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ABSTRACT

This volume contains the proceedings of the International Conference on Technology in Mathematics Education (TIME 2000). It includes papers, posters, and short oral reports. Research papers include: (1) "Implications of the Shift from Isolated Expensive Technology to Connected, Inexpensive, Ubiquitous and Diverse Technologies" (Jim Kaput); (2) "Developing Simulation Activities to Improve Students' Statistical Reasoning" (Beth Chance, Joan Garfield, and Robert delMas); (3) "Technology and Versatile Thinking in Mathematics" (David Tall); (4) "A Strategy for the Use of Technology to Enhance Learning and Teaching in Maths, Stats and OR" (Pam Bishop and Neville Davies); (5) "Technology and the Curriculum: The Case of the Graphics Calculator" (Barry Kissane); (6) "What Mathematical Abilities Are Most Needed for Success Beyond School in a Technology Based Age of Information?" (Richard Lesh); (7) "Mathematics in Action: Two New Zealand Case Studies" (David M. Ryan); (8) "What Can We Learn from Computer-Based Diagnostic Testing?" (John C. Appleby); (9) "Graphics Calculators in Mathematics Learning: Studies of Student and Teacher Understanding" (Michael Cavanagh and Michael Mitchelmore); (10) "Year 8 Students' Progression on an Integrated Learning System" (Tom Cooper, Rod Nason, Gillian Kidman, and Romina Jamieson-Proctor); (11) "Optional Use of Graphics Calculators in Applied Questions for High-School Calculus" (Patricia A. Forster and Ute Mueller); (12) "A Graphic Calculator Approach to Understanding Algebraic Variables" (Alan T. Graham and Michael O.J. Thomas); (13) "Mathematics Education and Learning Technology Research Program Trajectories of the National Science Foundation (USA)" (Eric R. Hamilton); (14) "Teachers' Thinking about Information Technology and Learning: Beliefs and Mathematical Outcomes" (Steven E. Higgins and David V. Moseley); (15) "SuperÐCalculators and Conceptual Understanding of the NewtonÐRaphson Method" (Ye Yon Hong and Michael O.J. Thomas); (16) "From Construction to Deduction: Potentials and Pitfalls of Using Software" (Celia Hoyles and Lulu Healy); (17) "When a Visual Representation is Not Worth a Thousand Words" (Gillian C. Kidman and

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PREFACE

TIME 2000 An International Conference on Technology in Mathematics Education, took place 11th–14th December, 2000 in Auckland, New Zealand. The conference was jointly sponsored by the Department of Mathematics, The University of Auckland and the Department of Applied Mathematics, Auckland University of Technology. The conference, the first of its kind in New Zealand, attracted papers and presentations from researchers around the world, including the USA, UK, Germany, Japan, Singapore, Finland, Hong Kong, Australia and New Zealand. Among those presenting papers were a good number of distinguished international authors and I would like to express my sincere appreciation for their support of the conference.

These proceedings were prepared following a refereeing process overseen by a committee comprising myself and Mohan Chinnappan (Wollongong University), with assistance from Sergiy Klymchuk. Each paper was initially sent to two referees who were asked to recommend that it be placed in one of three categories:

Accept as a full paper at TIME 2000
Accept as a short communication at TIME 2000
Do not accept for TIME 2000.

Referees were asked to supply reasons for their judgement on the acceptance or rejection of the paper using the following categories:

Theoretical Framework and Related Literature
Methodology and Data Analysis (where appropriate)
Statement and Discussion of Results
Clarity and Presentation
General comments including reasons for the recommendation

Where there was any disagreement in the categorisation the paper was given to a third referee to read and comment on. In a number of cases the paper was also read by one of the members of the committee. After this process was complete one or two authors requested an opportunity to revise their papers in the light of reviewers' comments and this was given. Finally the twenty-two papers in this volume were accepted as full papers for the conference, along with the invited plenary papers. In view of the care taken I am confident that the quality of the papers accepted meets an acceptable international standard for research publications.

There were a number of well written papers which were not considered suitable as full research papers and eight of these were presented as Short Communications at the conference. While the abstracts for these presentations form part of these proceedings, the papers themselves are included in a separate volume of papers associated with the conference, which also contains papers submitted in support of workshops. It should be emphasised that that volume does not comprise part of the proceedings of the TIME 2000 conference, and should not be cited as such.

I would like to thank all those who acted as referees for this conference. I am well aware of the increasing demands on the time and energy of all of us involved in this vital activity and acknowledge their contribution to making this a successful conference. I would also like to thank my colleagues both in the Mathematics Department and on the Conference Committee for their support, assistance and understanding during the last year.

I am conscious that in an undertaking such as this one it is highly unlikely that I have carried it out without making a number of errors. I would like to take this opportunity to apologise to any author who feels that their work, through my editing, has not been represented here in the manner they would have liked.

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IMPLICATIONS OF THE SHIFT FROM ISOLATED, EXPENSIVE TECHNOLOGY TO CONNECTED, INEXPENSIVE, DIVERSE AND UBIQUITOUS TECHNOLOGIES

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We examine the long term history of the development of fundamental representational infrastructures such as writing and algebra, and how they were physically implemented via such devices as the printing press and computers, in order to (1) gain insight into what is occurring today both in terms of representational infrastructure change and in physical embodiments, (2) obtain clues regarding what to do next, and (3) determine the kinds of questions that research will need to answer in the coming decade if we are to make optimal use of new diverse and connected classroom technologies.

Placing current changes in historical perspective: The evolution of representational infrastructures and their material technologies

I suggest that we need to have a sense of some of the major changes of the past in order to understand the technology-related transformations of the past few decades and their trajectory through this coming decade—especially those changes having to do with the representational infrastructures with which we think and communicate. I will briefly examine the evolution of writing systems and then the printing press, as well as glance briefly towards the histories of arithmetic and algebra, in order to gain a perspective on what is happening today. In no way are the historical abstracts below intended to be definitive or at all complete, since the formal study of these matters is well beyond the scope of this paper. Further, while I will be drawing general parallels between the early development of basic representational infrastructures and the physical means by which they could be made available to wider populations, historical analogies are fraught with difficulties, especially in the details. But it is exactly in these complicating details where we can get a sense of what needs to be done to exploit the apparent representational advantages of the computational medium in mathematics education. In particular, we will, in the third part of this paper, examine some specific examples of new directions, and in the fourth part of the paper examine the kinds of open questions needing exploration in the coming few years.

Part I: The Evolution of Representational Infrastructures in Static Inert Media

The Evolution of Written Languages—Alphabetic, Phonetic Systems

Across the past half-dozen or so millennia of human history since the gradual emergence of writing—the primary means by which humanity extended its biological mind—several major changes in representational infrastructure have occurred. Over several thousand years and in several societies in the Middle East, and at various later times in other places around the world (Woodard, 1996), writing began as an ideographic, non-phonetic system for expressing ideas. In the Middle East the information was often quantitative information (Schmandt-Besserat 1978; 1981; 1988; 1992; 1994) expressed for economic purposes. These latter systems, with roots in the impressing of physical tokens in clay, made large demands on human memory and interpretive skill, and hence were laboriously learned and used only by specialists—scribes. For example, approximately 15% of all the 100,000 existing cuneiform tablets were used to train scribes. This writing system used hundreds of non-phonetic symbols in a highly nonlinear and context-dependent way (Walker, 1987). Note also that the complex non-phonetic system and the lexical lists used to train scribes during the 3rd millennium B.C. remained essentially unchanged for more than 600 years—a hint that the conservative nature of education is not a recent development. Indeed, the time scale of evolution of writing systems is on the order of thousands of years, which suggests that the achievements did not come easily and were driven and constrained by many factors beyond the inventiveness of the scribes.

Over a period of almost 3000 years, the systems of writing in the Middle East, including Egyptian hieroglyphs, gradually evolved into more phonetic systems (although the early hieroglyphs were more pictorial than early cuneiform writing). This allowed the users to tap into the meaning-carrying and meaning-making resources of spoken language, which had evolved over the previous 300,000 years and hence had deep biological support in the muscular and neurophysiological structure of all normal humans (Deacon, 1997). However, there was never a direct map onto the sound-stream of speech, but rather a far more complex relationship that typically involved decoding the meaning of a symbol sequence prior to being able to specify the sounds associated with it (Davies, 1987). Similar evolutions occurred with many other writing systems across the world (Woodard, 1996).

So, while the move to a phonetic system improved expressiveness of the writing systems, it did not solve the problem of learnability. The solution, which appeared gradually, took the form of a small, efficient alphabet. From two to four thousand years ago, across the Middle East and Mediterranean basin, various Semitic scripts developed single-consonant syllabaries (none for vowel sounds) enabling increasingly consistent encoding of symbols for sounds that in turn carried the meanings expressed by spoken language. As happened elsewhere, these evolved over many hundreds of years in different Middle East locations to become the Arabic, Hebrew, Aramaic, and Phoenician alphabets, which provided genuinely phonetic mappings onto the sound stream—achieving a mapping of time onto space (Ong, 1982).

During the period between 700–1100 BC, the 22 consonants of the Phoenician alphabet were adopted by the Greeks, where they were extended to encode their Indo-European language by converting certain unused consonants to vowels, to take into account the sounds of their Indo European language, which gives more prominence to vowel sounds. So by reinterpreting certain Phoenician consonants as vowels, and adding three more consonants, the Greeks produced a version of the alphabet that, in various forms, we use in virtually all Western languages today. Hence, by using the sound-mapping rules of the language at hand, every written combination of these symbols can now be spoken aloud, whether or not it corresponds to the words of that language, and any idea that can be spoken in that language can be written down! As with the development of spoken language, this extraordinary achievement changed the nature of what it means to be human by changing cognition, culture and the societies in which writing ensued (Donald, 1991). As put by Goody & Watt (1968, p.9) and cited in Haas, (1996, p.11): "the notion of representing a sound by a graphic symbol is itself so stupefying a leap of the imagination that what is remarkable is not so much that it happened relatively late in human history, but that it ever happened at all" (Havelock, 1982).

Of course, there are logograms that carry meaning independent of speech sounds, including the combinations of Hindu-Arabic numerals that express numbers (whose pronunciation varies according to the phonetic language used), and various visual icons such as the circled picture of a cigarette crossed by a diagonal line that is internationally used as a "no smoking" sign. But these are at the boundary of an extraordinarily efficient symbol system that uses roughly two dozen alphabet marks with an amazing, almost infinite, range of expressiveness.

However, it was not until writing was able to tap into human speech capability *efficiently* via a highly compact alphabetic system that writing could have a chance to become universal (although the coding/decoding processes are decidedly complex and change quite radically as learning occurs, e.g., Langer, 1986). It became a fundamental representational infrastructure, learnable by most humans by the age of eight or nine, if given the opportunity. And it changed the means by which humans constructed their world individually (Nelson, 1996) and culturally (e.g., Cole, 1997; Donald, 1991). Humans became able to communicate, build, and accumulate knowledge (and all that comes with knowledge—including power and control) across time and space.

The Next Step: Making the Representational Infrastructure Widely Available via the Printing Press—An Evolutionary, Not Revolutionary, Impact

In the West, the next 1500 years saw the availability of the writing infrastructure limited by two apparently linked factors, the physical scarcity of written materials due to the lack of inexpensive reproduction technology and social conditions that limited the availability of literacy instruction primarily to elite males—elite by virtue of belonging to the ruling class or by virtue of belonging to the academic (religious) class (Kaestle, 1985; Kaufer & Carley, 1992; Resnick & Resnick, 1977). The Kaufer & Carley study shows how the widely used argument for the "revolutionary" impact of the printing press provided by Eisenstein's massive and widely accepted historical account (Eisenstein, 1979) ignores the evolutionary nature of the change that actually seems to have occurred across the three hundred years following Gutenburg. For example, for the first 100 years the traditional manuscript/scribe structure existed in parallel to the printing system, and it was not until the much faster (by 1–2 orders of magnitude) steam-driven press was available during the Industrial Revolution two hundred years later that printed materials became widely available for the kinds of documents (e.g., newspapers and political statements) that would have large social impact. Indeed, as pointed out by Haas (1996), Eisenstein's own arguments and detailed historical accounts help make the argument that the cultural, social, and political changes that accompanied the spread of printed materials—especially in vernacular languages—were gradual and related to multiple factors beyond the innovation of moveable type (which actually was used by the Chinese, along with paper, at least 500 years earlier). Furthermore, as argued in detail by Clanchy (1979), many preconditions seemed to be necessary that involved shifts in government and business needs from oral to written modes, acceptance of written records of events, distribution and communication channels, availability of paper (as opposed to expensive parchment), and other factors.

Hence, the physical technology of the printing press *did* have profound effects on society, *did* make the democratization of literacy possible, *did* lead to a standardization of language, and *did* participate in the deep cultural changes of the Renaissance leading to Modernism as outlined by Eisenstein, but its effects were intimately linked to a variety of other changes, both prior to and following its advent in the 15th century, including changes in the printing technology itself (Kaufer & Carley, 1992).

One important distinction should be kept in mind—the distinction between a change in representational infrastructure, such as alphabetic writing, and a change in the material means by which that infrastructure can be embodied, such as the printing press and inexpensive paper—which participates in a different kind of infrastructure, a combined technological and social infrastructure.

Two Additional Representational Infrastructures—The Operative Symbol Systems of Arithmetic and Algebra

The histories of arithmetic and algebra are well-known, and we will not recount them here. See (Hoyles, Kaput, & Noss, in press) for a more detailed account. However, it is worth noting that the evolution of each was a lengthy process, covering thousands of years before the achievement of an efficient symbol system upon which a human could operate. Unlike written language, which supported the creation of fixed records in static, inert media, the placeholder system of arithmetic that stabilized in the 13th–14th centuries supported rule-based *actions* by an appropriately trained human upon the physical symbols that constitute quantitative operations on the numbers taken to be represented by those symbols. This system and the algorithms built on it, seems to be optimal in an evolutionary sense similar to the way the alphabetic phonetic writing systems seem to be optimal. Each has remained relatively stable for many centuries and has spread widely across the world. The arithmetic system, although initially a specialist's tool—for accounting purposes—came to be part of the general cultural tool-set as needs for numerical computation arose in Western societies. Interestingly, the early algorithms developed for accounting in the 14th–15th centuries and that appeared in the first arithmetic training books at that time have remained essentially unchanged to this day, and continue to dominate elementary school mathematics (Swetz, 1987).

Algebra began in the times of the Egyptians in the second millennium BC as evidenced in the famous Ahmes Papyrus by using available writing systems to express quantitative relationships, especially to "solve equations"—to determine unknown quantities based on given quantitative relationships. This is the so-called "rhetorical algebra" that continued to Diophantus' time in the 4th century of the Christian era, when the process of abbreviation of natural language statements and the introduction of special symbols began to accelerate. Algebra written in this way is normally referred to as "syncopated algebra." By today's standards, achievement to that point was primitive, with little generalization of methods across cases and little theory to support generalization.

Then, in a slow, millennium-long struggle involving the co-evolution of underlying concepts of number, algebraic symbolism gradually freed itself from written language in order to support techniques that increasingly depended on working with the symbols themselves according to systematic rules of substitution and transformation—rather than the quantitative relations for which they stood. Just as the symbolism for numbers evolved to yield support for rule-based operations on symbols taken to denote numbers, where attention and mental operations guide actions on the *notations* rather than what they are assumed to refer to, the symbolism for quantitative relations likewise developed. Bruner (1973) refers to this as an "opaque" use of the notations rather than "transparent" use, where the actions are guided by reasoning about the entities to which the notations are assumed to refer. In effect, algebraic symbolism gradually freed itself from the (highly functional) ambiguities and general expressiveness of natural language in order that very general statements of quantitative relations could be very efficiently expressed.

However, the more important aspects of the new representational infrastructure are those that involve the rules, the syntax, for guiding operations on these expressions of generality. These emerged in the 17th century as the symbolism became more compact and standardized in the intense attempts to mathematize the natural world that reached such triumphant fruition in the "calculus" of Newton and Leibniz (more on this below). In the words of Bochner (1966, pp. 38–39):

> Not only was this algebra a characteristic of the century, but a certain feature of it, namely the "symbolization" inherent to it, became a profoundly distinguishing mark of all mathematics to follow. ... (T)his feature of algebra has become an attribute of the essence of mathematics, of its foundations, and of the nature of its abstractness on the uppermost level of the "ideation" a la Plato.

Beyond this first aspect of algebra, its role in the expression of abstraction and generalization, he also pointed out the critical new ingredient:

> ... that various types of 'equalities,' 'equivalences,' 'congruences,' 'homeomorphisms,' etc. between objects of mathematics must be discerned, and strictly adhered to. However this is not enough. In mathematics there is the second requirement that one must know how to 'operate' with mathematical objects, that is, to produce new objects out of given ones (*ibid*, p. 313).

Indeed, Mahoney (1980, p. 142) points out that this development made possible an entirely new mode of thought "characterized by the use of an operant symbolism, that is, a symbolism that not only abbreviates words but represents the workings of the combinatory operations, or, in other words, a symbolism with which one operates."

This second aspect of algebra, the syntactically guided transformation of symbols while holding in abeyance their potential interpretation, flowered in the 18th century, particularly in the hands of such masters as Euler, to generate powerful systems of understanding the world. But this operative aspect of algebra is both a source of mathematics' power, and a source of difficulty for learners. However, another learning-difficulty factor is the separation from natural language writing and hence the separation from the phonetic aspects of writing that support tapping into the many powerful narrative and acoustic memory features of natural language. Indeed, as well known via the error patterns seen in the "Student-Professors Problem", the algebraic system is in partial *conflict* with features of natural language (Clement, 1982; Kaput & Sims-Knight, 1983). For many good reasons, traditional character string-based algebra is *not* easy to learn.

Historical Analogies with Writing: Arithmetic and Algebra

Small, Elite Literacy Community.

Thus, over an extremely long period, a new special-purpose operational representational infrastructure was developed that reached beyond the symbolic operational infrastructure for arithmetic. However, in contrast with the arithmetic system, *the algebra system was built by and for a small and specialized intellectual elite* at whose hands, quite literally, it extended the power of human understanding far beyond what was imaginable without it. Importantly, it was designed and used by specialists without regard for its learnability by the population at large. The effect of these learnability factors did not really become felt until the latter part of the 20th century when education systems around the world began to attempt to teach algebra to the general population. Prior to the middle of the 20th century, the algebra literacy community was quite small, quite analogous to the small literacy communities of specialists associated with early writing.

The Growth of Societal Need for Writing, Arithmetic and Now, Quantitative Insight.

Early writing was a response to a social need, which grew over time to include broad expressiveness requirements to encode the rules and cultures of growing urban societies, which led to the phonetic system and eventually the alphabetic system. A similar growth occurred in the case of arithmetic, which was initially the province of specially trained "reckoners"—the accountants of the 15th and 16th centuries. (A needs-argument could be given for the initial development of the arithmetic system over the previous centuries.) But the need for arithmetic skill spread across the population over the next 200–300 year, which led it to become one of the core topics of universal schooling in most countries.

By the end of the 20th century, with the growth of the knowledge economies, the need for quantitative insight spread across the population of industrialized countries in a way analogous to the way it spread for arithmetic skill earlier. This general need combined with the politically driven need to democratize *opportunity* to learn higher mathematics, typically assumed to require knowledge and skill in algebra, has produced considerable tension in many democracies, especially the United States, where access to algebra learning has come to be seen as political right (Moses, 1995). The attempts to democratize access to traditional algebra in the Unites States have not been successful despite much work and energy. Algebra is in many ways analogous to early writing in its learnability and associated literacy community.

Deep Changes in the Nature of the Representational Infrastructure—Writing and Algebra Linking Into Existing Powerful Systems.

However, just as writing gradually tapped into another, previously established human system of meaning making and communicating, and became radically reconstituted in the process as it became phonetic, algebra may likewise be on the verge of doing so. In this case, instead of the auditory-narrative system, it is the visuo-graphic system. Although it may not have been Descartes' (or Fermat's) intent, anticipated by Oresme (Clagett, 1968), he laid the base for tapping into humans' visual perceptual and cognitive capacities previously employed only by geometry. I am reminded of Joseph Lagrange's comment "As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company, they drew from each other fresh vitality and thenceforward marched on a rapid pace towards perfection," cited in Kline, (1953, p. 159). From 350 years ago to our contemporary graphs of quantitative relationships, I see an analogy to the gradual transition that occurred as writing became more phonetic—the newer ways coexist with the old in various combinations as we graph algebraic functions which are defined and input into our graphing systems via character strings. And more recently, we have been able to hot-link these in the computer medium.

Will we be able to make the next transition that might make the representation and manipulation of quantitative relationships broadly learnable? In the case of writing, this required the invention of the alphabet. Below, I shall propose an analogous step for algebra.

The Development of Material Means By Which Access to Learnable Representational Infrastructures Might Be Democratized—Analogy With the Printing Press.

Anticipating the discussion below, I will suggest that, with the emergence of computers, we are involved in an extended process analogous to the evolution associated with the development of the printing press. The first stages involved expensive and hence rare central computers, what we have known as "mainframes" and "mini computers." Then came the microcomputer and networks—connectivity. More recently, we have seen the emergence of hand-helds and, even more recently, connectivity across device-types. For computation, we are heading towards the kind of ubiquitous access and full integration into life and work that was achieved by printed writing materials that eventually occurred by the time of the Industrial revolution. But, as our cursory examination of the printing press evolution suggests, the process will take time and will depend on many other changes taking place along the way.

Part II: The Birth of New Representational Infrastructures in Dynamic, Interactive Computational Media—the Case of Calculus

The Shift from Static, Inert Media to Dynamic, Interactive Media

The systems of knowledge that form the core of what was taught in schools and universities in the 20th century were built using some representational infrastructures that evolved (e.g., alphabetic and phonetic writing) and others that were somewhat more deliberately designed, mainly by and for a narrow intellectual elite (e.g., arithmetic and, to a greater extent, operative algebra). In all cases, they were instantiated in and hence subject to the constraints of the static, inert media of the previous several millennia. But, the *computational* medium is neither static nor inert, but rather, is dynamic and interactive, exploiting the great new advance of the 20th century, autonomously executable symbolic processes—that is, operations on symbol systems not requiring a human partner (Kaput & Shaffer, in press). The physical computational medium is based on a major physical innovation, the transistor, that, in turn, was the product of the prior knowledge system (Riorden & Hoddeson, 1997). The longer term development of the computational medium is reviewed in Shaffer & Kaput (1999).

Reflecting the fact that we are in the midst of a huge historical transition, I see three profound types of consequences of the development of the new medium for carrying new representational infrastructures:

Type 1: The knowledge produced in static, inert media can become knowable and learnable in new ways by changing the medium in which the traditional notation systems in which it is carried are instantiated—for example, creating hot-links among dynamically changeable graphs equations and tables in mathematics. Most traditional uses of technology in mathematics education, especially graphing calculators and computers using Computer Algebra Systems, are of Type 1.

Type 2: New representational infrastructures become possible that enable the reconstitution of previously constructed knowledge through, for example, the new types of visually editable graphs and immediate connections between functions and simulations and/or physical data of the type developed and studied in the SimCalc Project—to be described briefly below.

Type 3: The construction of new systems of knowledge employing new representational infrastructures—for example, dynamical systems modeling or multi-agent modeling of Complex Systems with emergent behavior, each of which has multiple forms of notations and relationships with phenomena. This is a shift in the nature of mathematics and science towards the use of computationally intensive iterative and visual methods that enable entirely new forms of dynamical modeling of nonlinear and complex systems previously beyond the reach of classical analytic methods—a dramatic enlargement of the MCV that will continue in the new century (Kaput & Roschelle, 1998; Stewart, 1990).

Tracing any of these complex consequences is a challenging endeavor, particularly since they overlap in substantive ways due to the fact that knowledge is co-constituted by the means through which it is represented and used—it does not exist independently of its representation (Cobb, Yackel, & McClain, 2000). Hence, we will limit our discussion to a few cases close to our recent work in the SimCalc Project involving the Mathematics of Change & Variation (MCV), of which a subset concerns the ideas underlying Calculus. Thus we will be focusing on a Type 2 change.

Calculus

While the Greeks, most notably Archimedes—whose extraordinary computational ability compensated for the weaknesses of the available representational infrastructure in supporting quantitative computation—developed certain mainly geometric ideas and techniques, the Mathematics of Change and Variation leading to what came to be called "Calculus" evolved historically beginning with the work of the Scholastics in the 1300's through attempts to mathematize change in the world (reviewed in Kaput, 1994).

The resulting body of theory and technique that emerged in the 17th and 18th centuries, cleaned up for logical hygiene in the 19th, is now institutionalized as a capstone course for secondary level students in many parts of the world, and especially in the United States. These ultimately successful mathematizing attempts were undertaken by the intellectual giants of Western civilization, who, in so doing, also developed the representational infrastructure of algebra, including extensions to infinite series and coordinate graphs, as part of the task. Their work led to profoundly powerful understandings of the different ways quantities can vary, how these differences in variation relate to the ways the quantities accumulate, and the fundamental connections between varying quantities and their accumulation. These efforts also gave rise to the eventual formalization of such basic mathematical ideas as function, series, limit, continuity, etc. (Boyer, 1959; Edwards, 1979).

Over the past two+ centuries this community's intellectual tools, methods and products—the foundations of the science and technology that we utterly depend upon—were institutionalized as the structure and core content of school and university curricula in most industrialized countries and taken as the epistemological essence of mathematics (Bochner, 1966; Mahoney, 1980) as noted above. This content has also been taken as the subject of computerization. That is, it has been the target of Type 1 reformulation.

As already noted, the SimCalc Project has been engaged instead in a Type 2 reformulation of the core content associated with Calculus, which we review briefly before turning to newer technological issues.

Summary of SimCalc Representational Infrastructure Changes

We summarize the core web of five representational innovations employed by the SimCalc Project, all of which require a computational medium for their realization. The fifth—not discussed below in detail—is mentioned for completeness, actually falls into Type 3. In order to save space, these innovations are illustrated in later examples, which will refer to this list. Cross-platform software, Java MathWorlds for desktop computers can be viewed and downloaded at http://www.simcalc.umassd.edu and software for hand-helds can be examined and downloaded from http://www.simcalc.com.

- **Definition and direct manipulation of *graphically defined* functions, especially piecewise-defined functions**, with or without algebraic descriptions. Included is "Snap-to-Grid" control, whereby the allowed values can be constrained as needed—to integers, for example, allowing a new balance between complexity and computational tractability whereby key relationships traditionally requiring difficult prerequisites can be explored using whole number arithmetic and simple geometry. This allows sufficient variation to model interesting situations, avoid the degeneracy of constant rates of change, while postponing (but not ignoring!) the messiness and conceptual challenges of continuous change.

- **Direct connections between the above representational innovations and simulations**—especially motion simulations—to allow immediate construction and execution of a wide variety of variation phenomena, which puts phenomena at the center of the representation experience, reflecting the purposes for which traditional representations were designed initially, and enabling orders of magnitude tightening of the feedback loop between model and phenomenon.

- **Direct, hot-linked connections between graphically editable functions and their derivatives or integrals**. Traditionally, connections between descriptions of rates of change (e.g., velocities) and accumulations (positions) are usually mediated through the algebraic symbol system as sequential procedures employing derivative and integral formulas—but need not be. In this way, the fundamental idea, expressed in the Fundamental Theorem of Calculus, is built into the representational infrastructure from the start, in a way analogous to how, for example, the hierarchical structure of the number system is built into the placeholder system.

- **Importing physical motion-data via MBL/CBL and re-enacting it in simulations, and exporting function-generated data to drive physical phenomena** LBM (Line Becomes Motion), which involves driving physical phenomena, including cars on tracks, using functions defined via the above methods as well as algebraically. Hence there is a two-way connection between phenomena and mathematical notations.

- **Use of hybrid physical/cybernetic devices embodying dynamical systems,** whose inner workings are visible and open to examination and control with rich feedback, and whose quantitative behavior is symbolized with real-time graphs generated on a computer screen.

The result of using this array of functionality, particularly in combination and over an extended period of time, is a qualitative transformation in the mathematical experience of change and variation. However, short term, in less than a minute, using either rate or totals descriptions of the quantities involved, or even a mix of them, a student as early as 6^{th}–8^{th} grade can construct and examine a variety of interesting change phenomena that relate to direct experience of daily phenomena. And in more extended investigations, newly intimate connections among physical, linguistic, kinesthetic, cognitive, and symbolic experience become possible.

Importantly, taken together, these are not merely a series of software functionalities and curriculum activities, but amount to a reconstitution of the key ideas. Hence we are not merely treating the underlying ideas of Calculus in a new way, treating them as the focus of school mathematics beginning in the early grades and rooting them in children's everyday experience, especially their kinesthetic experience, but *we are reformulating them in an epistemic way.* We continue to address such familiar fundamentals as variable rates of changing quantities, the accumulation of those quantities, the connections between rates and accumulations, and approximations, but they are experienced in profoundly different ways, and are related to each other in new ways.

These approaches are not intended to eliminate the need for eventual use of formal notations for some students, and perhaps some formal notations for all students. Rather, they are intended to provide a substantial mathematical experience for the 90% of students in the U.S. who do not have access to the Mathematics of Change & Variation (MCV), including the ideas underlying Calculus, and provide a conceptual foundation for the 5–10% of the population who need to learn more formal Calculus. Finally, these strategies are intended to lead into the mathematics of dynamical systems and its use in modeling nonlinear phenomena of the sort that is growing dramatically in importance in our new century (Cohen & Stewart, 1994; Hall, 1992; Kaput & Roschelle, 1998; Stewart, 1990).

In terms of our historical perspective, we see this current work as part of a large transition towards a much more broadly learnable mathematics of quantitative reasoning, where both the representational infrastructure is changing as well as the material means by which those more learnable infrastructures can be made widely available. Taken together, it may be that the kinds of representational innovations outlined above and illustrated in the next part of the paper constitute the development of a new "alphabet" for quantitative mathematics which might do for mathematical representation what the phonetic alphabet did for writing, particularly if coupled to the right kinds of physical implementations, which are examined below.

Part III: Affordances & Constraints of Diverse, Connected, and Ubiquitous Computing Devices

Revisiting the Analogy with Changes Made Possible by the Printing Press

As we saw above, it is one thing to have a powerful and learnable representational infrastructure, such as alphabetic writing, and it is entirely another matter to have broad access to that infrastructure. We traced briefly the gradual three-century impact of the printing press on the democratization of literacy. It is a major hypothesis of this paper that a similar change is now underway in the 21st century relative to the new representational infrastructures made possible by the computational medium. We have traced three types of changes and introduced an example of one of them targeting the Math of Change & Variation (MCV) as developed in the SimCalc Project. Others could be put forward as well—for example, Dynamic Geometry™ or dynamically manipulable data management and analysis systems for statistics.

However, it is one thing to instantiate representational infrastructure innovations on expensive and hence scarce computational devices. It is entirely another to render them materially available on inexpensive, ubiquitous devices well-integrated into the flow of life, work, and education. A message from the history of the printing press is that the change needed to democratize access to the new infrastructures will be slow and will complexly involve many aspects of our culture. Of course, "slow" in today's terms is relative to rates of change that are at least an order of magnitude faster than in previous centuries, so that 300 years may shrink to 30 or less—but not likely to 3. The integration of automobiles, telephones, and television each took about a generation to reach wide penetration in industrialized societies. Penetration of the world wide web into everyday life and intellectual work seems to have taken only about half that time.

In today's school climate, full-sized, desktop computers are a relatively costly and rare commodity, compared to, say, pencils or notepads. Therefore, most schools that have computers share their availability across many different uses and populations. Furthermore, despite falling costs for a given level of power and functionality, their maintenance cost, especially in network configurations, tends to be prohibitive for most schools to deploy them on a wide-scale basis except in the wealthiest communities. The school mathematics alternative has been the graphing calculator, which has been designed primarily to support Type 1 changes. It has typically taken the form of a full, open toolkit, isolated technologically from other computational devices, and independent of any particular curriculum, which has been supplied offline. This condition is now changing.

Illustrations of Representational Infrastructure Innovations across Multiple But Non-Networked Hardware & Software Platforms

We will now explore some of the changes that are occurring in the nature and configurations of the technologies that can support the new representational infrastructures that computational media make possible. In particular, we see a rapid growth in the availability of flexible, relatively inexpensive, and wirelessly networkable hand-held devices that can run independently produced curriculum-targeted software and hence support new kinds of teaching and learning opportunities. We will first provide a series of activity-examples that simultaneously illustrate certain of the representational innovations identified above, how they map onto radically different hardware systems, and second, how they can be extended to operate in a networked classroom.

Determining Mean Values—SimCalc Representational Infrastructure Innovations 1–3 Above

Figure 1 shows the velocity graphs of two functions, respectively controlling one of the two elevators on the left of the figure (graphs on the desktop software are color-coded to match the elevator that they control). The downward-stepping, but positive, velocity function, which controls the left elevator, typically leads to a conflict with expectations, because most students associate it with a downward motion. However, by constructing it and observing the associated motion (often with many deliberate repetitions and variations), the conflicts lead to new and deeper understandings of both graphs and motion. The second, flat, constant-velocity function in Figure 1 that controls the elevator on the right provides constant velocity. It is shown in the midst of being adjusted to satisfy the constraint of "getting to the same floor at exactly the same time." This amounts to constructing the average velocity of the left-hand elevator which has the (step-wise) variable velocity. This in turn reduces to finding a constant velocity segment with the same area under it as does the staircase graph. In this case the total area is 15 and the number of seconds of the "trip" is 5, so the mean value is a whole number, namely, 3. We have "snap-to-grid" turned in this case so that, as dragging occurs, the pointer jumps from point to point in the discrete coordinate system. Note that if we had provided 6 steps for the left elevator instead of 5, the constraint of getting to the same floor at exactly the same time (from the same starting-floor) could not be satisfied with a whole number constant velocity, hence could not be reached with "snap-to-grid" turned on.

The standard Mean Value Theorem, of course, asserts that if a function is continuous over an interval, then its mean value will exist and will intersect that function in that interval. But, of course, the step-wise varying function is *not* continuous, and so the Mean value Theorem conclusion would fail—as it would if 6 steps were used. However, if we had used imported data from a student's physical motion, as in Figure 3, then her velocity would necessarily equal her average velocity at one or more times in the interval. We have developed activities involving a second student walking in parallel whose responsibility is to walk at an estimated average speed of her partner. Then the differences between same-velocity and same-position begin to become apparent. Additional activities involve the two students in importing their motion data into the computer (or calculator) *serially* (discussed below) and replaying them *simultaneously*, where the velocity-position distinction becomes even more apparent due to the availability of the respective velocity and position graphs alongside the cybernetically replayed motion.

Note how the dual perspectives of the velocity and position functions, both illustrated in Figure 1, show two different views of the average value situation. In the left-hand graph, we see the connection as a matter of equal areas under respective velocity graphs. In the right-hand graph, we see it through position graphs as a matter of getting to the same place at the same time, one with variable velocity and the other with constant velocity. Depending on the activity, of course, one or the other of the graphs might not be viewable or, if viewable, not editable. For example, another version of this activity involves giving the step-wise varying position function on the right and asking the student to construct its velocity-function mean value on the left. This makes the slope the key issue. By reversing the given and requested function types area becomes the key issue. Importantly, by building in the connections between rate (velocity) and totals (position) quantities throughout, the underlying idea of the Fundamental Theorem of Calculus is always at hand.

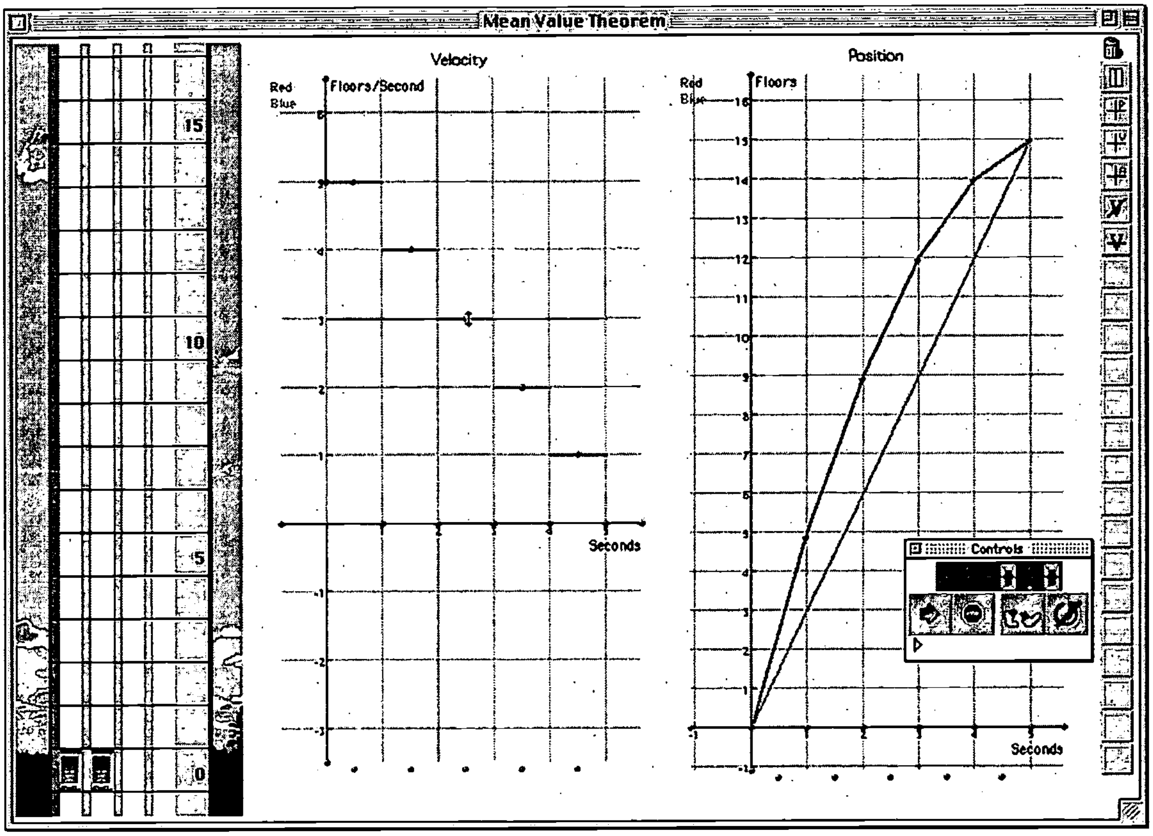

Figure 1. Averages from Both Velocity and Position Perspectives

Parallel Software and Curricula for a Graphing Calculator

Now, in the left two pictures of Figure 2 below are partially analogous software configurations—two elevators controlled by two velocity graphs. Instead of the clicking and drag/drop interface of the desktop software, most user interaction is through the SoftKeys that appear across the bottom of the screen which are controlled by the HardKeys immediately beneath them. The left-most screen depicts the Animation Mode, with two elevators on the left controlled respectively by the staircase and constant velocity functions to their right. The middle screen depicts the Function-Edit Mode, which shows a "HotSpot" on the constant-velocity graph. The user adjusts the height and extent of a graph segment via the four calculator cursor keys (not shown), and can add or delete segments via the SoftKeys. Other features allow the user to scale the graph and animation views, display labels, enter functions in text-input mode, generate time-position output data, and so on—very much in parallel with Java MathWorlds, but without the benefits of a direct-manipulation interface. The right-most screen shows a horizontal motion world with both position and velocity functions displayed (hot-linkable if needed, as with the computer software).

We have developed a full, document-oriented Flash ROM software system for the TI-83+ and a core set of activities embodying a common set of curriculum materials that parallels the computer software to the extent possible given the processing and screen constraints (96 by 64 pixels!). The parallelism is evident in the Calculator MathWorlds screens shown. We have also developed a prototype version of MathWorlds for the PalmPilot Operating System (see http://www.simcalc.umassd.edu to download it to a Palm device).

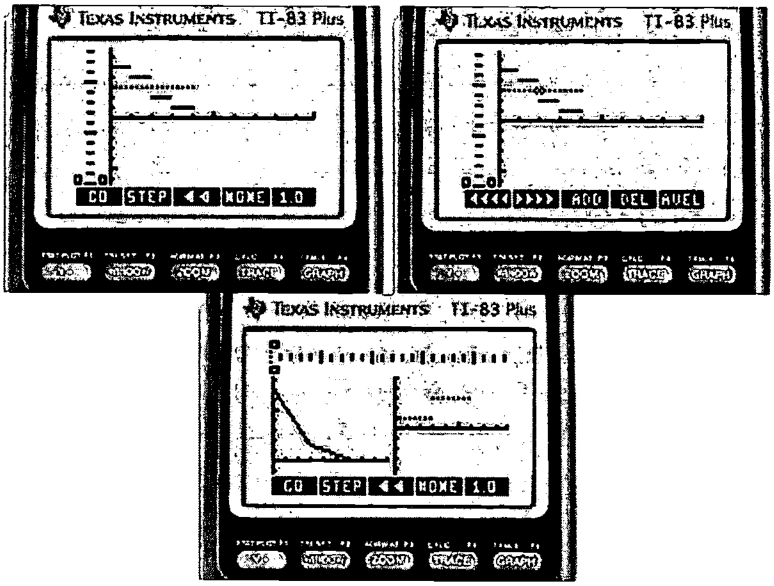

Figure 2. Calculator MathWorlds

Classroom/Homework Flexibility and Cognitive Flexibility

We are currently completing an Algebra and Pre-Calculus course for academically weak first year college students in which we are using a common set of activities that run on parallel versions of software for desktop computers and TI-83+ graphing calculators. The course takes place in a classroom with two computers per student and an overhead display panel for each kind of device. And each student also has a graphing calculator. Much of the classroom discussion uses the computers although some switching takes place—simply by exchanging the panel that sits on top of the overhead projector. Homework is usually assigned for the calculators, although frequently, the homework is an extension of what was begun in class on the computers. That is, the students might do the first few problems on the computer in class and then do the remaining ones with the graphing calculator at home. We imagine that there are likely to many situations where computers are only occasionally available, or where only the teacher has one, but where the students have access to hand-held devices. Hence parallel software *and* curricula can substantially expand the usability of curriculum-oriented technology in the classroom (as opposed to open tools, which likewise are useful and continue to be available in these dual-device scenarios).

Importantly, we have seen full student flexibility in switching between the two device types, despite the radically different interfaces. Careful analyses of students' discourse and gestural activity reveals that, when discussing problem solutions and difficulties, the language is primarily about the mathematical objects and relations rather than about the interface or device. Hence they refer to "the velocity graph" or "I need to increase the slope" or "I need to extend the domain" rather than, dragging a HotSpot, or pushing a certain key, etc. We have a suspicion, not justified at this point, that the crossing between interfaces may help in exposing the mathematical structure. After all, when a student only sees one device and one interface for working on a mathematical domain, we have every psychological reason to expect that, without reason to do otherwise, they will link their experience of that mathematics with the interface through which they learned it.

We have also found subtle perceptual carryovers from the computer to the calculator environments that may provide guidance on how to exploit the visual detail possible on the computer screen to compensate for limited screens of hand-helds. For example, despite the hard to read grid of the calculator screen, the students, who were sometimes presented activities using graph printouts based on the computer screens, seemed to treat the calculator screen as having visual attributes that were present only on the computer software. These kinds of potentially important phenomena need to be studied and documented in more detail, as do potential interference effects across the different environments.

Illustrations of Representational Infrastructure Innovations Across Multiple NETWORKED Hardware and Software Platforms

Simple Pedagogical Supports—Doing Old Stuff Better. Increasingly rich interactions are possible as connectivity increases between a teacher's computer or hand-held device, and a classroom of hand-helds. For example, a teacher can download sets of "documents" for homework or quizzes, and more interestingly, the students can upload their solution-documents as well as other data, which can then be aggregated in a variety of ways on the teacher's computer. With easy data-flow, teachers can ask diagnostic questions to 10 groups of 3 students each, such as the following (imagine the right-most part of Figure 2 above with its graphs hidden):

> The top car starts at 0 meters/second and accelerates to 3 meters/second in 6 seconds. Send me a position function for the bottom car so that it's motion matches the top car's motion exactly.

The teacher then shows all ten graphs on the same axes and then runs them. Each group's function is alive on the screen, so the diagnostic question illuminates everyone's understanding. Furthermore, patterns become evident, e.g., several groups might create a constant slope position function. Then, of course, there's a natural follow-up question—can you get your group's car to the same endpoint at exactly the same time, but at constant velocity?

New Learning Opportunities—Doing Better Stuff By Investing Individuals in a Collective Object. For example, groups of students can act out or choreograph a collective motion, say a dance, collectively, and then sit down to plan the coordination of their individual motions as mathematical functions that they will produce on their hand-held. They then upload their individual synthetically defined functions to the teacher's computer where the independently produced motions are aggregated into a simultaneously executed dance to be viewed by the entire class as in Figure 3. This kind of activity can quite literally pull students towards a parameter-based description of their motions because the motions differ in a quantitatively systematic way.

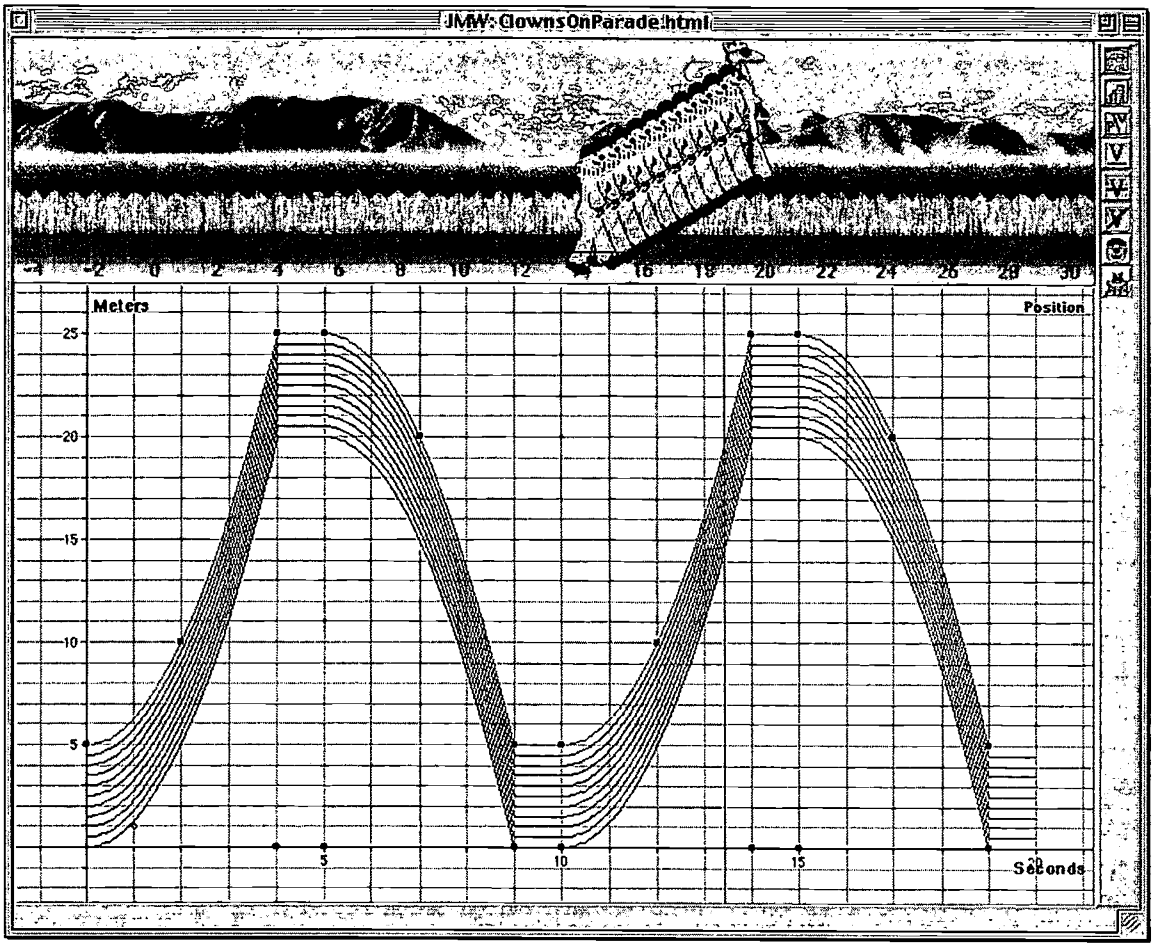

Figure 3. A Clown Parade—Staggered Initial Positions

Variations of this kind of aggregation activity can use CBL motion data input as well, where one character has its motion based on a student's actual physical motion imported into MathWorlds and where the other students create motions to participate in the parade. A wide variety of other aggregation and target activities is possible, for example, where each character's motion is imported from a serially produced dance. The kind of planning required for this kind of activity is exactly the kind of thinking that one wants in defining functions describing change. Another example involves building a "wave" action via delayed starting times as in Figure 4 where the dots hit the far "wall" at 18 meters and reflect backward. How could you make the same reflected wave motion with staggered starting positions?

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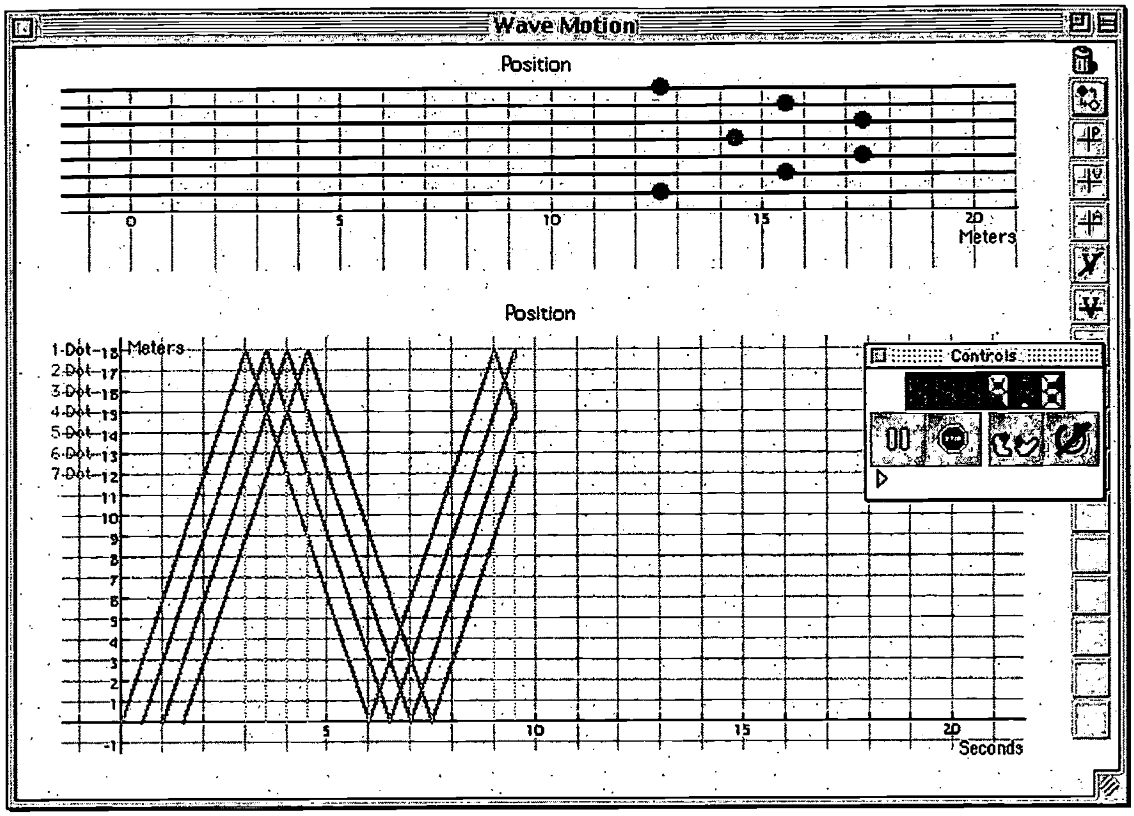

Figure 4. Wave—Staggered Starting Times

More Traditional Topics Using Participation in Shared Mathematical Objects. A standard student difficulty is in appreciating what it really means for a point to be on a line defined by an equation or other constraint. In the activity depicted in Figures 5 & 6, I "personalizes" this idea in a networked classroom where each student can send data to the teacher's computer (or calculator) as follows. First, students count-off in class, so each takes a number, which will serve as their x-coordinate. Then they are asked to make a point with their personal number as the x-coordinate but with whatever y-coordinate they wish, and send it up to the teacher's display. This results in the scattered points in Figure 5(a). Next, they are asked to make their y-coordinate *double their x-coordinate* and send this up. The result is the transformation depicted in Figure 5(b), where all the points assume their position on a line—which, of course, is the line we come to call "y = 2x". Naturally, any errors will show up as outliers.

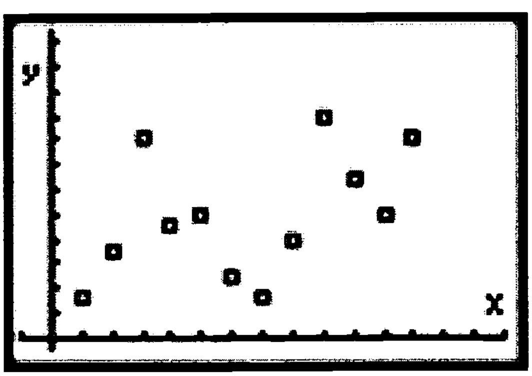

Figure 5(a).

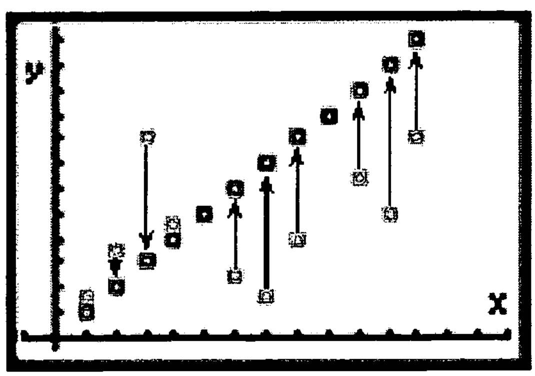

Figure 5(b).

The sequence is repeated in Figure 6, where a new scatter of points appears in (a) and then the students are asked to make the second coordinate of their point to be the square of half their number. Here, the order of halving and squaring is important, and anyone who squares first and then takes half will not lie on the line. Indeed, the issue of order of operations turns into the issue of the identity of the appropriate line. Then, when the line is finally determined, we discover that one person, Damien, who's number is 8, is on both lines! Why? This becomes a lead-in to the matter of simultaneous equations, where Damien satisfies both constraints—the double of her number is the same as half of her number, which is then squared. The fact that this is true only for her and nobody else (adopting zero and negative numbers comes a bit later) is a reflection of the fact that she and only she satisfies the equation $x/2 = (x/2)^2$. We can then ask, what could we do to the rule so that Jeri is on both lines? And how much algebra can be done in this kind of connected classroom?

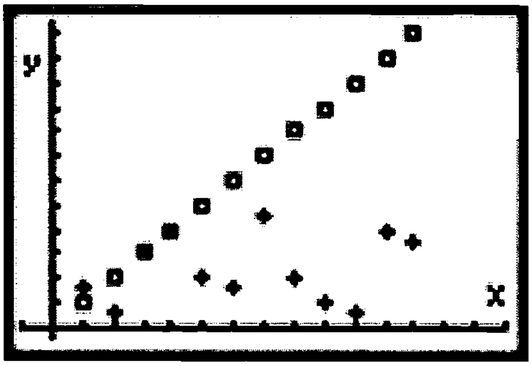

Figure 6(a).

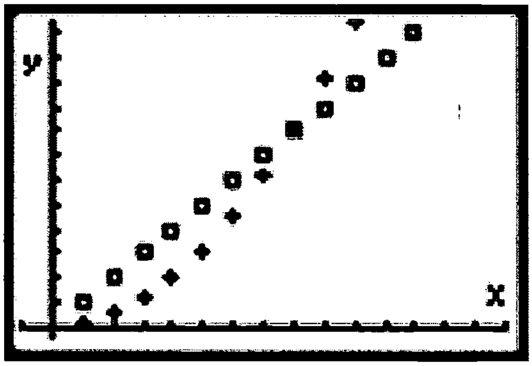

Figure 6(b).

We are currently examining, along with others, e.g., Stroup and Wilensky, how best to exploit the affordances of networked classrooms. The Participatory Simulations Project led by Stroup and Wilensky has been pursuing the opportunities for modeling emergent phenomena in dynamical systems, especially agent-based systems. For example, students can play the roles of predator or prey in a dynamic population model, or players in an economic model, and so on (Wilensky & Stroup, 2000). This work as well as that of another participatory simulations project based at the MIT Media Lab led by Resnick and Colella exploits parallel processing software that enables each participant to be represented as an independently controlled agent in the system.

Reflections on the Examples—Network-Based Activity Frameworks that Integrate Social Structure and Mathematical Structure

The above illustrations are only a small sliver of the possible range of activity structures in a networked classroom. Below is a further elaboration of the possibilities, limited due to space considerations.

Student-Teacher Activity Structures Using the Teacher's Computer as Publicly Viewable Common Ground.

(A) Students create functions on their own device and publicly upload them to a shared, publicly displayed object on the teacher's computer for a variety of purposes, including:

- Students contribute to an emergent object, where the properties or identity of the object are not well understood beforehand, and where the determination of the properties and identity is at the heart of the learning opportunity as in the last example involving emergent lines and intersections.

- Students contribute to an object of their own design where the design work is at the heart of the learning opportunity as in the parade and marching band examples.
- Students contribute to an object where the identity and properties of the object are known in advance and where they act as the scaffolding for learning of something else, as might occur when the parade design and motions are well established, but where one half the class is defining the motion of their actors via position functions and the other half is using velocity functions. In this case the target is well-defined, and the heart of the learning opportunity is in defining the means by which the target is reached.
- Students upload survey or other data (e.g., probability trials) to a common data-set on the teacher's computer that is then aggregated and downloaded as a data object subject to further analysis by the students on their local devices. This could take place inside the classroom or, depending on connectivity, engage students elsewhere. Indeed, this is a possibility in almost all the activity structures, although some would be more convenient if the students were close-at-hand.

(B) Target Activities between teacher and students where students upload responses to classroom challenges, where challenges and responses are shown on the teacher's display.

- Define a function to fit this data, or an equation for this curve, or a polynomial that has these roots, etc. How many of these Gray Globs can you hit with one quadratic function? (Dugdale, 1982). Define a velocity (or position or acceleration) function to match this motion, ...

(C) Miscellaneous teacher-directed activities that will utilize the teacher's existing repertoire of classroom moves, e.g., pool solutions to an open ended problem and investigate solutions for generalities, optimality, etc.

Student Target Activities Between Students or Small Groups of Students. Essentially all of the whole-classroom activities have student-student analogs, either between single students working in pairs or, more likely, between small groups of students.

We have only had a brief glimpse into the learning opportunity space opened up by classroom connectivity. Time will be needed for the technology to be tuned to the possibilities and to the realities of such classrooms. But even more challenging is the need to understand how the social structures of the classroom and the mathematical structures can be made to interact in fruitful ways. At this point, experience is very limited, and we inherit a long tradition of three modes of activity: (1) Teacher-centered classroom activity, (2) Small-group activity, and (3) Individual activity. In all three cases, the communication among participants is biased towards oral communication, typically of an indirect nature about the mathematical objects and relations being used or studied. The new connectivity obviously offers much more direct communication. We are currently pursuing research that includes several private sector partners to examine the affordances and constraints of networked mathematics classrooms employing mixes of hardware and software platforms.

Part IV: Looking Ahead—Research Questions and Agendas

Introduction: The Bigger Picture

After approximately a generation of growing computer use in the world of business, LAN and WAN connectivity *coupled with the integration of computation into all aspects of business practice* has paid off in surprising increases in economic productivity during the past decade, now approximately 4% annually in the U.S. And, of course, the connectivity embodied in the WWW has led to even more startling impacts on the world outside of education. Indeed, this wider connectivity has changed the conditions of innovation in ways that compound and accelerate change (Bollier, 2000). We are poised to begin a comparable application of connectivity in education. The missing ingredients are at-hand computation (see, for example, Becker, *et al.*, 2000) and connectivity at the epicenter of teaching and learning, the classroom. But of course, as was the case in business, and as our brief analysis of the printing press suggested, many changes must take place across many different dimensions before a new representation infrastructure delivery technology can have full impact. The classroom connectivity ingredient can pay off only if coupled with the integration of computation into educational practice.

The illustrations in Part III focused primarily on student learning, so we need also to address what connectivity among diverse inexpensive computing devices might mean, not only for learning, but for teaching, classroom management and assessment—the broader ingredients of teaching practice. Each is a complex matter, and we cannot delve into detail here, but will simply offer some major issues needing investigation.

Research Issues and Opportunities in Assessment, Learning & Teaching: Three Opportunity Spaces

We need to understand how new configurations and applications of connected devices can support or perhaps impede potentially profound progress in three opportunity spaces at the communicative heart of mathematics education in real classrooms:

Diagnostic assessment and evaluation: We need to study how teachers can use connectivity and analytic tools to exploit what we know about student thinking and learning in order to actively diagnose and efficiently respond to student thinking on a regular basis in the classroom.

Student learning and activity structures: We need to study classroom affordances and constraints of new activity structures that exploit the ability of students to design and pass structured mathematical objects (e.g. functions) and representations (e.g. graphs) among themselves and to the teacher—as illustrated briefly above.

Teaching and the classroom management of information flow: We need to study the classroom management implications of wirelessly connected hand-helds, with particular attention to teacher-specific tools to help organize the flow of the vast amount of information available to them (e.g. what each student is doing on their individual hand-held), decide among alternative actions (e.g. send every student an identical graph vs. call students' attention to a projected graph on a central display), and set policies on network communication (e.g., Can students send each other text messages? Only within their group? Only to an assigned partner?)

Specific Kinds of Concrete Research Questions Needing Investigation

Across these opportunity spaces, we need to gather and analyze data addressing the following broad kinds of questions in ways that span technology-specifics.

- What uses of mathematical notations and representations, when shared across devices, lead to deep, intense or efficient content-oriented interaction and meaning-making among students and between teacher and students?
- Which characteristics of networked, hand-held devices (e.g. screen size, lack of color, availability of stylus, ease of beaming data, ability to move about the room) strongly enable or impede the ease, comfort, and effectiveness of mathematical conversations in the classroom?
- In what ways do networked, hand-held devices most strongly engage learners' cognitive strengths and solve practical, important teaching problems, or conversely, distract learners from the task at hand and impose new burdens on the teacher?

Importantly, given the novelty of these environments and technologies, we cannot trap ourselves into study of phenomena that will disappear when the technologies become better established. Hence we need to analyze how the answers to the above questions change as both the teachers' experience with connectivity, and the technologies themselves, mature. These research questions reflect the belief that hand-held, networked devices will not necessarily have a simple causal effect upon learning outcomes. As has been the case historically, introducing technology into schools will not necessarily change practice (Cuban, 1986), and it is likewise well appreciated that many technological tools simply reinforce existing practice (Marx, *et al.*, 1998; Means, 1994). Similarly, as argued by Roschelle & Pea (1999) regarding WWW connectivity with resources outside the classroom, the promise may be ill-understood or near-sighted.

Hence, we need to begin building a framework that can adequately describe *uses* that are likely to distinguish effective from ineffective practice. Likewise, we cannot take for granted that manufacturers specifications (processor speed, communication speed, screen size) are the device characteristics that necessarily enable or impede use, and should seek to build a *conceptual analysis of device characteristics* that directly relates to observable classroom behaviors.

Finally, widespread impact from such devices is only likely if we identify the strongest ties to learner's strengths, solve difficult teaching problems, and introduce no serious *new* difficulties. Thus we need conceptual and analytic frameworks to clearly identify how these affordances and constraints play out in realistic classroom settings, and thereby to guide iterative design that magnifies the unique benefits and minimizes the newly introduced impediments. This will in turn inform the design of appropriate teacher development and support structures so that connectivity becomes a pedagogical support tool. As we begin to understand the classroom issues and technological issues we must immediately employ these understandings in the development of teacher development and support systems.

Last Words: Recognizing the Depth of the Changes that Are Underway

We are early in an exciting new era for technology in mathematics education. Both the representational infrastructures are changing and the physical means for implementing them are changing. We are seeing new alphabets emerging, new visual modalities of human experience are being engaged, and new physical devices are emerging—all at the same time. Much work needs to be done.

References

Becker, H. J., Ravitz, J. L., & Wong, Y.T. (2000). *Teacher- and teacher-directed student use of computers in american schools. Teaching, learning, and computing—1998 national survey.* Report #3. Center for Information Technology and Organizations, University of California, Irvine.

Bochner, S. (1966). *The role of mathematics in the rise of science.* Princeton, NJ: Princeton University Press.

Bollier, D. (2000). *Ecologies of innovation: The role of information and communication technologies.* A Report of the Eighth Annual Aspen Institute Roundtable on Information Technology. One Dupont Circle, Washington, DC: The Aspen Institute.

Boyer, C. (1959). *The history of calculus and its historical development.* New York, NY: Dover Publications.

Bruner, J. (1973). *Beyond the information given.* New York: Norton & Co.

Clanchy, M. (1979). *From memory to written record: England 1066-1307.* Cambridge: Harvard University Press.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13*(1), 16–30.

Clagett, M. (1968). *Nicole Oresme and the medieval geometry of qualities and motions.* Madison, WI: University of Wisconsin Press.

Cobb, P., Yackel, E., & McClain, K. (Eds). (2000). *Symbolizing and communicating in mathematics classrooms.* Mahwah, NJ: Erlbaum.

Cole, M. (1997). Cultural-historical psychology: A Meso-genetic approach. In L. Martin, K. Nelson & E. Toback (Eds.), *Socio-cultural psychology: Theory and practice of knowing and doing.* New York: Cambridge University Press.

Cohen, J., & Stewart, I. (1994). *The collapse of chaos: Discovering simplicity in a complex world.* New York: Viking Books.

Cuban, L. (1986). *Teachers and machines: The classroom use of technology since 1920.* New York: Teachers College Press.

Davies, W. V. (1987). *Egyptian hieroglyphs.* London: British Museum Publications.

Deacon, T. (1997). *The symbolic species: The co-evolution of language and the brain.* New York: W. W. Norton.

Donald, L. (1991). *The origins of the modern mind.* Cambridge, MA: Harvard University Press.

Dugdale, S. (1982). Green globs: A microcomputer application for graphing of equations. *Mathematics Teacher, 75*(3), 208–214.

Edwards, C. (1979). *The historical development of the calculus.* New York: Springer-Verlag, Inc.

Eisenstein, E. (1979). *The printing press as an agent of change.* (Two vols.) Cambridge: Cambridge University Press.

Goody, J. & Watt, I. (1968). The consequences of literacy. In J. Goody (Ed.), *Literacy in traditional societies* (pp. 27-68). Cambridge: Cambridge University Press.

Haas, C. (1996). Writing technology: Studies on the materiality of literacy. Mahwah, NJ: Erlbaum.

Hall, N. (1992). *Exploring chaos: A guide to the new science of disorder.* New York: W. W. Norton.

Havelock, E. (1982). *The literate revolution in Greece and its cultural consequences.* Princeton: Princeton University Press.

Hoyles, C., Kaput, J., & Noss, R. (in press). The evolution and impacts of efficient representational infrastructures in mathematics. To appear in L. English (Ed.), *International handbook of technology in mathematics education*. Kluwer.

Kaestle, C. (1985). The history of literacy and the history of readers. *Review of Research in Education, 12,* 11–53.

Kaput, J. (1994). Democratizing access to calculus: New routes to old roots. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 77–156). Hillsdale, NJ: Erlbaum.

Kaput, J., & Roschelle, J. (1998). The mathematics of change and variation from a millennial perspective: New content, new context. In C. Hoyles, C. Morgan, & G. Woodhouse (Eds.), *Mathematics for a new millennium* (pp. 155–170). London: Springer-Verlag.

Kaput, J., & Shaffer, D. (in press). On the development of human representational competence from an evolutionary point of view: From episodic to virtual culture. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education*. London: Kluwer.

Kaput, J., & Sims-Knight, J. (1983). Errors in translations to algebraic equations: Roots and implications. *Focus on Learning Problems in Mathematics, 5*(3), 63–78.

Kaufer, D., & Carley, A. (1992). *Communication at a distance: The influence of print on socio-cultural organization and change.* Hillsdale, NJ: Erlbaum.

Kline, M. (1953). *Mathematics in western culture.* New York: Oxford University Press.

Langer, J. (1986). *Children reading and writing: Structures and strategies.* Norwood, NJ: Ablex.

Mahoney, M. (1980). The beginnings of algebraic thought in the seventeenth century. In S. Gankroger (Ed.), *Descartes: Philosophy, mathematics and physics*. Sussex, England: Harvester Press.

Marx, R. W., Blumenfeld, P. C., Krajcik, J. S., & Soloway, E. (1998). New technologies for teacher professional development. *Teachering and Teacher Education, 14,* 33–52.

Moses, B. (1995). Algebra, the new civil right. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The algebra initiative colloquium* (pp. 53–67). Washington, DC: U.S. Department of Education.

Nelson, K. (1996). *Language in cognitive development: Emergence of the mediated mind.* Cambridge: Cambridge University Press.

Ong, W. J. (1982). *Orality and literacy: The technologizing of the word.* London: Methuen.

Resnick, D., & Resnick, L. (1977). The nature of literacy: A historical explanation. *Harvard Educational Review, 47,* 370–385.

Riordan M., & Hoddeson, L. (1997). *Crystal fire: The birth of the information age*. New York: W. W. Norton.

Roschelle, J., & Pea, R. (1999). Trajectories from today's WWW to a powerful educational infrastructure. *Educational Researcher, 8*(5), 22–25.

Schmandt-Besserat, D. (1978). The earliest precursor of writing. *Scientific American,* 238.

Schmandt-Besserat, D. (1981). From tokens to tablets: A re-evaluation of the so-called numerical tablets. *Visible Language, 15.*

Schmandt-Besserat, D. (1988). Quantification and social structure. In D. R. Maines & C. J. Couch (Eds.), *Communication and social structure*. Springfield, IL: C. C. Thomas.

Schmandt-Besserat, D. (1992). *Before writing: From counting to cuneiform* (Vol. 1). Houston, TX: University of Texas Press.

Schmandt-Besserat, D. (1994). Before numerals. *Visible Language, 18.*

Shaffer, D., & Kaput, J. (1999). Mathematics and virtual culture: An evolutionary perspective on technology and mathematics education. *Educational Studies in Mathematics, 37,* 97–119.

Stewart, I. (1990). Change. In L. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 183–219). Washington, DC: National Academy Press.

Swetz, F. (1987). *Capitalism and arithmetic: The new math of the 15th century.* La Salle: Open Court.

Walker, C. B. F. (1987). *Cuneiform.* London: British Museum Publications.

Wilensky, U., & Stroup, W. (2000). Networked Gridlock: Students enacting complex dynamic phenomena with the HubNet architecture. In *the Proceedings of the Fourth Annual International Conference of the Learning Sciences*, Ann Arbor, MI.

Woodard, R. (1996). Writing systems. In B. Comrie, S. Matthews, & M. Polinksy, (Eds.), *The atlas of languages: The origin and development of languages throughout the world*. London: Quarto Publishing, Inc.

DEVELOPING SIMULATION ACTIVITIES TO IMPROVE STUDENTS' STATISTICAL REASONING

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This paper describes a collaborative classroom–based research project at two American universities. Our research goal has been to investigate how technology can best help students understand, integrate, and apply fundamental statistical concepts, such as sampling distributions. We describe the three–year evolution of the software, activities, and assessment instruments we used to measure the impact of technology on students' conceptual understanding and to investigate effective implementation of such technologies. Key findings include the need to establish cognitive dissonance with student predictions. We hope our study serves as a model of classroom–based research for investigating the impact of technology on student learning.

Introduction

In recent years, there has been a shift in the focus of introductory statistics courses, emphasizing skills such as the ability to interpret, evaluate, and apply statistical ideas rather than procedural calculations. Calls for reform, similar to those in mathematics and science education, also emphasize that instruction should fully incorporate genuine data, technological tools, and active learning (e.g. Cobb, 1992). Technology offers us many ways of accomplishing these goals, from finding current data through the world–wide web to more authentic statistical analyses to use of interactive, visual computer simulations. In fact, numerous visualization programs are now available (e.g., ConStatS, Hyperstat, Visual Statistics, StatPlay).

Despite the availability of these technological resources, little is known about the impact on student learning of such technologies. There are no accepted methodologies for measuring what students are gaining from their interactions with these technologies and how they are affecting students' conceptual understanding (Hawkins, 1997). Part of the problem is the need for more informative methods of student assessment. Traditional assessment too often emphasizes the final answer over the process (Garfield, 1993) and may not provide informative data either for evaluation of student performance or for research studies on the effectiveness of new instructional techniques. Instead, we need more focus on *why* a particular interaction with technology works, *how* students' understanding and reasoning are affected by the learning experience, and *implications* for how teaching practice should be changed. In this paper, we provide an example of a collaborative classroom–based research study on the effectiveness of computer simulations in guiding student construction and visualization of one fundamental statistical concept in particular – the behavior of sampling distributions. We present not only results of this study, but also an example of how classroom–based research studies can effectively inform our understanding of students' interaction with technology.

The research question

Researchers and educators have found that students and professionals often misunderstand foundational statistical ideas. Many students develop a shallow and isolated understanding of important concepts such as sample, population, distribution, variability, sampling, and sampling variability. We were concerned that many students who pass a statistics course do not develop the deep understanding needed to integrate these concepts and apply them in their reasoning. A particularly difficult topic for our students has been the concept of sampling distributions. We found their failure particularly troublesome as this topic is the gateway to understanding the process of statistical inference. We felt that a visual simulation program could be an effective way to improve student learning about sampling distributions.

The *Sampling Distributions* program, developed by delMas (see website below), allows students to visually explore sampling distributions, in a dynamic, interactive environment. Students change parameters and then run simulations in order to directly see the effects of these changes. Development of this software was guided by literature on conceptually enhanced simulations (e.g. Nickerson, 1995; Snir, Smith, & Grosslight, 1995). An accompanying activity was developed to guide students through the interaction with the software based on ideas from literature in learning and cognition (e.g. Holland, Holyoak, Nisbett, & Thagard, 1987; Perkins, Schwartz, West, & Wiske, 1995). The three authors began using the software and activity in our introductory statistics service courses, allowing us to compare results from diverse institutions: a private liberal arts college, a College of Education, and a Developmental Education College. A wide variety of student majors and backgrounds enroll in these courses. In all three settings, students were expected to have read the appropriate textbook chapter on sampling distributions and the Central Limit Theorem prior to the activity. Students also engaged in a hands–on simulation demonstrating the Central Limit Theorem during the class period prior to using the program. Our goal was to document the learning gains of the students from use of the *Sampling Distributions* program, beyond what they learned from our normal textbook and lecture instruction.

Stage one

To assess the effects of the program and activity on students' conceptual understanding of sampling distributions, we initially focused on students' ability to demonstrate a visual understanding of the Central Limit Theorem's implications. Students were provided with a picture of a population distribution, and were asked to choose among several candidate graphs as a resulting (simulated) sampling distribution for a sample mean from that population. They then chose again for a different sample size. This was done for several population shapes. In pilot tests, students were asked to explain their selection. These open–ended responses were then categorized into several common explanations. In later tests, students were asked to choose among these potential explanations for their graph choice. From student responses and graph choices, we were able to identify several different types of reasoning.

- Correct Reasoning: Students chose the correct histograms and explanations.
- Good Reasoning: Students made reasonable choices (e.g., the sampling distribution for the larger sample size was more normal looking and had less variability than the sampling distribution for the smaller sample size) but demonstrated minor errors in their thinking (e.g., choosing a graph that looks like the population when n>1).
- Larger to Smaller Reasoning or Smaller to Larger Reasoning: Students attended to the change in variability but did not correctly predict the amount of variability or did not correctly pick the normal shape of the sampling distribution.

These categories covered about 80–90% of the responses for each problem, but there were also a variety of other, less frequent, responses (e.g. choosing the same histogram for both sample sizes). To determine the change in understanding due to interaction with the *Sampling Distribution* program, students were given a pre–test before using the program (but after standard classroom instruction), and then a post–test of comparable items.

After pilot testing the assessment instrument with students, revised instruments were administered to 79 students at the private college and 22 students at the College of Education during Winter, 1997. Eighty–nine students who gave responses to all pretest and posttest items were used for the analyses. (See delMas, Garfield, and Chance, 1999 for more details.) Over five different population shapes, the average percentage of "correct" or "good reasoning" choices on the pretest was 22%. This increased to 49% on the posttest. While this is considerable improvement, students were still demonstrating some definite misconceptions; e.g., confusion between the sample distribution and the sampling distribution, and interpretation of "variability." We learned that well–designed software with clear directions does not ensure sufficient student engagement or change in conceptual understanding.

Stage two

The above results led to alterations in the software and the accompanying activity. The main adjustment, inspired by a model of conceptual change (Posner, Strike, Hewson, and Gertzog, 1982), was to use the pre–test to guide student interaction with the software. Research indicates that people are generally resistant to change and are likely to find ways to either assimilate information or discredit contradictory evidence rather than restructure their thinking in order to accommodate the contradictions (Lord, Ross, & Lepper, 1979; Jennings, Amabile, & Ross, 1982; Ross & Anderson, 1982). Modern information processing theories (e.g. Holland, Holyoak, Nisbett, & Thagard, 1987) suggest that it may be necessary to direct attention toward the features of the discrediting experience in order for the contradictory evidence to be encoded. Left to their own devices, people will attend only to those features predicted by their current information structure. Adapting this approach, we had students make predictions on the pre–test, and then use the software to compare their answers by embedding the assessment instrument into the activity (e.g. students were asked to comment on how the correct graph compared to the graph they chose). When students discover that their prediction is incorrect, this creates cognitive dissonance between the students' current knowledge or expectation and what they are seeing. Students are then able to utilize the software to identify and correct their misconceptions.

Assessment results for a total of 141 students using the new activity at both schools showed that on average, students used correct or good reasoning on 16% of the pretest items (similar to before), but correct or good reasoning on 72% of the posttest items (delMas, Garfield, & Chance, 1999). These results agree with other research results that students learn better when activities are structured to help students evaluate the difference between their own believes and actual results (e.g. delMas and Bart, 1989). Furthermore, the activity allowed us to better track student misconceptions, and what knowledge was lacking in their understanding of sampling distributions. We then altered the activity to better address the most prevalent misconceptions.

Stage three

Our results indicated that students still struggled with the notion of sample, variation, and even histogram. We feel that without these concepts, students are not able to develop a deep understanding of sampling distributions. To help determine whether students are cognitively ready to learn about sampling distributions, we developed a pre–test of basic skills that highlights common misconceptions in prerequisite knowledge. For example, our studies had shown that students often confuse "bumpiness" of a histogram with "variability," and may not properly use statistical terminology such as "normal" vs. "even." The pre–test assessment allows the instructor to correct these misconceptions before using the *Sampling Distribution* software. We also embedded the activity into a contextual example in order to help students learn to apply the implications of the Central Limit Theorem. We again administered the post–test in our different institutional settings and compared post–test scores (55 students) on the graphic based questions for two population shapes to scores from previous versions of the activity. However, these results were not as impressive with only about 60% of students demonstrating good or correct reasoning. Some possible explanations include:

- Insufficient development and definition of sampling distributions in lecture prior to use of the computer program (this varied at the three schools).
- A decreased level of student engagement with the "prediction questions." In Stage Two, the pre–test questions were turned in to the instructor for marking before students used the program. In Stage Three, the activity relied on the student to invest sufficiently in the activity to create significant dissonance.
- The longer contextual activity may have required students to attend to more information than is feasible in one interaction with the software.
- The Stage Three activity did not include as many "prediction questions."

Stage four

Last year, interviews were conducted with students to gain a more in–depth understanding of their statistical reasoning about variability, samples, and sampling distribution (see also Garfield, 2000). The students were enrolled in a graduate–level introductory course in the College of Education and Human Development at the University of Minnesota. Interviews, which lasted from 45 to 60 minutes, asked participants to respond to several open–ended questions about variability and sampling and were guided through an interactive activity with the *Sampling Distributions* software. The interviews were videotaped, transcribed, and viewed many times as we tried to determine students' initial understanding of how sampling distributions behave and how feedback from the computer simulation program helped them develop an integrated reasoning of concepts. We found ourselves identifying stages that the students went through as they progressed from faulty to correct reasoning about sampling distributions. This led us to propose a framework that describes the development of students' statistical reasoning about sampling distributions. This framework is an extension of one developed by Graham Jones and colleagues to capture the statistical thinking of middle schools students (Jones, Langrall, Thornton & Mogill, 1997; Jones, Thornton, Langrall, Putt, & Perry, 1998; Tarr & Jones, 1997).

Level 1: Idiosyncratic Reasoning: The student knows words and symbols related to sampling distributions, uses them without fully understanding them, often incorrectly, and may scramble them with unrelated information.

Level 2: Verbal Reasoning: The student has a verbal understanding of sampling distributions and the Central Limit Theorem, but cannot apply this to actual behavior. For example, the student can select a correct definition, but does not understand how key concepts such as variability and shape are integrated.

Level 3: Transitional Reasoning: The student is able to correctly identify one or two dimensions of the sampling process without fully integrating these dimensions; e.g., the relationship between the population shape and the shape of the sampling distribution, the fact that large samples lead to more normal looking sampling distributions, the fact that larger samples lead to narrower sampling distributions.

Level 4: Procedural Reasoning: The student is able to correctly identify the dimensions of the sampling process but does not fully integrate them or understand the process. For example, the student can correctly predict which sampling distribution corresponds to the given parameters, but cannot explain the process, and does not have full confidence in predictions.

Level 5: Integrated Process Reasoning: The student has a complete understanding of the process of sampling and sampling distributions, coordinates the rules and behavior. The student can explain the process in their own words and predicts correctly and with confidence.

The current stage

Our current research focuses on the validation and possible extension of the above framework to other areas of statistical reasoning and to students at the secondary and tertiary level. We believe that in order for students to fully understand sampling distributions, they need to experience a variety of activities: text or verbal explanations, concrete activities involving sampling from finite populations, and interactions with computer–simulated populations and sampling distributions when the parameters are varied. This contradicts some of the psychology research that argues for teaching specific training rules.

We are currently developing activities that integrate the *Sampling Distribution* software earlier in the course. One aim is to provide the students with more familiarity with the program prior to the sampling distribution topic. We hope this will allow students to better focus on the statistical concept, having already learned the software. The second aim is to use the visualization capabilities of the program to between develop a correct and full understanding of foundational concepts, e.g. variation, sample distribution vs. sampling distribution. Students will construct prerequisite knowledge using a predict–and–test environment throughout the course. We are also trying to explore activities that help students develop the ideas "process" and "model" earlier and throughout the course. Finally, we are also expanding our collection of follow–up application questions to test students' ability to apply the knowledge gained from their interaction with the software in new settings.

Research and assessment

The above research presents an example of classroom–based research (e.g., Cross & Steadman; 1996, see also Kelly & Lesh, 2000) in the context of an introductory statistics course. We believe this is an exciting and productive model for research on the effects of technology as an instructional tool. Classroom–based research provides on–going, systematic evaluation in the classroom setting, narrowing the bridge between theory and practice. While classroom–based research is grounded in evidence, results are continually tied to existing theory and generative of new theory. It is a dynamic process that allows the questions to change in response to results and feedback, while simultaneously focusing on curricular development, instruction, and assessment.

While our students cannot be considered a random sample of all introductory statistics students, we have taken several steps to enhance the quality of our study. Working at different universities we have ensured multiple perspectives, diverse instructional settings and student audiences, and multiple time points. Our project, while focusing on our experiences as teachers, also combined our expertise in cognition, educational psychology, and statistics. We also brought in, and hopefully expanded, research results from other areas, such as cognition, learning theory, and information processing theory. While we have not identified a definitive approach to teaching sampling distributions, our research has provided substantial insight into students' misconceptions and their sources. We believe we are developing understanding about why an activity works, how students' understanding and reasoning are effected, and how prior knowledge affects their experience with the technology.

Furthermore our results have demonstrated the instructional uses of assessment. By embedding the assessment into the learning activity, we were able to strengthen the students' level of engagement with the technology. This assessment approach also takes advantage of the dynamic, immediate feedback nature of the technology. By indicating students' short–term and long–term understanding to the instructor, and by providing the students with more immediate feedback on their own understanding, assessment can provide a very powerful teaching tool.

Conclusion

Statistics instructors have been very excited about how advances in technology have dramatically changed what we can do in our courses. For example, shifting the computational burden to computers and calculators allows more time to focus on conceptual understanding and other reform goals. However, recent research is illustrating that quality programs and simulations are not enough to ensure cognitive change. For example, the establishment of cognitive dissonance appears to be a crucial component to effective interaction with technology, providing students with the opportunity to immediately test and reflect on their knowledge in an interactive environment. However, it is less clear what level of student engagement is necessary to promote cognitive dissonance. We also found that prerequisite knowledge plays a large role in students' ability to learn from technology. Indeed our sampling distribution research results have had numerous implications on instruction of topics earlier in the course (e.g. more emphasis on understanding of variability). We have also begun using a developmental model of reasoning to help us identify and improve a student's level of reasoning throughout the course.

As we continue to examine these issues, new assessment instruments need to be developed that better examine students' *process reasoning*, beyond their *verbal reasoning*. We also need to take full advantage of the role of assessment as an instructional and research tool. We encourage more classroom–based research done carefully, collaboratively, and over time, to effectively provide insight into why an interaction with technology works, improve understanding of the processes involved, and develop knowledge of similarities and differences across multiple instructional setting, while suggesting changes for improved teaching practice and ongoing research.

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References

Cobb, G. (1992). Teaching Statistics. In L. Steen (Ed.), *Heeding the call for change: Suggestions for curricular action* (pp. 3–43). Mathematical Association of America Notes, No. 22.

Cross, K. P., & Steadman, M. H. (1996). *Classroom research: Implementing the scholarship of teaching.* San Francisco: Jossey–Bass Publishers.

delMas, R. & Bart, W.M. (1989). The role of evaluation exercise in the resolution of misconceptions of probability. *Focus on Learning Problems in Mathematics*, 11(3), 39–54.

delMas, R., Garfield, J., & Chance, B. (1999). A model of classroom research in action: Developing simulation activities to improve students' statistical reasoning. *Journal of Statistics Education*, 7(3), online.

Garfield, J. (2000). The role of statistical reasoning in learning statistics. Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, April, 2000.

Garfield, J. (1993). An authentic assessment of students' statistical knowledge. In N. Webb (Ed.), *National Council of Teachers of Mathematics 1993 Yearbook: Assessment in the Mathematics Classroom* (pp. 187–196). Reston, VA: NCTM.

Hawkins, A. (1997). Myth–Conceptions! In J. Garfield and G. Burrill (Eds.). *Research on the Role of Technology in Teaching and Learning Statistics.* Voorburg, The Netherlands: International Statistical Institute.

Holland, J. H., Holyoak, K.J., Nisbett, R.E., and Thagard, P.R. (1987). *Induction: Processes of Inference, Learning, and Discovery.* Cambridge, Massachusetts: The MIT Press.

Jennings, D., Amabile, T., and Ross, L. (1982). Informal covariation assessment: Data–based versus theory–based judgments. In D. Kahneman, P. Slovic, & A. Tversky (Eds.), *Judgment Under Uncertainty: Heuristics and Biases* (pp. 211–230). Cambridge: Cambridge University Press.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Mogill, A. T. (1997). A framework for assessing and nurturing young children's thinking in probability. *Educational Studies in Mathematics,* 32, 101–125.

Jones, G. A., Thornton, C. A., Langrall, C. W., Putt, I. J., and Perry, B. (1998). *A framework for assessing and fostering students' statistical thinking.* Manuscript submitted for publication.

Kelly, A. & Lesh, R. (Eds.) (2000). *Handbook of Research Design in Mathematics and Science Education* (pp. 361–512). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lord, C., Ross, L., and Lepper, M. (1979). Biased assimilation and attitude polarization: The effects of prior theories on subsequently considered evidence. *Journal of Personality and Social Psychology* 37, 2098–2109.

Nickerson, R. S. (1995). Can technology help teach for understanding? In D. N. Perkins, J. L. Schwartz, M. M. West, & M. S. Wiske (Eds.), *Software Goes to School: Teaching for Understanding with New Technologies* (pp. 7–22). New York: Oxford University Press.

Posner, G.J., Strike, K.A., Hewson, P.W., and Gertzog, W. A. (1982). Accommodation of a scientific conception: Toward a theory of conceptual change. *Science Education*, 66(2), 211–227.

Ross, L. and Anderson, C. (1982). Shortcomings in the attribution process: On the origins and maintenance of erroneous social assessments. In D. Kahneman, P. Slovic, & A. Tversky (Eds.), *Judgment Under Uncertainty: Heuristics and Biases* (pp. 129–152). Cambridge: Cambridge University Press.

Snir, J., Smith, C., and Grosslight, L. (1995). Conceptually enhanced simulations: A computer tool for science teaching. In D. N. Perkins, J. L. Schwartz, M. M. West, & M. S. Wiske (Eds.), *Software Goes to School: Teaching for Understanding with New Technologies* (pp. 106–129). New York: Oxford University Press.

Tarr, J. E., and Jones, G. A. (1997). A framework for assessing middle school students' thinking in conditional probability and independence. *Mathematics Education Research Journal*, 9, 39–59.

ConStatS: http://www.tufts.edu/tccs/services/css/ConStatS.html

Hyperstat: http://davidmlane.com/hyperstat/index.html

Visual Statistics: http://www.mhhe.com/business/opsci/doane/

StatPlay: La Trobe University, Australia

Sampling Distributions: http://www.gen.umn.edu/faculty_staff/delmas/stat_tools/stat_tools_software.htm

TECHNOLOGY AND VERSATILE THINKING IN MATHEMATICS

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Today's technology gives us a great opportunity to complement the subtlety of human thought with the power and accuracy of modern computers. In this presentation I consider fundamental modes of human thinking to see how enactive, visual and symbolic methods can be used in a versatile way with the support of well-designed software. My analysis focuses on the use of symbols to think about mathematics and to do mathematical procedures and how visual enactive software can be used to enhance our conceptual thinking processes. In particular, I consider a theoretical cognitive development in arithmetic, algebra and the calculus and reflect on empirical research to show how the computer can be used both well and badly in supporting mathematical learning.

Introduction

The turn of a century is a time for looking forward and reflecting on the past. Given this is the turn of the millennium, perhaps we should begin by looking back long into our previous history. We stand now at a time of great innovation in technology, moving on at an exciting, even bewildering pace. But as we do so, we should reflect that the human brain which conceives these new ideas was not designed top-down to be able to process complex mathematical thoughts. It has evolved from the brain of an upright hunter-gatherer that lived in the grasslands of Africa over a hundred thousand years ago. Our sophisticated brain is therefore built on primitive perception and physical action. It may even be, as is contended by Lakoff and Johnson (1999) that all our thoughts are embodied, in that they build from our human interaction with the world.

The major part of our brain is taken up with vision, perception of space and causing actions on what we see and sense outside us. We have opposed thumbs that allow us to manipulate objects in subtle ways. By a happy accident, our upright stance moves our larynx into a position that enables us to complement our mental powers by making sophisticated sounds that give us language. This provides a formidable facility for constructing our thoughts and communicating them to others. In recent millennia, and in particular, in the period since the invention of calculus in the seventeenth century, modern *homo sapiens* has developed a method of writing and manipulating mathematical symbols. These symbols are peculiarly fitted to the working of the brain. They are compact tokens that allow us to think about them and the relationships between them, and they can be manipulated, using the brain's natural ability to practice a sequence of activities until they become automatic and need little conscious attention to carry them out. In this way, the symbols of arithmetic, algebra and symbolic calculus enable us both to *think about* mathematics and also to *do* mathematics by way of calculation and manipulation of symbols.

This sophisticated symbolic ability provides us with a sequential/verbal-symbolic mode of thinking complemented by our more primitive holistic visuo-spatial senses. The term *versatile thinking* was used by Tall and Thomas (1991, p. 130), following Brumby (1982), to refer to the "complementary combination of both modes, in which the individual is able to move freely and easily between them, as and when the mathematical situation renders it appropriate."

Technology offers new ways of operation to utilise our versatile thinking processes in more powerful ways. The computer can perform algorithms at enormous speed with great fluency and accuracy, but it lacks the brains multi-processing ability to put old ideas together in new ways. Computer software not only offers arithmetic computation and symbolic manipulation, the results of these calculations can be represented visually, allowing the versatile brain to use visually presented results to see conceptual linkages.

In this presentation, I therefore focus on various aspects of mathematics, involving symbolic concepts and processes on the one hand and global visual representations on the other. In particular I shall consider how arithmetic, algebra and the calculus benefit the individual thinker by being provided in a computer environment that enhances and encourages versatile thinking.

Symbols and technology

The algorithms of arithmetic proved very amenable to being programmed and we have had four-operation calculators with us for most of the last half of the last century. In the nineteen-eighties numeric calculation on computers became enhanced by symbolic manipulation. There was widespread belief that the computer could do away with all the unnecessary clutter of calculation and manipulation allowing the individual to concentrate more on the essential ideas. Computers and calculators in business remove much of the tedium of calculation. An individual with no arithmetic skills beyond typing in numbers can enter the cost of items in a shop and the machine will give the total and even issue the correct change. Furthermore entry of information can be simplified by scanning bar-codes and stock-control can be handled by referring the items sold to the stock database to allow replacements to be ordered automatically. Soon even more of the economy will be taken over by technological means.

However, simply *using* technology, does not necessarily mean we *understand* what is happening. We may even lose some of the facility we had before. Hunter, Monaghan and Roper (1993) found that a third of the students using a computer algebra system could answer the following question before the course, but not after:

'What can you say about u if $u=v+3$, and $v=1$?'

As the students had no practice in substituting values into expressions during the course, the skill seems to have atrophied. It is a warning given by the old adage 'if you don't use it, you lose it,' a saying supported by physical evidence in the brain that unused pathways will tend to decay.

Other evidence also suggests that the use of symbol manipulators to reduce the burden of manipulation may just replace one routine paper-and-pencil algorithm with another even more meaningless sequence of keystrokes. Sun (1993)—reported in Monaghan, Sun & Tall (1994)—describes an experiment in which nine highly able 16/17 year old students taking a 'further mathematics' course had unlimited access to the software *Derive*. When asked to find a limit such as

$$\lim_{x\to\infty}\frac{2x+3}{x+2},$$

eight out of nine Derive students used the software procedure to produce they answer '2'. They claimed they knew no other method, even though they had been shown the technique of dividing top and bottom by x to get

$$\lim_{x\to\infty}\frac{2x+3}{x+2}=\lim_{x\to\infty}\frac{2+(3/x)}{1+(2/x)}=\frac{2+0}{1+0}=2.$$

Meanwhile, in a comparable group of 19 students using only paper-and-pencil methods, twelve used the simplification method given above, three substituted various numbers and four left it blank. This slender evidence intimates two things. First, almost all the Derive students obtained the 'correct result', whereas a considerable minority of the others failed, showing the power of the software. Second, the Derive students appeared simply to be carrying out a sequence of button presses, showing the distinct possibility of using technology with a lack of conceptual insight.

This phenomenon was repeated when the students were asked to explain the meaning of

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Here both the Derive group and one of the paper-and-pencil groups had been part of a discussion of the meaning of the notation in their mathematics lessons. *All* of the non-Derive group gave a satisfactory theoretical explanation of the concept. *None* of the Derive group gave *any* theoretical explanations. Four of them gave examples by replacing $f(x)$ with a polynomial and going through the sequence of key-strokes to calculate the limit. Although the curriculum may include initial theoretical activities (and most introductions to calculus begin with some discussion of the limit concept), what the students learn is dependent on the way they sense and interpret those activities that they are actually involved with at the time. In the case of these students using symbol manipulators, they have routinized the button-pressing activities and carry them out as automatic routines.

From this data we can see that students learn what they do. If they press buttons, they learn about button-pressing sequences. What is therefore important is to build a sense of meaning through reflection on the underlying mathematics. It is here that a *versatile* approach may prove of real value.

Sensori-motor and visual aspects

Underlying the more conscious aspects of doing mathematics, there are other, deeper, human activities that provide an essential basis for all thought. The most primitive of these involve sensori-motor activity (physical sensations and bodily movement) and visual imagery. They play an important part in the computer interface. For example, the sensori-motor system allows decisions to be implemented intuitively using the mouse and keyboard.

These low-level cognitive actions also provide support for high-level theoretical concepts. Figure 1 shows software to build graphical solutions to (first order) differential equations by using the mouse to move a small line segment whose slope is determined by the differential equation. A click of the mouse deposits the segment and the user may fit line segments together to give an approximate solution.

Such an activity can be performed intuitively with little knowledge of the theory of differential equations. Yet it already carries in it the seeds of powerful ideas about possible existence theorems—that a typical first order differential equation will have a unique solution through each point, and following the changing direction will build into a global solution curve. By considering selected examples it is be possible to look at the wider view of what happens to a whole range of solution curves and to see their behaviour. In this way an intuitive interface can give advance organisers for formal theory, especially to those individuals who naturally build on visual imagery.

Symbolism as a mental pivot between process and concept

Symbols used in a range of mathematical contexts give *Homo sapiens* an incredibly simple way of dealing with quantities for calculation, problem solving and prediction. Many symbols simply act as a *pivot* between the symbol conceived as a concept (such as number) and a process (such as counting) (Figure 2).

symbol	*process*	*concept*
4	counting	number
3+2	addition	sum
−3	subtract 3 (3 steps left)	negative 3
3/4	sharing/division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)$ $\sum_{n=1}^{\infty} 1/n^2$	tending to limit	value of limit
$(x_1, x_2, \ldots, x_n)$	vector shift	point in n-space
__ S_n	permuting $\{1,2,\ldots,n\}$	element of S_n

Figure 2. Symbols as process and concept

Gray & Tall (1994) refer to the combination of symbol representing both a process and the output of that process as a *procept* (Figure 3).

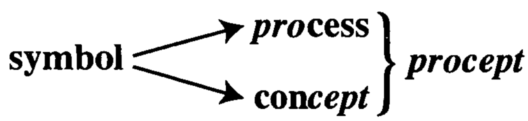

Figure 3: The symbol as pivot between process and concept forming a procept

The procept notion has been given increasingly subtle meaning since its first formulation (Gray & Tall, 1991). It is now seen mainly as a *cognitive* construct, in which the symbol can act as a *pivot*, switching from a focus on process to compute or manipulate, to a concept that may be *thought* about as a manipulable entity. We believe that procepts are at the root of human ability to use mathematical ideas in arithmetic, algebra and other theories involving manipulable symbols. They allow the biological brain to switch effortlessly from thinking about symbols as concepts to using them for doing mathematical processes in a minimal way.

There are several different ways in which the symbolism is used :

(a) a *procedure* consists of a finite succession of actions and decisions built into a coherent sequence. It is seen essentially as a step-by-step activity with each step triggering the next.

(b) the term *process* refers to when the procedure is conceived as a whole and the focus is on input and output rather than the particular procedure used to carry out the process. It may be achieved by n procedures ($n \geq 0$) and affords the possibility of selecting the most efficient solution in a given context.

(c) a *procept* requires the symbols to be conceived flexibly as processes to do and concepts to think about. This allows for more powerful mental manipulation and reflection to build new theories.

Different uses of symbolism give rise to differing levels of flexibility and ability to think mathematically (Figure 4). This is not to say that procedural thinking does not have its value. Indeed, much of the power of mathematics lies in its algorithmic procedures. However, a focus on procedures alone, without conceptual linkages between them, leads to increasing cognitive stress as the individual learns more and more disconnected pieces.

The difficulty in thinking conceptually seems to increase throughout the curriculum. My own perception of these difficulties is that the underlying procepts act in very different ways, so that the learner, who has internal methods of processing the ideas, finds new ideas strangely conflicting with inner beliefs. This, I believe, leads to a lack of connections and the desire to learn procedures solely to pass examinations.

This spectrum has been beautifully demonstrated by DeMarois (1998) who worked with students using graphing calculators in a remedial college pre-algebra course. They were asked to consider the problem given in Figure 5, relating to two

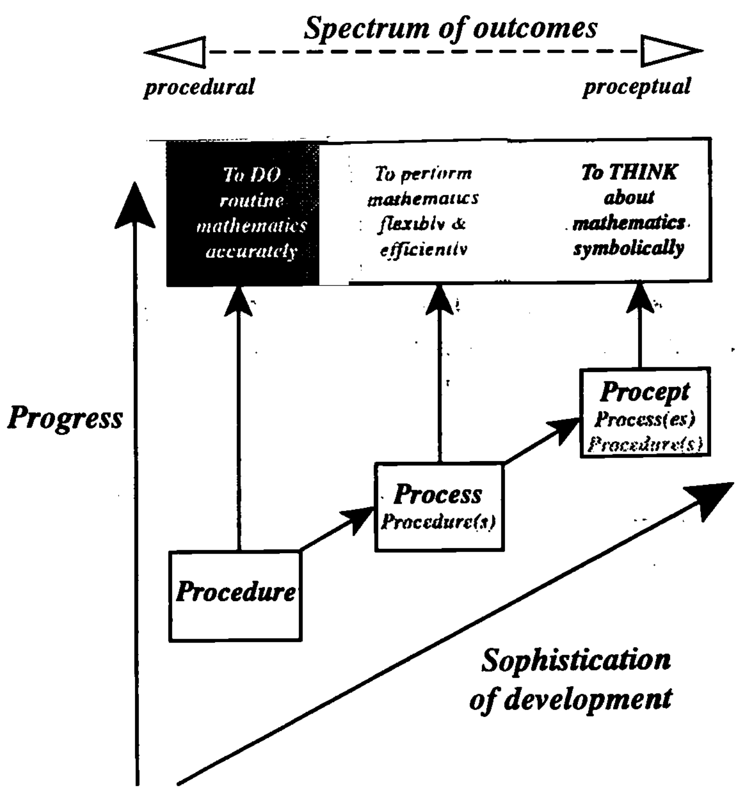

Figure 4. A spectrum of performance in the carrying out of mathematical processes

functions having different internal procedures but the same input-output relationship.

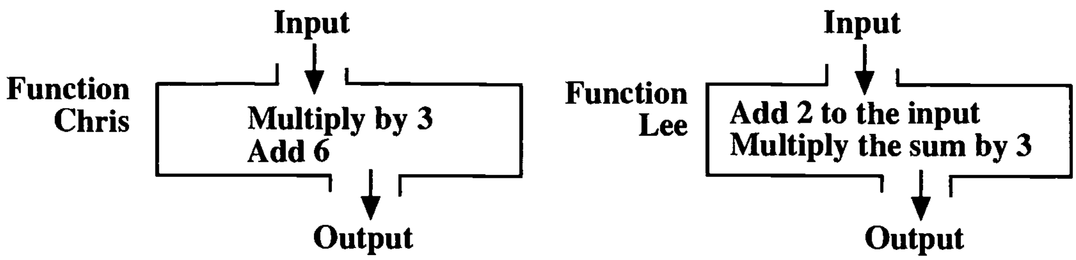

Figure 5. Write the outputs of these two function boxes and say if they are the same function.

The responses given by students achieving grades A, B, C showed characteristics of the three above distinct levels of sophistication. They were as follows:

•*Grade A Student*: **3x+6, 3(x+2)**
Yes, if I distribute the 3 in Lee, I get the same function as Chris. (*procept*)

•*Grade B Student*: **x3+6, (x+2)3**
Yes, but different procedures (corresponding to our notion of *process*)

•*Grade C Student*: **3x+6, x+2 (3x)**
No, you come up with the same answer, but they are different processes (corresponding to our notion of *procedure*)

(DeMarois, 1998, pp.171, 174–5)

The grade A student is responding in a manner which shows that she can manipulate the expressions and see that one is the same as the other. Her fluency of manipulation of the expression here and in other contexts reveals her flexible ability operating with expressions as *procepts*. Students B and C write the algebra in idiosyncratic ways, with B following the precise verbal order and C using the algebraic order for $3x+6$ yet writing the second function expression in a way which can be read as '$x+2$ (three times).' Although B and C use the terms procedure and process in their own way, B says 'yes' (they are the same function) suggesting a *process* interpretation and C says 'no' (they are not the same function) suggesting a *procedural* interpretation.

Fundamental differences in proceptual structure

Despite the power of procepts, students often have difficulty with them. This is in part often due to their working at a level which causes them greater cognitive stress, for example, working at a procedural level and being unable to put all the ideas together to be able to cope with more complex problems. However, there is also another highly significant factor. In various different parts of the curriculum, symbols as process and object operate in very different ways. Thus the student who may very well have a good grasp of the ideas in one context suddenly finds that things are confusing at the next stage. This may be illustrated by looking at some of the different types of procept in arithmetic, algebra and limits.

(1) ***arithmetic procepts***, 5+4, 3×4, $\frac{1}{2}+\frac{2}{3}$, 1·54÷2·3, all have built-in algorithms to obtain an answer. They are *computational*, both as processes and even as concepts. For instance in the sum 8+6, the concept 6 can be linked to the operation 2+4, which can be combined in the sum 8+2+4 to give 10+4 which is 14. A deeper analysis would reveal more differences between operations with whole numbers, negative numbers, fractions and decimals.

(2) ***algebraic procepts***, such as 2+3x, can only be evaluated if the value of x is known. Thus an algebraic procept has only a *potential process* (of numerical substitution) and yet the algebraic expressions themselves are expected to be treated as *manipulable concepts* using the usual algebraic rules.

Meaningful power operations such as

$$2^3 \times 2^2 = (2 \times 2 \times 2) \times (2 \times 2) = 2^5$$

can act as a cognitive basis for the power law

$$x^m \times x^n = x^{m+n}$$

valid for all real x and for whole numbers m, n. A student can have a meaningful interpretation of this symbolism, yet be totally confused when the symbols are used in a different way as:

(3) ***implicit procepts***, such as the powers $x^{1/2}$, x^0 or x^{-1}, for which the original meaning of x^n no longer applies, but is implicit in the generalised power law.

Attempting to give the 'same meaning' to these ideas does not work, for we can hardly speak of 'half an x multiplied together', or 'no xs multiplied together' (surely no xs must give zero). Thinking of x^{-1} as 'minus one xs' seems as foolish as talking about 'minus one cows'. These powers can, however, be given an implicit meaning by assuming the power law as an axiomatic basis for deduction. Thus, for $m = n = 1/2$, we get

$$x^{1/2} \times x^{1/2} = x^1 = x$$

from which we may deduce that $x^{1/2} = \sqrt{x}$. Of course, such a meaning is being deduced from a law that we do not *know* is true, but one that we *assume* as an axiom—a new way of doing mathematics that is powerful for those willing and able to follow it through, but puzzling for those who attempt to give the symbols the same meaning as before.

Later examples in the curriculum are:

(4) ***limit procepts***, such as $\lim\limits_{x \to a} \dfrac{x^3 - a^3}{x - a}$ or $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$. These have ***potentially infinite processes*** 'getting close' to a limit value, but this may not be computable in a finite number of computations. Limits are often conceived as 'variable quantities' which get arbitrarily close to a limiting value, rather than the limit value itself.

(5) ***calculus procepts***, such as $\dfrac{d(x^2e^x)}{dx}$ or $\int_0^\pi \sin mx \cos nx \, dx$. These are more familiar in the sense that they (may) have finite operational algorithms of computation (using various rules for differentiation and integration).

Although limit procepts are often introduced as the first idea in calculus, we shall see that this use does not fit with the embodied brain that invariably interprets the notion of limit as a potentially infinity process rather than a manipulable concept. Instead the students are much happier with calculus procepts that involve a process of computation and an *answer*, albeit in the form of an expression rather than a number.

The principle of selective construction

To reduce the cognitive strain on the learner in the mathematics curriculum using computers, I formulated the *Principle of Selective Construction* (Tall, 1993). This proposes the design of software that embodies selected aspects of the theory for the learner to explore whilst the computer carries out other essential processes internally. It can be accomplished in a range of ways. For instance, the approach to differential equations advocated earlier (Figure 1) provides a manipulable visual representation where the computer carries out the algorithms and draws the pictures leaving the student to concentrate on building a solution. The interface enables the user to gain an embodied visuospatial sense of the theory in which hand and eye coordinate to build the direction of the solution curve. Having gained this insight, the student could then profit by focusing on other aspects, such as the internal numerical procedures to produce approximate solutions, or the study of available symbolic methods to give solutions in particular cases.

Another use of the principle of selective construction arises in Michael Thomas's approach to the introduction of algebra (Thomas, 1988; Tall and Thomas, 1991). In one activity the pupils enact a physical game, away from the computer, using two large sheets of cardboard. One represents 'the screen' on which instructions are given. The other is 'the store' on which squares are drawn which can be labelled with a letter for the variable and the numerical value stored inside. Typical instructions might be to put A=2 (place a number 2 in a box marked 'A') then PRINT A+3 (find the number in the box marked 'A' and print out the sum of it and 3, to output 5). This concentrates on the *process* aspect of evaluation of expressions and uses an embodied approach that builds on human activity. On the other hand, writing simple programs on the actual computer, such as INPUT A: PRINT 2*(A+1), 2*A+2, gets the computer to carry out the process of calculation so that the student can concentrate on the *equivalence* of the same outputs from different procedures of calculation. This combination of separate focus on symbol as process and as (equivalent) concepts proved to give a long-term improvement in conceptualization of expressions as manipulable concepts (Tall & Thomas, 1991).

In the next few sections we look at the developing curriculum in arithmetic, algebra and the calculus. In each case we begin with the cognitive processes involved and then consider how the computer can be used to promote versatile thinking using the principle of selective construction.

Long-term considerations in arithmetic, algebra and calculus

Manipulating symbols in arithmetic

Whole number arithmetic has a fundamental proceptual structure in which the number symbol has a dual role as number concept and counting process (Gray & Tall, 1994). Addition is successively compressed from the triple 'count-all' process (count one collection, count the other, count the two together) through the transition of the double counting process of 'count-both' (count-one number then count-on the second) to the single 'count-on' process (count-on the second number starting from the first). Some facts are remembered (as 'known-facts') and may, or may not, be used flexibly to 'derive facts' from those already known (for instance, deriving 8+6 from 8+2+4=10+4=14 or 23+5=28 as 20+3+5, and so on). Gray Pitta, Pinto & Tall (1999) show how there is a bifurcation in strategy between those who cling to the counting operations on objects and those who build (to a greater or lesser extent) a conceptual hierarchy of manipulable relationships between process and concept. The former become imprisoned in thinking about manipulating objects in a way which does not generalise to larger numbers whilst the latter have an internal engine to derive new facts from old that forms a foundation for mental arithmetic.

The successive compression from triple-counting to the single count-on, to the flexible use of remembered known facts is a long journey taking several years to become efficient in arithmetic. The use of calculators and computers in early mathematics has been perceived by the British government and their experts as being less successful than it might be. So much so that, in the English National Curriculum, the use of calculators with young children has been discouraged in the hope that their absence will enable children to build mental arithmetic relationships.

Perhaps this is more to do with the misuse of the calculator (for performing calculations without having to think) than it is to any inherent defect in the apparatus itself. Used well, to *reflect* on mathematical ideas, the calculator can be very beneficial as Gray and Pitta (1997) showed in their work with a slow-learner having difficulties with arithmetic.

Emily, aged 8, was identified as one of the weakest four children in arithmetic in a year group of 104. She 'seemed to associate counting with fingers with the use of a particular sequence of fingers' and she found this difficult. She explained, 'sometimes I get into a big muddle with them ... I am not concentrating on the sum. I am concentrating on getting my fingers right which takes a while.' When she did simple arithmetic mentally, she imagined manipulating arrays of counters, for instance, she explained the sum 4–3, saying '... there's two dots above each other and then there's ... the first one, the one below and the next to it are being taken away and there is only the first one left.' (Figure 6.) With these direct manipulations of fingers or mental objects, she was under considerable cognitive stress handling small numbers and had great difficulty with numbers larger than ten. At this stage she clung to counting and was a candidate for long-term failure. However, Gray and Pitta planned a series of activities using a graphic calculator which displayed both the arithmetic operations to be carried out and the numerical results. (Figure 7).

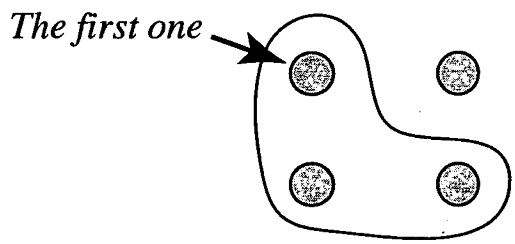

Figure 6. Emily's image of 4–3

4 + 5	
	9
4 + 4 + 1	
	9
3 + 2 + 4	
	9

Figure 7. Ways of making 9 on a graphic calculator

Emily was given a personalised workbook asking her to find ways to make 9, or to make 9 starting with 4, or with 10. Her combinations included standard ones such as 4 + 5, 6+9, but also more complex ones such as 4+4+1, 5+6–2, 5+1+1+2. In her interviews she was encouraged to talk about her discoveries without using the computer. Only once did she ever refer to seeing 'dots' and nine months later she was operating mentally, seeing numbers 'flash' into her mind. This insight shows her making strides towards compression that are enhanced by her calculator work. She was seeing relationships between *numbers* in her mind's eye.

The principle of selective construction is implicit in Gray and Pitta's work with Emily. The software performs the calculation and Emily can concentrate on the relationships without having to carry out the intervening counting processes. Were she to continue to focus on counting, it would have become too onerous for her to develop much further.

This imaginative use of the calculator shows the poverty of the UK numeracy strategy to discourage the use of calculators for young children. It is not the *technology* that is at fault, but the *use* to which it is put.

Proceptual problems in algebra

Long before the notion of 'procept' was formulated, students were noted to have difficulty conceiving an expression such as '$7+x$' as the solution to a problem—aphenomenon described by Collis (1972) as 'lack of closure'. Davis, Jockusch and McKnight (1978) remarked similarly that 'this is one of the hardest things for some seventh-graders to cope with; they commonly say "But how can I add 7 to x, when I don't know what x is?"' Matz (1980) commented that, in order to work with algebraic expressions, children must 'relax arithmetic expectations about well-formed answers, namely that an answer is a number.' Kieran (1981) similarly commented on some children's inability to 'hold unevaluated operations in suspension.' All of these can now be described as the problem of manipulating symbols that—for the students—represent potential processes (or specific procedures) that they cannot 'do', yet are expected to treat as manipulable entities. Essentially they see expressions as unencapsulated processes rather than manipulable procepts.

As mentioned earlier, this was tackled by Thomas (1988) using a combination of physical activity to give an embodied sense of the process and the use of the computer to focus on the relationships between the concepts.

In the work of McGowen (1998), DeMarois (1998) and Crowley (2000), American college students using graphic calculators to support their development of algebra show the spectrum of performance from procedural competence to flexible use of mental linkages. By getting her students to draw concept maps of their knowledge at various points in the course, McGowen found that the less successful simply focussed on the current work, attempting to use procedures as they were given, but the more successful built concept maps which built incrementally on previous work to give a highly integrated conceptual system. One successful student explained his methodology:

> While creating my [final] concept map on function, I was making strong connections in the area of representations. Specifically between algebraic models and the graphs they produce. I noticed how both can be used to determine the parameters, such as slope and the y-intercept. I also found a clear connection between the points on a graph and how they can be substituted into a general form to find a specific equation. Using the calculator to find an equation which best fits the graph is helpful in visualizing the connection between the two representations. I think it's interesting how we learned to find finite differences and finite ratios early on and then expanded on that knowledge to understand how to find appropriate algebraic models. (McGowen and Tall, 1999, p. 284).

There is evidence that this concentration on making links in the concepts in a course on straight line graphs using graphic calculators can radically affect performance on later courses. Crowley (2000, pp. 209, 210) found that those who continued to be successful 'had readily accessible links to alternative procedures and checking mechanisms' and 'had tight links between graphic and symbolic representations'. They succeeded even though they 'made a few execution errors.' Others who succeeded in the earlier course but 'had serious difficulties with the next,' had passed the first course whilst showing underlying weaknesses in conception. 'They had links to procedures, but did not have access to alternate procedures when those broke down. They did not have routine, automatic links to checking mechanisms. They did not link graphical and symbolic representations unless instructed to do so. ... They showed no evidence that they had compressed mathematical ideas into procepts.'

Proceptual problems in the concept of limit

It has long been known that students have difficulty coming to terms with the limit concept presented at the beginning of a calculus course. The early research on this topic is summarised in Cornu (1991). The conceptual difficulties can be clearly formulated using the notion of procept. At first a limit (say of a sequence of numbers) is seen as a *process* of giving a better and better approximation to the limit value. This process of 'getting close' but 'never reaching' the limit gives rise to the mental image of a variable quantity that is 'arbitrarily small' or 'arbitrarily close' to a fixed quantity. This may then lead to the construction of a mental *object* that is 'infinitely small'—a cognitive *infinitesimal.* Monaghan (1986) called this a 'generic limit'. It is a cognitive concept wherein the limit object has the same properties as the objects in the sequence which is converging to it. Thus in the sequence $(1/n)$, all the terms are positive, so the generic limit is positive. It is also arbitrarily small. This leads to a concept image of the number line that has infinitesimal quantities included and is therefore at variance with the formal definition of the real numbers.

Symbol manipulators use a variety of representations for numbers, including integers, rationals, finite decimals, radicals such as $\sqrt{2}$, $\sqrt[10]{7}$, special mathematical numbers such as π, e. Students conceive of different kinds of number in subtly different ways. For instance they may be feel secure in 'proper numbers' such as whole numbers and fractions. But they may regard infinite decimals, both repeating and non-repeating, as 'improper numbers' which 'go on forever' (Monaghan, 1986). Procept theory classifies these 'improper numbers' as 'potentially infinite' processes rather than as number concepts. We therefore see that many students do not have a coherent view of the number line. It is populated by a range of different kind of creatures, some familiar, some less familiar, and some downright peculiar.

Reconstructing these views of numbers is not something that has proved at all easy (Williams, 1991). My own consideration is that the potentially infinite process in the limit procept causes great conceptual difficulty and I have preferred to attack the design of the calculus curriculum by making the notion of limit *implicit* in the software, whilst encouraging the student to focus visually on what is happening.

The calculus

Technology offers new ways of approaching the calculus in a versatile way which are becoming widespread around the world. My own approach builds on using visual software to give an embodied sense of the underlying mathematical concepts. There are four procepts in the calculus: the notion of *limit* (in a variety of different forms), *change* (function), *rate of change* (derivative), *cumulative growth* (integral). However, given the evidence that the formal notion of limit is not a sensible place to start when students are learning calculus, I concentrate on the last three (Figure 8).

Procept		
Change: FUNCTION	doing	calculating values
	undoing	solving equations
Rate of change: DERIVATIVE	doing	differentiation
	undoing	anti-differentiation, solving differential equations
Cumulative growth: INTEGRAL	doing	integration
	undoing	fundamental theorem of calculus

Figure 8. Three procepts in functions and calculus

If we analyse these in greater detail, we find that there is a spectrum of approaches, from a real-world meaning of the terms, to various aspects that can be represented on the computer in graphic, numeric and symbolic ways, and on to the formal theory of analysis. Figure 9 shows my own analysis of the procepts and their representation in the calculus, taken from Tall (1997).

The first column (real-world calculus) has been exemplified in computer software by Kaput (see the *SimCalc* website http://tango.mth.umassd.edu). (The approach via SimCalc allows the movement of a child to be tracked by a sensor and then displayed as a graph to give a powerful embodied link between child and computer.) The final column does not concern us here as it relates to the formal theory discussed in an extension of this presentation (Tall, to appear) which carries the ideas of embodied thinking and the use of technology through to formal mathematical thinking. I will therefore concentrate on the three representations: graphic, numeric, symbolic.

Graphic representations on the computer already embody the principle of selective construction. The numerical calculations that produce the pictures are performed internally while the graphic software provides an environment for visuospatial exploration. In a logical development, the numeric and symbolic would almost certainly precede the graphic. In cognitive terms, a graphic approach to the calculus is part of a versatile learning sequence. It enables the learner to visualise and conceptualise the concepts in an embodied manner that can form a foundation for future development, be it a technical support in applications, formal epsilon-delta analysis, or the infinitesimal calculus of non-standard analysis.

In Figure 9, I have highlighted three parts of the graphic column: ***graphs***, ***visual steepness*** and ***area under the graph***. It would be a nice ending to my presentation to be able to show how a versatile use of these three conceptions can make the calculus deeply meaningful. However, although there are great benefits from such an approach, research has continually shown difficulties with the conception of function.

Mathematically the formal notion of function could not be simpler: there are just two sets A and B and, for each element x in A, there is precisely one corresponding element y in B which is denoted by the symbol $f(x)$.

The inner workings of our embodied brain colour this simplicity with a wide range of implied meanings linked to our perceptions and actions. This includes the idea that functions always seeming to have a formula, and a graph with a familiar smooth shape. In addition, experience with linear, quadratic, trigonometric and exponential functions focus on a wide range of different properties for each family of functions. For instance a linear function has an intercept and a slope that completely determine it. Study of the quadratic function involves factorization and 'completing the square'. Trigonometric functions involve radians, the sign of trig functions in each quadrant, relationships between sine, cosine and tangent, together with formulae for sin(A+B), cos(A+B), tan(A+B), and so on. Exponential functions have the power law and the relationship with logarithms and their properties. The learner is therefore bombarded more with the *differences* between all these examples than the underlying function properties that they share. Essentially the function concept itself is hardly ever the focus of attention. Instead the human brain makes links involving the special calculations and

		Representations				
		Visuo-spatial	**Graphic**	**Numeric**	**Symbolic**	**Formal**
Procepts		*Enactive* observing experiencing	*Qualitative* visualizing conceptualizing	*Quantitative* estimating approximating	*Manipulative* manipulating limiting	*Deductive* defining deducing
Change: FUNCTION	doing	distance, velocity etc. changing with time	**graphs**	numerical values	algebraic symbols	set-theoretic definition
	undoing	solving problems	visual solutions where graphs cross	numerical solutions of equations	solving equations symbolically	intermediate value & inverse function theorems
Rate of change:	doing	velocity from time-distance graph	**visual steepness**	numerical gradient	symbolic derivative	formal derivative
DERIVATIVE	undoing	solving problems e.g. finding distance from velocity	visualize graph of given gradient	numerical solutions of differential equations	antiderivative —symbolic solutions of differential equations	antiderivative —existence of solutions of differential equations
Cumulative growth: INTEGRAL	doing	distance from time-velocity graph	**area under graph**	numerical area	symbolic integral as limit of sum	formal Riemann integral
	undoing	computing velocity from distance	know area – find graph	know area —find numerical function	Symbolic Fundamental Theorem	Formal Fundamental Theorem
		REAL-WORLD CALCULUS	THEORETICAL CALCULUS			ANALYSIS

Figure 9. A spectrum of representations in functions and the calculus

manipulations that are actually the main focus of study.

The plight of the function concept in terms of graphs is sealed by the research of Cuoco (1994). The students he studied who had been taught using the traditional notion of function as formula and graph hardly *ever* saw the graph as a process of assignment (for given x, move up to the graph and across to the y-axis to read off $f(x)$.) On the other hand, he showed that programming functions in computer languages involves imagining inputs, carrying out an internal process and giving an output. The *name* of the function procedure allows it to be treated like a manipulable object.

The process usually associated with a graph is the process of drawing it, or tracing along the curve from left to right. The approach used in SimCalc, with *time* as the variable, therefore encourages a bodily sensation of seeing the covariation of, say, distance with time. This therefore gives an embodied approach to the graphs of functions, although there will need to be further development along the line to replace the variable time and its human sense of duration for other variables.

A *Graphic Approach to the Calculus* begins with the notion of a graph representing the relationship between an independent and dependent variable. The study of straight line functions is a necessary pre-requisite to be able to *see* the gradient of the function when the axes have equal scales and to be able to imagine that changing the scales on the axes might change the visual slope in the picture, but not the numerical value of the gradient as y-step over x-step.

Zooming in on a graph, retaining equal scales reveals that most of the graphs we know look less curved the more we zoom in, until they look like a straight line under high magnification. If a small square centred on a point on the curve reveals an approximate straight line when magnified, moving the square along the curve and noting the changing gradient of the curve appeals to the embodied idea of 'looking along the curve and *seeing* its changing gradient'. Thus, in one go, the student can look along the curve and see the gradient changing as a function of x. (In the embodied sense that, as I point at successive places on the graph, the gradient can be seen to change as x changes.)

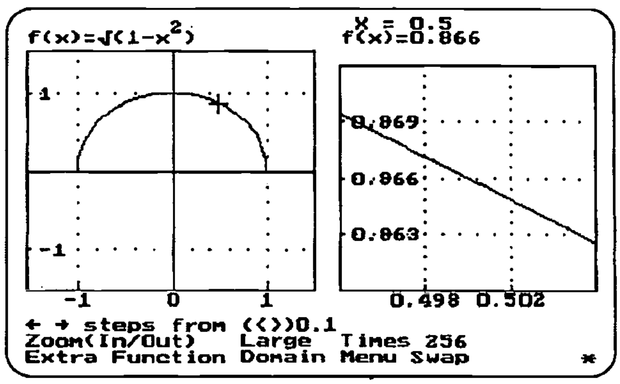

Figure 10. A circle is locally straight (drawn with Tall *et al.*, 1990)

Students are often quite surprised when they see that a circle–the archetypal 'curved' curve– is locally straight. (Figure 10), but then, as they realize that the curvature gets less as the circle gets bigger, this becomes less strange. Using software to magnify the curve can give graphic insight into the phenomenon.

Many mathematicians who hear me say things this way are puzzled and/or angry. For them the gradient must first be approached as a limit at a point, and then, when the limit is achieved, the point is varied and the limit value at the point for all points is the derivative function. For this reason, mathematicians often introduce the notion of 'local linearity' which means finding a linear function at a point on the graph which is the best linear approximation to the graph. Frankly, this is far more complex. Local linearity is defined first at a point in a functional symbolic manner, then the point is varied. Local straightness is a purely embodied visuo-spatial conception of the changing gradient of the graph itself.

Students who follow a locally straight approach are far better at drawing the gradient curve at a given point. They can just look along the curve and *see* the changing gradient and sketch the requisite curve. For instance, given the graph of cos$x$, as x increases from zero, the gradient of the graph starts at zero, moves increasingly negative until at $\pi/2$ the gradient is (about) -1, then it becomes less negative till the gradient is zero at π. Looking along the curve reveals the gradient function looking like the graph of sin$x$ upside down, suggesting the gradient is $-$sin$x$ (Figure 11).

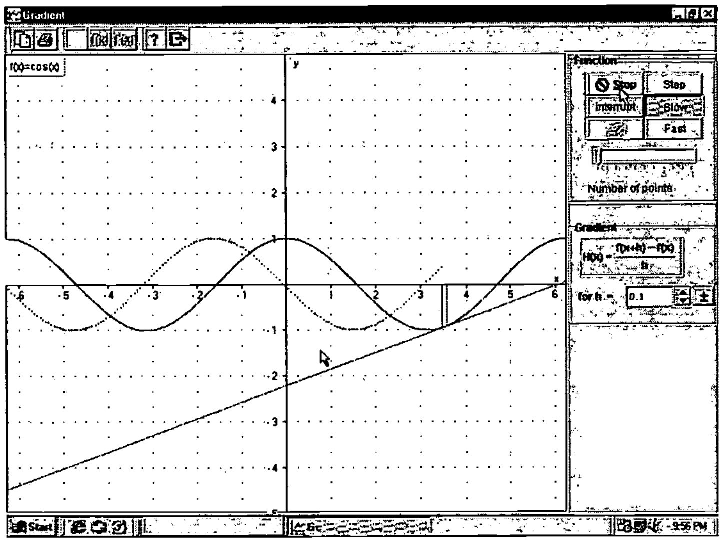

Figure 11. The gradient of cos$x$ (drawn with Blokland *et al.*, 2000).

The graphic approach to the calculus, first by magnification, and then by moving along the curve to trace out the changing gradient gives an *embodied* view of the rate of change. The principle of selective construction is being used again because the software computes the gradient accurately and allows the user to interpret the graphs using knowledge of standard graph shapes.

It fits exactly with the opening idea of solving a first order differential equation. If the first derivative of a function is given, then the function has a locally straight graph, so the original graph can be (approximately) reconstituted to build up a solution curve that has the given gradient.

I also include the graph of the *blancmange* function *bl(x)* and other similar functions which are fractals which reveal the same detail at successive levels of magnification. They never look straight at any magnification, and so they are *nowhere* differentiable. (This can be proved by an embodied visuo-spatial method that itself can be turned into a formal proof (see Tall, 1982)). I can even take a small copy of *bl(x)*, say $n(x)=bl(1000x)/1000$. The functions sin$x$ and sin$x + n(x)$ differ by less than 1/1000, so to a regular scale with the x-range, say from -5 to $+5$, will reveal no difference

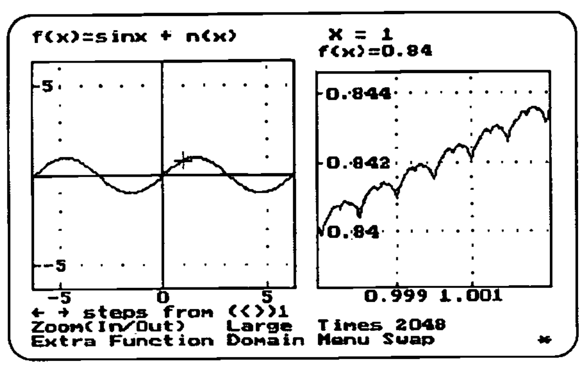

Figure 12. A 'smooth-looking curve' that magnifies 'rough'

between the two graphs on the computer screen. And yet the first is differentiable *everywhere* and the second is differentiable *nowhere*. Thus a graphic approach to the calculus offers insight into far deeper ideas about differentiability (Figure 12).

I do not have the space here to give full details of the extent to which a graphic approach to calculus can lead on to the most subtle of formal ideas. Suffice it to say that, by thinking carefully, one can develop embodied visual intuitions that can contain the seeds of highly subtle formal mathematics. It is mainly a question of looking at the right pictures in the right way. As Benoit Mandelbrot said:

> When I came into this game, there was a total absence of intuition. One had to create an intuition from scratch. Intuition as it was trained by the usual tools—the hand, the pencil and the ruler—found these shapes quite monstrous and pathological. The old intuition was misleading. ... I've trained my intuition to accept as obvious shapes which were initially rejected as absurd, and I find everyone else can do the same.
> (Mandelbrot, quoted in Gleich, 1987, p. 102).

In a publication associated with this conference (Tall, to appear), I follow up this quotation to see how visual intuition can support formal ideas in advanced mathematical thinking. This, and other related papers can be found on my website:

http://www.warwick.ac.uk/staff/David.Tall/

A selection of relevant papers includes:

Tall (1985, 1991, 1992, 1993, 1995, 1997) and Di Giacomo & Tall (2000).

Summary

In this presentation, I have presented a theory of mathematics built on the underlying vision and action of the human species, complementing theory with a range of empirical evidence. This shows that learning mathematics involves building up mental imagery in a sequence which is somewhat different from a formal logical development. In the last century, Kaput wrote:

> Anyone who presumes to describe the roles of technology in mathematics education faces challenges akin to decribing a newly active volcano – the mathematical mountain is changing before our eyes... (Kaput, 1992, p. 515.)

The volcano has been smouldering for a long time now and a shape of the future seems to be developing. In this future I am confident that what will make mathematics work to its best advantage are the qualities which make us human. But beyond that, I consider it foolish to attempt to say where the next millennium will take us, for even a decade is a long time at our present rate of development. Having taken part in previous crystal-ball gazing operations (such as the Mathematical Association Report on 'Computers in the Mathematics Classroom' (Mann & Tall, 1992), I now know that our vision of tomorrow soon becomes the history of yesterday. We then quoted the following as an example of forward thinking from a previous report:

> It is unlikely that the majority of pupils in this age range will find [a computer] so efficient, useful and convenient a calculating aid as a slide rule or book of tables.
> (Mathematical Association, *Mathematics 11 to 16*, 1974)

At this historical point I shall therefore refrain from suggesting what will happen even in the near future. But one thing seems sure. As we stand at the beginning of a new millennium (starting January 1st, 2001), being a mathematician and an educator has never been more exciting.

References

Blokland, P., Giessen, C, & Tall, D. O. (2000). Graphic Calculus for Windows. (see www.warwick.ac.uk/staff/David.Tall.)

Brumby M.N. (1982). Consistent differences in cognitive styles shown for qualitative biological problem solving, *British Journal of Educational Psychology*, 52, 244–257.

Collis, K. F. (1972). *A study of the relationship between formal thinking and combinations of operations*. University of Newcastle, NSW, Australia.

Cornu, B. (1991). Limits'. In D. O. Tall (Ed.), *Advanced Mathematical Thinking*, (pp.153–166), Kluwer: Crowley (2000).

Cuoco, A A. (1994). Multiple representations for functions, In E. Dubinsky, A. H. Schoenfeld & J. Kaput (Eds.), *Research Issues in Undergraduate Mathematics Learning*, MAA Notes 33, (pp.121–140). Washington DC: MAA.

Crowley, L. (2000). *Cognitive Structures in College Algebra*, PhD thesis, University of Warwick.

Davis, R. B., Jockusch, E., McKnight, C. (1978). Cognitive processes in learning algebra, *Journal of Children's Mathematical Behavior*, 2 (1).

DeMarois, P. (1998). *Aspects and Layers of the Function Concept*, PhD Thesis, University of Warwick.

Di Giacomo, S. & Tall, D. O. (2000). Cosa vediamo nei disegni geometrici? (il caso della funzione blancmange) *Progetto Alice* (to appear). [English version: What do we "see" in geometric pictures? (the case of the blancmange function) is available from www.warwick.ac.uk/staff/David.Tall.]

Gray, E.M. & Pitta, D. (1997). Changing Emily's Images, *Mathematics Teaching*, 161, 38–51.

Gray, E. M. Pitta, D., Pinto, M., & Tall, D. O. (1999). Knowledge construction and diverging thinking in elementary & advanced mathematics. *Educational Studies in Mathematics*. 38 (1-3), 111–133.

Gray, E. M. & Tall, D. O. (1991). Duality, ambiguity & flexibility in successful mathematical thinking, *Proceedings of PME XV*, Assisi, 2, 72–79.

Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic, *Journal for Research in Mathematics Education*, 26, 115–141.

Hunter, M., Monaghan, J. D. & Roper, T. (1993). The effect of computer algebra use on students' algebraic thinking. In R. Sutherland (Ed.), *Working Papers for ESRC Algebra Seminar*, Institute of Education, London.

Kaput, J. J. (1992). Technology and Mathematics Education. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, New York: MacMillan.

Kieran, C. (1981). Pre-algebraic notions among 12 and 13 year olds, *Proceedings of the Fifth Conference of the Psychology of Mathematics Education*, Grenoble, 158–164.

Lakoff, G. and Johnson, M. (1999). *Philosophy in the Flesh*. New York: Basic Books.

Mann, J. F. A., and Tall, D. O. (Eds.) (1992). *Computers in the Mathematics Curriculum, A Report of the Mathematical Association*. Leicester: Mathematical Association.

Mathematical Association, (1974). *Mathematics 11 to 16*. Leicester: Mathematics Association.

Matz, M. (1980). Towards a computational theory of algebraic competence, *Journal of Mathematical Behavior*, 3 (1), 93–66.

McGowen, M. & Tall, D. O. (1999). Concept maps & schematic diagrams as devices for documenting the growth of mathematical knowledge. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of PME*, Haifa, Israel, 3, 281–288.

McGowen, M. (1998). *Cognitive Units, Concept Images and Cognitive Collages*, PhD Thesis, University of Warwick.

Monaghan, J. D. (1986). *Adolescent's Understanding of Limits and Infinity*. PhD Thesis, University of Warwick.

Monaghan, J., Sun, S. & Tall, D. O. (1994), Construction of the Limit Concept with a Computer Algebra System, *Proceedings of PME 18*, Lisbon, III, 279–286.

Sun, S. (1993). *Students' Understanding of Limits and the Effect of Computer Algebra Systems*. M.Ed. thesis, Leeds University, U.K.

Tall, D. O., (1982). The blancmange function, continuous everywhere but differentiable nowhere, *Mathematical Gazette*, 66 11–22.

Tall, D. O. (1985). Understanding the calculus, *Mathematics Teaching* 110, 49–53.

Tall, D. O. (1991). Recent developments in the use of the computer to visualize and symbolize calculus concepts, *The Laboratory Approach to Teaching Calculus* , M.A.A. Notes Vol. 20, 15–25.

Tall, D. O. (1992). Visualizing differentials in two and three dimensions, *Teaching Mathematics and its Applications*, **11** 1, 1–7.

Tall, D. O. (1993). Interrelationships between mind and computer: processes, images, symbols In D. L. Ferguson (Ed.), *Advanced Technologies in the Teaching of Mathematics and Science*, (pp. 385–413). New York: Springer-Verlag.

Tall, D. O. (1995). Visual Organizers for Formal Mathematics. In R. Sutherland & J. Mason (Eds.), *Exploiting Mental Imagery with Computers in Mathematics Education*. Springer-Verlag: Berlin.52–70.

Tall, D. O. (1997) Functions and Calculus. In A. J. Bishop et al (Eds.), *International Handbook of Mathematics Education*, 289–325, Dordrecht: Kluwer.

Tall, D. O. & Thomas, M. O. J. (1991). Encouraging Versatile Thinking in Algebra using the Computer, *Educational Studies in Mathematics*, **22** 2, 125–147.

Tall, D. O. (to appear). Cognitive development in advanced mathematics using technology. *Mathematics Education Research Journal*.

Thomas M. O. J. (1988). *A Conceptual Approach to the Early Learning of Algebra Using a Computer*, Ph.D. thesis, University of Warwick

Williams, S. R. (1991). Models of Limit Held by College Calculus Students, *Journal for Research in Mathematics Education*, **22**, 237–251.

Note: Information about Graphic Calculus, and all the papers involving David Tall as author or co-author may be obtained from the website: **www.warwick.ac.uk/staff/David.Tall.**

A STRATEGY FOR THE USE OF TECHNOLOGY TO ENHANCE LEARNING AND TEACHING IN MATHS, STATS & OR

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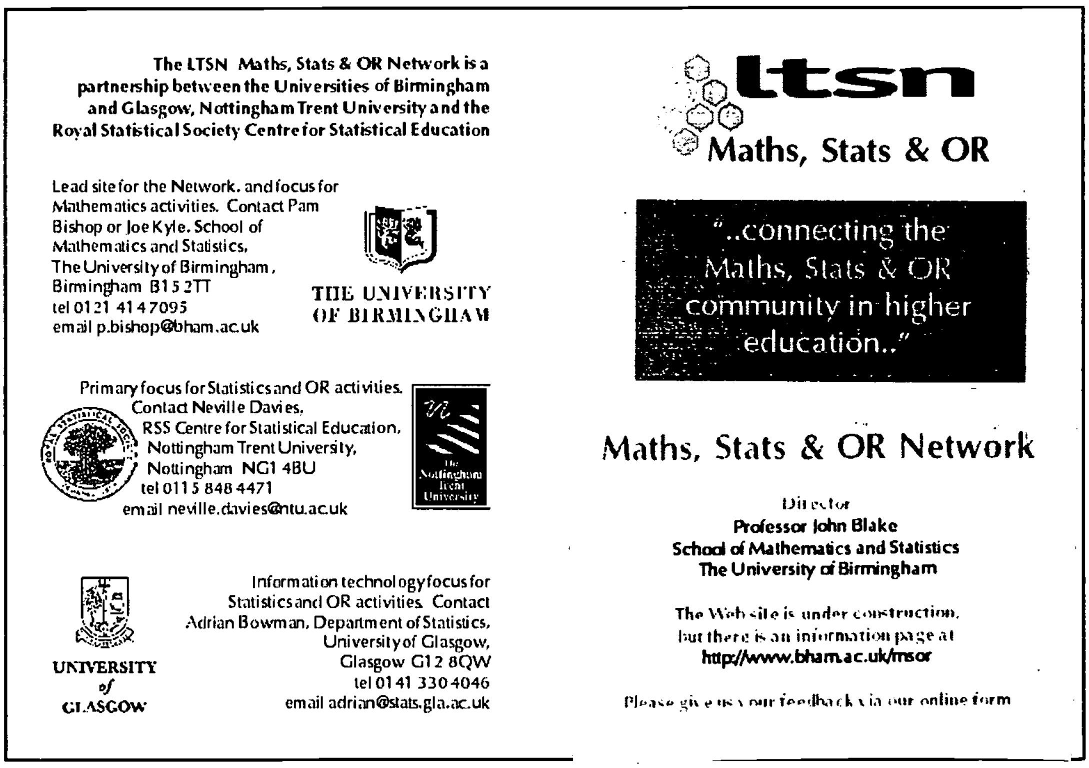
In January 2000 a network of discipline-based centres was established for higher education in the UK, including a Maths, Stats & OR Network which will:

- *support and enhance academic practice in teaching MSOR*
- *coordinate networks of MSOR academics*
- *disseminate innovation and good practice in learning, teaching and assessment*
- *create a forum for the exchange of information, ideas, philosophies and research findings*
- *exploit and harness change associated with new technology, integrating this into pedagogic developments*

The presentation will outline our current and future plans to take full advantage of technology to make teachers of Mathematics, Statistics and Operational Research more effective in Higher Education. The prize will be that students will be educated to their full potential in these subjects.


The LTSN Maths, Stats & OR Network is a partnership between the Universities of Birmingham and Glasgow, Nottingham Trent University and the Royal Statistical Society Centre for Statistical Education

Lead site for the Network, and focus for Mathematics activities. Contact Pam Bishop or Joe Kyle, School of Mathematics and Statistics, The University of Birmingham, Birmingham B15 2TT tel 0121 414 7095 email p.bishop@bham.ac.uk

THE UNIVERSITY OF BIRMINGHAM

Primary focus for Statistics and OR activities. Contact Neville Davies, RSS Centre for Statistical Education, Nottingham Trent University, Nottingham NG1 4BU tel 0115 848 4471 email neville.davies@ntu.ac.uk

The Nottingham Trent University

Information technology focus for Statistics and OR activities. Contact Adrian Bowman, Department of Statistics, University of Glasgow, Glasgow G12 8QW tel 0141 330 4046 email adrian@stats.gla.ac.uk

UNIVERSITY of GLASGOW

ltsn Maths, Stats & OR

"..connecting the Maths, Stats & OR community in higher education.."

Maths, Stats & OR Network

Director
Professor John Blake
School of Mathematics and Statistics
The University of Birmingham

The Web site is under construction, but there is an information page at http://www.bham.ac.uk/msor

Please give us your feedback via our online form

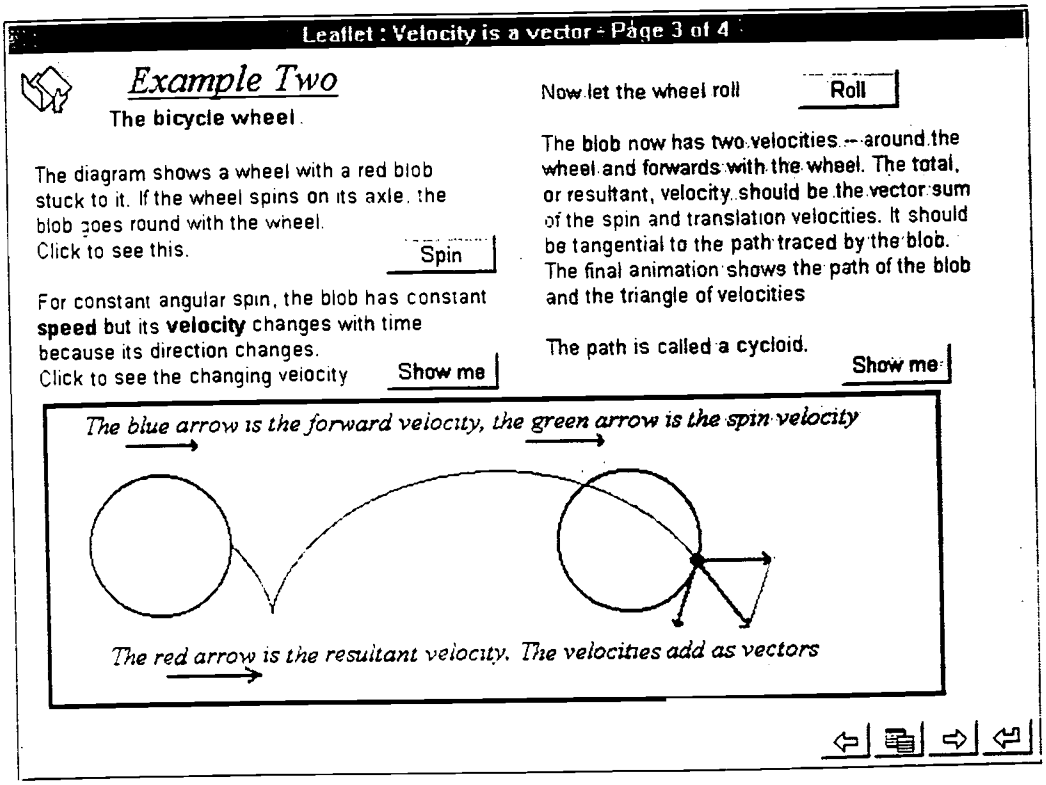
Introduction and history

The Maths, Stats & OR Network was established in January 2000 as part of the Learning and Teaching Support Network (LTSN), an initiative of the Higher Education Funding Councils in the UK. It is able to build on the resources of the earlier Computers in Teaching Initiative (CTI) centres in Mathematics and Statistics (1989-1999), and the Royal Statistical Society Centre for Statistical Education.

The CTI Centres focussed on ways that communications and information technology (C&IT) can enhance the teaching and learning of Mathematics and

Statistics. At their inception there was very little material available, and the compilation of a simple inventory was a valuable resource. Since then, in addition to the increased power of standard packages for carrying out symbolic manipulation or statistical analysis, the last decade has seen the arrival of some sophisticated systems for teaching ideas. Some of these have been commercial enterprises, including multimedia, flexible navigation and support systems. Others have been more freely available and targeted at specific areas such as service mathematics teaching or support for the use of a commercial statistical package. The Teaching and Learning Technology Programme (TLTP) was a particular stimulus to these developments in the early 1990s. Some innovative software was created and is still in use.

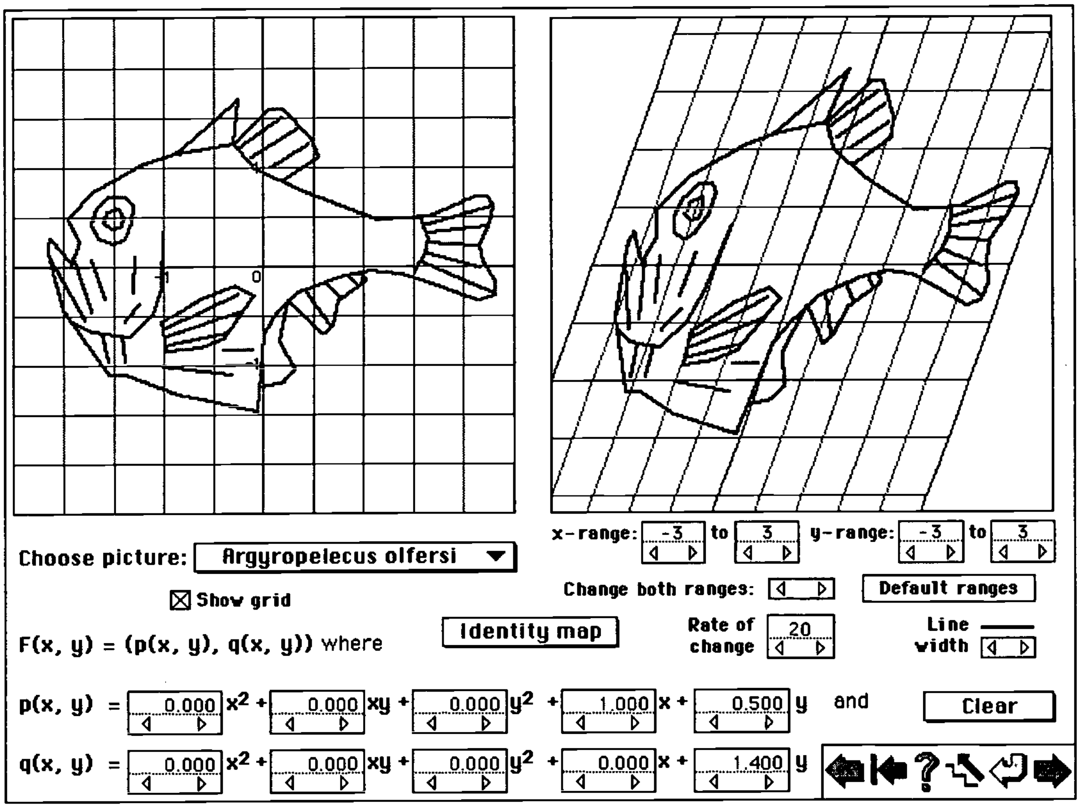
For example, Mathwise was developed by a consortium of over 30 UK universities as an integrated learning environment for teaching undergraduate mathematics, with modules based on the SEFI (European Society for Engineering Education) syllabus, comprising mathematical topics taught in pre-university and first year university, together with a number of key topics in second-year university Science and Engineering courses. The following illustration was clipped from an animation in the module on Basic Vector Algebra:



Some Mathwise modules have been published commercially, and a community of Mathwise users has been established. This group provides mutual support in the embedding of Mathwise into courses, is collating written and other support materials and making them available on a protected Web site, publishes a newsletter, runs workshops and staff development days and provides academic feedback to the publishers. An evaluation of the Mathwise project and the User Group can be found in Haywood et al (1999).

A different approach was taken by O'Connor and Robertson, who developed the system known as Mathematical MacTutor as a laboratory allowing students to conduct mathematical experiments. They use it to supplement conventional modes of teaching at the University of St Andrews in Scotland:

> We have concentrated on areas where we think that the computer, and particularly the superb graphics capabilities of the Apple Macintosh, can give insights not available in other ways. Thus, apart from the Calculus topics that one would expect to find in any mathematical software, MacTutor is particularly strong in Geometry, Algebra (and in particular, Group Theory), Graph Theory, Number Theory and the History of Mathematics. It has some interesting stacks on Statistics, Matrices and Complex Analysis.



The Statistical Education through Problem Solving (STEPS) project was based in the UK and a team of academics from seven universities developed computer based learning material using the software authoring product *ToolBook*. The STEPS software comprised problem-based material in computer modules that can be used as support material in a variety of courses. The aims of each problem context are self-contained and have a realistic scenario with easy-to-understand discipline matter. Most of them use introductory statistical concepts and tools as required that are relevant to the problem. The emphasis is on using statistics that can provide tools to aid problem-solving, rather than an approach that is generated by a curriculum. The material was reviewed by MacGillivray (1995).

Over the past eight years we have been involved in studies to decide the merits, or otherwise, of a technological approach to learning and teaching mathematical subjects. See, for example, Davies and Antcliffe (1996) and Bowman, Constable, Davies, Gilmore, Gilmore and Redfern (1998) where experience with learning and teaching statistics using computer based methods are reported. These, and indeed most other, studies show that there is no difference in performance in tests of different groups of students who do and do not use technology as part of their learning experience. Also the key points to emerge from results of surveys of students attitudes to using technology to learn are: Overall the feedback is very positive as students find that that carefully constructed computer-based learning modules are easy to understand and they enjoy using a computer to carry out learning tasks. Also, working at their own pace is attractive to them, and they invariably report that they feel their knowledge of the subject material improves with the use of a computer. Indeed it is rare for students to feel that that their understanding had worsened.

Many people are researching the use of technology in learning and teaching. The US web site http://www.learner.org/edtech/rscheval/ reports the results of research into the use of technology in education. As an example, the *Flashlight Project* develops survey items, interview plans, cost analysis methods, and other procedures that institutions can use to monitor the success of educational strategies that use technology. There are many useful links from that site.

Strategy of the Maths, Stats & OR Network

The Maths, Stats & OR Network has taken as its mission:

> to promote high standards in the learning and teaching of Maths, Stats and OR by encouraging knowledge exchange, innovation and enterprise, leading to an enhancement of the learning experience for students

and its aims are:

- to foster networks of academics in Maths, Stats & OR
- to disseminate innovation and good practice in learning, teaching and assessment
- to create a forum for the exchange of information, ideas, philosophies and research findings
- to support and enhance good academic practice in teaching Maths, Stats & OR
- to exploit and harness change associated with new technology, integrating this into pedagogic developments
- to run an efficient and effective Maths, Stats & OR Network as part of the wider Learning and Teaching Support Network

There are many activities we feel could be useful and productive for the Maths, Stats & OR community over the next five years. We have taken as our slogan 'connecting the Maths, Stats & OR community in higher education'. Existing practitioners have valuable skills, have tried out different teaching methods, have introduced flexible learning and assessment ideas and have found out what works and what does not. Collecting information about people and their skills, including best practice and reports of successes and failures, and making this information available to other practitioners will be the basis of our work. Activities enabling us to gather and disseminate this information will include local awareness raising workshops, where people can meet and exchange ideas, try out one another's materials and evaluate developments. This will enable us to identify champions of good practice in teaching in Maths, Stats & OR specialist departments and in service courses.

It will also be important to create information banks about existing and emerging resources for learning, teaching materials and effective instruments for assessment, as well as case studies of good practice using these resources. One of our first activities will be to take stock of the state of learning resources in the UK as reported by the Quality Assurance Agency (http://www.qaa.ac.uk) after its review of the subject areas of Mathematics, Statistics and Operational Research (MSOR) form 1998 - 2000, and to identify those departments with recognised good practice in Curriculum Design, Content and Organisation, Teaching Learning and Assessment, Student Progression and Achievement and Learning Resources.

During that period teams of reviewers visited more than 70 departments. At a general level good liaison between MSOR subject staff and library or IT staff enabled the determination of those resources required, including their effective acquisition.

There were many examples of the appropriate use of IT and appropriate software integrated into classroom activities, which led to the enhancement of the students' learning experience. Good practice included opportunities for students to give oral presentations, to acquire a range of IT skills, to work co-operatively with peers and to develop as independent learners, the latter often through project work.

Good practice in the use of technology was noted for a number of providers who make information such as lecture notes, past examination papers and administrative matters available on the Web, and there were instances of the use of interactive worksheets. Several providers included the development of computer based learning activities as part of their learning resource strategy.

It is clear that in UK higher education institutions the *integration* of technology into the curriculum, rather than employing bolt-on activities in the area, produce the most effective results. We intend to conduct an international survey of research into the use of technology for learning, teaching and assessment issues in our subject areas.

Action research will also be carried out, for example to establish good practice in teaching MSOR to large classes, or criteria by which different approaches to teaching MSOR can be compared for both specialist and non-specialist students. We will compare the learning gain achieved by traditional methods of teaching as compared with a mixture of traditional and C&IT methods, and contrast and compare different levels of the integration of C&IT approaches to teaching MSOR within courses, at both specialist and non-specialist levels. One issue is the effectiveness or otherwise of using "black box" packages in MSOR and how they affect student comprehension of difficult topics and concepts. Experiments will compare different approaches to teaching MSOR, examine the influence of assessment methods on what is taught and learned and look at how the use of a computer affects what needs to be known. We will investigate how and when we should teach the "big ideas" in MSOR and determine what these are for specialists and non-specialists, as well as guidelines on how to enable graduates in MSOR to be good communicators and good team members.

Dissemination of the above information will of course be vital if we are to enhance the student learning experience. We will contribute to and write reports of national findings in relevant areas such as using group teaching, including projects and co-operative learning; experiences with teaching MSOR to engineers; the relationships and synergies between pedagogy, content and technology in MSOR; and the effects of technology on learning and teaching in MSOR. Our own publications, both printed and electronic, will be freely available to the MSOR community, and we will help to disseminate the findings of other groups where these are relevant, such as the recent report from Hawkes and Savage (2000) on the mathematical preparedness of new undergraduates. But perhaps the main thrust of our dissemination activities will be face to face, through workshops and staff development activities, visits to departments and discussions with learning and teaching coordinators in our subject areas.

Using generic technology to enhance learning and teaching activities

Diagnostic testing has been an obvious candidate for computer support. Greenhow (2000) describes his use of objective tests using a commercially available product while pointing out the limitations of this approach and the lack of support for mathematical notation in such standard tools. Chickering and Ehrmann draw attention to the power of the computer with regard to more complex assessment processes:

> As we move toward portfolio evaluation strategies, computers can provide rich storage and easy access to student products and performances. Computers can keep track of early efforts, so instructors and students can see the extent to which later efforts demonstrate gains in knowledge, competence, or other valued outcomes. Performances that are time-consuming and expensive to record and evaluate — such as leadership skills, group process management, or multicultural interactions — can be elicited and stored, not only for ongoing critique but also as a record of growing capacity.

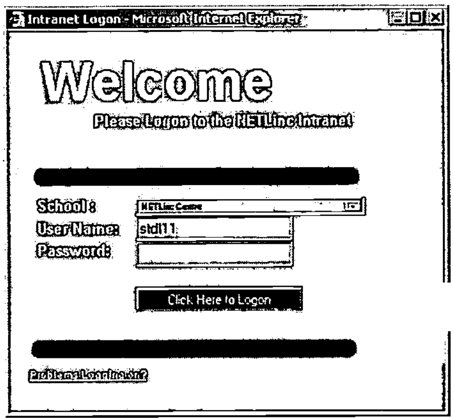
The use of email for submission of assignments or discussion of problems increases opportunities for contact between students and teachers and allows for prompt and flexible feedback. Conversely it also gives an opportunity for reflection not available in a traditional scenario, helping to overcome reticence and encouraging more thoughtful contributions. For example, one result of using electronic instead of paper forms for course evaluation at the University of Birmingham was that students added many more comments in the free text section.

Many departments are now setting up intranet sites where up-to-date course documentation is mounted for access at any time. An extension of this would be to share and exchange teaching materials between departments, such as happened in the Scottish MathPool project described by Maciocia (1997). When such sites are opened up to learners as well as teachers, an "electronic learning community" can emerge.

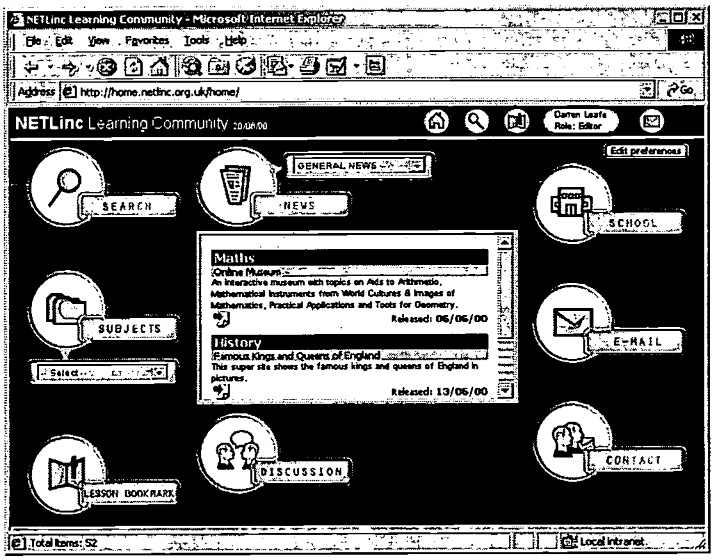
There are a number of projects at school level that are innovative and utilise web-based technology to the full. For example, an electronic learning community called NETLinc has been established for over 300 schools in Lincolnshire, a rural county in England. The provision is for all types of schools catering for pupils aged 6 – 18. Teachers are the main authors of material, but students are allowed to contribute as well. Editorial control is allocated to selected staff at each school and the resource base comprises all material from all schools.

The NETLinc system enables individual users to have their own view of the Intranet. This means that pupils will only see material that they have permission to see or that have been deemed as appropriate for them. When a pupil enters the Intranet, the system picks up their age and presents them with content suitable for their years of study. This means that materials are differentiated. Staff can view all pupil-based materials and additional resources that are classed as 'staff only'.

The following screen snaps show the secure logging on procedure and the opening screen that gives access to searchable resources.



Logon name and password give access to the NETLinc Learning Community.



Staff and Keystage 3, 4, & 5 students (ages 11 – 18) see this type of screen.

The resources are shared across all schools, with invisible demarcation between different schools. It has been very successful to date, mainly because it is very easy to use and authorised staff and pupils can continuously update the resource. All UK national curriculum subjects are catered for. We are negotiating with the authors of the NETLinc system to prototype a version that can be used as an electronic learning community for the Maths, Stats & OR Network at higher education level.

Using subject-specific technology

Powerful commercial packages carrying out computational, symbolic and statistical analysis have already transformed the teaching of MSOR subjects, especially in departments where there is a perceived need to prepare students for future employment. They permit the development of problem-solving skills, provide tools for investigational work and allow students to learn using research-related methods. The challenge is to create learning materials based on the packages that are relevant to students at different stages of their education. There is a great potential here for sharing such resources, and we expect the Maths, Stats & OR Network to play its part in encouraging collaboration in their production and dissemination.

One lesson learned from the TLTP was that the development of networked systems for learning and teaching ideas is extremely costly in resources. This cost can be spread if the products are adaptable and easily customisable for use in circumstances other than those envisaged by the original projects. Products such as Mathwise, MacTutor and STEPS contain valuable ideas that could be extended to other platforms or rewritten for delivery using the Web. Again we see a role for the Maths, Stats & OR Network in encouraging collaboration.

Some of the latest developments in web-assisted mathematics assessment in the UK were presented at a workshop organised by the Network earlier this year, published in its newsletter as a Workshop Report (2000). This included a system with more advanced support for mathematical input, partial credit and differential marking, as described by Jackson *et al.* (2000). One outcome of the workshop has been the joint development of a pilot diagnosis instrument for use in Midland universities at the start of the autumn term, with a database of exam questions/templates in a restricted number of topics, which will be extended to collaboration with a Belgian partner in the innovative use of Maple as an assessment engine.

The advent of MathML will mean that the benefits of Web technology will be fully extended to the MSOR community, since its goal is to enable mathematics to be served, received and processed on the Web, just as HTML has enabled this functionality for text. We hope to update an earlier Workshop Report (1999) following a visit to the MathML conference due to be held shortly before TIME 2000.

Conclusions

The final objective of the UK Maths, Stats & OR Network is, perhaps, the most important learning outcome we plan to achieve, namely that the students we teach will be better educated in all aspects of mathematics, statistics and OR. In trying to achieve this we will bear in mind that the use of communications and information technology for its own sake has been a drawback in many software products that we have seen over the past 10 years. It is our experience that students can be lulled into a false sense of security merely because they interact with the software through clicking, dragging-and-dropping and so forth. This can be what they remember most about the exercise and it can lead to an enjoyable feeling about what they have done.

However, even though students may have a high enjoyment factor when using clever technological wizardry, that does not mean they retain the material studied for an 'acceptable' length of time. Deep understanding of a subject may or may not come from the use of technology, and we feel that this is an area that deserves more carefully planned research.

Finally, it is our belief that, by creating an electronic learning community, with clearly identified champions and examples of good practice forming the building blocks of our Network, teachers of mathematics, statistics and OR will have a resource from which they will be able to give a much better service to our students.

References

Bowman, A., Constable, G., Davies, N., Gilmore, H., Gilmore, S., & Redfern, E. (1998). Computer based learning in statistics: a problem solving approach. *The Statistician*, 47.

Chickering, A. W. & Ehrmann, S. C., *Implementing the Seven Principles: Technology as Lever*, http://www.aahe.org/technology/ehrmann.htm

Davies, N. & Antcliffe, M. (1996). The effectiveness of technology for learning statistics. *Proceedings of the 4th International Conference on Technology in Mathematics Teaching*, Edinburgh: Napier University, 9–23.

Greenhow, M (2000) Setting objective tests in mathematics with QM designer, *Maths, Stats & OR*, 1.

Haywood, J., Anderson, C., Day, K., Land, R., Macleod, H., & Haywood D. (1999). *Use of TLTP materials in UK higher education.* Higher Education Funding Council for England.

Hawkes, T. O. & Savage, M. D. (2000). *Measuring the Mathematics Problem*, The Engineering Council.

Jackson, D. H., Fiddes, D. J. & Beevers, C. E. (2000). Assessment over the Web using CUE. *Maths, Stats & OR*, 3.

MacGillivray, H (1995). The STEPS material: Experiences from an Australian evaluator. *Maths&Stats*, 5.

Maciocia, A. (1997). MathPool, a web based resource system for university mathematics. *Maths&Stats*, *8*, 2.

O'Connor, J. & Robertson, E. F. *Mathematical MacTutor*, http://www-groups.dcs.st-and.ac.uk/~edmund/mactutor.html

Workshop Report (1999). Putting mathematical notation on the web. *Maths&Stats*, *10*, 4.

Workshop Report (2000). Computer-aided assessment in mathematics. *Maths, Stats & OR*, 3.

TECHNOLOGY AND THE CURRICULUM: THE CASE OF THE GRAPHICS CALCULATOR

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This paper outlines some of the complex relationships between the curriculum and technology, with particular attention on graphics calculators. Graphics calculators are of key significance for practical reasons of availability and affordability Until technology is available, curricula are not likely to be adjusted to accommodate them. Particular features of graphics calculators, including their recent extensions to algebraic calculators, are highlighted, and their significance for the curriculum evaluated. Curriculum change elements are interpreted in the light of calculators: the political process, materials development, assessment structures and professional development.

Introduction

The past decade has seen a number of technology conferences for mathematics teachers, mathematicians and mathematics educators in a number of countries. At the recent ICME-9 in Japan, the Working Group for Action concerned with technology attracted more participants than any other single group. Conferences of teachers generally have a large proportion of sessions concerned with the use of tèchnology of one kind or another in mathematics education. It is clear that issues of technology are prominent in the minds of many of us,. not only those of us who choose to attend a conference such as this one for which technology is clearly a major focus. This paper offers a brief outline of some of the relationships between technology and the mathematics curriculum, coloured somewhat by my experiences in Australia and the USA, and mindful that these relationships, like technology itself, are somewhat fluid. Partly because of my experiences, partly because of my interests, but mostly because of its central importance at this time, the focus is on the particular case of the graphics calculator, rather than other technologies. This is not to suggest, however, that similar kinds of analyses are not appropriate elsewhere.

What technology?

As this conference amply demonstrates, there are many kinds of technology considered relevant to school mathematics these days. These range from very powerful computer systems, such as *Mathematica* and *Maple* to much less powerful, but much more pervasive, technologies such as those involving paper and pencil. Although it is worth remembering that some traditional mathematical practices such as the deployment of the long division algorithm or the use of a table of critical values for a particular statistic are examples of technologies, the focus of this paper will rest upon *electronic* technologies in general and the graphics calculator in particular.

The developments of new technologies relevant to school mathematics over the past decade have been quite astonishing. In more prosperous countries, the microcomputer has developed from being an expensive device occasionally found in classrooms and sometimes found in laboratories in school to being a device that is common in many households. Computers have become smaller, easier to use and much less expensive. Some (few) schools and universities have even mandated the use of laptop computers for students. Well-constructed software for educational purposes, such as *Cabri Geometry*, *Geometer's Sketchpad* and *Fathom* have appeared, along with increasingly powerful computer algebra systems, comprehensive statistical packages and powerful spreadsheets such as Microsoft's *Excel*. Communications technologies such as the World Wide Web and electronic mail have moved from scientific teams in universities to the everyday world of many adolescents in their schools and homes. Such developments are exciting, and it is an important function of a conference such as this one to probe their significance for mathematics education.

The choice of graphics calculators is motivated mainly by the potential for them to be available to essentially all students all of the time, rather than to some students all of the time, to all students for some of the time or, worse, to only some students for only some of the time. In this respect, I regard them as an example of what Schumacher referred to as "... *intermediate technology* to signify that it is vastly superior to the primitive technology of bygone ages but at the same time much simpler, cheaper and freer than the super-technology of the rich." (1974, p.128). Many studies, in many (developed) countries over the past decade have reached similar conclusions regarding the penetration of computer technologies into the everyday world of mathematics classrooms. Recent local examples include Thomas (1996) who concluded that New Zealand mathematics teachers had limited access to computers for educational purposes and Norton (1999) who noted that, even when access problems were minimised, many mathematics teachers in Queensland made relatively little use of computers. A further testament to the significance of graphics calculators is the recent national conference of the Australian Association of Mathematics Teachers (Morony & Stephens, 2000), which took place in direct response to a mounting professional view that this form of technology was so critical to the various curriculum deliberations around Australia that nothing short of a national summit would suffice to allow for the free interchange of ideas, experiences, concerns and plans for the future.

Graphics calculators have been commercially available for longer than compact disk players, so it should not be necessary to describe their attributes in fine detail. The key physical features, however, are that they are small, portable, battery-powered and have a multi-line display on which graphic objects can be drawn. Operationally, they are programmable, with a constant memory and have a number of inbuilt mathematical capabilities. In the least sophisticated cases, these capabilities include all those of scientific calculators, together with data analysis and function graphing; in more sophisticated cases, the capabilities also include numerical equation solving, matrix arithmetic, complex number arithmetic, recursion and elementary hypothesis testing. Importantly, graphics calculators from the various manufacturers have now reached their third generation, and so are all relatively easy to use. Arguably, graphics calculators are the first example of an electronic technology specifically designed for education, mathematics education in particular. In the last few years, affordable algebraic calculators have appeared, best regarded as graphics calculators with the embellishment of relatively unsophisticated computer algebra systems (Kissane, 1999), thus further blurring distinctions between 'computers' and 'calculators'.

What curriculum?

Although there are many similarities, which more than justify the value of international meetings such as this one, the mathematics curriculum differs in content and structure between countries. Importantly, however, the curriculum has many forms within a country or, as in Australia, where curricula are state-owned rather than national, within a state.

Cuban (1992) distinguishes at least three levels and types of curriculum. The *intended* curriculum usually takes the form of a written document and has an official status. It describes what students are expected to know, understand and be able to do, and the circumstances under which these things will happen. In so doing, it represents a kind of consensus on what aspects of mathematics are most important. A second level concerns the *implemented* curriculum, which describes what actually happens when teachers attempt to deliver the intended curriculum to real students in real schools. It is frequently strongly affected by the materials such as textbooks and the classroom practices of teachers. Finally, the *attained* curriculum refers to what students actually learn as a result of being in the classroom. The attained curriculum is strongly related to what is tested and refers to the curriculum from the perspective of the learner. Each of these levels is important and deserves to be an object of study in its own right. Many research studies over recent years have suggested that there is a good deal of slippage between the ideals of the intended curriculum, the realities of the implemented curriculum and the attained curriculum that results.

Two strong, and related, school mathematics curriculum trends of the 1980's and 1990's are towards 'mathematics for all' and towards mathematics as a 'useful' subject. Even a casual glance at modern western curriculum materials such as student textbooks will reveal an apparent concern to highlight the applicability of mathematics to situations outside mathematics and clear attempts to broaden the appeal of mathematics to as wide an audience as possible. Both of these have some significance for the role of technology, and particularly for graphics calculators.

Partly because of the bewildering array of other things to choose, it now seems that the overwhelming majority of mathematics students opt for mathematics with a view to using it for some other purpose. In such a circumstance, an emphasis on a very practical view of mathematics, with less concern than many of us would like for aspects of mathematical thinking such as deductive reasoning and proof, is not unexpected; the availability of technologies that handle many of the practical and computational aspects of mathematical work is critically important. While studying mathematics for its own sake has never been a popular activity, it seems even less so nowadays, so that almost all students stop studying mathematics as soon as they are no longer obliged to opt for it. My estimate in the case of Western Australia, for example, is that much less than 1% of an age cohort of secondary students will ever study tertiary mathematics beyond the minimum requirements for their chosen profession. Almost all teaching of mathematics at many Australian universities, including mine, is described as 'service teaching', to meet the needs of graduates in other disciplines. Perhaps paradoxically, however, there are still many school mathematics curricula that seem to be framed on a premise that most students will pursue mathematics beyond the minimum.

Curricula and calculators

Until quite recently, technology was not regarded as a major influence on the mathematics curriculum. Although some official curricula acknowledged the invention of computers, there were few significant attempts to adapt mainstream secondary school mathematics curricula to technology before the 1990's, as Waits & Demana (2000) and several others have noted. In contrast, the recent revision of the NCTM *Standards* (2000) has highlighted technology in a 'technology principle':

> Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances student learning. ... Students can learn more mathematics more deeply with the appropriate use of technology. ... In

> mathematics-instruction programs, technology should be used widely and responsibly, with the goal of enriching students' learning of mathematics. The existence, versatility and power of technology make it possible and necessary to reexamine what mathematics students should learn as well as how best they can learn it. (pp. 24–25)

The *Principles and Standards* suggest that learning can be enhanced by technology in a number of ways: more examples can be accessed by pupils than formerly, more time is available for conceptualising, multiple perspectives are accessible and feedback is provided in novel ways. Teaching is enhanced because teachers can use technology to provide experiences to students that would not otherwise be available, such as mathematical modelling using probability simulation. Both what is taught and when it is taught are affected. For example, optimisation of functions can be treated prior to the formal methods of the calculus and more complex symbolic manipulations can be dealt with using an algebraic calculator than with paper and pencil methods.

An Enabling Technology

In exploring relationships between calculators and curricula, it seems natural to start by asking what can graphics calculator do? Earlier, a brief and incomplete catalogue of typical mathematical capabilities was given. What sorts of things do these *enable* for mathematics education?

In the first place, graphics calculators enable various kinds of guided explorations to be undertaken, which would have been too difficult without technology. As an example, pupils can examine directly whether or not a particular sequence appears to converge, the effects of changing parameters of a function on the shape of its graph, and the significance of outliers on various sample statistics. They can explore the relationships between gradients of pairs of lines and the lines themselves, the consequences of varying interest rates, the effect on a histogram of changing the widths of the intervals and the connections between equations and graphs. Enthusiastic claims have long been made for exploratory activity in mathematics; graphics calculators enable some of this to be realised in classrooms.

Secondly, graphics calculators can handle some–indeed many–of the routine, computationally intensive, aspects of elementary mathematical work. Examples include the solution of quadratic or simultaneous linear equations, testing an hypothesis involving two sample means or evaluating a definite integral. The mathematical concepts underpinning such procedures are rich and important for understanding. Yet all too frequently in the past it seems that pupils without calculators devote a great deal of time to calculation and correspondingly less to making sense of it. A shift in the balance of attention to concepts and skills might be possible and certainly would be desirable.

Thirdly, graphics calculators can offer new opportunities for pupils to encounter mathematical ideas not presently in the curriculum. One example is the use of simulation as a means of tackling some situations involving uncertainty. Another involves the use of iterative techniques to study some of the mathematics of chaos. Another involves dealing directly with real-world data collected electronically and transmitted to a calculator. Still others involve the use of elementary programs to study repetitive processes.

Empirical Evidence

A good deal of work has already been done by researchers and practitioners to study the effects of using graphics calculators in various ways. As might be expected in changing circumstances, some of the early work is located in a previous curriculum and educational context, with the graphics calculator externally imposed. Thus, in their excellent review of research up to that time, Penglase & Arnold (1996) note the methodological concern:

> The current state of research into the use and effects of graphics calculators, then, remains inconclusive. Few studies distinguish carefully between the use of the tool and the context of that use. Claims regarding the relative effectiveness of the tool are frequently based upon assessment procedures which equate "student learning" and "achievement" with performance upon traditional tests, and fail decisively to account for important influences upon attitudes and conceptual understanding. (p. 82)

Ruthven's (1995) analysis also repays careful study. He suggests that we need to look closely at how pupils actually use calculators (including graphics calculators).

> [A]t the level of principle, many important issues surrounding calculator use remain poorly conceptualised: our understanding would benefit from renewed curiosity, and a readiness to build connections with wider issues and other technologies. ... In many settings, calculators offer the most realistic prospects of transforming classroom mathematics within the medium term so as to incorporate considered use of computational technology. (p. 464)

There is not space in this paper for a detailed and critical review of recent research using graphics calculators, although such a review is now needed. However, recent work may redress some of the concerns expressed by Penglase & Arnold, and move in directions that Ruthven would agree are productive, as attention shifts to curriculum settings in which the calculator is better embedded. A good recent example is the study by Doerr & Zangor (2000), who identified a number of different (and mainly productive) ways in which pupils made use of graphics calculators in practice over a considerable time period with a competent teacher. An interesting finding of the study was that the small screens on graphics calculators tended to inhibit collaboration between pupils, while overhead projector versions of calculator screens encouraged powerful forms of classroom discussion and debate.

Empirical work of this kind will be important to accumulate and assess in the near future.

What is Now Important?

Until quite recently, there was no alternative to many of the analytical procedures associated with school mathematics. To find the stationary points of a function, differentiation was unavoidable. To find the solutions to a cubic equation, factorisation was unavoidable. But this is no longer the case. With a graphics calculator available, good enough numerical approximations to stationary points can be found visually, while numerical solutions to the cubic can be obtained quickly. In addition, the analytical solutions of such tasks can be supported by an algebraic calculator. The difficult curriculum decisions now involve deciding under which circumstances these new facilities should be ignored, whether or not they should be banned, and whether (and when) we should help pupils to use them. As Kennedy (1995) has argued so eloquently, a good deal of what is in the secondary school curriculum used to be necessary for any pupils who wished to progress further in mathematics. But it is less clear now that this is the case, and the time has come for a careful re-evaluation of what we have come to take for granted.

It seems inevitable that a reconsideration of what is important and central about mathematics ought to take place as technologies such as calculators become available. It is a reflection of the inertia and the natural conservatism of school curricula that they seem to be so resistant to change of this kind. Written partly in frustration, but mainly in a spirit of improving the mathematics curriculum, Ralston's (1999) paper concerned with the abolition of paper and pencil arithmetic offers a strong case, almost a quarter of a century past the introduction of affordable arithmetic calculators to schools.

The development of algebraic calculators, of no less significance for the secondary school than are arithmetic calculators for the primary school, suggests a need for further analyses of this kind; hopefully, this will not take another quarter of a century (Kissane, 1999). Examples of early analyses of what algebraic manipulations are important to be done by hand (or by head?) and what can be well left to a calculator are provided by Bjork & Brolin (1998) and Herget *et al.* (2000). Similarly, Goldenberg (2000) offers a tentative first list, suggesting that the mathematics education community ought be discussing such things now, if we are to find the necessary common ground from which to move forward. In presenting his list, Goldenberg notes:

> Why would anyone care whether students could solve problems like [those] above without using a symbolic calculator? After all, the calculator will do that stuff for them! I would make the case that one cannot make intelligent use of the technology with some such skills *without* the technology. The reason is that it is hard to know what computations to ask for without understanding what the computations will *do*. (p. 14)

Arcavi (1994) has provided a very nice analysis of 'symbol sense', worthy of careful study in order to help decide which aspects of algebra ought to occupy our attention most urgently. Thinking of this kind, together with empirical evidence of the kind likely to be available from the University of Melbourne project (McCrae *et al.*, 1999; Stacey *et al.*, 2000) will help to reach the sort of agreement Goldenberg seeks. One of our problems appears to be a lack of communication between teachers of mathematics at different levels. The resulting mismatches of expectations and emphases are problematic for pupils and deserve more attention.

Aside from the details of what is better left to machines and what is appropriately done otherwise, it seems clear that pupils need to learn how to use graphics calculators effectively, as part of the mathematics curriculum. Effective use involves much more than knowing about the detailed steps needed to operate a particular calculator, of course. It also includes learning to make good decisions about when to use and when not to use a calculator. It involves learning how to interpret calculator output with the right mixture of skepticism and acceptance. Such matters ought to be a conscious part of the implemented curriculum, as they are much too important to be left to chance. While some have expressed concern that we should be teaching mathematics and not technology, a view I share, it is still important that students learn enough about the technology to use it well. This is no different in principle from the need to teach pupils how to use tables of square roots, calculations via logarithms or deft use of a slide rule in days gone past. What has changed is that the technology has become more complex, pervasive and powerful, thus requiring more effort to master. Of course, only the most optimistic would expect that pupils will develop the critical discretionary expertise in situations in which calculators were often not available to them, or in which their use was controlled entirely by decisions of others, such as a teacher, a textbook or an examination paper.

Curriculum change

Adjustments to an official curriculum seem unlikely to happen until technology is (or can be) pervasive. A major reason for the significance of the graphics calculator is that it offers the best prospect for this to happen at the moment, certainly somewhat greater than is the case for the computer, the lap top or the Internet. Statutory authorities responsible for curriculum decisions are understandably likely to be very cautious at taking curriculum revision seriously in the light of technologies to which only some students have good access.

The Change Process

It is worth noting that curriculum change does not usually happen as a direct result of empirical research studies, but rather through a process involving a number of influences (among which, ideally and critically, research conclusions are one). Indeed, it is rare that empirical research provides an unequivocal, unambiguous direction for curriculum developers. Instead, the most critical functions of research are frequently to provide new perspectives for thinking about the curriculum and new questions to address. It is vital that good research be sponsored, conducted and reported, but naïve to expect that research results will be sufficient evidence upon which curriculum change will be built.

Statutory authorities legally responsible for making decisions about the official curriculum generally seek advice from a range of stakeholders, appropriately. At the senior secondary school level, this range is likely to include a mixture of tertiary mathematicians, secondary school mathematics teachers, representatives of key school systems and others (such as representatives of industry, professional societies and even parents). Perhaps curiously, and probably unfortunately, it is relatively rare for mathematics educators, researchers or curriculum developers to have much voice in official decision-making processes. By definition, such a grouping of people will make political decisions, based on the real and perceived power structures among them. Until recently, in many parts of the world, university mathematicians have traditionally been the main voices in curriculum decisions about secondary school mathematics, probably because a major goal of senior secondary mathematics is to prepare pupils for further study of mathematics and because university mathematicians are regarded as more qualified to identify what is most important about mathematics at present. When graphics calculators are considered, this presents a particular problem, as Tucker (1999) has noted:

> [M]ost college or university mathematicians have spent no time at all with a graphing calculator and are not inclined to spend the start-up time of an hour or two to learn, especially since they are unlikely to use graphing calculators on a regular basis outside the classroom. A bridge is needed for this gap between mathematics students (and secondary school teachers) on the one side and college faculty on the other. (p. 910)

As is well known, relationships among those with curriculum responsibilities have not always been harmonious, and seem to have been especially problematic in recent times in some quarters. The case of California and the so-called 'math wars' have received prominence, indeed notoriety, but there are tensions of a less public kind also to be found elsewhere, including Australia, when issues related to the place of technology in the curriculum are debated. Indeed, as noted earlier, we need to have more debates, even at the risk of exposing our differences.

Change processes of this kind are challenged with the very significant problem of the rate of change of technology. Put simply, the rate of change of graphics calculator technology realistically available to secondary school pupils is a good deal greater than the rate of curriculum change possible. While enthusiasts are understandably irritated by slow responses to exciting new opportunities, conservatives (or merely the less-enthused) are understandably anxious about changing too much too quickly, so that decision-making bodies advised by both kinds of people, among others, are unlikely to make changes that keep abreast of changing technologies. Some, such as Podlesni (1999) have suggested that calculator R & D teams may well have an undue influence on curriculum development in such circumstances:

> Are we getting to the point where technology companies are making de facto curriculum decisions for us? Are they paving the way, consciously or unconsciously, for their future leadership in that process by making calculators upgradeable–through their software, one presumes? ... Are we doing our job as teachers or relinquishing part of it to the electronics industry? Are we becoming unpaid salespeople for that industry with every new model? (p. 89)

Technology companies are necessarily driven mainly by the commercial market place rather than sound educational thinking and planning, even when they seek and obtain advice from people in the field. While such influence is mainly problematic, it can also serve to force us to confront big issues earlier than we would otherwise be likely to do. The best example of this is the development of algebraic calculators in recent times. Although computer algebra systems have been around for a good while now, it is their availability on hand-held devices, produced by commercial companies, that has demanded that our profession consider the complex issues involved with some urgency.

Curriculum Materials

As noted above, the curriculum materials provided by or to the teacher are a significant influence on the implemented curriculum. Ideally, a calculator-sensitive curriculum would be supported by appropriate materials. In fact, this is much more difficult to do than is commonly recognised, for a number of reasons.

In the first place, it is not an easy matter to produce good quality materials that integrate technology into the fabric of a course. Curriculum development projects such as the University of Chicago School Mathematics Project in the USA and Nuffield Advanced Mathematics in the UK have produced good exemplary materials, using the expertise of large teams of people. Such activity is very important to provide examples of what is possible, opinions about what is desirable and, hopefully, evidence about what are the consequences of using such materials in practice. However, many curriculum materials are produced with less resources and less access to specialist help and thinking. In Australia in recent years, for example, textbooks for secondary mathematics have begun to accommodate graphics calculators, but usually in the form of 'revised' versions of previous textbooks. In some cases, the revisions seem more concerned with appearance than with substance, as publishing companies are understandably anxious to be recognised as providing 'current' materials. Adding the occasional calculator screen dump or calculator activity does not constitute a sound form of revision. There seems to be no particular reason for expecting authors of textbooks written without consideration of technology, graphics calculators in particular, to be particularly capable of the necessary re-thinking. These circumstances are exaggerated by commercial imperatives to appear in the marketplace before the competition does.

At least in parts of Australia, there is a second powerful force militating against the production of quality materials. Many schools invest significant sums in purchasing instructional materials such as textbooks (and sometimes these days associated support materials), which are then loaned or hired to students. There are significant economic reasons for not changing text materials too often in such circumstances: school budgets will simply not permit this. As well as physical capital, many mathematics teachers have substantial intellectual and professional capital invested in the curriculum materials adopted by their school, so that too much is at stake to expect change to happen regularly. A consequence of tolerating very limited change over time is that at some point a more dramatic, and much more difficult, change is required to adapt curriculum to the resulting build up of new technological circumstances.

Anyone attempting to develop curriculum materials involving calculators will quickly confront the difficult question of how to deal with the fact that there are several different models and brands of calculators available. At one level, it seems unwise to embed key stroke commands into a text, unless there is some assurance that almost all readers, pupils and their teachers, are using the same calculator, which is of course increasingly unlikely. At a deeper level, there are differences in the functionality of calculators which are not always easy to accommodate. For example, to deal with solving a system of simultaneous linear equations, some calculator models require that coefficients be entered as a matrix, and that matrix operations be then performed. Others allow a matrix to be entered, but provide a solution immediately, if there is one, or an error if there isn't. Still others provide solutions using an RREF (row-reduced echelon form) command, which then requires an interpretation of the result. It is quite difficult to produce text material that sensibly accommodates to all of these at once. A partial solution is to provide supplements or appendices that tailor to particular graphics calculator models. Although this seems a bit clumsy, it may be the best available response in the short term. Developing curriculum materials is certainly not helped by regular changes to calculator capabilities, fuelled mainly by competition between manufacturers. Such changes provide further weight to the need to develop materials that are as device-independent as possible.

Curriculum materials do not always consist of pupil texts, of course. Another species consists of supplementary materials such as photo-reproducible masters for calculator activities or how-to books that provide detailed support for using a particular calculator, or suite of calculators, in an educational context. (Calculator manuals are rarely useful to beginners, unfortunately.) The latter may provide a temporary solution at an individual school with some control over which calculators are available to pupils, and may help teachers understand the possibilities and pitfalls of calculator use. However, the former run a grave risk of cementing in place a view of technology as an add-on, to be included at the behest of the teacher and the worksheet, rather than an integral part of the curriculum. By their nature, there seems to be a tendency, by no means universal, for supplemental materials of these kinds to focus on detailed key strokes, with the attendant risks that students might not be engaged in the desired kinds of mathematical thinking.

Assessment

As noted above, assessment is a key to understanding the attained curriculum. It is also a significant aspect of the implemented curriculum and of course an integral element of the official curriculum. This is especially so in the case of technology in general and graphics calculators in particular, since assessment structures that do not adequately incorporate technology will easily undermine any attempts to include technology in the curriculum. The evidence of the link between assessment regulations and calculator use in classrooms is overwhelming. In Australia, for example, graphics calculators are widely available and used by pupils in states which permit or mandate their use in formal assessment, and rarely used in those states which do not.

While a coherence between the everyday use of calculators and their availability in high-stakes assessment is clearly desirable, careful thought is required to bring this about effectively. Kissane, Kemp & Bradley (1996) outline and exemplify the range of possibilities in some detail. Essentially, assessment tasks need to be carefully designed to accommodate the use of technology, care needs to be taken to ensure that students are encouraged to learn mathematics, not just button-pushing, and equity issues should be borne in mind and minimised.

Recent changes to graphics calculators have thrust new challenges to assessment to the forefront again. Three of these are discussed in some detail by Kissane (2000): the availability of symbolic manipulation capabilities on algebraic calculators, the use of flash memory (which allows calculators to be upgraded and new capabilities added) and the expansion of available storage memory to a point at which very significant amounts of text information might be stored. One of the reasons that computers had limited impact on school mathematics curriculum is arguably that they have rarely been accommodated into assessment. Recent developments, reducing the now arbitrary (perhaps even merely linguistic?) gap between computers and calculators, are particularly interesting from this perspective. Research such as the University of Melbourne project (McCrae *et al*, 1999; Stacey *et al*, 2000) is especially important in this area.

Professional Development

Teachers are a crucial part of any process of curriculum change, and need to be supported in various ways. Through the lens of the implemented curriculum, the opinions and practices, beliefs and competencies of the teacher are a critical part of the curriculum. Curriculum developments that do not adequately provide for the legitimate needs of teachers do so at their peril.

It is vitally important to remember also the everyday working conditions of the great majority of mathematics teachers. In all countries, most teachers feel very considerable pressures of various kinds from their pupils, their school, administration, parents and the world at large. They frequently feel obliged to work within a curriculum context over which they have had little control, teaching more pupils than they would like, many of whom they feel should not be there at all. Many of their adolescent pupils have much more passionate interests than the mathematics classroom can satisfy, and indeed, derive more satisfaction from the social world of the school than its intellectual world. Increasingly, mathematics teachers are less well-qualified in mathematics itself. The pressures of external examinations, lesson preparation, motivating pupils, dealing with unruly pupil behaviour, marking and other responsibilities within the school all exact a toll on the proportion of the thinking and working time teachers have to adapt to new technologies such as graphics calculators. It is quite simply unreasonable to expect many teachers in these kinds of circumstances to shoulder the responsibility of adjusting the curriculum to the influence of graphics calculators without a good deal of help.

Morony & Stephens (2000, pp. 16–17) identify a number of dimensions of support in summarising the AAMT conference. These include the development of strategies to support practitioners who have reached different stages of comfort with graphics calculators, identified as 'novices', 'practitioners' and 'creators' in increasing order of sophistication. Good relationships among teachers both within and between schools need to be fostered and high quality resources for integrating calculators, going beyond the genre of worksheets and blackline masters need to be developed and disseminated.

Conclusions

In the past, many curriculum innovations have withered, and it is a common practice to talk of fads in education and to speak of pendulum swings of attention. It is not appropriate to approach technology in the mathematics curriculum from this perspective. Dangerous as it is to make predictions in the changing circumstances of technology, I suggest that we are not here dealing with another fad. Tucker (1999) expressed the new reality well:

> It pays to heed history. Technology always wins. The world may have been better when people walked instead of driving cars, but that is irrelevant. As long as there is gas, people will drive cars, and what I really care about is that they drive them sensibly. The mathematical world may have been better when people did arithmetic or graphed functions on paper or in their head instead of on a calculator, but that is irrelevant. As long as there are batteries, students will use calculators, and what I really care about is that they use them sensibly. ... pretending something doesn't exist is not a good teaching strategy. For many of my students, graphing calculators are as much a part of their intellectual constitution as pencil and paper, and I have to learn to deal with it. (p.910)

As many, such as Penglase & Arnold (1996), have suggested, the significance of the graphics calculator derives in the first place from its affordability and thus its accessibility. The world external to education has brought this situation about, and, in the near term, can be expected to continue it. We need to respond to that reality as soon and as well as we can, guided by sound research and good practice.

References

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics, *For the Learning of Mathematics*, 14 (3), 24–35.

Björk, L.-E. & Brolin, H. (1998). Which traditional algebra and calculaus skills are still important? In Z. Usiskin (Ed.) *Developments in school mathematics education around the world* (Vol. 4) Proceedings of the Fourth UCSMP International Conference on Mathematics Education, Chicago. (pp. 287–295) Reston, VA: National Council of Teachers of Mathematics.

Cuban, L. (1992). Curriculum stability and change. In P. Jackson (Ed.) *Handbook of research on curriculum*. (pp. 216–247). New York: Macmillan.

Doerr, H.M. & Zangor, R. (2000). Creating meaning for and with the graphing calculator. *Educational Studies in Mathematics*, 41(2), 143–163.

Goldenberg, P. (2000). *Finding some "common notions" about technology in mathematics learning.* Unpublished paper presented at the 9th International Congress on Mathematics Education, Makuhari, Japan.

Herget, W., Heugl, H., Kutzler, B. & Lehmann, E. (2000). *Indispensable manual calculation skills in a CAS environment*. [http://www.kutzler.com/article/art_indi/indisp.htm] 21 August.

Kennedy, D. (1995). Climbing around on the tree of mathematics. *The Mathematics Teacher*, 88(6), 460–465. [http://partners.mediacentre.com/grade9math/articles/tree.pdf]

Kissane, B. (2000). New calculator technologies and examinations. *Proceedings of 5th Asian Technology Conference on Mathematics*, Chiang Mai, Thailand: Asian Technology Conference in Mathematics. (in press)

Kissane, B. (1999). The algebraic calculator and mathematics education. In W-C Yang, D. Wang, S-C. Chu, & G. Fitz-Gerald (Eds.) *Proceedings of 4th Asian Technology Conference on Mathematics*, (pp 123–132) Guangzhou, China, Asian Technology Conference in Mathematics. [http://wwwstaff.murdoch.edu.au/~kissane/papers/ATCM99.pdf]

Kissane, B., Kemp, M. & Bradley, J. (1996). Graphics calculators and assessment. In P. Gomez & B. Waits (Eds.) *Roles of calculators in the classroom*, (Electronic proceedings of Topic Group 18), 8th International Congress on Mathematics Education, Seville, Spain. [http://ued.uniandes.edu.co/servidor/em/recinf/tg18/ArchivosPDF/Kissane.pdf]

Leinwand, S. (1994). Four teacher-friendly postulates for thriving in a sea of change. *The Mathematics Teacher*, 87(6), 393–5.

McCrae, B., Asp, G. & Kendal, M. (1999). Learning calculus with supercalculators. In J.M. Truran & K.M. Truran (Eds.) *Making the difference.* Proceedings of the 22nd annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, (pp. 359–364), Adelaide, SA: MERGA.

Morony, W. & Stephens, M. (2000). *Students, mathematics and graphics calculators into the new millenniunm.* Adelaide: Australian Association of Mathematics Teachers.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Norton, S. (1999). Secondary mathematics teachers' responses to computers and their beliefs about the role of computers in their teaching. In J.M. Truran & K.M. Truran (Eds.) *Making the difference.* Proceedings of the 22nd annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, (pp. 404–410), Adelaide, SA: MERGA.

Penglase, M. & Arnold, S. (1996). The graphics calculator in mathematics education: A critical review of recent research. *Mathematics Education Research Journal*, 8 (1), 58–90.

Podlesni, J. (1999). A new breed of calculators: Do they change the way we teach?, *Mathematics Teacher*, 92 (2), 88–89.

Ralston, A. (1999). Let's abolish pencil-and-paper arithmetic. *Journal of Computers in Mathematics and Science Teaching*, 18 (2), 173–94.

Ruthven, K. (1995). Calculators in the mathematics curriculum: The scope of personal computational technology. In A.J. Bishop, K. Clements, C. Kietel, J. Kilpatrick & C. Laborde (Eds.) *International handbook of mathematics education*, Volume 1, (pp 435–468), Dordrecht: Kluwer.

Schumacher, E. F. (1974). *Small is beautiful: A study of economics as if people mattered.* London, Sphere Books.

Stacey, K., McCrae, B., Chick, H., Asp, G. & Leigh-Lancaster, D. (2000). Research-led policy change for technologically-active senior mathematics assessment. In Bana, J. & Chapman, A. (Eds.) *Mathematics education beyond 2000.* Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia (pp 572–579) Perth: MERGA.

Thomas, M. O. J. (1996). Computers in the mathematics classroom: A survey. In P. C. Clarkson (Ed.) *Technology in Mathematics Education.* Proceedings of the 19th annual conference of the Mathematics Education Research Group of Australasia (pp. 556–563) Melbourne: MERGA.

Tucker, T.W. (2000). Reform, tradition and synthesis. *American Mathematical Monthly*, 106 (10), 910–929.

Waits, B. K. & Demana, F. (2000). Calculators in mathematics teaching and learning: Past, present and future. In M. J. Burke & F. R. Curcio (Eds.) *Learning mathematics for a new century: 2000 Yearbook* (pp 51–66) Reston, VA: National Council of Teachers of Mathematics.

WHAT MATHEMATICAL ABILITIES ARE MOST NEEDED FOR SUCCESS BEYOND SCHOOL IN A TECHNOLOGY BASED *AGE OF INFORMATION*?[1]

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In this paper, a central claim will be that one of the most important influences that technology should have on mathematics education is that many of the most important goals of mathematics instruction should consist of helping students develop powerful, sharable, and re-usable conceptual technologies for constructing (and making sense) of complex systems.[2] A second claim will be that these new conceptual tools don't involve introducing completely new topics into the mathematics curriculum as much as they involve dealing with old topics in new ways that emphasize mathematics-as-communication (description, explanation) more than mathematics-as-rules-for-symbol-manipulation. A third claim will be that, even though technology-based tools create the need to teach these new levels and types of understandings and abilities, in many cases, technology-based tools are not needed to teach them effectively. So, it is not necessary to have lots of classrooms full of educational technologies in order to provide learning experiences for students that are wise to the needs of a technology-base society.

How has technology influenced what's needed for success beyond schools in a technology-based *Age of Information*?

When people speak about appropriate roles for calculators, computers, and other technology-based tools in instruction, their comments often seem to be based on the implicit assumptions that: (i) the world outside of schools has remained unchanged – at least since the industrial revolution, and (ii) the main thing that new technologies do is provide "crutches" that allow students to avoid work (such as mental computation or pencil-&-paper computations) that they should be able to do without these artificial supports. But, technology-based tools do a great deal more than provide new ways to do old tasks. For example, they also create new kinds of problems solving situations in which mathematics is useful; and, they radically expand the kinds of mathematical understandings and abilities that contribute to success in these situations. In fact, one of the most essential characteristics of a technology-based *age of information* is that the constructs (and conceptual tools) that humans develop to make sense of their

[1] References for most of my comments come from two recent books that I co-edited: *Handbook of Research Design in Mathematics & Science Education* (Kelly & Lesh 2000) and *Beyond Constructivism: A Models & Modeling Perspective on Mathematics Problem Solving, Learning & Teaching* (Doerr & Lesh, in press). Each book contains chapters by more than thirty leading math/science educators. So, even though the views expressed here are my own, they were informed by a great many others.

[2] Here, I am using the term "complex system" is a somewhat more general sense than it is used in the newly emerging field of "complexity theory" in mathematics. Nonetheless, later in this paper, readers will see that the two uses of the term are closely related. Terms with fewer technical associations could have been used. For example, instead of calling these systems "complex", I could have called them "structurally interesting " or "mathematically significant". But, regardless what terminology is used, it should be understood that a system that is "complex" (or "structurally interesting" or "mathematically significant") is different for a child than for an adult.

experiences also mold and shape the world in which these experiences occur. Consequently, many of the most important mathematical "objects" that impact the everyday lives of ordinary people are complex, dynamic, interacting systems that are products of human constructions - and that range in size from large-scale communication and economic systems, to small-scale systems for scheduling, organizing, and accounting in everyday activities. Therefore, people who are able to create (and make sense of) these complex systems tend to enjoy many opportunities; whereas, those who don't risk being victimized by credit card plans or other systems created by humans.

To see evidence of the kind of changes that are being introduced into our lives by advanced technologies, look at a daily newspaper such as *USA Today*. In topic areas ranging from editorials, to sports, to business, to entertainment, to advertisements, to weather, the articles in these newspapers often look more like computer displays than like traditional pages of printed prose. They are filled with tables, graphs, formulas, and charts that are intended to describe, explain, or predict patterns or regularities associated with complex and dynamically changing systems; and, the kinds of quantities that they refer to go far beyond simple counts and measures to also involve sophisticated uses of mathematical "objects" ranging from rates, to ratios, to percentages, to proportions, to continuously changing quantities, to accumulating quantities, to vector valued quantities, to lists, to sequences, to arrays, or to coordinates. Furthermore, the graphic and dynamic displays of iteratively interacting functional relationships often cannot be described adequately using simple algebraic, statistical, or logical formulas.[3]

For a simple example to illustrate some of the impacts of the preceding new-uses-of-old-ideas on the everyday lives of ordinary people, consider the section of a local newspaper that gives advertisements for automobiles. Then, think about how these advertisements looked twenty years ago. They've changed dramatically! Today, it's often difficult to determine the actual price of cars that are shown. What's given instead of simple prices are mind boggling varieties of loans, leases, and buy-back plans that may include many options about down payments, monthly payments, and billing periods. Why have these changes occurred? One simple answer is: *graphing spreadsheets* (like the one shown in Figure 1).

Spreadsheets with graphs, like the one shown in Figure 1, provide dynamic and easily manipulable conceptual tools for describing and exploring relationships among time, interest rates, monthly payments, and the amount of money remaining to be paid (or that has been paid) at any given time. Therefore, such tools make it easy for car dealers to develop sophisticated buying, leasing and loan plans – based on a few "new ideas" dealing with iteraction, recursion, trends, and matrix-based organizations of information – but mostly based on new ways of using old basic ideas from elementary mathematics. Yet, these new ways of using old ideas emphasize mathematical under-

[3] It is now well known that several iteration of simple algebraic function can lead to a system that is essentially chaotic – with many characteristics that are unpredictable, with emergent characteristics that are not simply derived from characteristics of the interacting elements of the system, and often with feedback loops in which second-order effects often overwhelm the impact of first-order effects. Yet, new fields of mathematics, such as "complexity theory" or "discrete mathematics", don't call for new "foundation-level ideas and abilities" as much as they require much less narrow and shallow treatments of old topics in "elementary mathematics".

standings and abilities that are quite different than those that have been emphasized in traditional schooling. For example, instead of operating on pieces of information, operations often are carried out on whole lists of data. Instead of simple one-directional "input-output" rules, the kind of functions that are involved often are iterative and recursive (sometimes involving sophisticated feedback loops); and, the results that are produced often involve multi-media displays that include a variety of written, spoken, constructed, or drawn media.

Year	Amount Owed	Interest Rate	Payment/Month	Total Paid
0	$10,000	7.0%	$300	$0
1	$7,100	7.0%	$300	$3,600
2	$3,997	7.0%	$300	$7,200
3	$677	7.0%	$300	$10,800
4	($2,876)	7.0%	$300	$14,400
	($6,677)	7.0%	$300	$18,000

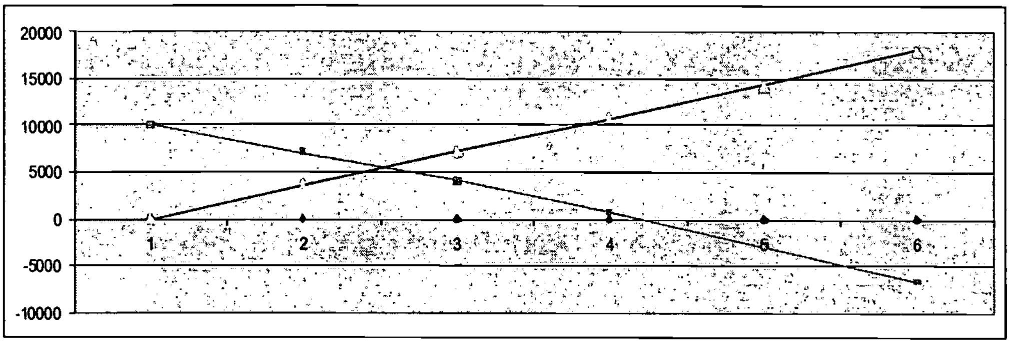

Figure 1. A Spreadsheet for Determining Interest Payments for Car Loans

Therefore, representational fluency is at the heart of what it means to "understand" many of the most important underlying mathematical constructs; and, some of the most important mathematical abilities that are needed emphasize: (i) mathematizing (quantifying, dimensionalizing, coordinatizing, organizing) information in forms so that "canned" routines and tools can be used, (ii) interpreting results that are produced by "canned" tools, and (iii) analyzing the assumptions that alternative tools presuppose – so that wise decisions will be made about which tools to use in different circumstances.

Figure 2 emphasizes another point about the kind of problem solving situations that occur with increasing frequency in everyday situations today. That is, unlike the kind of word problems that have been emphasized in traditional textbooks and tests, where the products that students produce are simply short answers to narrowly specified questions about specific situations, in more realistic situations where mathematics is useful, it's often the case that the construction of relevant conceptual tools is not simply a process on the way to producing "an answer". Instead, the conceptual tools ARE the products that are needed. --- For example, a textbook word problem might be about determining how much money to leave as a tip for the waiter at a restaurant if the bill is $23.52 and you want to give a 15% tip. Or, in a "real life" situation that's similar to the situation involving spreadsheets and automobile sales, the student might be asked to program a calculator so that, no matter what percent tip we want to give, and no matter how large the bill may be, the calculator will tell us how much money to give the waiter as a tip. --- In such as situation, the calculator routine or the spreadsheet provide conceptual tools that should be sharable, manipulable, modifiable and reuseable in a variety of situations. Furthermore, students need to go beyond thinking with these tools to also think about them - for example, by thinking about the assumptions that they implicitly presuppose.

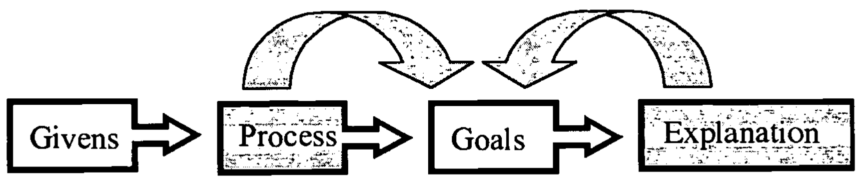

Figure 2. In real life situations where mathematics is useful, it's often the case that the process IS the product that's needed

An important fact to notice about the preceding kinds of conceptual tools is that, even when they reduce the burden of computing results, they often radically increase difficulties associated with describing situations in forms so that the conceptual tools can be used; and, they also my increase difficulties associated with interpreting the results that the tools produce. --- It's these facts that I've referred to when I say that "thinking mathematically" is about constructing, describing, and explaining at least as much as it is about computing.

Another example that will be given in the next section of this paper emphasizes the fact that describing situations mathematically may involve processes that range from quantifying qualitative information, to assigning "weights" to a variety of different kinds of qualitative and quantitative information, to operationally defining constructs (such as "productivity" for workers, or "cost-efficiency" for cars). For the purposes of this section, the main point that I want to emphasize is that the preceding kinds of mathematizing activities generally emphasize almost exactly the opposite kind of processes than those that have been emphasized in traditional word problems in textbook or tests. That is, in traditional word problems, what's problematic is (beyond the computational skills that such problems are intended to emphasize) that students must try to make meaning of symbolically described situations. But, in tasks that emphasize mathematizing activities, what's problematic is to make a symbolic description of meaningful situations.

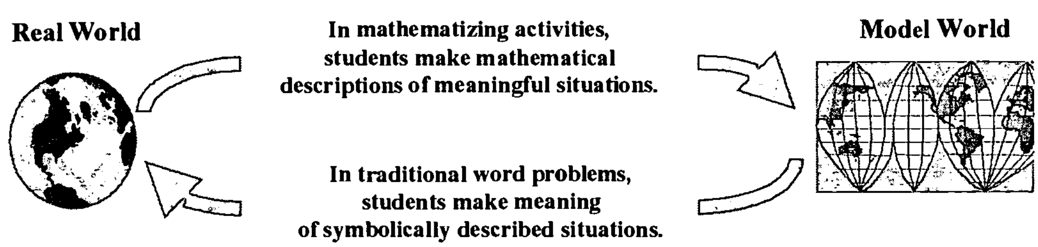

Figure 3. Mathematizing versus Decoding

What's an example of a mathematizing activity?

The *Summer Reading Program Problem* that follows is an example of a middle school version of a "case study" that I first saw being used at Purdue University's Krannert Graduate School of Management.

The Summer Reading Program

The St. John Public Library and Morgantown Middle School are sponsoring a summer reading program. Students in grades 6-9 will read books to collect points and win prizes. The winner in each class will be the student with the most reading points. A collection of approved books already has been selected and put on reserve. The chart below is a sample of the books in the collection.

Title	Author	Reading Level (By Grade)	Pages	Student's Scores on Written Reports	A Brief Description of the Book
Sarah, Plain and Tall	*Patricia MacLachlan*	*4*	*58*		*Note: On a "fold out" page, two or three sentences were given to describe each book. -- Was it a history book, a sports book, an adventure book, etc.*
Awesome Athletes (Sports Illustrated for Kids)	*Multiple Authors*	*5*	*288*		
A Tale of Two Cities	*Charles Dickens*	*9*	*384*		
Much Ado About Nothing	*William Shakespeare*	*10*	*75*		
Get Real (Sweet Valley Jr. High, No. 1)	*Jamie Suzanne & Francine Pascal*	*6*	*144*		

Students who enroll in the program often read between ten and twenty books over the summer. The contest committee is trying to figure out a fair way to assign points to each student. Margret Scott, the program director, said "Whatever procedure is used, we want to take into account: (1) the number of books, (2) the variety of the books, (3) the difficulty of the books, (4) the lengths of the books, and (5) the quality of the written reports.

Note: The students are given grades of A+, A, A-, B+, B, B-, C+, C, C-, D, or F for the quality of their written reports

YOUR TASK: Write a letter to Margaret Scott explaining how to assign points to each student for all of the books that the students reads and writes about during the summer reading program.

Notice that it's similar to many problems that occur when:

- business managers develop ways to quantify constructs like: the "productivity" of workers, or the "efficiency" of departments within a company, or the "cost-effectiveness" of a possible initiatives,
- teachers calculate grades for students by combining performance measures from quizzes, tests, projects, and laboratory assignments – or when they devise "scoring rubrics" to asses students' work on complex tasks, or

- publications such as *places rated almanacs* or *consumer guides* assessments (compare, rank) complex systems such as products, places, people, businesses, or sports teams.

We might refer to problems like the the *Summer Reading Problem* as "construct development problems" because, to produce the product that's needed, the basic difficulty that problem solvers confront involves developing some sort of an "index of reading productivity" for each participant in the reading program. This index needs to combine qualitative and/or quantitative information about: (i) the number of books read, (ii) the variety of books read, (iii) the difficulty of books read, (iv) the lengths of books read, and (v) the quality of reports written – as well as (possibly) other factors such as: (vi) the "weights"(or "importance values") that could be assigned to each of the preceding factors, or (vii) a "diversity rating" that could be assigned to the collection of books that each participant reads.

note: In the *Summer Reading Problem,* it might make sense to <u>multiply</u> the number of books by the difficulty level of each book. But, it might make sense to <u>add</u> scores from reading and scores from written reports. --- In general, to combine other types of information, students must ask themselves "Does it make sense to add, to subtract, to multiply, to divide, or to use some other procedure such as vector addition?" In other words, one of the main things that's problematic involves deciding which operation to use.

One important point to emphasize about "construct development problems" is that, even though such problems almost never occur in textbooks or tests, it's fairly obvious they occur frequently in "real life" situations where mathematics is used; and, it's also obvious that they represent just a small portion of the class of problems in which the products students are challenged to produce go beyond being a simple <u>numeric answers</u> (e.g., 1,000 dollars) to involve a the development of <u>conceptual tools</u> that can be:

- <u>used</u> to generate answers to a whole class of questions,
- <u>modified</u> to be useful in a variety of situations, and
- <u>shared</u> with other people for other purposes.

Furthermore, in addition to providing routines for <u>computations</u>, the tool also may involve:

- <u>descriptions</u> (e.g., using texts, tables, or graphs to describe relationships among variables),
- <u>explanations</u> (e.g., about how, when, and why to do something),
- <u>justifications</u> (e.g., concerning decisions that must be made about trade-offs involving factors such as quality and quantity or diversity), and/or
- <u>constructions</u> (e.g., of a "construct" such as "reading productivity").

Therefore, when mathematics instruction focuses on problem solving situations in which the products that are needed include the preceding kinds of conceptual tools, straightforward ways emerge for dealing with many of the most important components of what it means to develop deeper and higher-order understandings of the constructs that the tools embody.

Another important point to emphasize solving situations is that, when we observe students working on the preceding kinds of problems, the understandings and abilities that contribute to success often are quite different than those that have been emphasized in traditional textbooks and tests. For example, even though many of the same basic mathematical ideas are important (such as those involving rational numbers, proportional reasoning, and measurement), attention often shifts beyond asking *What computations can students do?* toward asking *What kind of situations can students describe (in forms so that computational tools can be used)?*

If we ask - What kind of mathematical understandings and abilities will be needed for success beyond school in a technology-based age of information? – the kind of examples that I've given so far should make it clear that the kind of mathematical conceptual tools that are needed often must be based on more than algebra from the time of Descartes, geometry from the time of Euclid, calculus from the time of Newton, and shopkeeper arithmetic from an industrial age. For example, mathematical topics that are both useful and accessible to students may include basic ideas from discrete mathematics, complexity theory, systems analysis, or the mathematics of motion – where the emphasis is on multi-media displays, representational fluency, iterative and recursive functions, and dynamic systems. Nonetheless, in general, to provide powerful foundations for success in a the new millennium, the kind understandings and abilities that appear to be most needed are not about the introduction of new topics as much as they are about broader, deeper, and higher-order treatments of traditional topics such as rational numbers, proportions, and elementary functions that have been part of the traditional elementary mathematics curriculum, but that have been treated in ways that are far too narrow and shallow for the purposes that concern us here.

At Purdue's *Center for Twenty-first Century Conceptual Tools* (TCCT), where I'm the Director, we enlist leaders from future-oriented fields ranging from aeronautical engineering, to business management, to computer technologies, to agricultural sciences to help us investigate:

- *What is* the nature of the most important elementary-but-powerful understandings and abilities that are likely to be needed as foundations for success in a technology-based *age of information*?

- *What is the nature of typical problems solving situations in which students must learn to function effectively when mathematics and science constructs are used beyond school?*

In these investigations conducted in the TCCT Center, it's noteworthy that participants are consistently reaching a consensus about the following claims.

- Some of the most important goals of instruction should be to help students develop powerful models and conceptual tools for making (and making sense of) complex systems.

- Some of the most effective ways to help students develop productive conceptual systems is to use "case studies" (or simulations of real life problem solving situations) in which students develop, test, and refine sharable and re-usable conceptual tools for dealing with classes of structurally similar problems.

- In problem-solving and decision-making situations beyond schools, the kind of mathematical and scientific capabilities that are in highest demand are those that involve: (i) the ability to work in diverse teams of specialists, (ii) the ability to adapt to new tools and unfamiliar settings, (iii) the ability to unpack complex tasks into manageable chunks that can be addressed by different specialists, (iv) the ability to plan, monitor, and assess progress, (v) the ability to describe intermediate and final results in forms that are meaningful and useful to others, and (vi) the ability to produce results that are timely, sharable, transportable, and re-useable. Consequently, mathematical communication capabilities tend to be emphasized, and so do social or interpersonal abilities that often go far beyond traditional conceptions of content-related expertise.

- Past conceptions of mathematics, science, reading, writing, and communication often are far too narrow, shallow, and restricted to be used as a basis for identifying students whose mathematical abilities should be recognized when decisions are made about hiring for jobs - or admissions for educational programs. This is because students who emerge as being especially productive and capable in simulations of "real life" problem solving situations often are not those with records of high scores on standardized tests. Therefore, new ways need to be developed to recognize and reward these students; and, these new approaches should focus on productivity, over prolonged periods of time, on the same kind of complex tasks that are emphasize in "case study" approaches to instruction.

In what ways does modern cognitive science have important implications for instruction focusing on the construction of powerful constructs (or conceptual tools)?

As Figure 4 suggests, humans have tended to explain the mind (and other complex systems) using their most recent advanced technologies as models. For example, during the twentieth century, psychology gradually moved from machine-based metaphors and factory-based models for the mind, beyond computer-based models, toward more organic models based on biotechnologies – from hardware, to software, to wetware.

From an Industrial Age using analogies based on hardware		**Beyond an Age of Electronic Technologies** using analogies based on computer software		**Toward an Age of Biotechnologies** using analogies based on wetware
where systems are considered to be no more than the sum of their parts, and where the interactions that are emphasized involve no more than simple one-way cause-and-effect relationships.	⇒	where silicone-based electronic circuits may involve layers of recursive interactions which often lead to emergent phenomena at higher levels which are not derived from characteristics of phenomena at lower levels	⇒	where neurochemical interactions may involve "logics" that are fuzzy, partly redundant, partly inconsistent, and unstable – as well as living systems that are complex, dynamic, and continually adapting.

Figure 4. Recent Transitions in Models for Making (or Making Sense of) Complex Systems

As a result, today, there is a growing recognition in mathematics education research that: (i) students, teachers, classrooms, courses, instructional programs, curriculum materials, learning tools and minds are all complex systems (taken singly, let alone in combination), and (ii) many of these complex systems cannot be explained using deterministic machine metaphors (even when they're embedded in silicone).

As *cognitive psychology* replaced *behavioral psychology* as the dominant way for educators to think about the nature of mathematics, learning, problem solving and teaching, many mathematics educators have adopted "constructivism" as an instructional philosophy. Two of the most basic constructivist claims are that: (i) constructs (cognitive structures, conceptual tools, and other complex systems developed by humans) must be constructed, and (ii) they can't simply be transmitted into children's minds in prefabricated forms (Steffe & Wood, 1990; Maher, Davis & Noddings, 1990: von Glassersfeld, 1991).

Unfortunately, it's far too easy for an educator to pledge allegiance to both of the preceding claims while continuing to cling to naïve software-based or machine-based metaphors for mind. For example, according to ways of thinking borrowed from the industrial revolution, teachers have been led to believe that the construction of mathematical knowledge in a child's mind is similar to the process of assembling a machine, or programming a computer. --- These "constructivists" might better be described as "assembly-ists" than "constructivists."

Assembly-ist constructivists tend to be easy to recognize. Their notion of the construction process is that teachers should use carefully guided sequences of questions that funnel students' thinking along pre-planned learning trajectory guided by the *teacher's* preferred way of thinking. They seldom put students in situations where the goal is for students to repeatedly express, test, and refine/revise/reject *their own* ways of thinking. Yet, what research at the TCCT Center is showing is that, in cases where ordinary students produce extraordinary results, the reason usually is because teachers devoted unusual attention toward getting students to express their ways of thinking in forms were testable – and encouraging students to refine their conceptual tools through multiple testing-and-revising cycles (Doerr & Lesh, in press).

Conclusions: Implications for productive uses of technology in mathematics education

Above all, what modern cognitive psychology does is to urge educators to focus on the developing conceptual schemes that humans use to make sense of structurally interesting systems. That is, mathematics is about *seeing* at least as much as it is about *doing;* it is about relationships among quantities at least as much as it is about operations with "naked" numbers (that tell "how much" but not "of what"); and, it is about making (and making sense of) patterns and regularities in complex systems at least as much as it is about calculations with pieces of data. It involves interpreting situations mathematically; it involves mathematizing (e.g., quantifying, visualizing, dimensionalizing, or coordinatizing) structurally interesting systems; and, it involves the <u>using</u> and <u>interpreting</u> an ever-expanding array of specialized languages, symbols, graphs, graphics, concrete models, or other representational media for purposes that range from construction, to description, or explanation. That is, *representational fluency* is at the heart of what it means to "understand" most mathematical constructs.

Some of the most important things that technology has done, both in education and in the world beyond schools, have been to radically increase the sophistication, levels, and types of systems that humans create. Another important thing that technology has done is create an explosion of representational media that can be used to describe, explain, and construct complex systems. Furthermore, at the same time that technology has decreased the computational demands on humans, it has radically increased the interpretation and communication demands. For example, beyond schools, when people work in teams using technology-based tools, and when their goals involve making (and making sense of) complex systems: (i) new types of mathematical quantities, relationships, and representation systems often become important (such as those dealing with continuously changing quantities, accumulating quantities, and iterative and recursive functions), (ii) new levels and types of understandings tend to be emphasized (such as those that emphasize communication and representation), and (iii) different stages of problem solving may be emphasized (such as those that involve partitioning complex problems into modular pieces, and planning, communicating, monitoring, and assessing intermediate results).

One of the most important consequences of the preceding trends is that, when broader ranges of mathematical abilities are recognized as contributing to success, broader ranges of people often emerge as being mathematically capable (Lesh & Doerr, 2000). In fact, in research in Purdue's *Center for Twenty-first Century Conceptual Tools* (TCCT), we're seeing that, in graduate schools in future-oriented fields ranging from aeronautical engineering, to business administration, to agricultural sciences, the kind of mathematical abilities that are needed are poorly aligned with (and poorly predicted by) abilities emphasized on traditional standardized tests.

Focusing on foundations for the future does not mean ignoring basics from the past. Abandoning basic skills would be as foolish in mathematics and science (or reading, writing, and communicating) as it would be in basketball, cooking, or carpentry. But, it's not necessary to master the names and skills associated with every item at Sears before students can begin to cook or to build things; and, New Zealand didn't become the famous home of the "All Blacks" in rugby by never allowing it's children to scrimmage until they'd completed twelve years consisting of nothing but drills on skills. What's needed is a sensible mix of complexity and fundamentals; both must evolve in parallel; and, one doesn't come before (or without) the other.

References

Davis, R.B., Maher, C.A., & Noddings, N. (1990). *Constructivist views on the teaching and learning of mathematics.* National Council of Teachers of Mathematics. Reston:VA.

Doerr, H. & Lesh, R. (Eds.) (in press). *Beyond Constructivism: A Models & Modeling Perspective on Mathematics Problem Solving, Learning, & Teaching.* Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Kelly, A. & Lesh, R. (2000). *Handbook of Research Design in Mathematics & Science Education.* Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R. & Doerr, H. (in press). Models & representations in mathematics teaching, learning, and problem solving,. In L. English (Ed.) special issue of *Mathematical Thinking & Learning: An International Journal.* Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R. & Doerr, H. (2000). Symbolizing, communicating, and mathematizing: Key components of models and modeling. In P. Cobb & E. Yackel. (Eds.) *Symbolizing, Communicating, and Mathematizing.* Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R. (1998) The development of representational abilities in middle school mathematics: The development of students' representations during model eliciting activities. In I. Sigel (Eds.) *Representations and Student Learning*. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R., & Lamon, S. (Eds.) (1992). *Assessment of Authentic Performance in School Mathematics.* Washington, DC: American Association for the Advancement of Science.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. & Wood, T. (Eds.) (1990). *Transforming Children's Mathematics Education: International Perspectives*

von Glasersfeld, E. (1991). *Constructivism in Mathematics Education*. Dordrecht: Reidel.

MATHEMATICS IN ACTION: TWO NEW ZEALAND CASE STUDIES

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Mathematics is playing an increasingly important role in business and industry. In this paper we present two case studies to illustrate the power and impact of mathematics in two important practical applications in New Zealand. The first case study describes the development of a mathematical optimisation model to maximise the value of aluminium produced at New Zealand Aluminium Smelters Ltd. The second case study describes the development and implementation of state of the art optimisation models and solution methods to solve all aspects of the crew scheduling problems for Air New Zealand Ltd.

Introduction

During the past fifty years the power and relevance of mathematics in business and industry have grown in both the variety of applications and the importance of its impact. Besides the development of the underlying mathematical techniques, an obvious reason for this exciting trend has been the development of computer technology and the associated computational techniques. In particular methods of mathematical optimisation including linear and integer programming have become widely used to model and solve many important practical problems. Scheduling and resource allocation decision problems occur in many business and industrial organisations. Often these problems involve valuable or scarce resources such as time or materials or people – finding optimal or near optimal solutions of these problems can provide millions of dollars of savings and provide a significant competitive business advantage.

The general linear programming (LP) model with m constraints and n variables has the form

$$\begin{array}{lll} \text{LP:} & \text{minimise} & z = \mathbf{c}^T\mathbf{x} \\ & \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \text{and} & \mathbf{x} \geq \mathbf{0} \end{array}$$

where A is an $m \times n$ real matrix and **b** and **c** are given real vectors of dimension m and n respectively. The mathematics of linear programming and the associated solution methods are interesting in their own right and in fact students taking Mathematics with Statistics at secondary school study linear programmes in two variables. Even beyond the beauty of the mathematics of LP, the relevance of the LP model in many practical applications makes linear programming one of the most important mathematical developments in the 20th century.

As a very special case of the general LP model, the set partitioning model provides an underlying mathematical model for many scheduling applications. The set partitioning problem (SPP) is a specially structured zero-one integer linear programme with the form

$$\begin{array}{lll} \text{SPP:} & \text{minimise} & z = \mathbf{c}^T\mathbf{x} \\ & \text{subject to} & A\mathbf{x} = \mathbf{e} \\ & \text{and} & \mathbf{x} \in \{0,1\}^n \end{array}$$

where $\mathbf{e} = (1,1,1,\ldots,1)^T$ and A is a matrix of zeros and ones and the solution vector $\mathbf{x}$ must take values of zero or one. Because of the computational difficulties in solving very large and practical instances of the set partitioning problem, many early attempts to use optimisation solution methods to solve scheduling problems were unsuccessful and researchers resorted to a variety of heuristic solution methods. While most heuristic methods are relatively easy to implement and may have reasonably inexpensive computer resource requirements, they suffer from two major disadvantages. Firstly they can provide no bound on the quality of any feasible solution that they produce, and secondly they are unable to guarantee a feasible solution will be found even if one exists. The heuristic methods may fail to find a feasible solution either because the heuristic method is inadequate or because the problem is truly infeasible. In contrast, an optimisation method can reliably detect infeasibility. During the past two decades, the development of optimisation methods and techniques for the solution of set partitioning problems, and the increase in computer power has meant that we are now able to solve realistic-sized models that arise in many practical scheduling problems.

In this paper, we present two case studies to illustrate the power and impact of mathematics in two important practical applications in New Zealand.

Case 1: Optimised Cell Batching at New Zealand Aluminium Smelters Ltd

New Zealand Aluminium Smelters Ltd operates a smelting facility at Tiwai Point near Invercargill. The smelter produces aluminium by the electrolytic reduction of alumina according to the reduction equation $2Al_2O_3 + 3C \rightarrow 4Al + 3CO_2$. This reaction which is called the Heroult-Hall process, is carried out as a continuous process in reduction cells constructed of an outer steel shell and a lining of refractory bricks. A carbon cathode is placed in the floor of the cell and carbon anodes are suspended above the cell on cast iron yokes. A very high direct current of approximately 190,000 amps is passed between the anode and cathode through a bath of molten cryolite at 960^0C which provides the electrical conductivity. The alumina is feed into the cryolite bath at regular intervals from a hopper that is located above the cell. The carbon required in the reduction reaction is provided by the carbon anode blocks which gradually reduce in size over a period of approximately twenty-seven days. When a block becomes too small, it is replaced by a new block. As the aluminium is produced it sinks to the bottom of the cell. Each day approximately 1260kgs of molten aluminium are tapped from the cell into a crucible by a vacuum siphoning system. A crucible is a large steel bucket lined with refractory bricks. Each crucible can tap the aluminium from three cells.

The cells are laid out in four lines. Three of the lines are each approximately 600 metres long and consist of 204 cells grouped into four tapping bays each made up of 51 cells. The fourth line 300 metres long was installed more recently and is made up of 48 cells of a newer technology in one tapping bay. All the cells in a tapping bay are tapped once each day and produce seventeen (or sixteen in the case of line 4) crucibles. Each bay is tapped either during the day shift or during the night shift. Once each crucible is filled with aluminium from three cells (always from the same tapping bay), it is transported from the reduction lines to furnaces in the Metal Products Division from where it is cast into finished products in the form of ingot or billet.

The purity of the aluminium varies from cell to cell depending on a number of factors including the age of the cell, the purity of the alumina feed and the manner in which the cell has been operated during its production life. The purity of the aluminium declines gradually as contaminants in the form of iron, silicon, gallium and other chemicals increase until at some stage a decision is made to cease production in the cell. The cell is then taken off-line and rebuilt before being brought back into production some days later. Each cell is assayed regularly to determine the percentage of aluminium, iron, silicon, gallium and other chemicals. Because high purity aluminum commands a premium price on the metals market, it is important that aluminium tapped into a crucible from high purity cells is not contaminated by tapping from low purity cells into the same crucible.

This Case Study describes the development of a set partitioning optimisation model to batch or group triples of cells so that the total value of metal produced is maximised. The main aim is to minimise the dilution of high purity (high value) metal by low purity (low value) metal. The optimised batches tend to group high purity cells together and leave the lower purity cells to be batched with other lower purity cells. Numerical results show that significant improvements in excess of 15% can be achieved in the value of metal by carefully batching cells.

In the following section of this paper, we will describe an optimisation model for cell batching and discuss the formulation of a natural objective to measure the solution quality. We will then discuss aspects of the solution process and in particular outline how integer solutions can be derived from continuous LP relaxation solutions. Some numerical results will be then be presented to show the benefits that mathematics can bring.

An Optimisation Model for Cell Batching

The cell batching optimisation can be formulated naturally as a set partitioning problem (SPP) which can be written as

$$\text{minimise } z = \mathbf{c}^T\mathbf{x}, \quad A\mathbf{x} = \mathbf{e}, \quad x_j = 0 \text{ or } 1$$

where A is a 0-1 matrix and $\mathbf{e}^T = (1,1,\ldots,1)$. Because cells in different tapping bays can never be tapped into the same crucible, the cell batching optimisation problem for each tapping bay can be considered independently of the other bays. For each tapping bay, the 51 constraints or rows of A correspond to cells in the tapping bay and ensure that each cell appears in exactly one batch or crucible. The columns of A represent all possible triples of cells (i.e. batches) which could be tapped into the same crucible. Each column then has exactly three nonzero unit values. For example, a batch made up of cells 1, 3 and 6 would be represented in the model by a column with zeros everywhere except for unit values in rows 1, 3 and 6. In general then the elements a_{ij} are defined as

$$a_{ij} \begin{array}{l} = 1 \text{ if cell } i \text{ is included in batch } j \text{ and} \\ = 0 \text{ otherwise.} \end{array}$$

In this basic unrestricted form of the model there are ${}^{51}C_3$ or 20825 columns or variables which can be easily enumerated. A solution of this SPP will be made up of exactly seventeen variables at unit value (representing the chosen batches) and all other variables will have zero value.

Spread Limitations on Cell Batches

Because each tapping bay is approximately 300 metres long, it is not practical to tap cells that are far apart into the same crucible. The actual tapping process involves the use of a gantry crane to carry the crucible and the human operator walks along the line from cell to cell. To avoid requiring the operator to walk large distances to fill each crucible, the usual practice is to tap cells that are within some specified maximum distance apart on the line. This is referred to as the spread of the batch. Spread can be defined simply as the difference between the maximum and minimum cell number in the batch. So three adjacent cells have a minimal spread of two. A batch made up of cells 1, 3 and 6 would have a spread of 5. In the enumeration of the columns or batches in the SPP model, the spread for each batch can be calculated. If the spread exceeds a specified limit, the batch can be rejected and not included in the model. Alternatively, the batch could be included in the model but marked as having a spread exceeding the maximum spread. During the optimisation process, these batches would be ignored unless the maximum spread was increased sufficiently. Such an increase could easily be included in a post-optimal investigation. In Section 3 we comment further on such investigations.

Batches exceeding the maximum spread can also be considered in a more useful manner. By adding an additional generalised SPP constraint to the basic SPP model, we could permit a limited number of batches with excessive spread to be included in the solution. This reflects the management view that a small number of batches (e.g. one or two of the seventeen batches) with excessive spread can be tapped provided they generate a sufficiently improved optimal solution when compared to the optimal solution using only batches that are within the maximum spread limit. Typical spread limits might be a maximum spread of 5 but up to two batches with a maximum spread of 10. In the enumeration of columns in the SPP model, all batches with a spread up to 10 would be generated but all those batches with spread between 6 and 10 would contribute to the additional constraint with a right-hand-side of 2. These spread restrictions which are included either implicitly (i.e. no spread exceeding 10) or explicitly (i.e. limited spread exceeding 6) significantly reduce the total number of variables in the SPP to something less than about 3000.

Alloy Codes and a Cell Batching Objective Function

A natural objective for the cell batching optimisation can be based on some estimate of the market value of the aluminium. This will obviously reflect the purity of each batch. Batch purity can be calculated as a simple weighted average of the known cell purities which make up the batch where the weights reflect the weight of metal tapped from each cell. Although in practice the actual cell tapping weights do vary a little, it is reasonable to assume before the tapping takes place that the tapping weights will be constant at 1260kgs for lines 1, 2 and 3 and 1480kgs for line 4. Given constant tapping weights and the cell assay values for aluminium, iron, silicon and gallium, the batch values are calculated as simple averages.

Table 1 defines eighteen aluminium alloy codes with their corresponding minimum aluminium percentage and maximum percentages for silicon, iron and gallium and an estimate of the corresponding alloy premium value. Generally speaking as the aluminium percentage increases (with corresponding decreases in the silicon, iron and gallium percentages) the alloy premium value increases rapidly. While this Table refers more particularly to finished product values cast in the particular alloy codes, we can classify the batch purity using the alloy specifications and then use the premium value as an objective coefficient for the batch. The negative premium values for the first three alloy codes reflect the fact that these grades of aluminium usually require purification by mixing with a higher grade metal and thus result in an actual loss of premium.

Table 1
Alloy Code Specifications and Premiums

Code	Min Al%	Max Si%	Max Fe%	Max Ga%	Premium
AA????	0.000	1.000	1.000	1.000	-50.00
AA150	99.500	0.100	0.300	0.100	-40.00
AA160	99.600	0.100	0.300	0.100	-25.00
AA1709	99.700	0.100	0.200	0.100	0.00
AA601E	99.700	0.100	0.080	0.100	40.00
AA601G	99.700	0.100	0.080	0.100	40.00
AA185G	99.850	0.054	0.094	0.014	15.00
AA190A	99.900	0.054	0.074	0.014	45.00
AA190B	99.900	0.050	0.050	0.014	50.00
AA190C	99.900	0.035	0.037	0.012	110.00
AA190K	99.900	0.045	0.055	0.034	100.00
AA191P	99.910	0.030	0.045	0.010	120.00
AA191B	99.910	0.030	0.027	0.012	139.00
AA192A	99.920	0.030	0.040	0.012	140.00
AA194A	99.940	0.020	0.040	0.007	150.00
AA194B	99.940	0.034	0.034	0.010	180.00
AA194C	99.940	0.022	0.027	0.009	200.00
AA196A	99.960	0.020	0.015	0.010	260.00

Off-line Cells

When cells are taken off-line to be rebuilt, they will not be tapped. This implies that one or more of the seventeen batches from the tapping bay will include fewer than three cells. It is important that the optimised solution determine which batches should be composed of fewer cells. One approach would be to generate all possible batches involving one and two cells and include them in the SPP model when the tapping bay has off-line cells. This results in a very large increase in the number of variables and causes further computational problems during the solution process. A much more attractive approach is to simply treat off-line cells as having a zero tapping weight. The off-line cells are then permitted to appear anywhere in the cell order during the enumeration of batches. In other words, the off-line cell can appear in any triple of cells without affecting either the spread calculation or the batch chemical composition. For example, if cell 50 is off-line, then a batch made up of cells 1, 2 and 50 would have a spread of 1 and the chemical composition would be determined entirely by cells 1 and 2. If this batch were included in the optimal solution, it would be interpreted as a batch involving just cells 1 and 2. However the SPP constraint for cell 50 would have been satisfied by this variable. The advantage of this approach is that all batches remain triples of cells including all triples involving off-line cells and the SPP model (at least for lines 1 to 3) is always made up with 51 cell constraints.

The Solution Process

The SPP model is solved by first solving the LP relaxation problem in which the integer restrictions are relaxed. The LP solution is trivial. With little extra effort it is possible to report on a sequence of LP solutions which gradually include variables with wider and wider spreads. During this initial optimisation phase, batches with spreads exceeding the maximum permitted spread are ignored or equivalently, the right-hand-side for the additional constraint described in Section 2.1 is set to zero thus preventing batches with excessive spread from contributing to the solution.

Handling Excessive Spread Batches

It is also simple to quantify the benefits of permitting a small number of batches with spreads exceeding the specified maximum spread as discussed in Section 2.1. After completing the initial LP solution, the right-hand-side for the additional constraint can be increased slowly to identify the potential benefits of using a limited number of batches with wider spread. All of these calculations are performed using the LP relaxation. While the LP solutions do exhibit some evidence of natural integer structure, most solutions involve fractional variables that must be forced to integer values using a branch and bound algorithm.

Constraint Branching in the Cell Batching Optimisation Model

The natural constraint branch (see Ryan and Foster, 1981) for this SPP is defined by any pair of cells that are not always included together in batches in the fractional solution. It is easy to show using balanced matrix theory (see Ryan and Falkner, 1988) that such a pair of cells must always exist in any fractional solution. The binary constraint branch can then be imposed by requiring on the one-side of the branch that the two cells always appear together in a batch and on the zero-side of the branch that the two cells always appear in different batches. The implementation of the branch involves removing sets of variables from the descendant LPs. On the one-side, batches in which the cells do not appear together are removed (effectively by setting their corresponding variable upper bounds to zero) and on the zero-side batches in which the cells appear together are removed.

While this constraint branching strategy is particularly effective, it is true that after imposing a sequence of constraint branches, the LP can become infeasible because a cell becomes isolated from its neighbours in such a way that no feasible set of batches can include the cell. When this happens it usually results in a long sequence of fathoming infeasible nodes and the branch and bound process can take a long time to find a feasible integer solution. For this reason we have implemented a heuristic integer allocation process at each node in the branch and bound tree in order to find integer solutions more quickly.

Integer Allocation Heuristics

Because of the special structure of the underlying SPP model in this application, it is easy to create infeasible LPs at nodes in the branch and bound tree. To avoid the computational problems that this causes, we have implemented an integer allocation process which is applied at each fractional node in the branch and bound tree including the root node. The process is heuristic in that it attempts to force an integer solution from the fractional solution by making a sequence of greedy decisions. First, all variables (batches) at value one in the fractional solution are fixed at that value. We then find the first cell which is not yet included in a batch and search amongst all variables (including those which are nonbasic) for the least cost batch including that cell and two other cells which are also yet to be covered. If no such variable can be found, the search is abandoned and the integer allocation fails. The allocation process can be applied using any order of the cells. In our implementation we apply the allocation process considering the cells in increasing and also in decreasing order.

Table 2

Optimised and Default Solutions for Tapping Bay 2AW

Optimised solution for tapping bay 2AW: [Cells 352 to 402] [Dataset d271197]												
Cell 15(-366) is off-line												
Cell 41(-392) is off-line												
A total of 51 cells found for bay 2AW; 2 cells off-line												
Generating with MINSPREAD 4; MAXSPREAD 10; MAXSPREADRHS 2												
Model size: 52 constraints; 2237 variables												
Bay	#	Weight	%AL	%SI	%FE	%GA	Code	Spread	Prem	Cells		
2AW	1	3840	99.800	0.044	0.095	0.016	AA1709	2	0	352	353	354
2AW	2	3840	99.810	0.040	0.105	0.016	AA1709	2	0	355	356	357
2AW	3	3840	99.737	0.041	0.167	0.015	AA1709	4	0	358	360	362
2AW	4	3840	99.847	0.042	0.080	0.014	AA601E	4	40	359	361	363
2AW	5	3840	99.760	0.041	0.132	0.016	AA1709	3	0	364	365	367
2AW	6	2560	99.835	0.034	0.079	0.015	AA601E	3	40	-366	399	402
2AW	7	3840	99.813	0.036	0.101	0.016	AA1709	2	0	368	369	370
2AW	8	3840	99.823	0.038	0.096	0.015	AA1709	3	0	371	372	374
2AW	9	2560	99.840	0.034	0.080	0.015	AA601E	2	40	373	375	-392
2AW	10	3840	99.820	0.038	0.095	0.016	AA1709	2	0	376	377	378
2AW	11	3840	99.833	0.035	0.087	0.016	AA1709	2	0	379	380	381
2AW	12	3840	99.837	0.037	0.079	0.015	AA601E	5	40	382	384	387
2AW	13	3840	99.817	0.037	0.093	0.016	AA1709	3	0	383	385	386
2AW	14	3840	99.837	0.033	0.080	0.016	AA601E	7	40	388	389	395
2AW	15	3840	99.753	0.075	0.126	0.015	AA1709	3	0	390	391	393
2AW	16	3840	99.843	0.037	0.080	0.015	AA601E	3	40	394	396	397
2AW	17	3840	99.827	0.036	0.092	0.016	AA1709	3	0	398	400	401
Integer objective: 240 (Branch and Bound time of 12.82 seconds)												
Default 2 spread solution for tapping bay 2AW: [Cells 352 to 402]												
2AW	1	3840	99.800	0.044	0.095	0.016	AA1709	2	0	352	353	354
2AW	2	3840	99.810	0.040	0.105	0.016	AA1709	2	0	355	356	357
2AW	3	3840	99.770	0.041	0.153	0.013	AA1709	2	0	358	359	360
2AW	4	3840	99.813	0.041	0.094	0.016	AA1709	2	0	361	362	363
2AW	5	2560	99.760	0.040	0.121	0.016	AA1709	1	0	364	365	-366
2AW	6	3840	99.793	0.038	0.119	0.016	AA1709	2	0	367	368	369
2AW	7	3840	99.827	0.035	0.093	0.015	AA1709	2	0	370	371	372
2AW	8	3840	99.830	0.038	0.090	0.015	AA1709	2	0	373	374	375
2AW	9	3840	99.820	0.038	0.095	0.016	AA1709	2	0	376	377	378
2AW	10	3840	99.833	0.035	0.087	0.016	AA1709	2	0	379	380	381
2AW	11	3840	99.820	0.040	0.090	0.016	AA1709	2	0	382	383	384
2AW	12	3840	99.833	0.034	0.083	0.016	AA1709	2	0	385	386	387
2AW	13	3840	99.820	0.035	0.087	0.016	AA1709	2	0	388	389	390
2AW	14	2560	99.715	0.093	0.144	0.015	AA1709	2	0	391	-392	393
2AW	15	3840	99.847	0.037	0.077	0.015	AA601E	2	40	394	395	396
2AW	16	3840	99.837	0.035	0.083	0.015	AA1709	2	0	397	398	399
2AW	17	3840	99.837	0.035	0.088	0.016	AA1709	2	0	400	401	402
Default 2-spread objective: 40												

The process is remarkably effective in that it often produces integer solutions with objective values very close to the LP upper bound value. Such an integer solution often becomes the best bound that fathoms the remaining live nodes in the branch and bound tree and the solution process terminates.

Some Numerical Results

We report here (see Table 2) some results that illustrate the performance of the cell batching optimisation and compare the results with a so-called default order solution.

Table 3

Solution Summary by Alloy Codes for all Tapping Bays

		Optimised solution		Default solution	
Code	Premium	# crucibles	premium	# crucibles	premium
AA160	-25	3.00	-75.00	3.00	-75.00
AA1709	0	27.67	0.00	73.67	0.00
AA190K	100	1.00	100.00	5.33	533.33
AA192A	140	31.67	4433.33	40.00	5600.00
AA194B	180	22.00	3960.00	16.00	2880.00
AA194C	200	1.67	333.33	1.00	200.00
AA196A	260	9.00	2340.00	3.00	780.00
Totals:		219.00	16011.67	219.00	12998.33
Percentage increase in total default premiums 23.18%					

The default order solutions, based on batching cells in the natural sequence such as (1,2,3), (4,5,6), (7,8,9) etc, actually form the basis of the manual solution method in which obvious poor value batches are modified locally by changing the allocations of cells to nearby batches to improve the solution. This is a difficult process for human decision making and even experienced operators are unlikely to produce optimal decisions. Table 2 includes cost values based on the alloy code premium for each batch. It can be seen that in tapping bay 2AW, the optimised solution produces 6 batches with premium value of 40.0 (alloy code AA601E) while the default solution produces just one batch with this premium value. Notice also that batches 12 and 14 in the optimised solution involve spreads of 5 and 7 respectively. All other batches have spreads not exceeding 4. The two off-line cells (366 and 392) appear in batches 6 and 9 respectively as negative cell numbers. The weights, spreads and chemical compositions of these two batches ignore the off-line cells.

In Table 3 we give a comparison of the overall results of the optimised cell batching applied to all tapping bays. The optimised results can be compared with the corresponding overall results of the default solution. The results are reported in terms of the number of batches (i.e. crucibles) produced in each alloy code and the total premiums generated by those batches. The optimised solutions show an improvement of approximately 23% over the default solutions.

Conclusions

While the cell batching optimisation has produced significant improvements in the value of metal produced from the reduction lines, this problem really forms part of a larger production scheduling problem at the smelter. The full problem involves first a decision about which products from the order book to produce during the day. This production must then be scheduled on the furnaces and casting machines in the Metal Products Division. A furnace production schedule is made up of periods during which the furnace is filled with suitable batches from the reduction lines followed by periods during which the chemical composition of the furnace metal is adjusted and stabilised before the required product is cast. Given a production schedule for each furnace, the allocation of batches from the cell batching optimisation to furnace fills must be decided. This particular problem is not trivial since it implies a time sequencing for the actual production of the batches. There are particularly important constraints on the production of batches that result from the limited capacities of the gantry cranes used during the tapping process. In the cell batching optimisation, these constraints were ignored as was the time sequencing of production of the batches. We are currently investigating this further scheduling problem to determine a feasible production sequence for the optimal batches to match a given furnace schedule.

Case 2: Optimised Crew Scheduling for Air New Zealand Ltd

In the mid 1980s, the scientific literature contained relatively few papers documenting the successful application of optimisation methods in the solution of airline crew scheduling problems although some heuristic methods were being applied. In fact, stories circulating in the airline industry at that time suggested that a number of larger airlines had tried and failed to implement optimisation based crewing systems. In fact, when I first approached Air New Zealand in the early 1980s to ask if I could obtain information about their crew scheduling problems, I was informed by a senior manager that he was aware of the failures of other airlines and he asked me what made me think I could solve crewing problems for Air New Zealand. I remember responding that I didn't know if I could solve the problems but I simply wanted to find out about the problems and obtain some data to try solving them. In 1984, Air New Zealand agreed to provide information about their planning or Tour of Duty (Pairings) problem for us to use in an Honours project for a student in Engineering Science at the University of Auckland. This initial project, involving a small part of the domestic (or internal) problem, was undertaken by Michelle Kunath. The results we presented to Air New Zealand at the completion of the six-month project provided the basis of a most productive and successful collaboration between the University of Auckland and Air New Zealand. Over the intervening period of more that 16 years, optimisation methods have been developed and implemented to solve all aspects of Air New Zealand's crew planning and rostering problems.

In 1984, all crewing decisions were made manually and the airline used no Operations Research (OR) techniques. Today the airline is totally dependent on state of the art optimisation based computer systems in the areas of crew planning and rostering. The airline now employs eight staff with backgrounds in OR. In this Case Study, we document the transition from dependence on manual methods to dependence on mathematical optimisation methods in New Zealand's national airline.

The Crew Scheduling Problems

Airline crew scheduling involves two distinct processes of Planning and Rostering. The Planning process (also referred to as the Pairings problem) involves the construction of a minimum cost set of generic Tours of Duty (ToDs) or pairings which cover all relevant flights (sectors) in an airline flight schedule. Each Tour of Duty begins and ends at a crew base and consists of an alternating sequence of duty periods and rest periods with duty periods including one or more sectors. At Air New Zealand, domestic (or National) ToDs (on B737 aircraft) are between one and three days in duration while for International airline operations (on B767 and B747 aircraft), ToDs can be up to fourteen days in duration. New ToDs are constructed for each new flight schedule and for day to day variations in an underlying schedule. It should be noted that the construction of ToDs does not involve any consideration of the crewmembers who will actually perform the ToDs.

The Rostering process involves the allocation of ToDs to each crew member of a rank so that all flights are crewed with the correctly qualified crew complements and each crew member has a legal feasible line of work over a given roster period. At Air New Zealand, domestic rosters are built over a fourteen-day roster period while international rosters cover a twenty-eight day period. Roster construction must take into account activities rostered in the previous roster period, which carry over into the current period. From a crew point of view, it is also important to provide high quality rosters that satisfy crew requests and preferences as much as possible.

During the past two or three decades airlines have invested heavily in the development of techniques to solve their crew scheduling problems. The main reasons for this focus on crew scheduling can be identified in the following major factors.

- Reducing Aircrew costs
  Aircrew costs are one of the largest operating costs faced by an airline (second only to fuel). However, the crewing problems of Planning and Rostering are very large and hard to solve. Manual and heuristic-based solution methods will almost never find minimum cost solutions due to the very large number of alternative solutions and the complex nature of the crew scheduling rules. Because small improvements in solution quality return large dollar savings, the use of optimisation techniques to solve the Planning problem has been the primary focus of much research within the Airline Industry for many years.

- Reducing solution time
  The time taken to create solutions manually meant that only one solution could be developed, and alternative proposals could not be evaluated quickly or accurately. A shorter planning cycle means that the airline can respond quickly to changes in the market and capitalise on opportunities. The flight schedule can be changed much more frequently and solutions must be continually updated. The challenge is to maintain crew productivity and the quality of the crewing solution while the flight schedule is changed from day to day.

- Compliance
  Crew scheduling is constrained by many complex and conflicting rules, further exacerbated by time-zone changes, daylight savings, and foreign currencies. Solutions must comply with legislative, contractural and operational rules. Airlines are required to have systems in place to ensure that rules are not violated. The complexity of the rules is also a factor in the time taken to solve these problems and manually produced solutions are not able to guarantee compliance.

- Reducing costs to construct and maintain the crew scheduling process.
  The complex nature of the rules and the experience required to achieve consistently high-quality manual solutions mean that staff changes could result in increased crewing costs.

Over the past sixteen years considerable research and development of underlying optimisation methods for crew scheduling has been undertaken at the University of Auckland in collaboration with Air New Zealand. This research has resulted in the development of seven optimisation-based computer systems to solve all aspects of both the Planning and the Rostering processes for both the National and the International Airlines. It should be noted that each business problem is characterised by quite unique aspects that prevent the development of a single common optimisation solver. The following Table shows the implementation dates of each system. The term "Technical Crew" refers to pilots (Captains and First Officers).

	Flight Attendants		Technical Crew	
	Planning	Rostering	Planning	Rostering
National	1986 revised 1997	1993	1986 revised 1997	1998
International	1998	1989 revised 1996	1996	1994

These systems incorporate state-of-the-art mathematical optimisation technology and provide Air New Zealand with sophisticated crewing solvers. In 1989 when the International flight attendant rostering system was implemented, Air New Zealand knew of no other airline worldwide with an implemented rostering systems based on optimisation and even today, few airlines use optimisation based rostering techniques. The optimisation solvers are fully integrated into other information systems at Air New Zealand. Further details of each of these systems and their integration are given below.

Many scheduling problems can be formulated mathematically using a Set Partitioning model. From a technical point of view, this specially structured zero-one integer linear programme which has relatively few constraints but a very large number of variables model is a. Both the Planning and the Rostering problems of airline crew scheduling can be formulated as set partitioning problems with special structure. Research conducted at the University of Auckland, in collaboration with Air New Zealand, has resulted in major breakthroughs in the solution of very large instances of set partitioning models which occur in practical applications. By recognising the special model structure and incorporating it in the solution methods, it is possible to solve optimisation problems that just ten years ago were considered far too difficult to solve. In particular, practical instances of the Rostering problem can be very large but it is still possible to produce high quality solutions. These important technical developments involving

- limited subsequence matrix generation
- constraint branching strategies for integer programming
- resource constrained shortest path column generation and
- anti-degeneracy and steepest edge pricing strategies

have been described by Ryan and Falkner (1988), Ryan and Osborne (1988), Ryan (1992).

The Tours of Duty Planning Model

In the basic ToD planning model, each column or variable in SPP corresponds to one possible ToD that could be flown by some crew member. Each constraint in SPP corresponds to a particular flight and ensures that the flight is included in exactly one ToD. The elements of the A matrix can then be defined as

$$
\begin{aligned}
a_{ij} &= 1 \quad \text{if the } j\text{th ToD (variable) includes the } i\text{th flight (constraint)} \\
&= 0 \quad \text{otherwise.}
\end{aligned}
$$

The value of c_j, the cost of variable j, reflects the dollar cost of operating the jth ToD. The calculation of c_j values is specified by the particular problem being considered but usually includes the cost of paid hours (both productive and unproductive), ground transport, meals and accommodation, and the cost of passengering crew within the ToD. Many authors (see Rubin, 1973 and Wedelin, 1995) model the ToD planning problem using the set covering formulation in which the equality constraints are replaced by "greater than or equals" constraints. The overcover of a flight permitted by the set covering constraint can be interpreted as passengering of the excess crew cover. This formulation results in fewer variables but it has the major disadvantage of making it difficult to model accurately the rules and costs associated with passengering crew. For example, passengering duty time limits are generally longer than operating limits. This cannot be correctly modelled using set covering constraints. In the Air New Zealand applications, set partitioning constraints are used which allows passengering to be accurately modelled. This results in additional columns which explicitly include passengering flights with $a_{ij} = 0$. It is also important to note that each column or ToD must correspond to a feasible and legal sequence of flights which satisfies the rules specified in civil aviation regulations or employment contracts or agreements. These rules or constraints can be thought of as being implicitly rather than explicitly satisfied in the ToD planning model. For example, rules imposing limits on total work time and rest requirements are embedded in the variable generation process.

The ToD planning model is usually augmented with additional constraints that permit restrictions to be imposed on the number of ToDs included from each crew base. Because these constraints typically have non-unit right-hand-side values, we describe the ToD planning model as a generalised set partitioning model.

In Air New Zealand, the ToD planning model is formed and solved independently for each crew type and the flights they operate. We will identify further specific variations and extensions of the basic ToD planning model in the detailed discussion of each of the Air New Zealand systems.

The Rostering Model

The rostering problem involves the construction of a LoW for each crew member in a rank so that each ToD is covered by the correct number of crew members from that rank. For each crew member we can generate a set of many LoWs from which exactly one must be chosen.

The rostering problem can also be modelled mathematically using a generalised version of the set partitioning model. Assuming there are p crew members and t ToDs, the model is naturally partitioned into a set of p crew constraints, one for each crew member in the rank, and a set of t ToD constraints corresponding to each ToD which must be covered. The variables of the problem can also be partitioned to correspond to the feasible LoWs for each individual crew member. The A matrix of the rostering set partitioning model is a 0-1 matrix partitioned as

$$A = \begin{bmatrix} C_1 & C_2 & C_3 & \cdots & C_p \\ L_1 & L_2 & L_3 & \cdots & L_p \end{bmatrix}$$

and $C_i = \mathbf{e}_i\mathbf{e}^T$ is a (p x n_i) matrix with $\mathbf{e}_i$ the i^{th} unit vector and $\mathbf{e}^T = (1,1,...,1)$. The n_i LoWs for crew member i form the columns of the (t x n_i) matrix L_i with elements l_{jk} defined as $l_{jk} = 1$ if the k^{th} LoW for crew member i covers the j^{th} ToD and $l_{jk} = 0$ otherwise. The A matrix has total dimensions of $m \times \sum_{i=1}^{p} n_i$ where $m = p + t$. The right-hand-side vector $\mathbf{b}$ is given by $b_i = 1$, $i = 1,...,p$ and $b_{p+i} = r_i$, $i = 1,..., t$ where r_i is the number of crew members required to cover the i^{th} ToD. We refer to the first p constraints as the "crew constraints", and the next t constraints as the "ToD constraints".

The cost vector $\mathbf{c}$ is chosen to reflect the relative "cost" of each LoW. Since most airlines do not use optimisation systems for rostering, there is no obvious or traditional measure which can be used to discriminate among feasible solutions in an optimisation. We define particular rostering objectives in the discussion of each of the specific rostering systems developed at Air New Zealand. Typically the rostering objective reflects either the "preferential bidding by seniority" (PBS) or the "equitable" rostering philosophy.

The rostering model has a special structure which deviates from pure set partitioning in that the right-hand-side vector is not unit valued and some constraints need not be equalities. The crew constraints of the A matrix also exhibit a generalised upper bounded structure which is not commonly found in set partitioning.

Crewing Systems at Air New Zealand

National (Domestic) Planning

The original Planning system for the National Airline covering both Flight Attendants and Technical crew was developed in 1984 and 1985 and implemented as a mainframe computer system in 1986. The system remained in production essentially in its original form until 1997 when it was replaced by improved optimisation methodology implemented on a Unix workstation. The current system generates optimised ToDs for all crew ranks and for three crew bases in Auckland, Wellington and Christchurch. It is also able to produce "fully-dated" solutions.

National (Domestic) Rostering

While involving relatively small crew ranks (at least compared to the International Airline), these problems are probably the most difficult of all the Air New Zealand optimisation problems to solve because of their combinatorial complexity. Two previous attempts to solve the problems in the late 1980s were unsuccessful but the problem for Flight Attendants was finally solved in 1993 by Dr Paul Day in his PhD research sponsored by Air New Zealand (see Day, 1996 and Day and Ryan, 1997). In 1998, the same solution methodology was adapted to produce Technical crew rosters under quite different operating rules. These two unique systems are now fully integrated into the Air New Zealand Genesis Rostering System and produce rosters of excellent quality for all crew ranks and all crew bases in less than four person days. Previous manual rostering methods involved 6 roster builders and took two weeks to complete the roster build. The actual optimisation runs themselves take less than one hour in total.

International Flight Attendant Rostering

This problem, involving 1500 flight attendants in four crew ranks, is the largest problem solved at Air New Zealand. The original system was implemented in 1989 and was revised to incorporate column generation methods in 1996. At the time of its implementation in 1989, the optimised solution demonstrated that it was possible to construct rosters with a 5% reduction in the number of flight attendants and at the same time, significantly improve the quality of the rosters from a crew point of view. The development and implementation involved representatives of the Flight Attendant Union who defined the issues of roster quality which are incorporated in the optimisation. The current system also incorporates a language assignment optimisation step (Waite, 1995) which ensures that flight attendants with relevant language qualifications are assigned ToDs requiring those language skills. This aspect of Flight Attendant rostering has important commercial benefits to Air New Zealand in that many of its passengers, particularly from Asia and Europe, are non-English speaking.

International Technical Crew Rostering

International technical crews in most airlines world-wide are rostered by systems based on preferential bidding by seniority (PBS). The algorithms are generally based on greedy sequential heuristic roster construction methods. PBS involves crew members bidding for work or days off and rosters are then constructed by satisfying as many bids as possible but considering crew members strictly in seniority order within the crew rank. During 1992 and 1993, a new optimisation model and solution method for PBS was developed (see Thornley, 1993). The solution method incorporates a unique "squeeze procedure" which violates the bids of more junior crew members in order to satisfy the bids of more senior crew members. This guarantees that the maximum number of bids can be satisfied in seniority order. Heuristic methods used by other airlines are unable to provide such a guarantee. The PBS system was implemented in 1994 and is now fully integrated into the Genesis Rostering System at Air New Zealand.

International Technical Crew Planning

Following the completion of his Masters research on the topic, Andrew Goldie (see Goldie, 1995) implemented the Technical Crew Planning system for Air New Zealand International in 1996. The system automatically generates "third pilot" ToDs which allow duty periods to be extended by including a third pilot on some relevant sectors. This feature is believed to be unique since we understand that Planning systems used by other airlines construct such ToDs in a subsequent step.

International Flight Attendant Planning

The International Flight Attendant Planning problem is a particularly difficult problem in that flight attendants are qualified to operate on all aircraft types. The added complexity arises because each aircraft type requires different numbers of crew. For example, a full B747 crew may split after a B747 sector and part of the crew may fly a B767 sector in their next duty period. The remaining part of the B747 crew could fly as passengers or could be combined with other crew members to make up a crew for some other sector. This crew splitting complication does not occur for Technical crew who are qualified to fly just one aircraft type. An optimisation solver for International Flight Attendant Planning has been developed by Chris Wallace in his PhD research. This system is again unique in that it automatically permits crew splitting. No other known Planning system incorporates this feature.

Implementation and Integration Issues

On-site development by a small team of developers, working closely with the users, has been central to the successful implementation of these systems. The complex rules-bound nature of the industry requires detailed understanding, and the optimisation solvers must be developed with constant reference to the planners and rosterers.

The optimisation solvers are able to find many solutions of similar dollar value, some of which are preferred by the users, and much effort has been spent developing control mechanisms for the users to interact with the solutions and so "shape" the solutions produced. For example, users may wish to fix a particular subset of a solution, or prevent particular undesirable characteristics from being included in a solution. Similar mechanisms have been developed to handle changes to inputs to the problem. For example, if a flight is re-timed in the flight schedule, the optimiser will minimise the changes required from the previous solution to the new solution.

The ToD optimisers are integrated into a purpose-built PC-based user interface, also developed as part of the project. The system receives flight schedule data for both proposed and published flight schedules in industry-standard formats from Airflight, the Schedules Management tool supplied by The Sabre Group. The solutions produced by the ToD optimisers can be viewed graphically, using a tool specifically developed for the purpose. Solutions may be electronically uploaded into the Air New Zealand Genesis Rostering System.

The Genesis Rostering System, which has been developed independently of this project, is used by all Rostering Staff to manage the construction of rosters for aircrew. It has replaced existing mainframe and PC-based systems and manual methods, previously used for roster construction. It provides a common user interface for managing ToDs, crew pre-assignments, training and crew requests. Genesis passes data to the Rostering optimisers where the roster is constructed, before it uploads and displays the optimised rosters in the graphical interface.

There are two important aspects associated with the implementation of sophisticated crewing systems: the first is concerned with the effects of new technology and the second is concerned with issues of technology transfer.

The introduction of high-technology mathematical optimisers has changed the nature of the jobs and the key competencies of staff in the Planning and Rostering areas. Staff now manage data and processes associated with the construction of ToDs and rosters rather than simply constructing ToDs or rosters. The optimisers allow staff to concentrate on meeting specific business requirements which might include provision of training or leave, or meeting special requirements of management or crew, or incorporating last-minute changes.

Because of the sophisticated nature of the mathematical optimisation models and methods, it is important that management, the crew and the planning and rostering staff develop trust and confidence in the solution methods and the quality of the solutions. This can only be achieved through close collaboration with these affected groups. The development of trust and confidence has been a major objective of this project since it is an essential requirement of successful technology transfer from a research and development environment to a production environment. We believe that this objective has been fully achieved in this project.

Economic and Other Benefits

The crew scheduling optimisers provide real dollar benefits to Air New Zealand, by directly reducing the cost of crewing in areas such as the total number of crew required and the number of hotel bed-nights, meals, and other expenses which must be paid to crew away overseas. Each optimisation application has reduced the costs of constructing and maintaining the crewing solution for the flight schedule. Over the past ten years, Air New Zealand's aircraft fleet and route structure has increased significantly in size yet the number of staff required to solve the crew scheduling problem has reduced significantly from a total of 27 in 1987 to 15 today.

A conservative estimate of the dollar savings (in New Zealand dollars) from the crew scheduling optimisers has been calculated as NZ$15,655,000 per annum. It is interesting to contrast the estimated total savings of NZ$15.6 million per annum with the estimated total development costs over 15 years of approximately NZ$2 million. It is also interesting to contrast the estimated total savings of NZ$15.6 million per annum with the 1999 net operating profit for the Air New Zealand group of NZ$133.2 million (excluding one-off adjustments).

In addition to the direct dollar savings, many intangible benefits are also provided by these optimisation systems.

1. Using the crewing optimisers, high quality solutions can be produced in a matter of minutes, compared with two or more days to create a manual solution. For example, a B767 pilot ToD problem can be optimally solved in approximately 60 minutes, while the B747-400 pilot ToD problem can be solved in less than 5 minutes on a Unix workstation. The international flight attendant rostering problem involving 550 crew in one rank is solved in less than six hours.

2. Crew schedulers can now focus on data preparation and validation, and the interpretation and evaluation of solutions. Their role has changed from the mechanical process of ToD and roster construction to one of an analyst.

3. There is a reduced dependence on highly skilled staff with an intimate knowledge of the employment contracts and scheduling rules. These contracts and rules are now embedded in the crewing optimisers.

4. The crewing optimisers can be used to investigate strategic decisions for crewing such as the evaluation of proposed rule changes and their cost impacts, and basing studies to determine ideal crew numbers at crew bases.

5. The relatively short build times for ToDs and rosters enables solutions to be produced much closer to the day of operation and with reduced lead times. This enables the company to accommodate late schedule changes and reduces rework.

6. The ToD optimisers make it possible to provide accurate and reliable feedback to the schedules planning group about proposed schedules and the impact on profit

from a crewing point of view. This information was very difficult to provide manually because of the time taken to produce manual ToD solutions.

7. Before the implementation of the national ToD optimiser in 1986, Air New Zealand operated a fixed six monthly winter and summer schedule. Now Air New Zealand is able to operate a much more flexible schedule that varies from week to week and allows the company to respond quickly to market opportunities.

8. The ToD optimisers also make it possible to repair solutions quickly when small changes are made to the schedule. The cost to maintain solutions is kept to a minimum.

9. The rostering optimisers reflect crew defined roster quality measures and crew are encouraged to identify "soft rules" to further improve roster quality. The involvement of the crew in the development of systems that affect their lifestyle is a very important benefit that cannot be provided adequately by manual systems.

10. The rostering optimisers have delivered significantly improved levels of crew request and bid satisfaction. Over 80 percent of all legal international flight attendant requests for ToDs and days off are consistently achieved, with even higher achievement of requests for the national pilot and flight attendant rostering systems.

11. The rostering optimisers can accurately identify roster infeasibility and can minimise the level of infeasibility. They can also identify any days on which there are insufficient crew.

12. Improved passenger service has been achieved directly though the use of the international flight attendant languages optimisation. Air New Zealand's high levels of customer service have been recognised through recent awards including the "Globe Award for the Best Airline to the Pacific" awarded by top British travel industry newspaper Travel Weekly in January 2000. Air New Zealand has received this award in four of the last five years. Air New Zealand was also ranked first for inflight service by the AB Road Airline Survey, in August 1999.

13. The crewing optimisers provide a guarantee that important legislative and contractual rules are satisfied. In many of these situations, Air New Zealand is required to demonstrate compliance through an audit procedure. Manual systems are unable to guarantee compliance without time consuming checking.

References

Day, P. R., & Ryan, D. M. (1997). Flight attendant rostering for short–haul airline operations, *Operations Research,* 45(5), 649–661.

Day, P. R. (1996). *Flight Attendant Rostering for Short–haul Airline Operations*, PhD Thesis, University of Auckland.

Goldie, A. P. (1995). *Optimal Airline Crew Scheduling Using Dynamic Column Generation*, Masters Thesis, University of Auckland.

Ryan, D. M. (1992). The solution of massive generalised set partitioning problems in aircrew rostering, *J. Op. Res. Soc.* 43, 459–467.

Ryan, D. M., & Falkner, J. C. (1988). On the integer properties of scheduling set partitioning models, *European Journal of Operations Research*, 35, 442–456.

Ryan, D. M., & Foster, B. A. (1981). An integer programming approach to scheduling, in A. Wren (Ed) *Computer Scheduling of Public Transport*, North Holland, 269–280.

Ryan, D. M. & Osborne, M. R. (1988). On the solution of highly degenerate linear programmes, *Math Prog*. 41, 385–392.

Thornley, M. E. (1993). *Crew Rostering Under A Seniority Preferential Bidding Environment Using Column Generation*, Masters Thesis, University of Auckland.

Tuck, S. (1997). *Optimal Cell Batching for New Zealand Aluminium Smelters Tiwai Point Facility*, Year Four Project, Dept of Engineering Science, University of Auckland.

Waite, J. (1995). *Language Assignment For Flight Attendant Rostering*, Masters Dissertation, University of Auckland.

RESEARCH PAPERS

WHAT CAN WE LEARN FROM COMPUTER-BASED DIAGNOSTIC TESTING?

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The use of diagnostic testing at Newcastle University is described, using the computer-based test DIAGNOSYS. Diagnostic and exam results are compared for different groups over several years. It is concluded that consistent testing can give useful information to individuals and for groups, but that both prediction of subsequent success and selection of groups for remediation must proceed with caution.

Introduction

Higher Education institutions, and other tertiary institutions, are experiencing rapid changes as they try to respond to changing intakes, In numbers, aptitudes and backgrounds, changing expectations from students and society, changing funding patterns, and changing knowledge, both within subjects and of education itself. Some of these issues reflect even greater changes in the school system. One of our common concerns is the background knowledge of mathematics of new students entering our institutions. What do our new students know? What curriculum is appropriate, in content and delivery? What can and should we do for those whose starting point is at one extreme — well- or ill-prepared? (LMS, 1995).

In this context, many institutions in the UK are doing, or considering, diagnostic testing of their intake of mathematical knowledge and skills (Edwards, 1996; Gatsby, 1999). Newcastle University has been doing so, especially in Engineering, since 1991, initially using paper tests, then the computer-based test DIAGNOSYS since 1993 (Appleby et al., 1995a,b, 1997a,b; Cambridge, 2000). We have had four aims: Help for the individual (self-remediation, perhaps with assistance), targetting groups with special needs, curriculum design for whole groups, assessment of selection policy. Computer testing was used to increase speed and to increase specific feedback, and offered much greater possibilities for analysis such as correlations of factors with each other.

This paper describes briefly the features and operation of the DIAGNOSYS maths test, and an analysis of its success with regard to individuals and groups. A large body of data accumulated from our own testing, and a very large group tested in Singapore (Lim & Tay, 1999), provide evidence of the influence of the interface, the test management, and of the question design on the reliability of results.

DIAGNOSYS - history and features, and other tests

Edwards (1996) and Cambridge (2000) describe several other diagnostic tests in use in the UK. *Mathwise* is a substantial CAL package for (mostly) first year maths teaching, and was a TLTP funded project (the Teaching and Learning Technology Programme was a major UK drive to create and promote CAL in UK Higher Education), and includes assessment facilities (Beevers, 2000). *CALMAT* is a package of similar intention, used in over fifty institutions (web reference given below). *Mathletics* is a testing package, created with Question Mark Designer (Greenhow, 2000). Hibberd (1996) describes a multiple-choice test, developed under another TLTP project at Nottingham and used elsewhere. Lawson (1997) has, in contrast, used a paper-based test for large numbers of students, with the test unchanged for some years, and thus can report on the changing intake.

The computer-based tests offer various facilities, including self-testing modes to be used prior to a more formal test. Some are limited to numerical or multiple-choice questions. Others permit one-line entry of algebraic answers, with optional re-display in mathematical form for user-checking.

DIAGNOSYS is actually a testing shell, designed to permit the implementation of tests in subject areas other than mathematics, but offering some features especially suited to maths. The actual test implementation is held in several text files. The two features that are notably different from the others mentioned are that it has a fully interactive, intuitive algebra interface, and it is an expert-system (or knowledge-based system), with question-setting adapting to the background of the student and to their previous answers. The design process, including how the test was validated, is described in Appleby et al (1997b), and a summary only is given here.

The expert system

Many mathematical topics and skills can be regarded as part of a hierarchical network, in which new topics taught have certain clear pre-requisite skills and knowledge. For example, knowledge of how to expand a double bracket depends on knowledge of how to expand a single bracket. If the first question asked is of an appropriate level, it will often be possible to ask just one of these two questions, rather than both, reducing the overall number of question asked, and also not discouraging the student (Figure 1).

	Expand $3x(x+1)$ _ Expand $(x+1)(x+3)$		
Weak entrant attempts first:	correct	→	attempt second
	incorrect	→	can't do second either
Strong entrant attempts second:	correct	→	can do first also
	incorrect	→	attempt first

Figure 1. Hierarchy of skills

Following analysis of results from a pen-and-paper test in 1991 and 1992, a whole network was created on this basis, later refined and extended (Figure 2). The ninety-two skills included cover basic numeracy, algebraic manipulation, miscellaneous graphical and trigonometric knowledge, and simple calculus and statistics. In practice, only part of this material is tested, as different student groups may be allocated to different sub-sections, following their answers to an initial 'administrative' question. A second question fixes their starting level according to their prior qualifications. A typical test takes about one hour, testing sixty skills by asking around thirty-five questions.

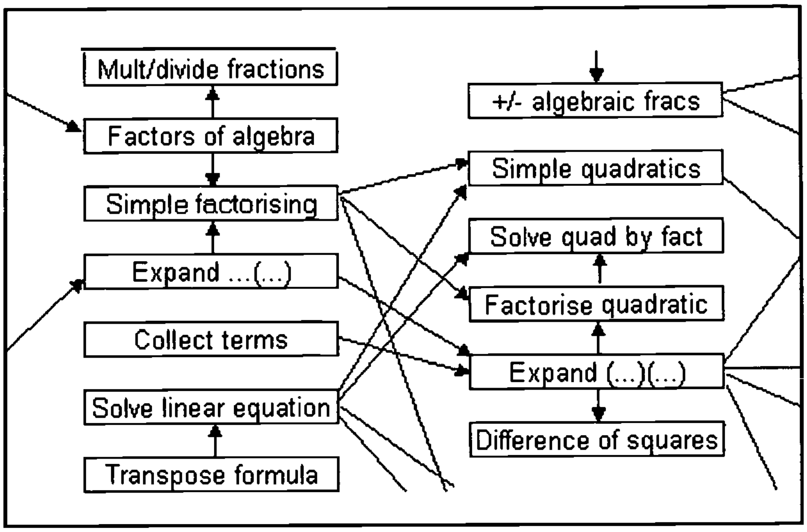

Figure 2. Part of the maths network

The algebra interface

DIAGNOSYS v1 had a very simple algebra input facility, in which the square of a variable could be entered by using the character 's'. *DIAGNOSYS* v2 used the up and down arrows to create indices, but had no special facility for fractions. Following an initial tutorial in how to use the interface, most students had little problem using this feature (Figure 3), where on-screen hints were also used. At a time when most packages were being written for Windows 3.1, we still used DOS as it made direct screen control for this kind of feature much more straightforward.

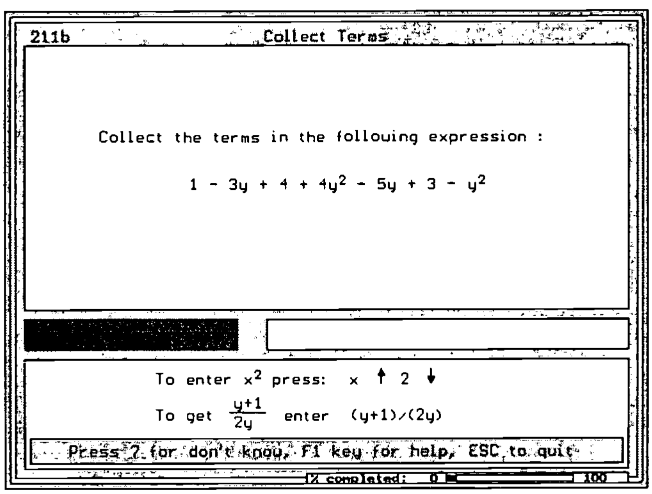

Figure 3. The algebra interface for *DIAGNOSYS* v2

Other features incorporated were the use of 'lives' (usually five) to permit second attempts at a limited number of questions where the student felt that they had simply made a slip, and 'free' re-entry of mathematically correct but inappropriate answers (e.g. unsimplified or incompletely factorised). The checking strategy is described below.

DIAGNOSYS v3, available from Autumn 1998, is a Windows 3.1 package with a fully interactive, intuitive algebra interface (Figure 4).

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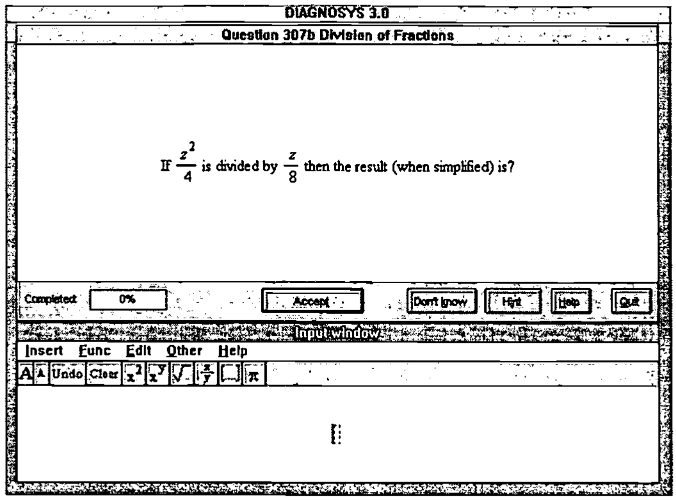

Figure 4. The interface for *DIAGNOSYS* v3

The input window resembles the Microsoft Equation Editor, but has an 'intelligent' output to the main program. Again, there is a preliminary tutorial in how to enter different types of answers (number, multiple-choice, algebra, etc.), and new students have little difficulty in most cases. The on-screen help is for the program and entry methods, not for the mathematics content. However, Figure 4 also shows a 'Hint' button, visible only in the Self-Test mode of v3.5, which *does* offer mathematical help. In this way the students can use *DIAGNOSYS* as a self-remediation package (choosing topic areas and starting level themselves in this case), perhaps doing the actual test again later.

Validity of the test - for individuals and groups

How is validity established?

There are several approaches that can be used to establish the validity of a new form of test, under two headings: self-consistency and comparison with other tests or information. The latter is examined in the next section.

Self-consistency

For the network to be not just valid, but also usefully discriminating, the success rate (including the inferences made) should decrease as we go from simpler to more advanced skills. For example, the Singapore group, scored 89% and 51% respectively for the expansion of single and double brackets.

Where more than one variant of a question is provided, the success rate on each variant should be similar. However, detailed examination of the actual answers given for a large data set (1218 students in Singapore) shows that this is hard to ensure. For example, the five (apparently similar) variants of expanding a double bracket are

$$(2+v)(3-v) \quad (r-4)(r+2) \quad (t-2)(t+4) \quad (1-s)(2+s) \quad (y+3)(y-5)$$

with respective success rates 42%, 73%, 69%, 38%, 75%, suggesting that this group of students has previously practised only on cases with positive quadratic term. Note that if random coefficients were used, it would be necessary to restrict the choice of coefficients as the sign and size of numbers also affects difficulty.

It is important to check the validity of the links in the network, since these are used to make inferences. A small experiment was done, in which part of the network was used without the links, so that we could examine whether the expected pre-requisites *were* essential. A high level of consistency was found.

Results for individuals and groups

Overall changes and predictability

Table 1 shows four years of results from First year and Foundation year students of Engineering at Newcastle. Comparisons can be made between years, between departments (not shown), and between diagnostic and exam results.

Table 1
Diagnostic and Exam Results for 1996/7 to 1999/0

Year	**1996/7**	**1997/8**	**1998/9**	**1999/0**
Test version	v2.4	v2.4	v3.0	v3.2 + 5 var'ts of qq
First year group				
Exam	2 hrs + ass	2 hrs + ass	2 hrs + ass	1 hr, ex only
Number with both marks	261	193	204	163
Diagnostic average	61	60	60	58
Exam average	47	50	54	52
R^2_{adj}	0.37	0.36	0.30	0.37
R^2 for five departments	0.99	0.42	0.19 (1)	—
Diagnostic time	68	58	44	55
Foundation Year group				
Number with both marks	41	33	37	28
Diagnostic average	44	48	44	44
Exam average	37	55	40	32
R^2_{adj}	0.47	0.19 (2)	0.33 (3)	0.31
Diagnostic time	59	70	61	62

The diagnostic averages for the four years show a slight decrease over time for essentially the same test each year, consistent with results obtained by Lawson (1997 and in Cambridge 2000). There is *no* sudden change in 1998/9, when the new version of the test was introduced, suggesting that the improved interface did *not* affect the students significantly, and also no significant change in 1999/0, when more question variants were introduced, even though their difficulty appeared to vary (see above). In contrast, the conventional examination, three months later, seemed to vary more in standard (though note that in 1999/0 a shorter exam with no included coursework was used). However, the *time* taken for the diagnostic test changed dramatically from year to year; whether because of what was allowed or what was needed is hard to say.

The correlation of diagnostic and exam results is fairly high in most cases, typically 0.6, which can be compared with the correlation between first and second maths exams of 0.77 for First year and 0.82 for one Foundation year, but also with 0.56 for a Foundation year exam and the assessed coursework.

However, for the individual student, the figure to compare is R^2 or, better, R^2_{adj}, which indicate the proportion of the variance in the second result predictable from the first result. For the First year groups, this figure is fairly constant around 0.36, but indicates that prediction for any individual from this figure alone may be very unreliable. For Foundation year groups, (notes 2, 3 in Table 1), the figure is more variable; it depends strongly on a few individual results for such small data sets. The figures of 0.19 and 0.33 become 0.35 and 0.42 if three and one 'odd' cases are changed respectively. Note, however, that since the main aim of the testing was to help the individual (or the group) *before* the first exam, one would hope that the group had levelled up to some extent if remediation (including self-remediation) was actually working. In this sense, accurate prediction is *undesirable*!

The test results are also compared with exam results for five departmental averages, taught in three lecture groups and tutored in about twenty smaller groups. In 1996/7, the correlation was almost perfect, indicating perfect prediction of cohorts of students! However, subsequent years gave much poorer results, probably because the test supervision was increasingly delegated and hence was more variable. The lowest figure of 0.19 (note 1 in Table 1) was probably attributable to one department allowing more time (taking 57 minutes on average instead of 44); if that diagnostic mark is decreased by just 2%, then R^2 =0.40, though the spread of *individual* marks is unaffected.

Table 2
Changes in First Year Diagnostic Results (bracketed figures are scaled to 100%)

Year	**1996/7**	**1997/8**	**1998/9**	**1999/0**
Test version	v2.4	v2.4	v3.0	v3.2, 5 vars
Number	141	125	171	118
Diagnostic time	69	59	42	52 (3)
Overall diagnostic mark	64 (100)	61 (95)	61 (95)	59 (94) (1)
Level 1 skills	99	96	99	97
Level 2 skills	95	91	94	92
Level 3 skills (start level)	66	62	63	61
Level 4 skills	29 (100)	25 (89)	21 (73)	22 (76) (2)
All algebra entry	—	66	66 (4)	—
All fractions entry	62	59	57	56 (5)

Changes in knowledge

Table 2 shows a more detailed breakdown of results for First year students entering with A-level Mathematics.

The overall results for this group show a slight decline over the four years, with no obvious evidence of a change as the new versions of the test were introduced (note 1 in Table 2), although the time taken for the test did vary significantly (note 3). However, the reduction in success was concentrated in the highest level skills (note 2), where significant declines were observed from 1996/7 to 1997/8 and again to 1998/9, of 11% and a further 14% respectively.

When individual question *types* were examined, there was again no evidence that the improved interface had made a significant difference to the students' success. The question types that should have been most affected were those requiring fractions input or algebra input, yet there was no observable change in success for either type (notes 4, 5).

Conclusions

The *DIAGNOSYS* test has been used for several years to test Engineering students at Newcastle, and also at several dozen other institutions including a large group in Singapore. It provides some rapid feedback to individual students on their weaknesses in background knowledge and skills, and helps staff to judge the content and delivery of the curriculum.

However, it is clear that the test can not be used alone as a reliable predictor of subsequent success for individual students, although it appears to correlate as well or better with exam results as does assessed coursework, and is often not much poorer in this respect than other exams. If the conditions under which the test is taken are uniform, there is probably greater reliability, particularly for whole group performance, and it may be that if it were used with more preparation, the unfamiliarity of the assessment mode, and the variable 'rustiness' of new students would become less significant.

Computer-based and paper-based diagnostic tests, because they may be unchanged from year to year, do offer a means of monitoring trends in knowledge of groups, including changes in detail in areas of knowledge. For our data, it was apparent that an overall slight decline in diagnostic performance was concentrated in higher level skills.

Although the interface used for student entry of mathematical answers is important, there was no evidence from this data that slight changes in methods used for algebra and fractions input, and changes to screen layout to improve readability, made a significant difference to overall success, though familiarisation was incorporated in both versions. The time taken by the students may have been more affected.

Remedial provision

For several years, one First year group and the Foundation year group had substantial remedial provision for students whose performance on the diagnostic test, coupled with known background qualifications, suggested they needed special help. For the First year group, their exam performance seemed to improve initially, but dropped back later once the remedial provision was completed (Table 3). Their overall ability was apparently similar to the group as a whole. In a climate where many departments are offering, or preparing to offer, such special courses to whole groups of students, it is important to evaluate such provision carefully (Edwards 1997).

Table 3
Success of Remedial Provision

Year	Number	1st exam	2nd exam	Overall
Remedial	13	52	43	54
Whole	95	52	51	53

Since the material tested is often well below the standard of the First year course itself, it is possible that the deficit is *not* simply of knowledge, nor necessarily of ability, but of *attitude*. A question worth asking is, "Why did these students not learn this material first time round?" If this is indeed the problem, then simply re-teaching the same material may have as little effect as these results seem to show.

References

Appleby, J. C., Anderson, A., & Fletcher, P. (1995a). Diagnosis and treatment of students from varied backgrounds. *IMA Conference: The Mathematical Education of Engineers, Loughborough, 1994.* (ISBN 0198511914)

Appleby, J. C., Anderson, A. *et al.* (1995b). Managing the transition to university education: Computer-based diagnostic testing of background knowledge. International Congress of Engineering Deans and Industry Leaders, Monash, 1995.

Appleby, J. C., & Anderson, A. (1997a). Diagnostic testing of mathematics and mechanics on entry *Mathematical Education of Engineers, Loughborough, IMA, 1997.*

Appleby, J. C., Samuels, P. C., & Treasure Jones, T. (1997b). *DIAGNOSYS* - a knowledge-based diagnostic test of basic mathematical skills. *Computers in Education*, 28, 113–131.

Beevers, C. (2000). Computer aided assessment in mathematics at Heriot-Watt University. *LTSN Maths & Stats Newsletter*, University of Birmingham, no.1, February 2000. (ISSN 0959-3950)

CALMAT - produced at Glasgow Caledonian University, information and availability from web address: http://www.maths.gcal.ac.uk/calmat/aboutcal.htm

Edwards, P. (1996). Implementing diagnostic testing for non-specialist mathematics courses. *Open Learning Foundation 1996,* (ISBN 1 86050 034 X).

Edwards, P. (1997). Just how effective is the mathematics diagnostic test and follow-up support combination? *Teaching Mathematics and its Applications*, 16, 3, 118–121.

Cambridge (2000). *Measuring the Mathematics Problem.* Cambridge seminar on diagnostic testing in mathematics, May 1999. Engineering Council: see www.engc.org.uk/gateway — Goldmine/Publications.

Greenhow, M. (2000). Setting objective tests in mathematics with QM Designer. LTSN Maths & Stats Newsletter, University of Birmingham, no.1, February 2000. (ISSN 0959-3950)

Hibberd, S. (1996). The mathematical assessment of students entering university engineering courses. *Studies in Educational Evaluation*, vol.22, no.4, 375–384.

Lawson, D. A. (1997). What can we expect of 'A' level mathematics students? *Teaching Mathematics and its Applications*, vol.16, no.4, 151–156.

Lim Khia Kiam & Tay Mui Hua (1999). Diagnosys - a diagnostic tool for the testing of basic mathematics. *Educational Research Associations of Malaysia & Singapore Conference*, November 1999.

LMS (1995). *Tackling the Mathematics Problem*. Joint report of LMS, IMA and RSS. LMS.

Availability of software and materials:

Ordering information, demo software and upgrades for *DIAGNOSYS* are available from:

http://www.staff.ncl.ac.uk/john.appleby/diagpage/diagindx.htm

GRAPHICS CALCULATORS IN MATHEMATICS LEARNING: STUDIES OF STUDENT AND TEACHER UNDERSTANDING. [1]

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This study examined how students and teachers interpret the graphical images produced on the screen of a graphics calculator. Clinical interviews were conducted with 25 Year 10 and Year 11 students as they used graphics calculators to study linear and parabolic graphs. 12 teachers were observed as they did more complex tasks at a workshop. Students had a poor understanding of scale and the decimal coordinates given when tracing, and often failed to recognise when the viewing window displayed an incomplete graph of a function. Both students and teachers showed very poor understanding of the procedures used by a graphics calculator to calculate and display the graph of a function.

Graphics calculators (GCs) were first developed in the mid-1980's and since then they have become steadily cheaper, more user-friendly, and more powerful. Most of the research on GCs reported in the educational literature consists of studies comparing the test results of classes that were taught the same topics using either GCs or a "traditional" approach. A few studies have also interviewed students about their difficulties, but only after instruction has been completed. In general, these methodologies tend to be superficial, not identifying the causes of changes in performance or understanding with any certainty (Dunham & Dick, 1994). Consequently, there is a need to identify those aspects of GC use which best facilitate student learning (Penglase & Arnold, 1996).

Background

The main focus of precalculus graphing concerns the graphs of linear and quadratic functions. Students learn to predict the shape of a graph from its equation (straight line, concave-up parabola, concave-down parabola) and come to recognise the important features of such graphs (intercepts, the gradient of a line, the vertex of a parabola). When drawing these graphs by hand, examples need to be carefully chosen so that distractions are minimised. Scales are almost always symmetric and marked in unit intervals, and functions whose main features lie close to the origin are commonly used. In addition, the coordinates of intercepts and vertices are usually integer or near integer values.

There have been a number of studies which incorporate the use of GCs in precalculus mathematics (e.g., Asp, Dowsey, & Stacey, 1993). Many of these studies argue that one of the main advantages of using the technology is that it allows multiple graphs to be displayed quickly thereby permitting students greater freedom to explore using many more examples than they could investigate if graphs were drawn by hand (Demana, Schoen, & Waits, 1993). However, these investigations inevitably lead to more complicated graphs where students have to deal with unsymmetrical scales, blank screens or partial views, and non-integer coordinates. Thus there is a real danger that the whole process might appear somewhat arbitrary or magical (Dion & Fetta, 1993) and that fundamental misconceptions might arise (Mueller & Foster, 1999).

A small number of studies have noted student misconceptions associated with the use of GCs. Goldenberg and Kliman (1988) identified students' poor understanding of scale as a major source of misconceptions. Tuska (1993) analysed the responses of first year college students on multiple choice examinations and identified misconceptions in students' understanding of the domain of a function, asymptotes, the solution of inequalities, and the belief that every number is rational. Williams (1993) lists student difficulties relating to the low resolution of the GC screen. Steele (1993) noted the ready acceptance by students of the initial graph shown in the default window of the GC. Vonder Embse and Engebretsen (1996, p. 508) describe a situation where a pair of perpendicular linear graphs do not appear at right-angles on the GC screen because of the unequal scales on the axes. The present study was designed to investigate such misconceptions in detail.

Method

Design. As part of an on-going study, both students and teachers were observed as they undertook GC tasks designed to challenge their understanding.

Student sample. Clinical interviews were conducted with 25 students, 5 students from each of 5 Sydney high schools (15 students were in Year 10: 8 girls and 7 boys; and 10 students were in Year 11: 5 girls and 5 boys). The students were drawn from the higher ability classes in each participating school because we felt that they might be better able to respond to the challenge of the interview tasks and articulate their ideas. It is assumed that the misconceptions of these students would also be found in those of lesser ability.

All the students had studied the graphs of straight lines and the gradient-intercept equation $y = mx + b$. They had sketched parabolas given in general form and had used the quadratic formula to locate the intercepts and vertex of a parabolic graph. The students had used a GC, the Casio *fx*-7400G, in their mathematics lessons for six to twelve months prior to the first interview. The majority of students had limited access to a class set of GCs. In one school, the students owned their own *fx*-7400G which they were expected to use in all lessons and examinations. In general, the students were novice users of GCs who had only used them to display straight lines and parabolas.

Each student was interviewed individually by the first author for fifty minutes on three separate occasions, each approximately two weeks apart. The students completed graphing tasks using the Casio *fx*-7400G and were asked to interpret the output provided by the calculator and explain their thought processes. All of the interviews were videotaped and selected segments were later transcribed. In particular, student responses were studied with the aim of understanding their conceptual and technical difficulties as they operated the GC.

Student tasks. Some tasks used in the student interviews are described here.

1. Draw a sketch of $y = 0.1x^2 + 2x - 4$. You may use the graphics calculator to help you.

The initial image produced by the GC is shown in Figure 1. Task 1 tested whether the students could use the symbolic form of the function to recognise that the calculator did not display a complete graph of the parabola, and if they would zoom-out to find it.

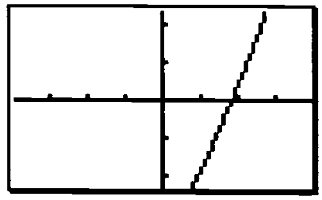

Figure 1. The graph of $y = 0.1x^2 + 2x - 4$ in the initial window.

2. Display the graph of $y = 0.75x^2 - 1.455x - 1$ on the graphics calculator. Find the intercept with the positive x-axis and the coordinates of the vertex.

Figure 2 shows the parabola. The roots of the quadratic function chosen for Task 2 were irrational to see how the students would interpret the decimal coordinates displayed on the GC. The graph also appeared with a flat line near the vertex rather than a single lowest pixel to investigate how the students would deal with that situation.

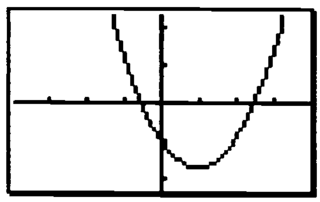

Figure 2. The graph of $y = 0.75x^2 - 1.455x - 1$

3. Explain why the graphs of $y = 2x + 3$ and $y = -0.5x - 2.5$ do not appear at right-angles on the screen. What could you do to make the lines look more perpendicular?

Figure 3 shows the screen for Task 3 with the x and y-axes displayed from -10 to 10. Students do not generally have much experience with axes which are not scaled symmetrically and so Task 3 was used to investigate how the students would deal with such a situation and discuss the students' concept of scale.

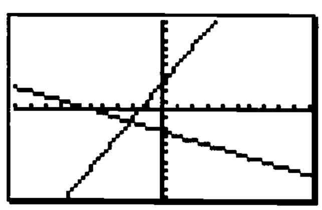

Figure 3. The graphs of $y = 2x + 3$ and $y = -0.5x - 2.5$

Teacher sample. Observations were made of 12 mathematics teachers (2 from each of 6 Sydney high schools) during a two-day workshop. The teachers had no previous knowledge of GCs but wished to implement them in their classrooms the following term. On the first day, the teachers were first instructed in the basic operations of the Casio *fx*-7400G. They then worked in small groups to complete a number of graphing tasks similar to those previously given to the student sample but presented at a higher level of mathematical sophistication. The second day was devoted to a discussion of student and teacher misconceptions and the planning of GC lessons.

Teacher tasks. Some of the tasks used in the teacher workshop are described here.

4. Display a graph of $y=\frac{e^x}{x}$ in the initial window. Is this a reasonable graph of the function? Can you explain why the graphics calculator displays the graph in this way?

The graph of $y=\frac{e^x}{x}$ in the initial window is shown in Figure 4. This task tested whether the teachers could recognise that the calculator did not display a complete graph of the exponential function. This task also investigated how the teachers would interpret the apparent illusion of the graph appearing to end at approximately $x=-2.2$

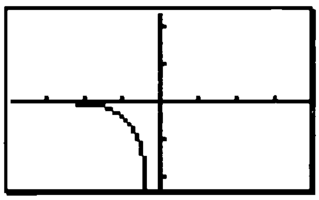

Figure 4. The graph of $y=\frac{e^x}{x}$ in the initial viewing window.

5. Display a graph of $y=1.5\sqrt{2-x^2}$ in the initial window. Is this a reasonable graph of the function? Can you explain why the graphics calculator displays the graph in this way?

Figure 5 shows the graph of the half-ellipse as it appeared in the initial window. The purpose of this task was to investigate how the teachers would interpret the fact that the graph did not meet the x-axis in this window.

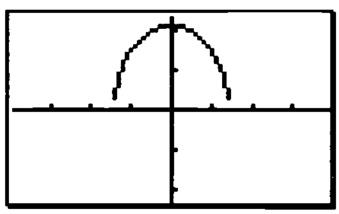

Figure 5. The graph of $y=1.5\sqrt{2-x^2}$ in the initial viewing window.

6. Display a graph of $y=\sin(60x)$ in the initial window. Is this a reasonable graph of the function? Can you explain why the graphics calculator displays the graph in this way?

Figure 6 shows the graph of $y=\sin(60x)$ in the initial window. The function was carefully chosen so that it appeared inverted and with a much larger period than would be expected from the formula. The graph of this function also demonstrated some of the issues concerning the ways the GC uses highlighted pixels to display graphs.

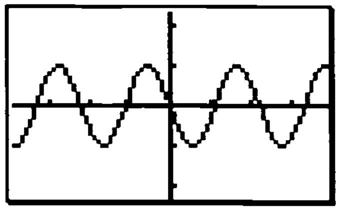

Figure 6. The graph of $y = \sin(60x)$ in the initial viewing window.

Results

Task 1. Seven students (28%) sketched a parabola; all commented that the x^2 term in the function indicated that the graph must be a parabola and then zoomed-out until they saw the U-shaped curve. The remaining 18 students (72%) drew a straight line as their sketch of the quadratic function. Of these, 5 students (20%) simply copied the straight line directly from the calculator screen; the other 13 students either zoomed-out ($n = 6$) or referred to the constant term in the function ($n = 7$) to find the y-intercept.

Task 2. 19 students (76%) averaged the x-coordinates of the pixels immediately above and below the x-axis to obtain an estimate for the intercept, despite the fact that the y-coordinate displayed for one pixel was considerably closer to zero than the other. In a similar way, 21 students (84%) explained that the x-coordinate at the vertex of the parabola could be found by averaging the x-values of the two centre pixels in the row at the base of the graph in Figure 2. They did not seem to realise that the y-coordinate of this point was greater than the y-coordinate of the lower of the two centre pixels. Only 3 students (12%) correctly stated that the zeros of the function were irrational and that the x-intercept was thus a non-terminating decimal, which could not be found exactly. One of these students also explained that the x-coordinate of the vertex of the parabola was a rational number, which could be expressed by a terminating decimal.

Task 3. Only 4 students (16%) explained that the angle between the lines shown in Figure 3 was due to the unequal scaling of the coordinate axes without being prompted. The remaining 21 students (84%) eventually recognised that the axes were not scaled equally and that this explained why the lines did not appear perpendicular.

Tasks 4-6. At the time of writing, it has only been possible to make a preliminary analysis of the teachers' responses to these tasks. In general, the teachers were better able to identify the mathematical inconsistencies in the graphs produced by the GC, which is not surprising given their broader knowledge and experience. For instance, they had no difficulty in recognising that the graph shown in Figure 4 was incomplete or that the graph in Figure 6 was incorrect. However, when an understanding of the basic operation of the GC was required, the teachers had difficulties. For instance, although they recognised that the graph shown in Figure 5 should meet the x-axis, none of the teachers could correctly identify that the values given to the columns of pixels in the viewing window was the source of the problem, nor could they explain how the GC assigned such values. Similarly, many of the teachers initially failed to realise that the exponential curve in Figure 4 was superimposed on the x-axis below $x = -2.2$ and did, in fact, continue to the very left-hand edge of the viewing window. Furthermore, none of the teachers could understand how the GC had produced the graph in Figure 6.

Discussion

The students' responses to Tasks 1-3 suggest four areas of difficulty in using GCs.

Scale. A proper understanding of scale is critical to interpreting graphs correctly (Goldenberg & Kliman, 1988) yet the scale concept was poorly understood by all of the students who were interviewed. The students exhibited a strong preference for symmetric scales and found graphs like those in Figure 3 difficult to interpret. They showed little awareness of the impact that the scale of the viewing window could have on the graphs displayed within it (Dunham & Osborne, 1991; Hodges & Kissane, 1994).

An important feature of the students' ability to comprehend scale was the distinction between what might be called a *relative* or *absolute* understanding of this concept. The former correctly regards scale as a ratio of distance to value, while the latter interprets scale solely as either the measure of the distance between adjacent markings on the axes or their value. It appeared that the vast majority of students had only developed an absolute understanding of scale. One reason seemed to be that they had only experienced graphs and scale drawings where the horizontal and vertical scales are enlarged by the same factor.

Approximations. Dick (1992) argues that students will need to acquire new kinds of numerical skills if they are to use GCs effectively as problem solving tools. The results of this study certainly support that view. The students who were interviewed often had great difficulty in making appropriate numerical estimates for the values they were looking for. The most common approach used by the students was to average the coordinate values on either side of a point of interest regardless of whether the actual value was closer to one side or the other. Students also regularly made a direct correlation between the greater number of significant figures given in a decimal and its accuracy. Somewhat paradoxically, however, the students showed a marked preference for integer or other "nice" values when locating intersection points or intercepts.

Students were aware of the pixel approximations due to the low resolution of the screen. For example, most students could explain why the calculator displayed the parabola shown in Figure 2 with a horizontal line of pixels near its vertex. However, the numerical approximations of the coordinates were harder for the students to recognise because they regarded the coordinates as exact values. This was particularly so as the number of significant figures increased. In general, the students thought that all points on a graph must always be expressed by finite decimal values because they represented exact distances from the origin on the number plane. Few students were aware of the differences in the decimal representations of rational and irrational numbers.

The strength of the visual image. As Smart (1995) has noted, visual images seem to have a much greater influence on the students' thinking than the corresponding algebraic and numerical representations of the functions they encountered. We too found this in our discussions with students during the interviews. It was most clearly shown in Task 1, where the specific appearance of the GC screen easily overpowered the students' general knowledge of quadratic functions and their graphs. The students tended to accept whatever was displayed in the initial window without question and did not relate the graphs they saw to the algebraic representation of the function.

How a graph is calculated and displayed. We have already commented on the errors teachers made because they did not understand how a GC calculates and displays the graph of a function. Students made many errors for a similar reason. For example, when zooming to search for the x-intercept of the parabola in Task 2, all of the students eventually reached a stage where the cursor was on the x-axis but the value of the y-coordinate was not zero. Although the students recognised the discrepancy, they based their answers almost exclusively on the visual appearance of the pixels.

For a more detailed discussion of the students' conceptual and technical difficulties, see Cavanagh and Mitchelmore (2000a, 2000b).

Implications and conclusions

The results of this study indicate that a stronger curricular emphasis on scale may be required. Students need more experience in controlling scale and watching for its effects on graphs, and greater use of asymmetric scales should also be encouraged. Number theory, particularly relating irrational numbers to their decimal approximations, may need more attention as well. Students also need to be aware of how the GC displays graphs and this will only happen if teachers themselves understand the technology better. The initial findings from the teacher workshop seem to suggest that if teachers are to make the best use of the GC then they will need fairly detailed instruction on how it operates to display graphs, and strategies for instructing students in this as well.

Given the misconceptions we have identified, the question arises: Should mathematics teachers attempt to structure the examples students work so that they avoid such difficulties? Or should they deliberately plan activities, which force students to face them? We believe that teachers should encourage students to confront the limitations of the GC because doing so would strengthen their basic understanding of the GC itself, encourage a more critical attitude to its output and a more discriminating use of the GC (Kissane & Kemp, 1999), but could also lead to so much interesting mathematics. Once they did begin to see how the GC operated to produce graphs, the teachers expressed more confidence in using the technology and a willingness to try examples which confronted its limitations with their students. It will be interesting to see whether they, in fact, do so.

[1] The project is supported by a Strategic Partnerships with Industry—Research and Training (SPIRT) grant from the Australian Research Council. The industry partner is Shriro Australia Pty Ltd (distributors of Casio calculators).

References

Asp, G., Dowsey, J., & Stacey, K. (1993). Linear and quadratic graphs with the aid of technology. In B. Atweh, C. Kanes & M. Carss (Eds.), *Contexts in mathematics education, Proceedings of the 16th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 51–56).

Brisbane: Queensland University of Technology.

Cavanagh, M., & Mitchelmore, M. C. (2000a). Student misconceptions in interpreting basic graphic calculator displays. *Proceedings of the 24th annual conference of the International Group for the Psychology of Mathematics*, Hiroshima, Japan, *2*, 161–168.

Cavanagh, M., & Mitchelmore, M. C. (2000b). Students' technical difficulties in operating a graphics calculator. Paper submitted for presentation at the 23rd annual conference of the Mathematics Education Research Group of Australasia, Perth, Australia.

Demana, F., Schoen, H. L., & Waits, B. (1993). Graphing in the K–12 curriculum: The impact of the graphing calculator. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 11–40). Hillsdale, NJ: Lawrence Erlbaum Associates.

Dick, T. (1992). Super calculators: Implications for the calculus curriculum, instruction, and assessment. In J. T. Fey & C. R. Hirsch (Eds.), Calculators in mathematics education: 1992 yearbook (pp. 145–157). Reston, VA: National Council of Teachers of Mathematics.

Dion, G. S., & Fetta, I. B. (1993). Calculator graphs: Magic? or mathematics? *Mathematics and Computer Education, 27(3),* 180–197.

Dunham, P., & Dick, T. (1994). Research on graphing calculators. *The Mathematics Teacher, 87(6),* 440–445.

Dunham, P. H., & Osborne, A. (1991). Learning how to see: Students' graphing difficulties. *Focus on Learning Problems in Mathematics, 13(4),* 35–49.

Goldenberg, E. P., & Kliman, M. (1988). Metaphors for understanding graphs: What you see is what you see. (Report No. ETC–TR–88–22). Cambridge, MA: Harvard Graduate School of Education, Educational Technology Center. (ERIC Document Reproduction Service No. ED 303 369).

Hodges, A., & Kissane, B. (1994). Learning about functions and graphs using a graphics calculator. In T. Herrington (Ed.), *New horizons, new challenges* (pp. 189–201). Adelaide: Australian Association of Mathematics Teachers.

Kissane, B., & Kemp, M. (1999). Graphics calculators and lower secondary mathematics. *The Australian Mathematics Teacher, 55(4),* 4–10.

Mueller, U., & Forster, P. (1999). Graphics calculators in the public examination of calculus: Misuses and misconceptions. In J. M. Truran & K. M. Truran (Eds.), *Making the difference, Proceedings of the 22nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 396–403). Adelaide: MERGA.

Penglase, M., & Arnold, S. (1996). The graphics calculator in mathematics education: A critical review of recent research. *Mathematics Education Research Journal, 8(1),* 58–91.

Ruthven, K. (1990). The influence of graphic calculator use on translation from graphic to symbolic forms. *Educational Studies in Mathematics, 21,* 431–450.

Smart, T. (1995). Visualising quadratic functions: A study of thirteen–year–old girls learning mathematics with graphic calculators. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th International Conference for the Psychology of Mathematics Education* (pp. 272–279). Recife.

Steele, D. (1993). The graphics calculator: Friend or foe? In J. Mousley & M. Rice (Eds.), *Mathematics: Of primary importance* (pp. 50–56). Melbourne: Mathematical Association of Victoria.

Tuska, A. (1993). Graphing calculators and misconceptions of precalculus students. In L. Lum (Ed.), *Proceedings of the Fourth Annual Conference on Technology in Collegiate Mathematics* (pp. 428–432). Reading, MA: Addison–Wesley.

Vonder Embse, C., & Engebretsen, A. (1996). Friendly windows for graphing calculators. *The Mathematics Teacher, 89(6),* 508–511.

Williams, C. G. (1993). Looking over their shoulders: Some difficulties students have with graphing calculators. *Mathematics and Computer Education, 27(3),* 198–202.

YEAR 8 STUDENTS' PROGRESSION ON AN INTEGRATED LEARNING SYSTEM

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This paper explores seven Year 8 students' progress on the core mathematics course of an integrated learning system (ILS) where individual students are presented with electronic worksheets in random form. It shows the students' frustrations, confusion and stress with the inflexible marking system of the ILS, the repetitive formats of the ILS worksheets and the unusual setting-out procedures in some worksheets. It presents a model that relates student, teacher, classroom and operational characteristics to performance and progression on the ILS.

One of the by-products of the growth of information technology in education has been the computer-based integrated learning system (ILS). These systems have substantial course content, aggregated learner record systems, and a management system which "will update student records, interpret learner responses to the task in hand and provide performance feedback to the learner and teacher" (Underwood, Cavendish, Dowling, Fogelman, & Lawson, 1996, p. 33). They can be marketed to schools as the answer to students' numeracy difficulties and teachers' reporting requirements.

This paper reports on a study in which students used the core mathematics course of an ILS. The course was a closed system, that is, the curriculum content and the learning sequences were not designed to be changed or added to by either the tutor or the learner (Underwood et al., 1996). It was endorsed by the manufacturers only as a tool for teachers to use to consolidate already introduced material and to diagnose student difficulties with this material. It was divided into a range of *strands* (e.g., numeration, addition, multiplication, fractions, measurement) which are then sub-divided into collections of tasks that are sequenced in terms of performance at different *levels*.

The mathematics tasks were in the form of individual, timed, and randomly-presented electronic worksheets. This allowed the ILS to place the students at their mastery level (70%) and to automatically raise them to the next level when they achieved high mastery (85%) which equated to 60% mastery at the next level. The tasks were generally attractive in their presentation and sometimes creative in the way they probed understanding. They provided 2-D representations of appropriate teaching materials in mathematics (e.g., base-10 blocks, place value charts, fraction and decimal diagrams) and online student resources (e.g., help, tutorial, toolbox, audio and mathematics reference). However, some tasks had novel presentations and solution formats (e.g., some algorithms required the ones to be typed first), other tasks had strict syntax and setting-out requirements (0.63 correct, 0.63 incorrect), and accessing help or tutorial automatically graded performance as incorrect.

The ILS management of the core mathematics course does not reflect current theories with respect to effective mathematics teaching and computer use. Students are passive and do not construct knowledge or investigate (Wiburg, 1995) their initiative and autonomy are removed and they do not receive activities in sequence or work in groups (Bottino & Furinghetti, 1996; Sivin-Kachala, Bialo, & Langford, 1997). As well, the closed nature of the ILS tends to marginalise the teachers' role (Bottino & Furinghetti, 1996). There is also a tendency for tasks to be closed, based on speed, and to have insufficient variety to prevent repetition.

There is very little evidence that the ILS core mathematics course improves student learning (Becker, 1992). In fact, there is some evidence that it is possible to progress on the ILS without knowledge (Baturo, Cooper & McRobbie, 1999a) in a similar manner to how the Individually Prescribed Instructional (IPI) packages of the 70s allowed progression without facilitating mathematics learning (Erlwanger, 1975).

Context of the study

This study is part of the second stage of a three-stage evaluation of the ILS. Stage 1 focused on Years 1 to 10 classes as cases to identify factors that influenced teacher endorsement (see Baturo, Cooper, Kidman & McRobbie, in press), and to measure student achievement and affect in relation to progress on the ILS (see McRobbie, Baturo, & Cooper, submitted). Figure 1 provides a summary of the findings with regard to the factors that impacted on teachers' endorsement of the ILS.

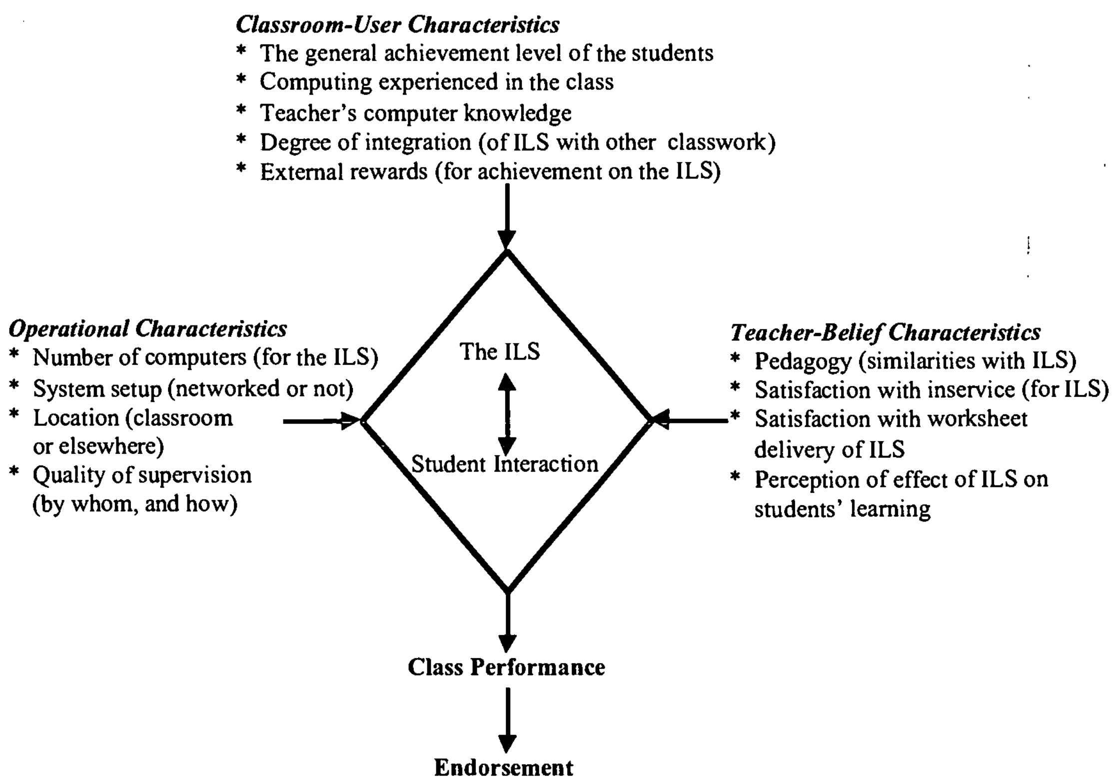

Figure 1. Factors that influence endorsement of the ILS (Baturo, Cooper & McRobbie, 1999b, modified)

Stage 2 focused in-depth on 14 individual Years 6 and 8 students who had never worked on the ILS to determine whether initial ILS placement was low in relation to mathematics knowledge and whether ILS progress reflected syntactic or semantic mathematics knowledge. (Resnick et al., 1989). The ILS was restricted in the mathematics topics it presented to allow in-depth diagnostic interviews. Stage 3 repeated Stage 2 but with a Year 6 and a Year 8 class that had been identified by the ILS distributors as the most successful users of their product in the Brisbane metropolitan area and with no restrictions on mathematics topics.

Method

An interpretative research methodology involving the use of persistent detailed observations and regular individual diagnostic interviews was used in this study to investigate seven Year 8 students' acquisition of knowledge from the core mathematics course of the ILS.

Subjects. The subjects were seven students who were selected from 15 volunteers in a Year 8 mathematics class in a Brisbane middle-class secondary school. The school and teacher volunteered for the project and neither the teacher nor the students had used the ILS before. The seven students were selected on the basis of their responses to a diagnostic mathematics test that comprised three sections relating to structural, representational and procedural knowledge (Baturo, 1998). Two students were low achieving, two medium, two high and one very high. The students were withdrawn from their classroom and rotated through three 15-minute sessions per week on computers. Because of the study's mathematics restrictions, there was no integration of the ILS mathematics with normal classroom mathematics.

Instruments. A pre- and post-test focusing on two crucial knowledge structures that need to be established in the middle years, namely *fractions* (common and decimal) and *multiplicativity* (× and ÷ operations, and area), was developed and administered. A pre- and post-questionnaire was also developed and administered to determine the students' affects and beliefs with respect to numeracy, to computers in general, and to the ILS in particular. Two semi-structured individual interviews were undertaken _ one at the end of the ILS placement period and one at the end of the trial. The first interview focused on the students' observed behaviours as they became familiar with the ILS environment; the final interview focused on the students' understanding of the concepts and processes they had "learnt" during the computer sessions, and their feelings about learning in this environment. ILS reports provided records of each student's time on the computer and gains in years as measured by the ILS.

Procedure. A laboratory of 4 computers was established in an adjacent classroom. The ILS was programmed to provide only those tasks that related to fractions (common and decimal), multiplication and division operations, and area measurement. Each student was enrolled in the ILS course at level 7.00. It was decided to enrol the students below their school year level as students in Stage 1 of the larger study had experienced early difficulties with either the ILS itself or the maths. The students were rostered for 3 ILS sessions per week during their mathematics lesson time. Video snapshots were taken at regular intervals of their ILS sessions. Students were withdrawn from the classroom and interviewed individually. The interviews were audio-taped. Notes were taken on a specially prepared response sheet.

Analysis. The video-tapes were viewed and instances of students' dilemmas as they got to know the ILS procedures were noted. These formed the basis of the items of the first interview, resulting in idiosyncratic first interviews contingent on the observed video behaviours. The observations and the interview results were translated into protocols and behaviours categorised. The questionnaire results were coded and these results combined with the test results and ILS reports from the management system to provide a profile of each student.

Results

The results for the diagnostic test and progress on the ILS (averaged across fractions, decimals, multiplication, division and area) are presented in Table 1. The highest test score (from a possible 185) was 165, while the lowest was 44. The results for the test were supported by comments about the students' abilities by their class teachers. Each student's progress on the ILS was managed by the ILS management system, which interpreted the student's response to each exercise, and provided performance feedback. The feedback was accessed at the conclusion of every third session.

Table 1
Students' Diagnostic Test Means (%) and Progress on the ILS

	Test results		Progress on the ILS in week					
Student	%	Rating	IP	3	6	9	12	15
Dan	89.2	VH	7.00	7.03	6.49	6.59	6.18	6.54
Sally	81.6	H	7.00	6.46	6.03	5.05	4.62	4.27
Louie	76.8	H	7.00	5.96	5.51	5.36	5.44	5.64
Leon	59.5	M	7.00	6.46	5.42	4.47	4.58	4.65
Suzie	59.5	M	7.00	6.46	5.49	5.52	5.03	4.91
Barby	31.9	L	7.00	5.94	3.01	3.32	3.34	3.69
Trent	23.8	L	7.00	5.94	4.92	4.48	4.58	4.69

Note. Rating = VH - very high, H - high, M - medium, and L - low achieving; IP = Initial Placement.

As is evident from Table 1, all students' performances decreased from the initial placement. However, this decrease was not uniform across the study– for example, Sally's performance steadily decreased, Louie's performance initially decreased and then began to steadily increase, whilst other performances showed erratic increases and decreases (e.g., Dan). Performance on the ILS appeared not to be related to mathematics knowledge. For example, Sally (81.6% test mean) finished on a lower ILS level than Trent (23.8% mean).

None of the students equated mathematics with work on computers; most equated computers with playing games or learning to type. Hence, using computers for learning mathematics was a novel activity for the students, and it was made more novel by some of the unique presentation and solution formats in the ILS. Each student dealt with these novel situations differently.

Dan, the student who scored highest on the test, appeared very able and mature. He was the only student of the seven who passed the Year 8 mathematics unit in classwork. Although he generally did not find the novel situations bothersome, he nevertheless lost motivation for using the ILS mathematics course. **Sally** welcomed the computer because she had limited use of her hands and found writing difficult. Her typing skills improved quickly and she learnt to do limited mouse functions. However, she became very frustrated with the forced setting out in most algorithms. Her correct answers were commonly marked incorrect due to not strictly following protocols or to errors in typing. She lost motivation and her performance deteriorated. **Louie** had a truancy problem and enjoyed the computer sessions because he was not in class. He strongly believed that the ILS would teach him more than his mathematics teacher would. He was very attentive to any explanations given by the ILS. His performance did not deteriorate to the extent that other students' performances did.

Leon was initially cooperative, but after 6 sessions he became bored and disruptive. He would allow the ILS to "time-out" (exceed allowable time). When he realised an exercise was a repeat, he simply used the Enter key until a new exercise was displayed. He complained the ILS was giving him "stuff that was too easy". Leon would ignore some exercises entirely. He said, "It was the font sizes fault, 'cause it keeps changing". **Suzie** was competent at long division, yet the forced setting-out procedure required by the ILS was different to the format she was accustomed to. This was confusing for her, resulting in errors that frustrated her.

Barby also became very frustrated with the ILS. She made many typing errors which the ILS interpreted as incorrect responses. On a number of occasions, she received the same exercise 3 times in succession. Barby was confused when decimal fractions needed to be recorded. For example, the ILS would deliver an exercise, she would respond appropriately (say, with .63) but would be assessed as being incorrect. A second attempt (0.63) would be assessed as being correct. This behaviour exemplifies the pedantic nature of the ILS. Moreover, Barby was a very impatient student – she wanted the computer to react instantly with either a correct or incorrect mark. The computer however, wanted to provide tutorials for Barby, and this infuriated her.

Similar frustrations were experienced by **Trent**. He found having to interact with the computer quite stressful and his behaviour changed over the semester from quiet and helpful to disruptive. This change in behaviour was mainly caused by the ILS's delivery of a full explanation of a concept, even though Trent had answered the exercise correctly. This confused Trent. At no time did it occur to him that the ILS may have a fault, he thought there was a problem with his mathematics. He quickly learnt to madly click the "done" button to get answers to avoid doing concepts he disliked (e.g., division). Another cause of huge frustration for Trent was the percentage correct provided by the ILS at the end of a session. Trent impatiently awaited the end of each session to see "how he went". However, he had no understanding of percentages so that he was often puzzled, and ultimately angry, by the percentage correct given, saying, "I should get 99% if I get 1 wrong, and 98% if I get 2 wrong". Trent was only completing on average 30 exercises per session so he couldn't understand why the percentage chart at the end of his session was sometimes different.

After 3 months, all seven students disliked the ILS intensely. Their frustration, confusion and stress were obvious, and the students were glad the study was over. They came not to enjoy being in the study; so much so that when it was their roster time, they would cringe and try to make excuses as to why they could not do their ILS session for that day. The net result was a negative attitude towards the ILS. This was highly evident for Dan, who didn't even attend the "thank you" party (lunch from a well-known fast food outlet) given for the participating students.

Discussion

The ILS in this study controlled what activities were presented to each student, the order in which they were presented, the teaching provided (whether required or not), what procedures should be used, and how answers were to be recorded. Thus, the students were placed in a totally passive role (Wiburg, 1995) with little chance to use initiative and autonomy (Bottino & Furinghetti, 1996; Sivin-Kachala, Bialo, & Langford, 1997). The students had to sit through what the ILS provided regardless of their knowledge and the worth of what the ILS was providing to them. Any use of initiative often resulted in their response to the task being marked as incorrect. This passive role caused frustration and stress, particularly when there was too much repetition. In one situation, the ILS repeated the same exercise three times in succession; in another situation, a tutorial on how to get the answer had still to be sat through although the correct answers was given.

The programming limitations of the ILS meant that there was repetition in worksheet formats and inflexibility in solution formats. This caused confusion, particularly when a correct answer was assessed as incorrect. There was a particular difficulty in recognising whether to include or omit the '0' in the ones place for decimal fractions between 0 and 1, (eg. 1/2 or 0.5 etc.). For example, the ILS marked the solution to one example incorrect when the 0 was omitted and another solution incorrect when the 0 was included. Another difficulty was determining whether to give solutions as common or decimal fractions. Overall, the forced setting-out of the ILS, especially in the division and multiplication strands caused confusion. In some solutions the ones had to be typed first. Students were really annoyed when they had the correct answer but were marked incorrect because they typed it in wrongly.

Some difficulties were the result of poor presentation. Some students found constantly changing font style and font size a problem. Other students were unimpressed with the graphical inaccuracies and technical glitches that regularly confronted them. Some drawings were open to other interpretations than the one the ILS marked correct. The computers often froze for a few minutes and students had to wait until they began again or were forced to reboot. Often, students were timed out and marked incorrect during these periods, which made the experience of the glitches even worse.

The inflexibility and lack of autonomy extended to the ILS timetable. The ILS recommended a minimum of three 15-minute sessions per week for each student. Achieving this was sometimes stressful to students. Although they were not rostered on to the ILS during art and sport (most students' favourite school subjects) like many students in other schools were (Baturo et al., in press), they still disliked missing part of three of their regularly scheduled mathematics lessons each week.

The inflexibility of the ILS affected different students differently. Less able students tended to be deterred by inconsistencies such as changes in font; more able students tended to be bored by too much repetition. Sometimes brighter students coped well, continuing to concentrate on answering the tasks while dismissing poor or inconsistent presentation as unnecessary and "slack" on the manufacturers part. Sometimes less able students did well, appreciating the repeating format. However, all students disliked being marked incorrect for not giving the required presentation and they hated having their typing expertise used as a determinant for their mathematics abilities.

Implications for the model

Figure 1 provided a model that showed the interactions between characteristics and the ILS that lead to teacher endorsement or non-endorsement. It showed that the success of the ILS depends on *operational*, *classroom-user* and *teacher-belief* characteristics. Many of these characteristics were negative in this study. There was little proactive supervision of the ILS sessions (the teacher taught the remainder of the class and the researchers restricted their interaction with the students to observing responses and managing behaviour). There was no integration of the ILS with other class work in mathematics and no systematic provision of external rewards (e.g., a chocolate bar for a certain time of committed interaction with the ILS). There was also a large difference between the practice worksheet activities of the ILS and the creative experiences most students had with computers when they were not in an ILS session. As Baturo, Cooper, Kidman & McRobbie (in press) argued, low supervision, integration and rewards and available creative alternatives are associated with lack of success of the ILS.

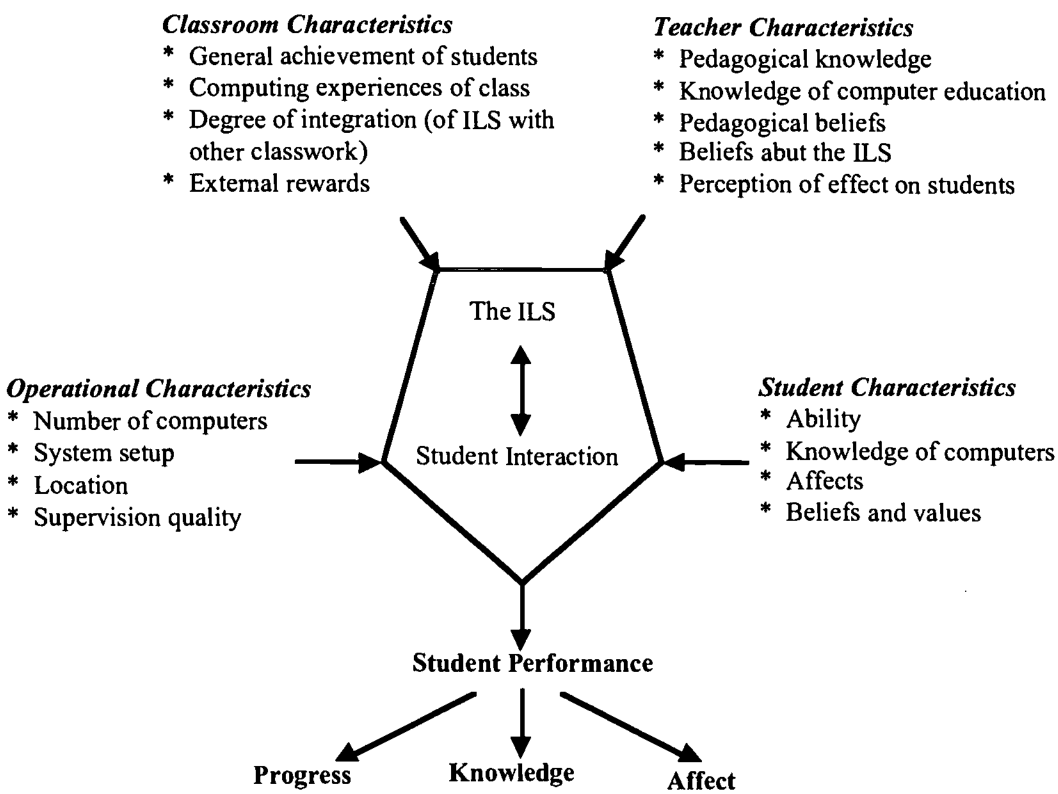

Figure 2. Factors influencing the students' performance on an ILS

It is possible for strengths in one characteristic of the Figure 1 model to overcome deficiencies in another. This is particularly the case for supervision; strong supervision means that the students receive assistance with the idiosyncrasies of the ILS in the beginning sessions, and encouragement to continue when unfamiliarity results in an incorrect mark for a correct solution. It is also possible for one characteristic to negatively affect other characteristics. A particularly important characteristic is teachers' pedagogic beliefs; if the teachers are negative towards the ILS and/or unfamiliar with the ILS and therefore unwilling or unable to overcome the deficiencies of some of its programming, the students became dissatisfied.

Unlike Stage 1 that focused on classes, Stage 2 focused on individual students. At this level of analysis, it is possible that student characteristic may make up for deficiencies elsewhere. Therefore, to provide the detail required when analysis comes down to the student, the model needs to be extended to that in Figure 2.

Thus, Figure 2 shows that dissatisfaction is, in part, dependent on the particular student. For example, higher-achieving students may be able to come to terms with the deficiencies of the ILS; more committed students may be able to overcome their frustrations (this can be seen in Sally and Louie's progress in Table 1. As actor-network theory (Gaskell & Hepburn, 1998) describes, the effect of software like the ILS depends on the actors within the class – its actions shapes these actors and, in turn, are shaped by these actors. Thus, the ILS can not exist in isolation; it requires the interactions of a network of other actors as it is taken-up, modified, used and/or ignored. Therefore, as Gaskell and Hepburn (1998) describe, computer software (like the ILS) and networks evolve together with the result being 'course networks' (Gaskell & Hepburn, 1998). Thus, when the focus of the analysis is the individual students, as it is in this paper, the model in Figure 1 has to be modified to include a focus on actors. Figure 2 illustrates the beginning of such an actor-based model. As is described in Baturo, Cooper, and McRobbie (1999b), this model begins to explain the actions of individual students with the ILS.

References

Baturo, A. R. (1998). Year 6 students' cognitive structures and mechanisms for processing tenths and hundredths. Unpublished doctoral dissertation, QUT, Centre for Mathematics and Science Education, Brisbane.

Baturo, A. R., Cooper, T. J., Kidman, G. C., & McRobbie, C. J. (2000). Factors influencing teachers' endorsement of the core mathematics course of an integrated learning system. Paper to be presented at the 24th annual conference of the *International Group for the Psychology of Mathematics Education*. Hiroshima, Japan: PME, *2*, 65–72.

Baturo, A. R., Cooper, T. J., & McRobbie, C. J. (1999a). Karen and Benny: Deja vu in research. In O. Zaslavsky (Ed.), *Proceedings of the 23rd annual conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 53–60). Haifa, Israel: PME.

Baturo, A. R., Cooper, T. J., & McRobbie, C. J. (1999b). *Lance and John get to know an integrated learning system*. Paper presented at the Annual Conference of Australian Association of Research in Education, Melbourne, Vic, Australia.

Baturo, A. R., Cooper, T. J., McRobbie, C. J., & Kidman, G. C. (1999). A case study of teacher endorsement of an integrated learning system. In J. M., & K. M Truran (Eds.), *Making the difference* (Proceedings of the 22nd annual conference of the *Mathematics Education Research Group of Australasia* (pp. 60–67). Sydney, NSW: MERGA.

Becker, H. J. (1992). Computer–based integrated learning systems in the elementary and middle grades:

A critical review and synthesis of evaluation reports. *Journal of Educational Computing Research, 8*(1), 1–41.

Bottino, R. M., & Furinghetti, F. (1996). Teachers' behaviours in teaching with computers. . In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th annual conference of the International Group for the Psychology of Mathematics Education,* (Vol. 2, pp. 129–136).

Erlwanger, S. H. (1975). Case studies of children's conceptions of mathematics. *Journal of Children's Mathematical Behavior, 1*(3), 157–283.

Gaskell, J., & Hepburn, G. (1998). The course as token: A construction of/by Networks. *Research in Science Education, 28*(1), 65–76.

McRobbie, C., Baturo, A., & Cooper, T. (submitted). Low achieving mathematics students' attitudinal and achievement changes as a result of using an integrated learning system. Paper to be presented at the *23rd annual conference of the Mathematics Education Research Group of Australasia*, Perth, WA.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education, 20*(1), 8–27.

Sivin–Kachala, J., Bialo, E.R., & Langford, J. (1997). *Report on the effectiveness of technology in schools: 1990–1997*. Washington, DC: Software Publishers Association.

Underwood, J., Cavendish, S., Dowling, S., Fogelman, K., & Lawson, T. (1996). Are integrated learning systems effective learning support tools? *Computers in Education, 26*(1), 33–40.

Wiburg, K. (1995). Becoming critical users of multimedia. *The Computing Teacher, 22*(7), 59–61.

OPTIONAL USE OF GRAPHICS CALCULATORS IN APPLIED QUESTIONS FOR HIGH-SCHOOL CALCULUS

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In this paper we discuss the extent and nature of students' use of graphics calculators in answering applied questions on a calculus tertiary entrance examination. The results show that the majority of students utilised the numerical integration capabilities of graphics calculators but fewer took advantage of graphical approaches on the technology when they were an option. Implications for teaching are that skills and knowledge specific to calculator usage need to be subjects of instruction.

Introduction

This paper is part of a longitudinal study inquiring into the introduction of graphics calculators on the Western Australian Calculus Tertiary Entrance Examination (TEE). The paper flows out of a critical analysis (Mueller & Forster, 2000) of Calculus TEE questions for 1996-1999, two years before and two years after the inclusion of the calculators. The analysis brought to light opportunities for optional and advantageous use of graphics calculators for solving 'Applications of calculus' questions in 1998 and 1999. The extent and nature of students' graphics calculator usage in four applied questions from the 1999 paper are the subjects of inquiry here. The findings have implications for teaching practice.

Flexibility in problem solving

Although research is not unanimous on the benefits of technology, there is evidence that sustained use of graphics calculators can result in superior understanding of functional relationships and in students being able to perceive functions in both process and product (object) terms (Hollar & Norwood, 1997; Schwarz & Hershkowitz, 1999). Understanding functional expressions both "as manipulable objects and as triggers to evoke mathematical processes" lends flexibility to problem solving and is a hallmark of more able students (Gray & Tall, 1994). It contrasts with the limited perception of some students that algebraic formulae are "mere strings of symbols to which certain well-defined procedures are routinely applied" (Sfard & Linchevski, 1994, p. 199).

First, this paper considers students' use of graphics calculators for the procedural aspects of problem solving, including their use for evaluation of area and volume using numerical integration. Second, other questions are discussed where students were required to determine properties of functions which modeled real situations. These called on viewing the functions in graphical form as objects or static structures (Sfard, 1991).

Calculator graphing allows students to consider the "graphical relationship unencumbered by a filter of symbolic manipulation" (Dick, 1996, p. 38) and allows graphical approaches to problem solving that would be impractical without the technology (Forster & Mueller, 2000). However, having the technology available does not guarantee its use, as was observed in particular in students' evaluation of limits in the 1998 Calculus TEE (Mueller & Forster, 1999). Nor does the availability of technology mean that students automatically have the skills that are necessary for interpretation of graphical displays on it (Boers & Jones, 1994). As a result asymptotes, discontinuities and other features of a graph might be missed (Boers & Jones, 1994; Forster & Mueller, 1999, 2000; Ward, 1997). Linkage with the algebraic form of a graphed function is sometimes needed to determine all the graphical features and entails integrated visual-algebraic reasoning (Dreyfus, 1994). The extent to which calculator graphing was used, and advantages and difficulties associated with it for answering optimisation, simple harmonic motion and rectilinear motion questions are discussed here.

Graphics calculators in TEE calculus

Calculators without symbolic processing and the Hewlett Packard HP38G with limited symbolic capabilities are approved for TEE purposes. Other than for a general instruction to show all working in sufficient detail to allow answers to be readily checked, a non-prescriptive approach has been taken to setting out. Specific instructions to use graphics calculators are not generally given. Four A4 pages (two sheets) of notes are allowed because the text storage capacities differ between calculator brands.

Research method

Four types of data were collected for the inquiry. First, three students from each of two schools and four students from another were interviewed about calculator processes they used in the examination. Students were selected by their teachers on the basis of being communicative. Their school assessment grades ranged from A to C (D is the lowest pass grade). The examination paper was used as a heuristic in the interviews. The second type of data was results for all candidates (marks per question), obtained from the Curriculum Council of Western Australia who administer the TEE examinations. Third, examination markers recorded data on a proforma, for a sample of their allocated scripts. Part marks were recorded and columns ticked to indicate if students' methods were traditional or graphics calculator based for seven examination questions where graphics calculator usage was expected; and columns were ticked to indicate specific graphical features in students' answers. Three of these questions are discussed in this paper. Nine out of 24 markers volunteered to be involved. All were experienced teachers and this resulted in data for 195 scripts. The fourth type of data consists of our own observations, in our role as examiners and markers. We recorded data while marking 240 scripts (from the total 1957 scripts--scripts are marked twice) to obtain a total sample of size 435. In addition we collected data for one further question which is the fourth question discussed here. The sample statistics are only a guide and not a definitive statement of graphics calculator usage--discerning graphics calculator usage from examination scripts is necessarily interpretative and the sample was not randomly selected. However, scripts from a school are distributed between the bundles for marking, and bundles are allocated to markers without preference.

Results

In 1999, there were seven questions from the 'Applications of calculus' component of the curriculum in the 20 question Calculus TEE paper. Four of the questions allowed use of graphics calculator capabilities, over and above scientific calculator capabilities, and are discussed below. The means quoted for each question are based on the scores of the students who attempted them. Every question on the paper was to be answered but some students did not attempt all of them.

Question 7: Suppose that $f(x) = \dfrac{3\cos x}{2+\sin x}, \quad -\pi \le x \le \pi$.

(a) Determine exactly the two zeroes r_1 and r_2 of f.
(b) Calculate the exact co-ordinates of the maximum stationary point.
(c) Determine the area of the region above the x-axis bounded by the x axis and the graph of f.

Solution:
(a) $x = \pm\,\pi/2$

(b) Max stationary point = $(-\pi/6, \sqrt{3})$

(c) $\int_{-\pi/2}^{\pi/2} 3\cos x/(2+\sin x)\, dx = 3\cdot 30$(2dp)

Results: For parts (b) and (c) markers were asked to indicate 'traditional method (working shown) or calculator approach (essentially the answer only)', together with part marks and whether exact or inexact values were given in part (b). The results are summarised in Table 1. Data were not gathered for (a) as the zeroes were simply those of the cosine function in the interval [-π, π] and no calculators (scientific or graphics) were needed to find them.

Table 1
Methods Adopted for Determining a Maximum Turning Point (7b) and Area (7c)

	Question 7(b)		Question 7(c)	
	Traditional	Graphics calculator	Traditional	Graphic calculator
[a]Number of students	311	99	73	300
Mean score	4.1(81%)	3.8(76%)	1.9(63%)	2.5(84%)
Number who scored zero	8	5	14	17
% who gave exact values	89%	74%	-	-

[a]n= 435, 410 were recorded to have attempted 7(b) and 386 for 7(c)

More than three-quarters of the students in the sample (see Table 1) chose to use (routine, procedural) differentiation to establish the maximum point in part (b). Judging by interview responses, one reason was the requirement of exact values. However, incomplete working with traditional methods together with rough sketches (see Figure 1 for a calculator generated graph) of the function on the domain indicated frequent use of graphics calculators for the final determination of the co-ordinates of the maximum stationary point. Adopting a graphical approach from the start meant that processing to identify the maximum point was completely off-loaded to the calculator, but then decimal values had to be converted to exact values. The lower incidence of exact values (see Table 1) is one explanation for the lower mean mark with the calculator approach, which suggests that the traditional (or combined) approach had merit. Other errors in part (b) were to omit the second co-ordinate and, when using differentiation, to locate the maximum point at π/6 instead of −π/6. A graphical approach on the calculator avoids this misidentification: the functional relationship is seen as a static whole and the location of maximum is seen at a glance (see Figure 1).

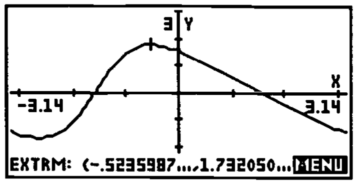

Figure 1. Graphics calculator display of $f(x) = (3 \cos x) / (2 + \sin x)$.

In part (c), over three-quarters of the students in the sample who attempted the question (see Table 1) used numerical integration to evaluate the area. Students scored a higher average with a calculator approach (see Table 2). Errors were to integrate $|f(x)|$ over the interval $[-\pi, \pi]$, instead of integrating $f(x)$ over $[-\pi/2, \pi/2]$, and there were isolated instances of answers consistent with calculator mode set to degrees instead of radians. The mean score for the population (N = 1937) for the question as a whole was 7.76 out of 10, a relatively high average result.

Question 8: A wood-turner turns a piece of wood to make a curved bowl with a solid cylindrical base. The base has a radius of 1 unit and thickness 0.1 unit. The curved part of the bowl can be described mathematically as the solid obtained by rotating about the x-axis the region in the first quadrant bounded by the y-axis, $y = \cos x$ and $y = \sin x$. For both parts of the bowl one unit represents 9 cm.

(a) Write an integral representing the volume of the curved part of the bowl.
(b) What is the volume of wood in the finished bowl?

Solution:

(a) $\pi\int_0^{\pi/4}(\cos^2 x - \sin^2 x)\,dx$

(b) Volume of the curved part $= \pi/2$ or 1.57 (2 dp)
Total volume $= (\pi/2 + \pi \times 1^2 \times 0.1) \times 9^3\text{cm}^3 = 1374\text{ cm}^3$

Results: While marking, it seemed to us that a correctly set up integral was less frequently accompanied by a correct answer when the answer only was given (implying processing on the calculator) than when students used traditional integration techniques. To test this conjecture, we recorded the nature of students' answers for the 240 scripts that we marked.

Table 2
Numbers of Students Using Traditional and Calculator Evaluation of an Integral

	Traditional		Graphics calculator		
	[a]Number	Correct evaluation	[a]Number	Correct evaluation	No evaluation
Correct setup	23	20	43	31	4
Incorrect setup	31	14	65	38	33

[a]sample size n = 240 of whom 41 did not attempt the question

The results in Table 2 support the conjecture above. In addition, of the 199 students who attempted the question, 108 chose to use their graphics calculator to evaluate the integral they obtained. Of these only 69 succeeded in obtaining a value consistent with their integral. This indicates that quite a large proportion (36%) of this group did not use their calculators correctly to process the integral, perhaps making entry and syntax errors. Only 70 (35% of 199) of the students who attempted the question found the correct integral expression with errors including misidentification of the limits of integration and of the integrand (to use $(\cos x - \sin x)^2$ instead of $\cos^2 x - \sin^2 x$) A further common problem was the adjustment of the volume units. The mean mark for the question as a whole was low at 47% (3.78 out of possible 8 marks, n = 1728).

Question 12: The difference between high and low water tide levels in one of the ports along the northwest coast of Western Australia is 6 metres. At the entrance of the port from the ocean, the level x of water is given by $\frac{d^2x}{dt^2} = -\frac{1}{4}x$ where t is measured in hours from the time of the low tide.

(a) Calculate the amplitude and the period of the motion of the tides.
(b) Write x as a function of t.
(c) A ship can enter or leave the port as long as there is at least one metre of water above the low tide mark. If the low tide takes place at 10.15am, what is the latest time a ship can leave the port on that day?

Solution:

(a) amplitude = 3m, period = 4π hours
(b) $x = -3\cos(t/2)$ is one possible answer.
(c) $-2 = -3\cos(t/2)$, $t = 10{\cdot}8842$, Time = 10:15am + 653·05mins = 9:08pm

Results: The apparent method of solution and mark awarded were recorded for 12(c). The data is summarised in Table 3. The other part questions were calculator inactive, except for checking in (b), so were not considered in the data collection.

Table 3
Results for Part (c) of the Simple Harmonic Motion Question

	Traditional	Graphics calculator
[a]Number of students	151	123
[a]Mean score out of 3	1.9(64%)	2.2(75%)
Number who scored zero	28	21

[a]n = 435, 293 attempted the question of whom 274 had a method recorded.

The results in Table 3 indicate a slight preference by students for using an inverse trigonometric function to answer part (c) of the question. A calculator alternative to this approach (as the first option or for checking) was to determine the time from the point of intersection of the graphs $f(t) = -3\cos(t/2)$, and $g(t) = -2$ (see Figure 2). Another method on the HP38G was to use the 'Solve' aplet for symbolic equations.

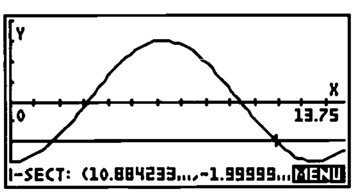

Figure 2. Graph of $f(t)$ = -3cos(0.5t) representing the tide, and $g(t)$ = -2.

Some students calculated the earliest time at which the ship can enter or leave port after low tide. This error is explained by students not realising that the required time was during the falling tide and as a consequence they did not explore multiple solutions. Determination of the correct time is strongly encouraged by a visual approach. The rise and fall of the graph represents the tide less abstractly than does the equation, and in-built capabilities of the calculator can produce the value for time. These aspects account for the higher average result with the calculator (see Table 3).

A common error in part (b) was to multiply by the period resulting in an incorrect coefficient for t, for example, to write $-2=-3\cos(4\pi t)$. Students using a graphical approach could have detected this error by relating the graph to the real situation. Adjusting the time t to allow for the 10:15 am start was another source of error.

The average mark for the question overall was low at 5.72 out of 11 (52%, n = 1811). The question was a good discriminator for examination purposes, with a correlation of 0.71 between students' scores on the question and their total examination scores. That is, students who did well on the question tended to do well in the examination and the sample data indicate that choosing to use a graphics calculator as first option might have been a factor of their success.

Question 14:
A particle is moving along a straight line that runs in an east-west direction. Its position function $s(t)$ at time t is given by $s(t)=\dfrac{t^2+1}{t^4+1}$.

(a) Determine the velocity function of the particle.
(b) The particle is moving in an easterly direction when the velocity is positive. Use the graph of the velocity function to decide when the particle is moving in a westerly direction.
(c) Use the graph of the velocity function to determine the maximum speed of the particle and when it is attained.
(d) Calculate the position of the particle at the time when the maximum speed is attained.

Solution:

(a) $s=\dfrac{t^2+1}{t^4+1}$, $\quad v=\dfrac{(t^4+1)2t-(t^2+1)4t^3}{(t^4+1)^2}$

$$v=\frac{2t-2t^5-4t^3}{(t^4+1)^2}$$

(b) $t>0.64$
(c) $t=1.095$ (3dp) speed = 1.045 (3dp)
(d) $s=0.90$ (2dp)

Results: Part marks for (a)-(c) were recorded and are summarised in Table 4.

Table 4
Results for the Rectilinear Motion Question

	14(a)	14(b)	14(c)
[a]Number. students who attempted the question	419	380	368
Mean mark for the question	2.6 (88%)	1.3(63%)	1.2(59%)
Number of students who attempted the question and scored zero	16	158	170

[a]n = 435

On average students were more successful in part (a) (see Table 4), which involved differentiation, than in parts (b) and (c). In our observation, the vast majority of students seemed to follow the instructions in (b) and (c) to use a graph and generated it on their calculators (see Figure 3), which was indicated by the lack of algebraic working in written answers. The graph would have been difficult to establish analytically so that graphical approaches on a calculator were effectively forced. Without the technology the graph and the necessary supporting working would have attracted high mark weighting. Here, parts (b) and (c) attracted two marks each—all of these were for interpretation of a graph and none for making it. Thus, the lower marks in parts (b) and (c) were associated with errors in graphical interpretation.

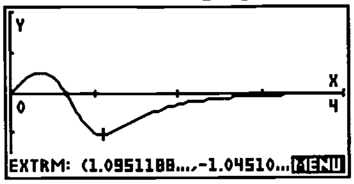

Figure 3. Graph of the velocity function.

Mistakes in calculating the derivative were followed through in the marking, which reduced the impact they had on students' scores in parts (b) and (c). Mistakes could have arisen in entering the expression into the calculator but other errors were not specifically attributable to technology usage. Recognising that the root was needed in part (b) was usually unproblematic and interviewed students indicated that they used in-built capabilities of their calculators to find it. Determination of the appropriate interval was seen to be of greater difficulty with students using = (attributable to misinterpreting the question) instead of >. There were isolated instances of $t < 0.64$, $t \leq 0.64$ and $0.64 < t < 1$, indicating confusion as to the properties of the velocity function.

For part (c), interviewed students again indicated that they used in-built capabilities on their calculators to determine the required time and speed (see Figure 5). However, mathematical errors were writing the maximum velocity instead of the maximum speed and reading the speed correctly from the minimum point but writing it as a negative quantity. Some students did not write the time, which might have been avoided by rereading the question. Combinations of these errors explain why a large number of students scored zero for parts (b) and (c) of the question (see Table 4). The mean score for the question for the population was 5.50 out of 8 (69%, n = 1891), which is greater than the mean score for some of the other applied questions but this was on account of part (a) which called on differentiation skills.

Concluding discussion

In three of the applied questions, some students used traditional, procedural approaches and others off-loaded the processing completely to their graphics calculators. The alternatives for 7(c) were differentiation and a graphical approach on a calculator, in 7(c) and 8 were symbolic integration and numerical integration on the calculator, and in 12(c) were solving a trigonometric equation symbolically or graphically on a calculator (or numerically on the HP38G). In 14(b), establishing a graph using differentiation was impractical, which the majority of students seemed to realised immediately, so that few students showed any working, indicating use of the technology.

The majority of students in the sample chose numerical integration in 7(c) and 8 in preference to traditional integration (see Table 5). Time benefits would have depended on students' familiarity with the integrals and competency with hand integration.

Table 5
Estimated Graphics Calculator Usage Among Students Whose Method was Recorded

	7(b)	7(c)	8	12(c)
% opting to use technology	24	80	67	45

Graphical approaches allowed 'big picture' views of the functional relationships, that is, allowed them to be perceived as static objects. This meant that, potentially, errors noted with procedural approaches could be avoided: students could identify maxima, points of intersection and roots easily but some experienced difficulty when more sophisticated interpretation was required, as was the case in identifying the interval and maximum speed in question 14. Less than half of the students in the sample who attempted questions 7(b) and 12(c) chose graphical methods on their calculators as their first option, but many showed flexibility in solving the problem in 7(b), combining or checking a traditional method with a graphical approach. In question 14, graphing on a calculator was widespread because of the nature of the function.

In view of the results reported here and assuming that increased opportunities for optional use of graphics calculators in calculus are not confined to our local context, implications for teaching are that skills and understanding specific to technology usage need to be subjects of instruction. These include familiarity with syntax for integration, checking entries for entry errors, converting decimals to exact values (for example by squaring), skills of graphical interpretation, including checking that conditions stated in the question have been met and checking traditional procedural approaches graphically on the calculator. Facility with technology might also include flexible use of the tables of values, which was called on in other questions in the 1999 Calculus TEE paper.

References

Boers, M.A.M. & Jones, P.L. (1994). Students' use of graphics calculators under examination conditions. *International Journal of Mathematics Education in Science and Technology, 25*(4), 491–516.

Dick, T. P. (1996). *Much more than a toy. Graphing calculators in secondary school calculus.* Paper presented at the eight International Congress of Mathematics Education, Spain.

Dreyfus, T. (1994). Imagery and reasoning in mathematics and mathematics education. In D. F. Roitaille, D. H. Wheeler, & C. Kieran (Eds.), *Selected lectures from the seventh International Congress on Mathematical Education.* Sainte–Foy: Les Presses de l'Université Laval.

Forster, P. A., & Mueller, U. (1999). Use of graphics calculators in the public examination of calculus: Reality and rigor. *Australian Senior Mathematics Journal 13*(2), 4–20.

Forster, P. A., & Mueller, U. (2000). *Graphics calculator usage in the West Australian tertiary entrance examination of calculus.* Paper presented at the annual conference of the American Educational Research Association. New Orleans.

Gray, E. M., & Tall, D. D. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*, 116–146.

Hollar, J. C. & Norwood, K. (1999). The effects of a graphing–approach intermediate algebra curriculum on students' understanding of function. *Journal for Research in Mathematics Education 30*(2), 220–226.

Schwarz, B. B. & Hershkowitz, R. (1999). Prototypes: Brakes or levers in learning the function concept? The role of computer tools. *Journal for Research in Mathematics Education 30*(4), 362–389.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics 22,* 1–36.

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification – the case of algebra. *Educational Studies in Mathematics, 26*, 191–228.

Ward, R. A. (1997). *An investigation of scaling issues and graphing–calculator associated misconceptions among high school students.* (Doctoral dissertation, University of Virginia, 1997). Dissertations Abstracts International, 58/06, 1.

A GRAPHIC CALCULATOR APPROACH TO UNDERSTANDING ALGEBRAIC VARIABLES

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Algebra is a difficult topic for many secondary school students. Discovering good ways of introducing algebra into the students' experience and to foster understanding has stimulated much discussion and research. It is our view that an understanding of the concept of variable is fundamental to further progress in algebra. This paper describes part of a study using a module of work based on a graphic calculator to provide an environment where students could experience some aspects of variables, and hence begin to build an understanding of them. The graphic calculator proved to be a motivating instrument for successfully achieving a significant improvement in student understanding, something which has often proved difficult.

Background

The concept of variable is one of the most important, yet all too rarely discussed, concepts of algebra. While it underpins all that students learn in algebra, Küchemann's (1981) research showed that extremely few students reach a working knowledge of variable. This has led to many students having great difficulty in understanding the algebra of generalised arithmetic, and failing to overcome obstacles in learning it (see e.g. Tall & Thomas, 1991; Stacey & MacGregor, 1997).

The fundamental nature of variables, and the failure to give them adequate prominence in teaching, has been well described by Schoenfeld and Arcavi (1988, p. 420):

> The concept of variable is central to mathematics teaching and learning in junior and senior high school. Understanding the concept provides the basis for the transition from arithmetic to algebra, and is necessary for the meaningful use of all advanced mathematics. Despite the importance of the concept, however, most mathematics curricula seem to treat variables as primitive terms which – after some practice, of course – will be understood and used in a straightforward way by most students.

Thus students should gain some understanding of variable before other algebraic concepts are introduced, if they are ever to progress beyond basic processes A major problem with variables is their sophisticated and multi-faceted nature. Wagner *et al.* (1981) have observed that mathematicians use letters – sometimes referred to as 'literal symbols' – in algebra in a wide variety of ways. We suggest that the use of literal symbols to generalise arithmetic relationships is too sophisticated to be useful as an introduction to the idea of variable and they are more likely to progress if they first have a firm foundation of 'letter as placeholder'. This idea has been central in our choice of a graphic calculator as a micro-world for helping students gain a more basic understanding of 'letter as placeholder'.

Processes and objects in algebra

Prior to the introduction of algebra, children become accustomed to working in an arithmetic environment where they solve problems by producing a numerical "answer" (Kieran, 1981), leading to the expectation that the same will be true for algebra. In algebra however, the expression $x + 1$, can be viewed as a *procept* (Gray & Tall, 1994).

In other words, it simultaneously represents an *object*, namely an expression (or function), and the *process* of adding one to an unknown value. In this case, however, it is important to encapsulate the generalised process of adding one as the object $x + 1$, because the process cannot be carried out directly unless x is given a value. This indeed is precisely the value of the symbolisation of variable (see Tall & Thomas, 1991).

However, many students only see the symbol $x + 1$ as a process and not as a mental object in its own right, capable of manipulation in an abstract form. They do not progress to the point where they can think in this *versatile* way. Instead, they are *process-oriented* (Thomas, 1994), thinking primarily in terms of mathematical processes, rather than *versatile* (Tall & Thomas, 1991; Hong & Thomas, 1998), having the ability to think both holistically about a concept or object, as well as sequentially about the process from which it has been encapsulated

We believe that students' success rate in early algebra can be significantly improved by giving them a coherent meaning for letters through environments in which they can manipulate examples, predict and test outcomes, and gain experiences on which higher-level abstractions can be built. The graphic calculator is an available, portable and affordable option for doing just that in many schools (Ruthven, 1990; Penglase & Arnold, 1996). The device has two extremely useful qualities that can be further exploited. First is the intrinsic use of variables in its operation, including the identification or labelling of its large number of in–built stores by the use of letters. Second is its multi-line display which can enable one to see, reflect on, and interact with, several previous input/output episodes, capturing something of the changing nature of a variable. The first quality, combined with the act of storing a number into and retrieving it from a lettered store provides a direct, dynamic analogue of the system of letter–use in early algebra. The research study described in this paper sought to apply these advantages of graphic calculators in the classroom, with the goal of improving student understanding of the concept of variable.

Method

'Tapping into Algebra' was a classroom–based research project comparing the teaching of variable in algebra with and without the use of the graphic calculator. It was a collaborative project involving students and teachers in both the United Kingdom (UK) and New Zealand (NZ), this having the advantage of trialing the ideas in two different lands and cultures. The United Kingdom based research is described here (see Graham & Thomas, 2000 for the results of the New Zealand research).

The algebra module

The basic premise of the module, which was designed to last about three weeks, was to use the graphic calculator's lettered stores as a model of a variable. Each store is represented as a box in which changing values of the variable come and go, and above which sits the label. This model had been used successfully in research using the computer (Tall & Thomas, 1991), and was fundamental to this study.

It was assumed that almost no students would have had previous experience of using graphic calculators, so the first section of the module comprised an introduction. For example, they were shown how to perform simple calculations using the four operations, to edit key sequences and to store and retrieve numbers using the letter keys.

This section has since been extended and published as two books in a five book series on calculator mathematics (Graham & Galpin, 1998).

On the graphic calculator (we used primarily the TI-80 and TI-82) sequences such as this were used:

3—›A (using the [STO▶] key)
A [+] 2 [ENTER]
A [+] 3 [ENTER]
or
A [+] 2—›B
B [ENTER]

This sort of activity can help students begin to formulate theories about the consistency with which any given language handles the symbols, and to build an understanding of their purpose. Figure 1 gives an idea of the layout used in the teaching module, illustrating the 'Press', 'See' and 'Explanation' features which were universally used.

Building on this introduction, a typical early exercise requiring students to reflect on the relationship between two variables and their contents was:

Store the value 2.5 in A and 0.1 in B.
Now predict the results of the ten sequences listed below.
Then press the sequences to check your predictions.

A+B, B+A, A – 5B, 2A + 10B, A/B

AB, BA, 2A + 2B, 2(A+B), 4(A+5B).

You can use letters as stores for numbers. Try the following:

Press	See	Explanation
4 [STO▶] [ALPHA] A [ENTER]	4 → A 4	The value 4 is stored in A.
[CLEAR]		This clears the display.
[ALPHA] A [ENTER]	A 4	This confirms that the number stored in A *is* 4.

Figure 1. An example of the layout of the work in the algebra module

One of the novel teaching aspects of the module was the use of *screensnaps*, where students were given a screen view and required to reproduce it on their calculator screen. Some examples of those included are given in Figure 2. These *screensnaps* have the advantage of encouraging beginning algebra students to engage in reflective thinking using variables. This is beneficial since they do not attempt to reproduce them by using algebraic procedures, but by assigning various values to the variables and predicting and testing outcomes.

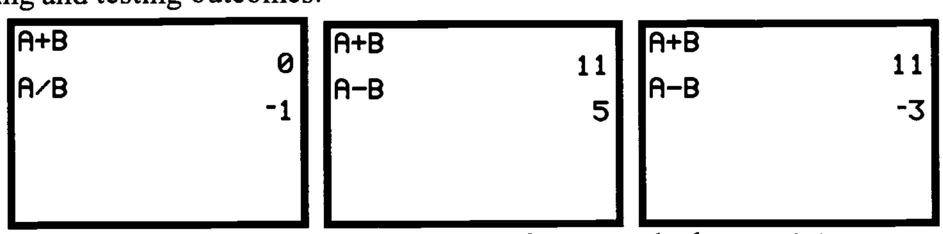

Figure 2. Examples of *screensnaps* from the algebra module

Other topics covered included squares and square roots, sequences, formulas, random numbers and function tables of values.

Results

Teachers from five United Kingdom schools volunteered to teach the module of work in algebra to one of their classes, based on the TI-80 graphic calculator. In addition, they chose a control group of pupils, similar in ability and background to the experimental group, against which to make a comparison. The control group received corresponding algebra work to that of the experimental group, but were taught by their usual teaching methods. Students were from year 8 (age 13 years) top and middle ability groups. The module was taught by the classroom teachers, each of whom had attended a weekend course run by one of the researchers to help them gain proficiency in the use of the calculator. The researchers were not present while the students were learning. The classroom groups were all given a pre-test and a post-test which comprised questions based on, and extending, the Küchemann (1981) research (see Tables 2 and 3 for examples). The two tests used were different, the pre-test having 28 questions and the post-test 68. This latter test was more difficult, containing 63.2% of level 3 and 4 questions (requiring an understanding of specific unknown and generalised number) compared with 53.6% for the pre-test.

A summary of the results of those students from the five schools who completed both of the tests is given in Table 1, although these results may be more easily compared by examining Figure 3, which gives the mean percentage scores in the pre- and post-tests. In each case, the relative improvement of the experimental students over the control students is clearly seen.

Table 1
A Statistical Analysis Using t-Tests of the Post-Test Results for each of the Five Schools

	Experimental means (SD)		Control means (SD)					
	Pre-test (max=28)	Post-test (max=68)	Pre-test (max=28)	Post-test (max=68)	N_e	N_C	t	p
School 1	5.48 (2.68)	26.68 (7.40)	5.65 (3.23)	24.61 (5.93)	22	23	1.03	n.s.
School 2	20.00 (3.35)	45.88 (8.68)	17.44 (4.12)	34.19 (8.18)	26	27	5.05	<.0001
School 3	8.75 (2.71)	28.96 (7.86)	8.0 (6.39)	22.24 (15.06)	29	24	2.08	<.05
School 4	2.04 (1.43)	10.17 (5.91)	3.24 (2.08)	6.88 (3.33)	23	17	2.23	<.05
School 5	3.32 (1.96)	27.93 (11.35)	2.6 (1.92)	21.21 (6.60)	31	29	2.73	<.005

With the exception of school 2, the groups did not differ at the pre-test. However, the post-test results of the experimental groups were significantly better than that of the controls for 4 of the 5 schools. The experimental group in school 2 performed significantly better than the controls on the pre-test ($m_E = 20.0$, $m_C = 17.4$, $t = 2.48$, $p < 0.05$), but their improvement from an average raw score of 20 to 45.9 was still significantly better than that of the controls, whose average raw scores went from 17.4 to 34.2. Since the tests were constructed so that they were a direct measure of the students' level of conceptual understanding of letter as specific unknown, generalised number and variable in algebra, we conclude that the graphic calculator module has improved the students' conceptual understanding of the use of symbolic literals.

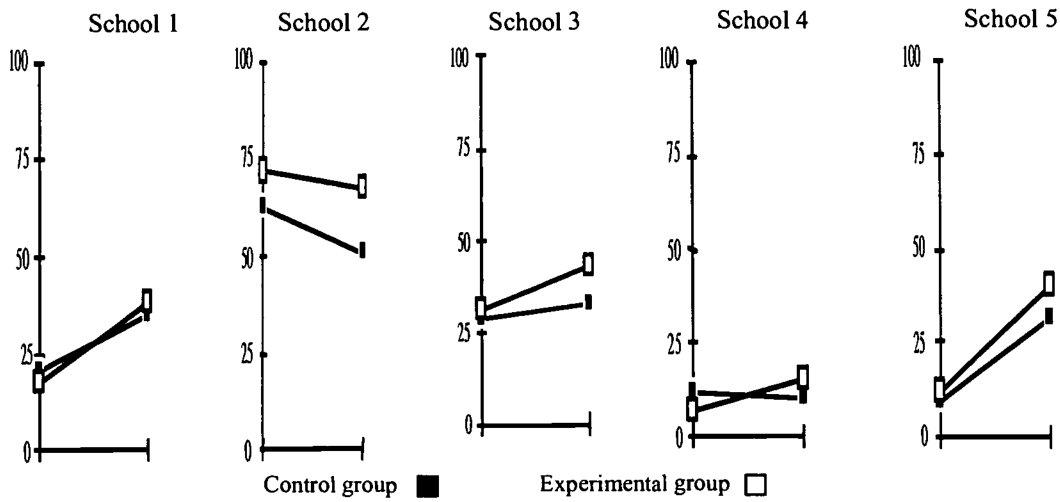

Figure 3. The mean percentages of correct answers in the pre- and post-tests for each of the five schools

In order to see the extent of this improvement, we analysed the performance of the two groups on those questions at levels 3 and 4 only (understanding letter as specific unknown and generalised number respectively), as described by Küchemann (1981). Considering these two levels only, there was some evidence that the experimental group students were again getting a higher proportion of questions correct (P_E = 0.27, P_C = 0.20, χ^2=3.74, p<0.1).

Table 2

A Comparison of Questions Examining Higher Levels of Understanding of Letter as Variable

Question	Level	Experimental proportion correct (N=131)	Control proportion correct (N=120)	χ^2	p
x-y=z-y — always, never, sometimes ... when?	4	0.28	0.15	6.42	<0.05
a+b=b — always, never, sometimes ... when?	4	0.33	0.13	13.2	<0.001
L+M+N=L+P+N, — always, never, sometimes ... when?	4	0.29	0.18	3.92	<0.05
3h=c+3 and h=2, then c=?	4	0.56	0.45	2.88	n.s.
r=s+t and r+s+t=30, then r?	3	0.34	0.14	11.2	<0.001
Area of rectangle 5 by e+2	4	0.15	0.08	3.10	n.s.
Which is smaller: 3q or 3+q? Explain	4	0.08	0.02	3.68	n.s.

Table 2 provides examples of specific questions (abbreviated) at levels 3 (specific unknown) and 4 (generalised number) where the understanding of the students who had used the calculators appeared better. Of the four questions shown where they did significantly better, three are at a level requiring an understanding of letter as generalised number. This seems to represent a very useful advance in understanding. The question of why the students were better equipped to deal with such questions after the calculator work is addressed later.

To check whether the calculator students had been disadvantaged in terms of their manipulative skills, we looked at the corresponding results on questions which fall into the more 'traditional' simplification skills category. What this showed (Table 3) was that the students who used the graphic calculators did at least as well on these questions in virtually every case, and significantly better in two of them.

Table 3

A Comparison of Questions Examining Procedural Skills

Question	Experimental proportion correct (N=131)	Control proportion correct (N=120)	χ^2	p
Simplify $(a+b)+a$	0.56	0.43	3.88	<0.05
Simplify $2a + 5b + a$	0.57	0.52	0.79	n.s.
Simplify $3a - b + a$	0.17	0.10	2.75	n.s.
Simplify $(a-b)+b$	0.18	0.08	6.42	<0.05
Simplify $3a - (b + a)$	0.41	0.38	0.22	n.s.
Simplify $(a+b)-(a-b)$	0.03	0.03	---	n.s.
$2a+2b=2(a+b)$ — always, never, sometimes ... when ?	0.42	0.46	0.44	n.s.

Since there had been no attempt to teach these skills explicitly, these results were very encouraging. It appeared that the students who had used the graphic calculators had gained a better understanding of the use of letters as variables than those who had not, without a detrimental effect on their basic ability to manipulate symbols.

The view of the students and teachers

While a statistical analysis certainly tells us something about the value of a teaching module such as the one we used, there are other factors which are equally important, if not more so. If teachers and students are not comfortable using the ideas in the classroom then they are of little practical value. In this research both the teachers and their students were asked to comment freely on their experiences with the graphic calculator teaching module and we present below a synopsis of their views.

Student comments

The majority of students felt that the experience of using the graphics calculator was of benefit in improving understanding and making the learning of algebra more palatable by providing a useful diversion, with typical remarks:

> The work we did using the graphics calculator was very interesting and it made algebra seem a little more fun. Algebra was a lot easier on the graphics calculators that it was doing it the ordinary way. . .The graphics calculators have also given me a better understanding of algebra. I think I understand algebra more after this course and if it worked for me it should work for almost anybody.
>
> I don't particularly like doing algebra and have never been that good at it. I think I knew a lot more than I did before the first lesson on algebra. The graphics calculators are fun to work with and easy to operate.
>
> I learnt more about algebra than I had at first. My understanding is greater. . .

Generally, the majority of the participating pupils enjoyed the experience and they found the worksheets clear and easy enough to follow. However, a small minority found some aspects of the module hard or unsatisfying, suggesting that a few students do not need such a prolonged introduction to variables, while others seem to find algebra difficult however it is approached.

Teacher comments

Each participating teacher was asked to submit a commentary on their own impressions of the project, including ways in which it could have been improved. The project was not designed to provide a set of comprehensive, polished or coherent classroom materials. Nevertheless the teachers were clearly interested in using these materials again in the future and made a number of useful suggestions on how they could be organised more effectively. For example, commenting on the practicality of the worksheets a teacher said, "I thought the worksheets were extremely well presented and the pupils were able to follow them easily. I gave the exercises involving predictions as homework to check their understanding."

Overall the teachers' comments were very positive and they felt that their pupils' algebra did benefit as a result of working on the calculator. One particular area mentioned centred around the primary purpose of the module, namely the idea of a letter as a store for a number and the value of a physical metaphor for this concept:

> I think it was useful to use the calculators for the idea of 'storing' a number. This was a concept that the children found easy to grasp. It was much easier to get this idea across with the calculators because the number was physically stored.

The comments from the teachers were encouraging since it has been suggested (Thomas *et al.*, 1996) that the most important element in the successful introduction of technology into the classroom is the attitude and support of the teachers. Without this, any initiative is likely to fail. We are extremely grateful to all the teachers who participated in this research study for their enthusiasm and hard work.

Conclusions

In elementary algebra the use of letters has often been poorly understood. Yet an understanding of 'variable' in mathematics seems to underpin all advanced work and so it is important that students gain confidence at handling variables. The evidence that we have presented from our study shows that students can obtain an improved understanding of the use of letters as specific unknown or generalised number from a module of work based on the graphic calculator. While the improvement occurred regardless of the ability level of the students, the gains were particularly noticeable, in terms of relative advancement, for the weakest students. Questions such as those requiring students to say when $L+M+N=L+P+N$. or $A+B=B$, will be very difficult for the student who sees the letters as concrete objects, since two different objects can never be the same! However those who are thinking in a versatile way see the symbols as procepts – encapsulated objects representing a range of values. Hence it is possible for the value the letter represents to be sometimes the same as that of another letter. The graphic calculator helps to build such a versatile view of letters through the physical experience of students tapping keys to place different numbers into the calculator stores, changing the values in those stores and retrieving values from them. With the assistance and support of classroom teachers, we believe that innovative strategies such as that proposed here can make a significant difference to students' progress in algebra and in their grasp of the subtle notion of 'variable'.

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References

Graham, A., & Galpin, B. (1998). *Calculator Maths* Corby, UK: A+B Books.

Graham, A. T., & Thomas, M. O. J. (2000). Building a versatile understanding of algebraic variables with a graphic calculator, *Educational Studies in Mathematics, 41(3)*, 265–282.

Gray, E., & Tall, D. O. (1994). Duality, ambiguity & flexibility: A proceptual view of simple arithmetic, *Journal for Research in Mathematics Education, 26*, 115–141.

Hong, Y. Y., & Thomas, M. O. J. (1998). Versatile understanding in integration, *Proceedings of the International Congress of Mathematics Instruction – South East Asian Conference on Mathematics Education*, Seoul, Korea, 255–265.

Kieran, C. (1981). Pre-algebraic notions among 12 and 13 year olds, *Proceedings of the Eighteenth International Conference of the International Group for the Psychology of Mathematics Education*, Grenoble, 158–164.

Küchemann, D. E. (1981). Algebra, in K.M. Hart (Ed).*Children's Understanding of Mathematics: 11-16* (pp. 102-119). John Murray.

Penglase, M., & Arnold, S. (1996). The graphics calculator in mathematics education: A critical review of recent research, *Mathematics Education Research Journal, 8(1)*, 58–90.

Ruthven, K. (1990). The Influence of graphic calculator use on translation from graphic to symbolic forms, *Educational Studies in Mathematics, 21*, 431–450.

Schoenfeld, A., & Arcavi, A. (1988). On the meaning of variable, *Mathematics Teacher, 81(6)*, 420–427.

Stacey, K., & MacGregor, M. (1997). *Learning to use algebra for solving problems: Research findings and their implications for Mathematics Education policy*, Department of Science and Mathematics Education, The University of Melbourne.

Tall, D. O., & Thomas, M. O. J. (1991). Encouraging versatile thinking in algebra using the computer, *Educational Studies in Mathematics, 22*, 125–147.

Thomas, M. O. J., & Tall, D. O. (1988). Longer-term conceptual benefits from using a computer in algebra teaching, *Proceedings of the 12th Conference of the International Group for the Psychology of Mathematics Education* Veszprem, Hungary, 601–608.

Thomas, M. O. J. (1988). *A Conceptual Approach to the Early Learning of Algebra Using a Compute*r, PhD Thesis, University of Warwick.

Thomas, M. O. J. (1994). A process-oriented preference in the writing of algebraic equations, *Proceedings of the 17th Mathematics Education Research Group of Australasia Conference*, Lismore, Australia, 599–606.

Thomas, M. O. J., Tyrrell, J., & Bullock, J. (1996). Using computers in the mathematics classroom: The role of the teacher, *Mathematics Education Research Journal*, April, *8(1)*, 38-57.

Wagner, S., Rachlin, S. L., & Jensen, R. J. (1984). *Algebra Learning Project – Final Report*, Department of Mathematics Education, University of Georgia.

MATHEMATICS EDUCATION AND LEARNING TECHNOLOGY RESEARCH PROGRAM TRAJECTORIES OF THE NATIONAL SCIENCE FOUNDATION (USA)

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This paper outlines selected developments in mathematics education and learning technology research sponsored by the USA National Science Foundation (NSF) through its Division for Research, Evaluation and Communication (REC). Learning technology-related funding capitalized on and spurred emerging network infrastructures for education and promoted the application of advanced technologies in education. Primarily focusing on mathematics and science education research, the investment helped produce important new technologies. The paper includes an overview of current investment strategies by discussing the evolution of programs at the agency to yield the current programs on Research in Learning and Education and the Interagency Education Research Initiative.

Introduction

The National Science Foundation (NSF) organization includes directorates that correspond to scientific domains for which the agency provides research support. The directorates include Geosciences; Computer and Information Sciences and Engineering; Engineering; Social, Behavioral and Economic Sciences; Mathematics and Physical Sciences; and Biological Sciences. The agency, now celebrating its fiftieth anniversary as an independent executive agency of the US government, annually invests approximately $3.3 billion in around 19,000 projects that advance its charter mission to help secure the growth and vitality of the nation's scientific and technological enterprise. Included in this budget is a seventh directorate, Education and Human Resources (EHR), which supports with an annual budget of approximately $690 million a comprehensive array of activities corresponding to organizational divisions in elementary, secondary and informal education; undergraduate and graduate education; human resource development; reform of educational systems; and research, evaluation and communication (REC). The REC division has maintained responsibility for managing most of the agency's investments in mathematics education, science education, and learning technology research. The paper outlines several of the intellectual and organizational factors that characterized the evolution of NSF's research investments in two of these areas, mathematics education and learning technology, leading to the two principal programs it currently supports, Research on Learning and Education (ROLE; NSF, 2000) and the Interagency Education Research Initiative (IERI; NSF, 1999). The paper includes occasional references to grants in science education in order highlight program features.

A discussion research program trajectories is necessarily abridged in a short paper, with some topics, contributors and themes underemphasized or omitted at the expense of others. (While the summary in this paper is inherently limited to a *small* sampling of projects from a pool of several hundred , summary self-reports covering topics such as research questions, methods, collaborators, and findings for most of the current REC portfolio appears at http://www.ehr.nsf.gov/repp_survey/.) Furthermore, this paper does not represent an official retrospective of the agency, but rather is the author's perspective as the Interim Director for the Division of Research, Evaluation, and Communication (REC). The discussion is grounded in the vehicles through which NSF has implemented its strategies, namely a) the program announcements through which the agency outlines education research and learning technology priorities and approaches, and b) the successful grant applications through which NSF has helped to build a portfolio of learning technology innovation. The funding programs are often referred to as an alphabet soup of activities, because of the commonplace use of acronyms to identify them. Table 1 sketches the programs, their acronyms, funding period, and brief relationships between them.

Technology programs (AAT, NIE, LIS, CRLT)

REC's support over the past ten years in the Applications of Advanced Technology (AAT) Program has been designed to spur R&D in educational applications of computer telecommunications, scientific instrumentation, and other information technology advances. Two of the goals for the program were "to lay research and conceptual foundations to advance knowledge in the use of technology in support of teaching, learning, cognition and problem-solving in science and mathematics for all students" and "to support experimentation on educational and scientific innovations that can significantly improve teaching and learning of increasingly complex content through the use of appropriate technology and technology-based methodologies," (NSF, 1993). The program welcomed, above all else, imaginative thought and talent to advance these goals. It explicitly encouraged proposals in four areas that included technology-based science and mathematics education, interactive tools and learning environment, telecommunications and distributed resources, and knowledge-based systems and intelligent tutors. AAT supported approximately 95 projects before it stopped funding new projects in 1996 (with all projects concluding funding by 1999). It is difficult to isolate samples for AAT or the other programs, because no small collection of awards represents all of the broad themes in the full portfolio of awards.

Among the awards it funded, AAT supported continuing work in software systems for algebra tutors (Anderson, 1992); the use of hypermedia learning environments in classrooms to demonstrate conceptual interrelationships among scientific concepts (Lesgold, 1991); and simulations to disclose -- in understandable and dynamic form -- underlying ideas of calculus, using computers and calculators (Kaput, 1996). Other AAT projects also supported innovations in distance education by Levin (1992) and telecomputing networks by Tinker (1992) in mathematics and science; and profound shifts in enabling learning to occur through rich and dynamic visualizations of scientific phenomena by Pea, Gomez, & Soloway (1992). These and other projects began to disclose to learners mathematical and scientific phenomena in ways that static, textual, or symbolic representations could not, and *produced general findings to the community that appropriately enabled, youngsters previously thought unable to learn challenging content were, to the contrary, quite capable.* The Networking Infrastructure in Education (NIE) program, beginning in 1994, was designed, in part, as a complement or partner to the AAT program, in order to "build synergy between technology and education researchers, developers, and implementers to explore networking costs and benefits, test self-sustaining strategies, and develop models of a flexible educational networking infrastructure that will speed the pace of educational innovation and reform." (NSF, 1996). Additionally, while AAT only funded researchers, NIE also funded activities requiring technology infrastructure.

NIE seeded an important direction for collaborative activities within NSF. While the AAT program funded a few projects with funds from the Computer and Information Sciences and Engineering (CISE) Directorate, NIE was explicitly managed both by the EHR Directorate and the CISE Directorate and became a model for the later cross-agency Learning and Intelligent Systems (LIS) and Collaborative Research in Learning Technologies (CRLT) Programs. NIE also explicitly referenced educational technology infrastructure programs that other agencies in the US government supported, including the Department of Commerce and the Department of Education. In this sense, NIE began a growing pattern of collaboration across NSF's seven directorates and with other agencies. Sample NIE awards include a number that evolved or graduated from AAT support. One of the USA's most prominent electronic K-12 educational communities is the Math Forum, under Eugene Klotz (1996), which provides students, teachers, parents, and other stakeholders with a burgeoning array of high-quality mathematics content and virtual community interaction over the Internet. The Math Forum successfully extended the pre-Internet Geometry Forum (1991) which developed a dialup database of geometry information and content. Soloway was another NIE awardee whose involvement in an AAT project (1995) led in part to his NIE grant. His NIE work involves the use of sophisticated computer and communications environments, equipped with software tools for scientific data collection, analysis, modeling and visualization, to engage students in low performing high schools in authentic scientific inquiry. This project has also led to the development of dynamic digital libraries for middle school students. The work of Klotz, Soloway, and a later REC grantee (Pea, Roschelle, & DiGiano, 1998) provided three models influential in NSF's National Science, Mathematics, and Technology Digital Library Initiative.

Table 1
Major Programs Funding Mathematics Education and Learning Technology Research. Programs overlap with successor programs to fulfil multi-year grant obligations

Earlier Program	Later Program
• Applications of Advanced Technology (AAT), 1991-1999	Provided part of framework and became partner program to Networking Infrastructure for Education (NIE), 1994-1999
• NIE was funded by education and computer science directorates of NSF	Provided cross-directorate funding model and part of framework for Learning and Intelligent Systems (LIS) funded across NSF directorates (1997-2001)
• LIS	Also funded three national centers for Collaborative Research in Learning Technology (CRLT), 1996-2000
• AAT • NIE • Research in Teaching and Learning (RTL) • Studies and Indicators (SI)	Were all subsumed in 1996 in a single program, Research in Education Policy and Practice (REPP), 1996-2001
• REPP	Was replaced in 2000 by Research on Learning and Education (ROLE), 2000-
• LIS, with an education focus	Now supported by ROLE
• Scalability research (including CRLTs) based on proven and technologically-rich approaches	Now supported by Interagency Education Research Initiative (IERI), 1999-

Parallel to the different learning technology advances that AAT and NIE helped to spawn, electronic communications, even before K-12 Internet use, encouraged educators to dream about technology ubiquity and access; this, along with the Internet explosion that the NSF-funded Mosaic-browser project (at the University of Illinois Center for Supercomputing Applications) began in the mid- 1990s, changed the dynamics and mandates of innovation. Under what conditions could access to high quality learning technology "reach scale" -- that is, become integrated into the normal school day or educational experience of all students? This question has been like an onion, giving way one layer at a time. The question coincided with a shift that was partly signalled by the AAT's explicit mention of increasing the quality of education for all students: the agency's EHR Directorate began explicitly departing from focusing primarily on increasing the pipeline of youngsters moving toward careers in mathematics, science and technology. While not discarding this critical goal, the agency began to recognize the moral and social dimensions of the nation's catastrophic failure in K-12 educational systems to insist on and provide high quality mathematics, science and technology instruction for all students. Much of the early 1990's rhetoric that sustained US reform movements throughout the decade has centered on the word "all." "All children can and all children must learn rigorous mathematics, science and technology" was the motto for the Chicago Urban Systemic Initiative, for example (Vallas, 1994). One of several specific examples of this progression involves Kaput (1996), who continued early AAT work and symbolizes important features that cut across a large portion of the portfolio. His NIE project, "SimCalc: Democratizing Access To The Mathematics Of Change" developed mechanisms in which the mathematics students learned would be *different,* partly because it was *deeper* but also because computer and hand-held technology afford different visual embodiments of the mathematics of variability. Indeed, by producing mathematical simulations on low-cost devices, and demonstrating that all youngsters could manipulate sophisticated mathematical objects, Kaput aimed to help dispel the prevalent US cultural myth that mathematics and science are domains for a subset of students rather than for all students.

Collaborative Research in Learning Technologies (CRLT), Learning and Intelligent Systems (LIS)

By 1996, with a mature set of projects in the AAT and NIE portfolio, the agency was already benefiting from a more collaborative and multidisciplinary approach to learning technology R&D. The pattern for increasing disciplinary collaboration was not confined to the funding agency internally, but extended to the guidelines for applicants. NIE (and AAT) not only led to REPP, but also contributed deeply to another new program, Collaborative Research in Learning Technologies (CRLT). CRLT explicitly sought multidisciplinary research teams and the new domains of inquiry and development that their inherently broader perspectives could formulate and advance. The program continues to fund three research testbed centers, all of which have roots in the AAT and NIE programs. One center, Learning Technologies in Urban Systems (Gomez et al, 1997) has effected a collaboration between two large and troubled urban school systems (Detroit and Chicago) and Northwestern University and the University of Michigan. The center, which was originally funded as an AAT testbed before being supported by NIE and then CRLT, explores how sophisticated learning technologies (such as those referenced above, in Pea et al's collaborative visualization project or Soloway's authentic scientific inquiry project) become intertwined in the curriculum reform central to large-scale urban systemic reform. Another activity under Pea (1997) involves a nationally distributed Center for Innovation in Learning Technologies (CILT), that seeks to seed small exploratory research efforts addressing educational problems in collaboration with high-tech firms, with a view to synergizing the partnerships that can accelerate the scaling of innovation. The CRLT Program was embedded in the agency-wide Learning and Intelligent Systems (LIS) program. The purpose of LIS, beginning in 1997, has been to support inquiry into learning and intelligence based on experimental and theoretical frameworks from a variety of disciplines. These include computer science and education, biological sciences, cognitive science, neuroscience, and mathematics, among others. LIS awards provide an important methodological foundation for research on learning and have led to significant breakthroughs (for example, in intelligent tutors, simulation agents, semantic analysis) for classroom computer technologies of the future. Among the most prominent of these in terms of classroom application involves another former AAT and NIE awardee, Roschelle (Pea, Roschelle, and Kaput, 1998), whose project has developed software extensions that enable production of a reusable, digital library of software components based on six of NSF's middle school mathematics curricula.

Education research progams (RTL and REPP)

Thus far, the alphabet soup (AAT, NIE, LIS, CRLT) has a technology focus, albeit one with deep connection to mathematics education, as several of the examples note. The Research in Teaching and Learning (RTL) Program was not primarily a technology program; instead, its primary focus was on research on effective classroom mathematics and science teaching and learning. RTL converged with the AAT and NIE programs relative to generating important and enduring frameworks for later research, however. The three programs were eventually merged in 1996; the new program, Research in Education Policy and Practice (REPP), also subsumed the Studies and Indicators (SI) Program, not discussed in this paper. Both RTL and REPP supported important mathematics education research trends.

Mathematics Education Research in RTL and REPP

Within the mathematics education research funding, themes of various granularity emerged and provide important directions for the future program, ROLE.[1] The COSMOS Corporation was contracted by REC to summarize themes and elements of awards in REC that concluded funding between 1996 and 1998. Their analysis (Kirkland, 1999) confirmed that a bifurcated individual (or radical) and social constructivism reverberated throughout the portfolio of mathematics education research, exemplified by the work of Steffe (1993) and Cobb (1993, 1998). Kirkland cites Cobb's analysis of the distinct approaches and frameworks that each take; NSF provided underlying funding for much of the research that developed and then was based on both paradigms and their amalgams. The mathematics education research community has broadly internalized constructivism in its underlying framework (PCAST, 1997); the framework has characterized more specific lines of inquiry that NSF has supported in mathematics learning. These include research in specific domains such as number understanding and development (Fuson, 1998), multiplicative reasoning at different ages (Confrey, 1994), rational number development, development of statistical concepts (Rubin, 1995; Thompson, 1998) and the mathematics underlying variability and change (Kaput, 1996). A recurrent theme in all of the domain research is the evolution of conceptual models, including the use of technologies either to advance or to represent knowledge construction (Kaput, 1996; Lobato, 1997).

Most research projects have heavily relied on teacher involvement as they have investigated student learning. A significant part of the portfolio explicitly focuses on teaching skills and approaches and on the adaptation of technologies such as video analysis in pedagogical development (Peressini, 1996; Nemirovsky, 1991). Another

[1] One challenge for NSF is to secure for its "rotator" program staff top-flight thinkers who will join the agency to help provide leadership to the research communities of which they are a pan, and help guide the agency in fulfilling its mission to spur research at the most productive frontiers. NSF benefited during the 1990s with researchers such as Richard Lesh, Chris Dade, Nora Sabelli and Eamonn Kelly who helped it maintain a freshness and constant re-examination in light of developments in technology and in the knowledge base of the fields that the research programs were supporting.

thematic overlay in the portfolio is research on curriculum development. Thus, work by Rubin (1995), Kaput (1996) and others tied research on learning to their respective projects' development of experimental curriculum and curriculum technologies (Kafai, 1996; Fuson, 1998). Others, like Romberg (1997), focused on one or more existing reform mathematics curriculum and developing analytic tools and constructs (e.g., opportunity-to-learn, or OTL) to garner insights about optimizing student learning in the ensemble of classroom activity. While most of the technology developed or employed in the portfolio focused on the student, many used technology to advance the research methods. Romberg, for example, led the use of data warehousing in mathematics education research. Nemirovsky not only has advanced video analysis, but also developed an early paradigm for a hyperlinked system of publications. The REPP program subsumed all AAT, NIE, and RTL funding with an explicit intent to increase research in systemic reform strategies -- these strategies included but were not limited to the uses high-quality technology can bring into educational settings to catalyze changes and make them more generally available. The expectation that a unified program could lead to a more coherent approach to reform invited new researchers, including those interested in policy components of educational reform, but also continued support for previous AAT and NIE grantees. REPP funded approximately 150 projects, the majority of which involved some innovation in the use of computer technology in education. REPP efforts often focused on altering educational systems and their approaches to technology. The broader issue of a pervasive commitment by NSF to enable fuller and equitable access to high-quality mathematics and science to those historically underserved characterizes the work of all the EHR Divisions. It has been a theme especially prominent in the development of new portable (hardware and software) technologies that REC has supported (e.g., Pea, 1998; Confrey, 1998) and also the explicit focus of researchers such as Nettles (1999), Barton (1997), and Campbell (1989).

Strategies and challenges leading to ROLE and IERI programs

Hamilton (1999) identified three key strategies that characterized the learning technology and REPP portfolio through the 1990s, including long-term and intellectually robust planning; willingness to sustain higher risk as necessary to secure higher gain; and capacity-building partnership within the research and education community. These strategies produced a number of large payoffs, and they remain benchmark principles for managing the agency's research program. But while REPP spawned a great deal of important research, developments concurrent to its early stages were already suggesting new steps for NSF to follow. The LIS program was producing evidence that interdisciplinary and agency-wide research on learning could yield new strategies for which NSF's organizational mechanisms still required significant evolution. Advances in cognitive neuroscience reinforced the interdisciplinary nature of some of the most consequential research in learning. These advances, however, also continued a pattern of focus on dysfunctionality or disease related research, with minimal attention to the neurophysiologic basis of learning functionality. We are at the very primitive stages of elucidating these phenomena; it has become increasingly the judgement of REC, for example, that research on learning in mathematics classrooms requires bridging to mechanisms that cognitive neuroscience and cognitive science are beginning to understand. While we concur with Bruer's (1999) conclusion that tying brain research to education may constitute a "bridge too far", we believe that now is the time to begin building many small bridges. The National Academy report, *How People*

Learn (Bransford, Cocking, & Brown, 1999), provides a strong, concurring voice to develop a far-reaching science of learning agenda.

The REPP Program, despite intentions to the contrary, also produced less research to uncover the mechanisms of effective educational system reform than the agency had hoped. The agency invests over $100 million per year in educational system reform; these funds leverage several hundred million dollars in additional resources that help spur reform in large urban centers and rural regions nationally; a productive stream of research is critical to the long-term vitality of this work. Absent strong contributions from the REPP portfolio, REC relied on interim invitations ("Dear Colleague Letters" or DCLs) to various research communities to conduct systemic reform research. Funding for devices such as DCLs are generally limited and tend to produce activities that are intellectually and organizationally distinct from the primary programs of their divisions (in this case, of the REPP Program). In general, beyond the need to build a strategic knowledge base in system reform, the marginalization of education research in important public policy debates added further impetus to build tighter and more consequential mappings between warranted research findings and public discourse on mathematics education.

The work of Wilensky (1996), Kaput (1996), Resnick (1993), and others also began to suggest new models for curriculum and for collaborative learning, such as participatory simulations. Such models extend constructivist notions, integrating social cognition and radical constructivist approaches. They also suggest that features of formal complex system theory may provide important new directions or explanatory enablements across the spectrum of the mathematics education interests that NSF holds. For example, the participatory simulation work suggests new curriculum approaches in science and mathematics learning, with far-reaching consequences in pedagogical theory and teacher professional development. Constructs of adaptivity, rule-based agency, levels of scale and emergent or self-organizing behavior not only provide a lens for youngsters to observe the world, but may provide highly useful approaches to educational system reform. Thus, the challenge for REC in managing the evolving of its research program was to respond in connected and coherent fashion to a series of influences that are apparently disconnected: continue to support the maturing education research and learning technology development; attend to advances in neuroscience; take advantage of and extend the interdisciplinary community development in the cross-agency LIS program, bringing the benefits of the LIS approach to education research; rethink the mappings between education research and practice; build a stronger base of knowledge in systemic reform; accommodate the more expansive views of learning and educational systems that complexity theory are beginning to elucidate.

The Research in Learning and Education (ROLE) Program (NSF, 2000) took shape as a successor to REPP; we employed a four-quadrant continuum to organize the program, including 1) brain research as a foundation for research on human learning; 2) fundamental research on behavioral, cognitive, affective and social aspects of human learning; 3) research on science, mathematics, engineering and technological (SMET) learning in formal and informal educational settings; and 4) research on SMET learning in complex educational systems. We state in its Program Announcement that "ROLE seeks gains at the intersections of these areas, where issues arising from research and educational practice can be reconciled, and hypotheses generated in one area may be tested and refined in others," (NSF, 2000). ROLE represents a significant shift for education research at NSF. It is explicitly transitional; while responding to each of the challenges above with what we believe is a stable and robust framework, the aggregate profile of new performers, new research and methods, maturing prior research, and new idea sets will lead to the next evolutionary step for the research program. In some ways, the portfolio of grants under ROLE may itself come to exhibit characteristics of a self-organizing complex system and produce non-trivial and unanticipated phenomena (Hamilton, in publication).

ROLE will continue to support cutting edge research and it is expected to produce fresh innovations and insights across the continuum of quadrants it supports. Funding for such advances, however, does not accommodate research on the scalability of the innovations. Parallel to ROLE is an activity jointly funded by NSF, the US Department of Education, and the National Institutes for Health. The purpose of the Interagency Education Research Initiative (IERI) is to support scalability research on the replicability and adoptability of technologically-rich innovations in mathematics, science, and reading education, innovations for which high thresholds of extant evidence suggest that a concerted effort by several federal agencies is warranted to bring benefits on large scale to as many learners as possible.[2]

References

Anderson, J. (1992). Methodology for Developing a Practical Algebra Tutor. NSF Award 9253161.

Bransford, J., Cocking, R., and Brown, A. (1999). *How people learn.* Washington, DC: National Academy Press.

Bruer, J. (1999). *The myth of the first three years.* New York: Free Press.

Campbell, P. (1989). University of Maryland, College Park. Improving the Mathematical Understanding of Minority Students. NSF Award 8954652.

Cobb, P. (1993). Vanderbilt University. Mathematizing, Modeling, and Communicating in Reform Classrooms. NSF Award 9353587.

Confrey, J. (1990). Cornell University. Splitting, Similarity and Rate of Change: New Approaches to Multiplication and Exponential Functions. NSF Award 9053590.

Confrey, J. (1998). U. of Texas, Austin. Algebra by Design. NSF Award 9896126.

Fuson, K. (1998). Northwestern University. Building Algebraic, Multiplication/Division/Fraction, and Measurement Understandings in Urban Classrooms in English and in Spanish. NSF Award 9806020.

Gomez, L. (1997). Northwestern University. Learning and Intelligent Systems: Center for Learning Technologies in Urban Schools. NSF Award 9720383.

Gomez, L. (1997). Northwestern University. The Living Curriculum Project. NSF Award 9720423.

Gomez, L., Marx, R., Clay-Chambers, J., and Burgess, C. (1997). Northwestern University. Center for Learning Technologies in Urban Schools. NSF Award 9720383.

Hamilton, E. (1999). Strategies for U.S. National Science Foundation (NSF) Investment in Learning Technology Research and Development. Advanced Research in Computers and Communications in Education: New Abilities for the Networked Society (G. Cumming, T. Okamoto, L. Gomez, Eds.). Association for the Advancement of Computing in Education.

Hamilton, E. (In publication) Organizing NSF's Education Research Program As A Complex System of Investigation. *Proceedings of the Third Internanational Conference on Complex Systems,* New England Complex Systems Institute (NECSI), May 2000.

Kafai, Y. (1996). U. of California, Los Angeles. Learning Science by Design. NSF Award 9632695.

Kaput, J. (1996). University of Massachusetts, Dartmouth. SimCalc: Democratizing Access to the

2 IERI emerged in large part in discussions that followed the 1997 *Report to the President on the Use of Technology to Strengthen K-12 Education in the United States* by the President's Committee of Advisors on Science and Technology (PCAST, 1997). Like ROLE, IERI is a new program.

Mathematics of Change. NSF Award 9619102.

Kirkland, E. (1999). GPRA and EHR: Research on Education, Policy, and Practice Program. NSF Contract to COSMOS Corporation 94-52970.

Klotz, E. (1991). Swarthmore College. The Geometry Forum. NSF Award RED 9148693.

Klotz, E. (1995). Swarthmore College. The Mathematics Forum. NSF Award REC 9618223.

Levin, J. (1992). University of Illinois, Urbana-Champaign. Computer-Based Networks for Teacher Education: Science and Mathematics Teaching Teleapprenticeships. NSF Award 9253423.

Lobato, J. (1997). San Diego State U. Generalization of Learning Mathematics Through Multimedia Environments. NSF Award 9733942.

National Science Foundation (1993). *Applications of Advanced Technology.* Publication 93-143.

National Science Foundation (1996). *Research in Education Policy and Practice.* Publication 96-13.

National Science Foundation (1999). *Interagency Education Research Initiative.* Publication 99-74.

National Science Foundation (1999). *Learning and Intelligent Systems.* Publication 99-29.

National Science Foundation (2000). *Research on Learning and Education.* Publication 00-17.

Nemirovsky, R. (1991). TERC. Student's Conceptions of the Mathematics of Change and Variation. NSF Award 9155746.

Pea, R., Roschelle, J., & DiGiano, C. (1998). SRI, Inc. Educational Software Components Of Tomorrow: A Testbed For Suitable Development Of Reusable, Interoperable Object For Middle School Mathematics Reform. NSF Award REC-9804930.

Pea, R. Bransford, J. Means, B. Linn, M., & Tinker, R. (1997). SRI. Learning and Intelligent Systems: Center for Innovative Learning Technology. NSF Award 9720384.

Peressini, D. (1996). Learning to Teach Secondary Mathematics in Two Reform-Based Teacher Education Programs. University of Colorado, Boulder.

President's Committee of Advisors for Science and Technology (PCAST) (1997). *Report to the President on the Use of Technology to Strengthen K-12 Education in the United States.* Washington DC: Office of Science and Technology Policy, Executive Office of the President.

Resnick, M. (1993). M.I.T. NSF Young Investigators. NSF Award 9358519.

Rubin, A. (1995). TERC. Through the Glass Wall. NSF Award 9555641.

Soloway, E. (1995). University of Michigan. Computational Support For Authentic Science Inquiry Computer-Based Support for the Learning of Software Design. NSF Award 9555719

Soloway, E. (1995). University of Michigan. Computational Support For Authentic Science Inquiry. NSF Award REC-9555719

Songer, N. (1992). University of Michigan. Kids as Global Scientists: The Utilization of the Internet for Middle School Atmospheric Science. NSF Award RED 9253464.

Thompson, P. (1998). Vanderbilt University. An Investigation of Multiplicative Reasoning as a Foundation for Teaching and Learning Stochastic Reasoning. NSF Award 9811879.

Tinker, R.. (1992). TERC, Inc. The Global Laboratory II. NSF Award 9250097.

Vallas, P. (1994). Chicago Public Schools. The Chicago Systemic Initiative. NSF Award ESR 9450600.

Wilensky, U. (1996). Tufts University. Participatory Simulations Network-Based Design for Systems Learning in Classrooms. NSF Award 9814682.

TEACHERS' THINKING ABOUT INFORMATION TECHNOLOGY AND LEARNING: BELIEFS AND MATHEMATICAL OUTCOMES

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Constructs about teaching and learning were elicited from teachers of Reception (4-5 year olds), Year 2 (6-7 year olds) and Year 4 (8-9 year olds). Self-ratings on 70 constructs relating to ICT and learning were compared with pupil outcome data, questionnaire information about classroom practice and teachers' self reported level of ICT skill. Internally consistent patterns of thinking emerged for the group of 75 teachers. Clear differences emerged by year group taught in the way these patterns predicted pupil progress. Teachers' attitudes towards ICT and ways of using it were significantly linked with pupil outcomes in literacy and mathematics.

Teachers' thinking and beliefs

There is a growing body of research literature which suggests that the beliefs teachers hold directly affect their perceptions and judgements about teaching and learning interactions in the classroom, and that these in turn affect their teaching behaviours (e.g Nespor, 1987; Eisenhart, Shrum *et al.*, 1988; Cole, 1990; Day 1996).

Particularly strong evidence comes from research into mathematics learning concerning the impact of teachers' beliefs on pupil outcomes (e.g. Thompson, 1984; Ernest, 1989). Fennema and Franke (1992) found that teachers' beliefs about teaching and learning mathematics were associated not only with their pedagogical practices but also with what their pupils learned. Similarly Askew *et al.* (1997) reported that: "What distinguished highly effective teachers from other teachers was a particular set of coherent beliefs and understandings which underpinned their teaching of numeracy."

Methodological difficulties arise, however, in any attempt to study teachers' thinking and beliefs. Beliefs are not directly observable in the classroom and they may be presented differently to different audiences. The approach that we used was based upon personal construct psychology developed by Kelly (1955, 1963). The theory has been applied in a number of fields including teaching and learning. Pope and Keen (1981) applied these ideas to education broadly, and other researchers have used the ideas in specific areas such as secondary teacher training (Diamond, 1985) and primary science teaching (Shapiro 1996).

We chose to elicit constructs through comparisons of familiar activities because the procedure generates a rich variety of language with no expectation that the interviewee will adopt a particular position or will organise aspects of their thinking into a coherent narrative. Pope and Denicolo (1986, 1993) have indicated some of the limitations of interview data alone and the value of other related methods such as repertory grids.

We found the activity helpful in that it provided a self-rating instrument based on the language and concepts which teachers use. We tested the findings of the construct elicitation and ranking exercise through discussion and interview. In some cases we were also able to triangulate patterns of thinking with teachers' responses to questionnaires and with classroom observations. We believe that the patterns of thinking which emerged provided valuable information to compare with other data gathered in the project and that the analysis raises important issues for the role of ICT in teaching and learning.

Constructs about teaching and learning

32 teachers in a Teacher Training Agency (TTA) funded research project (Moseley, Higgins *et al.,* 1999) took part in an activity designed to elicit their thinking about a range of teaching and learning activities and the relationship between ICT and teaching and learning. In the elicitation interviews there were 13 reception class teachers, 7 who taught Year 2 (Y2) pupils and 12 who taught Year 4 (Y4). Randomly grouped sets of three statements describing different teaching and learning activities were presented to the teachers individually on cards, in a process known as triadic questioning. The task was to identify in what way two of the activities were similar, yet different from the third. Teachers' responses were recorded and were taken to represent different ideas or 'constructs' about teaching and learning. These ideas and preliminary relationships were also returned to the teachers individually for discussion and validation and formed part of a further interview with a subset (19) of these 32 teachers.

Construct ranking exercise

The 70 constructs elicited were sorted and organised according to their opposites on a bipolar scale (constructs with more than one opposing idea were presented more than once with the different contrasting ideas). The teachers were then asked to express a preference for one end of the scale or the other (*'teacher-led* or *pupil initiated learning'*, *'open* or *closed activities'*). Following an exploratory factor analysis, twelve reliable composite scales were established for the self-ratings of 19 teachers on 60 of 70 elicited constructs. The twelve composite measures were then related to other data on the same teachers, taken from two questionnaires administered a year apart and from 4 classroom observations on each teacher. The results of this exercise were again presented to the teachers as part of a final interview so that they could give examples of how they thought their preferences affected their teaching and to check that the information was accurate. (Full details of these preliminary patterns and the 12 scales or dimensions of which they are composed, are given in Moseley, Higgins *et al.* (1999) Appendix 4.)

Expanding the sample

A number of further relationships emerged from this initial research when the construct data were compared with data from the questionnaires and with the relative pupil performance data from the Performance Indicators in Primary Schools (PIPS) project at Durham University. For example, more experienced teachers were likely to favour a more formal, subject-based approach, but there was no statistically significant link with the relative attainment of their pupils. The number of teachers involved in this aspect of the TTA project was small (19) and any conclusions could only be tentative, so the construct ranking exercise was sent to a random sample of 125 of the 250 teachers who had completed questionnaires as part of the early stages of the research. A return and consent rate of 45% (56) yielded a total set of 75 teachers for whom there was construct ranking data, as well as permission to access pupil performance data from the PIPS project. Of the sample of 75 teachers there were 29 who taught reception classes throughout. There were 21 other teachers who taught Y2 classes for part or all of the three years under consideration and there were 25 teachers who were predominantly teachers of Y4 pupils.

PIPS data

Improvements in reading, number and attitude made over the previous year (in reception) and over the previous two years (in Y2 and Y4) are assessed by PIPS project schools. In calculating 'value-added' scores, pupil and school contextual factors are taken into account, so that like is compared with like. For reception teachers, the value-added score refers to the effectiveness of a particular teacher in the previous year; in Y2 and Y4 classes, the gains made reflect (in most cases) the efforts of two different teachers over the previous two-year period. (Contact details for more information about this approach to monitoring pupil outcomes are given in the acknowledgments section at the end of the paper.) Value-added statistics from 1996-7 were provided by PIPS for 32 classes in which observations were carried out. These statistics were updated a year later with the data for the year in which classroom observations were conducted (1997-8) and made available for the teachers who completed the construct ranking exercise in 1999.

Patterns of thinking and belief

The next aim was to produce internally consistent groups of items which would apply across the year groups taught and would represent widely-shared dimensions or common patterns of thinking and belief. This was achieved by eliminating 11 items which in an exploratory factor analysis (principal components with varimax rotation) were found to constitute individual factors. The remaining 59 items were then grouped according to their highest loadings on the factors, but following the rule that each group should have at least three members. This resulted in seven non-overlapping groups of items, each internally consistent, but to varying degrees correlated with each other. These groups are listed in Table 1. Table 2 provides verbal summaries of the components of the identified patterns of thinking and belief. (We follow the convention of using a single asterisk to denote statistical significance at below the 5 percent level ($p<0.05$) on a two-tailed test, and two asterisks for the one percent level ($p<0.01$).)

Table 1

Pattern Descriptors with Cronbach's Alphas for the Total Sample of Teachers and for Three Subgroups Identified by Pupil Age

Descriptor	Rec. n=29	Y2 n=21	Y4 n=25	α for n=75
Teacher direction	0.86	0.89	0.90	0.87
Formal subject-based class teaching	0.75	0.71	0.85	0.80
Pupils coping with degrees of uncertainty and difficulty	0.78	0.87	0.88	0.86
Use of discussion rather than routine activities	0.79	0.85	0.78	0.80
Working with other pupils rather than alone	0.76	0.83	0.78	0.79
Preference for teaching mathematics rather than English	0.92	0.83	0.69	0.86
A positive attitude towards ICT	0.83	0.74	0.89	0.83

Comparisons by year group taught

Not only were there were broad similarities across the three year groups of teachers in their patterns of thinking: there were few consistent year-group-related trends and only one where a statistically significant difference appeared. Teachers of Y4 pupils indicated that they took a relatively formal approach, whereas reception teachers took the least formal position (one-way ANOVA with post-hoc Bonferroni comparison of means, $p<0.05$).

Table 2
Patterns of Teachers' Thinking, with Cross-Pattern Correlational Links

Pattern descriptor	Pattern components	Pearson r's with:	
Teacher direction (a_dir)	Teacher decides, directs, teaches, leads, uses structured activities, guides, instructs, organises, plans, does not use questioning, teaches strategies, and makes pupils react rather than reflect.	a_open	-0.62**
		a_form	0.55**
		a_coll	-0.54**
		a_ict	-0.46**
		a_diss	-0.29*
Formal subject-based class teaching (a_form)	Written or paper-based activities, formal, academic, subject-related. Emphasis on class teaching rather than on meeting individual needs. Pupils are expected to think, not to talk to each other.	a_dir	0.55**
		a_coll	-0.43**
		a_open	-0.26*
Pupils coping with degrees of uncertainty and difficulty (a_open)	Preferred tasks have more than one answer, often being open-ended, new and/or complex. Pupils explore problems, take initiatives, make sense, think, monitor their progress and learn more than one thing at a time.	a_dir	-0.62**
		a_ict	0.55**
		a_diss	0.41**
		a_form	-0.26*
Use of discussion rather than routine activities (a_diss)	Learning involves discussion, is purposeful and active, and has specific, public, observable outcomes.	a_open	0.41**
		a_dir	-0.29*
		a_ict	-0.28*
Working with other pupils rather than alone (a_coll)	Learning is collaborative and takes place in groups through social interaction.	a_dir	-0.54**
		a_open	0.49**
		a_ict	0.47**
		a_form	-0.43**
Preference for teaching maths rather than English (a_math)	Teaching maths/numeracy is preferred to teaching English/language work.		
Positive rather than negative attitude towards use of ICT (a_ict)	Classroom use of ICT, machines and computers is viewed favourably.	a_open	0.55**
		a_coll	0.47**
		a_dir	-0.46**
		a_diss	0.28*

Teachers' thinking and relative pupil attainment

Taking the sample as a whole (n=64), there were no significant correlations between the seven patterns of thinking and the relative pupil attainment data. When examining a correlation matrix without prior hypotheses to test, there is always a possibility of treating significant coefficients as meaningful and thereby making Type 1 errors (rejecting the null hypothesis as false when it is fact true). Sakoda *et al.* (1952) prepared tables for the evaluation of how many significant results are needed in a series of statistical tests for that number to be regarded as beyond chance expectation. These tables were consulted throughout our analyses, and due caution is expressed in relevant cases. However, we also used a triangulation approach when seeking support for the interpretation of statistically significant results, drawing on the questionnaire responses as well as on our knowledge of the teachers we had observed, interviewed and worked with as partners in action research.

Reception (4-5 year olds)

For 29 reception teachers, the only pattern of thinking which had a positive significant link with value-added was having a negative attitude towards using ICT (r=0.45* with the combined measure for mathematics and reading and r=0.50** for reading alone). According to Sakoda *et al.* (1954), this result is vulnerable because of the number of comparisons made. Confirmatory evidence was available, however, from the 1998 questionnaire in that overall value-added was significantly higher for reception teachers who believed print resources to be better than ICT resources (r=0.42*) and who believed their ICT skills to be inadequate for using ICT in their teaching (r=0.43*).

Computers are often used in reception classes as a reward or extension activity available to pupils who have finished other work. Our data suggest that this may be counter-productive so far as pupils' learning is concerned. On the positive side, there was some evidence that more focussed and cognitively demanding types of activity such as *'analysing patterns and interconnections'* on a computer can help develop mathematical attainment (r=0.45*).

To summarise: it seems that it is not a good idea for computers to be used in reception classes to keep some pupils busy, or simply as a motivational attraction (*'playing on the computer'*) or to learn how to find information simply as a computer skills exercise rather than for the purpose of using that information or understanding patterns and connections.

Year 2 (6-7 year olds)

For 16 teachers of Year 2 there were several significant links with the dimensions of thinking and the overall average value-added results. The open task preference dimension of these teachers' thinking correlated positively with pupils' progress in reading (r=0.58*), as did the pro-ICT dimension (r=0.50*). The relationship between progress in mathematics and teacher preference for open tasks was also positive, although not quite significant (r=0.49 ns, p=0.06), and there was a positive association between mathematics progress and a preference for discussion (r=0.50*). According to Sakoda *et al.* (1954), it is unlikely that these results are entirely chance effects, so we can safely infer that an effective Y2 teacher is one who promotes open activities.

High overall value-added was significantly associated with only one of the single-item constructs: a preference for exciting rather than routine activities (r=0.64**). This item was itself significantly linked with the four dimensions mentioned above as being indicators of value-added (the open, collaborative, pro-discussion and pro-ICT dimensions). One interpretation could be that to be effective with Y2 pupils, teachers need to engage their interest through enthusiastic teaching and exciting activities where the children have to make some decisions about what is expected.

For attitudes to ICT there were a number of significant correlations with value-added. For example the view that *'In my school there is not enough information about published educational software'* (r=-0.64**) and *'Most software is too complicated for my pupils to use'* (r=-0.61*) were correlated with reading value-added scores. A positive view that *'I think pupils like to learn with computers'* (r=0.53*) and disagreement with the view that *'Teachers do not need to use ICT to be effective'* (r=-0.56*) were significantly correlated with value-added in mathematics. This pattern of results is internally consistent and is statistically unlikely to be a chance result. We now have evidence from two sources of a positive link between value-added and a pro-ICT attitude in Y2 teachers - a link which is just the opposite of what we found in reception teachers.

The question now arises as to whether effective Y2 teachers differed from others not only in their attitude towards ICT but in their practice. Here we found that although the more effective teachers reported a significantly higher level of ICT provision (r=0.56*), they tended to make rather infrequent use of it. Although the levels of correlation did not reach significance, five out of six measures of class and pupil computer usage had inverse relationships with value-added. Moreover there was no evidence that any particular ways of using computers were better than others.

To summarise: we are left with the impression that for this sample of teachers, computers were important not so much because they were used, but because they formed part of a stimulating environment in which pupil enquiry and collaboration were encouraged by teachers who were themselves open to new ideas.

Year 4 (8-9 year olds)

For teachers of this year group there were no significant correlational links between value-added and any of the seven patterns of thinking or the 11 single-item constructs.

As we found for the sample as a whole (see Table 2) there was a clear tendency among the Y4 teachers for teacher direction to be linked with a negative attitude towards ICT. A directive approach was associated with a negative attitude towards ICT (r=0.46*) and with the belief that ICT resources are no better than print resources (r=0.80**). Teacher direction was linked with preference for a formal approach (r=0.66**) with little open-ended enquiry (r=-0.60**) and little interaction between pupils (r=-0.44*). A directive approach was generally preferred by the more experienced teachers (r=0.62**), but was not significantly related either to value-added in either reading (r=0.27, n.s.) or mathematics (r=0.27, n.s.).

There were, however, a number of significant correlations between the types of computer usage reported in 1997 and value-added outcomes (see Table 3).

Table 3

Four Types Of Computer Usage (out of 13) Where Usage was Significantly Correlated with Mean Value-Added Pupil Outcomes 1997-99 (n=19)

Usage descriptor	r with maths value-added	r with reading value-added
To demonstrate something to the whole class	0.48*	0.46*
As a reward	-0.34	-0.54*
As part of free-choice activities	-0.55*	-0.63**
For pupils to use when they have finished classwork	-0.47*	-0.46*

It appears that teachers of this age group who let children to use computers without particular learning targets in mind get much worse results than those who use a computer themselves as part of class teaching. In this case it is very likely that the results are not attributable to chance, as seven out of the complete set of 26 coefficients were statistically significant at or beyond the 5% level. According to Sakoda *et al.*, this is likely to happen by chance less than once in a thousand examples.

When the computer usage measures and the thinking patterns were entered into stepwise regression equations in order to predict value-added, the variables which came through consistently as predictors for both mathematics and reading outcomes were (1) not using computers as part of free-choice activities and (2) teacher direction. These two measures predicted overall value-added to a level of r=0.85**, which means that they accounted for 72% of the variance.

To summarise: it appears that for this age group reported computer use predicted pupil learning outcomes as measured by PIPS better than the construct ratings. The evidence supports the use of ICT for clearly-defined purposes, whether teacher-determined or pupil-determined.

Teachers' thinking and ICT

As discussed above, teachers' thinking and beliefs play an essential role in their classroom practices and affect their teaching and learning interactions. Honey and Moeller (1990) suggested that unless teaching practices change, technology will not be widely integrated into primary classrooms because of a mismatch between teachers' beliefs about teaching and learning and their perceived value of educational ICT. There is also growing evidence that existing approaches to teaching and learning have a powerful inertia and that schools tend to *assimilate* rather than *accommodate* new approaches, particularly with ICT (Tyack and Cuban, 1995). Research also shows that teachers who use computers do so because their conceptions of using ICT already fit within their existing notions of effective teaching practices (Hadley and Sheingold, 1993) or that they assimilate innovative practices using ICT according to these beliefs and thinking (Higgins and Miller, 2000).

Conclusions and implications

Distinct patterns emerged from our analyses of teachers' thinking. These patterns were common across the age ranges of pupils, although teachers of younger children preferred a significantly less 'formal' approach. Taking the sample as a whole there were no significant correlations with the PIPS relative pupil outcome measures. However, when these patterns were examined separately by year group taught, some associations were apparent. This suggests that while teachers of primary pupils (4-11 years) have similar beliefs about teaching and learning some of these beliefs are associated with the progress that their pupils make according to the year groups taught.

In reception a degree of scepticism about the value of ICT and purposeful use of computers was a positive indicator of pupils' progress. Teachers of Year 2 pupils who valued more open-ended tasks, favoured ICT (though were selective in how they used it) and favoured exciting teaching were likely to be more effective. For teachers of Year 4 the use of ICT for demonstration and purposeful use by pupils as well as the teacher direction dimension were associated with pupils' progress.

There are clear implications for professional development generally and with ICT in particular. Such professional development needs to take account of teachers' thinking about teaching and learning generally as well as their attitude and use of ICT. These considerations should include the year group taught by the teachers as predictable patterns are evident even within the primary age range.

Our data certainly support the view that an understanding of teachers' thinking and beliefs is vital. The investigation described above also suggests, however, that taking teachers' thinking and beliefs into account in developing effective professional development is a complex task. An important component in this process is information about pupils' progress so that beliefs about effective practices generally and about the effective use of ICT in teaching mathematics in particular are grounded in the impact that such beliefs and practices have on pupils' learning.

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References

Askew. M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective Teachers of Numeracy Final Report,* London: King's College, London.

Cole, A. L. (1990). Personal theories of teaching: Development in the formative years, *Alberta Journal of Educational Research,* 36, 203–222.

Day, C. (1996). Teachers beliefs and knowledge in D. C. Berliner and R. C. Calfee, (Eds.) *Handbook of Educational Psychology*, New York: Macmillan.

Diamond, P. (1985). Becoming a teacher: an altering eye in D. Bannister (Ed.) *Issues and Approaches in Personal Construct Theory,* (pp. 15–35) London: Academic Press.

Dwyer, D. C., Ringstaff, C., & Sandholtz, J. H. (1991). Changes in teachers' beliefs and practices in technology-rich classrooms, *Educational Leadership*, 48, 45–52.

Eisenhart, M. A., Shrum, J. L., Harding, J. R., & Cuthbert, A. M. (1988). Teacher beliefs: Definitions, findings, and directions, *Educational Policy*, 2(1), 51-70.

Ernest, P (1989). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching*, 15(1), 13–33.

Fennema, E. and Franke M. L. (1992). Teachers' knowledge and its impact, in D. A. Grouws (Ed.) *Handbook of Research on Mathematics Teaching and Learning* (pp. 147–164) New York: Macmillan.

Hadley, M., & Sheingold, K. (1993). Commonalities and distinctive patterns in teachers' integration of computers. *American Journal of Education*, 101, 261–315.

Higgins, S., & Miller, J. (2000). ICT as support for development in literacy and numeracy teaching *Professional Development Today,* 3(2), 49–56.

Honey, M., & Moeller, B. (1990). Teacher's beliefs and technology integration: Different values, different understandings, *CTE Technical Report*, Issue No. 6, August 1990.

Kelly, G. A. (1955). *The Psychology of Personal Constructs*, Volumes 1 and 2, New York: Norton.

Kelly, G. A. (1963). *A Theory of Personality,* New York: Norton

Moseley, D., Higgins, S., Bramald, R., Hardman F., Miller, J., Mroz, M., Newton, D., Tse, H., Thompson, I., Williamson, J., Halligan, J., Bramald, S., Newton, L., Tymms, P., Henderson, B., & Stout, J. (1999). *Ways forward with ICT: Effective Pedagogy using Information and Communications Technology in Literacy and Numeracy in Primary Schools*, Newcastle: University of Newcastle, September 1999.

Nespor, J. (1987). The role of beliefs in the practice of teaching, *Journal of Curriculum Studies*, 19, 317–328.

Pope, M. L. and Keen, T. R. (1981). *Personal Construct Psychology and Education* London: Academic Press.

Pope, M. and Denicolo, P. (1986). Intuitive theories – a researcher's dilemma: Some practical methodological implications, *British Educational Research Journal,* 8(1), 45-56.

Pope, M. L. and Denicolo P. (1993). The art and science of constructivist research in teacher thinking *Teaching and Teacher Education,* 9(5/6), 529–544.

Sakoda J. M., Cohen, B. H. and Beall, G. (1954). Tests of significance for a series of statistical tests, *Psychological Bullletin,* 51, 172–5.

Sandholtz, J. H., Ringstaff, C., and Dwyer, D. C. (1996). *Teaching with technology: Creating student-centered classrooms*, New York: Teachers College.

Shapiro, B. L. (1996). A case study of change in elementary student teacher thinking during an independent investigation in science, *Science Education,* 5, 535–560.

Thompson, A G (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, 15, 105–127.

Tyack, D. and Cuban, L. (1995*). Tinkering Towards Utopia: A Century of Public School Reform,* Cambridge, Ma: Harvard University Press.

SUPER–CALCULATORS AND CONCEPTUAL UNDERSTANDING OF THE NEWTON–RAPHSON METHOD

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One of the issues facing educators while using calculators in their mathematics classrooms is how to avoid an over–emphasis on procedural and computational aspects of calculator use. Whilst these are valuable facets of calculator use they may also be employed to enhance understanding of mathematical concepts. In this study we investigated the value of the TI–92 super–calculator for building understanding of some of the concepts associated with the Newton–Raphson method of finding zeros of functions, since this is a method which can easily become purely algorithmic with little understanding. The results of this study with 16–17 year olds students suggest that student understanding of concepts involved with the method improved, and that students developed a positive view of the role of the calculator.

Background

Within the past decade, mathematics educators have increasingly been concerned with finding ways to integrate graphic calculators, into the school mathematics curriculum in order to make use of the environments they provide. Graphic calculators are more accessible to students than computers in many schools (Kissane, 1995; Thomas, 1996) and this is a key advantage. There are, of course, still equity issues surrounding their use, based on cost, availability, etc. and these need to be addressed. However, many believe that such technology can change the way students think about mathematics, and can bring about opportunities for new content, new curricula, and new teaching techniques (Abramovich & Brown, 1999). In spite of this, much of school mathematics is still devoted explicitly to the manipulation of formal notational systems, despite considerable effort to change classroom practice to emphasise problem solving strategies, visualisation, pattern recognition, and other more conceptually oriented techniques (Goldin, 1998). Even with the introduction of calculators into a teaching programme there is often very little change in this situation. Continued research is necessary to provide evidence of whether, and where and how, graphic calculators can usefully be integrated into the mathematics curriculum. The growing body of research supporting their use in mathematics learning (see e.g. Ruthven, 1990; Dunham & Dick, 1994; Penglase & Arnold, 1996; Graham & Thomas, 2000), is still restricted in terms of the approaches used, the content areas covered, and the calculator facilities employed. The present research was considered part of the process of extending this body of evidence.

The use of graphic calculator provides an opportunity to promote investigation and discovery in mathematics classrooms through a multi–representational approach to problem solving. There are a number of different uses of the word representation in the literature but the one Kaput (1987, p. 23) proposes that "any concept of representation must involve two related but functionally separate entities. We call one entity the representing world and the other the represented world." is the one which resonates best with us. The super–calculators, such as the TI–89 and FX2 have particular value in investigations of functions, since they can represent them in a variety of dynamically linked ways, such as algebraic symbol form, ordered pairs, graphs and tables of values.

Translating between these various forms is something which an experienced teacher might take for granted, but to do so one needs to have an overview of the way the concept of function relates to each representation, and how sub-concepts, such as independent and dependent variables, one-to-one, roots, discrete, continuous, etc. are manifest in each representation (Chinnappan & Thomas, 1999; Hong, Thomas & Kwon, 2000). A precise view of these, and the processes which apply to each representation can help one make judgements about processes which, unlike concepts, may not travel well between representations. This research adopted the approach of students using the super-calculator as a means of improving concept formation across representations, developing mathematical reasoning, and linking different mathematical ideas via investigation.

Method

The Form 6 students (age 16~17 years) involved in this research project comprised an advanced class from a high school in Auckland, New Zealand. The class had previously covered all the required university entrance examination work, including calculus and the Newton–Raphson method (sometimes called Newton's method), which they covered as part of the Statistics course. Although experienced in using calculators and computers in their mathematics learning (the school supplies each student with a laptop computer) they had only studied the Newton–Raphson method using pencil and paper methods. The research project was carried out in the school during the period 16^{th}–26^{th} August, 1999, with the pre–test on August 16^{th}, the module of work in the following 7 days and the post–test and questionnaires on August 24^{th} and 26^{th}.

Instruments

A module of work, containing a description of the basic facilities of the TI-92 and addressing the concepts involved in the Newton–Raphson method, was prepared using a 'Press', 'See', and 'Explanation' format (see Figure 3). In the Newton–Raphson numerical method of solving equations, an initial value $x = x_1$ near a solution is chosen, the tangent to the graph at $(x_1, f(x_1))$ is drawn, and the next approximation is taken as the point where the tangent line intersects the x-axis as shown in Figure 1. In the module two methods were presented: first a visual method using the TI-92 and based on the idea that we can often get nearer to a root α by drawing tangents at estimates; secondly using the equivalence of the gradient of the tangent, $f'(x_1) = \dfrac{f(x_1)}{(x_1 - x_2)}$ to obtain a formula.

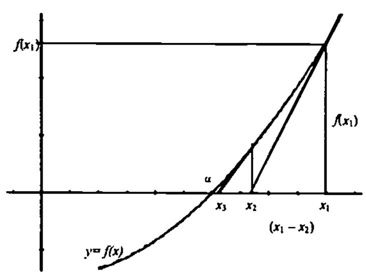


Figure 1. A diagram illustrating the Newton–Raphson method

Two tests comprising questions on the Newton–Raphson method were compiled, based on standard New Zealand school calculus textbooks. These tests were essentially parallel tests using different numerical values, divided into sections A and B, comprising questions highlighting process–oriented skills (Thomas, 1994) and conceptual understanding, respectively. The students used the super–calculators while sitting the tests. In order to assess students' knowledge of the background pre–requisites for understanding Newton's method the tests also included questions on differentiation, limit and the use of the Newton–Raphson formula $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, which the students were familiar with before the study. For example, section A comprised questions such as:

A2. Find: a) $\lim_{x\to 0}\frac{\sin 3x}{2x}$ b) $\lim_{x\to 0}\frac{\tan 2x}{3}$ c) $\lim_{x\to\infty}\frac{2x-1}{3x}$

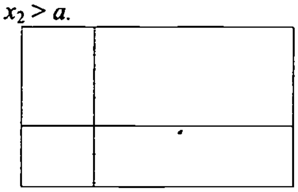
A3. Calculate the second approximation x_2 to the root of $f(x) = 0$, using the Newton–Raphson method, when $f(x) = \sin x - x$, and $x_1 = \frac{\pi}{2}$.

A4. What is the value of x at the point on the curve $y = x^2 + 7x - 8$ where the gradient is equal to 1?

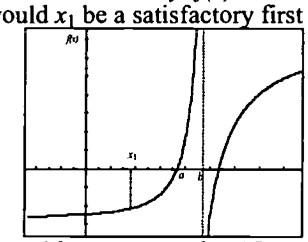
In contrast, section B sought to address students' thinking about the concepts when finding zeros or solving equations. We wanted to know if they knew how and why the method worked and could apply this understanding.

B3 If, in the Newton–Raphson method, for a function $y = f(x)$, $f(x) > 0$, $x_1 = 2$ and $f'(x_1) > 0$, where x_1 is the first approximation to the root, is $x_2 > x_1$ or is $x_2 < x_1$ where x_2 is the second approximation to the root? Explain your answer.

B6b) Draw a continuous function below where, if x_1 and x_2 are the 1st and 2nd approximations to the root $x=a$ using the Newton–Raphson method, then $x_1 < a$, and $x_2 > a$.



B4 a) Explain why x_1 in the diagram alongside is an unsatisfactory first estimate in the Newton-Raphson method for the root $x=a$ of $y=f(x)$.
b) When would x_1 be a satisfactory first estimate?



B7 How could you use the Newton–Raphson method to find the x-value of the intersection of the graphs of $y=2e^{-x}+\cos x$ and $y=2$? Explain your method clearly.

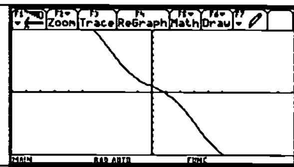
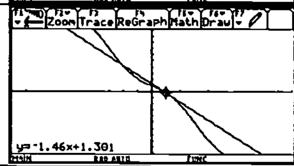
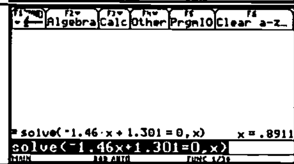
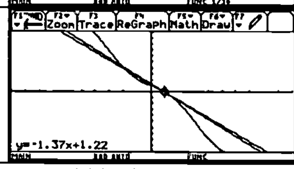
Figure 2. Some section B post–test questions on the Newton–Raphson method

A selection of four of the section B questions on the Newton–Raphson method is shown in Figure 2.

Procedure

The first researcher met twice with the students' normal classroom teacher to answer his questions and to make sure that he felt comfortable with the TI–92 calculator and the material to be presented. Following this the teacher taught the class for four lessons, covering basic facilities of the calculators including graphs and tables, how to find limits and a gradient function and how to implement the Newton–Raphson method both visually and symbolically on the TI–92. Each student had access to their own TI–92, which they kept with them for the whole of the time of the study, including their time at home. During lessons, which were observed by the researchers, the class teacher stood at the front and the class sat in traditional rows of desks. He demonstrated each step, employing a calculator viewscreen and projecting the image on his calculator using an overhead projector, while the students followed in the module and copied his working onto their own calculator. Following this the students, working individually but discussing progress with others, attempting the questions and investigations in the module. A section of the module illustrating the layout, the teaching approach used, and giving one of the two methods used for solving the equation $\sin x = 2x - 1$, is presented in Figure 3.

Example. Solve the equation $\sin x = 2x - 1$ using the Newton-Raphson method. Give the answer to 4 d.p.
The first step is to define the function $y = \sin x - 2x + 1$ and sketch its graph:

Press	See	Explanation
Method 1. • [Y=] SIN x) − 2x + 1 ENTER • [GRAPH]		The function is defined using $y1 = \sin x - 2x + 1$
F5 A:Tangent ENTER Tangent at? 1 ENTER		The first tangent line starts at the point $x_1 = 1$. The equation of the tangent is given as $y = -1.46x + 1.301$. This will be used to find x_2 (which should be closer to the root than x_1), by seeing where it crosses the x-axis.
• [HOME] F2 1 ENTER (-) 1.46 x + 1.301=0, x) ENTER		We can see that the next point $x_2 = 0.8911$
Return to [GRAPH] F5 A:Tangent ENTER Tangent at? 0.8911 ENTER		The equation of the tangent is given as $y = -1.372x + 1.218$ at the point $x_2 = 0.8911$. This will be used to find x_3 (which should be closer to the root than x_2).

This was then repeated until $x = 0.8882$ is obtained.

Figure 3. A section of the module on the Newton–Raphson Method showing the layout

Each session lasted 50 minutes, and after the teacher's explanation, the students spent the rest of the time working on the practice exercises and investigations while the teacher circulated and assisted with any problems. The final five minutes were used for a summary. An example of the type of investigative questions the students worked on using the calculators during the module is given below.

A function $f(x)$ is such that: $f(x)=0$ has only 2 solutions $x=a$ and $x=b$. Starting the Newton–Raphson method with $x_0>b$ it converges to $x=a$, where $b>a$. Can you (i) Sketch such a function $f(x)$? (ii) Find a possible formula for such a graph?

In order to answer a question such as this one can use basic concepts and principles to investigate possible solution functions, visualising their graphs. This requires linking data represented algebraically to a graphical representation, including tangents. Finally one has to model the solution symbolically, moving in the opposite representational direction. At the end of fourth tutorial the students were given the post-test, followed by an attitude test and a questionnaire about the teaching programme, their learning and the calculator.

Results

We can see from Table 1 that the 17 students who sat both tests performed significantly better in the post–test on both section A and B questions (section A; $m_{\text{pre}}=5.35$, $m_{\text{post}}=6.35$, $t=2.21$, $p<0.05$: section B; $m_{\text{pre}}=5.59$, $m_{\text{post}}=8.53$, $t=4.65$, $p<0.0005$). Thus the results on this small sample appear to support the fact that, overall the use of calculator had been a positive influence on the students' ability to answer questions involving conceptual understanding as well as more procedural ones.

Table 1

A Comparison Between the Overall, Section A and B Pre– and Post–Test Results

N=17		Mean	*t*	*p*
Overall	Pre–test	10.94	4.88	<0.0005
	Post–test	14.88		
Section A	Pre–test	5.35	2.21	<0.05
	Post–test	6.35		
Section B	Pre–test	5.59	4.65	<0.0005
	Post–test	8.53		

Comparing the performance on the individual questions, we can see from Table 2 that for questions on the Newton–Raphson method, namely A3, B3, B4, B6, there was a significant improvement between the tests (question B5 proved somewhat difficult even after the work).

Table 2

The Pre– and Post–Test Section A and B Mean Scores

Question Number (Max score)	Pre–test mean	Post–test mean	*t*	*p*
A1 (3)	1.59	1.82	1.07	n.s.
A2 (3)	1.18	1.82	3.39	<0.005
A3* (2)	0.82	1.03	1.1	n.s.
A4 (2)	1.76	1.68	0.55	n.s.
B1 (4)	0.35	0.82	1.73	=0.05
B2 (2)	1	1.53	1.85	<0.05
B3* (2)	0.76	1.29	2.31	<0.05
B4* (3)	1.12	1.59	2.22	<0.05
B5*(2)	0.47	0.47	0	n.s.
B6* (4)	1	1.94	3.24	<0.05
B7* (2)	0.88	0.88	0	n.s.

* Questions on the Newton–Raphson Method

However, in addition, the students also improved on the procedural questions A2 and B7, although they were able to use the calculator. A consideration of the working of individual students gives some idea of where changes in thinking may have occurred.

Individual students

In their pre–test solutions to question B3 students often appeared to guess whether the second approximation to the root was greater or less than the first approximation. Since a general function $f(x)$, rather than an explicit function, was given them, to answer the question they needed to apply the relationship between the first approximation and the second approximation to the root, using the given conditions on $f(x)$ and $f'(x_1)$. For example, student E1, whose mark was 7 (24%) in the pre-test but 12 out of 29 (41%) in the post-test, showed a considerable improvement in understanding of the role of the sign of the function in the Newton–Raphson method, as shown in Figure 4. From his responses to question B3 we can see that in the pre–test he attempted to use a procedural method based on the formula he had learned prior to the research study, but was unable to proceed due to the lack of an explicit function.

In the post–test he gave a full solution, along with a written description. This solution did not involve visualising the graph and the tangent but worked again with the algebraic representation, the formula. However, it was now based on general principles of the sign of the two functions $f(x)$ and $f'(x)$, as highlighted in the module.

Pre–Test Post–Test

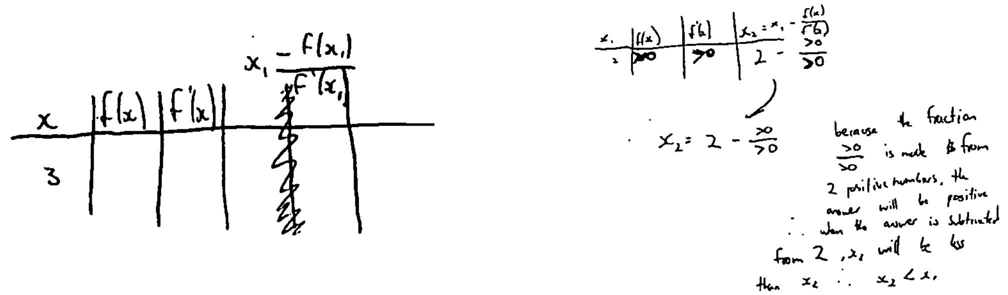


Figure 4. Student E1's pre–test and post–test working for question B3

His description of his thinking about the processes involved in the Newton–Raphson method and that he was now able to think visually about them in the graphical representation is apparent from his comments in the questionnaire. These show, for example, that he had understood the way the tangent influences the solution process.

Question: How does the Newton–Raphson method work?
Student E1: Tangents on the graphs bring you closer to the root of the graph.
Question: Has this tutorial affected your understanding of the Newton–Raphson method? If so how?
Student E1: Yes seeing the tangents going closer to the intercept. Also when solving the equation, helped me understood [sic].

He now appears to have a good mental image of the way that the tangent can be used to move one closer to the zero of the function. A second student, W3, who was unable to attempt question B3 on the pre–test also gave a correct answer on the post–test, as seen in Figure 5.

Pre–Test
No response

Post–Test

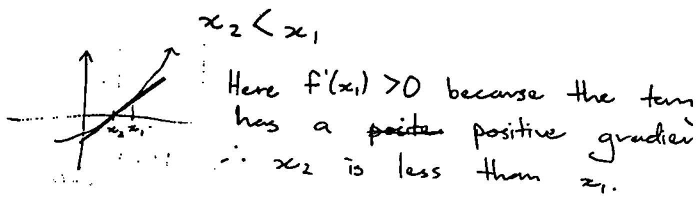


Figure 5. Student W3's pre-test and post-test working for question B3

He is able to connect the symbolic representation of the gradient function, *f'(x)*, with the tangent in the graphical representation and, working here, to see how this influences the second approximation.

This was supported by his questionnaire comment on the principle behind the method, where he replied:

Question: How does the Newton–Raphson method work?
Student W3: It finds a root of a graph by finding out where the tangent of a close guess cuts the *x*-axis.

Question B4 (see Figure 3) tested understanding that Newton's method does not always converge to the zero of the function. Rather, it is necessary to consider first the number of zeros and their position and to be aware that Newton's method might fail unless an appropriate starting value for each zero is chosen. Their experience in the TI–92 module of solving the equations $f(x) = e^x - x - 2$ and $f(x) = \begin{cases} \sqrt{x-2}, & x \ge 2 \\ -\sqrt{2-x}, & x \le 2 \end{cases}$ using $x =$

0 and 1 as starting values, helped to show them examples of where if the process is started where the graph is very curved (or horizontal), it may give another solution, or no solution at all.

Student F1 showed a noticeable improvement in this question, using the tangent in the given diagram, as Figure 6 shows. Apart from the incorrect use of the word gradient for tangent, student B4 was unable to explain in the pre–test why x_1 is unsuitable, or when it would be so.

Pre–Test

The gradient of x_1 cut the x-axis on the right side of b

Post–Test

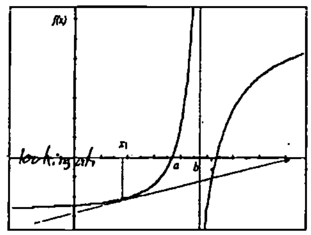


Figure 6. Student F1's pre-test and post-test working for question B4

However, in the post–test student F1 has seen in the graphical representation that the second approximation needs to lie on the left hand branch of the graph and has correctly transferred this to the symbolic domain, writing that this will occur when $x_2>a$ but $x_2<b$. This shows a grasp of how the concepts in the method relate to both the visual, graphical representation and the symbolic. On question B6b) (see Figure 3) they were asked to sketch a graph on the axes with the first approximation, $x_1<a$, and the second, $x_2>a$. Student L2 drew a graph (shown in Figure 7) and included the tangents, demonstrating again understanding of their role in deciding the position of the approximations relative to the zeros.

Pre–Test Post–Test

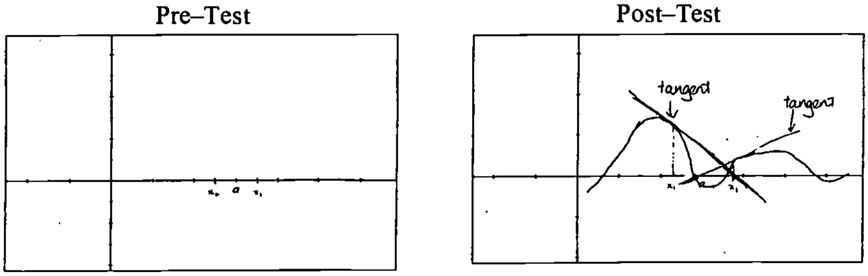


Figure 7. Student L2's pre-test and post-test working for question B6

Interestingly he has a function with 2 zeros, which was not required, and has then had to arrange the gradient of the tangent at x_2 to be small enough so that it crossed the axis near $x=a$ rather than close to the nearer zero. Confirmation of student understanding of importance of the choice of the first approximation to a root when using the Newton–Raphson method was seen in the following questionnaire responses.

Question: What is important about the choice of the first approximation to a root when using the Newton–Raphson method? Why?

Student E1: You can't choose a max or min point or else you won't cut the x-axis. Also the tangent could go towards the wrong root.

Student W3: So then its tangent will find the right root.

Student F1: If it is in the wrong place it will not work.

Student K1: You must choose a first approximation whose tangent cuts the x-axis. It can't be a stationary point.

Student L2: If it's too far away, it could give another root.

Student S1: If your approximation is insufficiently close then you may hit a point of inflection or arrive at the wrong root.

Student V1: It is very important that the approximation is close enough the root and not on a turning point. Otherwise you might be finding the wrong root.

Student W4: It must be close to the root so the tangent gives you the nearest value. Also you can't choose a stationary point as a first value.

Virtually all the responses include expressions involving 'tangents', 'cutting the axis', 'stationary point', 'close enough', 'far away' which all relate to the graph of the function, even though this is not mentioned in the question. This demonstrates the role of visual thinking in the minds of the students which was not generally apparent in the pre–test.

Student attitudes are important and the attitude scale responses confirmed a positive view of their experience with the calculators with an overall mean response to the calculators which was significantly positive (mean=3.27, t=2.43, p<.05 – integer score 1–5). Two scores in particular which showed their positive attitude were: More interesting mathematics problems can be done when students have access to calculators (3.9) and Mathematics is easier if a calculator is used to solve problem (4.05).

Conclusion

In contrast with mathematics instruction which often emphasises a procedural approach to the Newton–Raphson method this study focused on student development of inter–representational understanding of concepts, using the super–calculator to help students to make connections between sub–concepts across representations. The results suggest that this approach to learning can be more meaningful for each individual student, helping them to build a rich schema leading to versatility of thinking. It has sometimes been suggested that it is simply the novelty factor of the technology which produces changes in learning. We will have to wait until all students have had sufficient experience for the novelty to have worn off finally to discover if this is so. However, it should be noted that the students in this research were all very familiar with computers and graphic calculators in their mathematics learning, and so this was not a factor for them. Such graphics–oriented approaches to the learning of calculus can facilitate more challenging problems, with less focus on procedural skills, formulas, routine problems and exercises. The ability of all graphic calculators to present calculus concepts dynamically in various representations allows students to study processes and objects in numerical, algebraic, and graphical formats from different perspectives. In this way, processes of symbolic manipulation may become less important while graphical and visual solution processes may gain status. As we have tried to show for the Newton–Raphson method, the use of super-calculators such as the TI-92 and FX2 can complement pencil and paper methods, producing visualisation of mathematical ideas that strongly affects students' reasoning.

References

Abramovich, S. & Brown, G. (1999). From measuring to formal demonstration using interactive computational geoboards and recurrent electronic charts, Journal of Computers in Mathematics, and Science Teaching, 18(2), 105–134.

Chinnappan, M. & Thomas, M. O. J. (1999). Structured knowledge and conceptual modelling of functions by an experienced teacher, *Proceedings of the 22nd Mathematics Education Research Group of Australasia Conference*, Adelaide, 159–166.

Dunham, P. H. & Dick, T. (1994). Research on graphing calculators, *Mathematics Teacher,* 87(6), 440–445.

Goldin, G. A. (1998) Representational systems, learning, and problem solving in mathematics, *Journal of Mathematical Behavior*, 17(2), 137–165.

Graham, A. T. & Thomas, M. O. J. (2000). Building a versatile understanding of algebraic variables with a graphic calculator, *Educational Studies in Mathematics,* 265–282.

Hong, Y. Y, Thomas, M. O. J. & Kwon, O. (2000). Understanding linear algebraic equations via super–calculator representations, *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*, Hiroshima, Japan, 3, 57–64.

Kaput, J. (1987). Representation systems and mathematics. In C. Janvier (Ed.) *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 19–26). Lawrence Erlbaum: New Jersey.

Kissane, B. (1995). The importance of being accessible: The graphics calculator in mathematics education, *The First Asian Conference on Technology in Mathematics*, Singapore: The Association of Mathematics Educators, 161–170.

Penglase, M. & Arnold, S. (1996). The graphic calculator in mathematics education: A critical review of recent research, *Mathematics Education Research Journal*, 8(1), 58–90.

Ruthven, K. (1990). The influence of graphic calculator use on translation from graphic to symbolic forms. *Educational Studies in Mathematics*, 21, 143–50.

Thomas, M. O. J. (1994). A process-oriented preference in the writing of algebraic equations, *Proceedings of the 17th Mathematics Education Research Group of Australasia Conference*, Lismore, Australia, 599–606.

Thomas, M. O. J. (1996). Computers in the mathematics classroom: A survey, *Proceedings of the 19th Mathematics Education Research Group of Australasia Conference,* Melbourne, Australia, 556–563.

FROM CONSTRUCTION TO DEDUCTION: POTENTIALS AND PITFALLS OF USING SOFTWARE

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In this paper we present the principles underlying the design of two computer-integrated teaching experiments (one in algebra and one in geometry) which aim to help students make transitions between informal argumentation and experimentation and more formal logical deduction. We briefly present the final activities in the sequence and some preliminary interpretations of student responses.

Background

It is commonplace in mathematics to present proving in a hierarchy of levels in which the empirical precedes the deductive. Clearly students need to be able to distinguish empirical verification and pragmatic proof from deductive and conceptual proof (Balacheff, 1988) but the question remains as to how best this might be done. Is it a matter of development of the latter from the former or can links be forged between the two at every stage of schooling and throughout mathematical activity? Our rationale is that we need to design new learning contexts which require the use of clearly formulated statements and definitions and agreed procedures of deduction but which also allow opportunities for their connection with informal argumentation and empirical justification and the conviction these engender. Previous work points to what might be the constituent components of such contexts: ease of transition between an enactive or visual form of proof and a sequence of deductions (Tall, 1995); emphasis on explanatory proofs which "illustrate as well as dispel doubt" (Hanna, 1995); use of generic proofs where attention is focussed on a particular example chosen so as to be "typical of the whole class of examples and hence the proof is generalizable" (Tall, 1989).

Research in mathematics education has consistently highlighted students' difficulties in engaging with formally-presented, logical arguments and understanding how these differ from empirical evidence (see for example Balacheff, 1988, Chazan, 1993 and more recently Arzarello *et al.*, 1998). The National Curriculum for mathematics in England and Wales prescribes an approach to proving, partly as a response to these student difficulties, in which the introduction of proof is delayed until after students have progressed through stages of reasoning empirically and explaining their conjectures largely in the context of data-driven investigations (see Hoyles, 1997). Our research set out to describe how high-attaining students who had followed this curriculum conceptualise proving and proof in mathematics and to explore ways to address any difficulties they may have experienced through new teaching approaches using computers.

The research comprised two phases: a paper and pencil survey of the conceptions of proving and proof held by 2,459 students aged 14/15 years at a high level of mathematics attainment, (about the top 20%), followed by two computer-integrated teaching experiments in geometry and in algebra. This paper is concerned with phase 2 and aims to present the principles underlying the design of the teaching experiments. It incorporates some findings from phase 1 in so far as they informed the design and subsequent analysis of the experiments (for information on the survey, see Healy & Hoyles, in press).

Our phase 1 analysis showed that students, even in this high attainment band, had a limited view of proof and this had a significant and negative influence on their competence in proving. Yet the majority of students recognised that a valid proof should be general and valued arguments that they felt convinced and explained. Our results also suggested that despite their lack of experience with proof, the students had developed considerable expertise in conjecturing, arguing and explaining in 'everyday' language. Additionally, in contrast to previous studies, we found that most students were aware of the limitations of empirically-rooted arguments and recognised that a general, analytic justification was required: they simply did not know how to construct it. In devising our teaching experiments we aimed to build on the positive student attribute of confidence in explaining while guarding against the danger of sacrificing this in the pursuit of injecting more rigour. We wanted to introduce deduction and formality but alongside and connected to rather than instead of informal argumentation. We recognised that our aims were ambitious and that it was difficult to help students manage the path to proving. However we conjectured from a theoretical perspective that new technologies offered new approaches to these well-documented problems in learning to prove and it was therefore worthwhile to design teaching experiments in order to collect and analyse student response data to test out our conjecture.

The framework for the teaching experiments

We designed two teaching experiments, one in algebra and one in geometry. They set out to build on our knowledge of students' conceptions of proof as identified from the analysis of the Phase 1 and specifically to capitalise on the evident strengths of our students in narrative explanation. In general the experiments aimed to help students develop a more multi-faceted view of proof, which included verification, systematisation and deduction alongside explanation (see deVilliers, 1990), and which also introduced students to more formal proof without prescribing a particular mode of presentation, such as a two-column proof. To achieve these ends we integrated computer use.

Our starting point was that students should construct mathematical models on the computer, as it was we claimed through this constructive process together with reflection on the computer feedback that learning could take place (for more detail on this constructionist perspective, see Noss and Hoyles, 1996). Next we assert that to construct on a computer required *explicit* attention to the objects and processes used. The seeds of learning to prove could therefore be sown in a construction process which required an explicit description of properties and relationships – the 'givens' at the start of a proof in geometry or the mathematical model of the situation in algebra. The key insight was that we could build some of these initial assumptions into the fabric of the computer medium – the tools we chose to provide — thus shaping the types of actions that were possible for the students to perform. The level of what could be thought about, talked about, was thus notched up a rung or two as objects and relationships could be acted on and visualised. We also conjectured that computer interaction could motivate an *explicit* characterisation of the mathematical activity during which students *had* to attend to the relationships they were setting up and changing. We therefore hypothesised that students would be better able to formalise their explanations if they were derived from a basis of computer-based activity - they would of necessity have a clear grasp of the initial components in their models as they had built them themselves, and their experimentation with the software would provide a rationale for deciding which properties were necessary and which were not

Thus the activities in the teaching experiments we designed followed a common pattern: students *constructed* models for themselves on the computer, *conjectured* about the relationships between its elements, and *checked* the truth of their conjectures with the tools available. We anticipated that students would be helped to justify their conjectures by reflection on their constructions and the computer feedback, thus bringing constructing and proving together more closely in ways not possible without the appropriate technology. Deduction would become just one more facet of a proving culture, revitalised by the 'experimental realism' of the computer work (Balacheff and Kaput, 1996). Clearly the activities around the computer and the way a teacher fosters movement to and fro from formal to informal, rigorous to intuitive, are also crucial. We therefore designed a sequence of activities with pre-specified aims and as teachers set out to act as facilitators around the computer work while bringing the students together periodically to share ideas and to introduce new perspectives.

In algebra, we built a microworld in Microworlds Logo called Expressor to serve as a basis for the computer activities. A section of the activities were based on exploring the properties of the sums of consecutive numbers. Students could build up columns of dots to represent numbers by direct manipulation. Figure 1 shows the representation of 6, 7 and 8 formed by picking up the red dot and moving it to the right position and then writing a simple Logo program, col, to make a column of any size.

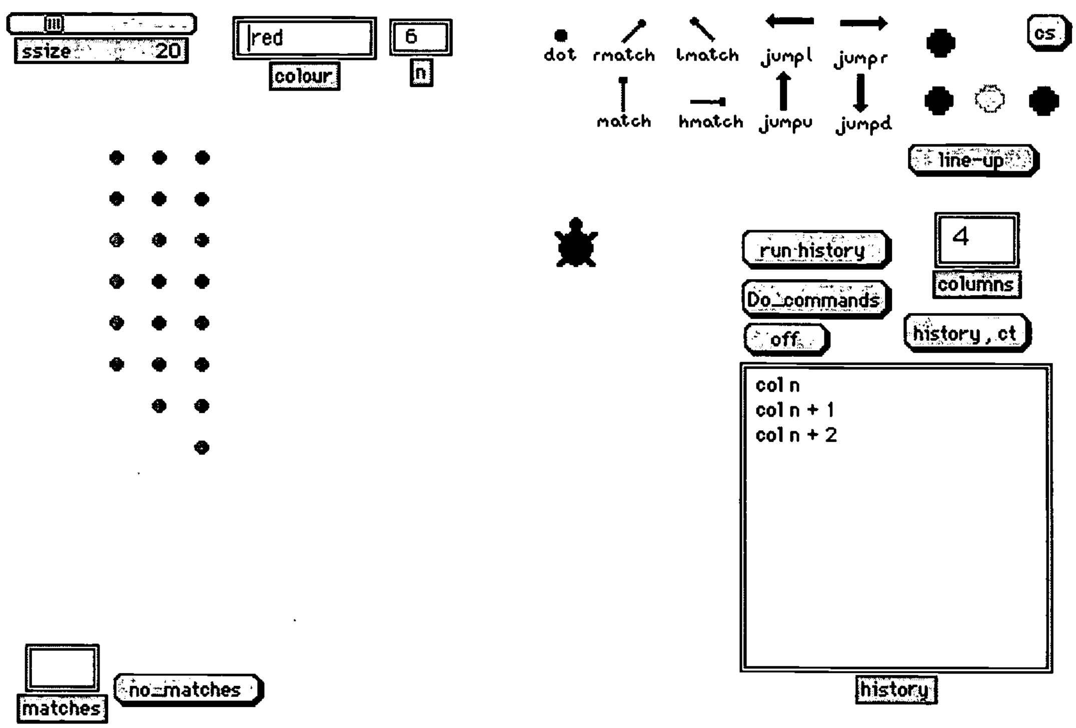

Figure 1. Modelling consecutive numbers with Expressor

Having set up the models, students could then move the dots about physically or symbolically to explore the sums of the columns of dots in order to help them come up with conjectures about the properties of these sums.

In geometry, we used the dynamic geometry system Cabri. Students were introduced to the standard menu items of creation and construction with attention focussed on the use of history (we used Cabri 1 which has this tool) so they were better able to reflect on their constructions - what they had done and in what order. We also introduced the students to trace and locus as again these tools would be useful in visualising properties of geometrical objects from which basis students could make conjectures.

Methodology

Each teaching experiment comprised 3 lessons and 3 homeworks and was conducted by the two authors. A total of 15 students, 5 from each school, undertook both experiments within a 5-month period[1]. The students were chosen by their teachers according to our criterion of high-attainment, but also so that the group experience would be beneficial both individually and collectively.

The teaching experiments were piloted with 6 students in three schools (one mixed, one boys, one girls) and modifications made on the basis of feedback from students and teachers. These included more precise procedures for the collection of systematic written data from the students on worksheets and the imposition of the common structure on both experiments; i.e. students were to construct mathematical objects on the computer, identify and describe the properties and relations that underpinned their constructions, use the computer resources to generate and test conjectures about further properties, and make informal explanations as to why they must hold. After the explanations had been generated, there followed a teacher-led introduction to writing proofs using paper and pencil examples, where students were helped to organise the arguments generated during the computer activity into logical deductive chains in an appropriate formal language. Finally, the students were given twin open-ended computer constructions to be explained and proved.

In the presentation, we will describe in more detail the sequence of computer-based activities in both experiments but focus attention on the two final constructions in each sequence. Both of the first constructions (Construction 1 in Table 1) were challenging. They aimed to provoke the students to use the computer tools with which they had become familiar to solve the problem and identify the required properties and then explain and prove them. The purpose of Construction 2 was to see if the students would use the properties they had identified in Construction 1 to explain why the construction proposed was impossible. The four activities are summarised in Table 1 below.

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[1] We planned to have 3 pairs of students per school, but in each case one student did not attend every session.

Table 1

Identifying properties through construction and using them to make predictions

Construction 1	Necessary Property (explained & proved)	Construction 2
Geometry: Construct with Cabri a quadrilateral in which the angle bisectors of two adjacent angles cross at right angles. Write down its properties and prove them.	One pair of opposite sides *must be* parallel. The quadrilateral *must be* a trapezium.	Predict if you can construct a triangle in which two adjacent angle bisectors cross at right angles. Predict yes or no, try to construct the triangle and then explain why your prediction was right or wrong.
Algebra: Construct 4 consecutive numbers in Expressor, write down any properties of their sum and prove them.	The sum of 4 consecutive numbers is even but *not divisible by 4*	Predict whether you can find 4 consecutive numbers that add up to 44. If yes, write them down, if no, explain why it cannot be done.

In both cases the activities proved as we anticipated to be by no means trivial. All of the students who faced the geometry Construction 1 had rather few intuitions as to the characteristics of the required quadrilateral. They were therefore forced to experiment with the software to see if they could find a quadrilateral that fitted the criteria and then 'see' its properties by experimentation with tools available - in particular using measuring and tracing while relaxing the perpendicularity constraint on the angle bisectors. In the algebra Construction 1, students found it easy to explore sums of consecutive numbers in the microworld and the multiple representations available helped students conjecture as to their properties - they could explore through direct manipulation, visual means and by changing the symbolic code. In the presentation, we will show in more detail how the tools we provided in both cases helped to scaffold the process of construction and proving for the students - yet also on occasions led them into cul-de-sacs from which it was hard to emerge.

In attempting Construction 2 in both cases we were surprised how frequently the implications of the properties identified in Construction 1 were ignored. Despite all the efforts students had made to complete the first constructions and find the necessary properties, they consistently failed to use these properties to make the correct prediction about the impossibility of the second construction. When they actually tried to do the construction on the computer they soon saw it was impossible, and knew immediately why this was the case. Thus, we suggest that although an appreciation of necessary properties could be separated from actions in the construction process while remaining deeply connected to the required outcome, there is still effort needed to transform the properties into theoretical constraints which can be used as a basis for logical argument.

References

Arzarello, F., Micheletti, C., Olivero, F., Robutti O., Paola, D., & Gallino, G. (1998). Dragging in Cabri and modalities of transition from conjectures to proofs in geometry. In A. Olivier, & K. Newstead, (Eds.) *Proceedings of PME 22* University of Stellenbosch, South Africa, 2, 32–39.

Balacheff, N. (1988). Treatment of refutations: aspects of the complexity of a constructivist approach to mathematics learning. In E. Von Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education* (pp. 89-110). Reidel.

Balacheff, N., & and Kaput, J. (1996). Computer–based learning environments in mathematics in A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.) *International Handbook of Mathematics Education* Part 1 (pp. 469–505). Kluwer.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, 24(4), 359–387.

deVilliers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, 7–24.

Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics*, 15(3), 42–49.

Healy, L., & Hoyles, C. (in press). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education.*

Hoyles, C. (1997). The curricular shaping of students' approaches to proof *For the Learning of Mathematics*, 17(1), 7–16.

Noss, R., & Hoyles, C. (1996). *Windows on Mathematical Meanings: Learning Cultures and Computers*. Dordrecht: Kluwer Academic Publishers.

Tall, D. O. (1989). The nature of mathematical proof, *Mathematics Teaching*, 127, 28–31

Tall, D. O. (1995). Mathematical growth in elementary and advanced mathematical thinking. In L. Meira & D. Carraher (Eds.), *Proceedings of the Nineteenth International Conference for the Psychology of Mathematics Education*, 1 (pp. 61–75). Recife, Brazil: Universidade Federal de Pernambuco.

WHEN A VISUAL REPRESENTATION IS NOT WORTH A THOUSAND WORDS

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In this paper, two visual representations typical of those found in four Integrated Learning Systems (ILSs) currently being evaluated in Eastern Australia are analysed using a set of principles generated from a review of the research literature from the fields of mathematics education, cognitive science, computer-aided learning, computer graphic design and semiotics. Using this set of principles, the authors explain why many of the visual representations contained within these four ILSs do not facilitate the construction of mathematical knowledge.

An Integrated Learning System (ILS) is basically a collection of tasks, presented as electronic worksheets, divided into a range of *strands* (e.g., subtraction, addition, fractions, and measurement). The tasks are visually attractive in their presentation and sometimes creative in the way they probe understanding. The visual representations may include Multi-base Arithmetic Blocks (MAB's), fraction and decimal diagrams. The manufacturers of Integrated Learning Systems (ILSs) endorse their products as tools to "develop and maintain mathematical skills, and to develop problem-solving skills" (Computer Curriculum Corporation, 1996, p.1) and as tools to diagnose student difficulties. In order to achieve these goals, ILSs presents "a mix of dynamically distributed exercises appropriate to a student's functional level, providing feedback and tutorial intervention when necessary and representing mathematical concepts through highly visual exercises" (Computer Curriculum Corporation, 1996, p. 1).

During the period 1997-2000, a three-stage evaluation of four computer-based ILSs has been conducted in 28 schools in Eastern Australia. In the first stage of the evaluation project, factors that influenced teacher endorsement of the ILSs were investigated (Baturo, Cooper, Kidman & McRobbie, in press), and student achievement in relation to progress on the ILSs was measured (McRobbie, Baturo, & Cooper, in press). The second stage focused on individual Year 6 and 8 students who had never worked on the ILSs. Stage 3 repeated Stage 2 but with classes that had been identified by ILSs' Australian distributors as the most successful users of their products. During the analysis of data from Stages 2 and 3, it became apparent that many of the visual representations utilised within the mathematics components of the ILSs were either not very effective or interfered with the children's construction of mathematical knowledge.

In order to identify why many of the visual representations utilised within the ILSs were not facilitating the construction of mathematical knowledge, the authors conducted a close analysis of 550 visual representations contained within the ILSs. This analysis was informed by a set of principles for analysing visual representations within mathematics education computer software that the authors generated from a review of the research literature from the fields of mathematics education, cognitive science, computer-aided learning, computer graphic design and semiotics. Our reason for generating a set of principles rather than utilising an existing set of principles was that during our review of the research literature, it became apparent that most current sets of principles did not take cognizance of the research findings from the fields of semiotics and computer-based representation as tools for supporting and organising thinking. We felt that many of the findings from these two fields of research have much relevance for gaining further insights into how computer-based mathematical visual representations influence the learning of mathematics.

Theoretical framework

A review of the research literature indicates that visual representations can play at least four different but interrelated roles in facilitating the construction of mathematical knowledge:

1. visual representations can provide the *source* for mathematical analogs (Charles & Nason, in press);
2. visual representations can act as tools to facilitate problem solving (Lowe, 1993);
3. visual representations can facilitate mathematical meaning-making (Lemke, 1999); and
4. computer-based visual representations can act as tools for supporting and reorganising thinking (Gordin, Edelson, & Pea, 1996; Pea, 1985).

Visual representations as sources for mathematical analogs

Many teachers and mathematics educators have grappled with the problem of having students attach meaning to mathematical concepts by the application of a wide variety of mathematical analogs during the process of teaching mathematics (Charles & Nason, in press). An analogy is the mapping of knowledge from one domain (the source) to another domain (the target) (Gentner, 1982). In a mathematical analog, a concrete/pictorial model acts as the source of the analog and the concept to be constructed acts as the *target* of the analog. In her seminal paper on analogies, Gentner (1982) listed four principles that she felt were necessary conditions for an analog to be effective:

1. The Clarity of Source Principle
2. The Clarity of Mapping Principle
3. The Principle of Conceptual Coherence
4. The Principle of Scope.

The first three of these principles have implications for the use of diagrams and other visual representations in the teaching/learning of mathematical concepts. The implication of the Clarity of Source Principle is that visual representations being applied during the course of teaching should be clearly displayed and explicitly understood by the student. The Clarity of Mapping Principle has two clear implications. First, the relationship between the visual representations and the mathematical concepts should facilitate clear, unambiguous conceptual mappings from the source to the target. Second, the visual representations should enable the student to focus on the deep structural rather than surface structural aspects of the problems being investigated. The implication of the Principle of Conceptual Coherence is that relations mapped from the visual representation to the target mathematical concept should form a cohesive conceptual structure.

Visual representations as tools for facilitating problem solving

Visual representations provide teachers and students with a means of externally representing mathematical ideas or concepts (Goldin & Kaput, 1996; Janvier, Girardon, & Morand, 1993). According to Janvier et al. (1993) and Fish and Scrivener (1990), external representations such as visual representations play important roles in the learning of mathematics and the solution of mathematical problems. They can do this by:

1. Helping students to recall knowledge and skills by making connections between prior internal representations and new situations (Dettori & Lemut, 1995; Lowe, 1993, Janvier et al., 1993);
2. Stimulating a relationship among the problem data (Dettori & Lemut, 1995);
3. Providing students with an external memory to store information temporarily during the process of problem solving (Fish & Scrivener, 1990); and
4. Providing environments for learners to abstract and understand mathematical concepts or a relation of information within problems (Fish & Scrivener, 1990).

Visual representations as tools to facilitate the mathematical meaning-making

According to Lemke (1999), mathematical meanings evolved historically to allow us to integrate two fundamentally different kinds of meaning making: meaning-by-kind and meaning-by-degree. Because the semantics of natural language with its system of possible meanings is primarily a categorical contrast system, a system of formal "types" or equivalence classes, natural language is well suited for representing meaning-by-kind. However, natural language is not good at giving precise and useful descriptions of natural phenomena in which matters of degree or quantitative variation are important. As Lemke pointed out, it is well nigh impossible to describe the exact nature of a mountain range, the precise difference between two colours or the exact movement through space by a fly solely using natural language. Meanings-by-degree such as these are more easily represented by motor gestures or visual representations. Because of this, Lemke argued that many mathematical ideas and concepts can never be completely translated into natural language statements and questions.

Mathematised visual representations thus have two very important roles to play in the process of mathematical meaning-making. First, they enable learners to explore and construct understandings about aspects of mathematical ideas and concepts that cannot be adequately represented in the semantics of natural language. Second, they also facilitate the process of translating between mathematical expressions and natural language.

Computer-based visual representations as tools for supporting and reorganising thinking

According to Pea (1985), the primary role for educational computing should not be to change how effectively we do traditional tasks. Instead, computers should change the tasks we do by reorganising our mental functioning. In order to do this, educational software needs to provide students with tools for supporting and reorganising their thinking.

According to Schwartz (1999), one of the most effective ways of doing this is to have appropriately crafted software environments that allow students to make and explore conjectures. Two good examples of software environments that allow students to do this are The Geometry Supposer (Schwartz, 1985) and the Scientific Visualisation (SciV) (Gordin, Edelson, & Pea, 1996). In both these software environments, visual representations play crucial roles in the construction of knowledge. An analysis of the visual representations in both of these software environments reveals visual representations are used for both *interpretative* and *expressive* activities. These activities are counterparts to one another, like reading and writing (Gordin *et al.*, 1996). Interpretation is the process of finding meanings from the visual representations. Expression is the process of crafting visual representations to convey meaning. Gordin *et al.* (1996) argue that in order for students to be able to interpret visual representations, they must also be given opportunities to create and/or modify visual representations.

Principles for analysing visual representations

The following set of principles was generated from the review of the literature presented in the Theoretical Framework:

1. Visual representations should be clearly displayed and explicitly understood by the student. This facilitates the process of stimulating relationships among the problem data and may also help students to recall knowledge and skills by making connections between prior internal representations and new situations.
2. Visual representations should enable the student to focus on the deep structural rather than surface structural aspects of the problems being investigated.
3. Visual representations should provide students with an external memory to display information temporarily during the process of problem solving. By doing this, the visual representation can reduce the working memory demands of the problem solving process.
4. Visual representations should provide physical/iconic environments for learners to abstract and understand mathematical concepts or a relation of information within problems.
5. Visual representations should facilitate the exploration and construction of understandings about aspects of mathematical ideas and concepts that cannot be adequately represented in the semantics of natural language.
6. Visual representations should facilitate the process of translating between mathematical expressions and natural language.
7. Visual representations should used for both interpretative and expressive learning activities.

Principle 1 was derived from a synthesis of the research literature on mathematical analogs (e.g., Charles & Nason, in press; Gentner, 1982) and visual representation of mathematical problems (e.g., Derrori & Lemut, 1995; Lowe, 1993; Janvier *et al.*, 1992). Principle 2 was derived from Gentner's (1982) Clarity of Mapping principle. Principles

3 and 4 were derived from Fish and Scrivener (1990). Principles 5 and 6 were derived from semiotic theories such as those espoused by Lemke (1999). Principle 7 had its genesis in the research into computer-based visualisation for supporting and reorganising thinking (e.g., Gordin, Edelson, & Pea, 1996).

Analysis of ILS visual representations

As was noted earlier, 550 visual representations of tasks from the ILSs were evaluated. 220 of these visual representations were found to be deficient in at least five of the seven principles. In order to illustrate the nature of the deficiencies found in the 220 visual representations, we will now present a detailed analysis of two of these visual representations. The deficiencies identified in these two examples (e.g., font problems, unclear purposes and layouts, forced formatting and poor links between natural language and visual representations) are typical of those found throughout the 220 visualisations.

The exercise in Figure 1 is presented to students working in the Number Concepts strand-Year 1 level (see upper left hand corner of Fig. 1: nc197 = number concepts, Year 1, 97 percent of the way through Year 1). The tool bar is red, blues, green, yellow, and grey coloured. The body of text and possible solutions are in black. The ILS 'correct response' (20 + 0) is shown by a red rectangle, while the ILS 'incorrect responses' are shown by yellow rectangles. The salient feature of the screen is the diagram of the Multi-base Attribute Blocks (MAB), which are a yellow colour outlined in black.

The exercise presented in Figure 2 comes from the repertoire of exercises contained in the Fraction strand-Year 8 level of the ILS (see upper left hand corner of Fig. 2: fr840 = Fractions, Year 8, 40 percent of the way through Year 8). The tool bar is red, blues, green, yellow, and grey coloured. The main body of text is black with the answer 1/8 in red. The instructions "Give your answer in simplest form" is coloured blue. The salient feature of the screen is the diagram of the right triangle, which is a coral pink colour.

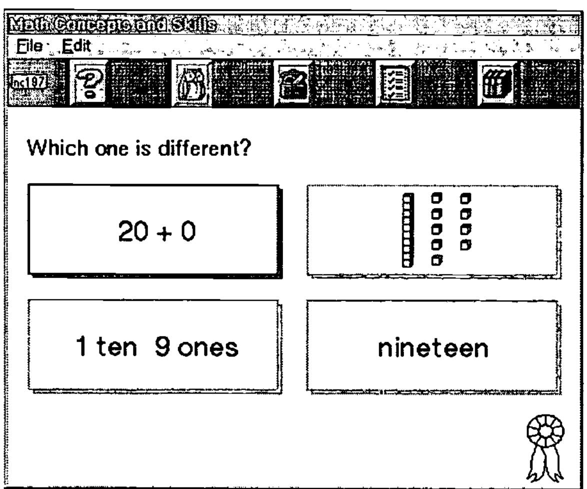

Figure 1. Number concepts-Year 1

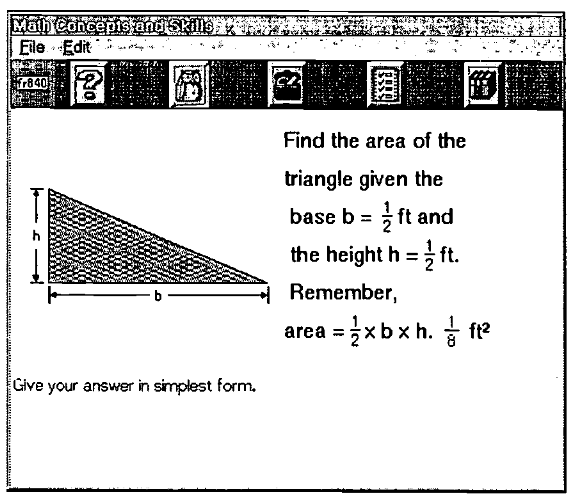

Figure 2. Fractions-Year 8.

The results of the analysis of the visual representation in Figure 1 are presented in Table 1. The results of the analysis of the visual representation in Figure 2 is presented in Table 2.

Table 1
Analysis of Figure 1

Principle #	Compliance	Commentary
1	Low	The font and the layout of the visual representation presented in this activity can be clearly read. However, because of the lack of definition of the term, "different", the focus of the activity is quite ambiguous. The four representations presented in Figure 1 can be categorised in at least two ways: ◆ pictorial representations v text and numbers representations ◆ magnitude of number This has the potential to hinder students' recall of relevant knowledge about grouping and place value.
2	Low	The feature that students need to focus on in this activity is the magnitudes of the numbers presented in the four representations. However, the salient feature in Figure 1 is the surface structural aspect of pictures v text and numbers. The layout of the four representations in Figure 1 also does not encourage students to utilise their intuitive knowledge about grouping and numeration concepts.
3	Low	Students are unable to add annotations to the visual representation. Therefore, unless they are instructed to make notes on paper as they proceed through this activity, much information has to be memorised. This has the potential to result in an overload in working memory capacity.
4	Low	The important notion that students need to abstract from this activity is that 20 + 0 is different from the number 19 represented in the other three cells because 20 consists of two groups of ten whereas 19 consists of one group of ten and nine ones. Even when the ILS presents the students with the "correct" answer, the representations are not utilised to bring out this important notion.
5	Low	The comparative relationship between 19 and 20 cannot be fully expressed in natural language. Because the representations in this activity do not meet Principle 4, exploration of relationship between 19 and 20 (i.e., 19 one less than two groups of ten or twenty) is not facilitated.
6	Moderate	The natural language representation of numbers (i.e., nineteen) is presented. The visual representation presented in this activity thus facilitates the process of translating between mathematical expressions and natural language. However, because of lack of compliance to Principles 1-5, only moderate compliance with this principle can occur.
7	Low	The visual representation allows for interpretation but because of the lack of compliance with Principles 1-5 and moderate compliance with Principle 6, only low levels of interpretation can occur. The visual representation cannot be used for expressive learning activity.

Table 2
Analysis of Figure 2

Principle #	Compliance	Commentary
1	Low	The font and the layout of the visual representation presented in this activity can be clearly read. However, the visual representation is difficult to understand because the triangle is not drawn to scale. Understanding of the visual representation is complicated by the accompanying text which gives the reader the impression that the focus of the activity is on area measurement rather than on fraction concepts. The visual representation plus its accompanying text thus may facilitate the recall of inappropriate knowledge about triangles and area measurement and hinder the recall of relevant previous knowledge about fractions.
2	Low	Because the visual representation does not relate the area of the triangle to the notion of the unit (i.e., a 1x1 square), the visual representation does not enable students to focus on deep structural knowledge. The visual representation also does not encourage the students to make use of their intuitive knowledge about fractions.
3	Low	Students are unable to add annotations to the visual representation. Therefore, unless they are instructed to make notes on paper as they proceed through this activity, much information has to be memorised. This has the potential to result in an overload in working memory capacity.
4	Low	The important notion that students need to abstract from this activity is that 1/2 x 1/2 x 1/2 is equivalent to 1/8 because if a half is halved, then a quarter is generated; and if a quarter is then halved, then an eighth is generated. Because the visual representation does not relate the area of the triangle to the unit square, the "Correct" answer is not conceptually linked back to diagram. The visual representation thus does not provide an adequate environment for learners to abstract and understand the important notion underlying this activity.
5	Low	The important notion underlying this activity is almost impossible to adequately represented in the semantics of natural language. The visual representation does not enable students to adequately explore this notion. For example, as was noted earlier, the visual representation does not enable students to relate the area of the triangle back to the unit square. It also does not enable students to explore different ways of generating 1/2 x 1/2 x 1/2. Therefore, students probably will not construct iconic understandings of relationship between 1 whole, 1/2, and 1/8.
6	Low	Because the visual representation contains little or no natural language, it does not facilitate links to natural language.
7	Low	The visual representation allows for interpretation but because of the lack of compliance with Principles 1-6, only low levels of interpretation can occur. The visual representation cannot be used for expressive learning activity.

Discussion

Both of the visual representations analysed in this paper are clear, visually appealing and relatively uncluttered. Therefore, we can assume that a student's attention will be attracted and possibly maintained by these visual representations. However, neither of the two visual representations facilitates the process of meaning-making. For example, neither visual representation does much to stimulate relationships between the problem data or the recall of relevant prior knowledge and skills. Furthermore, neither of the visual representations enables students to focus beyond the surface level aspects of the problems being investigated. Another serious limitation of the two visual representations that was identified during the analysis was that neither provides students with opportunities to modify and/or construct visual representations of the problem. Thus, neither visual representation provides students with the means to make use of external memory strategies to reduce the working memory demands of the problems or with the means to make and explore conjectures about the problems. A further limitation of the two visual representations is that they do not provide students with the means to construct a deeper appreciation of the concepts that goes beyond that which can be achieved through the semantics of natural language.

Because of these limitations, it is highly unlikely that either of the visual representations would do much to facilitate the construction of deep-level, principled knowledge about numeration or fractions. However, the effects of these limitations may more serious than this. First, the barriers to meaning-making created by these two visual representations could do much to make the process of learning mathematics a very stressful and frustrating experience for many students. For example, in the case of Figure 1, trying to ascertain that the purpose of the task is to classify by number and not by other (more salient) attributes is a most difficult task. Therefore, it is highly probable that many Year 1 students may never get around to working out exactly what they are being asked to do. Second, the barriers these visual representations place on students being able to make and explore conjectures about the mathematical concepts being investigated presents students with perceptions about the nature and discourse of mathematics which are contrary to those in most current mathematics curricula. Thus, instead of perceiving mathematics as a human, active, creative, investigative and cultural activity (Ernest, 1996) or as a co-equal partner with language in the analysis of natural and social phenomena (Lemke, 1999), students are more likely to perceive mathematics as an abstract, rote and absolutist phenomena.

As was noted earlier, the two visual representations analysed in this paper typified the 220 out of 550 visual representations that were found to be deficient in at least five of the seven principles. Indeed, many of the other 218 visual representations fared even worse when they were analysed. Mukherjee and Edmonds (1994) made the observation that visual representations in many Integrated Learning Systems often seem to have been developed in a vacuum by individuals or teams that have no background in graphic design or visual literacy. One suspects that this may have occurred during the development of the Integrated Learning Systems currently being evaluated.

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References

Baturo, A. R., Cooper, T. J., Kidman, G. C., & McRobbie, C. J. (in press). Factors influencing teachers' endorsement of the core mathematics course of an integrated learning system. Paper to be presented at the 24th annual conference of the *International Group for the Psychology of Mathematics Education*. Hiroshima, Japan: PME.

Charles, K. & Nason, R. (in press). Towards the specification of a hypermedia environment to facilitate the learning of fractions. *THEMES in Education*.

Dettori, G. & Lemut, E. (1995). Representations in Problem Solving Activities. In the Proceedings of *SEMPT Conference*, Prague (The Czech Rep.), 90–93.

Ernest, P. (1996). The nature of mathematics and teaching. http://www.ex.ac.uk/~PErnest/pome/pompart7.htm

Fish, J. and Scrivener, S. (1990). Amplifying the Mind's Eye: Sketching and Visual Cognition, *Leonardo* 23(1): 117–126.

Gentner, D. (1982). Are scientific analogies metaphors? In D.S. Miall (Ed.), *Metaphor: Problems and perspectives* (pp. 106–132). Atlantic Hughlands, NJ: Humanities Press.

Goldin, G. A., & Kaput, J. J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L .P. Steffe and P. Nesher (Eds.), *Theories of Mathematical Learning* (pp. 397–430). Mahwah, NJ: Lawrence Erlbaum Associates.

Gordin, D. N., Edelson, D. C., & Pea, R. D. (1996). Supporting Students' Science Inquiry through Scientific Visualization Activities. Paper presented as part of an interactive symposium entitled "*Scientific Visualization Tools in Science Classrooms*" at the Annual meeting of the American Educational Research Association, New York, April 8–12.

Janvier, C., Girardon, C., & Morand, J. (1993). Mathematical Symbols and Representations. In P. S. Wilson (Ed.) *Research Ideas for the Classroom: High School Mathematics* (pp. 79–102). Reston, VA: NCTM.

Lemke, J.L. (1999). *Mathematics in the middle: Measure, picture, gesture, sign and word.* http://academic.brooklyn.cuny.edu/education/jlemke/papers/myrdene.htm

Lowe, R. (1993). *Successful instructional diagrams.* London: Kogan Page.

McRobbie, C., Baturo, A., & Cooper, T. (in press). Low achieving mathematics students' attitudinal and achievement changes as a result of using an integrated learning system. Paper to be presented at the *23rd annual conference of the Mathematics Education Research Group of Australasia*, Perth, WA.

Mukherjee, P. & Edmonds, G. S. (1994). Screen design: A review of research. In D. G. Beauchamp, R. A. Braden, & J. C. Baca (Eds.), *Visual literacy in the digital age: Selected readings from the 25th Annual Conference of the International Visual Literacy Association* (pp. 112–118). Blacksburg, VA: International Visual Literacy Association.

Pea, R. D. (1985). Beyond amplification: Using the computer to reorganise mental functioning. *Educational Psychologist, 20*(4), 167–182.

Schwartz, J. L. (1985). *The Geometry Supposer.* Pleasantville, NY: Sunburst Communications.

Schwartz, J. L. (1999). *Some ethical and philosophical issues arising from the use of technology in education.* http://gseweb.harvard.edu/~faculty/schwartz/papers/papers570.htm.

THE INFLUENCE OF COMPUTER-RICH LEARNING ENVIRONMENTS ON MATHEMATICS ACHIEVEMENT

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This paper reports on a mixed-method study that investigated the influence of a computer-rich learning environment on mathematical learning. The mathematical achievement of Year 6 students at a computer-rich school was quantitatively compared with those of Year 6 students at four non-computer-rich schools. A case study of the Year 6 class at the computer-rich school illuminated the quantitative findings. The paper identifies a number of important factors that influence the educational efficacy of computer-rich learning environments.

It is now generally accepted that technological change is expected to transform teaching and learning. The Department of Education Queensland (1995) has argued that computer technology will change the nature of student learning, the roles of both teachers and students, and "enhance the achievement of educational goals across the P-12 curriculum" (p. 3). Further, the use of computers in classrooms has been linked to change in teaching strategies. The Australian Association of Mathematics Teachers (AAMT) (1996) contended that "Educators must be imaginative, flexible, and willing to renew their visions of teaching and learning if they are to fully realise the potential of educational technology" (p. 3).

Technology is often seen as the panacea for the problems of modern schooling. McArthur, Lewis and Rand (1995) argue that generic software (e.g., spreadsheets and graphing packages) has the potential to reverse the reported overall trend in classrooms which is towards content-focused pedagogy. They contend that by being able to access real databases, for example, students are able to generate their own hypotheses and then explore the merits of reported hypotheses or construct their own models. They reported that students who were engaged in this type of learning activity developed sophisticated inquiry and analytical methods. Further, they found that generic software added an element of realism to the students' work, because they were using real-world tools.

More specifically, there is a high expectation of the potential of computer technology to enhance the teaching and learning of mathematics (Gentile, Clements & Battisa, 1994). It has been suggested that computer use can improve mathematical modelling ability (Zbiek, 1998), increase construction of higher level conceptualisation in geometry (Gentile *et al.*, 1994), and motivate students by providing them with interesting learning activities, which help them better prepare for post-schooling activities (Sandholtz, Ringstaff & Dwyer, 1997). Some studies have also reported that computer rich environments improve high-level reasoning and problem solving (Baker, Geerheart & Herman, 1993; Sandholtz *et al.*, 1997), and are generally more effective in this than traditional learning environments (Cousins & Ross, 1993).

One explanation for the high expectation for computers to enhance mathematics understanding lies in the observation that learning power is increased each time we transform knowledge from one medium to another. This point was explored by Lemke (1996, p. 1) who argued that human communication normally deploys the resources of multiple "semiotic systems and combines them according to essentially functional principles". He contended that visual-graphical representations offered great potential in formulating concepts and relationships and found the ability to integrate textual and graphical displays to be an essential skill, partly because verbal text does not duplicate information presented graphically. As computers are able to represent information in both textual and visual forms, they have the potential to provide the multi-media environments for more effective learning. However, research on the implementation of educational reforms involving computers has indicated that maintaining student-performance standards during early stages of reform is difficult (Trotter, 1998). For example, when teachers began to implement the constructivist principles associated with technology-rich environments, they, like novice teachers, became preoccupied with discipline and resource management issues and experienced personal frustration (Dwyer, Ringstaff & Sandholz, 1989).

The high expectations for computers in education have lead to the establishment of schools with significantly more information and communication technology (ICT) resources than commonly found in most other schools. Such schools are able to offer their students computer-rich environments in which to learn.

The purpose of this study was to compare the mathematics achievement of Year 6 students in an ICT-rich school with that in schools which were representative of less ICT-resourced schools, and to explain those differences in achievement by investigation of the mathematics teaching and learning in the ICT-rich school.

Design and methods

A mixed-method design (Brewer & Hunter, 1989) was used in a longitudinal study of the impact of a computer-rich learning environment on students' achievement in mathematics.

The quantitative component of the study involved a quasi-experimental pre-test post-test nonequivalent control group design (Campbell & Stanley, 1963). It compared the mathematics achievement of a cohort of students moving through Year 6 at the computer-rich school (the treatment school) with corresponding student cohorts from four control schools, that were representative of other Queensland state primary schools with respect to student to computer ratio. The treatment school (School 1) had a student to computer ratio of 4:1, while the control Schools 2, 3, 4 and 5 had ratios of 12:1, 11:1, 16:1 and 12:1 respectively. The control schools were chosen to match the treatment school as closely as possible on student enrolment and the socioeconomic level of the school. Students were pre-tested using the Australian Council for Educational Research (ACER) Progressive Achievement Tests in Mathematics (PATMaths) 2A, in November 1998 (Year 5). They were then retested using PATMaths 2B in November 1999 (Year 6). These ACER instruments are standardised mathematics tests that have been normed on age-equivalent Australian students. They provide both a raw score (used in the following analyses) and a standardised score for the total test and each of the five sub-dimensions of the test: (1) Number with calculator, (2) Number without calculator, (3) Measurement, (4) Geometry (Space) and (5) Chance and Data. In order to compare mathematics achievement between the treatment school and the other four control schools, School (5) by Time (2) repeated measures multivariate analyses of variance

(MANOVA) were conducted.

The qualitative component of the study was a case study of mathematics learning in one Year 6 class in the computer-rich treatment school to investigate how this environment interacted with the students' learning and to provide some explanation of the quantitative results. This component was an interpretative case-study guided by the criteria for quality constructivist inquiry - trustworthiness, authenticity and the benefits of the hermeneutic cycle (Guba & Lincoln, 1989). The subjects of the study were the teacher and students from the Year 6 case study class of the treatment school. This school strongly espoused a philosophy of constructivism in its policy documents. The data sources were: video and audio recordings of classroom interactions; observations and field notes of lessons; interviews with selected students and their teacher; collection of classroom artefacts; and computer activity records. The researchers visited the classroom regularly throughout the year and attended all lessons for a unit on Chance and Data. The video and audio-recordings were transcribed and combined with the field notes, artefacts and computer records to give a rich description of the class. These data were then considered in relation to the quantitative comparisons to investigate supportive and inconsistent findings.

Results

Comparing mathematics achievement

When the means of the five schools were compared using a repeated measures analysis for the total mathematics achievement score, statistically significant differences were found between Schools ($F(4,241)=7.39$, $p=.00$) and across testing periods ($F(1,241)=50.29$, $p=.00$). School by Time interaction effects were statistically non-significant.

To further investigate these findings, similar analyses were conducted for each of the scale's sub-dimension scores. Only the sub-dimension of Measurement displayed a statistically significant School by Time interaction effect ($F(4,241)=3.00, p=.02$). Figure 1 plots the pre-test and post-test mean scores of the five schools for the Measurement sub-dimension.

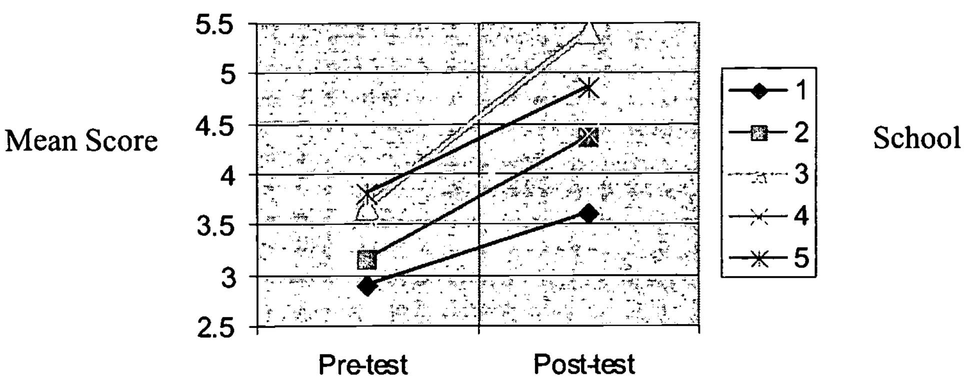

Figure 1. The pre-test and post-test mean scores of the five schools for Measurement.

To further investigate this interaction repeated measures analyses were conducted between School 1 (the treatment school) and each of Schools 2, 3, 4, and 5. A probability level of statistical significance of $p<=.01$ was used in these analyses to account for the potential for Type 1 errors. Only the School 1 - School 3 School by Time interaction was statistically significant ($F(1,179) = 6.74$, $p = .01$). Table 1 shows that School 3 mean scores improved 1.74 test score units from pre-test to post-test while the mean scores for the treatment school (School 1) increased 0.70 test score units, an enhancement in favour of the control school of about 0.8 SD which is an educationally significant effect (Cohen, 1969). That is, this interaction appears to be a result of the enhanced performance of School 3 relative to School 1.

Table 1

Means Scores (Standard Deviations) of the Five Schools for the Total and Sub-dimension Pre- and Post-tests Scores on the Mathematics Achievement Test. (N=246)

	School (*N*)				
Test	**1** (*N*=20)	**2** (*N*=34)	**3** (*N*=62)	**4** (*N*=62)	**5** (*N*=68)
Total Score (Max score=39)					
Pre-test	20.05 (6.44)	22.79 (5.50)	27.15 (5.23)	22.55 (6.80)	24.94 (6.86)
Post-test	23.70 (6.27)	25.21 (6.62)	31.52 (8.09)	26.50 (7.29)	30.47 (13.78)
Number with Calculator (12)					
Pre-test	7.10 (2.59).	7.76 (2.06)	9.22 (1.79)	7.46 (2.2)	7.91 (2.56)
Post-test	7.70 (2.89)	8.64 (2.13)	10.10 (1.75)	8.33 (2.57)	10.24 (13.22)
Number without Calculator (6)					
Pre-test	2.00 (1.62)	2.59 (1.37)	4.11 (1.46)	2.92 (1.69)	3.37 (1.55)
Post-test	3.10 (1.55)	3.38 (1.63)	5.19 (51.3)	3.84 (1.53)	4.40 (1.31)
Geometry (Space) (7)					
Pre-test	3.45 (1.64)	4.32 (1.47)	4.79 (1.32)	4.21 (1.75)	4.71 (1.63)
Post-test	5.05 (.94)	4.68 (1.59)	5.85 (.94)	5.26 (1.61)	5.62 (1.30)
Measurement (7)					
Pre-test	2.90 (1.17)	3.15 (1.13)	3.67 (1.23)	3.47 (1.53)	3.81 (1.37)
Post-test	3.60 (1.54)	4.35 (1.70)	5.41 (1.54)	4.39 (1.68)	4.84 (1.46)
Chance & Data (7)					
Pre-test	4.60 (1.67)	5.03 (1.45)	5.30 (1.49)	4.48 (1.80)	5.13 (1.51)
Post-test	4.25 (1.59)	4.55 (1.82)	5.64 (1.07)	4.82 (1.50)	5.46 (1.00)

The analyses also indicated statistically significant school main effects for Number without calculator ($F(4,241) = 8.35$, $p = .00$), Geometry ($F(4,241) = 5.21$, $p = .00$), and Chance and Data ($F(4,241) = 6.41$, $p = .00$). Statistically significant Time main effects were also obtained for Number with calculator ($F(1,236) = 4.88$, $p = .03$), Number without calculator ($F(1,241)=23.04$, $p=.00$), and Geometry ($F(1,241)=76.38$, $p=.00$). The main effects for the Measurement sub-dimension are not reported as they are meaningless in the presence of a statistically significant interaction (Pedhazur & Schmelkin, 1991). Table 1 summarises the pre-test (1998) and post-test (1999) mean scores and standard deviations for each school for total score and for scores on each of the five sub-dimensions of the achievement test.

In general, school mean scores improved across the year for all schools, except for the sub-dimension of Chance and Data. This general improvement needs to be interpreted with caution, however, as the pre-test and post-test were different instruments. Further, where the School by Time interaction was not statistically significant, there was a statistically significant main effect between the five schools for the Total and sub-dimension mean scores, except for Number with calculator. Post hoc Scheffe tests ($p<=.05$) for differences between the treatment school mean scores for these statistically significant main effects, showed that the mean scores for School 3 were statistically higher for each analysis, as was School 5 for the Total mean scores. These effect sizes would generally be considered as educationally significant also (Cohen, 1969).

Explaining mathematics achievement

In the case study component, the unit of work focussed on for intensive study in the treatment school related to Chance and Data. It was thus able to illuminate in some detail how the computer resources were interacting with the learning of the students while studying that topic and to provide some evidence to support the findings from the quantitative component of the study, particularly as they related to Chance and Data.

One of the activities in the Chance and Data topic was an activity called "Fastest Down the Mountain." In this game, two dice are thrown and the students predict which resulting sum of the die will enable them to climb down the mountain vertically in the fastest time. The board for this game is a "mountain" with varying heights relative to the possible sums for two standard dice (Figure 2). The height of the mountain varied from one unit for the sums 2 and 12 to six units for the sum 7.

This is a complex game mathematically. Seven is the most commonly occurring sum for throwing two dice; but 7 has to be thrown six times for the mountain to be descended while 2 or 12 have to be thrown only once. A complete analysis of the probabilities shows that either 2 or 12 are the best numbers to choose. It is more likely for 2 or 12 to occur once, than for 7 to occur six times. In each session, the students played the game for ten minutes and then used a computer program that simulated the dice throws for a stipulated number of throws, for example, 500 or 1000. The scores for the computer dice simulation were presented graphically on the computer screen.

The interviews with the teacher showed he did not understand fully the mathematics underlying this "game" and had greatly underestimated its mathematical complexity. Thus, he did not fully comprehend the degree of difficulty of the game and he failed to comprehend that the complexity of the probability knowledge required to correctly predict the most appropriate sum probably made the activity far too advanced for his Year 6 students. This was exacerbated by the fact that the teacher's inadequate understanding of the mathematics resulted in his being unable to provide adequate scaffolding for the students during both the computer-based and the dice-throwing activities. During the lessons, there were few attempts by the teacher to simplify the process of alternating between manual rolling and the computer simulation or to integrate the two types of activities. Further, there was no discussion tying the physical dice throwing with the computer simulation, nor theoretical explanations for the expected outcomes drawn from the activities. As well, there was no adequate discussion and reflection of the results that could have provided some insights for the students. Lesson observations and the student interviews showed that students were left to construct their own meaning.

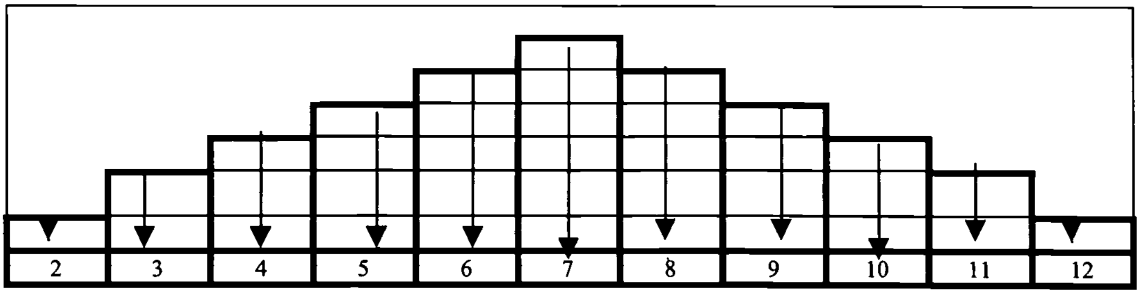

Figure 2. Fastest Down the Mountain Gameboard

Post activity interviews indicated that students had a range of differing learning outcomes. Some students did not think the simulation and playing the game by hand were linked. "It is different on the computer because it goes in a pattern on the computer, but it didn't go in a pattern the other way." Other students noted that it was "different on the computer but faster," while still others thought the computer program was more reliable "because the computer has a program and just does it automatically." Some students considered that the most likely outcome would be an alternative number, for example 4, "because I think it has more possibilities." Others would keep betting on 12 "because it is my lucky number," or 11 "because maybe the dice roll a bigger number," Few students even realised that 7 was the most likely sum. No student even came close to going beyond this to looking at the inverse relationship between mountain height and probability of dice sum.

Finally, the observations and analyses of this Chance and Data unit were typical of other computer-based mathematics activities observed throughout year.

Discussion

The results of this study do not support the contention that the mathematics achievement of students has been enhanced by working in the computer rich learning environment of the treatment school. Further, they give some indication of how the computer resources are being used in the class studied at the treatment school and why the potential for enhanced learning may not be being achieved.

As the selection criteria of the control schools matched the characteristics of the treatment school over several variables, the statistically significant difference in mean scores between the schools suggests that, overall, some of the control schools were achieving at a statistically and educationally significant higher level in mathematics than the treatment school. Other controls schools were achieving at a level which was not statistically significantly different from the treatment school.

The results of this study suggest that failure to attend to the following can result in limited development of understanding of mathematics in computer related activities.

Degree of difficulty. Many mathematics activities on computers use a game or problem format to motivate student activity in the expectation that observation of results will provide insight into a mathematical concept or process. However, such activities have to be carefully chosen to be within the students' zone of proximal development (Vygotsky, 1978) or the students will be unable to make the leap to the new knowledge.

Links to non-computer activity. Kaput and Rochelle (1997) argued that it is crucial for teachers to either simplify the process of alternating between computer-based and non-computer-based activities or integrate on and off computer activities.

Scaffolding. The mathematics behind an activity needs to be understood by the teacher so that the (s)he can provide the necessary scaffolding to assist the students' construction of knowledge from the computer activities (Bagley & Hunter, 1992). This scaffolding can be provided by focus questions and verbal feedback. It also can be provided by careful structuring of the computer activity and supporting off-computer activities.

Reflection. All learning is a consequence of reflection on activity (Davis & Rimm, 1998). Therefore, teachers should provide opportunities after computer activity for students to discuss and reflect on what they have done, what they have learnt, and where this new learning could lead.

These features were generally not apparent in the treatment school class studied. Attention to these features may have enabled the potential for enhanced learning from the integration of computer activities in mathematics learning to be achieved. As it was, the way the computer activities were implemented was consistent with the findings from the quantitative analyses which showed a decrease from pre-test to post-test for the treatment school class and that the mathematics achievement was not enhanced when compared to the results of the control schools. As this unit was found to be typical of the computer related activities generally in this class, the case study results also provide wider support for the broad quantitative analyses.

The findings from this study clearly indicate that unless the computer activities are meaningfully interpreted and integrated into the principles of the activities, it is highly unlikely they will enhance understanding. Such observations support the findings of Dwyer, Ringstaff and Sandholtz (1989) that there are special problems in the transition to the use of technology and the application of constructivist learning theories simultaneously. It also further supports the essential role of the teacher in effective use of technology to capitalise on its potential, and that the mere provision of computer resources is unlikely on its own to lead to enhanced learning.

Conclusions

A number of important conclusions can be drawn from the study reported in this paper. Firstly, the provision of computer-rich learning environments by themselves do not necessarily lead to enhanced learning in traditional areas such as mathematics. As was noted in the quantitative findings, the computer-rich learning environment did not result in increased rates of mathematical learning. Secondly, teachers have very important roles to play in ensuring that computer-rich learning environments actually do facilitate the construction of mathematical knowledge. As was noted in the case study, teachers need to have fully developed repertoires of knowledge about the mathematical concepts, processes and principles that underlie the computer-based mathematical learning activities. Without such repertoires of mathematical knowledge, they will be unable to: make sound selections of mathematical learning activities; provide adequate cognitive scaffolding to support the construction of knowledge by their students; or, provide conceptual links between computer and non-computer activities.

The final conclusion that can be drawn from this study is that qualitative case study investigations, following quantitative investigations, can provide researchers with data that will triangulate, complement, develop, initiate and expand the quantitative findings (Greene, Caracelli & Graham, 1989).

References

Australian Association of Mathematics Teachers Inc. (1996). *Statement on the use of calculators and computers for mathematics in Australian schools.* Adelaide, South Australia: Author.

Bagley, C., & Hunter, B. (1992). Restructuring, constructivism, and technology: Forging a new relationship. *Educational Technology, July*, 22–27.

Baker, E.L., Gearheart, M., & Herman, J.L (1993). *The Apple classrooms of tomorrow: The UCLA evaluation studies.* CST Technical Report 353. Los Angeles, CA: Center for the Study of Evaluation/Center for Technology Assessment, University of California.

Brewer, J. & Hunter, A. (1989). *Multimethod Research: A Synthesis of Styles.* Newbury Park, CA: Sage.

Campbell, D. T., & Stanley, J. C. (1963). Experimental and quasi–experimental designs for research on teaching. In N. L. Gage (Ed.), *Handbook on research on teaching* (pp. 171–246). Chicago, IL: Rand McNally.

Cohen, J. (1969). *Statistical Power Analysis for the Behavioral Sciences.* New York: Academic Press.

Cousins, J. B., & Ross, J. A. (1993). Improving higher order thinking skills by teaching "with" the computer: A comparative study. *Journal of Research on Computing in Education, 26*(1), 94–115.

Department of Education Queensland. (1995). Guidelines for the use of computers in learning. Brisbane, Qld: Queensland Government Printer.

Davis, G. A., & Rimm, S. B. (1998). *Education of the Gifted and Talented* (4th ed.). Boston, MA: Allyn and Bacon.

Dwyer, D. C., Ringstaff, C., & Sandholtz, J. H. (1989). *Teacher beliefs and practice. Part 1: Patterns of change.* Cupertino, CA: Apple Classrooms of Tomorrow Research Report [Report No 8].

Greene, J. C., Caracelli, V. J., & Graham, W. F. (1989). Toward a conceptual framework for mixed–method evaluation designs. *Educational Evaluation and Policy Analysis, 11*(3), 255–274.

Gentile, K., Clements, D., & Battisa, M. (1994). Effects of computer environments on students' conceptualization of geometric motions. *Journal of Educational Computing Research, 11*, 121–140.

Guba, E. G., & Lincoln, Y. S. (1989). *Fourth Generation Evaluation.* London: Sage.

Kaput, J., & Roschelle, J. (1997). Deepening the impact of technology beyond assistance with traditional formalisms in order to democratise access to ideas underlying calculus. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education.* Lahti, Finland: PME.

Lemke, J. (1996). Multiplying meaning: Visual and verbal semiotics in scientific text. In J. R. Martin & R. Veel (Eds.), *Reading Science* (pp. 87–113). London: Routledge.

McArthur, D., Lewis, M. W., & Rand, M., B. (1995). Esscots for learning: Transforming commercial software into powerful educational tools. *Journal of Artificial Intelligence in Education, 6*(1), 3–33.

Pedhazur, E., & Schmelkin, T. (1991). *Measurement Design and Analysis. An Integrated Approach.* Hillsdale, NJ: Lawrence Erlbaum.

Sandholtz, J. H., Ringstaff, C., & Dwyer, D. C. (1997). *Teaching with Technology: Creating Student–centred Classroom.* New York: Teachers College Press.

Trotter, A. (1998). *A question of effectiveness.* Http://www.edweek.org/sreports/tc98/intro/in–n.htm

Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes.* London: Harvard University Press.

Zbiek, M. (1998). Prospective teachers' use of computing tools to develop and validate functions as mathematical models. *Journal for Research in Mathematics Education, 29*(2), 184–201.

TEACHING MATHEMATICS TO ENGINEERS IN THE COMPUTER AGE

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As computers have become more powerful, the complexity of engineering problems which are being tackled is increasing. Since the pocket electronic calculator appeared in education some thirty years ago the increase in computing power available to students has been staggering. The course content of mathematics for engineers has not, in general, developed to reflect this change nor has the mode of delivery. It is important to harness this new tool in a planned manner to the benefit of student and teacher alike.

Introduction

In the last two decades or so one of the two most important issues which have arisen in the debate on the teaching of mathematics to engineers has been the role of the computer. Technological development is accelerating away at an ever-increasing rate from our capacity to adjust to the changes. We need time to think through the strategies which we might employ in order to harness the benefits that these developments might bring; thus far that time has not been granted to us.

While simple operations in arithmetic, algebra and calculus can (and should) still be carried out using 'pencil and paper' it is surely reasonable that calculators and computers should be used to handle the more tedious, lengthy and complicated operations. The rapid processing of data by computers has ensured that the time-scale required to perform those calculations which make a class-based modelling exercise both meaningful and worthwhile is now at a realistic level. This is an area which has, so far, not been developed to the extent that it merits.

Mathematics for engineers in the twenty-first century should have a syllabus which reflects the technological developments in computers in terms of both syllabus content and style of teaching. It is important of course to retain mathematical integrity, but it is no use defending the indefensible. What can sensibly be defended should, however, be so well guarded that it is virtually impregnable.

Even more than thirty years after the Organisation for Economic Co-operation and Development published its report "Mathematical Education of Engineers" (1966) it is still relevant to cite the reasons it gave for the central role of mathematics in the training of engineers. These can be summarised as follows:

Mathematics provides training in rational thinking.

It is the principal tool for deriving quantitative information.

It facilitates the analysis of natural phenomena.

It permits the engineer to generalise from experience.

It trains the imagination.

It is a preparation for adaptation to the future.

Whilst these aims are praiseworthy, it would be unwise to take it for granted that mathematics is the only, or even the most appropriate subject in the undergraduate curriculum by which they can be realised. Clements (1985) suggested that the aims were still relevant to the needs of the 1990's, but it is pertinent to ask whether the mathematics that we teach is helping to achieve them. Could we do better by modifying our teaching, both in content and style?

To a large extent, the content of our courses is outside our control. The widening of access to higher education and the changing nature of school mathematics has made the task of the lecturer of mathematics to engineers in the United Kingdom an increasingly difficult one. As if that were not enough to be going on with, there is the pressure from the engineering departments, which is largely financially driven, to reduce the time devoted to mathematics.

The pertinent question to ask is "Will an increased role for computers in the undergraduate engineering mathematics curriculum be beneficial?" Are we sufficiently altruistic to suggest that the only beneficiaries we need consider are the students, or do we include the mathematics staff whose employment depends largely on the continuation of the proportion of mathematics in the engineers' timetable?

The lessons of the past

Mankind has the luxury of being able to learn from history, but like so many luxuries it is often squandered. We have only to look back twenty-five years to see the mistakes which were made as the pocket electronic calculator progressed from an expensive, moderately powerful, cumbersome novelty to a cheap, very powerful, compact, and virtually indispensable aid. Little did we realise in the early nineteen-seventies that we were witnessing the beginnings of a potential monster. Today it has become so widespread and so easy to use that the younger generation turn to it for almost every calculation, no matter how trivial. Not only have today's students lost a feel for order of magnitude but also they do not appear to realise the merit in rounding off answers because of a lack of precision in the input data.

On the other hand, there are many who are concerned with primary and secondary education who take the view that not enough use is made of the pocket calculator at these stages of a pupil's development. Many of the assumptions which we would like to take for granted about the secure position held by some topics in the school curriculum are being challenged. For example, there are those who believe that less time could be spent teaching fractions in order to allow more time for other topics: the topic of fractions is difficult for children to grasp. With the universal use of the calculator decimals have superseded fractions in general practical calculations.

In the years since the O.E.C.D. report it is fair to say that a revolution has taken place in the extent of computer power which is widely available. The pocket electronic calculator has supplanted four-figure tables and slide rules, while batch processing on the mainframe has given way to on-line computing, which in turn has been superseded by the microcomputer and the PC. It might have been expected that such a revolution would have been matched by a similar revolution in the content and style of teaching mathematics. In some isolated instances moves have been made and are being made, but not to the extent that may have been envisaged.

There are some signs that the involvement of computers in the teaching of undergraduate engineering mathematics is beginning to gain momentum. It is perhaps relevant to ask whether this will bring the benefits that are claimed or whether it will do a disservice to those students whom it is intended to help. The evidence of the last twenty-five years or so is not encouraging. When metrication was introduced to the United Kingdom as a whole in 1971 its advocates assured the sceptics that it would make mensuration calculations easier and less prone to error. No need to worry about multiplying and dividing by 12, 3, 8, 14 and so on; 'it will all be done in tens'. There is precious little indication of that hope being fulfilled. Might it be that the mental agility developed by the need to calculate with all those numbers was of some benefit to children?

Then the pocket calculator arrived and we were told that there would be two advantages: first, the students would be relieved of hours of tedious calculation, leaving them free to concentrate on concepts and understanding; second, it was more likely that the calculations would be performed correctly. What do we find more than twenty years later? The most trivial of calculations is sub-contracted to the machine, the students have little or no feel for what they are actually doing or how precisely to quote their answers. What they can do is to obtain silly results more quickly and to more significant figures.

A more recent development is the advent of the graphic calculator. In many ways this represents a significant technological advance, enabling students to see the graphs of complicated functions and curves very readily and giving them the opportunity to compare easily the graphs of several related functions. Many students entering tertiary education are already accustomed to using a graphic calculator, a practice which is positively encouraged by many mathematics teachers. Not all investigators are convinced, however, that the use of a graphics calculator is an unqualified success story. But, the art of curve sketching was regarded by many teachers as a very good means of bringing together a variety of mathematical skills and of allowing the student to develop a deductive ability which was deemed to be important. Now all can be revealed by pressing a few buttons.

The use of the computer to tackle problems in geometry in an investigative manner is a rapidly-increasing area of development: a rising star is Cabri-Geometry which is being incorporated into some computers. And then, of course, there is calculus on the computer.

A development which should make all mathematics educators sit up with a jolt, be they passionate advocates of, or implacable opponents of, the increasing use of calculators and computers in the classroom is the super-calculator. The computing power and range of facilities that these machines offer is nothing short of amazing to those of us who grew up in the age of four-figure tables and the slide rule. The super-calculator is being promoted as having "the power of a computer laboratory with the independence of a calculator".

With the advent of the microcomputer and the PC a second chance to change radically the nature of the teaching of mathematics presented itself. This time, the stumbling-block was the lack of available quality software. Now, many hours of effort and much money are being invested with the aim of making the computer a more dominant factor in the learning of mathematics: the understanding of concepts and the acquisition of skills. It remains to be seen whether that investment will be justified.

What next?

As computers have become more powerful so the problems which are being tackled in the real world are becoming ever more complex, demanding more sophisticated numerical techniques. Problems which were taxing research mathematicians a few decades ago are now within the realms of possibility for the undergraduate engineer. With an ever-increasing demand on an already crowded curriculum the time available for practical computing is squeezed hard. As the mainframe has been partly supplanted by micros and PC's so the process of running a program has become more attractive. So too has the user-friendliness of many commercially available software packages. Therein lurks a danger.

Always anxious to reduce the contact hours allocated to mathematics, engineering colleagues point to the widespread availability of these packages. No need to spend time on teaching methods like Simpson's rule and Trapezoidal rule since they are not used in practice. Just use the packages as 'black boxes'. Ask a teacher of structural analysis whether he would want his students to use a framework analysis package without doing any 'hand calculations' on a simple framework. Or ask a lecturer in Fluid Mechanics whether he would prefer students not to do any laboratory work on simple ideal experimental situations. We must realise that this is the thin end of the wedge. Computer algebra packages are in wide circulation and the argument above can be extended to this area too. Why teach partial fractions when they can be produced via the package? Why indeed teach differentiation when it can be done by a computer package? Where will it all end? A handful of lectures on background work which could even be done by the engineer?

The changing nature of the upbringing of children over the last thirty years or so has had an influence on their learning capabilities which cannot be overlooked. Whereas books and radio featured large in the education of the children of yesteryear, they have paled into insignificance in comparison with the ever more dominant television. What does television offer? A series of visual images, some of which are memorable, and a background of words and sounds, most of which are never heard and very few of which are remembered. Watch a factual, documentary programme and take stock of what you have retained - words or pictures. Ask yourself what you have done to participate actively in the viewing process.

In the current age, television looms large from the earliest years. An effect which must not be ignored: the short concentration span which is developed by gazing at TV soap operas, where any action lasts a minute or two before lurching into a different scene. Detractors of the traditional lecture have criticised it from many angles, but have not been willing to concede that the training it offers in extended concentration is valuable in itself.

In the United Kingdom, attempts are being made to halt the alarming decline in the number of 18 year-olds taking Advanced Level examinations in Mathematics. Even allowing for the reduced numbers of 18 year-olds in the population, there is a decline relative to other subjects; indeed, the number taking English is increasing - at approximately the rate at which the number taking Mathematics is declining. Mathematics, it is being said, is perceived to be a 'hard' subject and something must be done to make it more attractive to 16 year-olds. There is a reform of the Advanced Level syllabuses currently taking place. One strand of activity is to re-define the "core" syllabus which all approved examination syllabuses must contain. A prime candidate for losing its prominent position is calculus.

So, the message is clear. If something is difficult then remove it so that the subject is more palatable, less demanding and, then, if it proves to be more popular the claim can be made that more over-sixteens are studying mathematics - but what mathematics?

Then we have the problems associated with widening access to Higher Education. First, re-name the polytechnics as universities to boost the numbers in universities and then encourage an expansion of student intake at very little extra cost. As a consequence some engineering departments complain that mathematics is the major stumbling-block to them increasing their intake.

Of course, it would be unthinkable if, with a wider undergraduate entry profile, failure rates were to increase dramatically. Engineering departments would be most aggrieved if the majority of failures were due to their students' inability in mathematics. And they wouldn't care too much about concepts: can the students do their basic sums? As for giving more time to allow the weaker student to develop the necessary mathematical proficiency, that is a pipe-dream. Time is being reduced and yet the criticisms will flow if the level of achievement declines.

Against this background what is the real role of computers in engineering mathematics education?

Pro-active or reactive?

We might well ask "What is left for us to teach ?" in this brave new world.

Suppose that we start from the premise that whatever can be calculated can be computed. If machines can do virtually all the mathematics that the average citizen might need then it is reasonable to ask whether there is any need for universal education in mathematics other than basic arithmetic. Roper (1994) argued that the increasing availability of powerful calculators could be liberating, because an individual will have access to a whole armoury of mathematical techniques which, in a suitable mathematical educational context, could give that individual great power and independence. It is therefore ever more important to define our aims in teaching mathematics to the wider school population.

Roper posed the question: if it is known that the statement $ab = 0$ implies that $a = 0$ or $b = 0$ or both are equal to zero, then is it necessary to be able to factorise polynomials when a symbol manipulator can perform the task for us? The argument originally put forward in favour of the calculator was that as a result of it removing the drudgery of calculation, pupils would be able to concentrate on solving problems from the world outside the classroom. Similar arguments are now put forward for the syllabuses of the future to be based on the symbol manipulator. Changes in the approach to examinations are required: for example, if graphic calculators are permitted in examinations then it is pointless to ask a candidate to sketch the curve of a given equation.

The Schools Curriculum and Assessment Authority (1993) stated that mathematical skills and understanding must include the component "appreciate how to use appropriate technology, such as computers and calculators, as a mathematical tool, and have an awareness of its limitations". Who decides what is 'appropriate'?

There are many mathematicians who believe that technological change makes us ask questions on whether the nature of mathematics itself is changing. If so, how should that change be reflected in the lecture room?

Hodgkinson (1987) expressed his concern over the role of computer algebra systems in the undergraduate engineering mathematics curriculum. He pointed out that there was often a large investment in time required in order to learn how to use such a system and it was difficult to find that time in a crowded syllabus. He stressed that there was a need to use the software sensibly: those engineering teaching staff who envisaged that software packages could, should and would replace traditional mathematics misunderstood their proper role. It might be added that those people also misunderstood the role of mathematics in engineering education, as espoused by the OECD Report (1966).

There is a danger that poorly-designed software, or software used in an inappropriate context, will become a replacement for vital elements of the learning process. To use algebraic manipulation packages without the necessary reinforcement of some 'hand' calculation on examples, albeit fairly elementary ones, cannot be wise. By all means use the package when some really involved factorisation is required, but at least give the student the experience of factorising some simple expressions, for example $x^2 - 5x + 6$, for himself. No doubt, there are those who will argue that this is a parallel with the case for doing long division by hand before using the ÷ key on the pocket calculator; the parallel, however, is a false one.

Planning for the future - Building a curriculum for the year 2000

It is vital that we start from a justification of the central position of mathematics in the school curriculum of the next century. Having surveyed a number of authors' attempts to support the prominence given to mathematics at the school level and beyond, Wain (1989) suggested a list of his own reasons:

(i) mathematics is useful

(ii) mathematics is important in our lives and its place needs to be understood

(iii) mathematics trains the mind

(iv) mathematics is a powerful means of communication

(v) mathematics is enjoyable and has aesthetic value.

It is perhaps cynical to suggest that the aesthetic element is hardly likely to cut any ice with all but a very small minority; cynical, but realistic. It sometimes seems that schoolteachers feel the need to elevate the enjoyment aspect above all others, in the belief that it is essential to capture and maintain pupils' interest and that this task is becoming more difficult. This, together with the coursework element in the 16+ examination in the UK means that students have become accustomed to an investigative approach to 'learning' mathematics. In the UK this approach is not really carried through into the teaching at sixth form level, and this change of emphasis is a source of much disappointment to the pupils.

Attempts have been made at the tertiary level to introduce the investigative approach into an already crowded curriculum, but these attempts are sporadic. Something has to give and that something is often from the analytical component of the curriculum. There are many academics who regret the diminution in the role of analytical methods, believing that, even in this computer age, it is more important to develop these skills than to carry out computer-based investigations.

What is clear is that those of us in the tertiary sector cannot ignore what is happening in our schools nor make lofty pronouncements about what ought to happen and then expect the schools to fall in line immediately and without question. Mathematics is perhaps unique in that it is so structured. Those children who will eventually become professional practitioners of the subject must be allowed, and encouraged, to develop at a suitable pace. The difficulty arises in devising a syllabus and a scheme of study which allows less able children to learn appropriate material at a speed appropriate to their abilities.

It is necessary to decide on a list of priorities and then produce a syllabus which incorporates as many of them, starting from the top of the list, as time permits.

The application of mathematics is as vital as many of the techniques which can be applied. It is surely part of the duty of those who are entrusted with teaching mathematics to engineering undergraduates to go further than merely exposing them to methods of solving equations, evaluating definite integrals, and so on. If we are to maintain an honest curriculum then we must develop the students' ability to apply these methods to the solution of meaningful problems in their engineering discipline. If we do not, then we may find that the teaching of mathematics is taken from us. 'Customer care' is a term which is currently in fashion; if we do not reform radically our syllabuses and style of teaching then we may find that we have no customers for whom to care.

We must embrace the technology of tomorrow in our design of a new syllabus and method of delivery. That is not to say that we should go overboard in accepting the software package approach to teaching, but that we should plan for its sensible role in the overall scheme of things. It is unwise for those of us who have had the benefit of a more traditional education in mathematics, which underpins our application of computer-based methods and our use of packages, to assume that today's students (and those of tomorrow) will be equally well-placed to make the best use of this technology without that underpinning. Too many changes in the mathematics curriculum have occurred (certainly in the UK) without adequate prior research.

If software packages are not used sparingly and sensibly then there is the risk that engineering departments, ever more conscious of financial strictures, will argue (perhaps with no little justification) that they could do the mathematics teaching just as competently as their mathematical colleagues. Many departments of mathematics depend for their very survival on the service teaching which they provide and many more would be severely wounded by its loss.

A further worry is that any attempt on a large scale to replace the teaching of techniques by case studies which demand computer-based solution might lead to criticism by engineering staff that the mathematicians are invading their territory. 'Your job is to teach the techniques and our job is to provide the case studies and applications' is a cry which has already been heard and may become more frequently voiced. The engineers might also make the comment that with ever-diminishing skills and knowledge being demonstrated by engineering freshmen, 'or so you keep telling us', it is more important that the mathematics teachers concentrate on making sure that the students can do basic sums.

Conclusions

The inevitable advance in computer technology means that school 'mathematics' will probably continue to move further away from the necessary grounding for our future mathematicians, scientists and engineers, *inter alia.*

It is surely reasonable to ask those who champion the cause of increasing the role of computers in the teaching and learning of undergraduate engineering mathematics to pause and reflect for a moment. The lack of provision of both software and hardware has slowed the apparently inexorable influence of the computer into the curriculum. This provides a rare opportunity to conduct an appraisal of where we are and where we are going before we are carried away with the fast-flowing river of change which seems to de-stabilise so many areas of education today. There is a real danger that important skills will be lost in the same manner as we have seen accompany the greater availability of the pocket electronic calculator.

We must examine carefully the reasons for teaching mathematics to engineering undergraduates, and we must do it honestly. Then we must decide on the role that computers can and should play in this process. For, unless we do so, we may find that we have lost for the foreseeable future the rightful role of mathematics in the education of our young engineers.

References

Clements, R. R. (1985). *Int. J. Math. Educ. Sci. Technol.*, 16, 233–238.

Hodgkinson, D. (1987). Algebraic Computing in Education, in *Proceedings of the Conference on Mathematics and Statistics Curricula in Higher Education for the 1990s*, The Polytechnic of Edinburgh, 197–217.

Organisation for Economic Co-operation and Development (1966). *Mathematical Education of Engineers* Paris, *O.E.C.D.*

Roper, T. (1994). The Future of Mathematical Education, *Issues in Teaching Mathematics*, 212–216, Cassell, London.

Schools Curriculum and Assessment Authority (1993). *GCE Advanced and Advanced Supplementary examinations subject core for mathematics*, London, SCAA.

Wain, G. (1989). Mathematics, in P. Wiegand, and M. Rayner, (Eds.), *Curriculum Progress 5–16,* Lewes, U.K, Falmer Press.

CAN A NEW-GENERATION INTELLIGENT COMPUTER-ASSISTED LEARNING PACKAGE COMPETE WITH THE "SCHOOL MATHEMATICS TRADITION" IN ENHANCING ALGEBRA PERFORMANCE?

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Computers have the potential to play a significant reforming role in mathematics teaching and learning. Intelligent computer assisted instruction packages (ICAI) have been trialed with mixed reviews and their potential to contribute to mathematics learning remains the subject of debate. This study contributes useful insights to the understanding of students' learning of algebra through the use of the ICAI software, The Learning Equation, *which was trialed in a Year 9 algebra unit. Pre-test and post-test results of students who worked with the software were compared with the results of students in control classes who received normal tuition. The results indicated that, although both groups of students progressed in their algebra knowledge, those who worked with* The Learning Equation *generally performed better in the written post-test.*

In the 1970s, ICAI systems were developed where problems and responses could be generated to react to individual student inputs (Barr & Feigenbaum, 1989). Carbonell (1970) outlined the idea of a computer that acted as a tutor that would engage the student and make inferences regarding student knowledge. These early computer programs modelled traditional instructional practices and materials (Kaput, 1992). That is the use of ICAIs did not appear to reinforcing a pedagogical shift towards active learning by students and that these forms of software had the potential to reinforce traditional approaches (Kaput & Thompson, 1994). Many researchers have advocated approaches to teaching and learning mathematics students construct mathematical understandings through interactions with the environment and engage in social interactions with peers (Cobb, Yackel & Wood, 1992; Lucking, 1999; Vygotsky, 1987). It was found that there were only negligible performance differences on standard tests between students with practice on such programs and those who were taught using traditional mathematics courses (Becker, 1990). More recent examinations of ICAI software continues to question their potential to help students to learn mathematics (Becker, 1994; Cooper, McRobbie & Baturo, 1998). The aim of this study is to compare the learning outcomes of students who worked with The Learning Equation (ITP Nelson, 1998) with the outcomes of students who were taught in the "school mathematics tradition" (Gregg, 1995, p. 443).

Background

The Learning Equation is a multimedia software suite designed to be the major teaching resource for the first four years of high school. It is a complete mathematics course which extends from Years 7 to 10 and covers the mathematics concepts usually covered in Years 8-10 mathematics programs including number concepts, number operations, patterns and relations, variables and equations (algebra), shape and measurement, transformations, and statistics and probability. The software uses a cyclic approach with each of these topic areas covered in each of the year levels. Each topic unit is divided into lessons. In the Year 9 topic under study (patterns and relations: variables and equations) there were thirteen lessons. This unit was selected because it fulfilled the needs of the study school at the time the software was to be trialed. *The Learning Equation* is a multimedia environment, with voice explanations, textual explanations, practice questions where text cues guide students who make mistakes, summary activities and self-tests. Each lesson unit begins with an application or mathematics-modelling situation where the key concept is related to an applied problem. This is followed by a guided explanation. That is, students are led through the logic behind the concepts and procedures by a series of prompts and explanations. Sometimes the students are then required to model the concepts using a "binomial grid explorer." In this environment students can manipulate linear and quadratic expressions which are displayed on a Cartesian grid with black tiles representing x^2, x and units and red tiles representing negative variables and units. This is the major opportunity for students to control variables. In other respects, the students are limited by the software to selecting the order in which they do activities or pace their progression through the various activities. Following the introduction phase, there are practice questions, word problems, the use of stimulus material (such as graphs and functions) and a terminology activity. When students make an error, they are prompted with clues and structure. Three errors in a row result in the answer and a brief explanation of the correct procedure being presented. Finally, there is a self-test where students are given a selection of the types of questions studied in the lesson unit. The students can see their own responses and can view correct solutions complete with detailed working-out steps. Thus the software projects a traditional and absolutist (Ernest, 1996) image of mathematics but the medium is dynamic and interactive rather than static and inert (Kaput, 1992). In this study, the Year 9 units on *Patterns and Relations*, *Variables* and *Equations* were trialed. The final test was constructed to reflect the content of the course which was the same in both test and control classes. The total course consisted of 26 lessons of 70 minutes.

Method

The study design was a randomised pre-test-post-test design with one class using the multimedia environment of *The Learning Equation* as the treatment and two control classes being taught using traditional methods.

Subjects and Contexts

The subjects were 54 Year 9 students in a secondary school of 650 students located in a middle class suburb in the Brisbane metropolitan area. The students were randomly allocated to one computer class (28 students) and two control classes (13 students in each). The test and control students studied the same content. The primary resource for the control students was a traditional textbook (*Mathematics 9*; Priddle, Davies & Pitman, 1991). The teachers were assigned to teach the unit prior to the study commencing; Murray (all names are pseudonyms) taught one of the control classes and the treatment (ICAI) class, Anna taught the other control class until the last fortnight when she took sick leave and Jack taught her class. All three were considered by their colleagues as competent senior secondary mathematics teachers and had over 10 years mathematics teaching experience.

Instruments

Instruments in this study included a pencil and paper pre-test and post-test, lesson observations, and unstructured interviews with the teachers and students. The pre-test was made up of arithmetic questions and questions designed to test the students' knowledge in the algebra content they were about to study. Analysis of the results of this test allowed students' final performance levels to be compared with their starting performance. The post-test was constructed to test students' abilities to perform the algebra studied, that is, it contained questions of the form that students encountered in the algebra course. The algebra section of pre-test and post-test included questions of an *operational nature*, that is, the problems could be solved using arithmetic means (Sfard, 1991). This was followed by questions on the *variable concept*. The third subset contained *structural* questions which referred to question types that involved thinking in terms of abstraction, that is, the algebraic term or expression was seen as an abstract object (Sfard, 1991). The fourth question type was *word* problems where the students were required to represent the given verbal and diagrammatic information in algebraic form, decide what operations were necessary and then perform these operations. It has been previously noted that word problems are particularly good in distinguishing between those students who have competency in familiar symbolic manipulation and those who have a deeper understanding of the structure of algebra (Sfard & Linchevski, 1994). One of the word problems could be solved by either using numerical methods to find a simultaneous solution or algebraic methods. Word problems were not included in the pre-test. Students completed the tests individually without access to resources. The use of scientific calculators was optional. The six lesson observations of each class were made to provided data on the learning environment of each classroom. The lessons were video taped and field notes taken. Before the lessons, the first named researcher spent several hours familiarising Murray with the software and discussing how it might best be used to enhance student learning. During the observations, informal interviews were undertaken with teachers and students whenever something of interest was noticed. These interviews were audio taped where possible or recorded in field notes.

Analysis

The test papers were collected and marked. Students who did not attempt to answer a question were given no marks. Students' responses for each of the items were analysed in terms of correctness and, where common mistake patterns were observed, they were described. Preliminary statistical analysis was used to guide the subsequent qualitative analysis. That is, the means and standard deviations and two tailed independent *t*-tests were calculated on the subsets of both pre-test (arithmetic and algebra) and post-test (operational algebra, variable concept, structural algebra and word problems) data. The pre-test data was included in this analysis because the sample sizes were small. Since there was no significant difference in results between the two control classes in the arithmetic and algebra pre-test and post-test their results were combined. The overall results were then analysed in terms of means and standard deviations for each subsection. Subsequently the number of students who correctly answered each question was analysed. This approach to data reporting was undertaken because it enabled error patterns in student performance to be examined. That is qualitative analysis had been used to examine the implications of student responses.

Results and discussion

Classroom observations

In the control classes, the teachers taught the lessons consistent with descriptions of "the school mathematics tradition" (Gregg, 1995, p. 443). That is, they started lessons by checking homework, clearly illustrated procedural application of new material on the white board, and then assigned students to work on activities from the text. While students worked on assigned tasks, the teachers moved among the students encouraging, and assisting those who indicated they were having difficulty with the tasks. In the treatment (ICIA) class, students sat and worked in pairs at small desks upon which the networked computers were located. For most of each lesson, they worked almost exclusively from the computers. This classroom was much noisier than the two control classes.

Test results

An overview of pre-test results are summarised in Table 1. The arithmetic component of the pre-test was 30 marks while the algebra component was 48 marks The post-test results have been summarised in Tables 2 and 3. Although there were 28 students in ICAI class, only the results of 24 are reported because of student absences.

Table 1

Summary of Results of the Pre-test

Test	N	Mean	SD	t
Control group (arithmetic)	25	22.50	3.89	1.38 (p=.98)
ICAI group (arithmetic)	24	23.27	2.32	
Control group (algebra)	25	4.18	2.61	.01 (p= .38)
ICAI group (algebra)	24	4.18	2.18	

As reported earlier the pre-test results indicated that no significant differences existed between the classes in either algebra or arithmetic performance. Both groups performed poorly on the algebra questions. Table 2 shows the average subtotal score for each concept area. Independent 2 tailed t-test significance has been calculated for the totals within each concept area (operational algebra, variable concept, structural algebra and word problems). These calculations take account of all marks including part marks.

Table 2

Post test Comparison

	Control		ICAI		
Subset	Mean	SD	Mean	SD	t
Operational algebra	5.08	2.26	6.34	1.79	2.09 *
Variable concept	4.16	1.40	5.36	1.30	3.03 **
Structural algebra	2.88	2.72	4.41	2.55	1.98
Word problems	3.62	2.89	5.84	3.31	2.46*

$^{*}p < .05$. $^{**}p < .01$

Although the ICAI student averages were higher on all subsets of the test this was not statically significant on the structural algebra subset. Table 3 shows the number of students in each group who answered each question correctly. Clearly both classes improved in their abilities to do these types of questions in this test; however, neither class improved as much as educators might have wished. In operational algebra the significant difference in mean scores (p=.04)is largely accounted for by the ICAI students having performed much better on the directed number question. That is, only 7 control class students correctly solved the problem, $18 = -3x + 3$. This contrasted with 14 students in the ICAI environment.

Variable concept. In both environments, there was substantial improvement on the questions relating to the variable concept. However, students from the computing environment were more successful than the students in the control classes in articulating an understanding of the variable concept (p=.004). This may have been due to the use of frequent simulation type problems that started every introduction of units in *The Learning Equation*. On these occasions, students were taken through the steps with visual as well as voice instructions and explanations.

Table 3
Summary of Results of Pre-test and Post-test

Item	Number of students who correctly responded			
	Control (N=25)		ICAI class (N=24)	
Operational Algebra	Pre	Post	Pre	Post
Item 1: Solve for x in $x+7 = 10$.	25	24	23	20
Item 2: Operational with directed number component, e.g., $18 = -3x + 3$.	1	7	0	14
Item 3: Collecting monomial terms.	10	17	12	15
Item 4: Simplify with directed nos and the distributive law, e.g., $(4a^2+3a) - (2a^2-a+5)$.	0	9	0	7
Sub total	37	56	23	58
Variable concept				
Item 5.What does "a" mean in 3a?	0	6	0	18
Item 6.Is a+a+a+a=4a? Explain.	10	18	16	21
Item 7.If a = 4 is 3a = 34 ? Explain.	10	18	7	22
Sub total	20	41	25	56
Structural Questions				
Item 8.Solve for a in $4a-4 = 3a + 6$.	2	8	1	12
Item 9. Solve for n in $\frac{4}{n} = \frac{48}{3}$	0	0	0	2
Item 10.Evaluate $12x^2 + 4$ if $x = 4$.	4	9	2	12
Item 11.Expand $(x + 4)(2x - 5)$.	1	4	0	3
Item 12. Simplify $\frac{16x^3y - 12x^2y^2}{4xy}$	0	2	0	6
Sub total	7	23	2	34
Word Problems.				
Item 13.Simple perimeter.		13		8
Item 14.Rectangle and square.		2		6
Item 15 Simultaneous solution.		8		11
Item 16. Proof.		0		3
Sub total		23		30

Structural algebra and word problems. The difference between the scores of the control and ICAI students was not statistically significant. Still the ICAI students did better on these items except Item 11. The difference between the two groups was statistically significant (p = .018). The first word question asked the student to find a simplified expression for the perimeter of a rectangle given side length of 3x and width of 2x. A labelled diagram was given. Thirteen students in the control classes correctly answered this question while 8 ICAI class students responded correctly and 5 gave incomplete responses of $2(3x + 2x)$. Had these students completed this expansion (not a conceptual error), the two groups would have performed equivalently. In both groups most of the student errors involved a lack of understanding of the difference between the attributes of linear and area measurements and confusions related to indices. The second word problem gave a square labelled with a side of 3k–2 cm and a rectangle labelled with a length of 3k+1 cm and k+1 cm. The students were told the "square and the rectangle and the square both have the same perimeter, find the value of k." This problem required students to understand the equal concept as well as to translate the problem into one of form (ax + b) = (cx + d). Herscovics and Linchevski (1994) have identified this type of problem as requiring structural understandings. Only 6 ICAI class students did not attempt the question compared with 14 students not responding in the control classes. Only 4 of the control class students recognised the need to equate the two perimeters before attempting to solve for k compared with 13 students who recognised this requirement in the ICAI class. Further, only 3 students in the computer class focused exclusively on one side compared to 7 out of the 11 who responded from the control classes. This observation indicates that the ICAI class students may have developed a better understanding of the equal sign concept. The third word problem was also a simultaneous equation problem:

> *The Westmount Video Shop offers two rental plans. The first plan costs $22.50 per year plus $2.00 per video rental. The second plan offers a free membership for one year but charges $3.25 per video rented. For what number of rental videos will these two plans cost exactly the same.*

One reasons for including this question was that it can be solved using either numerical or algebraic methods. Two students in the control classes used algebraic methods and completed the question and six students in the control classes completely solved the problem using a numerical method. This contrasts with the ICAI class where 9 students used algebra and correctly found the solution. Two other students in this class correctly found the solution using numerical methods. The final word problem involved proof (Kieran, 1992, used this problem as an illustration of the type of problem that few, less than 10%, of Year 10 students were able to do - according to Kieran, the problem was that most students were unable to construct the starting equations):

> *A girl multiplied a number by 5 and added 12. She then subtracted the original number and divided the result by 4. She noticed that the answer is 3 more than the number she started with. She said, "I think that would happen, whatever number I started with.". Use algebra to show she is right.*

Three students in the ICAI class correctly constructed the equations, recognised the significance of equality and reached a solution. Two other students in the ICAI class and one from the control classes constructed inappropriate equations. The remaining students in both classes failed since they did not realise that an equal sign was involved, which supports Kieran's (1992) finding. Although the students in both groups performed poorly on the word questions, the students who worked with *The Learning Equation* performed higher. Interestingly the ICAI students attempted more problems and tried to use algebraic rather than numerical methods.

Conclusions

The students who studied using the *Learning Equation* software outperformed the control class on all classes of questions and this was statistically significant on all except the structural algebra subset. In this regard the results of this study counters those reported earlier (Becker, 1990; Becker, 1994; Cooper, McRobbie & Baturo, 1998) Analysis of student responses indicated that this relatively strong performance was largely due to the ICAI group's greater understanding of the directed numbers (Item 2), abilities to apply order convention (Items 8 and 12), ability to articulate meanings of the variable concept (Items 5, 6 and 7). Further analysis of student responses to the word problems indicated that the ICAI students were more inclined to use algebraic methods over numerical methods and had a better understanding of the equal concept (Items 14, 15 and 16). In summary, even though the program had only run for 10 weeks, there appeared to be educationally significant differences in performance on this pencil and paper test. In both groups, students' poor performance was directly linked to their inability to apply arithmetic procedures and apply concepts associated with pre-algebra (directed number, order of operations, equal concept) within an algebraic context.

This study provides evidence that more of the students using the new generation ICAI software improved in their ability to recognise the structures of various algebra problems (and knew what procedures to apply) than the students from the traditional classes. There are number possible explanations for accounting for these findings. First, the novelty of the ICAI may provide stronger motivation and more attention to task than the traditional classes. Second the two-to-a computer environment with *The Learning Equation* may reduce distractions and provide the necessary discussion to assist students construct effective knowledge. Third the electronic medium combines visual images and voice, provides immediate feed back in the form of structural hints, and models answers to all problems unsuccessfully solved. The teacher in a traditional classroom may only be able to offer explanations to individual students occasionally. Student practice on questions in the tradition text, may not offer much modelling, and can not provide immediate feedback. These differences may be crucial for learning for some students. Finally, the nature of the discourse between the students and the software in the ICAI class may facilitate greater cognitive time on task, or more quality cognitive time on task, than the discourse in the control classes. In this way, the ICAI may be more effective in prompting learning. In summary, the program is based upon traditional absolutist images of mathematics (Ernest, 1996) and the belief that careful explanation and practice of problem types is the most appropriate way to learning mathematics. Thus, the learning equation software mimics pedagogy consistent with the school mathematics tradition (Gregg, 1995). However, when interacting with the electronic media which is dynamic rather than static it appears that pairs of students engage in social interactions with peers that facilitates their construction of mathematical understandings (Cobb, Yackel & Wood, 1992; Vygotsky, 1987). The critical difference between this form of learning mathematics the construction of knowledge from the environment (Lucking, 1999) is that the environment is virtual and the learning process is highly guided.

Although there is some evidence to support each of these explanations, further analysis of the lesson observations and students' comments on their learning environments is needed to verify their potential to explain the differences in the performance between the learning environments. Further evidence needs to be collected to determine the long-term effects of working with this program upon student cognition and affect. This study has indicated that carefully constructed ICAI software has the potential to foster some students' ability to perform both operational and structural algebra. In fact, it may well be more successful in this regard than teaching behaviours described as "the school mathematics tradition" (Gregg, 1995).

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References

Barr, A., & Feigenbaum, E. (1989). *Handbook of Artificial Intelligence (Vol 1).* Reading, MA: Addison–Wesley.

Becker, H. (1990). *When powerful tools meet conventional beliefs and institutional constraints: National survey findings on computer use by American teachers.* [Report No 49. Centre for research on elementary and middle schools]. Baltimore, MD: Eric document reproduction service ED 337 142.

Becker, H. (1994). How exemplary computer–using teachers differ from other teachers: Implications for realising the potential of computers in schools. *Journal of Research on Computing in Education, 26*(3), 429–459.

Carbonell, J. (1970). AI in CAI: An artificial intelligence approach to computer–assisted instruction. *IEEE Transactions on Man–Machine Systems. MMS*–11.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*(1), 2–33.

Cooper, T. J., McRobbie, C., & Baturo A. (1998). *CCC Successmaker Program Evaluation.* Brisbane Australia: Queensland University of Technology (90).

Ernest, P. (1996). The nature of mathematics and teaching [Http:\www.ex.ac.uk/~PErnest/pome/pompart7.htm] (24th November).

Gregg, J. (1995). The tensions and contradictions of the school mathematics tradition. *Journal for Research in Mathematics Education, 26*(5), 442–466.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*, 59–78.

ITP Nelson, (1998). *The learning equation mathematics 9. Ontario*, ITP Nelson.

Kaput, J. J. (1992). Technology and mathematics education. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515–556). New York: National Council of Teachers in Mathematics Education.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 390–419). New York: National Council of Teachers of Mathematics.

Luckin, R. (1999). TRIVAR: Exploring the "Zone of Proximal Development." [http://www.cogs.susx.acuk/users/rosel/europper.html] 20/09/99.

Priddle, A., Davies, T., & Pitman, P. (1991). *Mathematics 9.* Brisbane: Jacaranda Press.

Sfard, A. (1991). On the dual nature of mathematics conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1–36.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification – The case of algebra. *Educational Studies in Mathematics 26*, 191–228.

Vygotsky, L. S. (1987). *The collected works of L. S. Vygotsky Volume 1.* Plenum Press, New York.

COMPUTER AVOIDANCE IN A MATHEMATICS STAFF: INFLUENCES OF BELIEFS AND CULTURAL PRESS

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Research has reported that many mathematics teachers have not integrated the use of computers into their mathematics teaching even where the resources have been available. This study used qualitative research methods to explore why a secondary mathematics staff used computer resources to a very limited degree. It was found that there were relationships between teachers' use of computer technology, their beliefs about teaching and learning and their teaching practices. In particular, it was evident that assessment instruments and textbook selection maintained teacher-centred pedagogies. The findings have implications for professional development in relation to the use of computers, particularly exploratory software, in mathematics teaching.

Internationally, education researchers have expressed high expectations for the potential of computer technology to improve the teaching and learning of mathematics (Kissane & Kemp, 1998). In particular, researchers have supported the use of graphics calculators (McRae & Kendal, 1999), generic software such as spreadsheets (Abramovich, 1995), specialist mathematics software such as DERIVE 2.55 (Pierce, 1999), and Cabri geometry MS-DOS (Vincent & McRae, 1999). However, studies have indicated that mathematics teachers have been slow to introduce and use computers in their teaching even when hardware and software has been accessible (Becker, 1994). Studies on teachers' reluctance to engage in the use of computers in their mathematics teaching have included examining teachers' beliefs about how students learn (Sarama, Clements, & Henry, 1998) and the mismatch with existing teaching practices (Norton, 1999). Further, another study has documented the importance of peers and school micro-culture in influencing teachers' implementation of innovative teaching strategies (Rosen, & Weil, 1995). This study examines teachers' responses to the potential of computing power in the context of the interplay of teachers' beliefs about teaching and learning, and their existing pedagogical practices. The study examines this interplay within the social setting of a mathematics staff community in a technology rich secondary school.

Theoretical framework

Norton (1999) described two orientations to the use of computers in mathematics teaching and learning. The first is *calculational* where computers are primarily seen as instruments to carry out procedures for deriving results (e.g., calculating numerical results or quickly graphing data). The second is *conceptual*, where computers are used to build conceptual understandings by focusing students' thoughts upon the underlying mathematical ideas and relationships between these ideas. Two images are used in this study to describe teachers' dominant beliefs about teaching and learning. The first is *transmission* where knowledge is transmitted from the teacher to the learner (Crawford, 1996); in other words, the learner acquires knowledge by some kind of process of absorption (Hatfield & Bitter, 1994). The second is *social constructivism* where the learner is seen as actively constructing their own knowledge from the environment (internalising knowledge as schemata) through interaction with physical reality (Luckin, 1999) and social interactions with peers and the teacher (Vygotsky, 1987). Kuhs and

Ball (1986) provided a framework of four descriptions for classifying teachers' views on teaching practice. The first is *content-focused (performance)* where the teacher plans in terms of the content to be taught with an emphasis on improving student performance. The second category is *content-focused (understanding)* which is similar to content-focused (performance) but with an emphasis on understanding the ideas and processes rather than on the performance on procedures. The third is *classroom-focused*, where the main concern is that classroom activity be well structured and organised according to effective teacher behaviours. The final category is *learner-focused* which is an approach consistent with constructivist theories of teaching and learning.

Method

The study was a qualitative educational case study (Stenhouse, 1990) involving a hermeneutic approach (Denzin & Lincoln, 1994). The design of the study and the analysis of the data heeded the criteria for constructivist inquiry of Guba and Lincoln (1989). The data were analysed using non-numerical analysis procedures that involved "examining the meanings of peoples' words and actions" (Maykut & Morehouse, 1994, p. 112).

Subjects and contexts

The study was carried out in a technology rich secondary school (known by the pseudonym "Hill View"). The school was chosen because difficulties associated with access to hardware and software seemed not to be a major obstacle to the use of computers in previous reports (Becker, 1994; Rosen & Weil, 1995). The school had seven designated computer laboratories each with between 25 and 30 networked Pentium computers for a student population of 650. The school also had a digital projector and the staff had their own computer room with sufficient computers for ready access. The mathematics staff also had a class set of graphing calculators.

Eight of the ten teachers who taught mathematics at this school were surveyed with the five most influential of those teachers selected for more in-depth case study. The latter group comprised the senior mathematics coordinator Julie, the junior mathematics coordinator Eva, Peter who had an extensive background in the use of computer technology, Mary who was previously a mathematics subject coordinator at another school, and, finally, Emm who was responsible for compiling the statistics for student exit results. The interviews of the computer education coordinator and a previous mathematics staff coordinator confirmed that these 5 teachers had a large influence upon the teaching of mathematics in the school. They exercised this control by writing/overseeing the work-programs that stipulated (which concepts would be taught and when they would be taught) and the assessment instruments (assessment at all levels was internal to the school). The two coordinators (Eva and Julie) selected the resources (including textbooks) that all teachers would use, and thus had a particularly strong influence on the pedagogy adopted by teachers in the school (as argued by Crawford, 1996, and Kagan, 1992).

Data gathering methods

Data were gathered using survey instruments, structured interviews, lesson observations, collection of artefacts and member-checking interviews (Guba & Lincoln, 1989). The survey instruments and structured interviews explored the teachers' experience (including computing expertise), perceptions of the school's computer resources, beliefs about the potential of computers in teaching mathematics, beliefs about teaching and learning mathematics, and specific pedagogical practices. Collected artefacts included student work samples, test items, the mathematics work programs, and teachers' writings including their web publications. The interviews and lessons were audio taped and transcribed.

To probe the teachers' beliefs about the potential of computers in mathematics teaching the first author volunteered to teach the teachers' classes with the exploratory mathematics software *Maths Helper* (Vaughan, 1997). The first author introduced each teacher to the software and constructed lesson plans for the algebra and calculus lessons those teachers were about to teach. These were shown to the teachers for their comment. The author taught two of Mary's classes and one of Eva's, and conducted a two-hour professional development seminar on the use of computing technologies in mathematics teaching and learning. *Maths Helper* contains a number of mathematics tools that enable students to explore mathematics concepts and engage in mathematical modelling; for example, a line tool that can be placed on any curve to give the linear function of the tangent at that point, a maximum and minimum tool, an angle tool that gives readings in degrees and radians, a point tool, and an intersection tool. Students can use these tools to explore mathematical concepts, particularly algebra and calculus concepts. The software also contains a powerful symbolic manipulator. The teachers' reactions to the software provided, indirectly, information on their beliefs and values regarding the use of computers to teach mathematics, particularly their beliefs about using exploratory software.

The researcher also asked the teachers what they thought about particular textbooks, resources and instances in mathematics teaching. For example, the teachers commented at length on the text book series *Investigating Change: An Introduction to Calculus for Australian Schools* (Barnes, 1993) and *Access to Algebra* (Lowe, Willis, Kissane & Grace, 1994), and these responses revealed much about their beliefs. These text series have many activities that are learner-focused and encourage conceptual development through physical activity, discovering patterns and then formalising the mathematics. Although it was an indirect way of collecting data about beliefs, it is considered to be more effective than direct questioning (Kagan, 1992). It is acknowledged that classifying teachers' beliefs and practices in this way may be simplistic. However, as Ernest (1996, p. 5) noted, "simplified models can suggest the way that important theoretical factors impact upon the teaching and learning of mathematics".

Results and discussion

The results commence with an overview of the mathematics staff, followed by a summary of teachers' beliefs about teaching and learning and the potential of computers (including excerpts from the teachers' interviews), and a description of their teaching practices.

Overview of staff

The mathematics staff <u>was</u> very experienced with an average length of teaching service of 18 years. Julie was the only teacher who reported attending professional development external to the school in the previous two years. This indicated that the staff could have been pedagogically isolated from recent initiatives in mathematics teaching. All teachers except Peter reported that a lack of access to and lack of knowledge of suitable software were important factors in limiting their use of computers. This contradicts the statements of the computer studies coordinator who reported that one laboratory had only about 10% usage. He believed that access was not the main limiting factor restraining the mathematics staff usage of computers in their teaching, "I think it's the time to learn how to use them (computers) in their area and a lack of knowledge of what can be done with computers". He stated that the mathematics department did not plan in advance for the use of computers at the beginning of each semester when block bookings of rooms were made. The computing coordinator also explained that each subject area was given an allocation of money to be spent on software and the mathematics department had not spent their allocation in 1997 or 1998.

Summary of case study data

The descriptions in Table 1 represent the teachers' dominant beliefs about computer use at the time of the study. These descriptions of the teachers' beliefs about teaching, learning and pedagogy represented beliefs and practices when teaching good students. All teachers in the study indicated that they used *content-focused (performance)* approaches when teaching less able students. The teachers' statements indicated essentially *calculational* orientations for the potential use of computing technologies in mathematics teaching. For example, Peter stated, "in numerical analysis I believe in the use of technology ... (but) I don't think it (computing technology) should be a substitute for theoretical approaches". Eva noted, "I could use it on the big screen to show them ... but the problem is they don't get to work it out by themselves ... (so) they might not understand the fundamentals". As she explained, "by doing it that way (by hand) students can see the links ... with computers, things are hidden". Emm noted, "I think computers are good for demonstrating things quickly (functions) ... but I don't like the idea of relying on them to teach the concept". Emm went on to explain that computers could be used by the teacher to illustrate the shape of a function, but student understanding was best achieved by students listening to her explanations. These views contrasted with Mary's comment, "I really liked it when the kids worked it out for themselves (relationships between quadratic equations and shape of the curve). I thought they learnt a lot from working with the computers." Of the software and the lessons the author taught her classes, she stated, "I thought it was really terrific ... I really liked your questions". Mary's belief about the potential use of computers in mathematics teaching and learning was reflected in her encouragement of the first author to conduct a professional development session in the school, her attempts to encourage the coordinators to purchase *Maths Helper* for the school, and her organisation of the professional development course carried out by the first author. Of the case-study teachers, Mary, Eva and four other mathematics staff attended the professional development; Peter, Julie and Emm did not (they later explained that they were "too busy").

Teachers' *transmission* beliefs (Crawford, 1996) about teaching and learning were articulated in statements such as "I like to demonstrate very clearly (mathematics procedure) ... you need to give them step by step instructions or you loose them" (Eva), "I am a great believer in example ... you need to tell them" (Peter), and "I work very hard to explain the underlying concepts ... I don't think there is much point students wasting their time sharing their ignorance" (Emm). Each of these teachers expressed a low evaluation of the Barnes (1993) and Lowe *et al.* (1994) texts that contained investigative activities. By contrast, Mary favoured the use of these resources and favoured students "doing mathematics, exploring ideas themselves ... and talking to each other about it".

Table 1
Summary of Case Study Data

	Eva	Peter	Julie	Emm	Mary
Position	Years 8-10 coordinator	Teacher of Years 10-12 and technology gatekeeper	Years 11-12 coordinator	Year 10-12 teacher and respected mathematician	Years 8-12 teacher
Use of computers	Rarely	Did not use	Occasionally used graphics calculators	Did not use	Sometimes used graphics calculators and computers
Beliefs about the potential of computing technology	Calculational, could be used to demonstrate explanations	Calculational but should not be used	Calculational, could be used to demonstrate explanations	Calculational, could be used to demonstrate explanations	Calculational and conceptual
Main reason for not using computer technology	Lack of expertise and concern that student use of computers is not as effective and efficient as explanation of procedure	Student use of computers could hinder learning by denying them practice on procedure and his current explanations were more effective and efficient.	Lack of expertise and commitment. She was overtaxed and did not have the time to explore alternative approaches to mathematics teaching	Clear explanation was the most effective and efficient way to teach mathematics	Lack of expertise and lack of models on how to integrate computing technology into mathematic teaching.
Beliefs about teaching and learning	Transmission/ absorption	Transmission/ absorption	Transmission/ absorption	Transmission/ absorption	Social/ construction
Teaching practice	Content-focused (performance)	Content-focused (understanding)	Content-focused (understanding)	Content-focused (understanding)	Learner-focused (good students only)

Descriptions of teaching practices

Lesson observations confirmed that there was a high degree of concurrence between teachers' stated beliefs about teaching and learning and their classroom practices. That is, in the classrooms where the teachers had essentially *transmission* beliefs about teaching and learning, student activity was mostly listening to teacher explanations and practising demonstrated procedures. Mary's students on the other hand, worked in-groups on investigative activities and engaged in discussions about their solutions with Mary and each other. Clearly, Mary differed with the other mathematics teachers on key educational issues. The other teachers believed that their current teaching practices were more efficient and effective in meeting their educational goals of covering the syllabus and helping students to pass examinations than having students use computers. This low evaluation of the potential of computers to enhance secondary school mathematics learning found expression in their unwillingness to investigate its potential (Peter, Julie and Emm not attending the seminar on *Maths Helper*). By contrast, Mary's beliefs about learning were more closely aligned with *social constructivist* view of teaching and learning (Luckin, 1999; Vygotsky, 1987). Her pedagogy reflected this by expressing learner-focused views on teaching. Mary saw computers as a tool by which students could construct mathematical meaning and explore mathematics concepts.

Mary may have been an instigator of change in relation to the use of computers in this school. However, this potential was not realised (eighteen months after the case studies were undertaken, computer use in the school had changed little and the *Maths Helper* software had not been acquired). She had been able to engage in teaching strategies consistent with *learner-focused* views (while her peers taught with strategies that reflected *teacher-centred* beliefs). However, there were other hurdles in adopting the use of computers in her teaching and other pressures for compliance. First, as Mary noted, assessment was an important issue, "if you use it (investigative learning and computers) more than the others and they write the test and don't incorporate that way of thinking (in the test), you are up the creek". Second, she feared that if she managed to convince the coordinators to purchase the software and began using computers in mathematics teaching, she would be taking on an additional burden. As Mary explained, "If we get the software, they will expect me to write all the lessons for it." A year later, Mary stated that, "I have decided not to worry about it (increasing the use of computers in teaching). In the grand scheme of things is it worth getting all worked up about? Life has a lot more to offer."

Conclusions and implications

The data indicate relationships between teachers' beliefs about teaching and learning, their classroom practices and their responses to the potential of computers in mathematics teaching. The teachers with *transmission* beliefs (Crawford, 1996) and *teacher-centred* pedagogies (Kuhs & Ball, 1986) believed that explanations were central to developing students' understanding and favoured textbooks, which contained exercises that students could use to practice demonstrated procedures. They had an essentially *calculational* orientation about how computing technology could be used and demonstrated little interest in investigating the potential of computer use in mathematics. This was because they believed that their current practises were more effective and efficient in helping students learn mathematics than having their students work with computers. In contrast, Mary saw computers both as a calculating tool and an instrument by which students could construct understandings. She was initially enthusiastic to learn more about mathematics software and its use. However, she was forced into strategic compliance in relation to the use of computers by the cultural press that operated within the mathematics staff. The most visible instruments by which this press operated were her peers' control of assessment and selection of supporting resources (software and textbooks). This finding supports previous studies in relation to the mechanisms of cultural press (Becker, 1994; Crawford, 1996; Barnes, Clark & Stephens, 1996).

Thus, professional development providers who wish to encourage teachers to use computers, particularly programs with exploratory potential, might consider to the following recommendations. First, challenge teachers' beliefs that *transmission* and *teacher-centred* approaches are effective in teaching mathematics, particularly in the long term. Secondly, show how student use of computers can help the learning of mathematics concepts (and the effectiveness of concepts for long-term mathematics performance). That is providing a model of computer use that teachers can critique. This may challenge some teachers to reflect upon the use of computers as aids to conceptual understanding rather than calculating tools. Thirdly, take account of teachers' concerns that student use of computers is time ineffective. This may mean that teachers will need to be convinced that they can "cover the content" as quickly as with traditional *teacher-centred* methods. Alternatively, it may mean a readjustment of the amount and type of content covered. Fourthly , stress the need for variety in assessment procedures, particularly those that value the changes in understandings and skills associated with investigative use of computers. Finally, demonstrate suitable support resources (such as textbooks) that integrate the use of computers into mathematics teaching and learning.

This study supports the finding of Sarama *et al.* (1998) that teacher beliefs and practices need to be accounted for in effective professional development. Further, teachers must have the opportunity to evaluate *learner-focused* and conceptual use of computers and be supported with resources to modify their behaviour. Clearly teachers' beliefs and current practices are important in influencing their use of computers. Failure to take account of these recommendations may mean that many teachers will resist the use of computers in their teaching or use them in a limited way to support their existing practices. Finally, the cultural dynamics operating within a staff need to be taken into account since this study has illustrated that innovative teachers can be readily forced into strategic compliance particularly when the innovation runs conflicts with the beliefs of other senior colleagues.

References

Abramovich, S. (1995). Technology–motivated teaching of advanced topics in discrete mathematics. *Journal of Computers in Mathematics and Science Technology, 14*(3), 391–418.

Barnes, M. (1993). *Investigating Change: An Introduction to Calculus for Australian Schools.* Carlton Victoria: Curriculum Corporation.

Barnes, M., & Clarke, D., & Stephens, M. (1996). The impact of external assessment on teaching practice: Constraints on change in the classroom. In P. Clarkson, (Ed.). The proceedings of the *Mathematics Education Research Group of Australasia* (MERGA). *Technology in Mathematics Education* (pp. 65–71). Melbourne: The University of Melbourne.

Becker, H. (1994). How exemplary computer–using teachers differ from other teachers: Implications for realizing the potential of computers in schools. *Journal of Research on Computing in Education, 26*(3), 429–459.

Crawford, K. (1996). Vygotskian approaches in human development in the information era. *Educational Studies in Mathematics, 31*(1), 63–93.

Denzin, N., & Lincoln, Y. (1994). Part V the art of interpretation, evaluation, and presentation. In N. K. Denzin: & Y. S. Lincoln (Ed.), *Handbook of qualitative research* (pp. 479–483). Thousand Oaks: Sage Publications.

Ernest, P. (1996). *The nature of mathematics and teaching* [Http:\www.ex.ac.uk/~PErnest/pome/pompart7.htm] (24th November).

Guba, E., & Lincoln, Y. (1989). *Fourth generation evaluation.* Newbury Park, London: Sage.

Hatfield, M., & Bitter, G. (1994). A multimedia approach to professional development of teachers: A virtual classroom. In D. B. Aichele, & A. F. Coxford (Eds.), *Professional Development for*

Teachers of Mathematics (pp. 102–115). Reston Virginia: National Council of Teachers of Mathematics.

Kagan, D. M. (1992). Implications of research on teacher belief. *Educational Psychologist, 27*(1), 65–90.

Kissane, B., & Kemp, M. (1998). Graphics Calculators and Lower Secondary Mathematics. *Australian Mathematics Teacher, 54*(4), 4–10.

Kuhs, T., & Ball, D. (1986). *Approaches to teaching mathematics: Mapping the domains of knowledge, skills, and dispositions.* East Lansing, Michigan: Michigan State University, Center of Teacher Education.

Lowe, I., Willis, S., Kissane, B., & Grace, N. (1994). *Access to algebra.* Carlton, Victoria: Curriculum Corporation.

Luckin, R. (1999). TRIVAR: *Exploring the "Zone of Proximal Development".* [http://www.cogs.susx.acuk/users/rosel/europper.html] (20/09/99).

Maykut, P., & Morehouse, R. (1994). *Beginning qualitative research: A philosophical and practical guide.* Washington, DC: The Falmer Press.

McCrae, B., Asp G., & Kendal, M. (1999). Learning with supercalculators. In J. M. Truran & K. M. Truran (Eds.), Proceedings of the twenty–second annual conference of The Mathematics Education Research Group of Australasia.(MERGA) *Making the Difference.* (pp. 339–364) Adelaide: MERGA.

Norton, S. J. (1999). Secondary mathematics teachers' responses to computers and their beliefs about the role of computers in their teaching. In J. M. Truran & K. M. Truran (Eds.), Proceedings of the twenty–second conference of the Mathematics Education Research Group of Australasia *(MERGA) Making the difference* (pp. 404–410). Adelaide: MERGA.

Pierce, R. (1999). Computer algebra systems facilitate positive learning strategies. In J. M. Truran & K. M. Truran (Eds.), Proceedings of the twenty–second annual conference of The Mathematics Education Research Group of Australasia.(MERGA) *Making the difference.* (pp. 431–438) Adelaide: MERGA.

Rosen, L. D., & Weil, M. M. (1995). Computer availability, computer experience and technophobia among public school teachers. *Computers in Human Behavior, 11*(1), 9–31.

Sarama, J., Clements, D. H., & Jacobs–Henry, J. (1998). Network of influences in an implementation of a mathematics curriculum innovation. *International Journal of Computers for Mathematical Learning,, 3,* 113–148.

Stenhouse, L. (1990). Case Study Methods. In H. J. Walberg & G. D. Haertel (Ed.), *The International Encyclopedia of Educational Evaluation* (pp. 644–649). Oxford: Pergamon Press.

Vaughan, B. A. (1997). *Maths Helper 2.* Goodna: Teachers' Choice Software.

Vincent, J. & McCrae, B. (1999). Cabri Geometry: A catalyst for growth in geometric understanding. In J. M. Truran & K. M. Truran (Eds.), *Making the Difference: Proceedings of the twenty–second annual conference of The Mathematics Education Research Group of Australasia.(MERGA)* (pp 507–514). Adelaide, S.A.: MERGA.

Vygotsky, L. S. (1987). *The collected works of L. S. Vygotsky Volume 1.* New York: Plenum Press.

CHANGING THE RULES: CHILDREN, CREATIVITY AND COMPUTER GAMES

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Computer games are important in children's culture. They afford opportunities for fantasy, challenge, collaboration and competition, all essential elements of play and learning. Yet their relationship with learning remains tenuous. This paper will present some work that seeks to build computational 'playgrounds' where children will learn about rules, by designing, building and playing their own computer games.

Introduction

Computer games are important in children's culture. Yet many computer games achieve very low thresholds of accessibility at the price of considerable restriction in terms of what can be done. Current computer games typically cast children in the role of game-player, playing according to rules programmed by someone else - a situation which, however motivating, sets strong boundaries around what might be learned. We claim that the fascination of children with computer games is entirely at the level of interface. That is, manipulation of the game objects is fun, expressive and engaging. But for the most part, the interface is all there is: the level below the interface is the preserve of the programmers and designers, not the user. The aim of the Playground project is to place children in the role of producers as well as consumers of games, and in so doing change their relationship with the rules of the games. In this endeavor we are building on the work of Papert (1998), Kafai (1995), Harel (1988), Klawe (1995), and Rubin (1995). As far as learning is concerned, we build on our own work in developing microworlds for mathematics to explore students' meanings (Noss and Hoyles 1996), as well as others: in particular in relation to facilitating links across representations (Clements & Sarama, 1995; Kaput & Roschelle, 1999, & Hoyles *et al.*, 1999).

The Playground project is a 3-year collaborative project, funded by the European Union, which has been working since November 1998 to build 'playgrounds': computational worlds in which children can play games, take them apart, reconstruct and share them. The project partners are the Institute of Education, University of London, UK; Logotron, Cambridge, UK; CNOTINFOR, Portugal; Royal Institute of Technology, Sweden; and Comenius University, Slovak Republic.

Playground is developing two platforms on which children can create and play their own games: an animation-based computing formalism called ToonTalk (Kahn, 1999), and a new, concurrent object-oriented version of Logo. Our research has progressed simultaneously on three levels: questionnaire and interview survey of the games 5-8 year olds like to play on and off computer; case studies of children designing, building and changing games, the resources they use, the meanings they evolve and share, and the understandings they develop; and analyses of how children come to understand the implications of the rules that they program into their games. Our preliminary work has revealed the need for a clearer distinction between children's strategies for building games and for taking apart and reconstructing them, and a more precise specification of our learning objectives.

We have developed two types of resource: playground fragments (play objects such as bouncing balls, noises and pictures) and control components (such as dice and timers) that can be used to make games; and a collection of different types of games composed of simple modular reconstructable elements. A preliminary identification of the most useful categories of games, the tools available and the aspirations of our children have led us to distinguish action games, strategy games, and school games. Prototype examples in each of these categories have been built.

The latter concern has led us to define more clearly what we mean by *rules* in this context. In any game, the rules may model 'real' laws (the way a ball bounces), specify constraints (this place is out of bounds) or simply define how the game is played (if a certain object is hit 10 points are scored). If children design their own games all these types of rules are necessarily raised to a more overt plane, they become objects of reflection, something else with which to play creatively. A central part of our development work has been to identify the right grain-size of the objects that children will want to manipulate to specify the rules and goals of their games: the appearance of the play objects, the sounds they make, their behaviours and the rules that underpin interactions between them. Our achievement to date is that we have built simple games in such a way that the children have come into contact with deep ideas: how motion can be decomposed into components, the symmetry of events such as 'hitting', how different scoring mechanisms work and their implications for the fairness of the game.

In this paper we will focus on game evolution and how two children changed a simple game, its appearance and how they modified its rules so it fitted more closely what they wanted to play.

From pong to underwater fishing

We wrote a very simple pong game in ToonTalk which we gave to two girls, Harriet and Roberta (age 7 years) to play, redesign and change as they wished. Harriet and Roberta had both been working with ToonTalk in an after-school computer club for about three months, so were relatively familiar with the metaphor and the simple tools available.

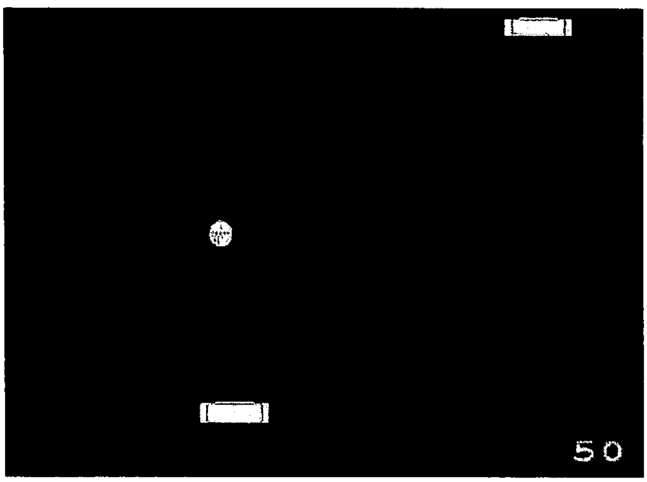

Figure 1. Original pong game

This original pong game was a two-player game, where one player controls the top paddle using the keys SHIFT and CTRL to move the paddle left and right, and the other uses the mouse to move the bottom paddle left and right (see Figure 1). The ball bounces around and the players must each try to hit it with their paddle. The score (bottom right hand corner) increases by 10 points whenever the top paddle hits the ball.

At first the two girls simply treated pong like a 'sport': as they put it, "it's like tennis". As they played, they soon worked out that the score was changed by the top paddle. They thought the game was rather boring, so we added more inducements to carry on playing - playing against the clock, trying to get the most points in 30 seconds.

Changing the Appearance of Bammers and Birds

We wondered if the girls wanted to change the game, which took us to the first phase of game evolution:

I: How can you change the game?
H: Could have two scores, one for bottom one for top.
H: Make it more colourful... it's a bit dark.
H: You could have... like the paddle as a fish.
R: I've got an idea: Bammer hits the thing down and hits the ball.

Out of these ideas they first implemented the change in colour - to light blue. This is trivial in ToonTalk and is achieved by hitting the space bar. They had decided to make the paddles look like Bammer - a special animated mouse in the ToonTalk world, so we asked:

I: How will you get Bammer to behave like the paddle?
H&R: I know — you stick the paddle on the back.

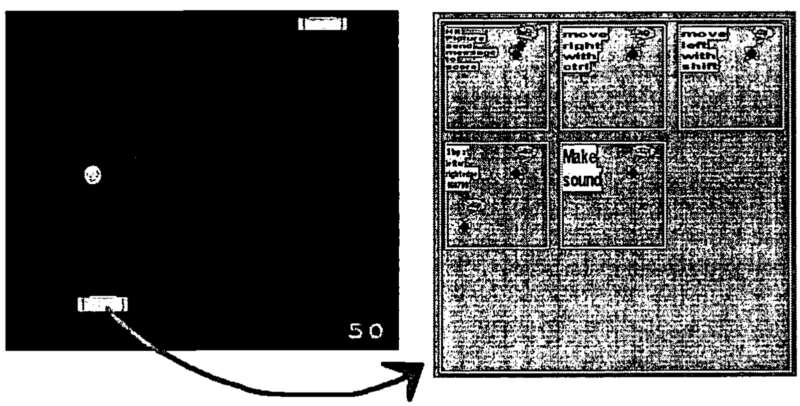

Figure 2. The paddle and behaviours on the back of it

The two girls were referring to a general method for exchanging behaviours in ToonTalk. On the back of any picture are its behaviours (as illustrated in Figure 2). The girls knew that if you flip over a new picture, flip over the paddle, and put the paddle on the back of the picture, then the picture will inherit the paddle's behaviours.

Next they changed the ball to look like a bird.
I: That's horrid! [i.e. the bird is being hit by a hammer!]
H & R: No, no it's flying up and down, up and down — it's OK.

This was achieved exactly in the same way as the change of appearance of the paddle. They also changed the colour of the background at the bottom to be yellow (see Figure 3).

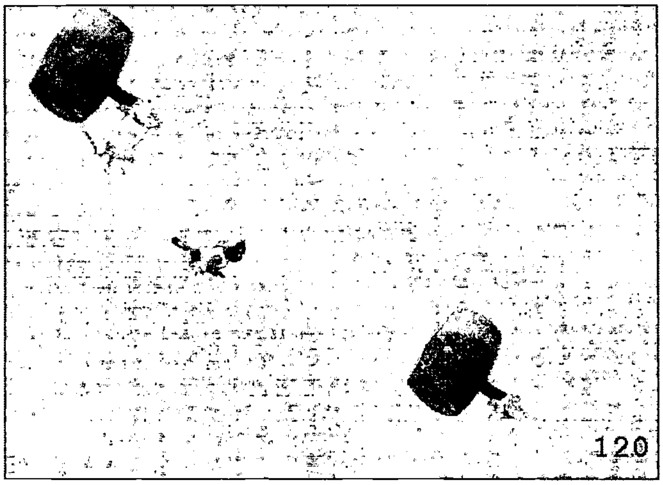

Figure 3. Changing the appearance of Bammers and Birds: Bammer is the animated mouse with a *large* hammer

The changes in colour stimulated more ideas, and it seemed to support the girls' inclination to build an underwater narrative which they had mentioned earlier when they said they wanted fish. They decided how far they were willing to suspend conventions of reality; a tacit agreement or compromise, both parts of make-believe play.

H: I know... that's like the sea and he's [Bammer] running down into it! Cos that's like there's a hill and there's sand going down.
R: There's a problem! He's walking on... the water!
H: It doesn't matter.

The new game is no less boring in terms of playability than the first version. Yet the girls found it far more compelling because *they* had made it. They also became less concerned about the scoring aspect of all their ideas for development but we only encouraged them to implement the (simple) picture changes at this point. This indicates the importance of the interactions with a more capable other (such as a teacher) who is able to judge what is possible as well as desirable in computer interactions at a particular time and for a particular group of children. It was clear that Harriet and Roberta needed extra playground elements such as pictures of fish and sounds. We developed these and stored them in libraries of play objects and control components for them to access later.

Going underwater

In the next session, we gave Harriet and Roberta some new pictures of fish and they picked out the shark picture for the paddles. They discussed more ideas: Roberta wanted to have lots of fish bouncing up and down, (an idea she had picked up from the boys who had made multiple balls for their game).

H: But if the fish reach the top or bottom then they make another fish?
R: And also they take away from your score.

This latter suggestion is a different sort of transformation incorporating a penalty scheme in addition to the scoring one. In fact they simply changed the paddles to sharks, now an easy maneuver as they had already changed the paddle to be Bammer. The girls were changing the appearance of the game and programming at what we term 'the picture level'. They simply put the paddle on the back of the shark picture. They also changed the ball to a fish by putting the 'ball with all its behaviours' on the back of a fish, reducing its size with a ToonTalk tool called Pumpy (an animated bicycle pump which makes things bigger and smaller), and then copying it many times with another tool, Wandy (which copies objects and all their inherited behaviours). Their game now looked like Figure 4.

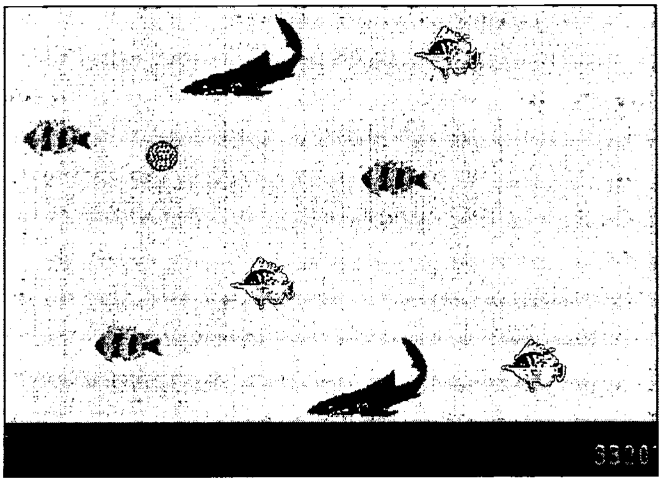

Figure 4. Changing the appearance to sharks and fish

From hitting fish to eating them

Once Harriet and Roberta played with the shark and all the fish, they immediately wanted to make more changes - changes in the rules of the game as well as its appearance.

R: The sharks are the paddles. And if one of those hit the sharks -
H: - any of them -
R: it goes like this [R chomps]
H: No it doesn't.
I: That's what you want?
H: This is what we want.
R: We really want to make more of the balls and when it comes they go [chomp] and if you press a button it spurts out again.

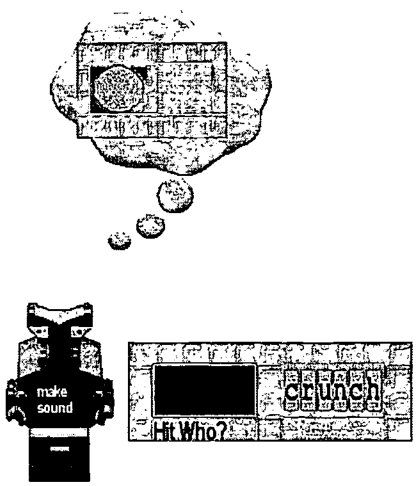

Figure 5. 'Make sound' robot and its new input box

So the girls wanted a new sound that would be played whenever the shark hit the fish. This was rather harder to manage: they needed to get further into the system, to reprogram a robot which is the mechanism for making a behaviour and lives on the back of a picture so it made the noise 'crunch' rather than 'beeyaw' if the shark and fish collide. They managed this since all they had to do was find the right robot and change its 'input box' to have the noise they liked, as illustrated in Figure 5.

Next, the girls tried to build a game that was more realistic by changing another rule: every time a fish hit a shark, instead of bouncing off, it would be eaten. This idea made the pair think of new ways to win - which shark eats the most fish!

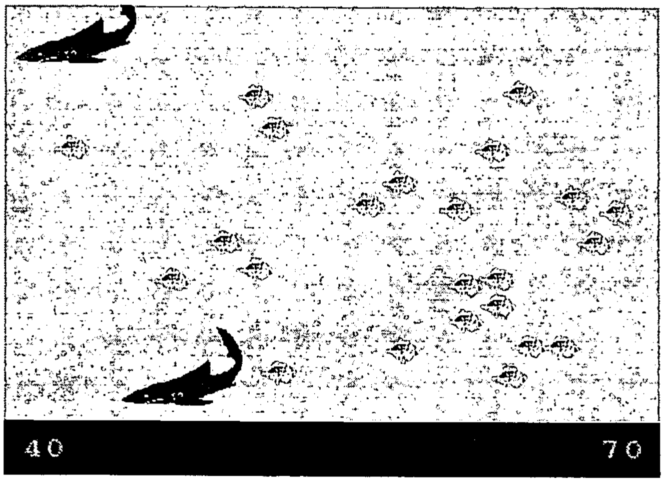

Figure 6. Shark game with two separate scores

In this paper it is not possible to illustrate all of the next phases of their game evolution. We simply summarize: first they made the shark eat the fish by making a 'disappear when hit by shark' behaviour; then they made many fish and changed the scoring system so the game had two scores, one for each shark. After help with this final phase, their game looked like Figure 6.

Conclusions

What this brief paper illustrates is how two girls were able to transform a game and the rules by which it was both played and won. It shows how they were able to program at the level of pictures, noises and behaviours: that is, swap these attributes and functions in ways that satisfied their own goals. In the next stage of the project, we are seeking to investigate in more depth the meanings that the children are attributing to their interactions in our playgrounds, and what exactly they see as the rules of their games. We know they are largely unaware of the rules of the computer games they play at home and school, and we know they are very aware of the events they wish to see and hear in their own games. What we have yet to discover is how far they are able to articulate these rules to others - face to face or over the web.

References

Clements, D. H., & Sarama, J. (1995). Design of a Logo environment for elementary geometry. *Journal of Mathematical Behaviour,* 14, 381–398.

Harel, I. (1988). *Software design for learning: Children's constructions of meanings for fractions and logo programming*. Unpublished doctoral dissertation. Cambridge MA: MIT Laboratory.

Hoyles, C., Morgan, C., & Woodhouse, G. (Eds.) (1999). *Rethinking the Mathematics Curriculum.* London: Falmer Press.

Hoyles, C., & Noss, R. (1999). Playing with (and without) words. *Proceedings of Eurologo 1999 Conference,* Sofia, Bulgaria, 18–30.

Kafai, Y. B. (1995). *Minds in play: Computer game design as a context for children's learning.* Lawrence Erlbaum Associates.

Kahn, K. (1999). Helping children learn hard things: computer programming with familiar objects and activities in A. Druin (Ed.) *The Design of Children's Technology.* San Francisco: Morgan Kaufman Publishers Inc. 223–241.

Kaput, J. J., & Roschelle, J. (1999). The mathematics of change and variation from a millennial perspective: New content, new context in C. Hoyles *et al.* (Eds.), *op cit.*

Klawe, M. M., & Phillips, E .(1995). A classroom study: Electronic games engage children as researchers. *Proceedings of Computer Support for Collaborative Learning '95* (CSCL), Bloomington, Indiana.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers.* Dordrecht: Kluwer.

Papert, S. (1998). Does easy do it? Children, games, and learning. *Game Developer,* June, 77–78.

Rubin, A. (1995). *Through the glass wall: Computer games for mathematical empowerment.* Massachusetts: TERC.

IMPLICATIONS OF USING DYNAMIC GEOMETRY TECHNOLOGY FOR TEACHING AND LEARNING

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This paper explores the implications of using Dynamic Geometry technology for teaching and learning geometry at different levels of education. Through example explorations and problems using the Geometer's Sketchpad I hope to provoke questions concerning how children might learn geometry with such a tool, and the implications for teaching geometry with such a tool. I draw on my own experiences and the experiences of other teachers and researchers using dynamic geometry technology with young children, adolescents and college students.

What is dynamic geometry technology?

This question is best addressed through demonstration. I include any technological medium (both hand-held and desktop computing devices) that provides the user with tools for creating the basic elements of Euclidean geometry (points, lines, line segments, rays, and circles) through direct motion via a pointing device (mouse, touch pad, stylus or arrow keys), and the means to construct geometric relations among these objects. Once constructed, the objects are transformable simply by dragging any one of their constituent parts. Examples of dynamic geometry technology include, but are not limited to the following:

Cabri-Geometry (Cabri I and Cabri II for desktop computers, and Cabri on the TI-92 calculator).

The Geometer's Sketchpad (Version 3 for both Windows and Macintosh computers, and the new implementation for the TI-92 calculator and the Casio Cassiopeia hand-held computer).

Geometry Expert (GEX, a new computer-expert system from China, Gao, Zhang & Chou, 1998).

Goldenberg & Cuoco, (1998) provide an in-depth discussion on the nature of Dynamic Geometry. A common feature of dynamic geometry is that geometric figures can be constructed by connecting their components; thus a triangle can be constructed by connecting three line segments. This triangle, however, is not a single, static instance of a triangle which would be the result of drawing three line segments on paper; it is in essence a prototype for all possible triangles. By grasping a vertex of this triangle and moving it with the mouse, the length and orientation of the two sides of the triangle meeting at that vertex will change continuously. The mathematical implications of even this most simple of operations was brought home to me when my seven-year old son was "playing" with the *Geometer's Sketchpad* software (hereafter referred to as *Sketchpad* or GSP). As he moved a vertex around the screen he asked me if the shape was still a triangle. I asked him what he thought. After turning his head and looking at the figure from different orientations he declared that it was. I asked him why and he replied that *it still had three sides!* He continued to make triangles which varied from squat fat ones to long skinny ones (his terms), that stretched from one corner of the screen to another (and not one side horizontal!). But the real surprise came when he moved one vertex onto the opposite side of the triangle, creating the appearance of a single line segment. He again asked me if this was still a triangle. I again threw the question back to him and his reply was: "Yes. It's a triangle lying on its side!" I contend that this seven-year old child had constructed for himself during that five minutes of exploration with the *Sketchpad* a fuller concept of "triangle" than most high-school students ever achieve. His last comment also indicates intuitions about plane figures which few adults ever acquire: That they have no thickness and that they may be oriented perpendicular to the viewing plane. Such intuitions are the result of what Goldenberg, Cuoco, & Mark (1998) refer to as "visual thinking."

Implications for elementary teaching and learning

Nathan's use of the dynamic drag feature of this type of computer tool illustrates how such dynamic manipulations of geometric shapes can help young children abstract the essence of a shape from seeing what remains the same as they change the shape. In the case of the triangle, Nathan had abstracted the basic definition: a closed figure with three straight sides. Length and orientation of those sides was irrelevant as the shape remained a triangle no matter how he changed these aspects of the figure. Such dynamic manipulations help in the transition from the first to the second van Hiele level: from "looks like" to an awareness of the properties of a shape (Fuys, Geddes & Tischler, 1988).

What Nathan did during the next 15 minutes with *Sketchpad* also indicates how such a tool can be used to explore transformational geometry at a very young age. I showed him how he could designate a line segment as a "mirror" using the TRANSFORM menu. We then selected his triangle and reflected it about the mirror segment. Nathan was delighted with the way the image triangle moved in concert with his manipulations of the original triangle. He quickly realised that movement toward the mirror segment brought the two triangles closer together and movement away from the "mirror" resulted in greater separation. I decided to add a second line approximately perpendicular to the first mirror segment and designate this second line as a mirror. We then reflected both the original triangle and its reflected image across this line, resulting in four congruent triangles. Nathan then experimented by dragging a vertex of the original triangle around the screen (see Figure 1). He was fascinated by the movements of the corresponding vertices of the three image triangles. He was soon challenging himself to predict the path of a particular image vertex given a movement of an original vertex. At one point he went to the chalkboard, sketched the mirror lines and triangles, and indicated with an arrow where he thought an image vertex would move. He then carried out the movement of the original vertex on the screen and was delighted to find his prediction correct. Note also that Nathan was not constrained by physical mirrors. He had no hesitation in crossing over the mirror lines! Goldenberg and Cuoco (1998) challenge us to think seriously about the educational consequences for children working in an environment in which such mental reasoning with spatial relationships can be provoked.

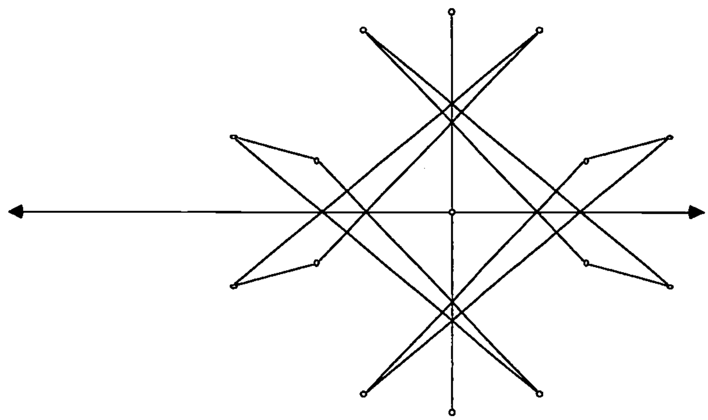

Figure 1. Double reflection of a triangle from *Geometer's Sketchpad*

Lehrer, Jenkins and Osana (1998) found that children in early elementary school often used "mental morphing" as a justification of similarity between geometric figures. For instance a concave quadrilateral ("chevron") was seen as similar to a triangle because "if you pull the bottom [of the chevron] down, you make it into this [the triangle]." (p. 142) That these researchers found such "natural" occurrences of mental transformations of figures by young children suggests that providing children with a medium in which they can actually carry out these dynamic transformations would be powerfully enabling (as it was for Nathan). It also suggests that young children naturally reason dynamically with spatial configurations as well as making static comparisons of similarity or congruence. The van Hiele (1986) research focussed primarily on the static ("looks like") comparisons of young children and did not take into account such dynamic transformations.

Implications for secondary teaching and learning

At the secondary level dynamic geometry environments can (and should) completely transform the teaching and learning of mathematics. Dynamic geometry turns mathematics into a laboratory science rather than the game of mental gymnastics, dominated by computation and symbolic manipulation, that it has become in many of our secondary schools. As a laboratory science, mathematics becomes an investigation of interesting phenomena, and the role of the mathematics student becomes that of the scientist: observing, recording, manipulating, predicting, conjecturing and testing, and developing theory as explanations for the phenomena.

The teacher intending to take advantage of this software, and change mathematics into a laboratory science for her students, faces many challenges. As Balacheff & Sutherland (1994) point out, the teacher needs to understand the "domain of epistemological validity" of a dynamic geometry environment (or microworld). This can be characterised by "the set of problems which can be posed in a reasonable way, the nature of the possible solutions it permits and the ones it excludes, the nature of its phenomenological interface and the related feedback, and the possible implication on the resulting students' conceptions." (p. 13) Such knowledge can only be obtained over a long period of time working with the software both as a tool for one's own learning as well as a tool for teaching mathematics. There are resources, however, that teachers can turn to. The publication "Geometry Turned On" (King and Schattschneider, 1997) provides several examples of successful attempts by classroom teachers to integrate dynamic geometry software in their mathematics teaching. Michael Keyton (1997) provides an example that comes closest to that of learning mathematics as a laboratory science. In his Honours Geometry class (grade 9) he provided students with definitions of the eight basic quadrilaterals and some basic parts (eg. diagonals and medians). He then gave them three weeks to explore these quadrilaterals using *Sketchpad.* Students were encouraged to define new parts using their own terms and to develop theorems concerning these quadrilaterals and their parts. Keyton had used this activity with previous classes without the aid of dynamic geometry software. He states:

> In previous years I had obtained an average of about four different theorems per student per day with about eight different theorems per class per day. At the end of the three-week period, students had produced about 125 theorems... In the first year with the use of *Sketchpad*, the number of theorems increased to almost 20 per day for the class, with more than 300 theorems produced for the whole investigation. (p. 65)

Goldenberg and Cuoco (1998) offer a possible explanation for the phenomenal increase in theorems generated by Keyton's students when using *Sketchpad.* Dynamic geometry "allows the students to transgress their own tacit category boundaries without intending to do so, creating a kind of disequilibrium, which they must somehow resolve." (p. 357) They go on to reiterate a point made by de Villiers (1994), that "To learn the importance and purpose of careful definition, students must be afforded explicit opportunities to participate in definition-making themselves." (Goldenberg and Cuoco, 1998, p. 357)

Keyton's activity with quadrilaterals stays within the bounds of the traditional geometry curriculum but affords students the opportunity to create their own mathematics within those bounds. Other educators have used dynamic geometry as a catalyst for reshaping the traditional curriculum. Cuoco and Goldenberg (1997) see dynamic geometry as a bridge from Euclidean Geometry to Analysis. They advocate an approach to Euclidean geometry that relates back to the "Euclidean tradition of using proportional reasoning to think about real numbers in a way that developed intuitions about continuously changing phenomena." (p. 35) This approach involves locus problems, experiments with conic sections and mechanical devices (linkages, pin and string constructions) that give students experience with "moving points" and their paths.

Both Cabri II and *Sketchpad* have the facility to trace the path of points, straight objects and circles. The locus of a moving point can be constructed as an object. Points can move freely on such constructed loci. These features enable the user to create direct links between a geometrical representation of a changing phenomenon and a representation of the varying quantities involved as a coordinate graph. As an example, consider the investigation of the area of a rectangle with fixed perimeter. Figure 2 shows a *Sketchpad* sketch in which the perimeter, length AB, height BC and area of the rectangle ABCD have been measured. A point has been plotted on *Sketchpad's* coordinate axes using the measures of AB and area of ABCD as (x, y). The trace of this point can be investigated as the length AB of the rectangle is changed by direct manipulation of vertex A. In Figure 2, the locus of point (x, y) as the length of base AB changes has been constructed. The rectangle with maximum area can be found by experiment to be when length and height are approximately equal (see Figure 3). The fact that the measures of base and height are not exactly the same in Figure 3 should lead to an interesting discussion, and a need to "prove" by algebraic means that the maximum area will, in fact, be when AB = BC.

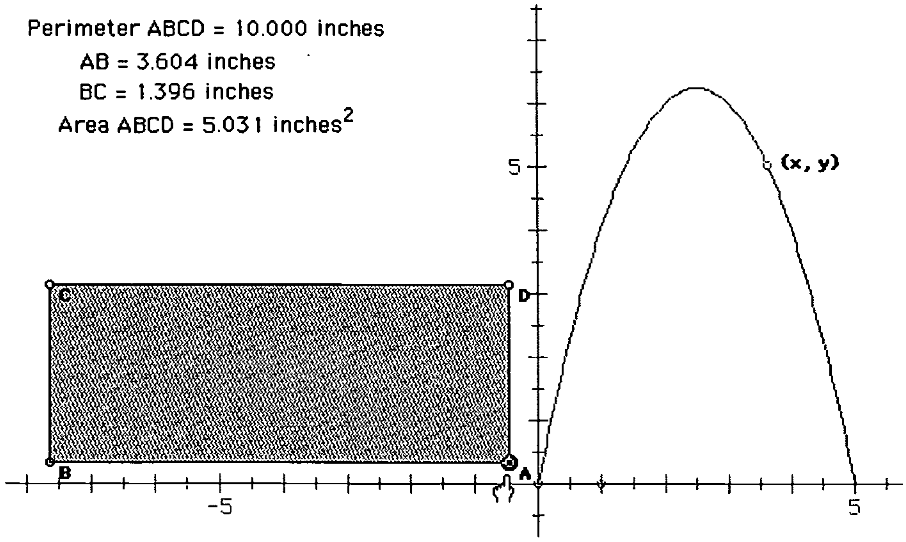

Figure 2. Plotting base against area of fixed perimeter rectangle

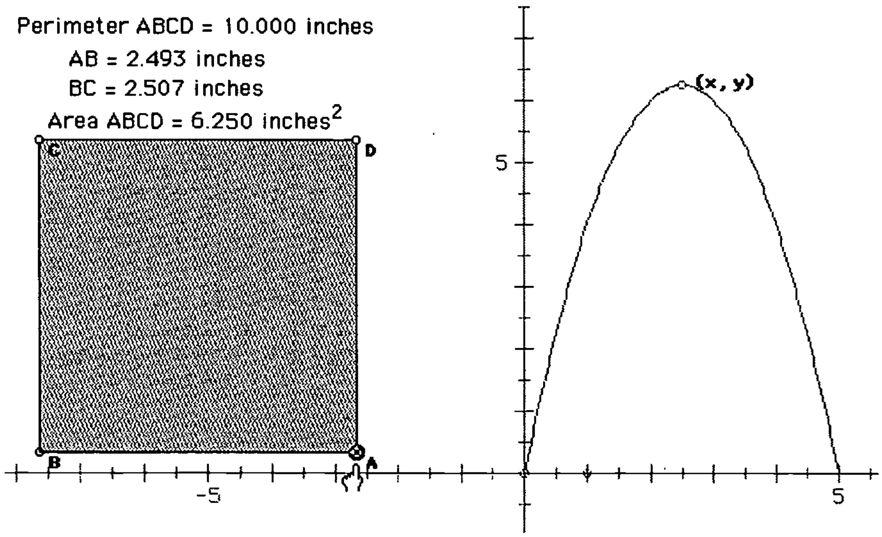

Figure 3. Rectangle with maximum area

Proof in Dynamic Geometry

The above example of finding a solution in dynamic geometry by experiment is analogous to finding roots of a polynomial using a graphing calculator. The solution can be found but the students still have a need to prove that the solution is valid. In the case of the rectangle with maximum area there is a need to prove that the conjecture (or hypothesis) that, for any rectangle with fixed perimeter, the maximum area will be achieved when the rectangle becomes a square. Manipulating rectangle ABCD in the sketch (moving vertex B will change the perimeter) can give convincing evidence that the generalisation is indeed true. There is a danger here that students may regard this "convincing evidence" as a proof. Michael de Villiers (1999) has addressed this concern through a thorough analysis of the role and function of proof in a dynamic geometry environment. De Villiers expands the role and function of proof beyond that of mere verification. If students see proof only as a means of verifying something that is "obviously" true then they will have little incentive to generate any kind of logical proof once they have verified through their own experimentation that something is always so. De Villiers suggests that there are at least five other roles that proof can play in the practice of mathematics: explanation, discovery, systematisation, communication, and intellectual challenge. He points out that the conviction that something is true most often comes *before* a formal proof has been obtained. It is this conviction that propels mathematicians to seek a logical *explanation* in the form of a formal proof. Having convinced themselves that something must be true through many examples and counter examples, they want to know *why* it must be true. De Villiers (1999) suggests that it is this role of *explanation* that can motivate students to generate a proof:

> When students have already thoroughly investigated a geometric conjecture through continuous variation with dynamic software like Sketchpad, they have little need for further conviction. So verification serves as little or no motivation for doing a proof. However, I have found it relatively easy to solicit further curiosity by asking students *why* they think a particular result is true; that is, to challenge them to try and *explain* it. (p. 8)

The following example from my own class of pre-service secondary mathematics teachers illustrates the explanatory function of a proof when solving a problem using *Sketchpad.*

The Power Plant Problem. A power plant is to be built to serve the needs of three cities. Where should the power plant be located in order to use the least amount of high-voltage cable that will feed electricity to the three cities? If the three cities are represented by the vertices of a triangle, ABC, then this problem can be solved by finding a point with minimum sum of distances to all three cities. In exploring this situation in *Sketchpad* students can measure the three distances from an arbitrary point P and the three vertices, A, B and C of the triangle. They can then sum these distances and move P around to find a location with minimum sum. When such a location appears to have been found, students can make conjectures concerning relations among P and the three vertices. Many students conjectures have been to see if any of the known triangle centres satisfy the minimum sum requirement (eg. incenter, centroid or cirumcenter). Some of these may well appear to work for certain triangles, but not for others. Eventually, some students will notice that the angles formed by the point P and each pair of vertices appear to be the same. Measurements would indicate that they are all close to 120 degrees.

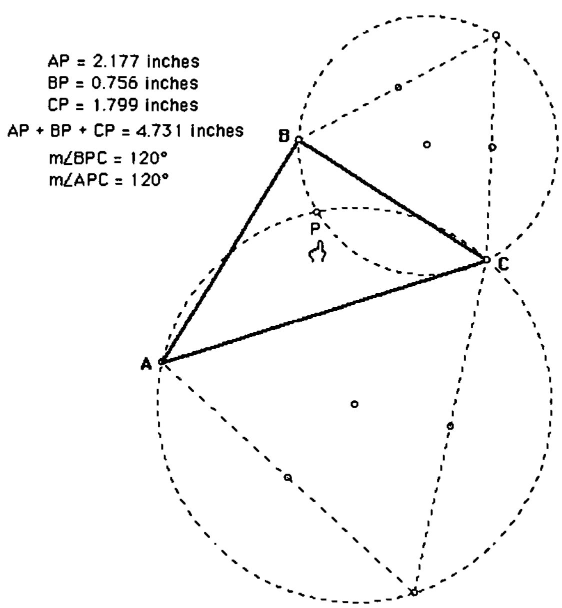

Figure 4. Constructing the location of the power plant, P.

Having discovered a possible invariant in the situation, students then look for a way to construct a point that subtends 120 degrees with each pair of vertices. Various construction methods arise. One way is to construct an equilateral triangle on two sides of the triangle ABC and then construct the circumcircles of these equilateral triangles. Where the circumcircles intersect will subtend 120 degrees with each side of the triangle (see Figure 4).

After the students have successfully located the position of the power plant and found a way of constructing that position, I ask them to explain *why* this point provides the minimum sum of distances to each vertex of the triangle formed by the three cities. This question challenges them to find a way of proving that their constructed point P must be the minimum point (at least for triangles with no angle greater than 120!). This proves to be a difficult challenge for my students but one they are willing and eager to engage. The proof that I find the most satisfying and explanatory is one that makes use of the fact that the shortest distance between two points is along a straight path between the two points. The proof involves a rotation of segments AP, AB and BP about vertex A by 60 degrees (see Figure 5).

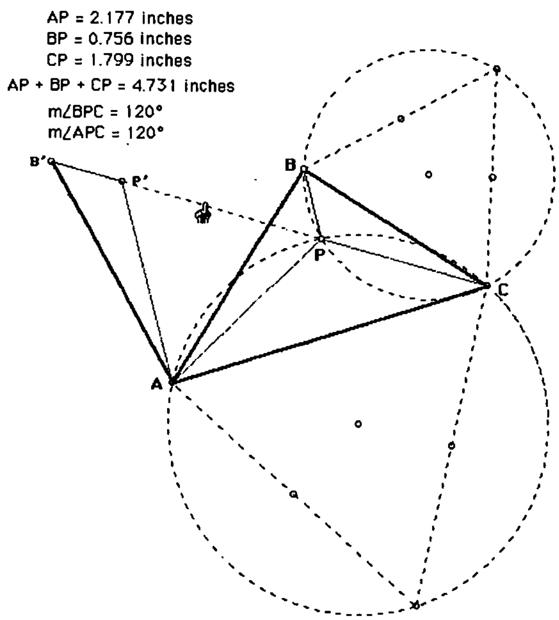

Figure 5. Rotation of ABP about A by 60° to form AB'P'

As rotation preserves length, AP' is congruent to AP and B'P' is congruent to BP. Thus AP'P is an isosceles triangle with an angle of 60° between the congruent sides. As base angles of an isosceles triangle are equal, and angle sum of any triangle is 180, the base angles must also be 60°. Thus triangle AP'P is equiangular, and, therefore, equilateral. Thus P'P is congruent to AP.

Thus by the above reasoning, the path B'P'PC is equal in length to the sum of the distances BP, AP and PC. The path B'P'PC will have a minimum length if, and only if, the path is a straight path, as the shortest distance between two points (B' and C) lies on a straight path. Rotation preserves the shapes of figures. Thus the angle relationships within triangle AP'B' are the same as in triangle APB. In particular, when angle APB = 120°, angle AP'B' = 120° and thus B'P'P will be 180° -- a straight angle. Also, when angle APC = 120°, angle P'PC will also be a straight angle. Thus B'P'PC will lie on a straight path when (and only when) P subtends angles of 120° with each side of the triangle ABC. Thus, this is the condition that provides the minimum sum of distances from P to the three vertices of the triangle ABC.

The above argument appears logical and rigorous as well as explanatory. It, therefore comes as a surprise to most students to learn that the power plant should be built in the centre of city B (rather than at P) when the angle ABC is greater than 120°! This "exception" to what they have just proved leads to an investigation of the implicit assumptions in their proof (eg. that P is in the interior of triangle ABC). The rotation used in the above proof also gives rise to an alternative construction for finding the location of P -- draw lines connecting the outer vertex of each equilateral triangle to the opposite vertex of the original triangle. Where these lines intersect will be the location of point P, also known as the *Fermat* point.

Concluding comments: The need for more research

While there have been many personal accounts of the powerful learning that can take place when students of all ages work with dynamic geometry technology (my own included), there have been very few, well designed research projects to study the effects on learning in such environments. A group in Italy headed by Ferdinando Arzarello (Arzarello et al, 1998a)has conducted investigations of students' transitions from exploring to conjecturing and proving when working with Cabri. They applied a theoretical model that they had developed to analyse the transition to formal proofs in geometry (Arzello et al., 1998b). They found that different modalities of dragging in Cabri were crucial for determining a shift from exploration to a more formal approach. Their findings are consistent with the examples given in previous sections of this paper. The different modalities of dragging that they classified are described as:

> (i) *wandering dragging*, that is dragging (more or less) randomly to find some regularity or interesting configurations; (ii) *lieu muet dragging*, that means a certain locus *C* is built up empirically by dragging a (draggable) point P, in a way which preserves some regularity of certain figures. (p. 3)

They also describe a third modality: *dragging test,* that is used to test a conjecture over all possible configurations. All three modalities were used in the Power Plant exploration, above.

The group at the University of Grenoble in France have been conducting research studies on the use of Cabri for many years (Laborde, 1992, 1993, 1995, 1998). They have focussed both on what students are learning when working with Cabri and the constraints both students and teachers face when teaching and learning with Cabri. Laborde (1992 & 1993) and Balacheff (1994) conclude that the observation of what varies and what remains invariant when dragging elements of a figure in Cabri, helped break down the separation of deduction and construction. Laborde (1998) points out that it takes a long time for teachers to adapt their teaching to take advantage of the technology. She reports three typical reactions that teachers have to the perturbations caused by the introduction of dynamic geometry software into the teaching-learning situation:

> reaction alpha: ignoring the perturbation
>
> reaction beta: integrating the perturbation into the system by means of partial changes
>
> reaction gamma: the perturbation is overcome and looses its perturbing character. (p. 2)

It is only in the last stage (reaction gamma) that teachers make an adaptation in their teaching that truly integrates the technology.

The efforts at researching the effects of technology use on students' learning has been hampered by the prevalence of reactions alpha and beta. As more teachers achieve reaction gamma we have both the opportunity and the responsibility to carefully research the effects of integration of dynamic geometry technology into the teaching and learning of geometry and mathematics in general.

References

Arzarello, F., Micheletti, C., Olivero, F., Robutti, O., Paola, D., & Gallino, G. (1998a). Dragging in Cabri and Modalities of Transition from Conjectures to Proofs in Geometry. In A. Olivier & K. Newstead (Eds.) *Proceedings of the 22nd Annual Conference of the International Group for the Psychology of Mathematics Education (PME 22).* Stellenbosch, South Africa, Volume 2, 32–39.

Arzarello, F., Micheletti, C., Olivero, F., Robutti, O., & Paola, D. (1998b). A model for analysing the transition to formal proofs in geometry. In A. Olivier & K. Newstead (Eds.) *Proceedings of the 22nd Annual Conference of the International Group for the Psychology of Mathematics Education (PME 22).* Stellenbosch, South Africa, Volume 2.

Balacheff, N. (1994). Artificial Intelligence and Real Teaching. In C. Keitel & K. Ruthven (Eds.), *Learning through computers: Mathematics and Educational Technology,* 131–158. Berlin: Springer Verlag.

Balacheff, N., & Sutherland, R. (1994). *Epistemological domain of validity of microworlds. The case of Logo and Cabri–géomètre.* Paper to IFIP, The Netherlands.

Cuoco, Albert A., & E. Paul Goldenberg. (1997). Dynamic Geometry as a Bridge from Euclidean Geometry to Analysis. In James R. King and Doris Schattschneider (Eds.) *Geometry Turned On!: Dynamic Software in Learning, Teaching, and Research.* Washington, D.C.: The Mathematical Association of America, 33–46.

de Villiers, M. (1994). The role and function of a hierarchical classification of quadrilaterals. *For the Learning of Mathematics,* 14(1), 11–18.

de Villiers, M. (1999). *Rethinking proof with the Geometer's Sketchpad.* Emeryville, CA: Key Curriculum Press.

Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in geometry among adolescents. *Journal for Research in Mathematics Education Monograph,* No. 3.

Gao, X. S., Zhang, J. Z., & Chou, S. C. (1998). *Geometry Expert* (computer program). Taiwan: Nine Chapter Pub. (in Chinese).

Goldenberg, E. P., & Cuoco, A. A. (1998). What is Dynamic Geometry? In R. Lehrer & D. Chazan (Eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space.* Mahwah, NJ: Lawrence Erlbaum Associates, 351–368.

Goldenberg, E. P., Cuoco, A. A., & Mark, J. (1998). A role for geometry in general education. In R. Lehrer & D. Chazan (Eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space.* Mahwah, NJ: Lawrence Erlbaum Associates, 3–44.

Keyton, M. (1997). Students discovering geometry using dynamic geometry software. In J. King & D. Schattschneider (Eds.) *Geometry turned on! Dynamic software in learning, teaching, and research.* Washington, D.C.: The Mathematical Association of America, 63–68.

King, J., & Schattschneider, D. (Eds.) (1997). *Geometry turned on! Dynamic software in learning, teaching, and research.* Washington, D.C.: The Mathematical Association of America.

Laborde C. (1992). Solving problems in computer based geometry environments: the influence of the features of the software. *Zentrablatt für Didactik des Mathematik,* 92(4) 128–135.

Laborde, C. (1993). The computer as part of the learning environment: The case of geometry. In: C. Keitel and K. Ruthven (Eds.), *Learning from Computers: Mathematics Education and Technology,* 48–67. Berlin: Springer–Verlag.

Laborde, C. (1995). Designing tasks for learning geometry in a computer–based environment. In L. Burton & B. Jaworski (Eds.) *Technology in Mathematics Learning – a bridge between teaching and learning,* (pp. 35–68). London: Chartwell–Bratt.

Laborde, C. (1998). *Factors of integration of dynamic geometry software in the teaching of mathematics.* Paper presented at the ENC Technology and NCTM Standards 2000 Conference. Arlington VA, June 5–6.

Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 137–168). Mahwah, NJ: Lawrence Erlbaum Associates.

van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education.* Orlando, FL: Academic Press.

USING TECHNOLOGY AS A LEARNING TOOL IS NOT ENOUGH

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Pre-service Secondary Mathematics Teachers (PST's) were exposed to web-based resources and used powerful computer tools to explore many mathematical topics in their mathematics and mathematics education courses during their junior and senior years. They were provided with Notebook computers during their field experiences and their student teaching and were encouraged to make use of the technology tools and resources in their work with high school students. Only three of the 14 students interviewed made significant attempts to use technology in their student teaching. While lack of access to technology in the cooperating schools, and lack of experience or desire to use technology on the part of their cooperating teachers were inhibitors, most PST's appear to require specific instruction on using technology as a Teaching Tool in order to use it in their own teaching.

Introduction

The research reported in this paper took place in the context of a project to evaluate the use of The Math Forum (TMF) World Wide Web site as a learning and teaching resource for pre-service secondary mathematics teachers[i]. The results, however, have implications for pre-service teachers use of technology in general.

The mathematics education community world-wide views the use of technology as a teaching and learning tool as a desirable and (in some cases) necessary strategy for the improvement of mathematics teaching and learning. The new *Principles and Standards for School Mathematics* in the USA (NCTM, 2000) includes "Technology" as one of its six guiding principles for the reform of school mathematics. While there has been a great deal of effort and research on the integration of technology into existing mathematics classrooms (Dwyer, Ringstaff and Sandholtz, 1990; Laborde, 1998; Olive, 1998, Olive and Ramsay, 1999) little attention has been given to the technological preparation of pre-service mathematics teachers. At the University of Georgia, however, it has long been an established practice within our mathematics education program to integrate the use of technology into our whole professional course sequence. This integration is now spreading over to some of the content courses that our students take in the mathematics department. Computers and calculators are used as ubiquitous tools for doing and learning mathematics in practically all of our pre-service teacher education and mathematics content courses. Our expectation, then, is that our students will use these same technological tools and resources in their own teaching. This study indicates that the expectation is not being met and offers suggestions for what may be needed to make the expectation a reality.

Design of the study

The major focus of the evaluation project was on our pre-service teachers' (PSTs) use of The Math Forum web site (TMF) as a resource for teaching and learning mathematics. The evaluation was aimed at addressing the following three major questions:

- What technology-related resources did PST's use during student teaching?
- How did PST's use TMF (and other technology) before and during student teaching?
- What factors influenced PST's use of TMF (and other technology) before and during student teaching?

In an effort to address these evaluation questions, we followed one cohort of PSTs during their junior and senior years in our program (1998 and 1999). We collected data on the extent and usefulness of the PSTs' technology-based activities and, more specifically, on their use of TMF. The data consisted of four parts: questionnaire responses from PSTs and from their school-based mentor teachers, interviews with PSTs, and various artefacts including a resource guide generated by the PSTs.

Data collection process

In June 1998, prospective mentor teachers (cooperating teachers for PSTs who would be student teaching during the spring 1999 semester) were asked to complete a questionnaire summarising their school's computer and Internet accessibility for themselves, their prospective PSTs, and their own students.

Near the end of the fall semester of 1998, the pre-service teachers completed a questionnaire designed to provide feedback on their use of technology and TMF. It was an adaptation of the questionnaire sent to us by Ann Renninger of the Forum Evaluation Staff. The questionnaire was divided into four main parts. Part 1 focused on the PSTs' interest and use of mathematics and computers in a variety of settings. Part 2 examined the PSTs' use of the Internet, including their general use and their use specific to course assignments. In part 3 PSTs were asked to describe their use of TMF. In part 4 of the questionnaire, the PSTs were asked to describe their anticipated general use of computers, including TMF, during their student teaching.

A 20 to 30 minute structured interview was conducted with each of eight PSTs during the post-student teaching seminar in April of 1999. The interviews focused on the PSTs' use of technology, and specifically TMF, during student teaching. Of particular interest was the elicitation of what circumstances facilitated or hindered their use of technology and TMF. The beginning of the interview was individualised for the PSTs as they were asked to elaborate on their questionnaire responses. In particular, they were asked to expand on any responses in which they had indicated an intention to use technology during their student teaching. The balance of the interview followed a structured format for all interviewees. The interviewer also had several opportunities to observe four of the PSTs during student teaching.

Description of the cohort

The cohort consisted of 21 pre-service teachers. Fifteen PSTs completed the questionnaire; all but one of these 15 students completed student teaching. Interviews were conducted with 8 of the 15 PSTs. Seven of the mentor teachers completed a questionnaire about computer and Internet access in their school. All PSTs in the cohort group were typical undergraduates, approximately 20-21 years old, who were completing their undergraduate program in mathematics education. Five of the 14 PSTs who both completed the questionnaire and student taught were males. Their mentor teachers were all experienced teachers; 11 were females, 3 were males.

Method of analysis

Questionnaire results for both PSTs and mentor teachers were tabulated to determine the breadth and range of responses. Further, a constant comparison method was used to analyse PSTs' questionnaire and interview responses in an effort to determine themes that emerged from their use of technology and TMF. Perceptions of the PSTs and their mentor teachers regarding the use of technology were compared and contrasted.

Description of mathematics education experiences

During the spring quarter of 1998 the mathematics education cohort of students were enrolled in two mathematics education courses (EMT 335 and EMT 468), as well as mathematics courses. The two EMT courses were planned as joint enrolments so that topics could be coordinated. The purpose of EMT 335 was to study mathematics found in the secondary curriculum from an advanced point of view, and to study criteria for good curricula. The focus of EMT 468 was on exploring these mathematical topics through the use of computing technologies. Our goals were to introduce students to powerful technological tools in ways that enhanced their own understanding of the mathematical topics, as well as provide models for ways in which they could use the technology to enhance students' understanding in their own teaching. Students made extensive use of *The Geometer's Sketchpad* (Jackiw, 1991) throughout the course, as well as graphing software, spreadsheets, *TesselMania!®* (Lee, 1994) and various other mathematical programs. They also used graphing calculators along with CBL data collection probes for collecting and analysing real data. A major requirement of EMT 468 was to create an instructional unit for a selected mathematical topic that integrated the use of technology in significant ways. These instructional units were developed by groups of four students working cooperatively.

All students in these two courses were required to have e-mail accounts. Assignments were provided electronically and work was turned in electronically. The students were introduced to the World Wide Web during the first week of classes. In particular, the different areas of The Math Forum were highlighted and explored. Both classes made explicit expectations that students would use the Web (and The Math Forum in particular) as a major source of information on specific topics, and as a resource for curricula activities, research, teaching suggestions, and discussions. Many of the students were already familiar with the Web and had been using it for personal as well as professional purposes. Once introduced and initially explored, it was up to the students to use TMF and the Web as they found these resources appropriate and useful for their various assignments.

During the fall semester of 1998 these same students enrolled in our methodology class in mathematics education and its associated field experience (EMAT 4360 and 5360). Both classes met in the Technology Enhanced Classroom during their on-campus period so that the technology tools and resources they had been using in their previous courses were available to them in these two courses. The students engaged in a specific exercise using the *Dr. Math* archives to evaluate responses to high-school students' questions submitted to this service of TMF. This was a planned intervention on our part to use the student/tutor interactions provided by the *Dr. Math* service as a stimulus for effective question answering techniques

During the field component of EMAT 5360 our students were able to access the internet from most of the schools and were able to remain in contact via e-mail. The opportunity to use technology with high-school students during these field placements, however, varied greatly among the different school sites. Only about a third of the cooperating teachers at the school sites were experienced users of technology. In-class access to technology was also very limited at most of the school sites. The same situations were encountered by our students during their student teaching in the spring semester of 1999. Thus, there were limited but varied opportunities for our students to use technology as a teaching tool during their student teaching experiences.

Results and conclusions

Mentor teacher questionnaire responses

Two middle schools and six high schools were used for field experiences and student teaching. Although several teachers seemed unsure of the Internet access in their schools, the aggregate responses from each school provided an overall view of the schools' accessibility. A most striking result is the wide disparity in level of access among schools. Middle School 2 had no student lab and only one computer in the school (located in the Media Centre) with Internet access. High Schools 4 and 6 appeared to be in a similar situation, although the teachers did not indicate how many computer stations had Internet access. High Schools 1 and 5 had student labs, but only limited Internet access. In contrast, High Schools 2 and 3 had Internet access at a substantial number of stations in their student labs as well as access on several computers in the media centre. Only High Schools 3 and 6 appeared to have Internet access in every classroom. However, a majority of the teachers indicated that they had access to the Internet at home. At the very least, every teacher indicated that they had some access to the Internet.

PST questionnaire responses

The PST questionnaire was administered towards the end of the Fall semester prior to their student teaching experiences in the Spring semester of 1999. The results are for all 15 respondents and so include the responses of the one student who did not participate in student teaching.

Computers and mathematics

Only four PSTs reported having a mathematics-related hobby. The fourth of these responses was different from the rest: "Life is a math and computer experience for me." Evidently this PST saw mathematics as an integral part of her life. Unfortunately, no specific details were provided and the PST was not available for an interview.

Nine PSTs reported the following computer-related hobbies: web surfing, e-mail, computer games, and computer software (eg. GSP, Excel, and Microsoft Word). Apart from web surfing, which could include a myriad of activities, the responses seemed to be either utilitarian (eg., use of e-mail or Microsoft Word) or casual, recreational use. The six PSTs who did not list computer-related hobbies failed to list a mathematics-related hobby as well.

Ten PSTs indicated that they had had computer-related work experience. The occupations included being a secretary, law intern, and computer installer for IBM. Five students indicated no experience.

When asked the question, "For how many years has mathematics been of interest to you and what sparked your interest?" all but two responded with answers ranging from 3-11 years. The remaining two PSTs had almost identical responses: "As long as I can remember so I have no idea what sparked my interest." The majority of the PSTs attributed their interest in mathematics to a good mathematics teacher. Several indicated that they became interested in mathematics because they were good at it. Two PSTs indicated that they became interested in mathematics after they had decided to become teachers. Since they were good in mathematics, they decided to teach mathematics.

When asked a similar question about computers, most PSTs indicated a more recently acquired interest, listing 1-8 years. Three responded that they had no interest in computers. Most of the PSTs indicated that the necessity of using computers in their school-work had initially sparked their interest. Others listed mathematical applications, games, and the Internet, one going so far as to say, "[My interest] began with word processor and progressed to eventual dependence on Internet."

Given their fairly recent enrolment in EMT 468 it is not surprising that, when asked to list several good examples of software, PSTs listed *Geometer's Sketchpad* (GSP) and Microsoft Excel, the two programs most used in that course. Aside from the rather common response—"GSP—great for geometry"—the following response was typical: "I think this program [GSP] really allows students to see the material and why it works the way it does. The program is also fun and can really keep the interest of the students." Other responses included the following: Microsoft Word, PrintMaster, Jasper (a video-disk mathematics program), and graphing calculators.

The PSTs listed technical difficulties and their own lack of computer knowledge as their primary frustrations with using computers. They usually received help from friends, family, or whoever happened to be "working in the room who knows more about computers than me." Four of them, however, said that no one helped them solve their problems; one PST put it this way: "I usually muddle through on my own." Only four PSTs indicated that others would ask them for help when working on computers, three citing Excel as the program in which this occurred.

Given these responses, it is not surprising that the majority of these PSTs would be reticent to incorporate computer technology into their teaching. They were concerned that their students might raise questions, even easy questions, that they would not be able to answer. The following comment captures the essence of this perspective, "I'm usually the one needing help!"

Use of the internet

Every PST checked that they used the Internet to do research for school projects. All but two surfed the Web and all but four used e-mail. About half indicated that they had used the Internet to find help with teaching problems or with solving mathematics problems. Only one PST had down-loaded software off the Internet; no one had created his or her own web page.

The frequency of Internet usage varied dramatically among the 15 PSTs. At the one extreme, one PST used it "three times a semester—at night" and another "2 times a month—at night." At the other extreme, one PST responded, "afternoon and evening—every day," and another added, "2-5 hrs/day—usually in the afternoon." When asked for ways that they had been using the Internet in their classes, most mentioned looking up lesson plans and doing research for papers and class projects. There is little indication from the questionnaire responses, and more particularly from the interviews, that they used TMF to do this searching.

The Math Forum, computers, and student teaching

All but three of the PSTs indicated that they were planning on using computers with their students during student teaching. Seven specifically mentioned GSP. Responses ranged from "computing numbers easier and visual stimulation" to "whenever possible, I hope to use the computers in my lesson plans to demonstrate a different way to teach and learn in class." A majority of the PSTs' responses, however, were vague and did not reflect a *strong* commitment to using technology with their students.

None of the PSTs gave firm indications that they planned to use TMF with their students, although three did indicate that they would introduce them to it. Most responses listed reasons why no such plans existed. These responses fell into three main categories: 1) They saw TMF as a teacher resource, not a student resource, 2) they felt they did not know how to use TMF with their students, and 3) they indicated that their schools did not have sufficient Internet access.

PSTs were asked how they thought computers might aid their students' work. The most common response was visualisation. Others talked about quicker and more accurate calculations, neatness and organization, and motivation and interest. One PST mentioned exploration and relational understanding, and another mentioned problem solving.

The final survey question asked the PSTs to identify specific areas (curricular or teaching) in which they planned to use TMF during student teaching. These responses were very similar to previous ones in that some PSTs were ambiguous about their intended use of TMF (four did not even respond) and others indicated that it might be used as a resource for ideas and activities. The following two responses were markedly different, however:

> Hopefully, I will be able to work on my ability to relate to my students and my ability to motivate them to investigate all facets of mathematics. I think the Math Forum can help them with this because in the Forum they can ask questions and find web sites about anything in mathematics that interests them.

> I plan to focus on supplementing my lesson plans with activities other than the ones in the book. I want to try and teach students so that they will be able to retain the information and use it to benefit them later in life. I think that the Math Forum will help me with my lesson plans and also with my mathematical questions.

The first comment suggests that the PST sees TMF as a basis for motivating students. The goal seems to be finding ways to make mathematics interesting and TMF provides a means for realising this goal. The second comment seems more utilitarian. The issue here may or may not involve motivating students but it does have something to do with the everyday practice of creating and using lesson plans.

Interview responses

Three themes regarding the PSTs' use of technology during student teaching emerged from an analysis of the interview data: use of the Internet and TMF as resources, perceived limitations of the Internet and TMF, and factors that limited PSTs' use of technology. Each of these themes is discussed.

Use of web-surfing to generate ideas

It appeared that surfing the Internet was a common experience among the PSTs. For most of them web surfing served the purpose of generating ideas for lesson plans, even though the material found was seldom directly related to the lessons they were teaching. The PSTs had hoped to find lesson plans on the same content that they were teaching, but this rarely happened. Rather, what they found had to be translated and adapted to fit their own particular lessons. Nevertheless, most of them found web surfing worthwhile and enjoyable. For a few PSTs, however, the inability to find lesson plans that directly fitted their needs raised in their mind the question of whether use of the Internet was a productive use of time.

Perceived limitations of the internet as a resource for student teaching

Perceived limitations of the Internet were expressed in a variety of ways. One PST chose not to use the Internet during her student teaching because she did not feel comfortable with it; she felt other available resources sufficed for her teaching of mathematics. Other PSTs indicated frustration with the use of the Internet in the following way:

> And, you know, honestly, every single time I would try and go anywhere on the internet, regardless of if it was the Math Forum or not, it just seemed like I spent so much time looking, it's like I could have come up with something on my own by the time I finished looking.... And like my favourite lessons, and activities, and games and stuff I created from my head. (PST6)

Overall impressions from the interviews

The PSTs appeared to view computers as more of a preparation tool than that of a teaching tool. Typing tests and work-sheets, keeping track of grades, and surfing the net were the most common preparation uses. Only PST6 and PST9 indicated that use of computers with students was an integral part of their classroom approach. In general, when PSTs did use computers with their students, they had to rise above the prevailing attitudes towards technology held by their mentor teachers or, in some cases, the school itself.

Consistent with their general use of computers, the PSTs used the Internet mainly as a preparation tool. They surfed the web in an attempt to generate ideas for teaching, mostly doing general searches in Yahoo! rather than searching TMF. Use of the Internet and TMF with students was virtually nonexistent.

Factors that limited PSTs' use of technology

We observed two factors that seemed to impede the PSTs' use of technology. The first was related to the school setting in which they did their student teaching. The second was related to their own particular expertise in using technology.

School setting

In High Schools 1,2,3 and 6, the availability of technology seemed conducive to the use of computers both in the classroom and as a teaching resource. However, the teachers and consequently the PSTs demonstrably under-utilized computer technology. In High School 4, which was not a particularly technology-rich school, use of computers was rather extensive, due largely to the initiatives of PST6, PST12, and PST14, who were strongly committed to the use of computer technology in the teaching of mathematics.

A factor that impeded the use of technology was the mentor teachers' reluctance to use technology in their teaching of mathematics, even at technology-rich sites. In some cases one could sense a direct link between the attitude of the mentor teachers (as perceived by PSTs) and the PSTs' use of computer technology: "I'm not bashing my mentor teacher, but I asked and it was, 'I've thought about it in the past. I think it's a waste of time. Let's don't do that.'" (PST1).

Expertise

The PSTs' coursework suggests that they had ample training on how to use technological tools such as GSP, and Excel to explore and solve mathematics problems. It seems fair to say that they felt comfortable with their own personal use of these tools.

However, some appeared uncomfortable with their ability to use technology as a tool for teaching mathematics. For example, one of the PSTs indicated that she did not feel confident in her use of computer technology when leading a class discussion. She expressed this concern in the following way.

> I didn't feel confident enough to know how to hook up an overhead thing, and borrow a power book and stuff. And I just imagined these horror stories of me getting there and it not working and I don't know what to do. So I was kind of fearful of going through that whole route. (PST8)

Her discomfort appears to have stemmed from her inexperience in using technology and an accompanying fear that it would not work. But it could also be that technology opens the door for questions about either mathematics or the technology itself that leads to a certain amount of uncharted waters, waters the PST is not comfortable traversing. This insecurity is heightened not only because of the technology but because any questions that invite problematic situations can be threatening to PSTs.

Final comment

The results of this study indicate that providing rich opportunities for pre-service teachers to use technological tools for learning and doing mathematics in their college courses is not sufficient preparation for their use of these same tools with their students when teaching. We need to provide examples of how to prepare and conduct exciting lessons for secondary students that integrate technology. We also need to provide field experiences where pre-service teachers can observe secondary students using these technologies as a regular part of their mathematics instruction. Until we have a cadre of available mentor teachers (for our student teaching placements) who are, themselves, experienced and comfortable with integrating these technologies into their own mathematics teaching, the opportunities for our student teachers will remain limited.

References

Dwyer, D. C., Ringstaff, C., & Sandholtz, J. H. (1990). The evolution of teachers' instructional beliefs and practices in high-access-to-technology classrooms: First-fourth year findings. *ACOT Report #8*. Cupertino, CA: Apple Computer, Inc.

Jackiw, N. (1991). *The Geometer's Sketchpad* (computer program). Emeryville, CA: Key Curriculum Press.

Laborde, C. (1998). Factors of integration of dynamic geometry software in the teaching of mathematics. *Paper presented at the ENC Technology and NCTM Standards 2000 Conference*. Arlington, VA, June 5–6.

Lee, K. (1997). *TesselMania!* (computer program). Minneapolis, Minnesota: MECC.

NCTM (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Olive, J. (1998). Opportunities to explore and integrate mathematics with the Geometer's Sketchpad. In R. Lehrer & D. Chazan (Eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space*. Mahwah, NJ: Lawrence Erlbaum Associates, 395–418.

Olive, J., & Ramsay, A. P. (1999). Technology and change in mathematics teaching: Three case studies of project LITMUS teachers. *Paper presented at the Research Pre-session of the 77th Annual Meeting of the National Council of Teachers of Mathematics*, San Francisco, CA.

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GOALS FOR A CAS-ACTIVE SENIOR MATHEMATICS CURRICULUM

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This paper canvasses options for goals of using computer algebra systems (CAS) in senior secondary mathematics subjects. CAS may be used in order to align school mathematics with the use of technology in the modern world, or to make students better users of mathematics, or to achieve deeper learning, or to change views of mathematics, or to provide curriculum space to introduce new topics, or some combination of these. The features of CAS making these feasible goals are outlined. To stimulate debate, we propose priorities amongst the goals and propose the relative likelihoods of the enabling mechanisms working in practice.

Introduction

This paper outlines some preliminary issues being addressed by a new study which aims to investigate the changes that regular access to CAS (computer algebra system) supercalculators will have on senior mathematics subjects and the associated assessment in Victoria, Australia. It will explore the feasibility of offering new mathematics subjects that use CAS extensively. In Victoria, schools provide Year 11 and 12 subjects for the Victorian Certificate of Education (VCE) under guidelines in a "Study Design" provided by the Victorian Board of Studies. Final grades, which are used for various purposes including selection to tertiary courses, are derived from a mix of tasks set at the school and external examinations. Subject to the continuing approval of the Board, the curriculum, teaching methods and formal assessment undertaken by students in three project schools will be altered, culminating with the trial in 2002 of an alternative VCE Study Design and examinations using CAS.

The study is funded from 2000–2002 by the Australian Research Grant Strategic Partnerships with Industry Scheme. The Chief Investigators of the project are Gary Asp, Helen Chick, Barry McCrae and Kaye Stacey from the University of Melbourne with David Leigh-Lancaster from the Board as a partner investigator. There are four industry partners: the Board and three calculator suppliers and manufacturers, Hewlett-Packard, Shriro (Casio) and Texas Instruments. The industry partners will supply CAS supercalculators to students in three schools for a three year program of classroom based research. All the schools will use CAS technology that is hand-held. Although this was not the original intention, it means that the broader issues of learning and assessing with full desk or laptop computer capabilities do not have to be addressed. Further details of the project are available from the authors and are given in Stacey, McCrae, Chick, Asp and Leigh-Lancaster (submitted), which also canvasses preliminary decisions about the role to be played by CAS in the formal year 12 assessment.

At this point in history, we believe that the most significant stimulus for change in senior school mathematics is new technology generally and, for the next few years specifically, the advent of affordable computer algebra systems (including symbolic algebra, graphics, statistics, calculus, matrices etc). CAS is acting as an agent of educational change in two different ways simultaneously: it necessitates some change and it provides an opportunity or stimulus for other change (see Figure 1). At least in the long term, the ubiquity of mathematically able software will necessitate change in the mathematical methods that are taught. Responsiveness to new technology may be slow but it will occur Finding good (i.e. simple, quick, routine, widely applicable etc) methods of calculation is critical for all branches of Mathematics. Calculation in a broad sense is a major obstacle to using mathematics and so significant new technology for performing routine procedures (whether abacuses, logarithm tables, slide rules, calculators or computers) will always be embraced for appropriate problems in the long term. CAS therefore necessitates changes to mathematical methods taught for solving problems, but it does not dictate a timetable for change.

Relationship of new technology to curriculum change
CAS necessitates change of methods taught (possibly slowly).
CAS provides an opportunity for other change.

Figure 1. CAS changes curriculum and teaching

On the other hand, the arrival of CAS provides an opportunity and a stimulus to move curriculum and teaching in certain desirable directions. These directions can accord with our values and beliefs about what mathematics is or should be (at this level and for these students), tempered by previous experience of what can be achieved in practice. However, exactly which directions for curriculum change are feasible depends on a practical assessment of the actual features likely to be available on affordable machines within a medium timeframe. In this way, there must be an interplay between the changes in senior mathematics that we may desire on the basis of beliefs about mathematics and good mathematics teaching on the one hand, and technical features of the new technology which is underpinning the change, on the other. In this paper we will present, for debate, the major goals that we think can be achieved by using CAS (Figure 2) and relate these to the features of CAS and teaching with CAS which make them feasible (Figure 3). This sets out a framework (Figure 4) that links the goals and mechanisms for achieving them.

In whose interest is curriculum change? A caution

In this paper, we present our preliminary thinking on one of the major policy issues to be resolved through the project: the way in which the curriculum will be adapted to use CAS. This is likely to be a highly contentious issue and so careful appraisal of the evidence relevant to various positions will be needed to guide policy. The main groups involved in this change are the students, the teachers, the curriculum authorities (in this case, the Board), the calculator industry and mathematics education researchers. There are clear benefits for some stakeholders. It is easy to see how the adoption of CAS will benefit the calculator industry. It also benefits researchers, because perturbation in the learning environment provides us with new ways of exploring fundamental issues about teaching and learning mathematics. Leading teachers will also benefit, because their professional lives are enriched by new challenges. We must be certain, however, that students will also benefit from change: both as individuals and as future citizens of a country which aspires to take a place in a competitive global economy based on technology.

Goals for a new curriculum

Why should CAS be used in senior mathematics, or, indeed, why should it not be used? And, when the goals are clear, how can CAS be used to achieve them? As noted above, deciding on the goals for introducing a new CAS-active mathematics subject requires consideration of both philosophies of mathematics teaching and technical features of CAS. In Figure 2, the goals that we have for the potential new mathematics subject are presented. Note that these goals are additional to the "normal" goals of a mathematics subject at Year 12. They are presented in a proposed order of priority: this order is not yet settled, but is presented as a point to debate. In fact, much of what follows is presented in this spirit, as preliminary thinking to be refined and then operationalised in the light of experiences of teaching with CAS, its impact on students and discussions with other stakeholders.

Goals for CAS-active mathematics
(in priority order – for debate)

1. To make students better users of mathematics
2. To increase congruence between real maths and school maths
3. To achieve deeper learning by students
4. To promote a less procedural view of mathematics
5. To introduce new topics into the curriculum

Figure 2. Goals for a CAS mathematics subject, presented in order of importance

Each of the goals listed in Figure 2 is shorthand for a complex of ideas. Goal 2, increasing congruence, is based on the principle that schools should prepare future citizens for a technologically infused society and that school mathematics should prepare them for mathematics as practised beyond school. In this working world, analysts such as Pea (1993) see intelligence itself not solely as a function of one person's mind but as distributed between members of a working team, technology and symbol systems. Successful professionals form a partnership with "intelligent technology" (Saloman, Perkins and Globerson, 1991), not expecting to be separated from it. Those who agree with this goal may nevertheless differ in their view of what "real world" mathematics is like: is it the "real world" of the mathematician or the engineer, what is the "real world" expert use of by-hand procedures etc? Moreover, an immediate pragmatic concern of teachers is that before taking up their long-term role in the workplace, most senior mathematics students undertake tertiary education and the transition to CAS there is still unclear—although as Galbraith, Haines and Pemberton (1999) note, incorporation of CAS is taking place at an increasing rate. In summary, if a general principle of alignment to the technology used in the "real world" is accepted, there are still many different views about what this might mean in practice.

Everyone would agree with the first and third goals, to make students better users of mathematics and to help them learn better, but again there will be debate on the nature of "better". Do we wish students to be able to solve a greater range of problems or do we want them to be able to solve problems in the current range more reliably? Can a CAS-active mathematics subject help them become more discerning and confident users of mathematics? And what is better learning? Is it more thorough learning of the same topics, or learning deeper properties? Does it involve being able to apply learning to an expanded range of unfamiliar problems? Becoming better users of mathematics will surely require relatively more emphasis on formulating problems in mathematical terms and interpreting results, and relatively less emphasis on methods of solving. Experience has shown, however, that formulating is a difficult step for students and the routine aspects of solving are easier—if formulating cannot be taught effectively, what will be left for average students to do?

The fourth goal relates to the values and beliefs about mathematics as a subject that students (and teachers) might hold. It is a commonly shared belief that Mathematics should be more of a "thinking subject" which encourages higher-order thinking and has less emphasis on performing routine procedures in over-rehearsed standard responses to exercises. School mathematics should mirror genuine mathematical activity and include both mathematical discovery and mathematical modelling. Students need the opportunity to "do mathematics", as well as to practise it. Whilst there is often general agreement on the goal of moving away from routine work, there can be quite different images of what this might mean. Again, the relative importance of proof (or at least justification) compared with a more empirical approach seems a key variable.

The final goal of introducing new topics into the curriculum must be raised as a possibility. If some routine procedures are to be omitted from a curriculum, should not something be put in its place? What might this be? Writing at the beginning of CAS technology on personal computers, Stacey and Stacey (1983) predicted that time saved on calculus techniques by using CAS might be better spent on studying the mathematics for computer science. Is this today's choice? Alternatively, can the time be used to teach the higher order skills of formulating and interpreting more successfully than in the past?

The intention of this section has been to flesh out some aspects of the breadth of each of the five goals listed in Figure 2, and to demonstrate some of the variation in what they may mean, so as to stimulate debate on what is most worthwhile. Amongst our team, there is support for each of goals 1, 2 and 3 as first priority.

Features of CAS enabling the achievement of new goals

As noted above, the goals for CAS-active mathematics cannot be arrived at without assessing the capabilities of the CAS systems that will be available to students in the next few years and the ways in which they can be used in classrooms. If CAS is to be the stimulus of change, then the direction of possible change is to some extent dictated by features of the technology. Fortunately, there is now a growing set of research reports that provide evidence about the likely effects of CAS use.

In Figure 3, we list (again welcoming debate) the major mechanisms by which using CAS can achieve the goals outlined earlier. Again, as a point of debate amongst our team and with others, we have tentatively put the list in order: this time in the order of likely widespread success. In passing, note that some mechanisms (i.e. 1, 2, 3, 4, 6) are a consequence of functional use of CAS (its capacity to produce answers) and others (1, 4, 5, 6, 7) are a consequence of pedagogical use of CAS (using it as a teaching tool). This distinction between functional and pedagogical use is outlined, with examples for basic calculators, by Etlinger (1974).

The first feature listed, providing multiple and easily linked representations of mathematical ideas, is extremely important and well known, featuring in almost all CAS curricula. It is both a functional and pedagogical feature. Not only does it enable students to solve problems by moving between representations (e.g to find a zero of a function graphically), but it provides the teacher with convenient capacity for supporting visualisation in Mathematics, including dynamic classroom demonstrations. Examples include

- how a family of functions changes as a parameter changes,
- how a secant approaches a tangent,
- how changing outliers affect a line of best fit,
- how an optimum value in a linear programming problem varies with parameters.

Such demonstrations can also be made available to students on their own machines for use whenever they want.

Mechanisms by which CAS use might help achieve new curriculum goals (in order of likely success)
1. By providing multiple, easily-linked representations of ideas
2. By freeing up curriculum time (for re-allocation)
3. By enabling less constrained real problem solving
4. By reducing curriculum emphasis on routine procedures
5. By providing "trainer wheels" for learners
6. By providing more options for teaching and problem solving
7. By promoting positive learning strategies

Figure 3. Mechanisms by which CAS use can achieve curriculum goals

There are, of course, unanswered questions about the use of multiple representations, especially whether students beginning in an area will be able to learn well across a range of representations and what choices teachers will make. Kendal and Stacey (1999) have some preliminary results. They studied the different choices of representation that teachers of introductory calculus made for teaching and how this impacted on students learning. Because CAS provides technological support for working numerically, graphically and symbolically, a range of methods (e.g. for finding a tangent) which were previously only available in theory (e.g. because they involve excessive calculations) become available in practice. There are more options for teaching and there are more options for problem solving (see also Tynan, Stacey, Asp and Dowsey, 1995). Kendal and Stacey found that teachers will choose to highlight attributes of CAS which support their own beliefs and values about mathematics. Teachers who value routine procedures can find on a CAS a plethora of routine procedures to teach students; teachers who value insight can find many ways in which they can demonstrate links between ideas better than ever before.

Many experimenters assume using CAS will free-up curriculum time so that time that is currently spent on less-valued objectives will be able to be reallocated to more valued objectives. Time is freed up in several ways. Firstly, some lesson time is saved when tasks which are presently done by hand, are sometimes done by CAS. For example, students can draw a lot of graphs quite quickly, freeing up time for class discussion. Secondly, some curriculum re-sequencing studies, which Heid (1997) reports have had positive results for calculus, operate on the principle that CAS enables concepts to be taught in advance of skills. This makes the later teaching of by-hand skills more efficient because they are learned on a sound conceptual footing. McCrae, Asp and Kendal (1999) report an Australian study on this principle. They decided to spend an allocated 20 lessons for introductory calculus by first building conceptual understanding very carefully. Ten lessons focussed on the concept of a derivative before differentiation methods were treated with much reduced practice time. The results of these Year 11 students were comparable with those of Year 12 students in the same school. In the initial concept building, however, graphical and physical (datalogging) tools were more useful than the symbolic capability of CAS.

Drijvers (1999) took a different approach to time re-allocation. He outlines two units taught in the Netherlands that featured imaginative problems requiring optimisation of complicated functions in real life situations. Time previously spent on practising by-hand skills was re-allocated to early CAS-supported investigations, from which students could see the purpose of the mathematics to be learned. Drijvers' experiment is also an example of how CAS enables less constrained problem solving.

More controversially, time can be freed up by not teaching certain by-hand skills at all. This is the option which will need the most careful consideration of costs and benefits, but which also offers the greatest potential for really creating space in the curriculum. We are yet to be convinced that teaching students to use a CAS will not absorb whatever additional time is created by other mechanisms. One important cost of not teaching certain skills is that students generally feel uneasy about using entirely "black-box" methods (at least at this stage of the change). Drijvers (1999) showed this, as have Australian studies. In the McCrae et al (1999) study, 15 of the 59 students made at least one written comment during the study that expressed some uneasiness or concern. Pierce (1999) working with tertiary students also finds this uneasiness, even though her students have unconstrained access to CAS in the examinations. For example, 13 students undertaking an introductory calculus unit were asked if they thought CAS would offer fresh hope to students with difficulties in mathematics and 6 expressed some sort of uneasiness including:

> Student 7: "No, I think you need to know what the computer is working out and how."
>
> Student 8: "CAS offers the student the answers without really needing the basics. I personally would have enjoyed more time spent on the basics."

"Trainer wheels" studies are designed around the theory that CAS use can support learners as they learn to carry out mathematical procedures for themselves. This can reduce cognitive load and enable students to recover from errors within an attempted solution. Tynan and Asp (1998) worked with younger students beginning algebra (48 Year 9 students at one school in two classes). They used CAS for two reasons: firstly as "trainer wheels" when students were learning to solve equations by the "do the same to both sides" method. Students could use the CAS to test the effect of actions (such as adding $2x$ to both sides), to check each line of their work and to recover when they had made mistakes. CAS was also used to show students the power of algebraic methods of solving problems, before they could reliably solve one variable linear equations by hand. All too often, students learn how to solve equations, but when faced with a problem situation they either do not realise that equation solving is relevant, or they feel they are unlikely to be successful with an equation, and so they resort to numerical methods or guess and check (Stacey & MacGregor, 1999). Tynan and Asp reported clear success in getting students to appreciate the power of algebraic methods. For example, on one item, 56% of student taught with CAS set up an equation and tried to solve it using algebraic "do the same to both sides" methods. In the class taught traditionally, only 26% did this.

There is now a significant body of research that supports the proposal that use of CAS can promote positive learning strategies, such as exploration of ideas, group work, discussion and negotiation of meaning. Pierce (1999), for example, provides evidence that these strategies are more prevalent in her CAS classes. Positive learning strategies are not, however, specific to computer algebra systems and may be more a consequence of computer use where there are shared computers and public displays.

How CAS can assist in achieving new goals

Figure 4 highlights for us the type of assumptions about CAS use that we make when we claim that CAS will enable us to better achieve certain objectives. It also provides guidelines for the way in which CAS will need to be built into a new curriculum and its teaching, learning and assessment.

Goals	CAS-related mechanisms for achievement
To make students better users of mathematics	By enabling less constrained real problem solving By freeing up curriculum time By providing multiple, easily-linked representations of ideas By providing more options for teaching and problem solving
To achieve congruence between real maths and school maths	*Use of CAS in teaching and assessment* By enabling less constrained real problem solving
To achieve deeper learning by students	By promoting positive learning strategies By providing multiple, easily-linked representations of ideas By freeing up curriculum time By providing "trainer wheels" for learners
To promote a less procedural view of mathematics	By reducing curriculum emphasis on routine procedures By enabling less constrained real problem solving
To introduce new topics into the curriculum	By freeing up curriculum time

Figure 4. Goals for CAS-active mathematics and how they might be achieved

There is not a simple correspondence between the goals of Figure 2 and the mechanisms for achieving these goals in Figure 3. In fact, several mechanisms will operate to achieve each goal. Figure 4, again for discussion and debate, sets out the main links. For example, Figure 4 shows how the first goal is feasible because CAS can reduce the constraints on the problems that can be considered at school (e.g. there is no longer such a need to have "nice" functions and numbers), curriculum time can be freed to re-allocate to more experience of problem solving, and multiple representations make more methods of solving problems feasible. The second goal of achieving congruence between mathematics at school and mathematics as it is used in "real life" (in the case of senior mathematics, in scientific and technical employment) is largely achieved simply by the presence and use of CAS in teaching and, importantly, in assessment. We assume that complicated algebraic calculation, accurate graphing, statistical calculations, etc will be carried out by machine in virtually all workplaces, supplemented by a fair degree of mental or by-hand skill. Since CAS presence is not really a mechanism, it is in italics in Figure 4. Figure 4 similarly shows the features of CAS use on which achievement of the other three goals depends.

Conclusion

This paper has outlined options for the goals of introducing CAS into senior secondary mathematics. CAS is seen as driving some change in some directions and providing an opportunity for other change. There is a range of goals that might be achieved and our project must prioritise them. The achievement of any of the new goals depends on functional aspects of CAS and pedagogical choices about CAS use. The paper has highlighted both the options and the choices to be made, and the connections between them.

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References

Drijvers, P. (1999). Students encountering obstacles using CAS. A developmental–research pilot study. Paper presented at conference of *Computer Algebra and Mathematics Education*. August 1 – 2, 1999, Rehovat, Israel.

Etlinger, L. (1974). The electronic calculator: A new trend in school mathematics. *Educational Technology*. December, 43–45.

Heid, M.K. (1997). The technological revolution and the reform of school mathematics. *American Journal of Education* 106 (November) 1997, 5 – 61.

Galbraith, P., Haines, C., & Pemberton., M. (1999). A tale of two cities: When mathematics, computers and students meet. In J. M. Truran and K. M. Truran (Eds.), *Making the Difference. Proceedings of 22nd Annual Conference of The Mathematics Education Research Group of Australasia* (pp. 215– 222). Sydney: MERGA.

Kendal, M. and K. Stacey (1999). CAS, calculus and classrooms. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* Vol 3. (pp. 129–136). Haifa, Israel: IGPME.

McCrae, B., Asp, G., & Kendal, M. (1999). Learning calculus with supercalculators. In J. M. Truran and K. M. Truran (Eds.), *Making the Difference. Proceedings of 22nd Annual Conference of Mathematics Education Research Group of Australasia*. (pp. 359–364). Sydney: MERGA

Pea, R. (1993). Practices of distributed intelligence and designs for education. In G. Saloman (Ed.) *Distributed Cognitions: Psychological and educational considerations* (pp. 47–87). Cambridge: Cambridge University Press.

Pierce, R. (1999). Computer algebra systems facilitate positive learning strategies. In J. M. Truran and K. M. Truran (Eds.), *Making the Difference. Proceedings of The Twenty–second Annual Conference of The Mathematics Education Research Group of Australasia* Incorporated. (pp. 446–453). Sydney: MERGA.

Saloman, G., Perkins, D. & Globerson, T. (1991). Partners in cogition: Extending human intelligence with intelligent technologies. E*ducational Researcher,* 20 (3), 2–9.

Stacey, K., McCrae, B., Chick, H., Asp, G., & Leigh–Lancaster, D. (submitted) Research–led policy change for technologically–active senior mathematics assessment. *23rd Annual Conference of the Mathematics Education Research Group of Australasia.*

Stacey, K., & MacGregor, M. (1999). Implications for mathematics education policy of research on algebra learning. *Australian Journal of Education*. 43 (1), 58–71.

Stacey, P., & Stacey, K. (1983). Upper school mathematics in the 21st century. In D. Blane (Ed.) *The Essentials of Mathematics Education.* Melbourne: Mathematics Association of Victoria.

Tynan, D., & Asp, G. (1998). Exploring the impact of CAS in early algebra. In C. Kanes, M. Goos, & E. Warren (Eds.) *Proceedings of the 21st Annual Conference of the Mathematics Education Research Group of Australasia (vol 2)*, (pp. 621–628) Brisbane: MERGA

Tynan, D., Stacey, K., Asp, G., & Dowsey, J. (1995). Doing mathematics with new tools: New patterns of thinking. In *Galtha: Proceedings of the 18th Annual Conference of the Mathematics Education Research Group of Australasia*, (pp. 543–549). Darwin: MERGA.

CALCULATOR TECHNOLOGY CAN OFFER SENSIBLE CHOICE IN COMPUTATION

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Simple four function calculators have been available in primary schools for over twenty years but have had little impact on primary mathematics. They have been the focus of debate and have been labelled as the cause of the apparent decline in computational standards. Reliance on calculators by children when faced with a calculation is cited as evidence of this decline in mathematical competence. This paper identifies reasons for children's computational choice and discusses how calculator technology might be used to offer children genuine choice when faced with a calculation.

Introduction

When calculators were introduced into primary schools in the late seventies and eighties the expectation was that they would transform mathematics. The euphoria that surrounded the Calculator Aware Number project (CAN) (Shuard *et al.*, 1991) appears to have been lost as the expected transformation in primary mathematics has not occurred. There are several possible reasons for this:

- computers appeared in schools soon after calculators. Computers tended to be accepted by parents and teachers more readily than calculators (Sparrow & Swan, 1997);
- teachers were not able to use calculators as a personal tool and were unable to use the technology in their classroom teaching (Sparrow & Swan, 1997);
- teacher beliefs about the use of this form of technology varied from acceptance of, to reluctance toward the use of calculators; (Ofsted, 1993 cited Duffin, 1997)
- the position of standard written algorithms as the preferred and dominant means of teaching in many classrooms (Duffin, 1991; Porter, 1988; Willis & Kissane, 1989);
- curriculum documents envisaged the use of calculators in a sensible manner (AEC, 1991; NCTM, 1989) but the enacted curriculum, based on text books, often neglected the use of calculators for anything but trivial uses (Sparrow & Swan, 1997);
- a gap between the beliefs of many mathematics education lecturers and the community in general.

Do children actually have a choice when they have to complete a calculation? The next section discusses evidence of computational choice shown by children from a class were calculators were easily accessible.

Children's choice of computation

It is a goal of many curriculum documents (AEC, 1991; Curriculum Council, 1998; NCTM, 1989) for children to develop a repertoire of computational strategies and have a range of methods to solve numerical problems. This implies that students will also adopt the most efficient method based on their ability for a particular question. However, a lack of meaningful use of calculators in the primary school appears to have stunted the development of computational choice for many children. Little is known about the computational choices made by children and the reasons they have for making these choices. Previous studies (Reys, Reys & Hope, 1993; Price 1997) tended to focus on the computational choices made by students but not the reasons for making these choices. Reys, Reys and Hope (1993) found that while written methods tended to dominate the thinking and choice of many students, mental methods were still employed in many cases, while calculator methods were the least common. Price (1997) found teacher presence to be a key factor in the computational choices made by children. These results appear consistent with the findings of Carraher *et al.* (1985) who found students using different methods of computation depending on whether the computations were carried out in school or in the markets.

Recent work by Sparrow and Swan (1999) and Swan and Bana (2000) indicated that some children are making poor computational choices. Sparrow and Swan (1999) surveyed 151 students in Years four to seven in three schools in the South West of Australia. They were interested to find out how 'fluent, competent and confident' primary children were with a range of calculating strategies given they had easy access to calculator technology. They wished to ascertain if, with the easy access to calculators, children have become reliant on this form of technology and were therefore making poor computational choices.

Typically children chose to solve basic number fact problems in their head. Some children referred to number tricks taught by their teacher such as multiplying by 99. Strings of simple numbers such as 5 + 5 + 5 + 5 + 5 and simple multiplications such as '3 x 221' were also offered as examples of mental calculations. The choice of whether to use a paper-and-pencil algorithm seemed to be based on two-digit by two-digit multiplication, division and 'large numbers'. Research by Price (1997) indicated that teachers have an influence over the computational choices children make, especially if the calculations were performed in a classroom setting.

The major reason identified by children for using calculator technology was that the numbers were too large or the operation was too complex; in their words the calculation was too hard to do in the head or on paper. What constituted 'hard maths' varied from student to student, although many referred to large numbers as a factor. Multiplication and division featured prominently among the reasons given for choosing to use a calculator. It appeared however, that the children did not actually think about the numbers involved or were unaware of 'easy' ways of working with some numbers. However, of concern were the students who suggested the following as problems that would be 'too hard to solve in my head or with pencil and paper'.

1000 x 3000	10 x 200	7 into 63	16 x 9
5 x 5	13 + 3 + 4 + 6 + 10 + 11 + 9 + 3	89 x 100	1000 + 153 – 2

The decision to use a calculator was summed up by one child who suggested that a calculator would be used if 'it is really hard! And if I did it in my head, my head would explode!'

More recent research carried out by Swan and Bana (2000) looked at the choices made by children in Years five to seven and the reasons they gave for making those choices. Over seventy students from Years five to seven in two primary schools were interviewed about their response to an eighteen-item instrument. The details have been reported elsewhere (Swan & Bana, 1999; 2000). One of the criteria for sample selection was that the children had easy access to calculators in the classroom. This meant in all cases that children kept a calculator in their desk. It became evident, however, that while children kept calculators in their desk they did not use them all the time. There were some calculations for which the teacher did not allow calculator use, for example basic fact calculations. The results of the study indicated that students had not become over-reliant on the use of calculators, with 27% of calculations overall being completed with the aid of a calculator. Mental methods were favoured in 35% of the calculations and written methods for 27% of the calculations. The remaining 11% of calculations involved the use of mixed or combined methods, such as the use of some mental and some informal jottings or else no method was recorded–the child could not answer the question. Little comparative data is available although some items from previous studies (Price, 1997; Reys, Reys & Hope, 1993) were used. Of the items common to all three studies, it may noted that children in this study made more use of calculators than children in the previous studies. A question remains as to why children selected a calculator method. This is the focus of the next section of discussion.

Why children chose calculator methods

When the computational choices made by the children and the reasons given for making those choices were examined most choices appeared reasonable. There were a few items, however, for which the choice of calculator appeared inappropriate. Of the eighteen items on the instrument, students showed a preference for calculator use in the following items.

item 3,	$369 \div 3$	item 8,	1000 x 945	item 9,	10% of 750
item 10,	$14 \times 9 \div 6$	item 17,	0.25 x 800		

Items three and ten involved the division operation that was a source of confusion for many students. Item ten also involved the use of two operations that, for many students, was a departure from normal classroom practice. Item nine, which was relatively easy to compute mentally, revealed a lack of understanding of percentage by children. Many chose to use a calculator simply because they knew there was a percentage key on the calculator. Many did not achieve the correct answer, as they were unaware of how the percentage key on the calculator worked. The decimal component of item seventeen served as a catalyst for choosing to use a calculator. Students who recognised 0.25 as one quarter had little difficulty calculating the answer mentally.

Item eight might suggest that calculator use is dulling the ability of children to calculate mentally. The result of multiplying 1000 by 945 is relatively easy to compute mentally, and yet the majority of children in the study chose to use a calculator. A brief look at the transcripts of children's responses to this question might help to illuminate the reason why calculator methods were favoured here and also explain why, in many cases, mental methods proved ineffective.

The involvement of zero

When asked to give reasons for choosing to use a calculator for this item the following student gave this explanation:

I: 1000 x 945
S: 945 000
I: Why did you do that one on a calculator?
S: Because I don't really know how to do long multiplication
I: Could you do that one mentally?
S: Yes, I could have gone 145 x 1 and add three zeros.

The technique of 'adding zeros' was referred to often by children. In most cases they experienced difficulty applying this strategy and did not appear to have any relational understanding of why it worked. The confusion caused by zeros was most apparent in the following example, which at first glance may appear to be a simple mental question. This particular problem was common to many students:

I: 1000 x 945?
S: 945 000.
I: This time you used the calculator. Why was that?
S: Because it was pretty big and pretty tricky because I get confused if it has lots of 0's.

While it might appear surprising that a student would use a calculator to solve an item such as 1000 x 945, clearly many students could only see the size of the numbers and failed to look at the numbers themselves. When discussing this item with the students it became apparent that they viewed the written approach involving the placing of zeros as too cumbersome.

Big numbers and time

Students often cited 'big numbers' as the reason for choosing not to use mental methods. In some cases the 'big numbers' prompted students to use a written method, while in other cases they used a calculator.

I: What about 369 ÷ 3?
S: 123
I: So you used the calculator there, why?
S: Because it's a big number, it's easier to use a calculator.
I: Could you do that in your head?
S: Yes, I'd do it like I was writing it in my head.

There was no evidence to suggest this student could successfully complete the division item in her head but the comment 'writing it in my head' was interesting. Several students described mental methods based on the use of the paper-and-pencil approach that they manipulated in their heads. It appeared that one of the reasons children chose not to use a mental method was they did not posses appropriate, efficient mental methods

In the following example the student used the 'big number' reason in conjunction with a comment about the amount of time the item would take to complete on paper to explain why he opted to use a calculator. The inflection in his voice appeared to suggest that speed was also a factor:

I: 1000 x 945
S: 945 000
I: Okay, good. I noticed that you used a calculator, why was that?
S: *Big sum.*

I: Could you do it any other way?
S: On paper, but it would take forever.
I: Any other way?
S: No.

In the following example the student also gave speed and the size of the numbers as reasons for using a calculator.

I: 1000 x 945?
S: 945 000
I: Why did you use a calculator?
S: I would definitely use a calculator because it takes too long on the paper, it's pretty big.
I: Could you do it mentally?
S: No

Expressions like 'its faster', 'its quicker' or 'it's quick and easy' were often given in support of calculator use. Speed was also cited as a reason for not using the written algorithm. Comments such as 'it would take longer' and it would take too long, were commonly used to explain why the written algorithm was not favoured.

Specific operations

Reference was made to specific operations such as multiplication and division. Students commented on 'table drills' when giving reasons both for using mental methods and not calculator options. If students were good at 'tables' then mental methods were more likely to be attempted but if they knew they had difficulties with tables then they would be less likely to choose mental methods and possibly opt for the calculator. The division operation tended to prompt the use of methods other than mental.

The teacher

The influence of the teacher and the teaching of standard written algorithms as a restriction to selecting the calculator may be noted in the following extract:

I: Alright but you prefer to write it down. Is there a reason for that?
S: I just was taught to do it and I've always been doing it that way. Its easier because when I do it on the calculator I think its not using your brain and you don't work things out better and you don't get smarter, so I write it down because its easier and I understand better.

Other students in the study expressed similar sentiments. This result concurs with the findings of Price (1997) where the influence of the teacher accounted for computational choice in paper-and-pencil methods.

Heavy emphasis on the four operations and correct setting out appears to influence the thinking of some children. In many classrooms the teacher will demonstrate a set method for performing a particular calculation. The 'teachers method' then tends to be adopted by default especially when the teacher states that the children should show all their working.

No alternatives

When the responses given by the children are looked at more closely it may be seen that the students did not simply select a calculator without thinking. In fact, they often thought about at least one alternative approach. Students who suggested that it would take a long time to complete the question on paper were correct in their thinking. Using paper-and-pencil methods to multiply 1000 x 945 is rather tedious and inefficient. A much better alternative would be to complete the computation mentally. Unfortunately many children in the study did not have efficient mental methods at their disposal. Many students who did try using a mental method attempted to use a strategy they did not understand. This reason highlighted a key reason for children choosing to use a calculator–that was they did not have a viable alternative.

The examples given above, while not typical, were sufficiently common to report here. Opponents of calculator technology use in primary school could point to these examples as a result of calculator technology being introduced into primary classrooms in the eighties and nineties. A different explanation, however, suggests that the problem lies not in the use of calculators but rather in the lack of alternative calculating tools and strategies available for children to make appropriate choices. Many children simply are not making sense of the numbers in front of them.

Making better use of calculators rather than more use

Calculator technology can be used in the classroom in many different ways – some good, some bad. Dick (1988) noted that "The effect of calculator use on the acquisition of basic skills has always been the main disagreement between the two sides [pro and anti calculator] (p. 38). He also acknowledged "Students blindly punching buttons while using neither estimation nor number sense in judging the results" is not a desireable outcome of the use of calculator technology in the primary classroom. He also believed that teachers have a role to play in helping children make better and possibly less, but better use of calculators in the classroom.

Several authors (Bobis, 1997; Rousham & Rowland, 1997) have suggested that one way to improve computational choice and hence make better use of calculator technology is for teachers to encourage more calculator use. Calculator use in this instance does not refer to completing long lists of sums or trivial calculators but rather making use of the calculator to explore numbers and to learn how numbers and calculators work.

Calculator technology is not used much in primary school as one might imagine, and, often when the calculator is used it is for trivial calculations (SCAA, 1997; Sparrow & Swan, 1997). McIntosh (1990) noted that "some may allow it for checking calculations and a few for some games of making words by turning the calculator display upside-down but by and large it is shunned as a dangerous and potentially debilitating beast" (p. 25). Evidence from the above reports suggests that the use of calculators in primary schools is often not accompanied by the development of number sense and the mental strategies necessary for sensible and informed computational choice.

Implications

Where there has been an introduction of calculators into the classroom there does appear to be some effect on computational choices made by children. In some cases, however, primary children are making inappropriate choices. Clearly there is often a novelty effect associated with the introduction of calculators where children use them for every calculation because they are new and exciting. This may be overcome by adopting the use of calculator at an early age and allowing easy access to calculators in the classroom so that the novelty is quickly passed. It is overly simplistic to suggest that banning calculators, thereby denying children access to technology, will solve the computation problems experienced by primary children.

Teachers often have not been supported in their efforts to integrate calculators into their teaching of mathematics. Therefore they have tended to relegate the use of calculators to trivial things like making words, as they know of no better tasks. Probably the greatest impact of the CAN and Calculators in Primary Mathematics (CPM) (Groves & Stacey, 1998) projects was the change in teaching style they generated and supported. The teachers in the CAN project found that "the use of calculators encouraged CAN children to talk more about the mathematics they are doing." (PrIME, 1989, p.22). The teachers also reported that they needed to move away from traditional styles of teaching and adopt more flexible approaches. While it cannot categorically be stated that the calculator was the catalyst for the change in teaching style it certainly caused teachers to reexamine their approach to teaching primary mathematics (PrIME, 1989). Teachers need further direction on how calculator technology may be implemented into the classroom. The perception by some people is that calculator technology is the cause of many problems children have with poor computation. This is not the case. Duffin (1994) noted that "the calculator does not inhibit thinking; it only does so when it is introduced without care to use it properly" (p.28). Unfortunately Duffin does not go on to describe how to 'introduce calculator technology with care'. Non-sensible use of paper-and-pencil algorithms has been noted in a similar way (Hope, 1986) so it is overly simplistic to suggest that the introduction of calculator technology has brought about a reduction in number sense and appropriate computational choice. Usiskin (1998) noted:

> . . .because algorithms are reliable if done correctly, answers are often blindly accepted. This is true regardless of whether one does the process mentally, with paper and pencil, or with technology ... students should have checks (p. 15).

The implication of this statement and the findings of the projects described is that teachers should be helping children not only make sensible computational choices but they should also be equipping them with strategies to check whether the strategy they employ produces a reasonable result. If students are to become sensible users of calculator technology then it would seem more appropriate that teachers make the criteria for calculator use explicit and allow children to make the computational choice based on these criteria. They should be using calculators to help children develop connections between numbers and establish a range of mental strategies that children can adopt at appropriate places. Calculator technology can help children make sensible choices when calculating.

It is overly simplistic to suggest that banning calculators, thereby denying children access to technology, will solve the computation problems experienced by primary children. Duffin (1996) noted that what should be banned is "bad use of calculators" (p. 47). Calculator technology is here to stay, what is needed is mathematics teaching that acknowledges good use of calculator technology to support sensible choice in computation. This will only be achieved if teachers are also supported so they can help children make sensible use of calculator technology.

References

Australian Education Council (1991). *A National Statement on Mathematics for Australian Schools.* Melbourne: Curriculum Corporation.

Bobis, J. (1997). Calculators: They can't do mathematics, but they can count. In G. Booker, J., Briggs, D. Bond & G. Davey (Eds.). *Teaching Primary Mathematics, 2nd Edition* (pp. 130–131). Melbourne: Longman.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology, 3*, 21–29.

Curriculum Council, (1998). *Curriculum Framework for Western Australian Schools*. Perth: Author

Dick, T. (1988). The continuing calculator controversy. *Arithmetic Teacher, 35*(8), 37–41.

Duffin, J. (1991). Written Algorithms. *Mathematics in School, 20*(4), 41–43.

Duffin, J. (1994). Calculators: Do they inhibit or enhance mathematical thinking? *Micromath, 10*(3), 25–28.

Duffin, J. (1996). Opinion, calculator imbalance. *Micromath, 12*(2), 46–47.

Duffin, J. (1997). The role of calculators. In I. Thompson (Ed.), *Teaching and Learning Early Number*. Great Britain: Open University Press.

Groves, S. (1994). *Calculators: A learning environment to promote number sense*. Paper presented at the American Educational Research Association. New Orleans

Groves, S. & Stacey, K. (1998). Calculators in primary mathematics. In L. Morrow and M. Kenney (Eds.). *The Teaching and Learning of Algorithms in School Mathematics*. Reston, Virginia: NCTM

Hope, J. A. (1986). Mental calculation: Anachronism or basic skill? In H. L. Schoen & M. J. Zweng (Eds.), *Estimation and Mental Computation* (pp. 45–54). Reston, VA: NCTM.

McIntosh, A. (1990). Becoming numerate: Developing number sense. In S. Willis (Ed.), *Being numerate: What counts?* (pp. 24– 43). Victoria: The Australian Council for Educational Research.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School mathematics*. Reston, Va: NCTM.

Porter, A. (1988). *A curriculum out of balance: The case of elementary school mathematics*. (Institute for Research on Teaching Research Series, No. 191). East Lansing, MI: Michigan State University.

Price, P. (1997). Teacher presence as a variable in research into students' mathematical decision making. In F. Biddulph & K. Carr (Eds.). *People in Mathematics Education*. (pp. 422–428). Waikato, NZ: MERGA.

Primary Initiatives in Mathematics Education (PrIME) Project. (1989). *The second year of CAN*. Homerton College Cambridge: National Curriculum Council.

Reys, B., Reys, R., & Hope, J. (1993). Mental computation: A snapshot of second, fifth and seventh grade student performance. *School Science and Mathematics, 93*(6), 306–315.

Rousham, L., & Rowland, T. (1997). Numeracy and calculators. In R. Mertens (Ed.) *Teaching numeracy in the primary classroom* (pp. 61–77). Leamington Spa: Scholastic.

SCAA (1997). *Discussion paper, no.9. The use of calculators at key stages 1 – 3*. London: School Curriculum and Assessment Authority.

Shuard, H., Walsh, A., Goodwin, J., & Worcester, V. (1991). *Calculators, Children and Mathematics*. London: Simon and Schuster.

Sparrow, L. & Swan, P. (1997). Should all children use calculators?. In F. Biddulph & K. Carr (Eds.), *People in mathematics education* (pp. 464–469). Waikato: MERGA.

Sparrow, L., & Swan, P. (1999). *Do children really have a choice when calculating?* Paper presented at The 22nd annual conference of the Mathematics Education Research Group of Australasia, Adelaide.

Swan, P., & Bana, J. (1999). Choices of computational strategies. In J. Truran, & K. Truran (Eds.). *Making the difference*. Adelaide: Mathematics Education Research Group of Australasia.

Swan, P., & Bana, J. (2000). Computational choice: The reasons behind the choices. In J. Bana, & J. Malone (Eds.). *Mathematics Beyond 2000*. Fremantle: Mathematics Education Research Group of Australasia.

Usiskin, Z. (1998). Paper–and–Pencil Algorithms in a Calculator–and–Computer Age. In L. Morrow & M. Kenney (Eds.). *The Teaching and Learning of Algorithms in School Mathematics.* Reston, Virginia: NCTM.

Willis, S., & Kissane, B. (1989). Computing technology in teacher education in mathematics. In *Department of Eemployment, Education and Training, Discipline Review of Teacher Education in Mathematics and Science, Volume 3, Appendixes.* (pp. 57–92). Canberra: Australian Government Publishing Service.

TEACHING PRIMARY MATHEMATICS IN HONG KONG: FROM TRADITION TO VORTALS

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Internet resources are useful for teachers, but few people pay attention to assisting primary mathematics teachers to find relevant resources via Internet. Although a general search engine can help us locate related curriculum resources on the Internet, it is very time-consuming to inspect every possible web sites. The major purpose of this paper is to describe a potential vortal site for helping primary mathematics teachers to search useful teaching resources effectively via linking relevant web sites. This paper attempts to describe the rationales of developing this vortal site. Hence some important principles for designing vortal site are recommended for readers' reference.

Introduction

In the 1960-1970s, teacher-centered learning was commonly used and accepted in local primary mathematics teaching. The pupils were only required to look at the teachers and listen carefully during the lessons (Wong & Lai, 2000). In lessons, because of the big class size and limit of teaching resources, most mathematics teachers only relied on "Talk and Chalk" and "Lecture Method" as their teaching strategies. Nearly all teaching contents were textbook oriented, quite conventional and boring, not creative at all. Therefore, during lessons, pupils did not have many opportunities to speak, to discuss or to express themselves. The communications among teachers and pupils were not interactive and only within the school context. In Hong Kong, the past decade has produced numerous changes in the fields of politics, economic, technology, community and education. Due to these changes and also because of the advancement of information technology (IT), schools must instill their pupils with specific knowledge, value of education and new way of communication. For educational reform, task-based learning approach was introduced in 1990. Primary teachers also requested to use IT for enhancing teaching and learning. Task-based learning is learner-centered, completely different from traditional teacher-centered learning. Wong and Lai (2000) also found that IT can be used to facilitate the task-based teaching-learning process. Actually, IT is considered as a powerful educational tool, resource and can play a catalyst role in facilitating the paradigm shift – from a largely textbook-based teacher-centred approach to a more interactive and learner-centred approach (Education and Manpower Bureau, 1998).

From classrooms to the internet

It is commonly believed that information technology (IT) has the potential to liven up classroom life by making teaching and learning more dynamic, interactive and innovative (Education and Manpower Bureau, 1998). Computer technology has dramatically expanded the scope, range, and format of information and communications with a flexible time and place component (ITiCSE'97 working Group on CMC in Collaborative Educational Settings, 1997). Common computer-mediated communication (CMC) tools such as email, newsgroups, WWW allow users to exchange information without the limitations of time and space. Therefore learning and teaching should not be confined in a small classroom, within schools. In fact, new communications technology is bringing our lesson out of a classroom to the Internet.

From resources sharing to collaborative learning, the Internet technology is quickly transforming the way in which education is delivered in schools.

From personal web pages to portal sites

The Web is changing the world of mathematics. It has changed how we can learn, how we can communicate, how we can publish, how we can present ourselves, how we can teach. (Klotz, 1997). Many mathematics educators have created web sites for publishing their teaching notes, lesson plans, teaching aids, software, interesting ideas, pretty proof and things that they want to share with others. For example, the "Cheng Wing Kuen's Home Page"(http://www.netvigator.com/~wingkei9) is a typical local web site which focuses on resources for teaching secondary mathematics. This is a personal web site maintained by a secondary mathematics teacher. One more example, some mathematics teachers in a local primary school build another popular mathematics web page, (http://www.plklht.school.net.hk), they hope to use interactive online mathematics games to make learning more interesting. Although these web pages can help teachers share their teaching resources or let pupils learn at their own pace at anywhere and anytime, they face the problem of limited manpower for site maintenance. This explains why most personal pages are not updated frequently and the contents are insufficient for classroom use.

Indeed more comprehensive sites can only be well maintained when having adequate financial supports. For example, larger educational web sites such as Mathematics Net (http://www.edp.ust.hk/math/), Hong Kong Cyber Campus (http://www.hkcampus.net), IT in Education Network (http://itied.ied.edu.hk) are all supported by Quality Education Fund. We can see that these web sites share a common vision. They tend to entertain a wide range of users. For example, the target audience of Mathematics Net are from teachers and students to all people interested in mathematics. In other words, these web sties aim to become an educational portal.

The portal concept is pioneered by *Yahoo*. *Yahoo* has attached a lot of popular Web applications to its basic Web directory, including free email services, shopping, personal finance data, maps, calendar, weather, news, instant messaging, and customization, and so. O'Leary (2000) pointed out that the portal concept has been quickly adopted by all of the other principal Web directories and it has also been rapidly spreading to many subject-oriented Web sites.

From horizontal portals to vertical portals

Portal sites such as *Yahoo* and those mentioned above are referred as *horizontal portals*. Initially, horizontal portals are attractive as they provide a good starting point and massive resources for mathematics teachers. However, Ward (1999) warned us that, for some people, horizontal portals are confusing, cluttered, and full of irrelevant content. Besides, if you use a search engine to carry out a search on "fraction", more than tens of different variant response will be generated. Some are in primary level, some in secondary level and even some are in tertiary level. In such situation, you need to spend a lot of time on filtering irrelevant information. Obviously, this problem is due to the lack of focus of horizontal portals.

To solve the problem, web experts quickly go to another direction. Rather than focusing on all subject areas for all audiences, the portals should be specialized (Notess, 2000). These sites are referred as vertical portals. They also invented the word "vortal", a contraction of vertical portal in 1999. Tom (2000) explained that "vortal" refers to a web site that aggregates disparate content and services of interest to a particular industry and makes it available to industry members. Peek (1999) pointed out that what distinguishes a vortal from a portal is its clearly defined focus, and community may be the attribute that distinguishes vortals from mere Web sites. O'Leary (2000) observed that vortals are also defined by the presence of the "three C's: Content, Community, and Commerce". In other words, vortals are much narrower in focus, but they tend to be more useful (Ward, 1999).

A vertical portal for local primary mathematics teachers

Vortals are quickly becoming an important source of business, professional and technical information, and tightly focused on a specific topic. It seems to be a new direction for mathematics web sites. As vortals are much narrower in focus, busy teachers can save a lot of time on filtering irrelevant masses of information and get appropriate contents quickly. A vortal also provides opportunity for people to form an online community. This encourages job alike or same category mathematics teachers to share their experiences and ideas with each other. Although the attribute *Commerce* cannot be applied to educational web sites directly, it will make them more attractive if primary mathematics teachers can find convenient access to suppliers in their special interests such as textbook publishers, teaching aids suppliers etc. Since learning is the end at which teaching aims, it seems appropriate to have more choices of teaching modals and teaching materials for catering different learning abilities pupils. Thus, for the purpose of enhancing mathematics teaching, we think that it is meaningful to build vortals for mathematics teachers. We can foresee that mathematics teachers will spend more time on surfing vortals than holding a chalk in the future. Nowadays, IT is a feature of our age. Not only teachers and students, nearly all people are aware that web pages can help people share their experience, can help people search their required information via Internet. For these reasons, IT cannot be ignored in schools.

Designing vertical protal for primary mathematics teachers: Some principles

To explore the usefulness of portals in mathematics teaching, we have started a pilot project on building a protal for local primary mathematics teachers. In designing the web pages, we identify the following principles:

1. The presentation should be clean, attractive, simple and straightforward.Therefore the web users can go to a specific domain directly from the main page.

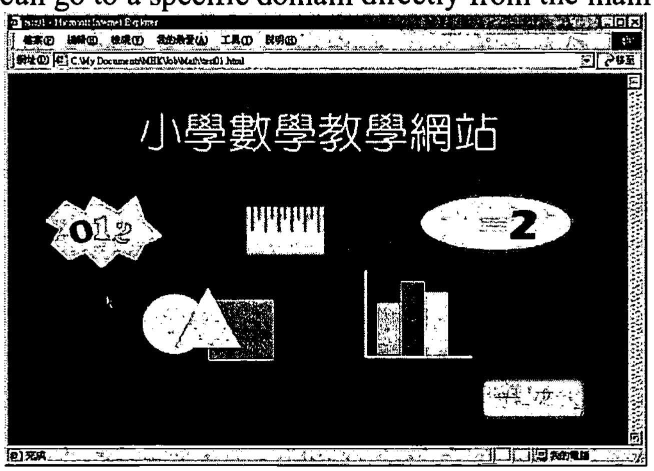

Figure1. The main page of the vortal site

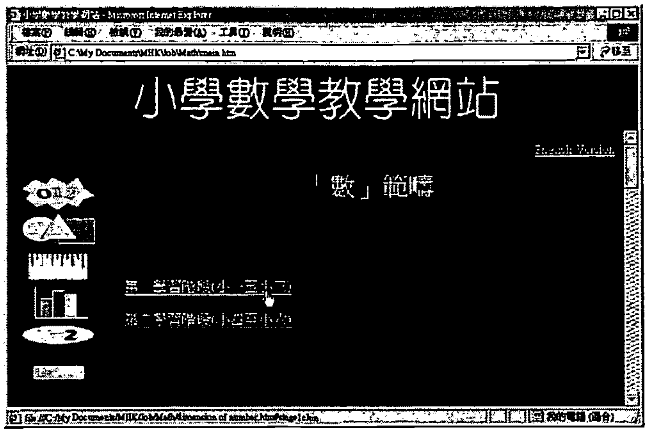

Figure 2. The page about numbers

2. *No search engine should be added.*

   Peek (1999) points out that unlike portals, vortals do not require a search engine to the outside world. It means that the contents should be well structured and straightforward as stated above.

3. *The contents only focus on primary mathematics teaching.*

   We should keep in mind that too much irrelevant information would discourage the teachers to visit the site again.

4. *A learning community should be built on the vortal.*

   Shrivastava(1999) stated that online learning communities provide a basis for anytime, anywhere learning, life-long learning, and workplace-focus learning. An online learning community can help primary teachers share their ideas and faces the changes under the impacts of education reform, language benchmarks, use of IT and so. Therefore the vortal should allow the teachers to communicate and collaborate through the use of email, bulletin boards, discussion groups, and chat rooms.

5. *Interactive web-based learning materials should be provided*

   Klotz(1997) predicted that using interactive web-based materials is the wave of the future. He has linked some live examples on learning mathematics with *Geometer's JavaSketchpad* and *Mathview* from his online article. Besides, instructional modules on the WWW can engage the user in dynamic activities and cause deeper cognitive processing of the content (Spikell, Aghevli, Bannan-Ritland & Egerton, 1998).

Conclusion

As CDC (1999, P. 6) has stated

> The culture of over-reliance on textbooks as the main teaching / learning resources will be changed through greater emphasis on inquiry learning and the introduction of diversified learning resources. The change is essential for providing the appropriate contexts for learning as well as meeting the changing needs of society and nature of knowledge for the real lifelong learning.

We can predict that vortals will become more and more important resources for every imaginable teaching area. You should not be surprised if you see some vortals called *PrimaryMathTeachers.com*, *TechnologyInPrimaryMathTeaching.org* on the Internet. However, the site management team should be careful if public funding supports the site. A vortal should serve for a target community in society and not your own institution. Moreover, as DiLima (1999) said, classroom learning will not disappear but increasing numbers will choose the Internet to complement traditional learning. Vortal is a useful teaching and learning tool and it only serves as a facilitator in the teaching and learning process, it can't be used to replace mathematics teachers.

References

Curriculum Development Council, Hong Kong (1999). *A Holistic Review of the Hong Kong Curriculum: Proposed Reforms*. Hong Kong: The Government Printer.

DeLima, F. (1999). Web–based learning more cost effective. *Computing Canada*, 25(27), 29.

Education and Manpower Bureau (1998). *Information technology for learning in a new era: Five–year strategy 1998/99 to 2002/03*. Hong Kong: The Government Printer.

ITiCSE'97 Working Group on CMC in Collaborative Educational Settings (1997). Computer–mediated communication in collaborative educational settings. *SIGCUE Outlook*, 26(4), 41–56.

Klotz, G. (1997). *Mathematics and the World Wide Web*. Retrieved May 9, 2000 from the World Wide Web: http://mathforum.com/articles/epadel/

Notess, G. R. (2000). On the net in 2000. *Online*, 24(1), 71–73.

O'Leary, M. (2000). Vortals on the rise. *Online*, 24(2), 79–80.

Peek, R. (1999). When is a Web site a 'vortal'? *Information Today*, 6(8), 36–37.

Spikell, M., Aghevli, B., Bannan–Ritland, B., & Egerton, E. O. (1998). *Dr. Super's Virtual Manipulatives Project: An Innovative web–based instructional strategy.* Retrieved May 25, 2000 from the World Wide Web: http://www.coe.uh.edu/insite/elec_pub/HTML1998/ma_spik.htm.

Spitzer, T. (2000). Vertical horizon. *Web Techniques*, 5(2), 50–56.

Ward, E. (1999). Find or make your portal now. *Advertising Age's Business Marketing*, 84(12), 40.

CHOOSING MODELS TO FIT THE DATA AND THE SITUATION

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Whilst it is tempting to take advantage of regression models, great care must be taken before deciding that a function model is appropriate to a given set of data. For example, how well does the model fit the situation, and how do we check for patterns in the data? This paper contends that knowledge of the underlying mathematical principles and the context of the problems being studied is required before students can sensibly decide between competing function models.

Introduction

The increased availability of hand-held and portable technology in the secondary mathematics classroom has contributed to an increased use of data-driven approaches in the study of algebra. One such approach is the use of functional models to explain the relationship between two variables. Tasks of this sort often follow this structure:

- Enter data collected or supplied for the two relevant variables;
- Create a scatterplot for the data;
- Find a functional model that fits the data well, either by guess/check/improve or by an automated method;
- Answer questions based on extrapolation and interpolation of the model;
- Comment on the limitations of the model.

While this format provides a refreshing alternative to tasks where a function rule is given in advance or is deduced from the situation, there is a temptation to rely on automated features such as regression at too early a stage as a quick and dirty method to find a rule which fits the data well.

Fitting a regression model – a quick and dirty method?

As an example of the ease with which models can be fitted with technology, consider the data set in Figure 1, which is collected from a simulation of throwing a ball with various speeds and measuring the horizontal range that the ball travels.

Speed (m)	25	30	35	40	45	50	55	60
Range (m)	63.8	91.8	125.0	163.3	206.6	255.1	308.7	367.3

Figure 1. Range of throw for various initial throwing speeds

It appears that the linear model fits the *data* well (see Figure 2), but does it fit the *situation*? Why wouldn't a quadratic, exponential or trigonometric model be more appropriate?

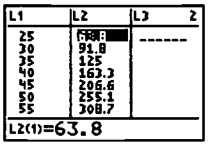
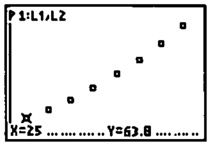
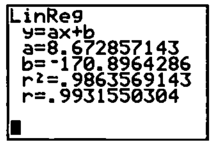
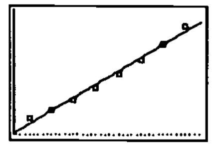
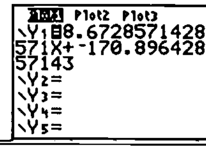
Enter the data	*Create Scatter Plot*	*Select regression rule*	*Draw rule*
			

Figure 2. Quick and dirty procedure for applying a least squares regression line of the form $y = ax + b$ to the data

There does not appear to be any research at the school level about the impact of automatic or 'black-box' regression procedures on student understanding of the concepts related to model fitting. As with other forms of technology, it is almost impossible to stop students using such tools. It is therefore incumbent on us to consider a range of related issues including the following.

- Are students mathematically ready to use such features?
  - Have they any understanding of the principles underpinning least squares regression?
  - Can they adequately interpret the values of r and r^2?
- On what basis should you choose one regression model over another?
  - What clues does the data provide about which model (if any) is appropriate?
  - What clues does the situation provide about which model may be appropriate?
- Does our assessment permit/encourage the use of such features?
  - Do the questions we ask encourage students to think critically about fitted models – or are students just asked to use the model they have chosen?

There is some evidence that student understanding of functions benefits from by-hand construction of graphs. Some initial focus on the underlying principles in the construction of computer or calculator generated function graphs is also useful (Tynan & Dowsey, 1997). With this in mind, what experiences do students need before being 'rewarded' with the use of automated curve fitting methods such as regression?

Clues revealed by the data

There are some methods that we can use to help students become more aware of clues that the data reveal about what models may be appropriate.

For example, what does a data set with a linear relationship look like? What are the clues in the following data set (Figure 3) that the relationship between x and y is linear?

x	1	2	3	4	5	6	7	8
y	11	14	17	20	23	26	29	32

Figure 3. A textbook linear data set?

A student who says that it is linear because "when x goes up by 1, y goes up by 3" seems on the way to understanding that linearity occurs when a constant change in one variable is associated with a constant change in the other variable.

Mathematics texts often present students with sanitised versions of x-y data sets, with the x data nicely ordered, and they usually restrict examples to perfect linear cases. It is therefore not unusual that many students will suggest linear models as appropriate.

What if the data is subject to error (due to random fluctuation or imperfect measurement) and not perfect? In such cases, it is more difficult for students to determine the best model.

Consider the ball throw data displayed in Figure 1. It seems that as the speed increases the range also increases – but is it linear? By tabulating 'Change in speed' versus 'Change in range', as seen in Figure 4, students can explore the data further.

Change in speed	5	5	5	5	5	5	5
Change in range	28.0	33.2	38.3	43.3	48.5	53.6	58.6

Figure 4. A table of the '1st order differences' for the ball throw data

Although the change in speed is constant, the change in range is not and so linearity is not present. So despite the apparent goodness of fit of the linear regression, this simple analysis shows that a linear model is not so good. [An alternative approach, if linear regression is used, is to draw a scatterplot of residuals to determine whether they are random or patterned.]

If we look more closely at the change in range, we should notice that successive differ by an amount which is roughly constant (5.2, 5.1, 5.0, 5.2, 5.1, 5.0) – these are the 2nd order differences. What does this tell us about a possible alternative model?

Problems like the above can lead to an informal introduction to the use of nth order differences as indicators of a nth-order polynomial model.

What if there is an exponential relationship between two variables – what hints are there in the data about this? Looking for constant ratio between successive values of the dependent variable will help in this case.

The procedures mentioned above (regression, finite differences, successor ratios) are available via a graphics calculator, either as native features or as extended features via a program. In this context, the program AMODEL (written by the authors) provides an array of tools for checking data for patterns and possible relationships. It also allows the user to apply transformations that may linearise the data (for example, square or log transformations) and to plot the resulting fits (see Figure 5).

Main menu — *Fit Polynomial menu* — *nth order differences* — *Plot of L2 vs L1^2*

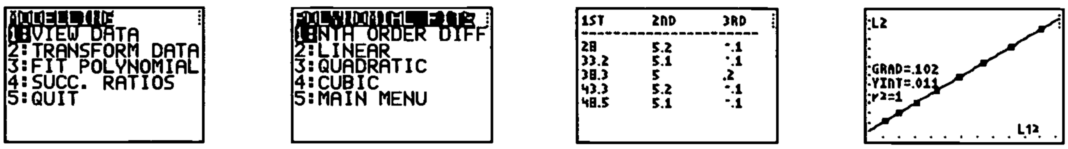


Figure 5. Using the AMODEL program to examine the ball throwing data

Understanding least squares regression

Before using least squares regression, students should be encouraged to understand the basic principle by which it operates. When used to fit a linear model, the method aims to find the straight line for which the sum of the squares of the vertical deviations is a minimum.

A simple spreadsheet (see Figure 6) can be set up to illustrate the basic principles. Student attention can be focused on:

- trying various values of m and c to help visualise which line is 'closest' to the data;
- the reason for using the sum of the square deviations rather than the sum of the deviations;
- the impact of outliers on the suitability of the method.

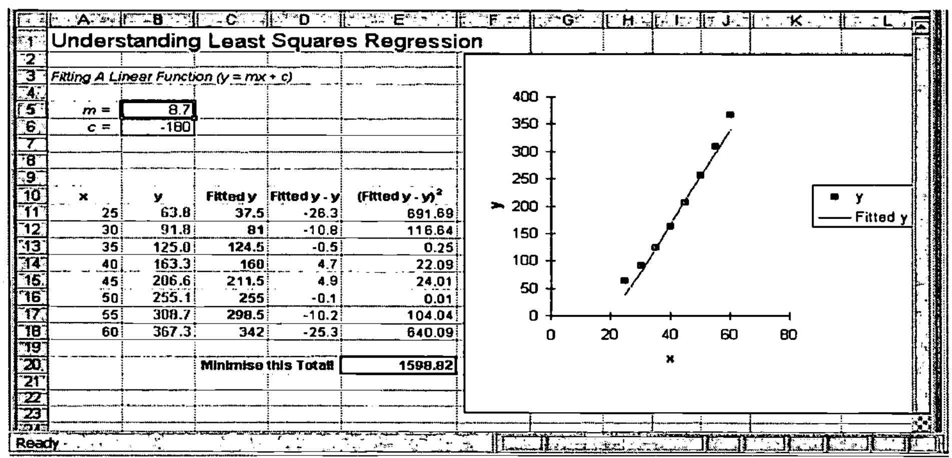


Figure 6. Using a spreadsheet template to illustrate least squares regression

A graphics calculator program can also be used to develop the principles. Figure 7 illustrates how the program LSREGR can be used by students in a search for the m and c values which minimise the sum of the squared deviations. Note that, like the spreadsheet, it provides both visual and numeric feedback to students about their estimates.

LSREGR main menu | *Trying a fit* | *The result*

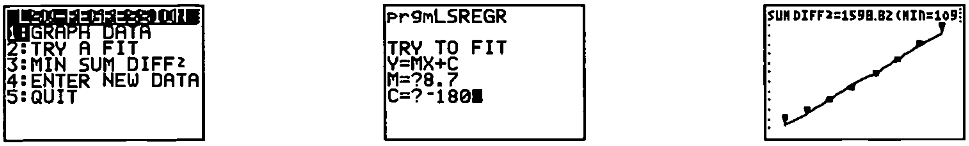


Figure 7. Using the LSREGR program to illustrate the principle of least squares regression

Fitting the data to the situation

Even when students are ready to use regression as an automated procedure, they need to need to appreciate that care must be taken when applying it. Consider the following problem.

The Cooling Law Problem

A temperature probe is placed in a saucepan filled with hot water and the temperature is recorded at 30 second intervals for 5 minutes. The ambient room temperature is 18.5°. Plot the data and try to fit a rule that will successfully model the temperature of the water after 30 minutes, justifying your choice.

Time (t)	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
Temperature (T)	98	91.5	87.5	84	80.5	78	75.5	73	71	69.5	68

Figure 8. Temperature vs. Time data for water in a cooling saucepan

Various regression models can be tried (see Figure 9) and evaluated. Here the fits are shown for the data provided (Automatic Window) and an expanded window extrapolating to $t = 30$. A reference line at T=18.5 is added in the latter cases (recall that 18.5° is the ambient temperature).

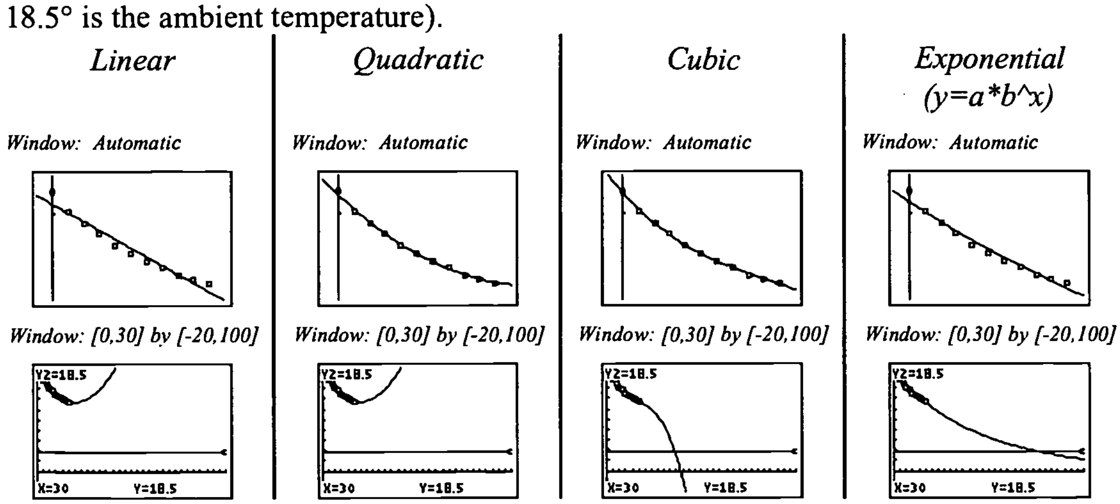


Figure 9. Regression models which fit the data, but do they fit the situation?

All of these models predict that the water will either fall below the ambient temperature, or re-boil! The poor fit to the situation raises one of the concerning aspects of using regression models mindlessly. Thus for example, even though an exponential relationship seems plausible, the suggested one has limited predictive power. Indeed, because the exponential regression procedure is fitting a curve of the form $y = ab^x$, it is not surprising that this curve is asymptotic to $T = 0$.

The *situation* suggests that using the *excess* temperatures (that is, the difference between each temperature and the ambient temperature) is likely to lead to a better model. The calculator lists can be readily used to transform T values to (T-18.5) values, and the exponential regression model can be reapplied. While the resulting fit to the data is no better than many of the earlier fits, this model fits the situation as well and can be usefully employed for predictive purposes.

The lesson from the above examples and discussion suggests that the same care must be taken when using automatic regression features as is taken with other technology based tools or software. Simply knowing how to apply a regression procedure is not enough. A knowledge of the underlying mathematical principles and the context of the problems being studied is required to allow an appropriate analysis.

Reference

Dowsey, J., & Tynan, D. (1997). The VCE Graphics Calculator Project – Reflections on a statewide professional development project. In N. Scott, & H. Hollingsworth, (Eds.) *Mathematics – Creating the Future.* (pp. 335–344). Proceedings of the 16th Biennial Conference of the Australian Association of Mathematics Teachers (Melbourne), AAMT.

Note - programs illustrated here are available from http://www.edfac.unimelb.edu.au/DSME/TAME/

MECHANICAL LINKAGES, DYNAMIC GEOMETRY SOFTWARE AND MATHEMATICAL PROOF

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Mechanical linkages abound in everyday objects as well as in historical curve-drawing mechanisms. Understanding the operation of these linkages, aided by dynamic geometry software models, provides motivation for mathematical proof. Such study also creates opportunities for re-forging links between algebra and geometry, and remedying the demise of geometry and visualisation in mathematics education over recent decades.

Dynamic geometry and proof

The development and increasing use of dynamic geometry software, such as Cabri Geometry II™ and The Geometer's Sketchpad®, has led to much debate about the role of proof in school mathematics. De Villiers (1998) maintains that in a traditional approach to the teaching of geometric proof, doubt is created in the minds of students in order to justify and motivate the need for proof. Such attempts at creating doubt include illustrations of pattern failure, optical illusions and false conclusions. He suggests that, in contrast, dynamic geometric software may be used to arouse students' curiosity, motivating them to ask why. Proof thus becomes an act of explanation rather than verification. De Villiers also notes that for mathematicians, proof is often a means of exploration, analysis and further discovery. Scher (1999, p. 24) believes that "dynamic geometry can provide not only *data* to feed a conjecture, but *ideas* to feed a proof."

Visualisation and mathematics

Historically, the concept of function developed from the creation of a curve from a geometric or physical action, for example, a mechanical linkage. Only after its construction was the curve analysed by means of its geometric properties to produce an algebraic representation. Descartes (1637), for example, used hinged linkages to construct curves, then, by analysis of the geometry of the construction, he was able to derive an equation relating pairs of coordinates.

Figure 1a shows Descartes' hinged mechanism for a hyperbola. The mechanism has been reproduced in Cabri (Figure 1b), with parameters a, b and c and coordinates x and y as designated by Descartes. The reversed x and y coordinates, and positive y-direction, in Descartes' diagram pre-date the present-day Cartesian convention and provide students with an interesting historical link.

Using similar triangles CBL/GAL and NLK/CBK, together with the fact that AL = AB + BL = x + BL, Descartes derived the equation $y^2 = cy - \frac{cxy}{b} + ay - ac$, i.e. $x = \frac{b}{c}((a+c) - y - \frac{ac}{y})$, for the hyperbola. In the Cabri construction, a, b and c can be easily varied to observe their influence on the graph and *both* halves of the hyperbola are generated. For example, in Figure 2, $a = 5$, $b = 0.5$ and $c = 1$, giving the hyperbola with equation $x = 3 - y - \frac{5}{y}$ (or $y = 3 + x + \frac{5}{x}$ with conventional coordinates). Isoda

(1998) gives Cabri constructions of other curves studied by Descartes.

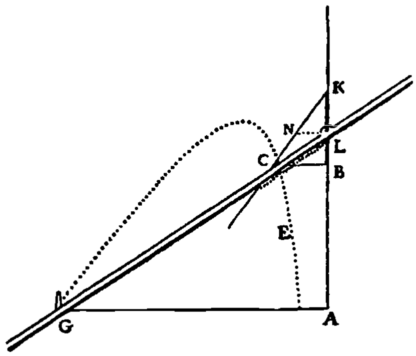


Figure 1a. Descartes' mechanism for construction of a hyperbola. (From Descartes, 1637, p. 52)

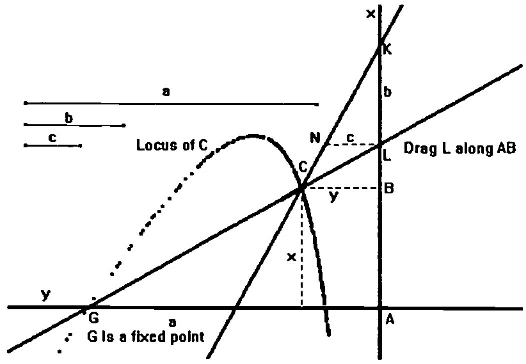


Figure 1b. Cabri construction of Descartes' mechanism

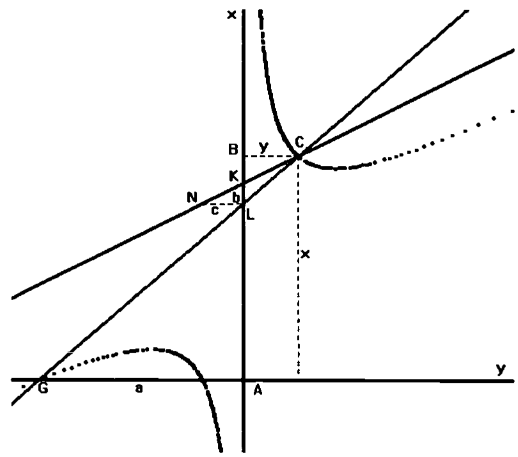


Figure 2. Cabri generation of a full hyperbola

Cunningham (1991) notes that the growth of symbolic, formal mathematics in the late 19^{th} and early 20^{th} centuries resulted in an almost total commitment to symbolic work and a discrediting of visual approaches. In discussing the demise of geometry and visual education over recent decades, Zimmermann and Cunningham (1991, p. 4) suggest that "the intuition which mathematical visualization seeks is not ... a superficial substitute for understanding, but the kind of intuition which penetrates to the heart of an idea", giving depth and meaning to understanding, guiding problem solving and inspiring creative discoveries. They express optimism that the "renaissance of interest in visualization", largely driven by computer graphics, will result in a more balanced view of mathematics.

Goldenberg, Cuoco, and Mark (1998, p. 5) observe that in most mathematics curricula "geometry represents the only visually oriented mathematics that students are offered". They express concern that this visual impoverishment has serious implications for the retention of students in mathematics courses and that

> for some students, a visual approach may be absolutely essential. For such students, including many who consider themselves to be poor at mathematics, visual approaches are *access*. Thus, for many students, visualization and visual thinking serve not only as a potential hook, but also as the first opportunity to participate. (p. 6)

Modelling mechanical linkages with dynamic geometry

A mechanical linkage is a system of hinged rods, which can move in relation to each other or to fixed points. Scher (1999, p. 24) suggests that the modelling and investigation of mechanical linkages with dynamic geometry illustrates "the constant interplay between deductive reasoning and software-supported experimentation". Steeg, Wake, and Williams (1993), Bruce, Wolf, and Sutherland (1996) and Laborde (1995) also discuss the unique capability of dynamic geometry in modelling mechanical linkages. Bartolini Bussi (1993, p. 97) describes "mathematical machines" as lying "in the intersection between the field of mechanical experience and the field of geometrical experience". She contrasts the handling of actual linkages by students in the classroom with a computer environment. Although both situations involve interaction between verbal-logical and visual-image activity, she notes the absence of visual-tactile activity in the computer environment, and suggests the need for research in the role of such activity in advanced geometrical thinking. Hazzan and Goldenberg (1997, p. 264), however, suggest that dynamic geometry gives the user

> a strong feeling of operating a smoothly functioning mechanical device – one in which the mathematical objects behave as if they were physical, obeying the laws of mechanics and conforming to our intuitions about movement in a continuous space.

In addition, constructing dynamic geometry models requires students to analyse the essential geometric properties. Steeg, Wake, and Williams (1993, p. 28) observed that Cabri enabled students

> to visualise the effects of variation of parameters within a mechanism, firstly in a qualitative way and ultimately quantitatively.... Within their mathematical experience pupils benefited greatly from seeing a practical motivation for using geometry, whilst gaining an insight into loci.

Irrespective, though, of whether it is an actual linkage or a dynamic geometry software model, explaining why a linkage functions the way it does can challenge students' understanding of geometric relationships and add meaning to mathematical proof.

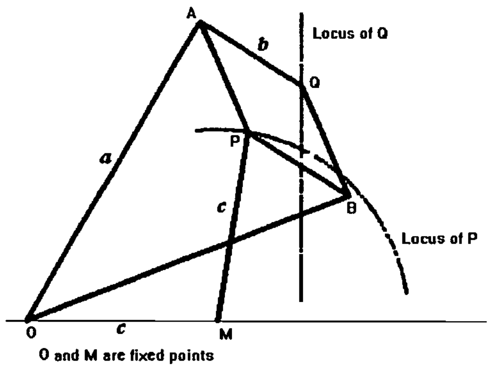


Figure 3a. Construction of Peaucellier's linkage

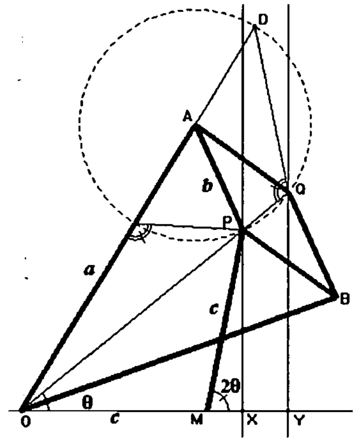


Figure 3b. Extra constructions to assist proof

The flourishing of mechanical invention which occurred as a result of the industrial revolution led to the development of a number of linkages for converting circular motion into linear motion. Figure 3a shows a Cabri construction of Peaucellier's linkage (Hilbert and Cohn-Vossen, 1932) comprising a kite OAQB and a rhombus PAQB, with O and M fixed. As P moves in a circle (centre M, radius c), Q moves on a linear path. While construction of the linkage is relatively simple, more challenging is the proof (with the help of the extra constructions in Figure 3b) that (OP)(OQ) = $a^2 - b^2$, and hence that OY= $\frac{a^2-b^2}{2c}$ — that is, OY is constant. This requires a combination of geometry, trigonometry and algebra knowledge.

Sylvester's linkage for producing linear motion (Bolt, 1991) is based on two similar kites, ACDE and DFBC, with sides in the ratio 2:1. Points O and F are fixed and OABF is a parallelogram. In the Cabri construction in Figure 4, as B is dragged on a circular path about the centre F, E moves on a straight line through F, perpendicular to AC. The proof involves using angle sums to show that EF is perpendicular to OF.

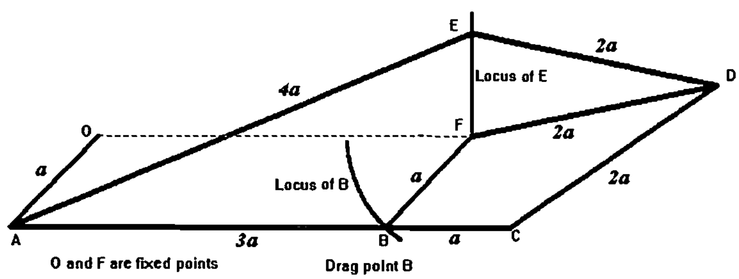


Figure 4. Cabri model of linkage of Sylvester

The rhombus pantograph of Sylvester (Scher, 1999) consists of six equal rods, with angles BAP and BCP' fixed and equal (Figure 5a). The dynamic geometry construction of the linkage enables students to drag point P and observe the invariant relationships: OP = OP' and ∠POP' = ∠BAP (= ∠BCP'). P' traces out the same locus as P, but is rotated through ∠BAP. The proof of these observations can then be generalised to a parallelogram version of the linkage (Figure 5b), as used by Bartolini Bussi (1993) with Year 11 students.

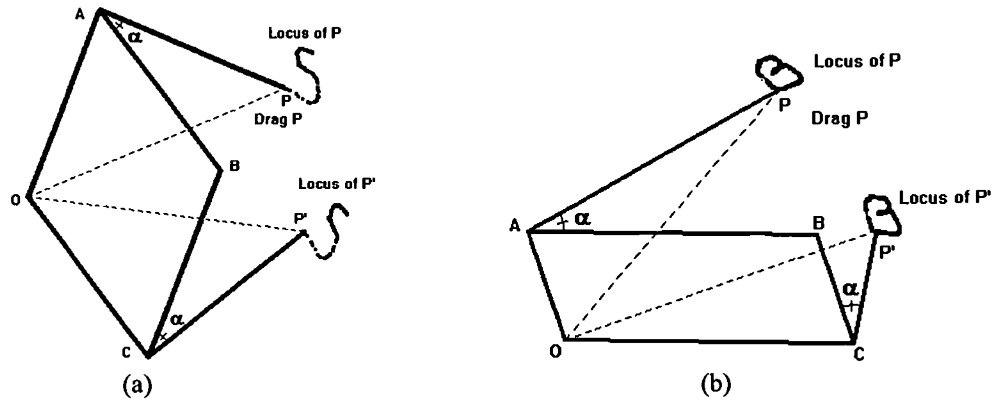


Figure 5. Cabri constructions of two versions of the pantograph of Sylvester

Isoda, Matsuzaki, and Nakajima (1998) suggest that student 'inquiry', involving *both* exploration and proof, can be enhanced if technology is used to provide multiple representations of mechanisms. They describe the study of a crank mechanism (see Figure 6) by year 11 students. Using a LEGO model, the teacher explained the operation of the crank mechanism. The students then were asked to describe, both verbally and graphically, how the length of OA changed when the crank OP was rotating. The majority of students based their response on an image of the motion, suggesting a "wave shape" and predicting circular function behaviour. Data from the LEGO model was then entered into a graphics calculator for curve fitting, with the result that many students abandoned the "wave shape" notion, as the discreteness and inaccuracies of the plotted data tended to obscure the underlying shape. Next, the students developed the following equation to represent the motion: OA = $r\cos\theta + \sqrt{L^2 - r^2\sin^2\theta}$, where OP = r and AP = L. By assigning values to r and L (initially in the ratio 1:2) they were able to fit the graph with this equation to the data. The ratio $r{:}L$ was then varied to observe the effect on the graph of the function. Referring to the actual crank mechanism then helped the students understand the discontinuities that occurred in the graphs for some ratios.

Use of a Sketchpad model (Figure 6) avoids data errors and the need for curve fitting, as well as providing instant feedback as the values of r and L are varied. Bruce, Wolf, and Sutherland (1996) not only examine the motion of a similar crank-slider mechanism using dynamic geometry software, but also use it to model the velocities of the ends of the con-rod AP.

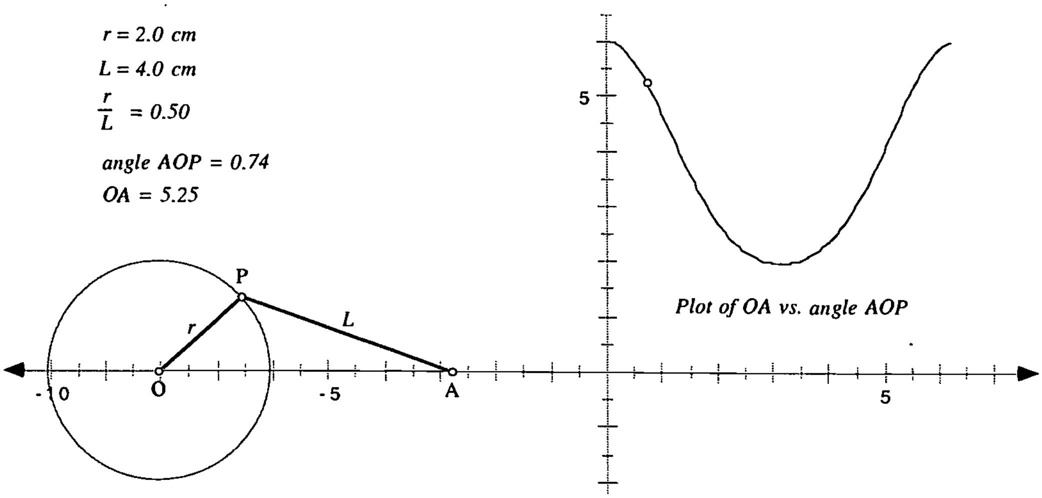
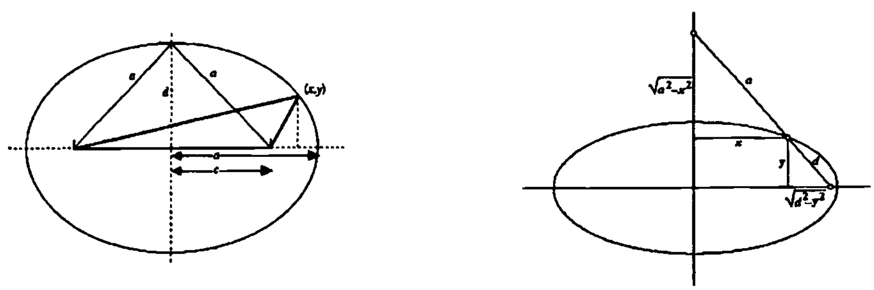


Figure 6. Modelling the crank mechanism with Sketchpad

Dennis and Confrey (1998) explore the linking of algebra and geometry through the use of physical curve-drawing devices. They describe the case of one student, Jim, who "expressed a strong preference for geometry over algebra" and "openly admitted that he easily got lost in algebra and that he found it very boring" (p. 302). Jim's drawings and equations for loop-of-string and trammel ellipse-drawing devices are shown in Figure 7. Jim received a great deal of satisfaction from showing that his two equations were equivalent. His confidence in applying algebra was greatly enhanced by his experience of the simplicity of the equation derivations, prompting him to comment "It makes me feel good to get that" (p. 314). Dennis and Confrey note that "Although Jim disliked algebra, the experience of connecting and confirming geometric experience with algebraic expression was both engaging and satisfying" (p. 316). They suggest that such a geometric approach to functions might change the attitudes of many students toward algebra. The additional involvement of dynamic geometry would provide students with the challenge of producing constructions that replicate the behaviour of the physical devices, as well as offering ready control over parameters such as the string and trammel lengths.



(i) $$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$
i.e. $$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

(ii) $$\frac{a}{d}=\frac{x}{\sqrt{d^2-y^2}}$$
i.e. $$\frac{x^2}{a^2}+\frac{y^2}{d^2}=1$$

Figure 7. Jim's drawings and equations for (i) loop-of-string and (ii) trammel devices (From Dennis and Confrey, 1998, Fig. 12.7, p. 308; Fig. 12.9, p. 310)

Linkages are found in many everyday objects, for example: the parallelogram linkages of expanding tool boxes (Figure 8) and construction equipment (such as a 'cherry picker'), the crossed quadrilateral linkage of pedal bins, and isosceles triangle linkages of car jacks. The behaviour of many of these objects is easily explained mathematically — for example, why does the orientation of each tool box remain invariant? It is postulated that, with the aid of dynamic geometry software, students will show more interest in understanding and deriving such explanations than has been the case with conventional geometrical proofs.

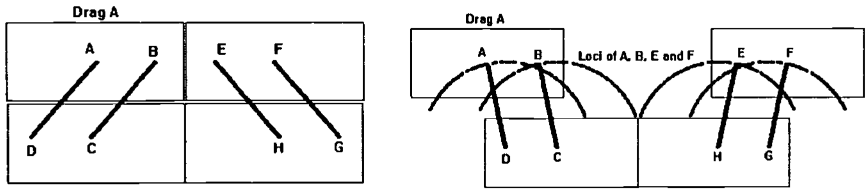


Figure 8. Cabri model of an expanding toolbox

A comparison of car jacks based on variable base isosceles triangles illustrates how related linkages can have a different structural form yet can be used to achieve the same function. In a simple isosceles triangle linkage, the vertex will rise as the base is reduced. For a given arm length, a doubling of the rise is achieved in the rhombus-shaped car jack (Figure 9a) by having two isosceles triangles, while the same result is achieved in the extended isosceles triangle jack (Figure 9b) by doubling the arm length PA. The linkages contain much interesting mathematics, for example: proving why the jacks rise vertically, and explaining the rate of increase in height as the screw is turned at constant rate (Figure 10).

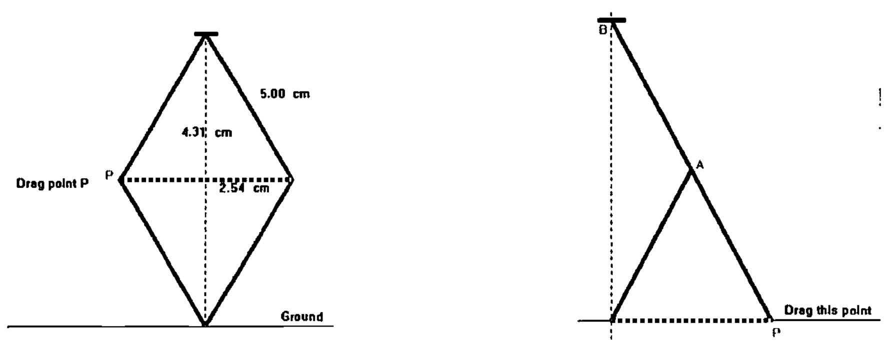


Figure 9. Modelling (a) rhombus and (b) extended isosceles triangle car jacks in Cabri

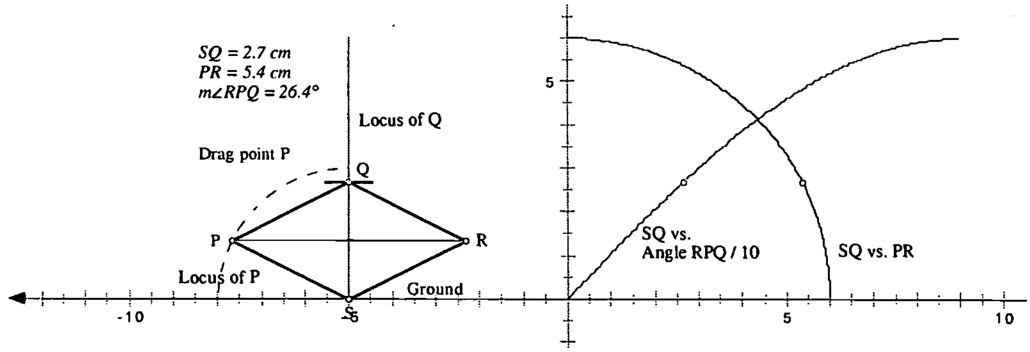


Figure 10. Sketchpad investigation of a rhombus car jack

Laborde (1995) uses a 'zig-zag' corkscrew based on a linkage of four rhombuses as an illustration of the modelling of linkage systems with Cabri, noting that while a static drawing requires only recognition of four congruent rhombuses, the dynamic construction in Cabri requires an analysis of the geometrical relations as well as identification of the hierarchy of dependence between the points. Nevertheless, several different Cabri constructions which simulate the operation of the corkscrew are possible, as is often the case, but some may use geometric constructions such as reflection of points which would not be used in physical construction. This need not be regarded as a limitation of the software model, though, as developing different construction methods challenges students' creativity and encourages flexibility in their thinking.

Steeg, Wake, and Williams (1993), however, note that there is a problem in modelling some linkages where a two-way function may be required, such as a pantograph. For example, the linkages in Figure 5 will not operate in reverse (that is, by dragging point P') because of the dependencies involved. Steeg, Wake and Williams (p. 28) suggest that perhaps

> we have stretched the software [Sketchpad] beyond its intended range of application, and we should be looking for a new design tool for such tasks. However, we see a number of benefits for pupils in their experience with both technology and mathematics in using this one package. In technology the package allows pupils to visualise the effects of variation of parameters within a mechanism, firstly in a qualitative way and ultimately quantitatively. ... Within their mathematical experience pupils benefited greatly from seeing a practical motivation for using geometry, whilst gaining an insight into loci.

Conclusion

Mechanical linkages in everyday objects, as well as historical mechanisms, provide a context to enhance the development of students' visualisation skills. They contain a wealth of geometry appropriate for secondary school mathematics, and modelling them with dynamic geometry software enables students to explore relationships and make conjectures. Explaining *why* the linkages work the way they do challenges students' geometric understanding, encourages deductive reasoning and provides motivation for mathematical proof.

References

Bolt, B. (1991). *Mathematics Meets Technology.* Cambridge University Press.

Bruce, J., Wolf, A., & Sutherland, R. (1996). Geometric modelling with Cabri–géomètre. *Micromath,* 12 (3), 32–33.

Bartolini Bussi, M. (1993). Geometrical proofs and mathematical machines: An exploratory study. In I. Hirabayashi, N. Nohda, K. Shigematsu & F–L Lin (Eds.) *Proceedings of the 17th Psychology of Mathematics Education Conference* 2, (pp. 97–104). Tsukuda, Japan: Program Committee of the 17th PME Conference, Japan.

Cunningham, S. (1991). The visualization environment for mathematics education. In W. Zimmermann & S. Cunningham, *Visualization in Teaching and Learning Mathematics* (MAA Notes 19), (pp. 67–76). Washington, DC: Mathematical Association of America.

Cuoco, A., & Goldenberg, E. P. (1997). Dynamic geometry as a bridge from Euclidean geometry to analysis. In J. King, & D. Schattschneider (Eds.). *Geometry Turned On: Dynamic Software in Learning, Teaching, and Research* (MAA Notes 41), (pp. 33–44). Washington, DC: Mathematical Association of America.

Dennis, D., & Confrey, J. (1998). Geometric curve–drawing devices as an alternative approach to analytic geometry: An analysis of the methods, voice, and epistemology of a high–school senior. In R. Lehrer & D. Chazan (Eds.). *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 297–318). New Jersey: Lawrence Erlbaum.

Descartes, R. (1637). The geometry. In D. E. Smith & M. Latham (1954). *The Geometry of Rene Descartes with a Facsimile of the First Edition.* New York: Dover Publications.

De Villiers, M. (1998). An alternative approach to proof in dynamic geometry. In R. Lehrer & D. Chazan (Eds.). *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 369–393). New Jersey: Lawrence Erlbaum.

Goldenberg, E. P., Cuoco, A., & Mark, J. (1998). A role for geometry in general education. In R. Lehrer & D. Chazan (Eds.). *Designing Learning Environments for Developing Understanding of Geometry and Spac,* (pp. 3–44). New Jersey: Erlbaum.

Hazzan, O., & Goldenberg, E. P. (1997). Students' understanding of the notion of function in dynamic geometry environments. *International Journal of Computers for Mathematical Learning,* 1, 263–291.

Hilbert, D., & Cohn–Vossen, S. (1932). *Geometry and the Imagination.* Translated by P. Nemenyi (1952). New York: Chelsea Publishing Company.

Isoda, M. (1998). Developing the curriculum for curves using history and technology. In W–C Yang, K. Shirayanagi, S–C Chu & G. Fitz–Gerald (Eds.), *Proceedings of the Third Asian Technology Conference in Mathematics* (pp. 82–89). New York: Springer–Verlag.

Isoda, M., Matsuzaki, A., & Nakajima, M. (1998). Mathematics inquiry enhanced by harmonized approach via technology. In *Proceedings of the First International Commission on Mathematics Instruction: East Asia Regional Conference on Mathematics Education* 3, (pp. 267–278). Korea: Korea National University of Education.

Laborde, C. (1995). Cabri–géomètre in one lesson. In R. Hunting, G. Fitzsimons, P. Clarkson & A. Bishop (Eds.), *Proceedings of ICMI Conference: Regional Collaboration in Mathematics Education. Melbourne*: Monash University, 59–68.

Scher, D. (1999). Problem solving and proof in the age of dynamic geometry. *Micromath,* 15 (1), 24–25.

Steeg, T., Wake, G., & Williams, J. (1993). Linkages. *Micromath,* 9 (2), 26–29.

Zimmermann, W., & Cunningham, S. (1991). What is mathematical visualization? In W. Zimmermann & S. Cunningham (Eds.), *Visualization in Teaching and Learning Mathematics* (MAA Notes 19), (pp. 1–7). Washington, DC: Mathematical Association of America.

SHORT COMMUNICATIONS

TEACHING UNDERGRADUATE STATISTICS: GRAPHICS CALCULATORS VERSUS COMPUTER PACKAGES

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Statistical computer packages have become commonplace in most university statistics units taught to first year science students. Consequently, the emphasis for the assessment of students' knowledge of hypothesis testing has moved away from the mechanics of number manipulation towards an understanding of which test is appropriate and how the ensuing results can be interpreted. The latest models in graphics calculators now incorporate hypothesis testing amongst their many features. As many students entering Murdoch University in 1999 possessed such calculators, it was decided to incorporate their use into a statistics unit and investigate which form of technology the students preferred. Students with the calculators were very strongly in favour of using them.

DIFFERENTIAL EQUATIONS IN HIGHER EDUCATION -THE BENEFITS OF COMPUTER ALGEBRA SYSTEMS

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This paper deals with the problem to enable students of the master course "Applied Systems Science" to solve Ordinary and Partial Differential Equations in order to discuss the behaviour of the solutions. The students of the master course in their schedule have much less lessons in Mathematics than the MSc candidates in Mathematics. The goal is achieved by making intensive usage of the Computer Algebra Systems Mathematica at all stages of the master course.

SUPPORTING DISTANCE MATHEMATICS STUDENTS

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In this paper we give a report on the progress of electronic tuition as used in the Open University's mathematics courses. The particular problems of communicating mathematics are avoided by the use of a computer algebra system. The study confirms several known features of computer–mediated communication, in particular the high level of investment of tutorial time needed to ensure quality. Nevertheless the conclusion is one of optimism: appreciation by both tutor and student is high.

COMPUTER ALGEBRA SYSTEMS AS VISUALISATION AND EXPERIMENTATION TOOLS IN MATHEMATICS EDUCATION

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In this paper we will discuss the role of a computer algebra system as a visualisation and an experimentation tool in mathematics education. Two types of visualisation methods are considered, static and dynamic. Quite often, visualisations lead to experimentation. The computer algebra systems such as Maple or Mathematica are good experimentation tools because of their built-in programming languages. These programming languages provide the user enough flexibility to experiment with mathematical or physical phenomena.

SOFTWARE DESIGN "WITH LEARNING IN MIND": ONE ROAD LEADS TO FATHOM™

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We developed Fathom Dynamic Statistics™ software from the beginning to be good for learning about data analysis in particular and mathematics in general. What does that mean in practice? This paper reviews some of the choices we thought we made at the time, and looks back over the six-year development trek to see, in retrospect, that many of the most important choices focused on synchrony, visibility, and generality.

AN ANALYSIS OF "MATHEMATICAL MUSEUMS" AND MATHEMATICS EDUCATION

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Nowadays, science museums play an important role to extend various kinds of knowledge of sciences and technologies to the public. For the same reason, the realization of "mathematical museums" might be expected. They could offer an opportunity of informal mathematics education. In this paper, we discuss a possibility of "mathematical museums" by analyzing their effects to the mathematical education. This discussion is related to the new curriculum of mathematics education prepared for 2002 in Japan. We also discuss exhibits of the mathematical museums from viewpoints of the history of mathematics.

USING TECHNOLOGY TO SUPPORT TEACHING STATISTICS TO A LARGE CLASS OF BUSINESS STUDENTS

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Teaching statistics to business students in a large class is always a challenge. Technology has emerged as a valuable educational pedagogical resource, with the expanding use of computers shifting the teaching-to-learning paradigm. Innovative methods are needed to overcome the challenges of teaching statistics to business students and shifting the teaching-to-learning paradigm of this so-called 'dry' subject. The desired objective is to motivate students to enjoy learning through the use of learner-based technology. This paper will share the experiences of the authors in introducing and utilizing technology as a pedagogical tool in the delivery of a university business school statistics course

DEVELOPING NATIONALLY BASED MATHEMATICAL SOFTWARE, THE UKMCC (MATHWISE) PROJECT

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The United Kingdom Mathematics Courseware Consortium (UKMCC), created some years ago was a major nationally based project which set out to provide the UK academic community with a substantial pool of multimedia learning materials. The principal product of the consortium was a set of modules together known as mathwise. This paper will trace the history of the project from its initial conception, through the design and authoring stages to the commercialisation of the courseware and the setting up of the successful mathwise user group (MWUG). The work of the MWUG to date will be outlined as will possible future developments.

DISCRETE MATHEMATICS: AN INTERACTIVE SOFTWARE PACKAGE

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This paper describes an interactive software package designed to teach basic concepts in discrete mathematics. The package is programmed using Mathematica but only requires students to interact with buttons and palettes. Examples of exercises and technical details are included.

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