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ABSTRACT

This set of transparencies shows how the manipulation of combinatorial structures in the context of modern combinatorics can easily lead to interesting teaching and learning activities at every level of education from elementary school to university. The transparencies describe: (1) the importance and relations of combinatorics to science and social activities; (2) how analytic/algebraic combinatorics is similar to analytic/algebraic geometry; (3) basic combinatorial structures with terminology; (4) species of structures; (5) power series associated with any species of structures; (6) similarity of operations between power series and the corresponding species of structures; (7) examples illustrating where the structures are manipulated; and (8) general teaching/learning activities. (KHR)

Manipulating Combinatorial Structures

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A combinatorial structure is a finite *construction* made on a finite set of *elements*. In real-life situations, combinatorial structures often arise as "skeletons" or "schematic descriptions" of concrete objects. For example, on a road map, the elements can be cities and the finite construction can be the various roads joining these cities. Similarly, a diamond can be considered as a combinatorial structure: the plane facets of the diamond are connected together according to certain rules.

Combinatorics can be defined as the mathematical analysis, classification and enumeration of combinatorial structures. The main purpose of my presentation is to show how the manipulation of combinatorial structures, in the context of modern combinatorics, can easily lead to interesting teaching/learning activities at every level of education: from elementary school to university.

The following pages of these proceedings contain a grayscale version of my (color) transparencies (two transparencies on each page). I decided to publish directly my transparencies (instead of a standard typed text) for two main reasons: my talk contained a great amount of figures, which had to be reproduced anyway, and the short sentences in the transparencies are easy to read and put emphasis on the main points I wanted to stress.

The first two transparencies are preliminary ones and schematically describe: a) the importance and relations of combinatorics with science/social activities, b) how analytic/algebraic combinatorics is similar to analytic/algebraic geometry. The next 7 transparencies (numbered 1 to 7) contain drawings showing basic combinatorial structures together with some terminology. Collecting together similar combinatorial structures give rise to the concept of *species of structures* (transparencies 8 to 12). A power series is then associated to any species of structures enabling one to count its structures (transparencies 13 to 15). Each operation on power series (sum, product, substitution, derivation) is reflected by similar operations on the corresponding species of structures (transparencies 16 to 18). The power of this correspondence is illustrated on explicit examples (transparencies 18 to 26) where the structures are manipulated (and counted) using various combinatorial operations. Transparencies 27 to 30 suggest some general teaching/learning activities (from elementary to advanced levels) that may arise from these ideas. I hope that the readers will agree that manipulating combinatorial structures constitutes a good activity to develop the mathematical mind.

References

The main references for the theory of combinatorial species are:

Joyal, A. (1981). Une théorie combinatoire des séries formelles, *Advances in Mathematics*, 42, 1–82.

Bergeron, F., Labelle, G., & Leroux, P. (1998). Combinatorial Species and Tree-like Structures. *Encyclopedia of Math. and its Applications*, Vol. 67, Cambridge University Press.

PRELIMINARIES

The dynamic connections between

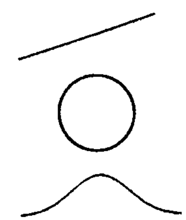
combinatorial / discrete mathematics and science / social activities

will greatly increase during the coming decades :

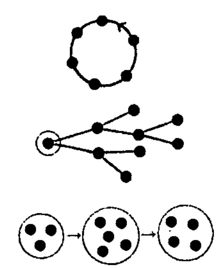
Combinatorics

- **Computer science** (analysis of algorithms, organization and compression of data structures, etc)
- **Chemistry** (classification, symmetries of molecules, isomers, etc)
- **Physics** (classification, symmetries of atoms, quantum theory, Feynman integrals, etc)
- **Biology** (analysis of DNA, genetic code, classification of species, etc)
- **Mathematics** (analysis, calculus, algebra, geometry, graphs, groups, etc)
- **Human activities** (public keys methods of encryption, encoding of compact discs, secure transactions, communications, design of plane flights, shortest paths, teaching and learning basic combinatorial stuctures, etc)

ANALYTIC / ALGEBRAIC GEOMETRY

Geometrical figures	$\rightleftarrows$	Mathematical formulas
		$2x - 6y + 7 = 0$ $x^2 + y^2 = 9$ $y = e^{-x^2}$

ANALYTIC / ALGEBRAIC COMBINATORICS

Combinatorial structures	$\rightleftarrows$	Mathematical formulas
		$y = \ln(\frac{1}{1-x})$ $y = x + xy^2$ $y = e^{3x}$

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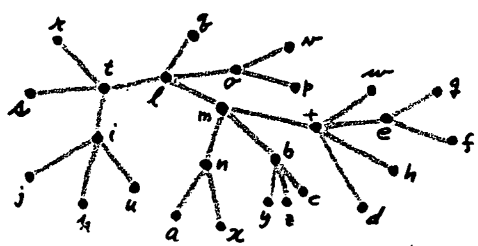
MANIPULATING COMBINATORIAL STRUCTURES

Gilbert Labelle / LaCIM - UQÀM

In Combinatorics, a structure, s, is a finite construction made on a finite set U.

We say that : U is the underlying set of s
or : U is equipped with (the structure) s
or : s is (built) on U
or : s is labelled by the set U

Example 1. A tree on $U = \{a, b, c, \ldots, x, y, z, +\}$



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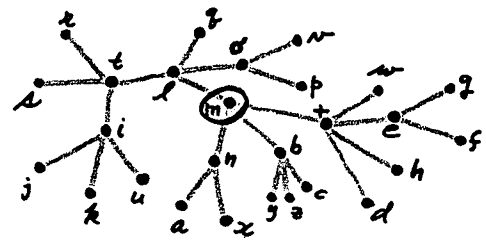
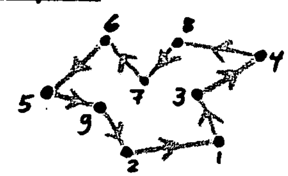
Example 2. A rooted tree on $U = \{a, b, c, \ldots, x, y, z, +\}$

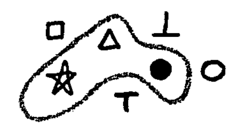


Example 3. An oriented cycle on $U = \{1, 2, \ldots, 9\}$



Variants:
even (oriented) cycle
odd (oriented) cycle

Example 4. A subset on (of) $U = \{\square, \circ, \triangle, \star, \top, \bot, \bullet\}$
(or bicoloration)



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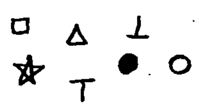
Example 5. A (the) set structure on

$U = \{\square, \circ, \triangle, \bigstar, \top, \bot, \bullet\}$



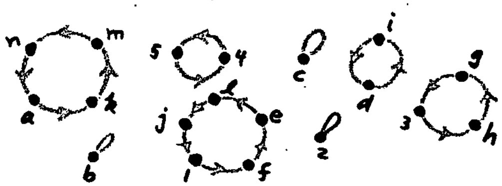
Variants:

even set

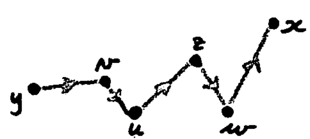
odd set

Example 6. A permutation on

$U = \{1,2,3,4,5, a, b, \ldots, n\}$



Example 7. A linear order on $U = \{u, v, w, x, y, z\}$



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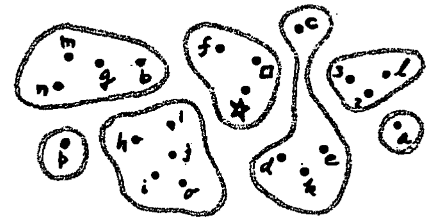
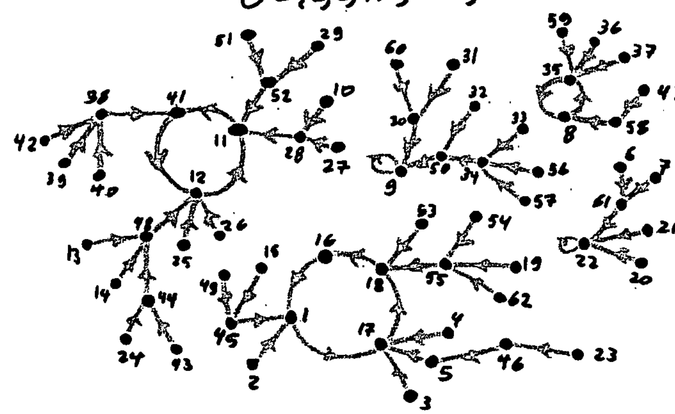
Example 8. A partition on

$U = \{\square, \bigstar, 1, 2, 3, a, b, \ldots, p\}$



Example 9. An endofunction on

$U = \{1, 2, \ldots, 62\}$



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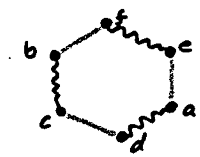
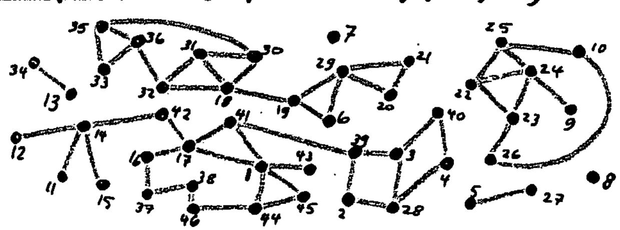
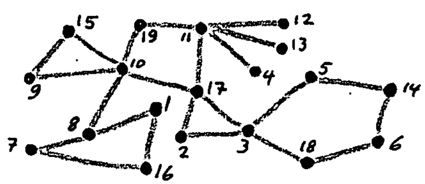
Example 10. A bicolored polygon on $U = \{a, b, c, d, e, f\}$



Example 11. A graph on $U = \{1, 2, \ldots, 46\}$



Example 12. A connected graph on $U = \{1, 2, \ldots, 19\}$



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Example 13. A singleton on $U = \{a\}$

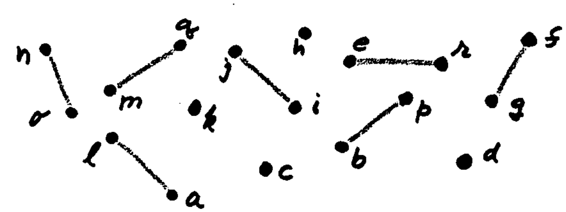
(There is no singleton structure on U if $|U| \neq 1$)

Example 14. An empty set structure on $U = \phi$

{ }

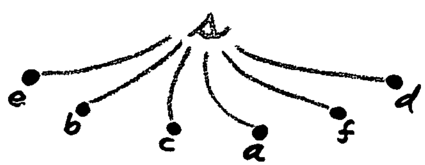
(There is no empty set structure if $U \neq \phi$)

Example 15. An involution on $U = \{a, b, \ldots, r\}$



ETC, ETC, ETC, ...

CONVENTION : An arbitrary structure s on U will sometimes be represented by



Here, $U = \{a, b, c, d, e, f\}$

A structure s on U is said to be even (odd) if U contains an even (odd) number of elements.

Two structures s and t are said to be disjoint if their underlying sets are disjoint.

Structures can be studied and classified by collecting them into SPECIES OF STRUCTURES :

For example, we can consider

the species $\mathcal{a}$ of all trees (arbres),
the species A of all rooted trees (arborescences),
the species $\mathcal{C}$ of all cycles,
the species $\mathcal{C}_{even}$ of all even cycles,
the species $\mathcal{P}$ of all subsets (of sets),
the species E of all sets (ensembles),
the species E_{even} of all even sets,
the species E_{odd} of all odd sets,
the species S of all permutations,
the species L of all linear orders,
the species B of all partitions,
the species End of all edofunctions,
the species P^{blc} of all bicolored polygons,
the species $\mathcal{G}ra$ of all graphs,
the species $\mathcal{G}ra^{c}$ of all connected graphs,
the species X of all singletons,
the species 1 of the empty set,
the empty species 0 (containing no structure),
the species Inv of all involutions,
⋮
the species F of all F-structures.

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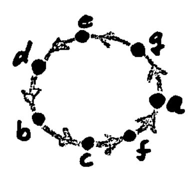
By convention, for any finite set U we write

$F[U]$:= <u>the set of all F-structures on U</u>

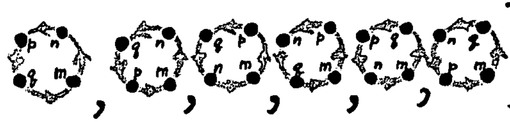
Example. If $F = C$ = the species of <u>cycles</u>, then



is a <u>C-structure</u>

and if $U = \{m, n, p, q\}$, then

$C[\{m, n, p, q\}]$

$= \{$  $\}$

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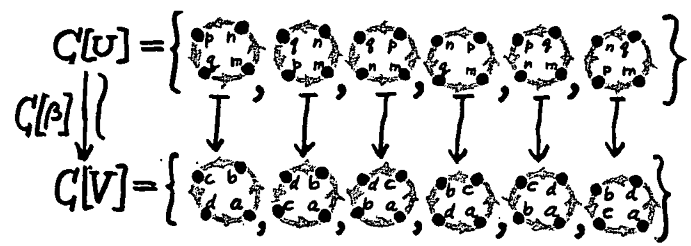
Fundamental properties satisfied by the preceeding example:

<u>PROPERTY 1.</u> For every finite set U, the set $C[U]$ is always <u>finite</u>.

<u>PROPERTY 2.</u> Every <u>bijection</u>

$$\begin{array}{ccl} U & = & \{m, n, p, q\} \\ \beta \downarrow\wr & & \;\;\downarrow\;\;\downarrow\;\;\downarrow\;\;\downarrow \\ V & = & \{a, b, c, d\} \end{array}$$

induces <u>another bijection</u>,



which <u>relabels via β</u> each C-structure on U to form a corresponding C-structure on V.

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Note. Two successive relabellings via β and β' amounts to a single relabelling via $\beta' \circ \beta$:

$$\begin{tikzcd} U \arrow[d, "\beta"'] \arrow[dd, bend left, "\beta'\circ\beta"] \\ V \arrow[d, "\beta'"'] \\ W \end{tikzcd} \Longrightarrow \begin{tikzcd} \mathcal{C}[U] \arrow[d, "{\mathcal{C}[\beta]}"'] \arrow[dd, bend left, "{\mathcal{C}[\beta']\circ\mathcal{C}[\beta] = \mathcal{C}[\beta'\circ\beta]}"] \\ \mathcal{C}[V] \arrow[d, "{\mathcal{C}[\beta']}"'] \\ \mathcal{C}[W] \end{tikzcd}$$

DEFINITION (André Joyal, UQAM, 1979)

A combinatorial species is a rule F which

1) generates, for each finite set U, another finite set $F[U]$,

2) generates, for each bijection $\begin{tikzcd} U \arrow[d, "\beta"', "\wr"] \\ V \end{tikzcd}$, another bijection $\begin{tikzcd} F[U] \arrow[d, "{F[\beta]}"', "\wr"] \\ F[V] \end{tikzcd}$

in a coherent way: $F[\beta'] \circ F[\beta] = F[\beta' \circ \beta]$.

- Any s in $F[U]$ is called a F-structure on U.
- Any bijection $F[\beta]$ is called F-relabelling via β.

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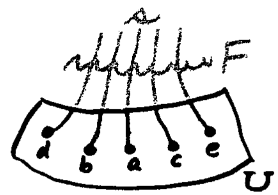
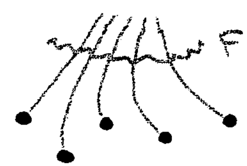
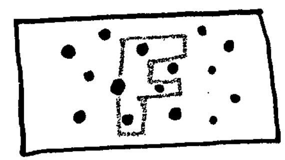
CONVENTION. An arbitrary F-structure s on U can be represented by diagrams like



or



or



ETC

Counting structures of a given species

The properties of relabellings imply that the number of F-structures on U depends only on the number of elements of U.

- That is, $|F[U]|$ depends only on $|U|$.

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Hence, if we want to simply count F-structures, it is sufficient to consider only the cases

$$U = \underline{n} = \{1,2,\dots,n\}, \quad n = 0,1,2,\dots.$$

DEFINITION. Let F be a species and

$$f_n = |F[\underline{n}]|$$

$=$ number of F-structures labelled by $\{1,2,\dots,n\}$.

The generating series of the species F is

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}.$$

NOTE. We will see that generating series give rise to a dynamic connection between

COMBINATORICS and ANALYSIS.

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$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$$

Example 1. S : the species of permutations

$$S(x) = \sum_{n\geq 0} n!\frac{x^n}{n!} = \sum_{n\geq 0} x^n = \frac{1}{1-x}$$

Example 2. $\mathcal{P}$: the species of subsets

$$\mathcal{P}(x) = \sum_{n\geq 0} 2^n \frac{x^n}{n!} = e^{2x}$$

Example 3. E : the species of sets

$$E(x) = \sum_{n\geq 0} 1\,\frac{x^n}{n!} = e^x$$

Example 4. E_{even} : the species of even sets

$$E_{even}(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \cosh x$$

Example 5. E_{odd} : the species of odd sets

$$E_{odd}(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sinh x$$

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Example 6. L : the species of linear orders

$$L(x) = \sum_{n\geq 0} n!\,\frac{x^n}{n!} = \sum_{n\geq 0} x^n = \frac{1}{1-x}$$

Example 7. C : the species of cycles

$$C(x) = \sum_{n\geq 1} (n-1)!\,\frac{x^n}{n!} = \sum_{n\geq 1} \frac{x^n}{n} = \ln\left(\frac{1}{1-x}\right)$$

Example 8. C_{even} : the species of even cycles

$$C_{even}(x) = \sum_{n \text{ even} \geq 1} (n-1)!\,\frac{x^n}{n!} = \sum_{k\geq 1} \frac{x^{2k}}{2k} = \ln\sqrt{\frac{1}{1-x^2}}$$

Example 9. End : the species of endofunctions

$$End(x) = \sum_{n\geq 0} n^n \frac{x^n}{n!}$$

Example 10. X : the species of singletons

$$X(x) = 0 + 1\frac{x}{1!} + 0\frac{x^2}{2!} + 0\frac{x^3}{3!} + \cdots = x$$

Example 11. Gra : the species of graphs

$$Gra(x) = \sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}$$

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COMBINING and MANIPULATING SPECIES

We know how to
add, multiply, substitute, and differentiate
series to obtain other series :

If $F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$, $G(x) = \sum_{n\geq 0} g_n \frac{x^n}{n!}$

and $H(x) = \sum_{n\geq 0} h_n \frac{x^n}{n!}$

then we have the following table

OPERATION	COEFFICIENT h_n
$H(x) = F(x) + G(x)$	$h_n = f_n + g_n$
$H(x) = F(x) \cdot G(x)$	$h_n = \sum_{0\leq k\leq n} \frac{n!}{k!(n-k)!} f_k g_{n-k}$
$H(x) = F(G(x))$, $G(0) = g_0 = 0$	$h_n = \sum_{\substack{0\leq k\leq n \\ n_1+n_2+\cdots+n_k=n}} \frac{n!}{k!\,n_1!\cdots n_k!} f_k g_{n_1} \cdots g_{n_k}$
$H(x) = \frac{d}{dx} F(x)$	$h_n = f_{n+1}$

This table suggests corresponding combinatorial operations for species :

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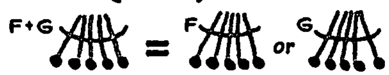
DEFINITION. From given species F and G other species can be built:

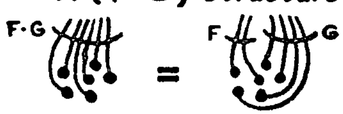
The species $F+G$

A $(F+G)$-structure is [a F-structure or (distinct copies) a G-structure



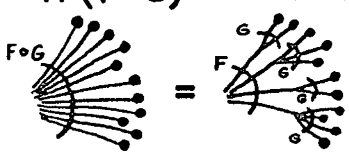
The species $F \cdot G$

A $(F \cdot G)$-structure is [an ordered pair of a F-structure and a disjoint G-structure



The species $F(G) = F \circ G$

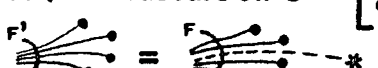
A $(F \circ G)$-structure is [a F-structure built on a finite set of disjoint G-structures



(here, $G[\phi]=\phi$)

The species F'

A F'-structure on U is [a F-structure on $U + \{*\}$



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THEOREM. If F and G are species, then

$$(F+G)(x) = F(x) + G(x),$$
$$(F \cdot G)(x) = F(x) \cdot G(x),$$
$$(F \circ G)(x) = F(G(x)), \quad (\text{if } G[\phi]=\phi)$$
$$F'(x) = \frac{d}{dx}F(x).$$

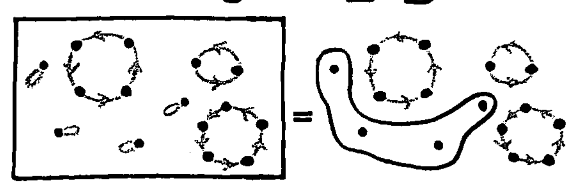
EXAMPLES / APPLICATIONS

1) $E = E_{even} + E_{odd} \Rightarrow E(x) = E_{even}(x) + E_{odd}(x)$

that is, $e^x = \cosh x + \sinh x$

2) Let D be the species of derangements, i.e., permutations without fixed points, then

$$S = E \cdot D$$



$\Rightarrow$ $S(x) = E(x) \cdot D(x)$ i.e., $\frac{1}{1-x} = e^x D(x)$

$\Rightarrow$ $D(x) = e^{-x}\frac{1}{1-x}$

$\Rightarrow$ d_n = number of derangements of n objects $= n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

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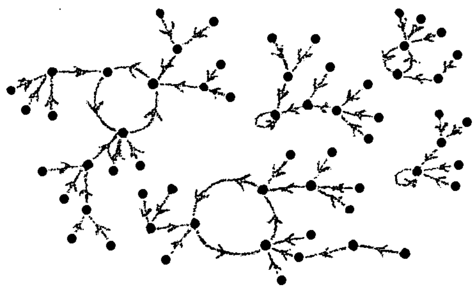
3) $\text{End} = S \circ A$



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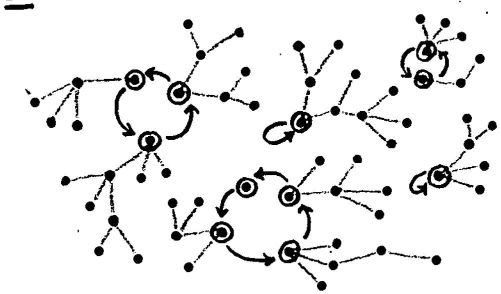

$$\Rightarrow \text{End}(x) = S(A(x)) \Rightarrow \sum_{n\geq 0} n^n \frac{x^n}{n!} = \frac{1}{1-A(x)}$$

$$\Rightarrow A(x) = \frac{\sum_{n\geq 1} n^n \frac{x^n}{n!}}{\sum_{n\geq 0} n^n \frac{x^n}{n!}}$$

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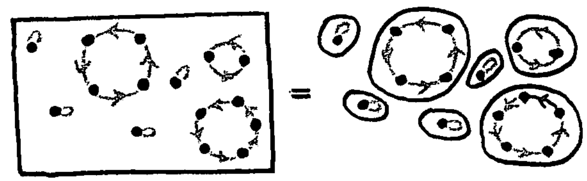
4a) $S = E \circ C$



$$\Rightarrow S(x) = E(C(x)) \Rightarrow \frac{1}{1-x} = e^{C(x)}$$

$$\Rightarrow C(x) = \ln\left(\frac{1}{1-x}\right), \quad c_n = (n-1)!$$

4b) Let $S^{\langle k \rangle}$ be the species of permutations having k cycles

and E_k be the species of k-sets,

then $S^{\langle k \rangle} = E_k \circ C$, $\quad E_k(x) = \frac{x^k}{k!}$

$$\Rightarrow S^{\langle k \rangle}(x) = E_k(C(x)) = \left[\ln \frac{1}{1-x}\right]^k / k!$$

$\Rightarrow s_n^{\langle k \rangle} =$ number of permutations of n points having k cycles

$=$ absolute value of Stirling number of the first kind

$$= \frac{1}{k!} \sum_{\substack{n_1+\cdots+n_k=n \\ n_i>0}} \frac{n!}{n_1 \cdot n_2 \cdots n_k} = |s(n,k)|$$

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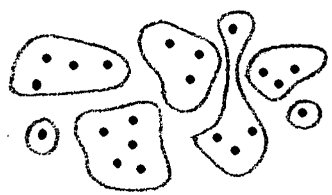
5a) $\mathbb{B} = E \circ E_+$

Where E_+ is the species of non-empty sets.

$E_+(x) = e^x - 1 \Rightarrow \mathbb{B}(x) = E(E_+(x))$



$= e^{e^x - 1}$

$= \frac{1}{e} e^{e^x}.$

It follows that

$b_n =$ number of partitions of a set of n elements

$= \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^n}{i!}$ (This is Dobinski's formula for the n th Bell number)

5b) Let $\mathbb{B}^{\langle k \rangle}$ be the species of partitions having k classes

and E_k be the species of k-sets,

$\Rightarrow \mathbb{B}^{\langle k \rangle} = E_k \circ E_+ \Rightarrow \mathbb{B}^{\langle k \rangle}(x) = [e^x - 1]^k / k!$

$\Rightarrow b_n^{\langle k \rangle} =$ number of partitions of n points into k classes

= Stirling number of the second kind

$= \frac{1}{k!} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i > 0}} \frac{n!}{n_1! n_2! \cdots n_k!} = S(n,k)$

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6) $\mathcal{P} = E \cdot E$

$\Rightarrow$

$\mathcal{P}(x) = (E(x))^2 = e^{2x}$

$\Rightarrow$

$p_n =$ number of subsets of a set of n elements

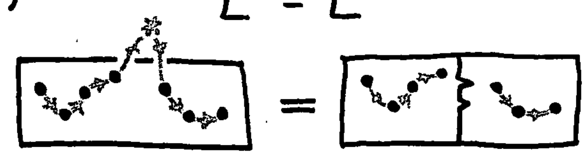
$= 2^n$ (known!)

7) Let $\hat{S}$ be the species of permutations having only even cycles: $\hat{S} = E \circ C_{even}$

$\Rightarrow \hat{S}(x) = e^{C_{even}(x)} = e^{\ln \sqrt{\frac{1}{1-x^2}}} = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \hat{A}_n = \begin{cases} 1^2 \cdot 3^2 \cdot 5^2 \cdots (2k-1)^2 & \text{if } n = 2k \\ 0 & \text{otherwise} \end{cases}$

8) $L' = L^2$



$\Rightarrow L'(x) = (L(x))^2 \Rightarrow \frac{d}{dx} \frac{1}{1-x} = \left(\frac{1}{1-x}\right)^2$ (known!)

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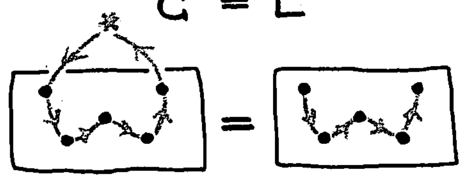
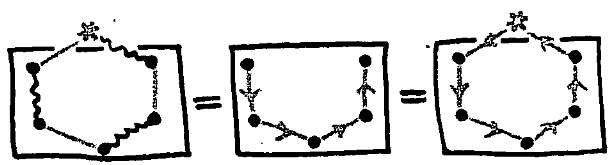
9) $C' = L$



(again!)

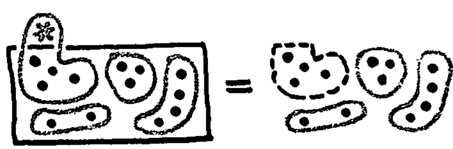
$\Rightarrow \frac{d}{dx}C(x) = L(x) \Rightarrow C(x) = \int L(x)\,dx = \int \frac{dx}{1-x} = \ln\left(\frac{1}{1-x}\right)$

10) $P^{bic\,\prime} = L_{odd} = C'_{even}$



but $P^{bic} \neq C_{even}$!!!

11) $B' = E \cdot B$



(recurrence)

$\Rightarrow \frac{d}{dx}B(x) = E(x)B(x) = e^{x}B(x) \Rightarrow b_{n+1} = \sum_{k=0}^{n}\binom{n}{k}b_k$

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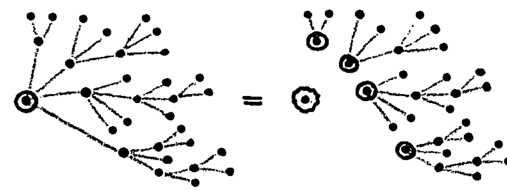
12) $A = X \cdot (E \circ A)$



$\Rightarrow A(x) = X(x) \cdot E(A(x)) = x e^{A(x)}$

$\Rightarrow A(x)e^{-A(x)} = x \Rightarrow a_n = n^{n-1}$

13) Let $\mathcal{V}$ be the species of vertebrates
(a vertebrate is a pointed rooted tree)

$\mathcal{V} = L \circ A$

$\Rightarrow \mathcal{V}(x) = L(A(x)) = \frac{1}{1-A(x)} = End(x)$

$\Rightarrow v_n = n a_n =$ number of endofunctions on n elements $= n^n \Rightarrow a_n = n^{n-1}$ (Joyal)

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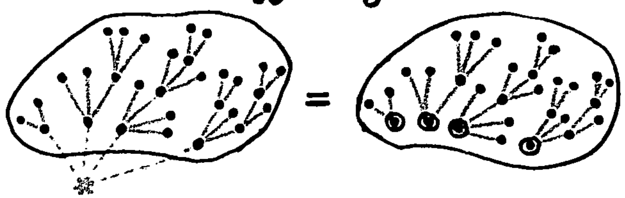
14) $\mathcal{F}$ = the species of forests of rooted trees

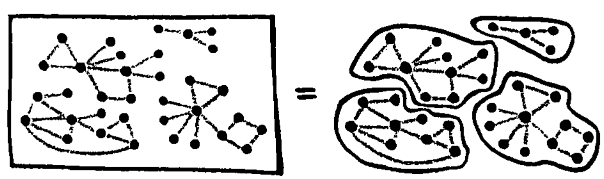
$$\mathcal{A}' = \mathcal{F}$$



$\Rightarrow \quad \mathcal{F}(x) = \frac{d}{dx}\mathcal{A}(x) = \frac{d}{dx}\sum_{n\geq 1} n^{n-2}\frac{x^n}{n!}$

$\Rightarrow \quad \varphi_n$ = number of forests of rooted trees on n points $= (n+1)^{n-1}$

15) $$\mathcal{G}ra = E \circ \mathcal{G}ra^c$$



$\Rightarrow \quad \mathcal{G}ra(x) = e^{\mathcal{G}ra^c(x)}$

$\Rightarrow \quad \mathcal{G}ra^c(x) = \ln \mathcal{G}ra(x) = \ln \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}$

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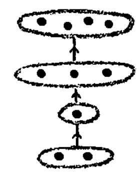
16) Let $\mathcal{B}al$ be the species of ballots.



$$\mathcal{B}al = L \circ E_+$$

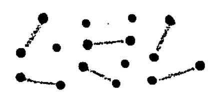
$\Rightarrow \quad \mathcal{B}al(x) = L(E_+(x)) = L(e^x - 1)$

$= \frac{1}{1-(e^x-1)} = \frac{1}{2(1-\frac{1}{2}e^x)} = \frac{1}{2}\sum_{k=0}^{\infty}(\tfrac{1}{2}e^x)^k$

Hence, the number of ballot outcomes for n candidates is

$$\sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} \quad \text{(always an integer !!)}$$

17) $\mathrm{Inv} = E \circ (X + E_2)$



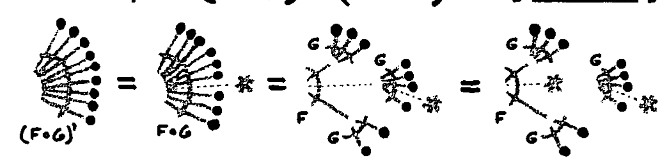
$\Rightarrow \quad \mathrm{Inv}(x) = e^{x + \frac{1}{2}x^2} = e^x \cdot e^{\frac{1}{2}x^2}$

$\Rightarrow$ the number of involutions on n elements is

$$\sum_{0 \leq k \leq [n/2]} \binom{n}{2k} 1\cdot 3\cdot 5 \cdots (2k-1)$$

18) Classical formulas of Calculus hold

For example: $(F \circ G)' = (F' \circ G) \cdot G'$ [chain rule]



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POSSIBLE TEACHING / LEARNING ACTIVITIES
(from elementary to advanced)

- To find, build or draw combinatorial structures from real-life situations or from concrete / abstracts objects.
- To decide whether two structures of a given species are equal or not.
- To make a complete and non-repetitive list of structures belonging to a given species, labelled by a given (small) set.
- To make a complete and non-repetitive list of unlabelled structures belonging to a given species on n indistinguishable elements (n small).
- To find combinatorial equations associated to species (and vice-versa).
- To find enumerative or structural properties of species from their combinatorial equations (via series expansions, for example).

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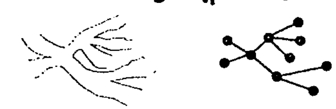
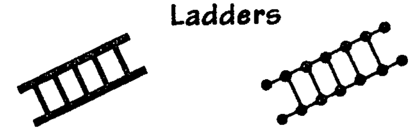
To find, build or draw combinatorial structures from real-life situations or from concrete / abstract objects.

Rivers making a (plane) tree

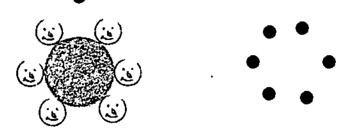


Ladders



Flowers

Sitting around a round table



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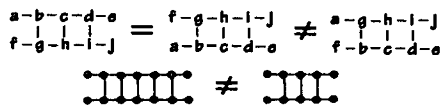
To <u>decide</u> whether two structures of a <u>given</u> species are <u>equal or not</u>.

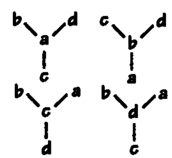


To make a <u>complete and non-repetitive list</u> of structures belonging to a given species, <u>labelled</u> by a given (small) set.

The 16 trees labelled by {a, b, c, d}:

a–b–c–d, a–c–b–d, a–b–d–c, a–d–b–c, a–c–d–b, a–d–c–b

b–a–d–c, b–d–a–c, b–a–c–d, b–c–a–d, c–a–b–d, c–b–a–d



To make a <u>complete and non-repetitive list</u> of <u>unlabelled</u> structures belonging to a given species on n <u>indistinguishable</u> elements (n small).

The 2 unlabelled trees on 4 identical elements:

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To find <u>combinatorial equations</u> associated to species (and vice-versa).

To find <u>enumerative</u> or <u>structural properties</u> of species from their combinatorial equations (via series expansion, for example).

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