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ABSTRACT

One hundred and seven 5th-8th graders were tested on spatial rotation multiple-choice items to determine age and gender differences in spatial ability. Thirty-one of them were subsequently interviewed. They were asked to explain their reasoning when solving 4 of the tested items and a problem-solving task. Features of visual and non-visual strategies used were identified, however students did not make consistent use of only one type of strategy across tasks. They switched between visual and non-visual strategies or made combined use of both. Particular task characteristics may have influenced the choice of strategy in each case. Older students seemed to perform better on items involving three-dimensional manipulation but no other age and gender differences in strategy use and in spatial ability were detected. (Contains 27 references.) (Author/MM)

Running head: SPATIAL REASONING IN ROTATION TASKS

Students' solution strategies in spatial rotation tasks

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Abstract

One hundred and seven 5th-8th graders were tested on spatial rotation multiple-choice items to determine age and gender differences in spatial ability. Thirty-one of them were subsequently interviewed. They were asked explain their reasoning when solving 4 of the tested items and a problem-solving task. Features of visual and non-visual strategies used were identified, however students did not make consistent use of only one type of strategy across tasks. They switched between visual and non-visual strategies or made combined use of both. Particular task characteristics may have influenced the choice of strategy in each case. Older students seemed to perform better on items involving three-dimensional manipulation but no other age and gender differences in strategy use and in spatial ability were detected.

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Introduction

Applying transformations and using visualization and spatial reasoning are two of the principal standards for geometry in the standards for school mathematics (National Council of Teachers of Mathematics, 2000). In educational practice though, rules and formulas, procedures and analytical thinking, are dominant elements in the mathematics curriculum. School geometry is taught in a formal manner, while visualization and intuitive sense about space do not receive much attention.

Spatial reasoning "consists of the set of cognitive processes by which mental representations for spatial objects, relationships, and transformations are constructed and manipulated" (Clements and Battista, 1992, p.420). Spatial rotation is one form of transformation involving mental manipulation of a figure or an object, appreciating how it would appear from a different viewpoint, how would it seem if it were turned around, but not when reflected. As with many other kinds of problems, rotation problems can be approached in two general ways: intuitive/visual or analytic/non-visual (Bruner, 1971; Gorgorió, 1998).

In contemporary research and development in mathematics education, one of the most important challenges and opportunities is to "[o]bserve carefully how students *actually* think about the topic under discussion, and build upon this process" (Davis, 1992, p.724, emphasis in original). Such an approach requires knowledge of the students' mental representations and how they operate upon them. This study investigates characteristics of the visual and non-visual strategies that children employ when solving problems of spatial rotation through the following research questions:

1. What are the characteristics of the visual and non-visual strategies students employ when solving spatial rotation problems?

2. What are the difficulties students encounter in this process?
3. Are there any age and gender differences in tested spatial ability and spatial reasoning?
4. Are there any differences in spatial reasoning between students whose tested spatial ability is high or low?

Insights on the repertoire of students' spatial thinking strategies, which may well vary from those of the teacher, can reveal the difficulties they encounter and the errors they make. From a didactic perspective, strategies are more useful as a construct than classifications of students according to levels of abilities, or preferences in processing modes. Independently of each student's cognitive style, "strategies can be shared and therefore taught" (Davis, 1992, p.226), thus enriching their available problem solving tools. Teachers may utilize them for more effective teaching, by offering new ideas about different models of thinking and concept formation. There are also implications for the widely implemented formal instruction of geometric concepts; it is a common belief that if instruction is not sufficiently connected with children's preconceptions to be meaningful, learning obstacles result. From a developmental point of view, knowledge about the influence of age on spatial reasoning can guide the design of instructional and assessment material suitable for specific age groups.

Theoretical perspectives

Piagetian research on the child's representation of space deals with the developmental nature of spatial and geometric concepts, as an indispensable part of the logical growth of the child. In 'The Child's Conception of Space', Piaget and Inhelder (1956) described the progressive transition of spatial reasoning – from topological, to projective, and later to Euclidean concepts. At the end of the sensori-motor stage young children become capable of mental imagery, which remains static during early childhood and no mental operations may be performed upon it. The intellectual structure of pre-operational children limits their ability to shift perspectives on spatial judgment tasks. This capability of retaining the static configuration of an object is named figurative and is distinct from and a prerequisite of the operative capability of transforming the configuration. As children become less egocentric and more able to co-ordinate several dimensions simultaneously, for example coordinating right and left, front and back, their ability to perceive scenes from alternative perspectives improves. The child is capable of more active manipulation of objects during the concrete operational stage, when reversible operations and decentration are accomplished. Finally, in the formal operational period the space realm can be represented and manipulated in the abstract. Even when a child can make use of mental representations of space, it may still be difficult to express these representations it in other formats, for example, verbally or pictorially.

The Van Hiele model is concerned with the developing structures of geometrical thinking and suggests a 5-level-sequence proceeding "from a Gestalt-like visual level through increasingly sophisticated levels of description, analysis, abstraction, and proof" (Clements and Battista, 1992, p.426). Young children recognize figures by their global appearance. Then, they can analyze the properties of figures, but only later do they become capable of relating properties with the respective figures. At the two highest levels, students develop sequences of statements to deduce one statement from another and analyze various deductive systems rigorously. Students' progression through stages depends on instruction, so the theory proposes phases of learning analogous to the levels of thinking.

The nature and development of spatial abilities have been studied in other research traditions as well: in psychometrics research on intelligence, in brain research and hemispheric specialization, in neo-Piagetian theories of cognitive development (Demetriou et. al., 1992), and in Gardner's (1993) theory of multiple intelligences.

In a more educationally relevant framework Krutetskii (1976) tried to reveal students' mental processes, through individual interviews and individually administered tests after experimental instruction. From his research on mathematically gifted students he proposed a typology of "preferred processing mode" with three types of students according to the way they interpret the world mathematically, and their preferences in problem solving. A small proportion of children showed a general preference in applying analytico-deductive and verbal-logical (analytic type), or visual-pictorial (geometric type) procedures when solving mathematical problems. The majority of the gifted children used both abstract and pictorial representations (harmonic type), according to the context of the problems.

Assuming that most children operate in a versatile way combining different strategies for a task or for different tasks, Gorgorió (1998) proposed a distinction between processing strategies. A strategy is considered visual when one can elicit from the student's explanation and observation that visual images had been used as an essential part of the solution. Verbal characteristics of visual strategies are not very specific or detailed and are often accompanied by gestures in an effort to describe a mental movement. Representations are often treated as a whole. On the other hand, a non-visual or analytic strategy is one in which visual images are not used, rather an argument is put forward to justify the solution process. Properties of the representations are used in non-visual strategies - e.g. left-right and up-down parts, counting of features, relative position to the configuration - to support an argument.

Attention is given to specific parts of the representation. Little has been published on the strategies present in the solution of geometric tasks. There is not much evidence comparing the effectiveness of the two processing strategies, but there is generally an agreement that any or both could be employed to solve a problem.

On a more specific level, the choice of a certain strategy may be provoked by the characteristics of the task (Gorgorió, 1998). In multiple-choice items, visual strategies are used when the manipulated object is simple and non-visual when it is more complex. When the choices are clearly different, students tend to focus on the object as a whole (global approach). A different approach (partial) is identified when they concentrate on the relative position of the objects' parts.

Spatial tasks are often cited as examples of gender differences in performance. There is evidence in favor of males in some spatial tasks (Fennema and Sherman, 1978); the difference however is identified after children enter adolescence (Maccoby and Jacklin, 1974). Also boys have been reported to be more efficient in using non-verbal modes in contrast to females, who preferred verbal modes (Clements and Battista, 1992).

Methodology

The study was conducted in grades 5 to 8 in two middle-class schools in a large town in Cyprus: an elementary school and a gymnasium. A test was constructed and administered to a randomly selected sample of 107 10- to 14-year-old students, to assess the spatial ability of mental rotation and determine any age- and gender-group differences in achievement. A large number of multiple-choice items obtained from standardized tests on figural or two-dimensional and block or three-dimensional rotation (Eliot and Macfarlane Smith, 1983) were piloted to determine items not extremely easy or difficult for students in those grades. The items were translated into Greek, and some were simplified and adjusted to the age and experiences of the children. The administered test consisted of 19 multiple-choice items. Figure 1 demonstrates one test item.

Insert Figure 1 about here

Four test items (including the one shown in Figure 1) and a problem-solving task were administered as interview tasks. The multiple-choice items asked students to single out the rotated representation that was the same as (or different from) an unrotated prototype. A smaller sample of 31 students was selected for individual interviews to study their solution strategies, the more private and personal ability for visual processing (Bishop, 1983). While preserving randomness in the selection procedure, equal numbers of males and females, elementary school and gymnasium students, and students scoring above and below the test mean were interviewed. The interviewer was in all cases the researcher himself, aiming to preserve consistent conduct of the interviews. All interviews were tape-recorded, any notes the students made were collected and the interviewer took notes on actions or gestures, which might suggest the strategy used. Students were first encouraged to explain their reasoning while solving the tasks. In case they were not articulate about their solution strategy they were prompted with specific follow-up questions:

- Is the figure/object manipulated as a whole or is the student concentrating on parts of it?
- Is he/she imagining any mental movement of the figure/object?
- If he/she concentrating on parts-characteristics of the object/figure, which are

these and how is he/she dealing with them?

In the end, the students were asked to confirm whether they had used a strategy with the characteristics of a visual or an analytic approach. From the collected information, it was determined which strategy each student used in each task.

Results

Analysis of the test[2] data showed no significant differences in achievement between gender groups, or between the two age groups (primary school and gymnasium), even though the mean of the males and of the gymnasium students were slightly higher than those of the females and the primary-school students, respectively (Table 1). Dividing the overall score into subscales of items with representations of two versus three dimensional objects older children scored significantly higher than the younger ones on the three-dimensional subtest ($t = 2.14$, $p < 0.05$). This may suggest that the effect on the overall score derives more strongly from the three-dimensional items. Overall performance on the two-dimensional items was higher than on the three-dimensional, but the difference was not statistically significant. The respective mean scores were 7.38 out of 9 (SD = 1.67), compared to 7.23 out of 10 (SD = 1.88) and the 95% confidence interval for the difference between the two proportions was (-0.277, 0.471). The two subscales were correlated ($r = 0.46$ and with correction for attenuation $r_{disatt.} = 0.86$). Item p-values ranged from 0.55 to 0.97 ($\overline{p} = 0.77$).

Insert Table 1 about here

The analysis of the interview data showed that there were no students who

[2] Cronbach's α reliability coefficient for the test was 0.71.

consistently used one single kind of strategy in all five tasks. In fact, what often occurred was that all students had switched between visual and analytic strategies for different tasks or combined both at some tasks. Table 2 shows the number of students who had used visual, non-visual, or combined both strategies for each task.

Insert Table 2 about here

Tasks 1 and 2 involved three-dimensional objects and almost all students had used a non-visual strategy or a non-visual with references to a visual image movement in their explanations. Very few occasions were identified as merely visualizing the rotation. The opposite happened for Tasks 3 and 4, the rotation of two-dimensional figures. Almost all of the students had implied the use of visual manipulation of the figures. Some of the older students had in some stage referred to analytic features, such as the left and right of the figure, or the direction it was "pointing to". On these four task students performed very well. They gave correct answers quickly and straightforward explanations of their solution. Tasks 1 and 2 seemed to be a bit more demanding than 3 and 4, with some students getting confused by some of the alternative choices at the beginning, but then changing their responses, producing reasonable explanations.

The problem-solving task (task 5) was notably different from the rest. Three dice were presented as in Figure 2, with one number missing. The task was to find the missing number without looking at the other sides of the dice. Seven students did not give a relevant response. Most students' reasoning was incomplete, while there was no clear tendency towards using any one of the visual, non-visual, or combined strategies. No particular strategy seemed to lead to a complete solution, either correct, or wrong. The five students who eventually reached the right answer came from all four grades, four of them were boys and four had scored high on the test.

Insert Figure 2 about here

Features of visual strategies

Visual strategies were not systematized, difficult to verbalize clearly and accompanied by gestures. Students using visual strategies concentrated on the representation as a whole, and relied on a general intuitive impression. For example:

MP: C is not a rotation.
INTERVIEWER: Why?
MP: Because when it turns, it will not do this... when the original shape is turning it will not... let's say it will go like this [makes a circular motion with her finger and points at multiple-choice option B], it will transform to this [D], then this [A] and finally it will make a circle.
INT: That is, it will not take this position [C] at all?
MP: It depends. No. First, I will do how this [original] shape is rotated ... Then, I form in my mind every movement; when it turns, what movement will be made.
INT: Yes...
MP: And then I try to see which one of those is not like the rotation (F, 7, low, Task 3).

INT: When you say that you take the original shape and you rotate it, what do you mean?
SO: I am turning it to take different positions in my mind (F, 6, low, Task 3)[3].

Another main characteristic of visual stategies is an intuitive claim of obviousness about the answer.

INT: How did you find C? You saw only that one, or did you check all of them?
GI: No, my eye just fell on that immediately... (M, 8, high, Task 3).

AKY: I saw A and B and found out that they are not [the same as the original]...
INT: Why?
AKY: Because they have fewer cubes.
INT: Did you count them?
AKY: It seems so (F, 6, high, Task 1).

Occasionally, in order to "see" the rotated representation, a real movement was

[3] Four characteristics of the student are reported in parenthesis: gender, grade, performance on the test, and number of task she or he is answering.

suggested instead of a mental one.

MP: We have to go on the back (F, 7, low, Task 2).

Features of non-visual strategies

Non-visual strategies focused on examining parts and special features of the objects sequentially, were well justified verbally and provoked logical arguments and comparisons, rather than mental manipulation of images. They were identified mainly by the use of local properties of the representations as reference. Students concentrated on specific parts and not the whole and justified their reasoning referring to changes in these parts. Indications of mental images or movement were absent. Three-dimensional representations of blocks of cubes provoked non-visual thinking in many instances. Specific actions identified included: counting of cubes in rows or in columns, concentration on the left and right edges of the objects and relative positioning of particular features, such as the top cube in relation to those below it.

AG: There is a little box missing here [A], here there are a lot of boxes missing [B], here they are the same [C]... Basically, I am counting the little boxes (M, 8, high, Task 1).

INT: How would it seem from the back?
AAG: The square on the top on the third column, would be on the third column from the back, but counting from the right (F, 7, low, Task 2).

In general, the number of dimensions in which the representation appeared and the nature of the pictures used seemed to be associated with the choice of strategy. Three-dimensional representations were more likely to invoke a non-visual approach involving several steps of analyzing features of the picture, such as counting blocks of cubes. Two-dimensional figures had distinct features, such as an arrow tip, or a small square in one edge on which students could rely on for rotating them, although a non-visual strategy was plausible.

A view from the back was the purpose for Task 2 (Figure 1). Although the object can be rotated in depth in order to view the back, most students preferred to rely on analytical elements like the ones described in the previous quote. Almost all of them recognized, explicitly or not, that the change of perspective implied the left-right reversal of the object.

SK: The shape will be reversed.
INT: What do you mean reversed?
SK: Since it is turning to the back, the little box on top, the first one, will be in the opposite direction. (F, 7, low, Task 2).

SE: Yes, those on the right will go to the left, and those on the left, to the right. (M, 6, high, Task 2).

GI: The opposite side, as it would seem in the mirror. (M, 8, high, Task 2).

Some students found it difficult to perceive an object from a different perspective; they were not able to distinguish that the front and the back view of an object are not exactly the same, but reversed.

EA: I think it is C, because as it is on the first side it will be the same and from the back.
INT: We are looking at an object from the front; if we view it from the back will it look the same?
EA: I think it will be the same (F, 5, low, Task 2).

The belief that, the "back is the same as the front side" (AKY, F, 6, high, Task 2) was the main cause for the erroneous answers in this task.

Spatial reasoning combining visual and non-visual components

Apart from the two-dimensional tasks in which visual strategies were dominant, students made use of both visual and non-visual strategies quite often. They seemed to change easily from one strategy to the other, but usually the visual component came first and the analytical followed. One student said:

AAN: I start and look at the little squares, if they are right when I turned them on the other side. I can see that A is the right one…
INT: What happens when you turn it on the other side?
AAN: This one goes there, the right goes to the left and the left moves to the right.
INT: Yes…
AAN: Then I look at the squares, if they are correct [same to] with the shape in the box (M, 8,

low, Task 2).

Confirmation was one function this procedure seemed to serve. A mental rotation takes place in the beginning and gives a clue about the decision. Then an argument is put forward, based on the characteristics of the parts of the object, either to add confidence to the correctness of the choice, or to justify its rejection.

Task 5: The rotated dice

Task 5 proved to be very difficult for most of the students. Three hand-made paper dice were presented to support the solution process (see Figure 2 for the exact presentation of the dice), but it was difficult for them to build up a sequence of steps to find the hidden number on the third die by using the numbers on the rest of the faces. Although many students had insightful ideas, they did not elaborate adequately on them. A typical idea was to find a common number on all three dice, and then the three dice were rotated in the "right position" to match each other.

AAN: We want a number, which exists on all three dice.
INT: Is there such a number?
AAN: No... If we turn them all on the straight, on the right position, we will be able to find the numbers.
INT: Which is the right position of the die?
AAN: The 3 must be on top...
INT: That is like the first two dice...
AAN: And 2 must be in front... Eh, 6 in front (M, 8, low, Task 5).

What often occurred was that most students used both a visual manipulation to rotate a die and proximity between numbers (which number is next to which), as components in their solution strategies.

INT: Which was the way you thought?
KM: It was to move them [the dice] so as to... If I only said that this is next to that one, I would be confused, while if I move them in my mind, I think it is easier.
INT: Do you prefer to move them in your mind? Or do you use both ways?
KM: Eh, sometimes I did it with numbers next to each other and it made things easier (F, 7, high, Task 5).

A "medium step" which proved to be crucial for the successful solution of the problem was the reconstruction of a die, the discovery of all the numbers on its faces. Most of those who managed to find the place of all six numbers, and only them, reached eventually the correct answer. AK reached this medium step:

> AK: We can see on the second die number 3 as on the first die. And we are able to see what is there at the back [of the second die]. Here are the 4 and 2 dots we can see on the first die; they are not on the [front faces of the] second die, that is they must be at the back... 4 and 2 must be at the back on the second die.
> INT: So?
> AK: We have found all faces apart from the bottom... We have the 3, 4, 2, 6 and 1 dots. The 5 dots are left [for the bottom face] (M, 6, low, Task 5).

But, even though this followed the clever visual manipulation of the third die giving the position of the hidden face on the constructed die, which follows, he did not manage to combine the two findings together and complete his thinking.

> AK: On the third die we have 1 dot, on its right 4 and on its left the paper [hiding the unknown number]... On the second die, the 1 dot is on the right. Next to it and on the left, there is a 6. That is, the position of the paper [hiding the number] is on the right.
> INT: On the right of 1?
> AK: Yes... Rather on the bottom should the paper be and 4 on the [back] right (M, 6, low, Task 5).

The use of a visual strategy to find the missing number had the same characteristics as the visual strategies in the other tasks. Simple words, together with hand movements pointing at various faces of the dice constituted the responses. Two 5th graders worked out the task successfully using visual reasoning.

An interesting explanation of the answer was given by AG, with the aid of a drawing. After trying ways, which did not enable him to reach a non-conflicting answer, he made recourse to taking notes, in order to keep track of the changes on the faces of the die. He first drew the three dice (see Figure 3), and noted all six numbers on the second one. Then, he drew a separate die to depict the second die after rotating it, with number 1 coming on top of it, 5 and 6 on the front faces (overwritten in his drawing by the final rotation) and 3, 4 and 2 at the back and bottom faces. Finally, he rotated it once more in the direction of the arrow seen at the bottom-right edge, so that it matched the third die, and noted the rotated faces (numbers 4 and 5) over the previous ones (5 and 6). He concluded that 5 was the hidden number.

Insert Figure 3 about here

A few students did not manage to articulate any solution strategy, or proposed irrelevant operations such as addition of numbers on respective faces, guessing, subtraction of numbers, and knowledge about the actual position of numbers on the die.

Discussion of the results

The analysis of the interview data suggests that the differential use of solution strategies was not made on a consistent personal preference basis. No student employed only one kind of thinking strategy throughout all tasks, in such a way so as to identify types of mathematical cast of mind as Krutetskii (1976) defines them. Each student often used visual and non-visual strategies for different tasks, or combined both components for the same task.

Some fundamental features of the strategies were identified according to the dichotomy between visual and non-visual thinking. Visual strategies seemed to adopt a holistic approach, were more intuitive, less systematized, difficult to explain verbally and accompanied by gestures. Students using non-visual strategies could articulate sound logical arguments and comparisons for their solutions, avoided reference to mental manipulation of representations and focused on parts and special features of the pictures sequentially (Bruner, 1977).

One way to interpret the choice of strategy is as a function of the characteristics of the task (Gorgorió, 1998). Two characteristics were found to relate to the implemented strategies: whether the representation was in two or three dimensions and the features of the iconic representation. A three-dimensional representation is more complex, calls for a more sequential processing, thus making the use of an analytic strategy more likely than a visual one. Summing up findings from hemispheric specialization research Corballis (1982) mentions that a more complex and sophisticated task may be accomplished in an analytic fashion, rather than a holistic one, with the processing carried out more sequentially. Figural rotation, in contrast, is simpler and more quickly processed as a whole, rather than requiring the additional operation of "partitioning" the image. The way of reasoning becomes more detailed in complicated multi-step problems, but for simple problems, it is possible to get "curtailment or shortening of reasoning" (Krutetskii, 1976, p.336). Hence, the students gave short and similarly phrased responses, implying visual strategies, especially in figural rotations in Tasks 3 and 4.

The other task characteristic, which might have influenced the choice of strategy, is the nature of the pictures used. The objects in the three-dimensional items consisted of blocks of cubes. It was easy for the students, and occurred most frequently, to count the cubes, instead of just mentally rotate the object. Students could rely on distinct features that the two-dimensional figures had for rotating them. A non-visual strategy for the figural rotation based on the left-right or top-down positioning of the arrow tip, or of the small square, which was also plausible, scarcely occurred. It could be said that the effect of the task characteristics was so influential, that it could explain the absence of age or gender effects on the choice of strategy.

The combined use of the two strategies, with the visual occurring first and the analytic following could be interpreted as a way of adding confidence to the student's original, intuitive sense of "wrongness" or "rightness" with more systematic and rigorous techniques (Bruner, 1971), a metacognitive skill which students develop - probably as a result of schooling, drawing on the formal logico-analytical approaches of school mathematics (Clements and Battista, 1992; Dickson et. al., 1984) - where the first solution is cross-checked for increased confidence. The frequent occurrence of combined strategies could alternatively be attributed to the questions' format. The multiple-choice format encourages attempts to confirm one answer, by eliminating the rest of the choices. For such a comparison it is more plausible to mobilize the use of analytic thinking to locate differences between pictures in a systematic and orderly manner rather than grasping a representation immediately as a whole. A different open-question format might require the construction of an answer in a more creative and holistic way.

The main difficulties were two. Some students were not able to distinguish that the front and the back view of an object are not exactly the same, but reversed. This misconception appeared in Task 2 as a declarative statement, when usually a mental rotation was not used. In these instances they could not indicate an ability to perceive an object from a different perspective. The other difficulty appeared in Task 5. They could not coordinate their actions to proceed through multiple steps to the solution of the task. It required a substantial use of memory in order to remember the previous transformation before proceeding to the next. When they were channeled into a strategy, what often happened was that they failed to correlate and make use of their former findings, since their reliance on memory only seemed to be dysfunctional.

Students' hesitation to make any written notes during the solution of the tasks supports the view that they could tackle most of the tasks relatively easily. In the case of the last one though, they were expected to rely on other methods than just their memory. It was very complicated to keep all the conditions and transformations resulting from the rotation of the dice in mind. This is perhaps an answer to Bishop's question (1983) on how memory affects visualization. AG's use of a drawing to manage the last task may be considered as a metacognitive skill, which served two cognitive functions (Ruthven, 1998). It augmented the working memory by recording each new item of information before proceeding to the next. As a result, he could have them ready for the next transformation. He also represented schematically the real objects on the page and then carried out a sequence of actions to reach his final purpose.

The manipulation of the representation of objects in the interview was not as easy and fast as the manipulation of two-dimensional figures. Transforming representations in two dimensions is accomplished earlier than in three (Gross, 1985). Mental rotation of three-dimensional shapes is considerably slower and presumably more difficult than rotating flat patterns (Corballis, 1982). Therefore, a difference in the degree of difficulty is anticipated, since the mere interpretation of the representation of the former constitutes an additional effort. On the other hand, the low degree of difficulty, especially for the two-dimensional items, might have caused a "ceiling effect" and all students scored very high, thus not allowing much variability in two- versus three-dimensional test scores.

The absence of gender differences in both the test performance and the tasks' solution strategies is consistent with Gorgorió's (1998) and Fennema and Sherman's (1977, 1978) claims of occasional occurrence of gender differences only in specific spatial tasks. However, as it has been noted earlier, it could be that the features of each task were a much more decisive factor in the choice of strategy, than any group characteristic such as gender, age within the age range of the research, or achievement level in the test. Irrespective of these three variables, students employed either, or both of the strategies concurrently in various tasks.

Conclusions

The present research project aimed at describing solution strategies in rotation problems from different standpoints. Concerned mainly with the study of the mental representations students have and of their way of thinking in this area, it employed task-based interviews as a means to achieve its main goal augmented with a test to investigate any group differences in the sample.

Students had not been taught the topic of rotation in school. They were however capable of rotating objects and figures, without distorting their properties, but preserving their characteristics and shape – Van Hiele Level 1 (Hoffer, 1983). They made fluent use of proximity and mental rotation, which are topological and projective properties respectively in Piaget and Inhelder's terms (1956).

In current research in mathematics education, the classification of individuals according to preference modes, or individual differences is not as appealing as the investigation of the diverse ways of thinking in particular curriculum topics. Apart from psychological differences, different branches of mathematics impose different demands on students, thus influencing their reaction strategies (Krutetskii, 1976). Moreover, the curriculum and the broader school context is possibly demanding or enforcing a particular mode of thinking. Many researchers agree that schooling favors a rigorous, analytic way of thinking, over an intuitive visual one (Barwise and Etchemendy, 1991; Tall, 1991; Bishop, 1980; Bruner, 1977).

Task-based interviews have been successful in showing that students have a repertoire of thinking strategies in the area of spatial reasoning. By differentiating task features, the availability of more than one strategy was revealed. The interactive dialogue during the interview moved beyond the original idea to a more justified and detailed description of the underlying reasoning. Subsequently, the description of the strategies and an assessment of their effectiveness were made possible. The gain from adding knowledge of strategies is that this knowledge can be shared, in contrast to other categorizations based on individual preferences and traits of spatial abilities (Gorgorió, 1998). Without denying the existence of individual traits, strategies can be taught to all students, thus enriching their problem solving tools.

The variability in the implementation of different strategies did not seem to be influenced by group characteristics. The effect of gender however at least in some spatial tasks cannot be doubted (Connor and Serbin, 1985). Age - when a wider age range is considered - might also increase the variability in performance and spatial reasoning. What matters though is that a range of ways of spatial reasoning appeared within each gender, age, and test achievement group (Gorgorió, 1998; Clements and Battista, 1992). The study of this variability in cognitive profiles should constitute an objective for current research. Also, contextual factors that potentially influence the understanding and development of spatial concepts and transformations, such as culture (Bishop, 1983), technology, language and formal instruction need to be explored.

Although the test had been constructed keeping in mind the necessity for construct validity and adjustment to the educational setting, it might not have been sensitive enough to grasp any group differences. The high facility of some items did not allow the emergence of any substantial group differences. Another limitation for the study emerged from the fact that the interviews aimed chiefly at spotting the thinking strategies, their characteristics and the difficulties students had in the tasks.

As a result, many strategies were classified as combining both visual and non-visual components, rather than one or the other. In the attempt to ascertain the range of available strategies within the time limits of the interview procedure, only a few tasks were set and the task characteristics as a variable was not controlled. The nature of each task was critical for the choice of strategy and was probably a more influential factor than gender, age and level of achievement. In addition, more creative response tasks could have given a richer picture of spatial abilities and reasoning compared to the multiple-choice tasks, which required selected responses (Eliot and Macfarlane Smith, 1983).

No single type of solution strategy, visual or non-visual, is best. There are many effective strategies for solving most problems and students are likely to be familiar with more than one and can certainly use then in a complementary way. Any preference for visual strategies that children have in solving a task should not be discouraged in favor of a logico-analytic, school-type way of thinking. Both visual and non-visual strategies can be effective. Even if visualizing in mathematics is considered intuitive and lacking rigor, it is a fundamental source of ideas and meaning making (Tall, 1991; Zimmermann and Cunningham, 1991) and a powerful tool for understanding various mathematical topics. "[T]he aim of a balanced schooling is to enable the child to proceed intuitively when necessary and to analyze when appropriate" (Bruner, 1971, p.83). Students should develop a rich synthesis of visual and non-visual strategies and left to adopt anyone they prefer (Zazkis et. al., 1996). Extending from Dowker's (1992) findings in the area of estimation, success in problem solving may be related to a flexible and the versatile use of strategies.

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APPENDIX

TABLE 1

Test Descriptive Statistics

Student Group	n	Mean number-correct score	Std. Deviation	Range (max.=19)
5th grade	26	14.12	2.89	12
6th grade	27	14.00	3.34	13
7th grade	27	15.00	2.97	10
8th grade	27	15.33	2.87	11
Primary school	53	14.06	3.10	13
Gymnasium	54	15.17	2.90	11
Male	49	14.73	3.32	13
Female	58	14.52	2.81	12
Overall sample	107	14.62	3.04	13

TABLE 2

Kinds of strategy identified in each task

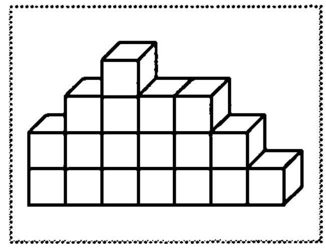
Task	Task description		Visual	Non-visual	Combination
	Type	Dimensions			
1	Multiple-choice	3	1	17	13
2	Multiple-choice	3	2	20	9
3	Multiple-choice	2	29	-	2
4	Multiple-choice	2	25	-	6
5	Problem-solving	3	12	3	9*

* The 7 missing cases in Task 5 are students who did not attempt any solution to the problem, or mentioned irrelevant operations.

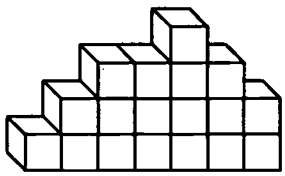
FIGURE 1

An item that appeared in both the test and the interview protocol

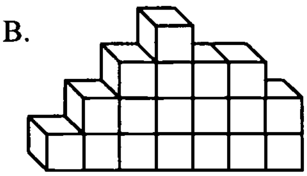
How would the shape in the box seem if we could see it from the back?



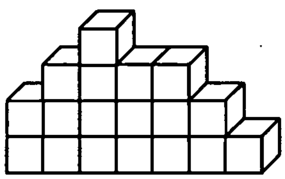
A.



B.



C.



D.

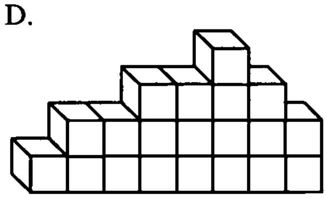


FIGURE 2

The three dice presented in Task 5

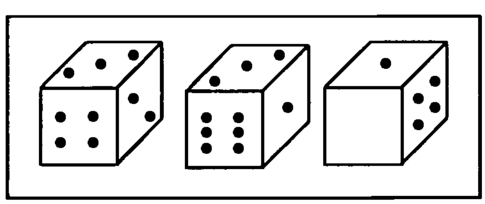
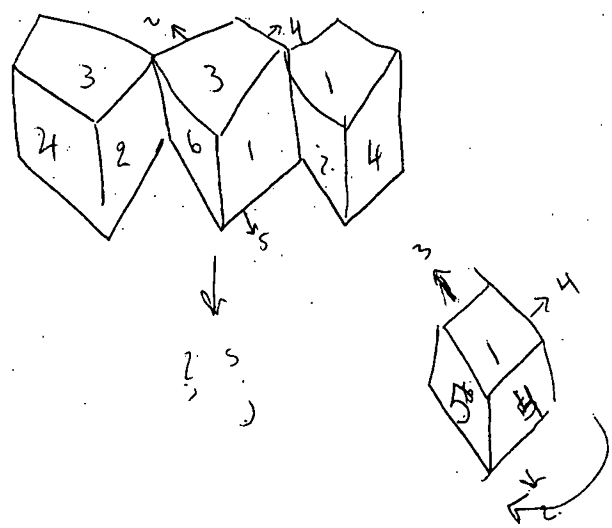


FIGURE 3

AG's drawing in Task 5



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