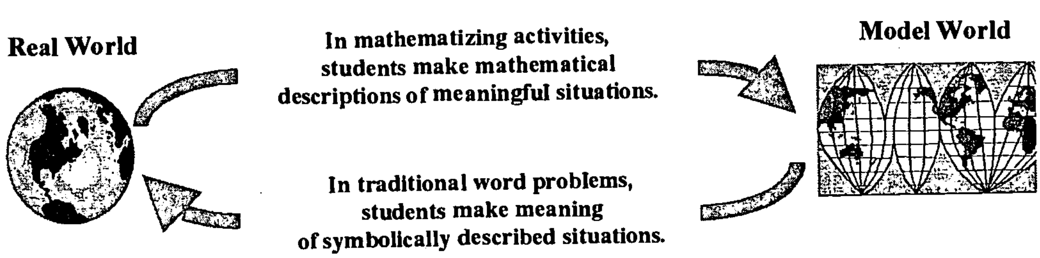

Figure 3. Mathematizing versus Decoding

What's an example of a mathematizing activity?

The *Summer Reading Program Problem* that follows is an example of a middle school version of a "case study" that I first saw being used at Purdue University's Krannert Graduate School of Management.

The Summer Reading Program

The St. John Public Library and Morgantown Middle School are sponsoring a summer reading program. Students in grades 6-9 will read books to collect points and win prizes. The winner in each class will be the student with the most reading points. A collection of approved books already has been selected and put on reserve. The chart below is a sample of the books in the collection.

Title	Author	Reading Level (By Grade)	Pages	Student's Scores on Written Reports	A Brief Description of the Book
Sarah, Plain and Tall	*Patricia MacLachlan*	*4*	*58*		*Note: On a "fold out" page, two or three sentences were given to describe each book. -- Was it a history book, a sports book, an adventure book, etc.*
Awesome Athletes (Sports Illustrated for Kids)	*Multiple Authors*	*5*	*288*		
A Tale of Two Cities	*Charles Dickens*	*9*	*384*		
Much Ado About Nothing	*William Shakespeare*	*10*	*75*		
Get Real (Sweet Valley Jr. High, No. 1)	*Jamie Suzanne & Francine Pascal*	*6*	*144*		

Students who enroll in the program often read between ten and twenty books over the summer. The contest committee is trying to figure out a fair way to assign points to each student. Margret Scott, the program director, said "Whatever procedure is used, we want to take into account: (1) the number of books, (2) the variety of the books, (3) the difficulty of the books, (4) the lengths of the books, and (5) the quality of the written reports.

Note: The students are given grades of A+, A, A-, B+, B, B-, C+, C, C-, D, or F for the quality of their written reports

YOUR TASK: Write a letter to Margaret Scott explaining how to assign points to each student for all of the books that the students reads and writes about during the summer reading program.

Notice that it's similar to many problems that occur when:

- business managers develop ways to quantify constructs like: the "productivity" of workers, or the "efficiency" of departments within a company, or the "cost-effectiveness" of a possible initiatives,
- teachers calculate grades for students by combining performance measures from quizzes, tests, projects, and laboratory assignments – or when they devise "scoring rubrics" to assess students' work on complex tasks, or

- publications such as *places rated almanacs* or *consumer guides* assessments (compare, rank) complex systems such as products, places, people, businesses, or sports teams.

We might refer to problems like the the *Summer Reading Problem* as "construct development problems" because, to produce the product that's needed, the basic difficulty that problem solvers confront involves developing some sort of an "index of reading productivity" for each participant in the reading program. This index needs to combine qualitative and/or quantitative information about: (i) the number of books read, (ii) the variety of books read, (iii) the difficulty of books read, (iv) the lengths of books read, and (v) the quality of reports written – as well as (possibly) other factors such as: (vi) the "weights"(or "importance values") that could be assigned to each of the preceding factors, or (vii) a "diversity rating" that could be assigned to the collection of books that each participant reads.

note: In the *Summer Reading Problem,* it might make sense to multiply the number of books by the difficulty level of each book. But, it might make sense to add scores from reading and scores from written reports. --- In general, to combine other types of information, students must ask themselves "Does it make sense to add, to subtract, to multiply, to divide, or to use some other procedure such as vector addition?" In other words, one of the main things that's problematic involves deciding which operation to use.

One important point to emphasize about "construct development problems" is that, even though such problems almost never occur in textbooks or tests, it's fairly obvious they occur frequently in "real life" situations where mathematics is used; and, it's also obvious that they represent just a small portion of the class of problems in which the products students are challenged to produce go beyond being a simple numeric answers (e.g., 1,000 dollars) to involve a the development of conceptual tools that can be:

- used to generate answers to a whole class of questions,
- modified to be useful in a variety of situations, and
- shared with other people for other purposes.

Furthermore, in addition to providing routines for computations, the tool also may involve:

- descriptions (e.g., using texts, tables, or graphs to describe relationships among variables),
- explanations (e.g., about how, when, and why to do something),
- justifications (e.g., concerning decisions that must be made about trade-offs involving factors such as quality and quantity or diversity), and/or
- constructions (e.g., of a "construct" such as "reading productivity").

Therefore, when mathematics instruction focuses on problem solving situations in which the products that are needed include the preceding kinds of conceptual tools, straightforward ways emerge for dealing with many of the most important components of what it means to develop deeper and higher-order understandings of the constructs that the tools embody.

Another important point to emphasize solving situations is that, when we observe students working on the preceding kinds of problems, the understandings and abilities that contribute to success often are quite different than those that have been emphasized in traditional textbooks and tests. For example, even though many of the same basic mathematical ideas are important (such as those involving rational numbers, proportional reasoning, and measurement), attention often shifts beyond asking *What computations can students do?* toward asking *What kind of situations can students describe (in forms so that computational tools can be used)?*

If we ask - What kind of mathematical understandings and abilities will be needed for success beyond school in a technology-based age of information? – the kind of examples that I've given so far should make it clear that the kind of mathematical conceptual tools that are needed often must be based on more than algebra from the time of Descartes, geometry from the time of Euclid, calculus from the time of Newton, and shopkeeper arithmetic from an industrial age. For example, mathematical topics that are both useful and accessible to students may include basic ideas from discrete mathematics, complexity theory, systems analysis, or the mathematics of motion – where the emphasis is on multi-media displays, representational fluency, iterative and recursive functions, and dynamic systems. Nonetheless, in general, to provide powerful foundations for success in a the new millennium, the kind understandings and abilities that appear to be most needed are not about the introduction of new topics as much as they are about broader, deeper, and higher-order treatments of traditional topics such as rational numbers, proportions, and elementary functions that have been part of the traditional elementary mathematics curriculum, but that have been treated in ways that are far too narrow and shallow for the purposes that concern us here.

At Purdue's *Center for Twenty-first Century Conceptual Tools* (TCCT), where I'm the Director, we enlist leaders from future-oriented fields ranging from aeronautical engineering, to business management, to computer technologies, to agricultural sciences to help us investigate:

- *What is* the nature of the most important elementary-but-powerful understandings and abilities that are likely to be needed as foundations for success in a technology-based *age of information*?
- *What is the nature of typical problems solving situations in which students must learn to function effectively when mathematics and science constructs are used beyond school?*

In these investigations conducted in the TCCT Center, it's noteworthy that participants are consistently reaching a consensus about the following claims.

- Some of the most important goals of instruction should be to help students develop powerful models and conceptual tools for making (and making sense of) complex systems.
- Some of the most effective ways to help students develop productive conceptual systems is to use "case studies" (or simulations of real life problem solving situations) in which students develop, test, and refine sharable and re-usable conceptual tools for dealing with classes of structurally similar problems.

- In problem-solving and decision-making situations beyond schools, the kind of mathematical and scientific capabilities that are in highest demand are those that involve: (i) the ability to work in diverse teams of specialists, (ii) the ability to adapt to new tools and unfamiliar settings, (iii) the ability to unpack complex tasks into manageable chunks that can be addressed by different specialists, (iv) the ability to plan, monitor, and assess progress, (v) the ability to describe intermediate and final results in forms that are meaningful and useful to others, and (vi) the ability to produce results that are timely, sharable, transportable, and re-useable. Consequently, mathematical communication capabilities tend to be emphasized, and so do social or interpersonal abilities that often go far beyond traditional conceptions of content-related expertise.

- Past conceptions of mathematics, science, reading, writing, and communication often are far too narrow, shallow, and restricted to be used as a basis for identifying students whose mathematical abilities should be recognized when decisions are made about hiring for jobs - or admissions for educational programs. This is because students who emerge as being especially productive and capable in simulations of "real life" problem solving situations often are not those with records of high scores on standardized tests. Therefore, new ways need to be developed to recognize and reward these students; and, these new approaches should focus on productivity, over prolonged periods of time, on the same kind of complex tasks that are emphasize in "case study" approaches to instruction.

In what ways does modern cognitive science have important implications for instruction focusing on the construction of powerful constructs (or conceptual tools)?

As Figure 4 suggests, humans have tended to explain the mind (and other complex systems) using their most recent advanced technologies as models. For example, during the twentieth century, psychology gradually moved from machine-based metaphors and factory-based models for the mind, beyond computer-based models, toward more organic models based on biotechnologies – from hardware, to software, to wetware.

From an Industrial Age using analogies based on hardware		**Beyond an Age of Electronic Technologies** using analogies based on computer software		**Toward an Age of Biotechnologies** using analogies based on wetware
where systems are considered to be no more than the sum of their parts, and where the interactions that are emphasized involve no more than simple one-way cause-and-effect relationships.	⇒	where silicone-based electronic circuits may involve layers of recursive interactions which often lead to emergent phenomena at higher levels which are not derived from characteristics of phenomena at lower levels	⇒	where neurochemical interactions may involve "logics" that are fuzzy, partly redundant, partly inconsistent, and unstable – as well as living systems that are complex, dynamic, and continually adapting.

Figure 4. Recent Transitions in Models for Making (or Making Sense of) Complex Systems

As a result, today, there is a growing recognition in mathematics education research that: (i) students, teachers, classrooms, courses, instructional programs, curriculum materials, learning tools and minds are all complex systems (taken singly, let alone in combination), and (ii) many of these complex systems cannot be explained using deterministic machine metaphors (even when they're embedded in silicone).

As *cognitive psychology* replaced *behavioral psychology* as the dominant way for educators to think about the nature of mathematics, learning, problem solving and teaching, many mathematics educators have adopted "constructivism" as an instructional philosophy. Two of the most basic constructivist claims are that: (i) constructs (cognitive structures, conceptual tools, and other complex systems developed by humans) must be constructed, and (ii) they can't simply be transmitted into children's minds in prefabricated forms (Steffe & Wood, 1990; Maher, Davis & Noddings, 1990: von Glassersfeld, 1991).

Unfortunately, it's far too easy for an educator to pledge allegiance to both of the preceding claims while continuing to cling to naïve software-based or machine-based metaphors for mind. For example, according to ways of thinking borrowed from the industrial revolution, teachers have been led to believe that the construction of mathematical knowledge in a child's mind is similar to the process of assembling a machine, or programming a computer. --- These "constructivists" might better be described as "assembly-ists" than "constructivists."

Assembly-ist constructivists tend to be easy to recognize. Their notion of the construction process is that teachers should use carefully guided sequences of questions that funnel students' thinking along pre-planned learning trajectory guided by the *teacher's* preferred way of thinking. They seldom put students in situations where the goal is for students to repeatedly express, test, and refine/revise/reject *their own* ways of thinking. Yet, what research at the TCCT Center is showing is that, in cases where ordinary students produce extraordinary results, the reason usually is because teachers devoted unusual attention toward getting students to express their ways of thinking in forms were testable – and encouraging students to refine their conceptual tools through multiple testing-and-revising cycles (Doerr & Lesh, in press).

Conclusions: Implications for productive uses of technology in mathematics education

Above all, what modern cognitive psychology does is to urge educators to focus on the developing conceptual schemes that humans use to make sense of structurally interesting systems. That is, mathematics is about *seeing* at least as much as it is about *doing;* it is about relationships among quantities at least as much as it is about operations with "naked" numbers (that tell "how much" but not "of what"); and, it is about making (and making sense of) patterns and regularities in complex systems at least as much as it is about calculations with pieces of data. It involves interpreting situations mathematically; it involves mathematizing (e.g., quantifying, visualizing, dimensionalizing, or coordinatizing) structurally interesting systems; and, it involves the using and interpreting an ever-expanding array of specialized languages, symbols, graphs, graphics, concrete models, or other representational media for purposes that range from construction, to description, or explanation. That is, *representational fluency* is at the heart of what it means to "understand" most mathematical constructs.

Some of the most important things that technology has done, both in education and in the world beyond schools, have been to radically increase the sophistication, levels, and types of systems that humans create. Another important thing that technology has done is create an explosion of representational media that can be used to describe, explain, and construct complex systems. Furthermore, at the same time that technology has decreased the computational demands on humans, it has radically increased the interpretation and communication demands. For example, beyond schools, when people work in teams using technology-based tools, and when their goals involve making (and making sense of) complex systems: (i) new types of mathematical quantities, relationships, and representation systems often become important (such as those dealing with continuously changing quantities, accumulating quantities, and iterative and recursive functions), (ii) new levels and types of understandings tend to be emphasized (such as those that emphasize communication and representation), and (iii) different stages of problem solving may be emphasized (such as those that involve partitioning complex problems into modular pieces, and planning, communicating, monitoring, and assessing intermediate results).

One of the most important consequences of the preceding trends is that, when broader ranges of mathematical abilities are recognized as contributing to success, broader ranges of people often emerge as being mathematically capable (Lesh & Doerr, 2000). In fact, in research in Purdue's *Center for Twenty-first Century Conceptual Tools* (TCCT), we're seeing that, in graduate schools in future-oriented fields ranging from aeronautical engineering, to business administration, to agricultural sciences, the kind of mathematical abilities that are needed are poorly aligned with (and poorly predicted by) abilities emphasized on traditional standardized tests.

Focusing on foundations for the future does not mean ignoring basics from the past. Abandoning basic skills would be as foolish in mathematics and science (or reading, writing, and communicating) as it would be in basketball, cooking, or carpentry. But, it's not necessary to master the names and skills associated with every item at Sears before students can begin to cook or to build things; and, New Zealand didn't become the famous home of the "All Blacks" in rugby by never allowing it's children to scrimmage until they'd completed twelve years consisting of nothing but drills on skills. What's needed is a sensible mix of complexity and fundamentals; both must evolve in parallel; and, one doesn't come before (or without) the other.

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