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ABSTRACT

This paper examines the question of whether the introduction and use of the function machine representation as a scaffolding device helps undergraduates enrolled in a developmental algebra course to form a rich, foundational concept of function. It describes students' developing understanding of function as an input/output process and as an object, tracing the internalization of the function machine concept as it relates to various representations of functions. (Author/MM)

USING THE FUNCTION MACHINE AS A COGNITIVE ROOT [1]

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We examine the question of whether the introduction and use of the function machine representation as a scaffolding device helps undergraduates enrolled in a developmental algebra course to form a rich, foundational concept of function. We describe students' developing understanding of function as an input/output process and as an object, tracing the internalization of the function machine concept as it relates to various representations of functions.

Introduction

Students' use of expressions, tables, and graphs in understanding functions has been studied extensively over the past several decades. Much of the literature on students' concepts of function examines what they do not understand and their misconceptions, offering explanations as to why this might be so (Goldenberg, 1988; Janvier, 1987; Kaput, 1989; Tall & Bakar, 1992; Thompson, 1994). Two recent studies suggest that the introduction of the function machine as an input/output box enables students to have a mental image of a box that can be used to describe and name various processes, often without the necessity of having an explicit process defined. Other forms of representation may be seen as mechanisms that allow an assignment to be made (by a table, by reading a graph, by using a formula, or by some other assignment procedure). The evidence of our research suggests that the function machine provides a powerful foundation and is a cognitive root for developing understanding of the concept of function. A cognitive root (Tall, McGowen, and DeMarois, 2000) is a concept met at the beginning of a curriculum sequence that:

(i.) is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence.

(ii.) allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction.

(iii.) contains the possibility of long-term meaning in later theoretical development of the mathematical concept.

(iv.) is robust enough to remain useful as more significant understanding develops.

1. McGowen, M., DeMarois, P. and Tall, D. (2000). Using the Function Machines as a Cognitive Root. *Proceedings of the 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: 247-254.

Students' internalisation of the function machine concept are examined against these criteria, addressing the question of whether use of the function machine representation leads to a rich, foundational understanding of function.

Data from two previous studies (DeMarois, 1998, McGowen, 1998) are examined for evidence of the function box as a cognitive root. The subjects of these studies were undergraduate students enrolled in developmental Introductory or Intermediate Algebra courses that do not carry general education credit. Many students had encountered the content before, so these studies used a restructured curriculum centred on the concept of function using function machines. The two studies include: (a) quantitative methods of data collection used to indicate global patterns generalizable across populations to document changes in students' understanding and to measure improvements in their mathematical competencies; and (b) qualitative methods that add depth and detail to the quantitative studies that allowed the researchers to focus on individual students within the context of the quantitative studies.

All students were given pre- and post-course surveys to establish what they knew about functions initially and after sixteen weeks. Several students from each course participated in interviews subsequent to the course. Data routinely collected in the Intermediate Algebra study also included student work and concept maps collected at five-week intervals, as well as mid-term and end-of-course student interviews. Growth in students' understanding and improved flexibility of thought was documented in descriptions/explanations of their work in terms of an input/output process and in their improved ability to (i) interpret and use ambiguous function notation, (ii) translate between and among various function representations, and (iii) view a function as an object in its own right. Various types of triangulation were used including data triangulation, method triangulation and theoretical triangulation (Bannister, et. al., 1996).

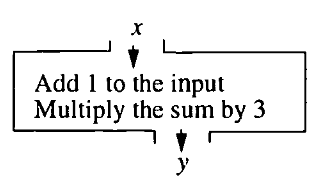
Examination of Data

A question asked on the pre- and post-course Introductory Algebra survey was:

Consider the diagram:

a. What are the output(s) if the input is 7?

b. What are the input(s) if the output is 18?



The summary of responses in Table 1 indicates that two-thirds of Introductory Algebra students were able to interpret a function machine diagram flexibly at the beginning of the course, an indication that the function machine representation is an accessible starting point for many students.

Table 1: TABLE 1: FUNCTION MACHINE INPUT AND OUTPUT (Introductory Algebra)

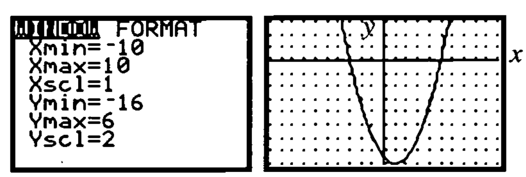
Question	Pre-course (n = 92) number (%) correct	Post-course (n = 92) number (%) correct
a) Function machine: input given	62 (67%)	79 (86%)
b) Function machine: output given	44 (48%)	64 (70%)
Function machine: both parts correct	43 (47%)	61 (66%)

Students in both studies were asked on pre- and post-course surveys to find output given a graph and input. They were also asked to find input given a graph and output. The questions on the two surveys differ in some respects. The Introductory Algebra question displays a window indicating scale and the graph of a parabola. A correct response includes recognition that there are two answers to part (b). Given the form of the question, students are not required to interpret function notation in order to solve the problem though they were required to switch their train of thought to answer part (b). Students' responses to both parts were considered a measure of their improved ability to think flexibly.

Consider the viewing window and graph copied from a TI–82 graphics calculator.

a. What are the output(s) if the input is 3?
Answer:________

b. What are the input(s) if the output is 0?
Answer:________



The Intermediate Algebra survey question asked students to determine output given the graph of a piece-wise function and an input, then to determine the input, given an output, using the same graph. The form of the question requires students to interpret function notation as well as to change the direction of their train of thought to answer part (b).

Given the graph

(8) Indicate what y(8) = ________
What first comes to mind:

(9) If $y(x) = 2$, what is x?________
What first comes to mind:

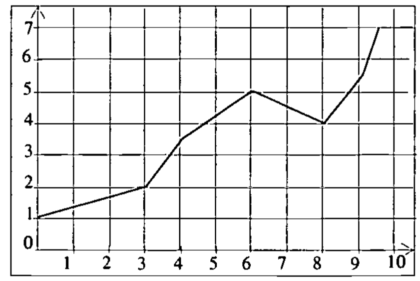


Table 2 displays the results of student responses to the survey questions:

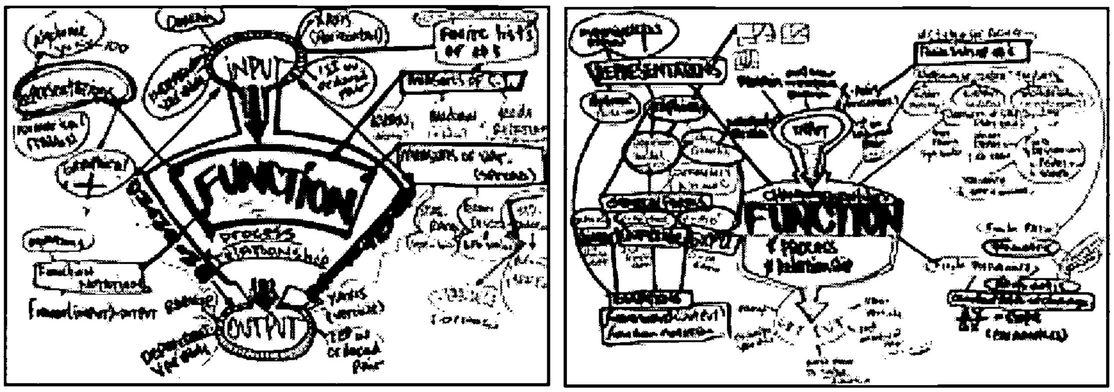
TABLE 2: GRAPH: INPUT AND OUTPUT:

Survey Question	Beginning Pre-course (% correct) n = 92	Beginning Post-course (% correct) n = 92	Intermediate Pre-course (% correct) n = 52	Intermediate Post-course (% correct) n = 52
Graph: input given	1% (1/92)	41% (38/92)	38% (20/52)	71% (37/52)
Graph: output given	0% (0/92)	22% (20/92)	17% (9/52)	46% (24/52)
Graph: pair correct	0% (0/92)	21% (19/92)	6% (3/52)	40% (21/52)

The results indicate that only 41% of Introductory Algebra students were able to find output given input and only 22% were able to reverse the process at the end of the semester. This suggests that, even when understanding of functions based on a function machine representation was demonstrated, these students demonstrated little connection between function machines and graphs (DeMarois, 1998). By the end of the semester, 71% of Intermediate Algebra students were able to find output given input and 46% were able to reverse the process. Only 21% of Introductory Algebra students and less than half (40%) of the Intermediate Algebra students were able to do both processes by the end of the semester. However, when one considers students' initial responses and their improved flexibility of thought over the sixteen weeks, the results are encouraging. The average change in correct responses for the Intermediate Algebra students was statistically significant (two-tailed paired t-test, $p < 0.001$).

Concept maps done throughout the semester document how the function machine idea allowed initial development through cognitive expansion of Intermediate Algebra students' developing understanding of function. Figure 1 illustrates how a student's concept image of function developed and was impacted by use of the function machine representation.

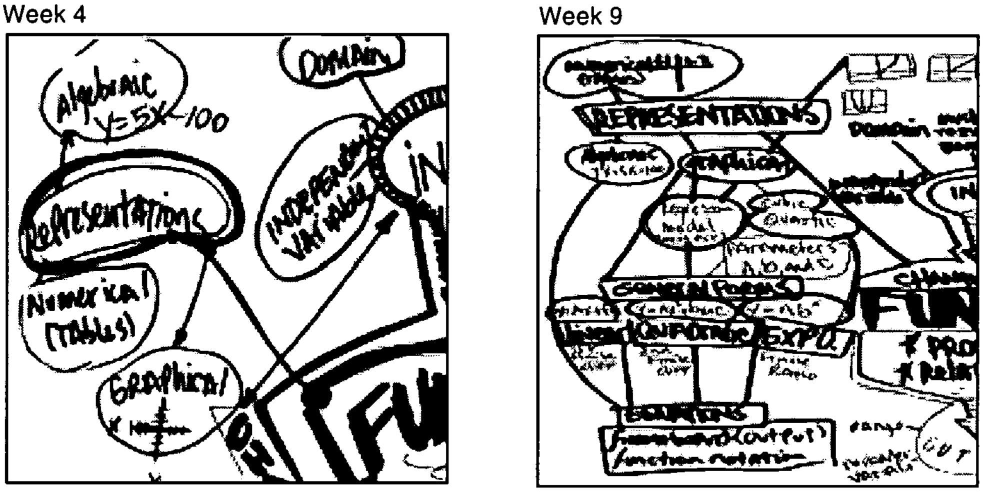
FIGURE 1: CONCEPT MAPS (Week 4 and Week 9): COGNITIVE EXPANSION



A closer examination of these maps in Figure 2 documents the student's growing understanding of representations that has occurred over time.

FIGURE 2: CONCEPT MAP CLOSE-UPS (Week 4 and Week 9): REPRESENTATIONS

Week 4 Week 9



In an interview at mid-term, the student describes his growing ability to make sense of and interpret functional notation in terms of input and output:

> I feel that I have really made sense of input and output when dealing with function notation. Problems on the Unit II individual test used to look so unfamiliar to me, but now make perfectly good sense.... I'm learning how these algebraic models are set up and what the variables that they contain represent. I'm no longer just blindly solving for x, but rather understanding where x (input) came from and how it was found from the data given. Through this kind of learning I have developed an understanding for the use of function notation [$f(x)$ = output] and how it replaces the dependent variable, y.

By Week 15 the student internalised the function-as-process concept. Evidence of the input/output cognitive root was still present in his final map, which was colour-coded to indicate concepts connected with input or output. By the end of the semester, the student was able to translate flexibly and consistently among various representational forms (tables, graphs, traditional symbolic forms and functional forms. He expressed confidence in the correctness of his answers. In his final interview of the semester, the student spoke of his understanding of function notation:

> I think the most memorable information from this class would be the use and understanding of function notation. A lot of emphasis was put on input and output which really helped me comprehend some algebraic processes such as solving for *x*.

Conclusion

The evidence presented suggests that the function machine is a cognitive root for the function concept for the subject population and that function machines provide a foundation on which to further develop the function concept. Function machines impacted students' thinking and learning as evidenced in their work and in their written and oral statements. They were able to interpret the instructions in a function machine diagram flexibly at the beginning of the courses, an indication that the function machine representation is an accessible starting point for many at the beginning of a learning sequence that made sense as representative of the func-tion input/output process.

Further analysis of the data documents the profound divergence that occurred over time between the most successful and least successful students. References to input and output occur in the work and interviews throughout the semester of students who were successful. They used the function machine notion to organise their thinking as they worked problems and inter-preted notation. Axes on graphs were labelled in terms of input and output, as were questions using symbolic notation. The function machine representation provided students with access to the function concept and became a meaningful unit of core knowledge upon which they built subsequent understanding about functions. Their concept maps document the cognitive expansion that occurred over time and provide evidence that the function machine as cognitive root is robust enough to remain useful as more significant understanding develops.

Strikingly, the least successful students generally did not make use of the function machine notion except in limited instances. In contrast to the more successful students, the least successful students made very few references to function machines in their work or in the vocabulary they used. The least successful students demonstrated little or no improvement in their ability to thinking flexibly. Such rigidity of thought extended to arithmetic computational processes. Their ability to reverse a train of thought appeared frozen, regardless of which representation was used. On the other hand, the most successful students demonstrated flexibility of thinking in their ability to use various representations. They were able to translate among representations, intelligently choosing among alternative procedures.

The usefulness of function machine as a cognitive root continues to be examined as students attempt to deal with the function concept at the College Algebra level. We are investigating the question of how their development of the function concept compares with that of developmental algebra students. The search for possible cognitive roots for other mathematical concepts is also on going.

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