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ABSTRACT

This paper presents a synthesis of recently published case-studies in mathematics education research on students' mathematical thinking, and provides analyses, with respect to that literature, of video vignettes of high school students' mathematical activities as they participate in a two-week precalculus institute. We questioned whether emergent themes from the literature review we conducted were evidenced by student performances as they worked together on open-ended mathematics tasks during the precalculus institute. Case study articles were selected for the literature review according to specific criteria. The dominant strands of students' mathematical thinking that emerged from the literature review were understanding, proof or justification, and representations. The video episodes analyzed here are selected excerpts from the fourteen-year ongoing study of students' mathematical thinking and proof-making conducted by Rutgers University under the direction of Dr. Carolyn Maher and funded by the National Science Foundation. Analyses of the institute participants' mathematical thinking provided evidence that, under certain circumstances, students appropriately construct understanding, develop logical arguments, and design meaningful representations. (Contains 44 references.) (Author/MM)

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Abstract

This paper presents a synthesis of recently published case-studies in mathematics education research on students' mathematical thinking, and provides analyses, with respect to that literature, of video vignettes of high school students' mathematical activities as they participate in a two-week precalculus institute. We questioned whether emergent themes from the literature review we conducted were evidenced by student performances as they worked together on open-ended mathematics tasks during the precalculus institute. Case study articles were selected for the literature review according to specific criteria. The dominant strands of students' mathematical thinking that emerged from the literature review were understanding, proof or justification, and representations. The video episodes analyzed here are selected excerpts from the fourteen-year ongoing study of students' mathematical thinking and proof-making conducted by Rutgers University under the direction of Dr. Carolyn Maher and funded by the National Science Foundation. Analyses of the institute participants' mathematical thinking provided evidence that, under certain circumstances, students appropriately construct understanding, develop logical arguments and design meaningful representations.

Theoretical Framework

Many classroom teachers recognize that multiple choice and short answer questions are insufficient indicators of students' mathematical thinking. Teachers who effectively assess student understanding also interpret student representations and evaluate student justifications. According to Davis (1994), "it will not be easy for teachers to shift to the new role – working alongside students, trying to be aware of the student's thinking, working to help the student modify that thinking in an appropriate way..."(p. 17). Therefore, if teachers are to effectively instruct under this paradigm, teachers must be cognizant of the complexities of students' thinking.

Students' mathematical thinking is a broad topic that may be viewed from many perspectives and may contain many research emphases. One perspective is to view students' mathematical thinking according to representations, proof, and understanding. Central to doing mathematics is the construction of mental representations. The representations built by students are carried forth, revisited, used, and extended over time. As students explain, justify and convince others of their ideas, representations are often re-examined and certain features of the representations emerge. The students' discourse and work, which involves their constructed representations, makes it possible to study the growth of mathematical understanding (Maher, 1997).

The NSF study conducted by Rutgers University, A Longitudinal Study of the Development of Proof Making in Students[1], under the direction of Dr. Carolyn A. Maher,

[1] **This work is supported in part by National Science Foundation grants #MDR-9053597 (directed by R. B. Davis and C. A. Maher) and REC-9814846 (directed by Carolyn A. Maher) to Rutgers, The State University of New Jersey. Any opinion, findings, conclusions, or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation."**

followed a group of fourteen students through two years of secondary school mathematics. Some of the goals of the project are: " (1) to provide in-depth case studies of the development of proof making in high school students; (2) to investigate the relationship of students' earlier ideas and insights to later justification and proof building; (3) to trace the origin, development, and use of representations of student ideas, explorations, and insights relating to explanation, justification, and proof building" (Maher, 1997, p. 3).

In 1989, The National Council of Teachers of Mathematics put forth their *Curriculum and Evaluation Standards for School Mathematics* that included the idea of proof making by learners. The longitudinal study has provided mathematics education with important information and research in the quest for a more complete understanding of proof making by students. (See for example: Dann, E., Pantozzi, R., and Steencken E., 1995; Martino and Maher, 1999; Maher and Speiser, 2001) A literature review provides teachers and researchers with a composite of individual research perspectives. This examination of student performance on a specific task as part of the longitudinal study, situated within the current literature about students' mathematical thinking, provides teachers and researchers with a broader, applied perspective of student representations, proof making, and understanding.

Methods of Inquiry

Mathematics education research may be viewed from many perspectives and may contain many research emphases. We began here, with a dual purpose. We had an interest in identifying emergent themes in recent research surrounding secondary students, and we were interested in how the literature paralleled or contrasted with the performance of students who participated in a summer mathematics institute as mentioned previously. Significant research in students' mathematical thinking has historically concentrated on young children and has been theoretical in nature. Scarcity of research focusing on secondary school students and beginning college students became evident as we began this inquiry.

Articles were selected for inclusion in the literature review based on well-defined criteria: case-studies of secondary mathematics students published between 1998-2000 in one of five refereed journals: Educational Studies in Mathematics; For the Learning of Mathematics; Journal for Research in Mathematics Education; Journal of Mathematical Behavior; Mathematical Thinking and Learning. We defined secondary students to include students in post-primary or above, grade six or older. Studies of post-secondary students were included if the mathematics was equivalent to secondary level curricula. We conducted an exhaustive search of each journal by reading titles and abstracts of each article published during the identified time period. If an abstract identified an article as a report on secondary students, copies of the articles were obtained for further classification as case studies on students' mathematical thinking. One article from Review of Educational Research and one article from International Journal of Mathematical Education in Science and Technology were also included, but an exhaustive search of articles over the three-year period described above was not conducted for these two journals. We recognize that restricting our review to case studies published in journals may leave out some of the most current research presented at conferences or contained in doctoral dissertations.

Following the tentative initial identification of articles, each of us independently read each article to gain a sense of the authors' emphases and orientations. Next, we individually classified the articles according to mathematical content and thematic strands. We compared our individual classifications, organized a classification chart from consensus on each of the articles, and formed three mathematical content groupings consistent with the ordering sequence traditionally employed in schools--algebra and geometry, precalculus and calculus, and discrete mathematics. Thematic strands that emerged were identified—understanding, proof, and representation. These three strands appeal to us because they implicitly include reasoning, logic, argumentation, inscription, notation, and problem solving. Also, they are unique enough to allow us to draw distinctions among case studies concerned with one or the others. Since these three strands are intertwined, it is possible that a study, which we classified as concerned with one strand, may be classified differently by another group of researchers. We assert that these strands are representative of the myriad dimensions of mathematical thinking in the age range considered, and that our classifications have been validated through first individually evaluating the articles, and then jointly confirming such evaluations.

The last organizational choice was to base our analysis on strands rather than content areas. This was in large part due to the fact that the strands spanned content areas and age levels. After sorting articles according to thematic strands, each of us selected a strand to review.

Data Sources

The videotape data come from a two-week summer institute that was a component of a longitudinal study on the development of proof making in students. High school students worked in groups on a series of open-ended precalculus mathematics problems. This paper focuses on one of the problems the students were given.

To demonstrate the interrelationship between the three strands—understanding, justification, and representation— identified in this corpus of literature, we analyzed video clips of students working on the Catwalk problem (Speiser and Walter, 1994). Students were presented with a scaled-down photocopy of a plate of twenty-four still-photo frames of a cat in motion, first walking then running, taken from Eadweard Muybridge's work *Horses and Other Animals in Motion* (see Figure 1).

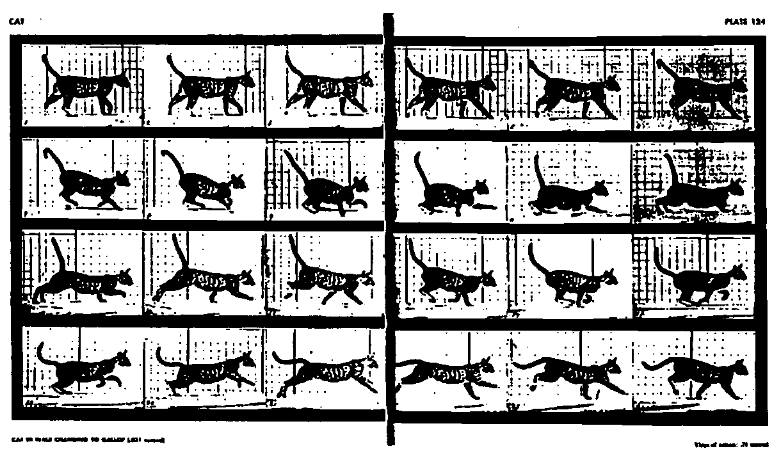

Figure 1.

The time interval between successive frames was .031 seconds, and the cat was moving in front of a grid whose lines were five centimeters apart. Students were asked to determine how fast the cat was moving in frame 10, and then how fast the cat was moving in frame 20.

The specific video episodes included here come from the Private Universe in Mathematics Project[2], which airs on the Annenberg/CPB Channel. The Annenberg tape is an overview of the students' work, which also includes individual interview segments.

Results

The video episodes analyzed here are exemplary pieces that demonstrate our findings in the literature review. The students' words and actions provide emic perspective. In this first episode, one student is interviewed about the initial work done on the task:

Episode 1:

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Jeff: We went in.  We had the two main components in finding out the speed or
      velocity.  And we went in and we all scribbled it down.  And we're like
      alright you know that's it.  What's next.  And then we start talking and
      then one of the people would bring something up.  And we'd be like oh we
      didn't think about that.  And then as you start looking into it, you kind
      of got to push what you did away.  And you start over and just get into
      some different things and see where it would take us.
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As students began to work on the "catwalk" task, their notions of speed or velocity initially centered on "two main components"—distance and time. Their solution strategies began with an algebraic inscription of the familiar formula for velocity, velocity equals distance divided by time, which students had learned from prior experience in school mathematics classes. Student discourse led to additional and alternative representations. Current research literature indicate that students will initially choose algebraic representation methods, and that algebraic representations are the predominant choices and have more status in analysis than geometric representations (Habre, 2000; Knuth, 2000). However, in contrast to the work shown by the students in the longitudinal study, the literature suggest that students are either unsuccessful in moving between representations or show little indication of awareness of alternative representations (Even, 1998; Booth and Thomas, 2000).

The events of this episode would not be considered by most as a proof, so we may refer to it as an argument that the students will construct. Knowledge needed for this argument consisted of the students' content beliefs about velocity and speed; the meaning of such terms and the algebraic representations of the same; as well as recollections of encounters with similar mathematical and academic situations.

In the beginnings of the encounter with the material the students processed the problem using reasoning based on established experiences that may be mathematically superficial (Lithner, 2000). They may have attempted to imitate procedures used previously on problems having a resemblance to this problem. Tirosh and Stavy (1999) have suggested "that students' responses to given mathematical and scientific tasks are often affected by common, external features of these tasks" (p. 63). However, after some reflection, the students in this setting considered other factors and used what Lithner

[2] The Private Universe Project in Mathematics is a professional development workshop series for K-12 teachers of mathematics produced by the Harvard-Smithsonian Center for Astrophysics in.collaboration with the Robert B. Davis Institute for Learning, Rutgers University. Any opinion, findings, conclusions, or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the Harvard-Smithsonian Center for Astrophysics or Robert B. Davis Institute for Learning, Rutgers University.

referred to as plausible reasoning based on the mathematical properties of the situation. Such plausible reasoning "is meant to guide towards what probably is the truth, without necessarily having to be complete or correct" (p. 167).

Edwards (1999) advocated the use of open-ended problems such as the Catwalk due to the lack of a 'right' or 'wrong' answer. This type of problem causes students to consider the reasonableness of the answers they arrive at and supports the use of exploratory processes.

Authors writing about understanding used a framework of levels of understanding. Some authors viewed understanding from a three level perspective, while others used a two level perspective. Based on this structure of understanding, this episode demonstrates the first level described by the authors. Watson and Moritz (2000) defined the beginnings of understanding, unistructural, as focusing on a single thing to solve a problem. When the student said they went in, tried some numbers and they were done, they matched this description. According to the students, they have just performed some manipulations without exploring any further. This description fit the operational level of understanding used by Hollar and Norwood (1999), which is called action conception by Slavit (1998) and is classified by Forster and Taylor (2000) as the interiorization stage of understanding. Similarly, Hung (2000) would state that these students are at the symbol level of understanding, meaning they have a numerical and computational understanding. This student stated that only after communicating with others, did they start to think about other aspects of the problem. So, in this episode, the students exhibited the lowest level of understanding. At this level a student is only able to work in a local arena and does not have the ability to generalize. When students are using symbols, they typically work on a manipulative basis.

In this second episode, a student presented, to the entire group, an explanation of his algebraic and graphical interpretations about the cat's movement throughout the frames of the picture.

Episode 2:

Michael: What this graph, everybody. It's it's not the velocity, it's just the change in distance. But what I, but what I see here is a certain velocity and another one. So the time would be as you go along, you know, and the distance is the difference between each and every one. So, that means he's walking at a constant rate for a very small amount of time. Then when this changes, that's around when he starts crouching down and he starts walking. Like [frame] ten is around here. He starts going at this speed. This is, it's not accelerating here, he's actually going the same speed that's why all these guys are like the same apart. That's why you see a line right here. So it's two lines. It may be a little off in there, but basically what it is he's walking in the beginning, going this speed, and he's running at the end, going a faster speed.

Students working on the catwalk problem provided wonderful examples of the development of student models linking multiple representations. In this video clip, Michael presented a modification of the original symbolic representation—velocity equals distance divided by time, and provided an oral interpretation of the calculator display of the distance with respect to time data that the students collected. He noted the slopes of the two "lines" connecting the discrete data points in a continuous curve represent the two different velocities of the cat in motion. Michael also described the change in distance as the difference between the values of different coordinates.

The literature overwhelmingly demonstrated that students lack facility in linking representations except through their own modeling development of multiple representations. Eighth grade students were found to have more sophisticated, high-level representational abilities, and conceptual knowledge from iterative representation work than graduate students' work on the same problem (Roth & McGinn, 1998). Booth and Thomas (2000) found that students chose the most familiar or least abstract representation when solving problems. Even (1998) established that students tended not to devise a second representation to identify or resolve any dissonance from an inefficient or inaccurate algebraic representation. Graphic calculators and computers provide graphical representations of algebraic systems and serve as visualizing tools to generate shared representations (Doerr & Zangor, 2000). There were opposing findings regarding whether significant differences in student performance occurred when students had access to multi-representational technology (Berger, 1998; Graham & Thomas, 2000). Doerr (2000) identified sequencing in students' modeling development of representations in an unfamiliar content domain. Students linked multiple representations as pivots for cognitive growth (Barnes, 2000).

This student was asked to share his group's progress thus far with the class. They had attempted to interpret the grid upon which the cat was photographed and in turn to interpret their measurements of change in the cat's position over time in reference to the grid. The students worked to translate the data into more meaningful everyday terms. This work was conducted based upon previous knowledge and was viewed through the students' filter that the mathematics should make sense. This filter is a motivating force in the students' search for meaning and justification of their ideas and potential solutions (Maher & Martino, 1996). They were searching for the feeling of "It must be so," "I feel it must be so" (Fischbein, 1982, quoted in Segal, 1999, p. 193).

The graphical representation of the data clearly shows a pattern that Michael presented using terms or phrases such as distance, change in distance, velocity, constant rate, time, speed, accelerating, line, walking, and running. Each of these terms or phrases had come from previous experience and was used in this context for the purpose of explaining the patterns illustrated by the graph. The student presented the graph as a valid representation and interpretation of the physical movement of the cat. According to the levels of proof proposed by Miyazaki (2000), this argument contained some sophistication in that the presenter processed at a higher level using the abstraction in lieu of the actual pictures of the cat. However, the presentation still lacked the functional language of demonstration.

In this episode, the students moved on to a new level of understanding about the cat's movement. Hung (2000) defined the problem level of understanding as viewing the problem as a whole. Similarly, when the new concept is seen as a whole, Forster and Taylor (2000) stated the condensation stage occurred. The students were working at this level because there was an examination of the movement of the cat throughout all the frames. Therefore, the students considered the problem as a whole, rather than focused on the frames surrounding frame ten and frame twenty. This examination of the entire movement of the cat allowed them to increase their understanding. Watson and Moritz (2000) suggested the students are at the multistructural level of understanding, because they mentioned several aspects of the task. For the authors, who used three levels to define understanding, this episode demonstrates that the students were working at the authors' definitions of a second level of understanding.

The third episode is an interview with a student about the group's decision to mimic the movement of the cat in the hallway. Students placed masking tape on the hallway floor to create a scale model of the cat's movement through all the frames. Each student took a turn moving along the tape measure to specific intervals representing the cat's distance traveled between frames. The interview presented here discusses why the students chose to perform this task.

Episode 3:

Romina: It made sense of it. Like we had the numbers there, but we, they didn't make sense to us. They, we didn't understand how something could change speed so fast. What was going on? Why it would change speed so fast? But when you do it like, a like real life version of it, you can see what the cat is doing. So you understand it. And like our graphs would go up all of a sudden and like fly down. But then when we did it, we could see, like, it was accelerating, it came to a peak, and then it was slowing down, and, like it just made sense of all the math.

Mental schemas affect students' abilities to appropriately represent the mathematization of problem situations. Students in the summer institute tried to construct mental schemas to mathematize and understand a problem situation. They developed appropriate algebraic and graphical representations to demonstrate correct relationships. They engaged in extended classroom discussion and utilized numerical measurement data that they collected and organized in tables and displayed graphically on the calculator. Each iteration in representation work influenced students' mental representations of the cat's velocity at a particular point in time. When students kinesthetically experienced representing the cat's motion themselves, Romina explained, "it just makes sense of all the math."

Students' mental schemas may interfere with algebraic and graphical attempts to represent correct relationships. If symbols are simply manipulated in a mechanical fashion, inappropriate expressions and equations follow (Cifarelli, 1998). Janvier (1998) determined that the notion of chronicle—time dependent variable—acts as a barrier to graphical representation of non-chronicles and encouraged teachers and programs to expand the context of graphical representations beyond chronicles. Secondary school students should be provided with opportunities to engage in classroom discussion and to compare through multiple representations the graphs of the same system as relevant parameters change (Janvier, 1998).

Students evidenced three different perceptions of what an equation represents: a formula, a narrative describing operations yielding a result, a description of essential relationships (Stacey & MacGregor, 2000). The literature indicated that some students translated narrative into expressions of mathematical symbols, rather than equations, and a substantial number of students' 'equations' were narratives of operations, rather than statements of equality. Stacey and MacGregor (2000) and Villarreal (2000) concluded that such arithmetic patterns of thinking act as barriers to algebraic representation. High performance on process-open tasks may be due to students' advantageous use of concrete drawings or lists rather than generalized, symbolic approaches (Cai, 2000).

Here we see the students' attempt to make sense of the problem by becoming the cat on a line. This search for meaning speaks to the students' sources of conviction or beliefs about what is true or valid in mathematics, which was internal in this case, appealing to such things as consistency, logic, and empirical evidence. Szydlik (2000) found that students with internal sources of conviction believed that since mathematics is consistent and logical, they should be able to figure it out on their own. This is in contrast to the external sources of conviction found in many students in the typical classroom. External sources of conviction include the teacher, the textbook, the answer key, another student, or anything else that the student considers an authority. Students with external sources of conviction "will not make sense of mathematics unless their sources of conviction are shifted; after all, to them mathematics is not *supposed* [original emphasis] to make sense" (p. 273).

While the students' reenactment of the cat on the line may seem simplistic, it was a personal construction in an attempt to organize the data, and was a necessary step in their understanding of the problem and in their development of meaningful answers to the questions at hand. This experience provided the students with "the opportunity to engage in less formal mathematical reasoning, by being asked to discover, describe, and explain simple mathematical regularities" (Edwards, 1999, p. 501). Such opportunities facilitate the students' development of the ability to work with greater abstraction and more complex and demanding arguments and proofs. After all, the "main function [of proof] in mathematics education is surely that of explanation" (Hanna, 1995, p. 47).

In the final demonstrative video episode, a student presented her group's conclusions about whether they would "bet" on how fast the cat was moving at frame 10 and frame 20. The researchers requested the students qualify how convinced they were that their solutions were correct by asking if students would be willing to bet on their solutions to the task.

Episode 4:

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Romina:  What we did is, we started off with like measuring like with our ruler
         and like where the cat's, um, nose moved. But then that wasn't working
         for us cause the measurements were getting too small. So we kind of
         eyeballed it. And we said that from [frame] nine to ten he moved about
         half a box. So that's 2.5 centimeters. And we divided that by point
         031, we got 80.
Teacher: You used the nose
Romina:  Yeah. Our numbers go from 80 to 200. That's like a big difference. So
         to take an average, I wouldn't, we just said we wouldn't bet on an
         average.
Angela:  Frame 10 is right in the middle of those two, so it gonna be the
         average.
Victor:  You have to look at what's he's doing
Matt:    It doesn't mean it's the average.
Romina:  Yeah but in frame 10 he's go like 80. There's a lot going on in frame
         10. I just don't think you should bet on frame 10. Frame 10 is too
         like. Frame 10, too much is going on in frame 10. Like, like cause if
         your going like technical, the guy could be like well I meant at the
         beginning of frame 10 what his speed was.
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In episode four, a student presented her group's answers to the questions about the cat's speed in frames ten and twenty. The purpose of her presentation was not only to display the answers, but also to discuss the accuracy of these answers. Her justification was in the form of a narrative argument. Narrative arguments are arguments "in which mathematical relationships and reasoning [are] described in everyday narrative or in pictures… examples are presented as evidence to convince and are followed by a discussion, in words perhaps illustrated by pictures, of why the statement is true" (Healy & Hoyles, 2000, p. 415).

This argument began in her small group where the members of the group found that it satisfied their need for personal conviction. Then the same argument was presented to all assembled where it needed to be validated in the setting of the community. Researchers found that expectations in these two realms were very different for students (Healy & Hoyles, 2000; Segal, 1999). Students often considered arguments that were personally convincing to not be formal or thorough enough to be used to convince others. "The literature provides evidence that students may believe that the validity of an argument equates to its having a particular 'look and feel'" (Segal, 1999, p. 195). In addition they favored clear, simple arguments for themselves, while presenting the most complicated and representationally abstract arguments to others, perhaps in an attempt to impress or bedazzle.

Edwards (1999) reminded us that only when a student believes that a statement or answer is true, will she earnestly seek out a justification for that statement or answer. This is why Healy & Hoyles (2000) appealed to the role of examples. Examples bridge the gap between the abstract statement of a problem and the concrete knowledge and experience of the students. Through examples the students are able to experiment and explore and gain a sense of "it must be so" (Fischbein, 1982, quoted in Segal, 1999, p. 193).

This episode shows that students were beginning to discuss ideas beyond the boundaries of this problem. The discussion now focused on the idea of average and what it means to take an average. When Romina stated that the speed of the cat could have been at the beginning of the frame or the middle, her interpretation of the problem had grown. Forster and Taylor (1999) described reification as the concept being separated from the process used to generate it. When students related the problem to other problems and concepts, Hung (2000) stated the students were at the situational level. The students generated a number answer for the speed of the cat. At this point though, they were not discussing how they arrived at the number. They were debating the validity of their solutions and if they had met the true definition of average. While this episode does not provide conclusive evidence that these students reached the final level of understanding, it is certainly obvious that they were approaching a relational level by referring to all the required elements needed to solve a problem (Watson & Moritz, 2000).

Conclusions

This paper applied a synthesis of current research literature to ongoing longitudinal research on students' mathematical thinking. The literature review provided a framework for analysis of students' thinking. Application of this framework to the longitudinal study provided evidence to support the various perspectives of the authors in the literature review.

Our model, presented here, uses a literature review as a point of reference for alternate examinations of existing studies. Using this model, mathematics educators have an opportunity to see how their own work is situated within the larger body of mathematics education research. Situating one's own work can offer different perspectives and raise alternate questions for consideration.

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