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ABSTRACT

This theoretical paper outlines a conceptual framework for examining growth in prospective teachers' mathematical understanding as they engage in thinking about and planning for the mathematical learning of others. The framework is based on the Pirie-Kieren (1994) Dynamical Theory for the Growth of Mathematical Understanding and extends into the pedagogical realm by assuming that growth in mathematically specific understanding for teaching is a dynamical, leveled but not linear, transcendentally recursive process of reorganizing ones knowledge. Data from a preliminary case study is shared to illustrate how the framework can be used to provide a lens for examining growth in mathematical understanding within the context of learning to teach high school mathematics. (Author)

GROWTH IN MATHEMATICAL UNDERSTANDING WHILE LEARNING HOW TO TEACH: A THEORETICAL PERSPECTIVE

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This theoretical paper outlines a conceptual framework for examining growth in prospective teachers' mathematical understanding as they engage in thinking about and planning for the mathematical learning of others. The framework is based on the Pirie-Kieren (1994) Dynamical Theory for the Growth of Mathematical Understanding and extends into the pedagogical realm by assuming that growth in mathematically specific understanding for teaching is a dynamical, leveled but not linear, transcendentally recursive process of reorganizing ones knowledge. Data from a preliminary case study is shared to illustrate how the framework can be used to provide a lens for examining growth in mathematical understanding within the context of learning to teach high school mathematics.

Determining ways to challenge and extend prospective teachers' ideas about *school mathematics* (the mathematics they will teach) is recognized as one of the most important matters to be considered by mathematics educators today (Conference Board of Mathematical Sciences (CBMS), 2001; Sowder et al., 1998). To break the perpetual cycle of inadequate knowledge of school mathematics, teacher preparation leaders were advised to modify their programs of study to provide better preparation for future teachers in both *university mathematics* (typical college mathematics courses) and school mathematics. This situation leaves teacher preparation program leaders faced with deciding exactly how mathematics preparation can be accomplished both efficiently and meaningfully without neglecting the much-needed time for pedagogical development.

Research results (Berenson & Cavey, 2000; Bowers & Doerr, 2001; Cavey, Berenson, Clark, & Staley, 2001; Clark, 2001; Ma, 1999) suggest that engaging prospective teachers in *teaching tasks* (thinking about and planning for the mathematical learning of others) may be an effective way of addressing both content and pedagogical developmental needs. Bowers and Doerr (2001) engaged prospective and practicing teachers in computer-based activities as learners of the mathematics of change (rate of change) and then as teachers of rate of change and observed both mathematical and pedagogical insights made by the prospective and practicing teachers as they were engaged in both types of activities (as learners & as teachers). Berenson and Cavey (2000), Cavey, Berenson, Clark, and Staley (2001), and Clark (2001) conducted preliminary studies to investigate the plausibility of using an experimental curriculum, "lesson plan study" (LPS), to promote growth in prospective teachers' understanding of school mathematics and teaching strategies. As LPS participants, prospective secondary mathematics teachers engaged in multiple conversations with others (researchers and peers) about teaching a specific secondary mathematics topic. Such conversations occurred over five weeks on the same lesson topic and seemed to promote growth in prospective teachers' understanding of school mathematics (Berenson & Cavey, 2000) and teaching strategies (Cavey et al., 2001; Clark, 2001).

The use of these teaching task methodologies leads to theoretical questions about prospective teachers' growth in mathematical understanding. As such, a conceptual framework was developed to provide a lens for examining prospective teachers' growth in mathematical understanding as they participated in LPS. Here, I briefly describe each component of the conceptual framework, the first activity of LPS, and provide an example to illustrate how the framework can be used.

The Pirie-Kieren Model: Growth in Mathematical Understanding

The Pirie-Kieren (1994) Dynamical Theory for the Growth of Mathematical Understanding depicts understanding as a dynamical, leveled but not linear, transcendentally recursive process of reorganizing ones knowledge. The model is a theory *for* the growth of understanding of a specific mathematical topic by a specific 'person'[i] over time and is comprised of layers of sophistication in thinking that describe the mental activities necessary for growth in mathematical understanding of a particular topic. It is assumed that a learner comes to a particular learning situation with primitive knowledge [all knowledge *not* related to the particular topic] as well as some knowledge of the particular topic, identified by some outer layer of thinking. The seven outer layers of thinking are: Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring, and Inventising.

Pirie and Kieren (1994) asserted that each of the seven outer layers "is composed of a complementarity of *acting* and *expressing*" where "acting encompasses all previous understanding, providing continuity with inner levels, and expressing gives distinct substance to that particular level" (p. 175). Both acting and expressing involve mental as well as physical actions. Through acting, the learner may reflect on how their previous understanding applies to a new learning situation. When expressing, however, the learner makes it clear to oneself or others what knowledge was gained. The terms used for acting/expressing complementarities within the image making, image having and property noticing layers are doing/reviewing, seeing/saying, and predicting/recording.

Critical to the theory is the idea of recursion—that learners revisit layers of thinking in the process of extending their mathematical understanding. Layers are revisited with more sophisticated thinking along one thread of a particular topic in an attempt to broaden and deepen knowledge of that topic. Folding back is the term used to describe the mental activity of accessing one's more primitive knowledge to construct mathematical understanding at an outer layer of thinking. This notion exemplifies the idea that an individual's mental activities do not move in one direction. Rather, an individual functioning at an outer level of understanding will repeatedly return to an inner level to extend their mathematical understanding (Martin, 1999). For example, in formalising his or her understanding of fractals, a mathematician may fold back to his or her knowledge of complex numbers to understand why fractals portray self-similarity. In comparison, an elementary student may fold back to his or her knowledge of whole numbers to develop a rule for finding a common denominator while formalising his or her understanding of the fraction concept.

A deeper look at folding back (Martin, 1999) reveals the complex nature of this process. In particular, not all acts of folding back are necessarily effective in extending mathematical understanding. Research results indicated that the effectiveness of folding back depends on both the structure of the environment and the individual learner and that folding back tends to be more effective when the learner is prompted to fold back to collect specific information (Martin, 1999). This type of folding back, collecting, "occurs when students know what is needed to solve a problem, and yet their understanding is not sufficient for the automatic recall of useable knowledge" (Pirie & Martin, 2000, p. 127).

Extending the Pirie-Kieren Model into the Pedagogical Realm

The framework is extended into the pedagogical realm by assuming that *teaching mathematics understanding* (understanding enacted when making decisions about the mathematical learning of others) draws upon three mathematically specific understandings: mathematics, mathematics teaching strategies, and mathematics learning. While it is obvious that teaching mathematics understanding draws upon other primitive knowledge types, such as understanding human behavior, the research for which this framework was developed is primarily focused on examining the development of the aforementioned mathematically specific knowledge domains.

It is assumed that growth in such understandings can be modeled as Pirie and Kieren (1994) modeled growth in mathematical understanding and that prospective teachers begin their teacher preparation programs, specifically their first methods course, with some level of understanding of mathematics, mathematics teaching strategies, and mathematics learning. Indeed, beginning prospective teachers typically have an outer layer understanding of mathematics and mathematical learning developed through coursework and other life experiences and images of teaching strategies developed through observation. Such outer layer understandings can constrain additional growth in prospective teachers' understanding of school mathematics (Berenson & Cavey, 2000) when prospective teachers do not see the need to revisit certain mathematical ideas. See Figure 1 for an illustration of how the Pirie-Kieren model is extended into the pedagogical realm.

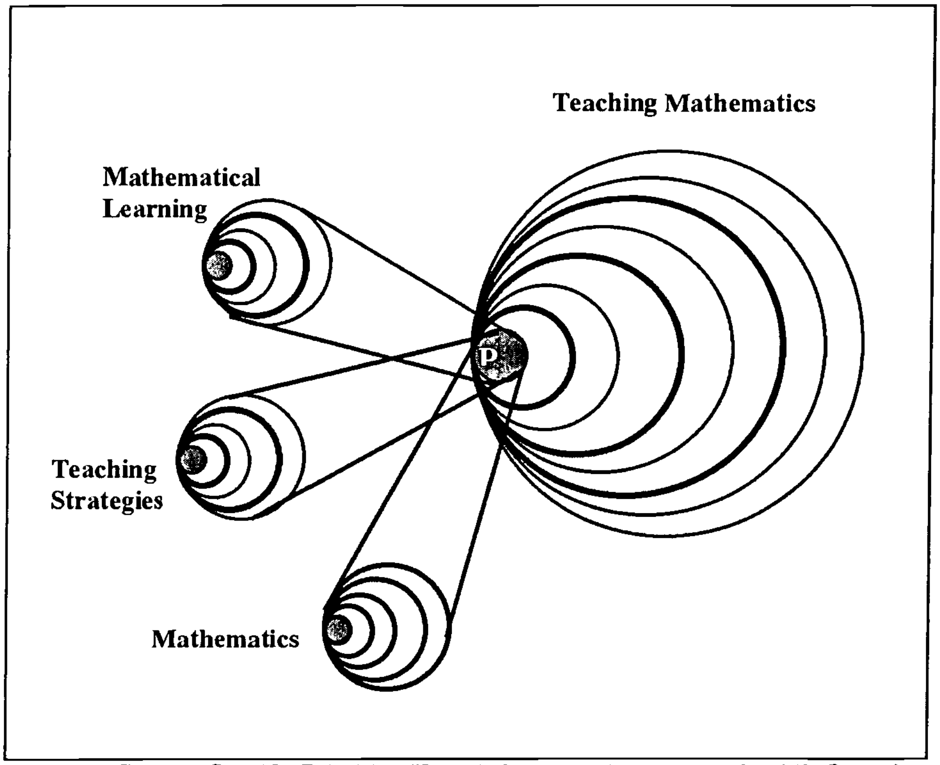

Figure 1. Content-Specific Primitive Knowledge Domains Accessed While Learning To Teach Mathematics.

Right Triangle Trigonometry LPS: The Framework In Action

Data from a preliminary case study is used to illustrate how the framework provides a lens for examining growth in mathematical understanding while one prospective teacher participated in a LPS. This LPS was conducted at the beginning of an introductory methods course for prospective secondary mathematics teachers at a large public university and focused the participants on teaching right triangle trigonometry. Data for one participant, Molly (pseudonym), are shared from the initial planning activity [preliminary interview, lesson planning, and post-planning interview], in the form of videotaped interviews and written artifacts. Table 1 contains a summary of the tasks included in the first activity of LPS focused on teaching right triangle trigonometry.

Table 1. Initial Planning Activity of the Right Triangle Trigonometry LPS

Task	Description	Duration
Preliminary Interview	Videotaped conversation between a researcher and a prospective teacher about what the prospective teacher remembered learning about right triangle trigonometry and preparation for lesson planning.	20 min
Creating Plan 1	Immediately after the preliminary interview, the prospective teacher was left alone with resources (texts, manipulatives, etc.) to plan a lesson to introduce right triangle trigonometry to a high school geometry class.	45 min

Initial Growth in Understanding Right Triangle Trigonometry

Image Saying

Even though Molly had difficulty remembering what and how she learned right triangle trigonometry, she shared two images of her understanding during her first meeting with the interviewer (the preliminary interview). One image was based on the memorization of an acronym whereas the other image was centered on using right triangle trigonometry to solve for an unknown angle measure. When Molly was asked about what she remembered learning about right triangle trigonometry, the following dialogue occurred.

Molly: I remember cosine, sine, and tangent.

I: Okay. Go ahead and write that down while you're explaining this. I want to know everything that you remember. [Molly wrote 'cos', 'sin', and 'tan' on her paper, with 'tan' directly below 'sin' and 'sin' directly below 'cos'.]

Molly: I don't remember all that much.

I: That's fine.

Molly: One thing that stands out is they told me something like socatoa. I'm trying to remember which one goes with which one. [Molly first wrote 'SOCA', then erased 'CA' and wrote 'HCAHTOA' so that 'SOHCAHTOA' was written across the top of her paper.] If I have this right, then this would mean that if you use sine, then you have opposite over hypotenuse, and if you use cosine, you have adjacent over hypotenuse, and if you had tangent, you had opposite over adjacent.

The subsequent conversation indicated that Molly recalled using right triangle trigonometry to determine an unknown angle measure. When asked about how right triangle trigonometry is used, Molly stated, "If you have a right triangle, the sides of it to try to figure out the angles, is all I remember." She seemed to be recalling a way of using right triangle trigonometry to determine an angle measurement by using the measurements of two sides of a right triangle.

Molly was clearly not confident in her ability to recall these or other ideas related to right triangle trigonometry. When asked about what she remembered about what her teacher showed her, Molly remarked, "All I remember is that sohcahtoa thing that stands out to me. I just remember cosine and sine." Molly recognized her limited memory and planned to 'fix' the problem through review. After Molly received instructions for the lesson-planning component, she asked, "Can I use these books to refresh?" Such an inquiry indicates that Molly planned to fold back and collect information about right triangle trigonometry for the purpose of planning her lesson.

Folding Back to Collect

As Molly created her first plan, she folded back to collect mathematically precise information concerning many ideas related to right triangle trigonometry. As indicated by her lesson plan notes, Molly folded back to collect 1) the definition of a right triangle, 2) the Pythagorean theorem, 3) definitions for sine and cosine, 4) the relationships between side lengths for 30-60-90 & 45-45-90 triangles, and 5) a definition for similar triangles. All items collected, except for the definition of similar triangles, were incorporated into her first plan.

Molly's collecting indicates that she valued mathematically precise definitions in her own learning and provides insight into her growth in understanding of right triangle trigonometry. Just the fact that four items were collected, in addition to the definitions of sine and cosine, indicates that Molly's right triangle trigonometry understanding thickened to include connections to the Pythagorean theorem, right triangles, relationships for special right triangles, and similarity. Collecting more precise definitions for sine and cosine indicates additional thickening in her understanding. By considering how these items were incorporated into her first plan, I gained additional insight on how her understanding of right triangle trigonometry changed.

Molly's first plan seemed to be geared towards solving for missing parts of right triangles. By starting the lesson with the definition of a right triangle, she set the stage for the context of the lesson. The Pythagorean theorem was viewed by Molly as a way "to calculate the unknown measure of a side of a right triangle given the measures of the other two sides." For Molly, this led directly to the definitions of sine and cosine since she thought of right triangle trigonometry as a way to find an angle measure based on the measures of two sides. In other words, she thought of both the Pythagorean theorem and right triangle trigonometry as means for 'solving for missing parts of right triangles'. Molly thought of the Pythagorean theorem as a means for using two side lengths to determine the third side length, whereas right triangle trigonometry was thought of as a means for using two side lengths to determine an angle measure. The relationships among the sides of triangles with angle measures 30-60-90 and 45-45-90 triangles seem to have been included to simplify the process when working with these types of triangles. Hence, Molly's understanding of right triangle trigonometry was thickened by connecting to mathematically precise information in relation to 'solving for missing parts of right triangles'.

Folding back to collect mathematically precise definitions for sine and cosine thickened Molly's understanding of right triangle trigonometry further. In her written notes, sine and cosine were referred to as ratios of the measures of sides. She wrote, "Sine is the ratio of the measure of the leg opposite the acute angle to the measure of the hypotenuse." She also referred to sine and cosine as ratios during the post-planning interview, indicating that she was image saying and had indeed remade her image of sine and cosine.

In essence, by the time she finished planning her first lesson, Molly collected mathematically precise definitions and relationships for topics she understood to be immediately connected to 'solving right triangles'. Collecting this information and incorporating the ideas into her lesson thickened Molly's knowledge of right triangle trigonometry.

Growth in Understanding Similarity

Image Saying

In her notes for plan 1, Molly wrote, "Similar triangles have 3 angles of 1 triangle congruent to 3 angles of another triangle and the measures of their corresponding sides are proportional." However, she had not incorporated the idea into her first plan and she struggled to explain the concept when prompted to do so during her second meeting with the interviewer. When Molly was asked how she might help a student understand similarity, she suggested a plan that started with giving the student the definition and then showing some examples. When asked to describe similarity she stated, "It's when two triangles have angles that are similar to one another." Molly was not confident in this response and quickly added, "I would have to look that up. I'm not sure to be honest with you," which indicates that Molly was interested in collecting more information on similarity.

Folding Back to Collect

The interviewer immediately prompted Molly to consider two triangles drawn on a separate piece of paper and asked how she would know if they were similar. Molly drew two right triangles and the following dialogue occurred.

I: Do they have to be the same size to be similar?

Molly: [Pause] No.

I: If they don't have to be the same size, what makes them similar?

Molly: If they are both right triangles, with maybe their angle measures the same, 45-45 or 30-60. As long as all three angles are the same, then they're similar, but they can have different lengths.

This dialogue seemed to help Molly rethink the relationship between corresponding angles of similar triangles, thereby remaking her image of similar triangles. It appears that she folded back to mentally collect part of the definition for similar triangles she had written in her lesson plan notes.

Summary of Molly's Growth in Mathematical Understanding

During the first meeting with the interviewer, Molly exhibited a limited understanding of right triangle trigonometry. In fact, Molly started the LPS with two images of right triangle trigonometry, the SOHCAHTOA acronym and the image of using right triangle trigonometry to determine angle measures. However, opportunities to share her mathematical images seemed to help Molly become aware of gaps in her own mathematical understanding and subsequently plan to collect mathematical information. In addition, as Molly made images for teaching right triangle trigonometry, she folded back to her mathematics understanding to collect, revise, and construct images of mathematics related to right triangle trigonometry. In addition, during subsequent LPS components, Molly repeatedly folded back to revise her understandings of right triangle trigonometry and related primitive knowledge domains. In particular, Molly revisited the ideas of ratio, similarity and the Pythagorean theorem in relation to right triangle trigonometry. An illustration of the primitive knowledge domains that Molly accessed while extending her mathematical understanding during LPS is provided in Figure 2.

Discussion and Implications

By placing the learning of school mathematics within the context of teaching tasks, prospective teachers rely on their teaching mathematics understanding and are thereby forced to fold back to their understandings of mathematics, teaching strategies, and mathematical learning to make decisions about the mathematical learning of others. However, there is much to be learned about the effectiveness of using teaching tasks in adding depth, breadth, and thoroughness (Ma, 1999) to prospective teachers' mathematically specific knowledge for teaching. Particularly, more research is needed to ascertain the effectiveness of using LPS to extend prospective teachers' mathematics teaching understanding.

The framework presented here provides a lens for examining growth in mathematically specific components of teacher knowledge (mathematics, mathematics teaching strategies, and mathematics learning). Initially developed to examine growth in prospective teachers' mathematical understanding as they participated in LPS, the framework can also be used to examine growth in prospective teachers' understanding

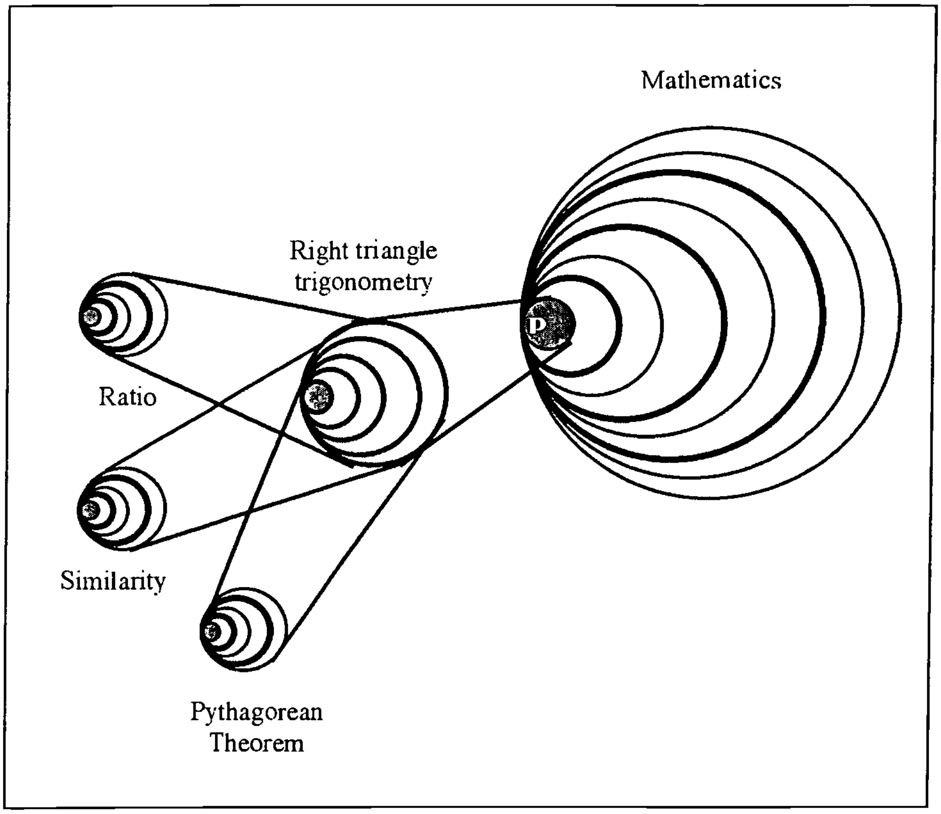

Figure 2. Primitive knowledge domains in mathematics understanding that Molly accessed during LPS.

of teaching strategies and/or mathematics learning. In addition, the framework can be used to examine growth in teaching mathematics understanding for any learner engaged in making decisions about the mathematical learning of others. Applying the framework to other teacher-learning situations is expected to contribute to understanding how teachers learn to teach mathematics. At a very minimum, future applications of this framework must consider the thoughts and actions of prospective teachers when interacting with k-12 students.

Note

[i]'Person' can refer to an individual learner or any size group of learners. Essentially, it can be thought of as a 'learner unit', comprised of any number of individuals that are jointly learning.

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