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AUTHOR	Yackel, Erna
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ABSTRACT

Interest in mathematics learning that focuses on understanding, mathematical reasoning, and meaning-making underscores the need to develop ways of analyzing classrooms that foster certain types of learning. The goal of this paper is to show that the constructs of social and sociomathematical norms, which grew out of taking a symbolic interactionist perspective, provide a means to analyze aspects of explanation, justification, and argumentation in mathematics classrooms. (Contains 23 references.) (DDR)

EXPLANATION, JUSTIFICATION AND ARGUMENTATION IN MATHEMATICS CLASSROOMS

Erna Yackel
Purdue University Calumet

Abstract: *Current interest in mathematics learning that focuses on understanding, mathematical reasoning and meaning-making underscores the need to develop ways of analyzing classrooms that foster these types of learning. In this paper, I show that the constructs of social and sociomathematical norms, which grew out of taking a symbolic interactionist perspective, and Toulmin's scheme of argumentation, as elaborated for mathematics education by Krummheuer, provide us with a means to analyze aspects of explanation, justification and argumentation in mathematics classrooms, including means through which they may be fostered. In addition, I use the example of current research in a university-level differential equations class to show how these notions can inform instruction in higher-level mathematics.*

In his plenary address to PME in 1994, John Mason said,

> I have long concluded that it is very hard to say anything new that has not been said more eloquently elsewhere.... I see working on education not in terms of an edifice of knowledge, adding new theorems to old, but rather as a journey of self discovery and development in which what others have learned has to be re-experienced by each traveller, re-learned re-integrated and re-expressed in each generation. (p. 177)

He went on to say that "All you can do, if you really want to be truthful is to tell a story" (p. 177).

My purpose in this paper is to tell a story that intends to capture something of what I have experienced and learned about explanation, justification and argumentation from a variety of mathematics classrooms that I have studied. At the same time, I intend to explain why I am telling you this story—what significance this story might have for someone else. In this regard, I am following the approach that Streefland (1993) took when he said that by analyzing his experiences, in his case in instructional design, his goal was to take what was an *after*-image for him and make it possible for that to become a *pre*-image for someone else working in the same arena. Similarly, my intention is to specify various aspects of explanation, justification and argumentation that have emerged from my analyses of classrooms so it becomes possible for others to use these aspects to inform their future efforts in mathematics education research and practice.

Background

The research activity that my colleagues and I have been engaged in is classroom based and involves conducting classroom teaching experiments that range from six weeks to an entire school year or university term. It involves developing instructional sequences and approaches as well as investigating teaching and learning as it occurs in the classroom. In this type of research, called developmental or design research (Cobb, Stephan, McClain, & Gravemeijer, in press; Gravemeijer, 1994), the researchers conduct ongoing analyses of classroom activity and use the results to inform instructional planning and decision making. It also involves retrospective analyses that attempt to explain the nature of the learning that took place and to explicate significant aspects of the learning situation.

Through analyzing these teaching experiments, we[1] have come to understand the importance of taking into account the social aspects of learning, including social interaction (Cobb, Yackel, & Wood, 1989; Yackel, Cobb, Wood, Wheatley, & Merkel, 1990; Yackel & Rasmussen, in press; Yackel, Rasmussen, & King, 2001). We have developed an interpretive framework for analyzing classrooms and we have explicated theoretical constructs within that framework in terms of our experiential base (Cobb & Yackel, 1996; Yackel & Cobb, 1996). Two constructs that are particularly relevant to issues of explanation, justification and argumentation are social norms and sociomathematical norms. Our investigations have pointed to the importance of these two constructs in clarifying both the functions that explanation, justification and argumentation serve and the means by which they might be fostered in the classroom. Further, the constructs of social and sociomathematical norms have been useful in clarifying how we might think of explanation, justification and argumentation in mathematics classrooms where understanding and meaning-making are the focus of instruction. In addition, in previous work, I have documented that children as young as second grade engage in sophisticated forms of explanation and justification and that their understanding of explanation advances as the school year progresses. Finally, I have argued (Yackel, 1997) that we can use Toulmin's argumentation scheme, as elaborated for mathematics education by Krummheuer (1995), as a methodological tool to demonstrate how learning progresses in a classroom.

All of these ideas were originally developed as a result of research in elementary school classrooms. A legitimate question is whether or not any of these results, including such constructs as social and sociomathematical norms, are useful in analyzing higher-level (including university-level) mathematics instruction in which the constraints are (or are perceived to be) different from those in the elementary grades. This question is particularly relevant since the amount of research at higher levels of mathematics instruction remains relatively small.

In this paper, I first discuss symbolic interactionism as a theoretical framework. Next, I explain what I mean by explanation and justification and discuss normative aspects of mathematics classrooms relative to explanation and justification. Then I describe how Toulmin's argumentation scheme can be used as a methodological tool

[1] In using *we* I am referring to various colleagues that I have worked with to conduct and analyze teaching experiments over the past decade and a half including Paul Cobb, Terry Wood, Koeno Gravemeijer, Diana Underwood Gregg, Chris Rasmussen, and Karen King.

to document learning in the classroom. I then discuss how this approach to argumentation can be used to explicate the teacher's role in the classroom. Finally, I show how these ideas, which are after-images of research conducted in mathematics classrooms at the elementary-school level, are being used as pre-images to inform current classroom-based research in a university-level mathematics class at my university. In doing so, I demonstrate their utility at the upper levels of mathematics learning. At the same time, the crucial role of explanation, justification and argumentation in mathematics learning from the lower to the higher levels of mathematics becomes apparent. Thus, I answer in the affirmative a question that was raised by Duval (2000) in his plenary lecture to PME last year when he asked "[I]s there something similar in the process of mathematics learning at the first levels and at upper levels?"

Theoretical Framework

In previous attempts to investigate explanation, justification and argumentation in mathematics classrooms, we have found it useful to take a symbolic interactionist perspective. Of the various approaches to social interaction, we have taken symbolic interaction as a theoretical lens for two reasons. First, it is compatible with psychological constructivism, which forms the theoretical basis for our investigation of individual learning (Cobb & Bauersfeld, 1995). Second, as Voigt (1996) points out, the symbolic interactionist approach is particularly useful when studying students' learning in inquiry mathematics classrooms[2] because it emphasizes both the individual's sense-making processes and the social processes without giving primacy to either one. Thus, we do not attempt to deduce an individual's learning from social processes or vice versa. Instead, individuals are seen to develop personal meanings as they participate in the ongoing negotiation of classroom norms.

The theory of symbolic interactionism has its roots in the work of George Herbert Mead, John Dewey and others and has been developed extensively by Herbert Blumer (1969). One of its defining principles is the centrality given to the process of interpretation in interaction. To put it another way, the position taken by symbolic interactionism is that in interacting with one another, individuals have to take account of (interpret) what the other is doing or about to do. Each person's actions are formed, in part, as she changes, abandons, retains, or revises her plans based on the actions of others. In this sense, social interaction is a process that *forms* human conduct rather than simply a setting in which human conduct takes place. As Blumer (1969) stated, "One has to *fit* one's own line of activity in some manner to the actions of others. The actions of others have to be taken into account and cannot be merely an arena for the expression of what one is disposed to do or sets out to do" (p. 8). Blumer further clarified that the term *symbolic* interactionism refers to the fact that the interaction of interest involves interpretation of action. Attempts to

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[2] I follow Richards (1991) in using the label *inquiry* mathematics classrooms to describe those classrooms in which students engage in genuine mathematical discussions with each other and with the teacher. See Yackel (2000) for a detailed discussion of the social and sociomathematical norms that characterize inquiry mathematics classrooms.

genuinely communicate involve understanding the meanings of another's actions (Rommetveit, 1985), and so involve symbolic interaction.

In addition to interpreting actions of others, individuals engaged in interaction attempt to indicate to others, through their actions, what their own intentions are. Thus, actions have meanings both for the person making them and for the person(s) to whom the action is directed. In this sense there is a joint action that arises by the articulation of the participating actors' activity. Blumer (1969) emphasized the collective nature of such joint action as follows:

> A joint action, while made up of diverse component acts that enter into its formation, is different from any one of them and from their mere aggregation. The joint action has a distinctive character in its own right, a character that lies in the articulation or linkage as apart from what may be articulated or linked. Thus, the joint action may be identified as such and may be spoken of and handled without having to break it down into the separate acts that comprise it. (p. 17)

Blumer further pointed out that it is important to recognize that "the joint action of the collective is an interlinkage of the separate acts of the participants" (p. 17). As such, it has to undergo a process of formation and, even though it may be well established as a form of social action, each instance of it has to be formed once again. Consequently, the meanings and interpretations that underlie joint action are continually subject to challenge. As a result, both individual actions and the joint (collective) action of a group can change over time. Furthermore, this view of joint action supports the position that social rules, norms and values are upheld by a process of social interaction and not the other way around.

A second defining principle of symbolic interactionism, in addition to the centrality of interpretation, is that meaning is seen as a social product. Blumer (1969) elaborated this point as follows:

> It [symbolic interactionism] does not regard meaning as emanating from the intrinsic makeup of the thing that has meaning, nor does it see meaning as arising through a coalescence of psychological elements in the person. Instead, it sees meaning as arising in the process of interaction between people. The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing. Their actions operate to define the thing for the person. Thus, symbolic interactionism sees meaning as social products, as creations that are formed in and through the defining activities of people as they interact. (pp. 4,5)

This view of meaning has important implications for how we interpret the results of classroom discursive activity. Since meanings grow out of social interaction, each individual's personal meanings and understandings are formed in and through the process of interpreting that interaction. Nevertheless, normative understandings are constituted. It is as these normative understandings are constituted that students develop their own interpretations of them. What we mean by saying that these understandings are normative is that there is evidence from classroom dialogue and activity that students' interpretations are compatible. It is in this sense that we say that students' interpretations or meanings are "taken-as-shared" (Cobb, Wood, Yackel, & McNeal, 1992).

Explanation and Justification

In this paper my interest is in explanation and justification as social constructs rather than as individual constructs. In this case, they are considered to be aspects of the discourse that serve communicative functions and are interactively constituted by the teacher and the students. Explanation and justification are distinguished, in part, by the functions they serve. Students and the teacher give mathematical explanations to clarify aspects of their mathematical thinking that they think might not be readily apparent to others. They give mathematical justifications in response to challenges to apparent violations of normative mathematical activity (Cobb et al., 1992). For example, consider the task "How can you figure out 16 + 8 + 14?" If a child responds, "I took one from the 16 and added it to the 14 to get 15 and 15; then I added the 15 and 15 to get 30, and the other 8 to get 38," we would infer that she was explaining her solution to others. However, a challenge that "you first have to add the 16 and the 8 and then add 14 to that sum" is a request for a justification.

Classroom norms that relate to mathematical explanation and justification are both social and sociomathematical in nature. Norm is a sociological construct and refers to understandings or interpretations that become normative or taken-as-shared by the group. Thus, norm is not an individual but a collective notion. One way to describe norms, in our case, classroom norms, is to describe the expectations and obligations that are constituted in the classroom.

By analyzing data from our initial teaching experiments, we were able to identify a number of social norms that characterized classroom interactions. These include that students are expected to develop personally-meaningful solutions to problems, to explain and justify their thinking and solutions, to listen to and attempt to make sense of each other's interpretations of and solutions to problems, and to ask questions and raise challenges in situations of misunderstanding or disagreement. In saying that these social norms characterized the classroom interactions, I mean that these ways of acting and of interpreting the actions of others became taken-as-shared. In subsequent classroom teaching experiments we had the constitution of these norms as an explicit goal. It is evident that each of these norms relates specifically to explanation and justification when taken as social constructs, as described above.

We were also able to identify normative aspects of interactions that are specific to mathematics. These we called sociomathematical norms (Yackel & Cobb, 1996). Normative understandings of what counts as mathematically different, sophisticated, efficient and elegant are examples of sociomathematical norms. Similarly, what counts as an acceptable mathematical explanation and justification is a sociomathematical norm. The distinction between social norms and sociomathematical norms is subtle. For example, the understanding *that* students are expected to explain their solutions is a social norm, whereas the understanding of *what counts* as an acceptable mathematical explanation is a sociomathematical norm.

We might ask how notions such as what counts as an acceptable explanation come to have meaning for students. To answer this question, we return to the symbolic interactionist position on meaning. This position is that meaning arises through interaction. Accordingly, the meaning of acceptable mathematical explanation is not something that can be outlined in advance for students to "apply." Instead, it is formed in and through the interactions of the participants in the classroom. As with all normative understandings, both explicit and implicit negotiations contribute to developing these understandings.

Elsewhere, I have documented second-grade children's evolving understanding of mathematical explanation and justification (Yackel, 1992; Yackel & Cobb, 1996). That analysis showed that initially children had to learn that their explanations and justifications needed a mathematical, rather than a social, basis. In that instance, the teacher initiated explicit discussions with the children about the basis for their explanations. For example, early in the school year a pupil changed her answer when the teacher asked the class if they agreed with her. In the subsequent discussion she revealed that she had interpreted the teacher's question as indicating that she had made an error. The teacher then used this as a paradigm case to discuss his expectation that students' answers should be based on mathematical reasoning with their explanations reflecting that reasoning. As the school year progressed students' explanations increasingly took on the character of descriptions of actions on objects that were experientially real for them. This expectation was both constituted and sustained, in part, through challenges that the students and the teacher made to explanations that described procedures. For example, when adding quantities such as 13 and 12 to get 25, explanations such as, "One and 1 are 2 and 3 and 2 are 5," were challenged by remarks such as, "That's a ten and that's another ten, and that's 20. And the answer is 25." Finally, by the end of the school year, some students took explanations as explicit objects of reflection and made comments such as, "How can someone understand what you mean? They don't know what you're referring to." The students making these challenge were taking the explanations as entities in and of themselves and were commenting on their overall potential as acts of communication within the classroom discourse. As with social norms, in subsequent teaching experiments we had the constitution of specific sociomathematical norms as an explicit goal, for example, that explanations should describe actions on objects that are experientially real for the students.

Argumentation

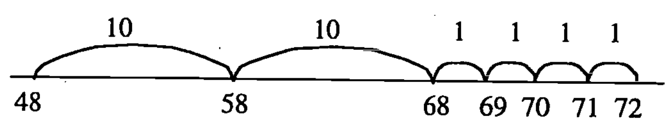
As noted earlier, my interest is in mathematical explanation and justification as interactional accomplishments and not as logical arguments. The focus is on what the participants take as acceptable, individually and collectively, and not on whether an argument might be considered mathematically valid. In this sense my interest is in what Toulmin (1969) calls substantial rather than logical argument.

Krummheuer's work on argumentation (1995) provides the background for the approach I take. In his study of the ethnology of argumentation, Krummheuer analyzes argumentation using Toulmin's scheme of conclusion, data, warrant, and backing. According to this scheme, the conclusion is a statement that is made as though it is certain. It is a claim. The support one might give for the conclusion is the data. Warrant refers to the rationale that might be given to explain why the data are considered to provide support for the conclusion. Backing provides further support for the warrant, that is, the backing indicates why the warrant should be accepted as having authority (Toulmin, 1969). Krummheuer uses the notions of conclusions, data, warrants, and backing to explicate how argumentation is an interactive constitution of the participants. For him, an argumentation in any given situation "contains several statements that are related to each other in a specific way and that by this take over certain functions for their interactional effectiveness" (Krummheuer, 1995, p. 247). Statements do not have a function apart from the interaction in which they are situated. Thus, what constitutes data, warrants, and backing is not predetermined but is negotiated by the participants as they interact.

I have demonstrated previously (Yackel, 1997) that this approach to argumentation is useful as a methodological tool for documenting the collective learning of a class because it provides a way to demonstrate changes that take place over time. Further it helps to clarify the relationship between the individual and the collective, that is, between the explanations and justifications that individual students give in specific instances and the classroom mathematical practices[3] that become taken-as-shared. As mathematical practices become taken-as-shared in the classroom, they are beyond justification and, hence, what is required as data, warrants and backing evolves. Similarly, the types of rationales that are given as data, warrants, and backing for explanations and justifications contribute to the development of what is taken-as-shared by the classroom community. For example, in one second-grade classroom, explanations of solution methods that involved using thinking strategies became more cryptic over time. For a problem such as $5 + 6 = \_\_$, students initially gave explanations such as, "I know that 5 and 5 is 10 and 6 is just one more; it's just one more on the 5, so the answer is 11; one more than 10." Later a typical explanation was "5 and 5 is 10 so its 11." In this case, when explanations of the second type are no longer questioned, we can assume that the warrant and backing provided in the earlier type of explanation are now taken-as-shared.[4] At the same time, it is through explanations of the earlier type that the taken-as-shared understandings develop.

[3] The construct of *classroom mathematical practices* refers to the collective development of the classroom community. See Cobb (1998) for a discussion of this construct and for an illustrative example.
[4] It could be argued that students might not question this type of solution because they are disinterested or simply do not want to ask a question. We can rule out these possibilities in this case because of the classroom social norms that were

Finally, this approach to argumentation is useful for analyzing classroom dialogue to investigate the nature of the contributions made by various participants in the interaction. In particular, our analyses of second-grade classrooms show that the teacher frequently serves the role of calling for or herself making explicit data, warrants, and backing that may be only implicit in the explanations and justifications that students give (Yackel, 1997). Here again, I give an example from a second-grade class. In explaining how he figured out how much to add to 48 to get 72, Louis said, "It's 24. How I got it was I said, 48, 58, 68, 69, 70, 71, 72." The teacher was aware that approximately half of the children in the class did not yet know that the difference between 48 and 58 is 10. Therefore, the data that Louis gave to support his conclusion would not have explanatory relevance for them. A warrant would be needed. Without waiting for anyone to ask a question, she asked Louis to elaborate by saying, "You said 48, 58, how many is that?" Using the language of argumentation, we would say that the teacher was prompting Louis to give a warrant. After Louis responded "Ten," she continued, "And then you went to 68, and how many was that?" Louis replied, "Twenty." The teacher went on to say, "That was 20 in all and then you went by ones didn't you? You went 69, 70, 71, 72. ... So you went a ten and another ten and four ones and that's how you got 24." In this episode we see the teacher working together with Louis to develop the needed warrant. In addition, as she and Louis were developing the verbal warrant, the teacher drew the following diagram on the chalkboard.



This action of drawing the diagram linked Louis' solution to the empty number line,[5] a pedagogical notation that was being used regularly in the class at that time. In general, the overall intention was to use the empty number line as a way to notate and record students' thinking with the goal that it might later become a tool for students' reasoning. In this case, the numerals written below the line recorded the information that Louis gave as data while the arcs and numerals written above the line recorded the additional information supplied in the warrant. In this sense, the diagram took on the form of a record of an argumentation because it included a record of both the data and the warrant.

In this example, the teacher initiated both the diagram and the verbal elaboration of Louis' solution. Her actions accomplished several goals. One was to call attention to the argumentative support for the conclusion. Second, she contributed to the class' understanding of what is taken as argumentative support. Earlier, I noted that what

operative. In this class students assumed the obligation of asking questions when they disagreed or did not understand. It is for this reason that we are able to interpret lack of questions as indicating taken-as-sharedness.

[5] The *empty* number line has no units marked on it. Therefore, unlike the (marked) number line that is often used in elementary school classrooms as a means for computing answers, the empty number line cannot be used for computing directly. Its power comes from the imagery that students use when they think about moving forward and backward from one number to another. Information about the distance between two numbers cannot be ascertained directly from the empty number line.

constitutes data, warrant, and backing is not predetermined but is negotiated by the participants as they interact. In this case, Louis' initial comments served as data for him. The teacher was aware that the only way these could serve as data for certain other students in the class was for additional information to be given, information that explained the relevance of Louis' comments. Finally, the teacher contributed to the goal that the empty number line might become a tool for reasoning by linking the data and warrant to the number line notation.

Explanation, Justification and Argumentation in a Differential Equations Class

In this section of the paper, I use the example of a differential equations class to demonstrate how the notions of explanation, justification and argumentation can be used to analyze the nature of the activity in a university-level mathematics class.

Data and Method

Over the past four years my colleague, Chris Rasmussen, has been investigating the teaching and learning of differential equations using his own classes as sites for investigation.[6, 7] During this time he has conducted classroom teaching experiments for all or part of the semester on three separate occasions. In each case, I have been an observer in the classroom and have participated in regular project meetings to discuss students' ways of reasoning, potential instructional activities and social aspects of the classroom.

Data from each teaching experiment consist of videotapes of each class session, including the small group work of two groups, field notes made by the observer(s) and the instructor, copies of students' work, and records of instructional activities and instructional decisions. Students' work includes in-class work, homework assignments, weekly electronic journal entries and reflective portfolios that students submitted twice in the semester. In addition, data include videotapes and artifacts from individual student interviews that were conducted with selected students at various times throughout each teaching experiment to gain initial information about their concepts related to functions and rates of change, to assess their understanding of key concepts of the course, and to inquire about their beliefs about mathematical activity and mathematics learning.

Social and Sociomathematical Norms Relating to Explanation and Justification

A specific goal in each teaching experiment was to constitute social and sociomathematical norms characteristic of inquiry instruction. Regarding explanation and justification, these include that students explain and justify their thinking, that they listen to and attempt to make sense of the explanations of others, and that explanations describe actions on objects that are experientially real for

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[6] Students in the differential equations classes at this university are typically engineering or mathematics majors. The typical class size is from 25 to 35 students.

[7] Rasmussen's work is supported by the National Science Foundation under grant number REC-9875388. I wish to thank him for permission to use episodes from his classes for this paper and to thank Jennifer Olszewski for making transcripts of selected episodes from videotapes.

them. Analysis of data from the first teaching experiment shows both *that* these norms were constituted and *how* the instructor initiated the negotiation of these norms (Yackel & Rasmussen, in press; Yackel, Rasmussen, & King, 2001).

The following example, taken from the second class session of the first teaching experiment, illustrates the constitution of social norms that foster explanation. The instructor began the class with a brief statement of the expectations he had for the students' mathematical activity. Then he orchestrated a whole class discussion of approximately twenty minutes in which he and the students discussed the rationale behind a differential equation they had used in the prior class session to indicate the rate of change of the recovered population in an infectious disease situation. The crucial aspect of this segment for purposes of this paper is the explicit attention the instructor gave throughout to the negotiation of social norms. Most of his comments were explicitly or implicitly directed toward expectations. For example, he said things such as:

Okay, can you explain to us then why it was 1/14 times I?
What do the rest of the people think about that?
Is that similar to what you were thinking?
Anyone want to add to that explanation? Expand on it a little bit?
So let's put that question out. So your question is ... Is that what I heard you say?

In making these remarks, the instructor was attempting to influence the interpretations students made of how to engage in the discussion. From this perspective, it might seem that the teacher is the only one who contributes to the renegotiation of social norms. However, norms are interactively constituted as individuals participate in interaction. In this case, as the episode evolved, students contributed to the negotiation of the norms by increasingly acting in accordance with the expectations. As the discussion progressed, students not only responded to the instructor's questions, they initiated comments that showed that they were beginning to change their understanding of the classroom participation structure. For example, a few minutes into the discussion one student said, "I didn't quite understand what he [another student] said" and a few seconds later explained what he did understand and said, "What I don't understand, what I was asking about ..."

As the episode continued, several students asked questions, offered explanations, and asked for elaboration and clarification. In doing so, they too were contributing to the ongoing constitution of the social norms that students are expected to explain their thinking to others and ask questions and raises challenges when they do not understand. We find it encouraging that as early as the second class session understandings such as these were becoming normative. Further, analyses of classroom interactions throughout the semester provide evidence that these norms became well established. It became routine for students to explain their thinking, to ask questions and raise challenges, and to elaborate their explanations and justifications spontaneously, without prompting from the instructor. Furthermore, although we have not provided examples here, it became normative that explanations were about actions on (mathematical) objects that were experientially real for the students.[8] Thus, we would say that the class could be characterized as following an inquiry mathematics tradition. Classroom observations verify similar results in the subsequent teaching experiments.

Argumentation

When I observed Rasmussen's class in the spring of 2001 it was immediately apparent that his instructional approach had evolved to have an even stronger emphasis on justification and argumentation than previously observed. In addition, he had developed a unique approach to collaborative learning. Further, these two aspects were intimately intertwined.

In a typical collaborative learning situation the instructor poses a task or problem, students work in groups and after some time, often sufficient for most groups to complete the task, the class engages in a discussion of the solution methods students developed in their small groups. This is, in fact, the approach that Rasmussen took in the first two differential equations teaching experiments. The dramatic difference I noticed in this most recent class was that groups often worked for little more than two minutes before sharing their thinking. Whole class discussion might then continue for as much as 15 minutes before another short segment of two to four minutes of small group work took place. Typically this cycle was repeated four to five times in an 80-minute class period. Furthermore, the students' task during small group work was typically to "think about" some question or issue rather than to solve a specific problem. Because of the continual emphasis on reasoning, whole class discussions resulted in the *emergence* of key concepts, including slope fields, bifurcation diagrams and phase planes. In this sense, the instructional approach seems to have considerable potential for in-depth conceptual development that grows out of students' discursive activity.

The following example, which is taken from the second session of the 2001 class, is used to clarify and illustrate the approach and to explain how it contributed to the emphasis on argumentation. During the 80-minute class session, there were four small group segments of two to four minutes each, interspersed between five whole class discussions. The first 20 minutes of the class was devoted to continuing a discussion from the previous class about a predator-prey situation described by a pair of differential equations. Students did not yet have any analytic techniques for recovering the solutions to the differential equations but used informal and qualitative reasoning to "make sense" of the situation. Throughout the discussion the instructor repeatedly asked the students for the reasons for their claims. When students gave reasons, he did not evaluate their validity but instead asked other students what they thought and solicited other arguments. For example, he made remarks such as, "What do you think of his idea?" and "Okay, that's nice. Let me

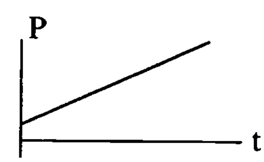
[8] I have limited the discussion to social norms for explanation. Similar analyses for sociomathematical norms have been conducted but space does not permit including them here. See Yackel and Rasmussen (in press) and Yackel, Rasmussen and King (2001) for a discussion of sociomathematical norms.

hear another argument for ..." From the point of view of norms, we would say that the instructor was initiating the expectations that students are to provide arguments to justify their claims, that they are to attempt to make sense of others' arguments, and that there might be more than one argument to support a given claim. From the point of view of argumentation, we would say that the discussion focused on data, warrants, and backing to support conclusions but not on the conclusions themselves.

The instructor then posed a simplified version of the predator-prey problem that involved only one species, in this case fish in a lake, with the assumptions that there were unlimited resources, no competitors, and there was a continuous growth rate. With no additional information, except that the initial population was 10 units, students were instructed to "take just a few minutes, sketch for yourself and talk about it with the folks in your group what your graphs [for the number of fish over time] look like and the reasons for why you think the graphs look like that."

Most students quickly drew either a straight line graph with positive slope or an exponential graph, beginning on the vertical axis at a height of 10, and made remarks to their peers such as "I thought it looks something like an increasing exponential curve" and "They're unbounded so it's just gonna continue to grow." After less than two minutes the instructor called the class back together.

Instructor: All right. Let me ask you a question. ... Suppose you were in a group and that's the way that someone offered. (The instructor draws the graph shown below on the chalkboard.) What would you—would you agree or disagree and what reasons would you give for agreeing or disagreeing with that?



In the ensuing discussion several students offered reasons why they disagreed or agreed with a linear graph. Some of the reasons given were directly related to thinking about the scenario, for example, that fish lay eggs and typically the number of eggs is large. Other reasons were related more to thinking about how to interpret the assumption of continuous reproduction. It is important to note that the students did not have any equations or functional expressions to inform their reasoning. Instead, they were in the position of having only the verbal scenario as a basis for thinking about what issues might be relevant and how to take those issues into account in developing arguments for one graph or another. As a result several important issues related to logistic growth problems were brought up by students including that population growth is dependent on the existing population and that it is reasonable to think about a growth rate parameter for a particular species.

The instructor then posed the specific case for population growth described by the differential $dP/dt = 0.2*P$. He asked what appeared to be a simple question. "What are the units of the 0.2 in the equation? How do you think about what the two is?"

When one of the students he called on said that she did not know how to put it in words, the instructor asked the class to "Take a second here, talk with the person next to you and say, well how would I think about that? What is that two? What kind of quantity is that two and how do I think about what two is here?" Once again, after only about two minutes, in which students verbalized their ideas to their group partners, the instructor called the class together for a whole class discussion.

What became apparent from observing the class over many weeks and from preliminary analysis of classroom data is that these brief small-group segments resulted in extremely productive subsequent discussions. The nature of the small group task was not to solve a problem but to "think about" something and "develop reasons to support your thinking." Even though students had little time to explore their reasons with one another in depth during the short time allotted to small group discussion, these "interruptions" in the class discussion obligated students to pursue their own ideas momentarily. Furthermore, because of the social norms that were operative in the class, students accepted the obligation and did engage in thinking about the issues at hand and in sharing their thinking within their groups. As a result, the students had a basis for participating meaningfully in the subsequent class discussion. Further, since the discussion inevitably focused on their reasons, students were in the position to compare and contrast their reasons with those of others.

In the class session discussed here, several other important ideas emerged as a result of the small group "thinking" and the subsequent whole group discussions. These included that the placement of the P (population) axis is arbitrary. Shifts to the right or left indicate different starting times. A related idea that developed is that for autonomous differential equations solution functions are horizontal shifts of one another. Before the class session ended students were beginning to think about the relationship between slope marks that they might make on a coordinate graph (a foreshadow of the slope field) and the graph of the function of population over time. In each case, these notions emerged from the students' reasoning. However, it is important to note that the instructor took a very proactive role and had a number of semesters of prior experience on which to draw as he posed questions that he anticipated would bring forth various types of reasoning.

I have suggested that the emphasis in this class was on reasons, rather than on conclusions. Using Toulmin's language of argumentation, the emphasis was on data, warrants, and backing. Students were less engaged in solving problems than they were in reasoning or "thinking." Preliminary analysis of excerpts from class dialogue indicates that what became constituted as data, warrants, and backing was not fixed or predetermined by the content or by the instructor but was negotiated by the participants as they interacted. It is in this sense that we use the label "group thinking" to refer to the collective argumentation that develops as the students engage in reasoning interactively, both in their small groups and in the whole class setting. We have adopted this label from a student in the class who used it in his second journal. As the following quotation from his journal indicates, students became keenly aware of the powerful nature of discussions for their learning.

> One way of thinking about a particular problem from class that helped me enlarge my thinking is *group thinking* [emphasis added]. You are the first professor I had who actually tries to make us understand the material and not just spit out equations and answers to you. The *group thinking* lets me communicate and speak my mind. My reasoning to a particular problem might be different than someone else's but another person in the class might also think the same way I do.
>
> A specific problem I liked was the predator-prey problem. Everyone had a different idea about it, which made everyone have to think. The *group thinking* helps me sort ideas out. Also, *group thinking* helps me put in words what I am trying to say. *Group thinking* in a math class is new to me, but I like it so far.

I would argue that the reason *group thinking* is so powerful for students' learning is that it emphasizes what most mathematicians and mathematics educators consider to be the essence of mathematics—mathematical reasoning and argumentation.

Conclusion

When investigating teaching and learning in the mathematics classroom there are many aspects that one might focus on. In this paper, I have pointed to aspects that relate to explanation, justification and argumentation. I have argued that a symbolic interactionist perspective gives us a way to make sense of the social aspects of the classroom such as social and sociomathematical norms. I have argued further that Krummheuer's approach to argumentation which uses Toulmin's scheme in a collective sense provides us with a way to explain why an emphasis on explanation and justification in a mathematics classroom leads to mathematics learning that emphasizes reasoning.

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