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ABSTRACT

Many mathematics educators have lost sight of the critical importance of the mathematical understanding which underlies procedural competence, in part because we do not have a language to refer to this kind of understanding. The modal way of categorizing mathematical knowledge--conceptual and procedural knowledge--is limited in that: (a) it is almost exclusively focused on the mathematics of elementary school, and (b) in studies using this framework. Knowledge of procedures and concepts are assessed in very different ways. Three students' solutions to a linear equation are presented and compared. This paper shows that the difference between what these three students know about linear equation solving can be framed in terms of students' planning knowledge of the procedures--in other words, conceptual knowledge about procedures. (Author)

RE-"CONCEPTUALIZING" PROCEDURAL KNOWLEDGE IN MATHEMATICS

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Abstract

Many mathematics educators have lost sight of the critical importance of the mathematical "understanding" which underlies procedural competence, in part because we do not have a language to refer to this kind of understanding. The modal way of categorizing mathematical knowledge -- conceptual and procedural knowledge -- is limited in that: (a) it is almost exclusively focused on the mathematics of elementary school, and (b) in studies using this framework, knowledge of procedures and concepts are assessed in very different ways. Three students' solutions to a linear equation are presented and compared. I suggest that the differences between what these three students know about linear equation solving can be framed in terms of students' planning knowledge of the procedures -- in other words, conceptual knowledge about procedures.

By many accounts, the U.S. is beginning to experience a backlash against current mathematics education standards-based reform efforts, with many advocating a renewed emphasis on "basic skills". Many mathematics educators recognize the similarities between this current push toward basic skills and prior backlashes against mathematics reforms in years past. Why does this oft-commented-on pendulum between “skill-based” and “reform-based” curricula appear to be swinging yet again? One reason for the emergence of the current back-to-basics movement is the failure of mathematics education research to adequately study and thus appropriate the topic of "skill". Current reform efforts and research are primarily focused on the type of understanding commonly referred to as "conceptual understanding". In doing so, we have lost sight of the critical importance of another kind of mathematical understanding -- that which underlies procedural competence. Current work lacks an emphasis on doing mathematics -- in other words, using and understanding the mathematical procedures and skills that are an essential part of our discipline. In this theoretical paper, I suggest a way in which mathematics education research can return to this issue of procedural competence -- not in place of a focus on "conceptual understanding" but in addition to it.

Conceptual and procedural knowledge. Particularly since the publication of Hiebert's book (1986) on the topic, the terms "conceptual knowledge" and "procedural knowledge" have served as a widely-used framework for thinking and analyzing mathematical knowledge. Hiebert and Lefevre define conceptual knowledge as "knowledge that is rich in relationships" (1986, p. 3). They note the following example of conceptual knowledge: the construction of a relationship between the algorithm for multi-digit subtraction and knowledge of the positional values of digits (place value) (Hiebert & Lefevre, 1986). Procedural knowledge is defined as "rules or procedures for solving mathematical problems" (Hiebert & Lefevre, 1986, p. 7). Hiebert and Lefevre (1986) write that the primary relationship in procedural knowledge is 'after,' in that procedures are step-by-step, sequentially ordered, deterministic instructions for how to solve a task. They note the following examples of a procedure: the adding of two fractions of unlike denominators (Hiebert & Lefevre, 1986).

These two "types" of knowledge are assumed to be distinct yet related. Much effort has been spent examining the relationship between them, particularly determining which optimally comes first. On this issue, Rittle-Johnson and Siegler (1998) conclude that there is no fixed order in the acquisition of mathematical skills versus concepts. In some cases, skills are acquired first; in other situations, the order is reversed.

Two main criticisms of Hiebert's conceptual/procedural knowledge framework emerge from a more thorough review of this literature (see Star, 1999). First, almost all of the studies underlying this framework (both in the 1986 book and in subsequent work) are from the topic areas of counting, single-digit addition, multi-digit addition, and fractions -- all areas of study in elementary school. Notably absent are studies of the development of procedural and conceptual knowledge in algebra, geometry, and calculus. Second, knowledge of concepts and knowledge of procedures are typically assessed in very different ways. Knowledge of concepts is often assessed verbally and through a variety of tasks. This would suggest that conceptual knowledge is complex and multi-faceted. By contrast, procedural knowledge is assessed uni-dimensionally and non-verbally by observing the execution of a procedure. A student either knows how to do a procedure (and therefore can execute it successfully and automatically) or does not know how to do the procedure.

Elsewhere I have examined the conceptual/procedural framework in more depth, as well as other terminological distinctions among knowledge types (Star, 1999). Here I present an example of an alternative conception of how a procedure can be known as a way of arguing that the current framework's treatment of procedural knowledge is inadequate.

A "procedural" example. Consider the following equation, which was given to middle school students in a unit on solving linear equations: $4(x+1)+2(x+1)=3(x+4)$. What are some different ways that students successfully solved such an equation?

One student, Joanne, performed this sequence of steps: 1. Use the distributive property on the left and right sides of the equation to "clear" the parentheses. 2. Add (-4) to both sides. 3. Add (-2) to both sides. 4. Add ($-3x$) to both sides. 5. Combine like variable terms on the left side and constants on the right. 6. Divide both sides by 3, yielding $x=2$. Upon completing her solution, Joanne was asked if she could solve the same equation using a different order of steps; she said that she could not.

A second student, Kyle, used this sequence of steps: 1. Use the distributive property on the left and right sides. 2. Combine like variable and constant terms on the left side. 3. Add ($-3x$) to both sides. 4. Add (-6) to both sides. 5. Combine like variable and constant terms on both sides. 6. Divide both sides by 3, yielding $x=2$. Upon completing the problem, Kyle was asked if he could solve the equation using a different order of steps; he said yes and generated a solution which was identical to the one used by Joanne. Kyle was then asked if he could solve the equation in yet another order of steps. He said that steps 3 and 4 of his initial solution could be done in the opposite order, but he knew of no other alternative sequence of solution steps.

A high school sophomore, Leah, initially solved the problem using Joanne's sequence of steps. Upon being prompted for a different solution, she used Kyle's sequence. Upon further prompting, she generated this sequence: 1. Combine the two terms on the left side, yielding $6(x + 1)$. 2. Divide both sides by 3, yielding $2(x+1)=x+4$. 3. Use the distributive property on the left side. 4. Add (-2) to both sides. 5. Add $(-x)$ to both sides. 6. Combine like variable and constant terms, yielding $x=2$. When prompted for more, alternative orders, she said (in a somewhat exasperated tone) that there are lots and lots of different orders, but that they are all the same.

It is difficult to draw broad conclusions about what these students know about equation solving from these limited anecdotes. However, it is possible to make some preliminary inferences about the differences in what these students know about the operators of the domain of equation solving. Joanne was only able to solve this equation using one ordering of steps. Her knowledge of the operators of this domain is relatively inflexible. She knows that applying operators in a particular order is very likely to lead to solution; she perhaps has not considered the question of whether or not any variation in her ordering will also be successful. Kyle knows and is able to use the same series of steps as Joanne. However, in addition, Kyle realizes that one can add constants or variable terms to both sides in any order without affecting the successful solution of the equation. He also knows that one can choose to simplify each side individually before any "moving" of terms. Leah realizes that there are a very large number of ordered arrangements of steps that can lead to a successful solution. She is able to look at the specific details of a particular problem and choose an ordering of steps that works. For example, if a term of the left hand side of the original problem were changed from $4(x+1)$ to $4(x+2)$, Leah would realize that her first two solution orderings (identical to Joanne's and Kyle's) would work, but that her third ordering would not.

It is possible to frame these differences between Joanne, Kyle, and Leah in "conceptual" terms, perhaps referring to each students' relative conceptual understanding of the commutative or distributive properties. However, it is equally plausible that one could characterize the differences between these students' knowledge in terms of what they know about the procedures and operators of this domain. The procedure for solving linear equations has a limited number of possible operators which, when correctly applied in particular combinations, lead to the solution. Kyle knows more than Joanne (and Leah knows more than Kyle) about how these operators fit together to achieve particular goals (e.g., getting the numbers to one side and the variables to the other side, transforming the equation to the form ax=b), what each operator does, and under what conditions operators and chains of operators can be used and to what end.

This kind of knowledge about a procedure has been referred to elsewhere as teleological semantics (VanLehn & Brown, 1980). The teleological semantics of a procedure is "knowledge about [the] purposes of each of its parts and how they fit together. ... Teleological semantics is the meaning possessed by one who knows not only the surface structure of a procedure but also the details of its design" (p. 95). VanLehn and Brown (1980) note that a procedure can be cognitively represented on a very superficial level (as a chronological list of actions or steps) or on a more abstract level (incorporating planning knowledge in its representation). Planning knowledge includes not only the surface structure (the sequential series of steps) but also "the reasoning that was used to transform the goals and constraints that define the intent of the procedure into its actual surface structure" (p. 107). In other words, planning knowledge of a procedure takes into account the order of steps, the goals and subgoals of steps, the environment or type of situation in which the procedure is used, constraints imposed upon the procedure by the environment or situation, and any heuristics or common sense knowledge which are inherent in the environment or situation.

Within Hiebert's framework, this type of knowledge does not fall nicely into either conceptual knowledge or procedural knowledge. In essence, teleological semantics is conceptual knowledge about a procedure -- it is both procedural and conceptual knowledge. It is knowledge which is rich in relationships, but where the relationships in question are abstract and are among procedural steps.

Conclusions. Brown and colleagues (Brown, Moran, & Williams, 1982; as cited in Nesher, 1986) argue that procedures should not be assumed to be rote but rather as objects with several different sources of meaning. The example described above illustrates a case where students may possess rather sophisticated knowledge about a procedure and its operators. Neither of the terms "procedural knowledge" or "conceptual knowledge" seems appropriate to describe the understandings that students such as Leah have and can use.

Especially given the renewed emphasis among anti-reformers on "basic skills", it behooves us as a community to be more explicit about what we mean by understanding. As Ohlsson and Rees (1991) note, a complete theory of understanding includes both an understanding of concepts and an understanding of procedures. At present, the terminological framework that is mostly widely used in the field makes it difficult to investigate the understanding of procedures.

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