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ABSTRACT

This paper on classroom factors influencing students' proof construction ability reports findings from the data collected in the first two years of a three-year National Science Foundation (NSF)-funded project. Four different classrooms, two from each participating school, were involved in the project. Data sources included videotaped classroom episodes, interviews with the participating teachers and with focus students from each class, as well as students' responses to items on the Proof Construction Assessment instrument. Student ability to construct proof was interpreted in the context of the classroom microculture (Cobb & Yackel, 1996). The results show that students performed poorly on items that required them to write a formal proof with no support. They also had difficulty on items that required students to make a single deduction from a given piece of information. (Author)

AN INVESTIGATION OF CLASSROOM FACTORS THAT INFLUENCE PROOF CONSTRUCTION ABILITY

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This paper on classroom factors influencing students' proof construction ability reports findings from the data collected in the first two years of a three-year National Science Foundation-funded project[1]. Four different classrooms, two from each participating school, were involved in the project. Data sources included videotaped classroom episodes, interviews with the participating teachers and with focus students from each class, as well as students' responses to items on the *Proof Construction Assessment* instrument. Student ability to construct proof was interpreted in the context of the classroom microculture (Cobb & Yackel, 1996). The results show that students performed poorly on items that required them to write a formal proof with no support. They also had difficulty on items that required students to make a single deduction from a given piece of information.

Objectives

Although proof and reasoning are seen as fundamental components of learning mathematics, research shows that many students continue to struggle with geometric proofs (Chazan, 1993; Harel & Sowder, 1998; Hoyles, 1997). In order to relate students' understanding of geometric proof to pedagogical methods and other classroom experiences, our three-year project investigates two components of student understanding of proof, namely, students' beliefs about what constitutes a proof and students' proof-construction ability.

In this paper we focus on students' proof-construction ability. We summarize findings from our second year of data collection that connect student ability to construct proof in geometry to classroom factors that may influence that ability. More specifically, we address two objectives:

1. To characterize the psychological aspects of students' evolving proof-construction ability in proof-based geometry classes in order to update and expand existing research in this area;
2. To link students' geometric proof-construction ability to aspects of the classroom microculture as well as to teachers' pedagogical choices.

Perspectives and Theoretical Framework

Existing research documents students' poor performance on proof items and identifies common, fundamental misunderstandings about the nature of proof and generalization in a number of mathematical content areas (Chazan, 1993; Hart, 1994; Martin & Harel, 1989; Senk, 1985). In particular, Senk's seminal study has set the benchmark for high school students' performance on geometry proofs. Other researchers (Balacheff, 1991; Harel & Sowder, 1998) have proposed frameworks that describe increasingly sophisticated strategies used by students to construct proofs. At the least sophisticated level, students appeal to external authorities for mathematical justification. At the next stage, students base their justifications on empirical evidence. Finally, students are able to use more abstract and mathematically appropriate techniques when proving statements. We have used the existing research to help us identify student misunderstandings and to characterize their strategies for constructing proofs.

The theoretical lens through which we view the classroom activities and students' mathematical development as participants in the community of the classroom is associated with the emergent perspective as described by Cobb and Yackel (1996). This perspective is useful in that it attempts to describe individual learning in the social context of the classroom. Thus student understanding of proofs (their beliefs about proofs and their ability to construct proofs) is seen as constructed on both a social level and a psychological level. In other words, the students' interactions with the teacher and peers lead to the development of taken-as-shared knowledge, or an understanding of social and sociomathematical norms. These interactions also influence individual students' developing understanding of proof. Our research focuses on individual student performance as well as the classroom microculture and teachers' pedagogical choices. By classroom microculture, we refer to social and sociomathematical norms, as well as classroom mathematical practices as defined by Cobb and Yackel (1996). We define pedagogical choices to include the choice of mathematical tasks, the ways the teacher allocates time for activities, the instructional strategies (direct instruction, cooperative learning, investigations), and the teacher's expectations about student ability that may be reflected in choices.

Methods of Inquiry and Data Sources

During the two years of the project, we captured the nature of proof instruction, as well as classroom interactions and student activities, through daily observations, video recordings, and written observer field notes. This allowed us to characterize the four participating teachers and their proof-based geometry classes. An initial pair of teachers participated for two years (Mrs. A and Mrs. B). A second pair of teachers participated for only the second year (Mrs. C and Mr. D). The multiple sources of data provided information about the context for the development of proof-construction ability in order to interpret this information and connect it to the classroom norms and mathematical practices.

Student proof construction ability was determined using three types of data collected during the two project years. First, we designed and administered a performance assessment instrument to measure students' varying levels of ability to engage in formal logical reasoning. This *Proof Construction Assessment* instrument includes items in which students must construct partial or entire proofs, as well as generate conditional statements and local deductions. In addition to some original items, the instrument include d items modified from Healy and Hoyles (1998), Senk (1985) and from the Third International Mathematics and Science Study (TIMSS) (1995). Second, data was collected during classroom observations, including video recordings, field notes, and student written work. Third, selected focus students participated in clinical interviews with researchers. The video-recorded interviews focused on some aspects of the Proof Construction Assessment and required focus students to create at least one original proof during the session.

Results and Conclusions

Results of the Proof Construction Assessment provide information about individual student ability to construct proofs. Analysis of the videotapes and field notes illuminate aspects of the social context in which individual learning developed. In addition, we discuss how the classroom social context may have influenced the individual understanding constructed by the students.

Proof Construction Assessment Results

Table 1 summarizes results from the Proof Construction Assessment from year two. The table provides a brief description of each item on the assessment instrument as well as measures of student performance for all four classes (n=84). Proof items were scored using detailed rubrics similar to those used in scoring the TIMSS (1995) and the National Assessment of Educational Progress (National Assessment Governing Board, 1994). The items were scored using rubrics of 5, 3, or 2 points, depending on the item. The table shows the average score for all participating students on each item. These average scores are also reported as percentages of available points, to facilitate comparison of student performance among items scored with differing numbers of available points. The last six columns display the percentage of students receiving a particular score on each item.

Item 1 on the Proof Construction Assessment required students to fill in statements or reasons in a two-column proof. The proof was a justification for "supplements of congruent angles are congruent." In general, students in all four classes did relatively well on this item. However, focus students in Mrs. C's and Mr. D's classes expressed frustration with the limitation of a pre-structured proof. Many of these students said that they would prefer to write their own proof.

Students also did relatively well on Items 3a, 3b, and 3c on the Proof Construction Assessment which required translation from an informal conjecture to a formal con-

Table 1. Proof Construction Assessment Results: Year 2

Item	Title of Item (Content Area)	Average Score (%)	% of Students Receiving Score 0	1	2	3	4	5
1	Fill-in proof (Supplementary angles)	3.29/5 (66)	0	3	20	37	23	17
2	Analytic proof without hints (Midpt of parallelogram)	2.54/5 (51)	16	20	10	19	21	14
3a	Conditional statement (Similar triangles)	2.18/3 (73)	2	25	25	48		
3b	Stating the *given* (Similar triangles)	1.25/2 (63)	3	68	29			
3c	Stating the *prove* (Similar triangles)	1.92/2 (96)	2	4	94			
4	Synthetic proof without hints (Isosceles, overlapping triangles	2.55/5 (51)	0	43	8	13	23	13
5	Synthetic proof with hints (similar triangles)	3.49/5 (70)	2	14	11	13	24	36
6a	Local deductions (Segment midpoint)	1.71/3 (57)	18	30	15	37		
6b	Local deductions (Intersecting segments)	0.64/3 (21)	70	10	6	14		
6c	Local deductions (Congruent triangles)	1.89/3 (63)	13	25	21	41		
6d	Local deductions (Non-parallel lines)	0.67/3 (22)	71	7	5	17		

ditional statement. The items also required the identification of "given" and "prove" statements. Students in all four classes had the most difficulty specifying all required conditions in the conditional statement. However, they had very little difficulty constructing "given" and "prove" statements that correctly corresponded to their conditional statement or to the original conjecture.

Items 6 a-d, which required students to draw a conclusion from a given set of conditions, proved to be much more difficult for all students than we expected. For these

items, students were provided with written statements describing given conditions, none of which were accompanied by diagrams. Average student performance ranged from 21% correct on part B to 63% correct on part C. Because each part only differed by the geometric content, it appeared that the combination of item type and context were influential in determining item difficulty. Students had very little difficulty concluding that a point equidistant from and between two other points was a midpoint, but had much more difficulty drawing any correct conclusion from two intersecting segments. During clinical interviews, many of the focus students noted that they were uneasy with the open nature of the local deduction items. They claimed to be much more comfortable proving a "fact," than making one or two deductions (for which they had to provide reasons) of their own choosing.

Although all four teachers emphasized the importance of drawing a diagram and marking it with given information before beginning a proof, this approach was not found to be useful for students in solving items 6b and 6d. For example, of the 11 focus students (out of 16) who chose to draw a diagram for Item 6b, none of them provided strong conclusions (using all given conditions) with valid reasons and only two of them gave valid but weak conclusions (using only some of the given conditions) with valid reasons. In addition, for Item 6d, out of the 12 focus students who used a diagram only four of their conclusions (strong or weak) and reasons were valid. Some of the incorrect conclusions drawn by students were restatements of the given information or answers left blank. Other incorrect conclusions involved assuming additional information that was not provided (such as assuming that *intersecting* segments *bisect* each other) or carelessly choosing one conclusion from a family of conclusions, without checking for the validity of the conclusion (such as assuming that some relationship between special angles associated with parallel lines would prove that the lines were parallel). Figure 1 shows an incorrect response given by a student who did not draw a diagram. By using a diagram, the student may have been more likely to check the validity of her conclusion or reason. Another common incorrect response is shown in Figure 2. In this case, the student came to the conclusion that P was the midpoint of the line segments XY and ZW. Two of the participating teachers conjectured that the students may have assumed that the segments bisected each other, because point P was said to be between the endpoints of *segments* rather than between two points on given *lines*. Although we are not able to determine the exact cause of the misconception, it is clear that students did not attend to the precise meaning of the symbols and words that were used to specify the given conditions.

6c. Given: ΔLMN and ΔPQR. $\angle L \cong \angle P$. $\overline{LM} \cong \overline{PQ}$. $\angle N \cong \angle R$.
Conclusion: Triangle LMN is congruent to triangle PQR
Reason: Angle-side-angle postulate

Figure 1. Sample student response to item 6c.

6b. Given: $\overline{XY}$ intersects $\overline{ZW}$ at point P. Point P is between X and Y.
Point P is between Z and W.
Conclusion: XP=PY and ZP=PW
Reason: P is a Midpoint of XW and ZW

Figure 2. Sample student response to item 6b.

Item 5, which required students to write a multi-step synthetic proof based on given information, a diagram, and a collection of hints (essentially an outline of the proof), was less difficult for students than we expected. Although students in Mrs. C's and Mr. D's classes said that they ignored the hints (preferring, again, to do things their way), the average score for all four classes was about 70% correct. The item required students to prove that two triangles were similar that were embedded in a diagram with two pairs of parallel lines. Some of the incorrect responses included unnecessary steps, which may have taken students off-track. These students did not appear to have a sense of the direction of the proof, despite the outline provided with the hints. Some students wrote little more than the given information.

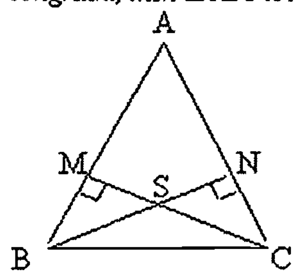
The full proofs, without hints, (Items 2 and 4) were very difficult for most students. Very few (14% and 13%, respectively) students gave fully correct responses. An additional 21% and 23%, respectively, wrote proofs that were satisfactory, containing roughly 4/5 of a correct argument. Correct responses tended to be mostly two-column proofs, but paragraph proofs were not uncommon, particularly from students of Mrs. C or Mr. D, who often used paragraph proofs in class.

For Item 2, the analytic proof, correct proofs either relied on the midpoint formula or properties of a parallelogram. Incomplete or incorrect proofs lacked justification for using the midpoint formula or lacked enough information to make a formal argument. This was particularly true in Mrs. A's and Mrs. B's classes in which students rarely encountered analytically represented figures associated with a proof. When figures were presented on a coordinate axis, it was in the context of the application of a theorem. These problems only required students to perform computations, without providing justification. During clinical interviews, students in Mrs. A's and Mrs. B's classes said they were unsure of what was required beyond the computations.

For Item 4, students were asked to show that a triangle was isosceles if the altitudes from the two base vertices were congruent (see Figure 3). Several students attempted to prove the two smaller triangles MSB and NSC congruent, although there was insufficient information to do so. Students either left out steps in their proofs, provided erroneous reasons for statements, or left reasons blank (perhaps, hoping for partial credit).

One of our focus students, Kevin who earned one out of a possible five points on Item 4, gave the following proof (copied from original script).

4. Consider the conditional statement and the accompanying diagram.
"If two altitudes, $\overline{BN}$ and $\overline{CM}$, in ΔABC intersect at point S and are congruent, then ΔABC is isosceles."



Write a proof of the statement.
Give geometric reasons for the statements in your proof.

Figure 3. Item 4 from Proof Construction Assessment.

Statement

1. BN ≅ CM (Given)
2. ∠NBC ≅ ∠MCB (Opp. ∠'s of ≅ opp. seg. are ≅)
3. triangleNBC ≅ triangleMBC (ASA)
4. ∠NBC + ∠MSB = ∠MCB + ∠NSC (Angle add.)
5. ∠ABC = ∠ACB (Substitution)
6. ∠ABC ≅ ∠ACB (If ∠'s are =, then they are also ≅)
7. AC ≅ AB (Opp. seg. of ≅ ∠'s are ≅)
8. triangle ABC is isos. (Def. of isos. triangle)

It can be noted that Kevin has a beginning and an end in his proof, but he writes in between does not make much sense. For example, he makes a hasty conclusion, with insufficient reasons, that triangle NBC and triangle MCB are congruent. It shows that Kevin focused on many irrelevant details and left out essential ones.

In the next section, we provide examples of teachers' pedagogical choices and describe aspects of the classroom microculture. We use this data to further explore possible connections between the social environment of the classroom and students' ability to construct proof.

Analysis of Teachers' Pedagogical Choices and Classroom Microculture

As noted earlier, we conjectured that the classroom teachers' pedagogical choices would have an impact on the students as they began to construct an understanding of geometric proofs. It might be more accurate to say that the teachers' pedagogical choices influenced the classroom microculture, including students' expectations for acceptable and valid proofs, hence influencing their understanding of how to construct valid proofs.

In some ways, the four participating teachers were alike. For instance, all four teachers followed the chosen textbook quite closely for structuring daily lessons and for assigning homework. In addition, the teachers and students used very little technology (one to two days a semester for Mrs. A and Mrs. B, four to five days a semester for Mrs. C and Mr. D) and very few hands-on investigations to help students explore geometric ideas that they studied in class. All teachers allowed the students to work with partners or groups on occasion to discuss proofs or other related problems. However, one major pedagogical difference between the classes was apparent. Mrs. A and Mrs. B chose to use geometric proofs as applications of the theorems and concepts they studied in class. In other words, Mrs. A and Mrs. B would introduce a new concept or theorem, demonstrate the concept, then show how to use the concept in a proof. In fact, this often led to students learning a particular kind of proof for a particular concept. In contrast, Mrs. C and Mr. D were more likely to use proofs to introduce new concepts or theorems to students. As a result, proofs assigned by Mrs. C and Mr. D were often the basis of the next day's lesson, whereas proofs assigned by Mrs. A and Mrs. B were rote applications of proof-writing procedures with limited student autonomy in problem solving and proof writing. This may explain the poorer performance of students in Mrs. A and Mrs. B's classes on the Proof Construction Assessment. In particular, the students in these two classes had more difficulty than other students in writing unsupported proofs (Items 2 and 4). Only with the guidance of the interviewers were the focus students from Mrs. A's and Mrs. B's classes able to make progress on these items.

A sociomathematical norm that appeared to be accepted in at least three of the four classrooms was the expectation that all mathematical problems can be solved in a relatively short period of time. Teachers contributed to the perception that mathematical problems can be solved quickly by providing examples that were always provable and usually in a few steps. As a result, students developed very little perseverance, in terms of reasoning ability, and gave up quite quickly on challenging tasks. Students rarely spent very long on a particular proof. This may be the reason why some of the students' responses to the proof items without hints (items 2 and 4) on the Proof Construction Assessment tended to be brief.

To determine what the students' deemed as valid mathematical proofs, the focus students from each class were given three different proofs of the same statement that

were constructed by other students. The focus students were asked to examine and grade the proofs. Most of the focus students claimed that they were able to follow the reasoning provided although many were not able to detect mistakes in the logic of the proof. Three separate norms were revealed as students from the various classes expressed the reasoning for grading the proofs as they did. For instance, three of the four focus students from Mrs. C's class commented on the overall creativity of the proofs. They were willing to assign a higher grade to a proof they saw as creative or one that was built on an idea that they did not come up with themselves. Similarly, the focus students from Mr. D's class commented on the elegance or lack of elegance in the proofs, which is something Mr. D stressed with his students during the school year. The focus students from Mrs. A's and Mrs. B's classes appeared to be more concerned with the level of detail in the proofs. These students were less likely to consider the overall structure of the proof. Again, this seems to be in concert with the choices Mrs. A and Mrs. B made to spend class time critiquing the details of students' proofs, rather than the choice of Mrs. C and Mr. D to also reflect on the overall proof as a way of making sense of the geometric concepts involved.

A classroom practice that appeared to have an influence on students' ability to construct proofs was the taken-as-shared perception that drawing and marking a diagram was a necessary prelude to constructing a proof. All four teachers frequently reiterated the importance of this step in class by marking diagrams when they wrote out formal proofs and when they "talked through" a proof without recording statements or reasons. Diagram marking gave students an opportunity to make some progress on just about every proof. The practice of redrawing complicated diagrams, also emphasized by the teacher, was not very well followed by the students on the Proof Construction Assessment. The students who used this strategy were generally successful in the problem with overlapping triangles (Item 4, as shown in Figure 3 above).

Interviewer: Do you usually draw your own diagram or do you use the diagram given?

Aaron: Well, like if we are trying to prove overlapping triangles then I'll usually separate the triangles out, ... And then I'll also look at the one given ... so I could see the reflexive property, because if you break it apart you can't really see it. So I draw them out and I also use the diagram given.

Interviewer: So you mark things [on the diagram] as you are going through?

Aaron: Mrs. A usually tries to preach to us ... that it would probably help us to mark stuff down to like ... mark the givens and mark the ... things we see down, and write it out before we go through and start the proofs. I've caught on to that and it really helps.

A second mathematical practice that was noticed in both Mrs. A's and Mrs. B's classes was that students seem to know that they should identify the *given* and *prove* for the geometry proofs presented in their homework or other class discussions. This practice helped the students to articulate the logical beginning and the end of their written proofs. Some students who often failed to go beyond the beginning and end perceived this practice negatively. This was noted during clinical interviews when a couple of the focus students indicated that they believed their proof was valid because they knew where to start and end and they had some statements and reasons in between (which may not have been logically connected). This may be tied to the "follow-the-pattern" proof writing that was emphasized by Mrs. A and Mrs. B.

Two of the teachers (Mrs. A and Mrs. B) openly encouraged students to memorize definitions, theorems, and postulates. This became a taken-as-shared method of learning these important facts of geometry. Even so, the students in these classes often claimed to not remember the substance of the definitions, theorems and postulates and demonstrated this by incorrectly recalling the statement of the theorems when asked. For the Proof Construction Assessment, the students were provided with a list containing the definitions, theorems, and postulates as stated in their textbooks. Surprisingly, most students claimed they did not use this as a reference. The dialogue below occurred during an interview with one of the focus students from Mrs. A's class.

Susan: Cause like when she told us about Angle-Angle and all the ones that you can get with right angles, like Hypotenuse-leg, I never understood it. And I never remembered any of it...

Interviewer: Did you look at the sheet that was given that had the theorems and postulates listed?

Susan: It wasn't helpful, because I was kind of like Ok, so what does that say?

Interviewer: So as you were reading them off the sheet, that was difficult for you?

Susan: Yeah, since I didn't grasp the concept the first time when she taught it to us, it didn't really even matter that it was on the paper.

Although the other two teachers (Mrs. C and Mr. D) did not explicitly state the need for memorization of definitions and theorems, they expected the students to begin to learn these through frequent use. Students were encouraged to write out theorems in their proofs, rather than to write a title for the theorem, such as the Angle-Angle Similarity Theorem as referenced in the dialogue above. At this point in our research, it is not clear how familiarity with and understanding of theorems and definitions influence student proof writing ability. It is becoming apparent, however, that the teachers' pedagogical choices greatly influence the students' views of what constitutes a valid

proof, as shown in the discussion of the interview questions related to students grading proofs done by peers. The students' growing sense of what constitutes a valid proof appears to have played a role in the students' proof construction.

As the analysis of our rich data continues, we hope to identify more clues related to the teachers' choices and aspects of the classroom microculture that influence student understanding of proof. At this point, our findings support those of Senk (1985) who identified aspects of proof construction that were difficult for students. It was most difficult for students to write formal proofs of statements with no hints given. We are beginning to notice, however, that teachers' expectations play a key role in student proof construction ability. In particular, it appears that if teachers focus on the overall structure and need for proofs in understanding the underlying mathematical concepts, students will also develop a better sense of the need for proof. On the other hand, if teachers expect students to learn to do proofs in a more mechanistic way, the students are likely to see proofs as just another exercise or application and will not develop a more complete understanding of proofs and how to construct proofs. These findings are supported by Battista and Clements (1995) who suggest that the teaching of formal proof should follow from helping students make sense of mathematical ideas. In other words, proofs should be seen as a way to establish the validity of ones ideas. Further analysis of our data is needed to determine how the teachers' expectations influenced student ability to construct proofs.

One surprising result was the difficulty students had with the local deductions problems. Although it is not part of our current plan to investigate this further, these open-ended deduction items may be a key step in creating formal proofs. Another avenue of analysis will be to investigate the students' proof construction ability in terms of the proof schemes described by Harel and Sowder (1998). These proof schemes are defined as what the student believes to be a valid way of ascertaining truth for herself or himself as well as persuading others of the truth of a situation or observation. For those students in Mrs. A's and Mrs. B's classes, it is unlikely that they hold very strong proof schemes, since from interviews with focus students, they appear to view proofs as exercises and not necessarily a means of ascertaining truth.

Note

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