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ABSTRACT

This paper gives a brief introduction to a discipline called the cognitive science of mathematics. The theoretical background of the arguments is based on embodied cognition and findings in cognitive linguistics. It discusses Mathematical Idea Analysis, a set of techniques for studying implicit structures in mathematics. Particular attention is paid to everyday cognitive mechanisms such as image schemas and conceptual metaphors. Some implications for mathematics education are discussed. (Contains 36 references.) (DDR)

MATHEMATICAL IDEA ANALYSIS: WHAT EMBODIED COGNITIVE SCIENCE CAN SAY ABOUT THE HUMAN NATURE OF MATHEMATICS

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ABSTRACT:
This article gives a brief introduction to a new discipline called the cognitive science of mathematics (Lakoff & Núñez, 2000), that is, the empirical and multidisciplinary study of mathematics (itself) as a scientific subject matter. The theoretical background of the arguments is based on embodied cognition, and on relatively recent findings in cognitive linguistics. The article discusses Mathematical Idea Analysis*—the set of techniques for studying implicit (largely unconscious) conceptual structures in mathematics. Particular attention is paid to everyday cognitive mechanisms such as* image schemas *and* conceptual metaphors, *showing how they play a fundamental role in constituting the very fabric of mathematics. The analyses, illustrated with a discussion of some issues of set and hyperset theory, show that it is (human) meaning what makes mathematics what it is: Mathematics is not transcendentally objective, but it is not arbitrary either (not the result of pure social conventions). Some implications for mathematics education are suggested.*

Have you ever thought why (I mean, really *why*) the multiplication of two negative numbers yields a positive one? Or why the empty class is a subclass of all classes? And why is it a class at all, if it cannot be a class of anything? And why is it unique? For most people, including mathematicians, physicists, engineers, and computer scientists, the answers to these questions have a strong dogmatic component (try these questions with your own colleagues!). It is common to encounter answers such as "well, that's the way it is", or "I don't know exactly why, but I know it works that way", and so on.

Within the culture of those who practice mathematics professionally, the dogmatic answers to these questions usually follow from definitions, axioms, and rules, they don't necessarily follow from genuine understanding. In those cases, the validation of the answer is provided by proof, not necessarily by *meaning*. This profound difference between determining *that* something is true and explaining *why* it is true, can be seen in the following historical anecdote.

Benjamin Peirce, one of Harvard's leading mathematicians in the 19th century (and the father of Charles Sanders Peirce), was once lecturing at Harvard on Euler's proof that $e^{\pi i} + 1 = 0$. In teaching this famous equation and its proof, he remarked,

> "Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved

it, and therefore we know it must be truth." (cited in Maor, 1994, p. 160)

Of course Peirce was not the only mathematician (or mathematics teacher) to fail to understand what $e^{\pi i} + 1 = 0$ means. Even today, relatively few mathematics teachers and students understand what the equation actually *means*. Yet generation after generation of mathematics teachers and students continue to go uncomprehendingly through one version or another of Euler's proof, understanding only the regularity in the manipulations of the symbols, but not the *ideas* that make it true. This is hardly an isolated example. Meaningless truth and meaningful sense-making are fundamental components of many debates involving the nature of mathematics.

In this plenary address, I want to show that it is meaning (i.e., human meaningful ideas), what makes mathematics what it is, and that this meaning is not arbitrary, not the result of pure social conventions. My arguments will be based on contemporary embodied cognitive science. More specifically, I intend to show the following:

1. That the nature of mathematics is about human *ideas*, not just, formal proofs, axioms, and definitions (proofs, axioms, and definitions constitute only a part of mathematics, which are also realized through precise sets of ideas).
2. That these ideas are grounded in species-specific everyday cognitive and bodily mechanisms, therefore making mathematics a human enterprise, not a platonic and transcendental entity.
3. That because of this grounding, mathematical ideas are not arbitrary, that is, they are not the product of purely social and cultural conventions (although socio-historical dimensions play key roles in the formation and development of ideas).
4. That the conceptual (and idea) structure that constitutes mathematics can be studied empirically, through scientific methods.
5. That a particular methodology based on embodied cognitive science —*Mathematical Idea Analysis*—can serve this purpose.

Most of the material I will present here is based on the work I have been developing for several years in close collaboration with the cognitive linguist George Lakoff in Berkeley (Lakoff & Núñez, 1997, 1998, 2000; Núñez & Lakoff, 1998).

THE CONTEMPORARY STUDY OF IDEAS: FROM ARMCHAIR PHILOSOPHY TO SCIENTIFIC UNDERSTANDING

Throughout history, many mathematicians have tried to answer the question of the nature of meaning, truth, and ideas in mathematics. In the last century or so, various influential mathematicians, such as Dedekind, Cantor, Hilbert, Poincaré, and Weyl, to mention only a few, suggested some answers which share important elements. They all considered, in one way or another, human intuition as a fundamental starting point for their philosophical investigations: Intuitions of small integers, intuitions of collections, intuitions of movement in space, and so on (see

Dedekind, 1888/1976; Dauben on Cantor (1979); Kitcher on Hilbert (1976); Poincaré, 1913/1963; Weyl, 1918/1994). They saw these fundamental intuitions of the human mind as being stable and profound to serve as basis for mathematics.[1]

These philosophical insights tell us something important. They implicitly say that the edifice of mathematics is based on aspects of the human mind that lie *outside* of mathematics proper (i.e., these intuitions themselves are not theorems, axioms or definitions). However, beyond the philosophical and historical interest these insights may have, when seen from the perspective of nowadays' scientific standards, they present important limitations:

- First, those mathematicians were professionally trained to do mathematics, not necessarily to study ideas and intuitions. And their discipline, mathematics (as such), does not study ideas or intuitions. Today, the study of ideas (concepts and intuitions) itself is a scientific subject matter, and it is not anymore just a vague and elusive philosophical object.
- Second, the methodology they used was mainly *introspection*—the subjective investigation of one's own impressions, feelings, and thoughts. Now we know, form substantial evidence in the scientific study of intuition and cognition, that there are fundamental aspects of mental activity that are unconscious in nature and therefore inaccessible to introspection.

The moral here is that pure philosophical inquiry and introspection—although very important—give, at best, a very limited picture of the conceptual structure that makes mathematics possible. What is needed, in order to understand the nature and origin of mathematics and of mathematical meaning, is to study mathematics *itself* (with its intuitive grounding, its inferential structure, its symbol systems, etc.) as a scientific subject matter. What is needed is a cognitive science *of* mathematics, a science of *mind-based mathematics* (Lakoff & Núñez, 1997, 2000). From this perspective, the answers to these issues should be in terms of those mechanisms underlying our intuitions and ideas. That is, in terms of human cognitive, biological, and cultural mechanisms, and not in terms of axioms, definitions, formal proofs, and theorems. Let us see what important findings are helpful in providing those answers.

Embodied Cognitive Science and Recent Empirical Findings about the Nature of Mind

In recent years, there have been revolutionary advances in cognitive science—the multidisciplinary scientific study of the mind. These advances have an important

[1] But, they didn't think of these intuitions and basic ideas as being "rigorous" enough. This was a major reason why, later, formalism would explicitly eliminate ideas, and go on to dominate the foundational debates. Unfortunately, at that time philosophers and mathematicians didn't have the scientific and theoretical tools we have today to see that human intuitions and ideas are indeed very precise and rigorous, and that therefore the problems they were facing didn't have to do with lack of rigor of ideas and intuitions. For details, see Núñez & Lakoff, 1998, and Lakoff & Núñez, 2000).

bearing on our understanding of mathematics. Among the most profound of these new insights are the following:

1. *The embodiment of mind.* The detailed nature and dynamics of our bodies, our brains, and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason.
2. *The cognitive unconscious.* Most cognitive processes is unconscious—not repressed in the Freudian sense, but simply inaccessible to direct conscious introspection. We cannot through introspection look directly at our conceptual systems and at our low-level cognitive processes. This includes most mathematical thought.
3. *Metaphorical thought.* For the most part, human beings conceptualize abstract concepts in concrete terms, using precise inferential structure and modes of reasoning grounded in the sensory motor system. The cognitive mechanism by which the abstract is comprehended in terms of the concrete is called *conceptual metaphor*[2]. Mathematical thought also makes use of conceptual metaphor, as when we conceptualize numbers as points on a line, or space as sets of points.

In what follows I intend to give a general overview of how to apply these empirical findings to the realm of mathematical ideas. That is, while taking mathematics as a subject matter for cognitive science I will ask how certain domains in mathematics are created and conceptualized. In doing so, I will show that it is with these recent advances in cognitive science that a deep and grounded *Mathematical Idea Analysis* becomes possible (for details, see Lakoff & Núñez, 2000). Keep in mind that the major concern then is not just with *what* is true in mathematics, but with what mathematical ideas *mean*, and why mathematical truths are true by virtue of what they mean.

At this point it is important to mention that when I refer to cognitive science, I refer to contemporary embodied oriented approaches (see, for instance, Johnson, 1987; Lakoff, 1987; Varela, Thompson, & Rosch, 1991; Núñez, 1995, 1999), which are radically different from orthodox cognitive science. The latter builds on dualist, functionalist, and objectivist assumptions, while the former has explicitly denied them, especially, the mind-body split (dualism). For embodied oriented approaches any theory of mind must take into account the peculiarities of brains, bodies, and the environment in which they exist. Because of these reasons analyses of the sort I will be giving below were not even imaginable in the days of orthodox cognitive science of the disembodied mind, developed in the 1960s and early 1970s. In general, within the traditional perspective, which under the form of *neo-cognitivism* (Freeman & Núñez, 1999) is still very active today, thought is addressed in terms of the manipulation of purely abstract symbols and concepts are seen as literal — free of all biological constraints and of discoveries about the brain.

I mention this, because, unfortunately, within the mathematics education community, for many, cognitive science is synonymous with the orthodox view.

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[2] As we will see later, this is a technical term.

Because of the various limitations that this traditional view has manifested over the years, many researchers in mathematics education concerned with developmental, social, and cultural factors have rejected cognitive science as a whole, assuming that it had little to offer (Núñez, Edwards, & Matos, 1998). I want to make clear then, that Mathematical Idea Analysis comes out of embodied oriented approaches to cognitive science. For a deeper discussion of the differences between orthodox cognitive science and recent embodied oriented cognitive science, see Núñez (1997), Lakoff & Johnson (1999), and Núñez & Freeman (1999).

Ordinary Cognition and Mathematical Cognition

Substantial research in neuropsychology, child development, and animal cognition suggests that all individuals of the species *Homo Sapiens* are born with a capacity to distinguish among very small numbers of objects and events (e.g., subitizing) and to do the simplest arithmetic—the arithmetic of very small numbers (for recent reviews on these and related issues, see Dehaene, 1997, and Butterworth, 1999). These findings are important for the understanding of the biological rudiments of basic arithmetic. However, they tell us very little about the full complexity and abstraction of mathematics. There is a lot more to mathematics than the arithmetic of very small numbers. Trigonometry and calculus are very far from "3 minus 1 equals 2". Even realizing that zero is a number and that negative numbers are numbers took centuries of sophisticated development. Extending numbers to the rationals, the reals, the imaginaries, and the hyperreals requires an enormous cognitive apparatus that goes well beyond what babies and animals and a normal adult without instruction can do. So the question of the nature, origin, and meaning of mathematical ideas remains open: What are the embodied cognitive capacities that allow one to go from such innate basic numerical abilities to a deep and rich understanding of, say, college-level mathematics?

George Lakoff and I have addressed this question, using methodologies from the growing field of cognitive linguistics and psycholinguistics (more about this below). According to what we have found to date, it appears that such advanced mathematical abilities are not independent of the cognitive apparatus used outside of mathematics. Rather, it appears that the cognitive structure of advanced mathematics makes use of the kind of conceptual apparatus that is the stuff of ordinary everyday thought such as image schemas, aspectual schemas, conceptual blends, and conceptual metaphor[3]. Indeed, the last one is one of the most important ones, constituting the very fabric of mathematics. It is present in all subfields of mathematics, as when we conceptualize functions as sets of points, infinite sums as having a final unique resultant state, or dynamic continuity as being static preservation of closeness (Weierstrass's ε–δ criteria).

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[3] Because of the scope of this presentation, here I will refer only to image schemas and conceptual metaphor. I will describe them in the next section.

Let us now have a look at the theoretical background of Mathematical Idea Analysis.

MATHEMATICAL IDEA ANALYSIS

Extending the study of the cognitive unconscious to mathematical cognition, implies analyzing the way in which we implicitly understand mathematics as we do it or talk about it. A large part of unconscious thought involves implicit rather than explicit, automatic, immediate understanding—making sense of things without having conscious access to the cognitive mechanisms by which we make sense of things. Ordinary everyday mathematical sense-making is not in the form of conscious proofs from axioms nor is it always the result of explicit, conscious, goal-oriented instruction. Most of our everyday mathematical *understanding* takes place without our being able to explain exactly what we understood and how we understood it. What Lakoff and I have done is to study everyday mathematical understanding of this automatic unconscious sort and to ask the following crucial questions:

- How much of mathematical understanding makes use of the same kinds of conceptual mechanisms that are used in the understanding of ordinary, nonmathematical domains?
- Are the same cognitive mechanisms used to characterize ordinary ideas also used to characterize mathematical ideas?
- If yes, what is the biological or bodily grounding of such mechanisms?

We have found that a great many cognitive mechanisms that are not specifically mathematical are used to characterize mathematical ideas. These include such ordinary cognitive mechanisms as those used for basic spatial relations, groupings, small quantities, motion, distributions of things in space, changes, bodily orientations, basic manipulations of objects (e.g., rotating and stretching), iterated actions, and so on.

Thus, for example:

- Conceptualizing the technical mathematical concept of a class makes use of the everyday concept of a collection of objects in a bounded region of space.
- Conceptualizing the technical mathematical concept of recursion makes use of the everyday concept of a repeated action.
- Conceptualizing the technical mathematical concept of complex arithmetic makes use of the everyday concept of rotation.
- Conceptualizing derivatives in calculus requires making use of such everyday concepts as motion, approaching a boundary, and so on.

From a nontechnical perspective, this should be completely obvious. But from the technical perspective of cognitive science, there is a challenging question one must ask:

> *Exactly what everyday concepts and cognitive mechanisms are used in exactly what ways in the unconscious conceptualization of technical ideas, such that they provide the precise inferential structure observed in mathematics?*

Mathematical Idea Analysis, depends crucially on the answers to this question. We have found that mathematical ideas, are grounded in bodily-based mechanisms and everyday experience. Many mathematical ideas are ways of mathematicizing ordinary ideas, as when the idea of subtraction mathematizes the ordinary idea of distance, or as when the idea of a derivative mathematicizes the ordinary idea of instantaneous change. I will illustrate these findings in more detail with some examples taken from set theory and hyperset theory. But because of the technicalities involved we must first go over some basic notions of cognitive linguistics, necessary to understand those examples.

Some Basic Notions of Cognitive Linguistics and the Embodied Mind

Recent developments in cognitive linguistics have been very fruitful in studying high-level cognition from an *embodiment* perspective (e.g., natural language understanding and conceptual systems). In particular, cognitive semantics (Sweetser, 1990, Talmy, 1999), conceptual integration (Fauconnier, 1997; Fauconnier & Turner, 1998) and conceptual metaphor theory (Lakoff, 1993; Lakoff & Johnson, 1980, 1999; Gibbs, 1994) have proven to be very powerful. These approaches offer the possibility of empirically studying the conceptual structure of vast systems of abstract concepts through the largely unconscious, effortless, everyday linguistic manifestations. They provide an excellent background for the development of Mathematical Idea Analysis.

Conceptual metaphor

An important finding in cognitive linguistics is that concepts are systematically organized through vast networks of *conceptual mappings*, occurring in highly-coordinated systems and combining in complex ways. For the most part these conceptual mappings are used unconsciously and effortlessly in everyday communication. An important kind of mapping is the one mentioned earlier, *conceptual metaphor*.

It is important to keep in mind that conceptual metaphors are not mere figures of speech, and that they are not just pedagogical tools used to illustrate some educational material. Conceptual metaphors are in fact fundamental cognitive mechanisms (technically, they are *inference-preserving cross-domain mappings*) which project the inferential structure of a *source domain* onto a *target domain*, allowing the use of effortless species-specific body-based inference to structure abstract inference. For example, humans naturally conceptualize Time (target domain) primarily in terms of Uni-dimensional Motion (source domain), either the motions of future times toward an observer (as in "Christmas is *approaching*") or the motion of an observer over a time landscape (as in "We're *approaching* Christmas"). That is, our everyday concept of Time is inextricably related to the experience of uni-dimensional motion. There are, of course, many more important details and variations of this general Time As Motion mapping but their analyses would go beyond the scope of this presentation. The point here, is that these conceptual metaphors (and conceptual mappings in general) are irreducible, they are extremely precise (e.g., in the Time As Motion example, their inferential structure preserves transitive relations), they are used extensively, effortlessly, unconsciously, and they are ultimately bodily grounded (for details, see Lakoff & Johnson, 1999, Chapter 10; Núñez, 1999).

Contrary to what some people think, conceptual metaphors (and conceptual mappings in general) are not mere arbitrary social conventions. They are not arbitrary, because they are structured by species-specific constrains underlying our everyday experience—especially bodily experience. For example, in most cultures Affection is conceptualized in terms of thermic experiénce: Warmth (as in "He greeted me *warmly*", or as in "send her my *warm* helloes"). The grounding of this mapping doesn't depend (only) on social conventions. It emerges from the correlation all individuals of the species experience, from early ontogenetic development, between affection and the bodily experience of warmth. It is also important to mention that a huge amount of the conceptual metaphors we use in everyday communication, such as Affection Is Warmth, is not learned through explicit goal-oriented educational intervention.

Research in contemporary conceptual metaphor theory indicates that there is an extensive conventional system of conceptual metaphors in every human conceptual system. As I said earlier, unlike traditional studies of metaphor, contemporary embodied views *don't see conceptual metaphors as residing in words, but in thought.* Metaphorical linguistic expressions thus are only surface manifestations of metaphorical thought. These theoretical claims are based on substantial empirical evidence from a variety of sources, including among others, psycholinguistic experiments (Gibbs, 1994), cross-linguistic studies (Yu, 1998), generalizations over inference patterns (Lakoff, 1987), generalizations over conventional and novel language (Lakoff and Turner, 1989), the study of historical semantic change (Sweetser, 1990), of language acquisition (C. Johnson, 1997), of spontaneous gestures (McNeill, 1992), and of American sign language (Taub, 1997). Conceptual mappings thus can be studied empirically, and stated precisely.

In what concerns mathematical concepts, Lakoff & Núñez (2000) distinguish, three important types of conceptual metaphors:

- *Grounding metaphors*, which ground our understanding of mathematical ideas in terms of everyday experience. In these cases, the target domain of the metaphor is mathematical, but the source domain lies outside of mathematics. Examples include the metaphor Classes Are Container Schemas (see below) and other conceptual metaphors for arithmetic.
- *Redefinitional metaphors*, which are metaphors that impose a technical understanding replacing ordinary concepts (such as the conceptual metaphor used by Georg Cantor to reconceptualize the notions of "more than" and "as many as" for infinite sets).
- *Linking metaphors*, which are metaphors within mathematics itself that allow us to conceptualize one mathematical domain in terms of another mathematical domain. In these cases, both domains of the mapping are mathematical. Examples include Von Neumann's Numbers Are Sets metaphor, Functions Are Sets of Points, and as we will see later, the Sets Are Graphs metaphor.

The linking metaphors are in many ways the most interesting of these, since they are part of the fabric of mathematics itself. They occur whenever one branch of mathematics is used to model another, as happens frequently. Moreover, linking metaphors are central to the creation, not only of new mathematical concepts, but often to the creation of new branches of mathematics. Such classical branches of mathematics as analytic geometry, trigonometry, and complex analysis owe their existence to linking metaphors.

Spatial relation concepts and image schemas

Another important finding in cognitive linguistics is that conceptual systems can be ultimately decomposed into primitive spatial relations concepts called *image schemas*. Image schemas are basic dynamic topological and orientation structures that characterize spatial inferences and link language to visual-motor experience (Johnson, 1987; Lakoff and Johnson, 1999). As we will see, an extremely important feature of image schemas is that their *inferential structure* is *preserved* under metaphorical mappings. Image schemas can be studied empirically through language (and spontaneous gestures), in particular through the linguistic manifestation of spatial relations.

Every language has a system of spatial relations, though they differ radically from language to language. In English there are prepositions like *in, on, through, above,* and so on. Other languages have systems that often differ radically from the English system. However, the spatial relations in a given language decompose into conceptual primitives (image schemas) that appear to be universal.

For example, the English word "on," in the sense used in "The book is *on* the desk" is a composite of three primitive image schemas:

- The *Above Schema* (the book is *above* the desk)
- The *Contact Schema* (the book is *in contact* with the desk)
- The *Support Schema* (the book is *supported* by the desk)

The Above Schema is orientational; it specifies an orientation in space relative to the gravitational pull one feels on one's body. The Contact Schema is one of a number of topological schemas; it indicates an absence of a gap. The Support Schema is force-dynamic in nature; it indicates the direction and nature of a force. In general, static image schemas fall into one of these categories: orientational, topological, and force-dynamic. In other languages, the primitives may combine in very different ways. Not all languages have a single concept like English *on*. For instance, even in a language as close as German, the *on* in *on the table* is rendered as *auf*, while the *on* in *on the wall* (which does not contain the Above Schema) is translated as *an*.

A common image schema that is of great importance in mathematics is the *Container Schema*, which in everyday cognition occurs as the central part of the meaning of words like *in* and *out*. The Container Schema has three parts: an Interior, a Boundary, and an Exterior. This structure forms a gestalt, in the sense that the parts make no sense without the whole. There is no Interior without a Boundary and an Exterior, no Exterior without a Boundary and an Interior, and no Boundary without sides, in this case an Inside and an Outside. This structure is topological in the sense that the boundary can be made larger, smaller, or distorted and still remain the boundary of a Container Schema.

The schemas for the concepts In and Out, have a bit more structure than the plain Container Schema. The concept In requires that the Interior of the Container Schema be *profiled*, that is, that it must be highlighted over the Exterior and Boundary. In addition, there is also a figure-ground distinction. For example, in a sentence like "The car is in the garage," the garage is the ground, that is, it is the landmark relative to which the car, the figure, is located. In cognitive linguistics, the ground in an image schema is called the *Landmark*, and the figure is called the *Trajector*. Thus, the In-Schema has the following structure:

- Container Schema, with Interior, Boundary, and Exterior
- Profiled: The Interior
- Landmark: The Interior

Image schemas have a special cognitive function: they are both *per*ceptual and *con*ceptual in nature. As such, they provide a bridge between language and reasoning on the one hand and vision on the other. Image schemas can fit visual perception, as when we see the milk as being *in* the glass. They can also be imposed on visual scenes, as when we see the bees swarming *in* the garden, where there is no physical container that the bees are in. Because spatial relations terms in a given language name complex image schemas, image schemas are the link between language and spatial perception.

We can now analyze how the inferential structure of image schemas (for example, the Container Schema) is preserved under metaphorical mappings to generate more abstract concepts (such as the concept of Boolean class). We shall see exactly how image schemas provide the inferential structure to the source domain of the conceptual metaphor, which via the mapping is projected onto the target domain of the metaphor to generate Boolean-class inferences.

Image schema structure and metaphorical projections

When we draw illustrations of Container Schemas, we find that they look rather like Venn Diagrams for Boolean classes. This is by no means an accident. The reason is that classes are normally conceptualized in terms of Container Schemas. For instance, we think (and speak) of elements as being *in* or *out* of a class. Venn Diagrams are visual instantiations of Container Schemas. The reason that Venn diagrams work as symbolizations of classes is that classes are usually metaphorically conceptualized as containers — that is, as bounded regions in space.

Container Schemas have a logic that appears to arise from the structure of our visual and imaging system, adapted for more general use. More specifically, Container Schemas appear to be realized neurally using such brain mechanisms as topographic maps of the visual field, center-surround receptive fields, gating circuitry, and so on (Regier, 1996). The inferential structure of these schemas can be used both for structuring space and for more abstract reason, and is projected onto our everyday conceptual system by a particular conceptual metaphor, the Categories (or 'Classes') Are Containers metaphor. This accounts for part (by no means all!) of our reasoning about conceptual categories. Boolean logic also arises from our capacity to perceive the world in terms of container schemas and to form mental images using them.

So, how do we normally conceptualize the intuitive premathematical notion of classes? The answer is in terms of Container Schemas. In other words, we normally conceptualize a class of entities in terms of a bounded region of space, with members of the class all *inside* the bounded region and nonmembers *outside* of the bounded region. From a cognitive perspective, intuitive classes are thus metaphorical conceptual containers, characterized cognitively by a metaphorical mapping —a grounding metaphor — the Classes Are Containers Schemas metaphor. The following is the mapping of such conceptual metaphor.

Classes Are Containers

Source Domain Container Schemas		Target Domain Classes
Interiors Of Container Schemas	→	Classes
Objects in Interiors	→	Class members
Being an Object in an Interior	→	The Membership Relation
An Interior of one Container Schema within a Larger One	→	A subclass in a Larger Class
The Overlap of the Interiors of Two Container Schemas	→	The Intersection of Two Classes
The Totality of the Interiors of Two Container Schemas	→	The Union of Two Classes
The Exterior of a Container Schemas	→	The Complement of a Class

This is our natural, everyday unconscious conceptual metaphor for what a class is. It is a *grounding metaphor*. It grounds our concept of a class in our concept of a bounded region in space, via the conceptual apparatus of the image schema for containment. This is the way we conceptualize classes in everyday life.

We can now analyze, how conceptual image schemas (in this case, Container Schemas) are the source of four fundamental inferential laws of logic. The structural constraints on Container Schemas mentioned earlier (i.e., brain mechanisms such as topographic maps of the visual field, center-surround receptive fields, gating circuitry, etc.) give them an inferential structure, which Lakoff and I called "Laws of Container Schemas" (Lakoff & Núñez, 2000). These so-called "laws" are conceptual in nature and are reflections at the cognitive level of brain structures at the neural level (see Figure 1). The four inferential laws are Container Schema versions of classical logical laws: *Excluded Middle*, *Modus Ponens*, *Hypothetical Syllogism*, and *Modus Tollens*. Let's see the details.

Inferential Laws of Embodied Container Schemas:

- *Excluded Middle*. Every object X is either *in* Container Schema A or *out of* Container Schema A.

- *Modus Ponens*: Given two Container Schemas *A* and *B* and an object *X*, if *A* is *in B* and *X* is *in A*, then *X* is *in B*.
- *Hypothetical Syllogism*: Given three Container Schemas *A*, *B* and *C*, if *A* is *in B* and *B* is *in C*, then *A* is *in C*.
- *Modus Tollens*: Given two Container Schemas *A* and *B* and an object *Y*, if *A* is *in B* and *Y* is *outside of B*, then *Y* is *outside of A*.

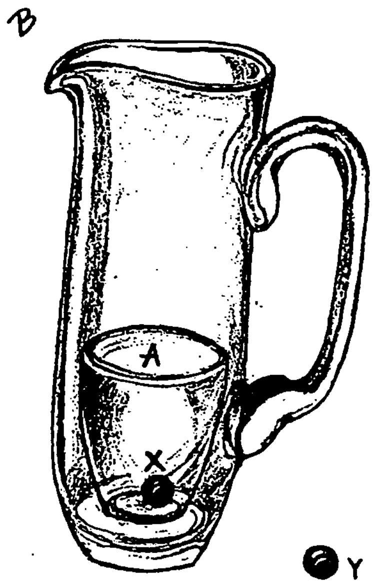

Figure 1. The "laws" of cognitive Container Schemas. The figure shows one cognitive Container Schema, *A*, occurring inside another, *B*. By inspection, one can see that if *X* is in *A*, then *X* is in *B*, and that if *Y* is outside of *B*, then *Y* is outside of *A*. We conceptualize physical containers in terms of cognitive containers. Cognitive Container Schemas are used not only in perception and imagination but also in conceptualization, as when we conceptualize bees as swarming *in* the garden. Container Schemas are the cognitive structures that allow us to make sense of familiar Venn diagrams.

Now, recall that conceptual metaphors allow the inferential structure of the source domain to be used to structure the target domain. So, the Classes Are Containers Metaphor maps the inferential laws given above for embodied Container Schemas (source domain) onto conceptual classes (target domain). These include both everyday classes and Boolean classes, which are metaphorical extensions of everyday classes. The entailment of such conceptual mapping is the following:

Inferential Laws for Classes Mapped from Embodied Container Schemas

- *Excluded Middle*. Every element *X* is either a *member of* class *A* or *not a member of* class *A*.
- *Modus Ponens*: Given two classes *A* and *B* and an element *X*, if *A* is a *subclass B* and *X* is a *member of A*, then *X* is a *member of B*.
- *Hypothetical Syllogism:* Given three classes *A*, *B*, and *C*, if *A* is a *subclass* of *B* and *B* is a *subclass of C*, then *A* is a *subclass of C*.

- *Modus Tollens*: Given two classes A and B and an element Y, if A is a *subclass of* B and Y is *not a member of* B, then Y is *not a member of* A.

The moral then is that these traditional laws of logic are in fact cognitive entities and, as such, are grounded in the neural structures that characterize Container Schemas. In other words, these laws are part of our bodies. Since they do not transcend our bodies, they are not laws of any transcendent reason. The truths of these traditional laws of logic are thus not dogmatic. They are true by virtue of what they mean.

This completes our brief and general overview of some crucial concepts of cognitive linguistics. Let us now see how this background can be used to apply a Mathematical Idea Analysis to some specific mathematical domains, Set theory and Hyperset theory.

ARE HYPERSETS, SETS?
A VIEW FROM MATHEMATICAL IDEA ANALYSIS

Consider the following question in modern mathematics: Are hypersets, sets? If not, what are they? We will now see, what embodied cognitive science can say about this. Since hypersets and sets are human (technical, mathematical) ideas we can provide an answer through Mathematical Idea Analysis. This is what we can say.

Sets

On the formalist view of the axiomatic method, a "set" is any mathematical structure that "satisfies" the axioms of set theory as written in symbols. The traditional axioms for set theory (the Zermelo-Fraenkel axioms) are often taught as being about sets conceptualized as containers. Many writers speak of sets as "containing" their members, and most students think of them that way. Even the choice of the word "member" suggests such a reading, as do the Venn diagrams used to introduce the subject. But if you look carefully through those axioms, you will find nothing in them that characterizes a container. The terms "set" and "member of" are both taken as undefined primitives. In formal mathematics, that means that they can be anything that fits the axioms. Here are the classic Zermelo-Fraenkel axioms, including the axiom of choice, what are commonly called the ZFC axioms.

The axiom of extension: Two sets are equal if and only if they have the same members. In other words, a set is uniquely determined by its members.
The axiom of specification: Given a set A and a one-place predicate, $P(x)$ that is either true or false of each member of A, there exists a subset of A whose members are exactly those members of A for which $P(x)$ is true.
The axiom of pairing: For any two sets, there exists a set that they are both members of.
The axiom of union: For every collection of sets, there is a set whose members are exactly the members of the sets of that collection.
The axiom of powers: For each set A, there is a set $P(A)$ whose members are exactly the subsets of set A.

The axiom of infinity: There exists a set A such that (1) the empty set is a member of A, and (ii) if x is a member of A, then the successor of x is a member of A.

The axiom of choice: Given a disjointed set S whose members are nonempty sets, there exists a set C which has as its members one and only one element from each member of S.

You can see that there is absolutely nothing in these axioms that explicitly requires sets to be containers. What these axioms do, collectively, is to *create* entities called "sets," first from elements and then from previously created sets. The axioms do not say explicitly how sets are to be conceptualized.

The point here is that, within formal mathematics, where all mathematical concepts are mapped onto set-theoretical structures, the "sets" used in these structures are not technically conceptualized as the Container Schemas we described above. They do not have container-schema structure with an interior, boundary, and exterior at all. Indeed, within formal mathematics, there are no concepts at all, and hence sets are not conceptualized as anything in particular. They are undefined entities whose only constraints are that they must "fit" the axioms. For formal logicians and model theorists, sets are those entities that fit the axioms and are used in the modeling of other branches of mathematics.

Of course, most of us do conceptualize sets in terms of Container Schemas, and that is perfectly consistent with the axioms given above. However, when we conceptualize sets as Container Schemas, a particular entailment follows automatically: *Sets cannot be members of themselves*, since containers cannot be inside themselves. But strictly speaking, this entailment does not follow from the axioms themselves, but rather from our metaphorical understanding of sets in terms of containers. The above axioms do not rule out sets that contain themselves. Indeed, an extra axiom was proposed by Von Neumann to rule out this possibility:

The Axiom of Foundation: There are no infinite descending sequences of sets under the membership relation. That is, $\ldots S_{i+1} \in S_i \in \ldots \in S$ is ruled out.

Since allowing sets to be members of themselves would result in such a sequence, this axiom has the indirect effect of ruling out self-membership.

<u>Hypersets</u>

Technically within formal mathematics, model theory has nothing to do with everyday understanding. Model-theorists do not depend upon our ordinary container-based concept of a set. Indeed, certain model-theorists have found that our ordinary grounding metaphor that Classes Are Container Schemas gets in the way of modeling kinds of phenomena they want to model, especially recursive phenomena. For example, take expressions like

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}$$

If we observe carefully, we can see that the denominator of the main fraction has in fact the value defined for x itself. In other words the above expression is equivalent to

$$x = 1 + \frac{1}{x}$$

Such recursive expressions are common in mathematics and computer science. The possibilities for modeling such expressions using "sets" are ruled out if the only kind of "sets" used in the modeling must be ones that cannot have themselves as members. Set-theorists have realized that a new non-container metaphor is needed for thinking about sets, and have explicitly constructed one (see Barwise and Moss, 1991).

The idea is to use graphs, not containers, for characterizing sets. The kinds of graphs used are Accessible Pointed Graphs, or APGs. "Pointed" indicates an asymmetric relation between nodes in the graph, indicated visually by an arrow pointing from one node to another—or from one node back to that node itself (see Figure 2). "Accessible" indicates that there is a single node which is linked to all other nodes in the graph, and can therefore be "accessed" from any other node.

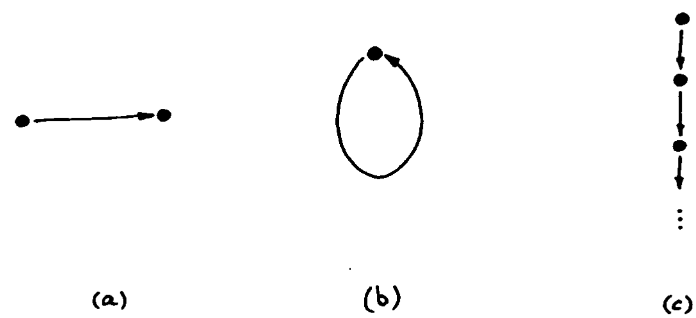

Figure 2. Hypersets: Sets conceptualized as graphs, with the empty set as the graph with no arrows leading from it. The set containing the empty set is a graph whose root has one arrow leading to the empty set (a). Illustration (b) depicts a graph of a set that is a "member" of itself, under the Sets Are Graphs Metaphor. Illustration (c) depicts an infinitely long chain of nodes in an infinite graph, which is equivalent to (b).

From the axiomatic perspective, they have replaced the Axiom of Foundation with another axiom that implies its negation, the "Anti-Foundation Axiom." From the perspective of Mathematical Idea Analysis they have implicitly used a conceptual metaphor, a linking metaphor whose mapping is the following:

The Sets Are Graphs Metaphor

Source Domain Accessible Pointed Graphs		Target Domain Sets
An APG	→	The Membership Structure of a Set
An Arrow	→	The Membership Relation
Nodes That Are Tails of Arrows	→	Sets
Decorations on Nodes that are Heads of Arrows	→	Members
APG's With No Loops	→	Classical Sets With The Foundation Axiom
APG's With or Without Loops	→	Hypersets With the Anti-Foundation Axiom

The effect of this metaphor is to eliminate the notion of containment from the concept of a "set." The graphs have no notion of containment built into them at all. And containment is not modeled by the graphs.

Graphs that have no loops satisfy the ZFC axioms and the Axiom of Foundation. They thus work just like sets conceptualized as containers. But graphs that do have loops model sets that can "have themselves as members." They do not work like sets that are conceptualized as containers, and they do not satisfy the Axiom of Foundation.

A "hyperset" is an APG that may or may not contain loops. Hypersets thus do not fit the Axiom of Foundation, but rather another axiom with the opposite intent:

- *The Anti-Foundation Axiom*: Every APG pictures a unique set.

The fact that hypersets satisfy the Zermelo-Fraenkel axioms confirms what we said above: *The Zermelo-Fraenkel axioms for set theory—the ones generally accepted in mathematics—do not define our ordinary concept of a set as a container at all!* That is, the axioms of "set theory" are not, and were never meant to be, about what we ordinarily call "sets", which we conceptualize in terms of Container Schemas.

So, What are sets, really?

Here we see the power of conceptual metaphor in mathematics. Sets, conceptualized in everyday terms as containers, do not have the right properties to model everything needed. So we can now metaphorically reconceptualize "sets" to exclude containment by using certain kinds of graphs. The only confusing thing is that this special case of graph theory is still called "set theory" for historical reasons.

Because of this misleading terminology, it is sometimes said that the theory of hypersets is "a set theory in which sets can contain themselves." From a cognitive point of view this is completely misleading because it is not a theory of "sets" as we ordinarily understand them in terms of containment. The reason that these graph theoretical objects are called "sets" is a functional one: they play the role in modeling axioms that classical sets with the Axiom of Foundation used to play.

The moral is that mathematics has (at least) two quite inconsistent metaphorical conceptions of sets, one in terms of Container Schemas (a grounding metaphor) and one in terms of graphs (a linking metaphor). Is one of these conceptions right and the other wrong? There is a perspective from which one might think so, a perspective that says that there must be only one literal correct notion of a "set". But from the perspective of Mathematical Idea Analysis these two distinct notions of "set" define different and mutually inconsistent subject matters, conceptualized via radically different conceptual metaphors. This situation is much more common in mathematics than the general public normally recognizes. It is Mathematical Idea Analysis that helps us to see and analyze these situations, by making explicit what is cognitively implicit.

EPILOGUE: SOME SPECULATIONS ABOUT THE IMPLICATIONS OF MATHEMATICAL IDEA ANALYSIS FOR MATHEMATICS EDUCATION

I would like to close my presentation, making some general remarks about possible implications of Mathematical Idea Analysis for mathematics education in general, and for the psychology of mathematics education in particular. This is by no means an exhaustive list. It is simply an open list to be taken as a proposal for discussion during the various sessions of the PME-2000 meeting.

In a nutshell, I could say that the deepest implication that Mathematical Idea Analysis provides, is the kind of philosophy of mathematics and of mathematics education that it brings forth. The approach presented here gives a portrait of mathematics that is fundamentally human. Concepts and ideas are human, and the truths that come out of them are relative to human conceptual systems. This includes mathematics. It follows from this perspective that teaching mathematics implies teaching human meaning, and teaching *why* theorems are true by virtue of what the elements involved actually mean. From this perspective at least the following implications can be mentioned:

- Mathematics education should demystify truth, proof, definitions, and formalisms. Although they are relevant, they should be taught in the context of the underlying human ideas. Therefore questions like those in the first paragraph of this article should be taken *very* seriously in the educational process.
- Mathematics Education should also demystify the belief that meaning, intuition, and ideas are vague and (purely) subjective. Human ideas and meaning have an impressive amount of bodily grounded constrains that make them non-arbitrary.
- Mathematics should be taught as a human enterprise, with its cultural and historical dimensions (which shouldn't just be a presentation of dates and a chronological list of events). These human dimensions should include those moments of doubts, hesitations, triumphs, and insights that shaped the historical process of sense-making.
- New generations of mathematics teachers, not only should have a good background in education, history, and philosophy, but they should also have some knowledge of cognitive science, in particular of the empirical study of conceptual structures and of everyday unconscious inferential mechanisms. They should know what is the implicit conceptual structure of the ideas they have to teach.
- The so-called "misconceptions" are not really misconceptions. This term as it is implies that there is a "wrong" conception, wrong relative to some "truth". But Mathematical Idea Analysis shows that there are no wrong conceptions as such, but rather variations of ideas and conceptual systems with different inferential structures (sometimes even inconsistent with each other, as we saw for the case of sets and hypersets).
- From a pedagogical point of view then, it would be very important to identify what are exactly the variations of inferential structure that generate the so-called misconceptions. By making this explicit, a pedagogical intervention should follow

for inducing students to operate with the appropriate conceptual mappings that bring forth the inferential structure required by the mathematical idea in question.

- When applied properly, Mathematical Idea Analysis can serve as a tool for helping people (especially adolescents and adults) to become aware of the organization, limitations, and potentials of their own conceptual systems, making explicit (and conscious) what in everyday life is implicit (and unconscious).
- "Being good at mathematics" doesn't necessarily mean being good at doing calculations and running algorithms. It means knowing how to keep one's metaphors straight, when to operate with the appropriate metaphors, when to shift from one to another one, when to combine them, and so on. Teaching how to master this conceptual gymnastics should be a goal for mathematics education.
- Beyond the mathematical content as such, the empirical study of conceptual systems can also give important insights into the attitudes and beliefs, students have about mathematics. The detailed study of students' conceptual structures underlying their linguistic expressions can reveal the origin of difficulties, lack of motivation, anxieties, and so on, that may be interfering with the learning of mathematics.

As you can see, this is far from being an exhaustive list. I believe that the cognitive science of mathematics, and Mathematical Idea Analysis in particular, provide a rich and deep tool, with a solid theoretical background, for bringing back human meaning into mathematics. The invitation is then extended for exploring how this can be accomplished in the process of teaching this astonishing conceptual structure called mathematics.

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