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AUTHOR	Weinberg, Suzanne Levin
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ABSTRACT

Fractions and division are two topics that receive considerable attention in elementary school mathematics, particularly in grade three and beyond. Despite this attention, fraction and division concepts and skills prove difficult for many middle school students to learn and for teachers to teach. This paper reports on part of a larger study involving 387 middle school students with regard to their performance on an assessment focusing specifically on concepts and skills relating to division and fractions. The study sought to determine how middle school students conceptualize fractions and division. The study also examined to what extent middle school students can recognize, identify, and apply the concepts and skills that stem from the mathematical definition of a fraction found in James and James' Mathematics Dictionary. It concludes that in order to help students and teachers and to refocus mathematics education research and textbook presentations of fractions and division, there must be a new conceptualization that includes fractions and division and the connections between them. Evaluation instrument is appended. (Contains 23 references.) (ASK)

Is There A Connection Between Fractions And Division? Students' Inconsistent Responses

A Presentation for the Annual Meeting of the
American Educational Research Association
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12 April, 2001

Introduction

Fractions and division are two topics which receive considerable attention in elementary school mathematics, particularly in grades three and beyond. By the time students reach the middle school years, they have had a minimum of three years of instruction in both fractions and division, and the NCTM *Principles and Standards* (2000) expect mastery and flexibility with the content of division (p. 148) and fractions (p. 214). Yet despite this attention, fraction and division concepts and skills prove difficult for many middle school students to learn and for teachers to teach. This paper reports on part of a larger study involving 387 middle school students with respect to their performance on an assessment focusing specifically on concepts and skills relating to division and fractions. The study sought to determine how middle school students conceptualize fractions and division. They study also examined to what extent middle school students can recognize, identify, and apply the concepts and skills that stem from the mathematical definition of a fraction found in James & James' *Mathematics Dictionary* (1992):

> "**fraction**, n. An indicated quotient of two quantities. The dividend is the **numerator** and the divisor the **denominator** (the numerator of $\frac{3}{4}$ is 3 and the denominator is 4.)" (p. 170) [emphasis as shown in text]

In other words, $\frac{a}{b} = a \div b$.

Fractions

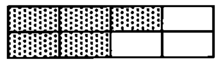
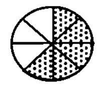
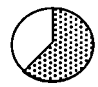
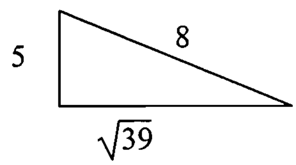
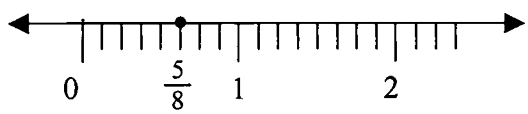
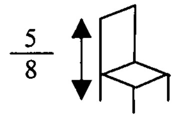
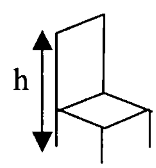
Fractions and division are complex topics. Fractions have a multitude of meanings and applications (see Behr, Lesh, Post, & Silver, 1983; Kerslake, 1986; Kieren, 1980, 1976; Levin, 1998; Novillis, 1976; Ohlsson, 1988). The research dealing with fractions falls into three main categories. The first category deals with philosophical discussions about how to think about fractions, including pictorial models and situational models. These articles describe fractions from a theoretical and mathematical point of view. The discussion of fraction does not necessarily include how *students* reason about fraction, nor does it include recommendations for how *teachers* might teach fractions. Additionally, articles in this first category do not focus upon operations on fractions, but rather more specifically what is meant by a single fraction. The works cited most commonly in these broad conceptualizations include the work from the Rational Number Project (see for example: Behr, Lesh, Post, & Silver, 1983) and two articles by Kieren (see Kieren, 1980, 1976). Other research conceptualizations mentioned included Kerslake (1986), Novillis (1976), and Ohlsson(1988).

A second category encompasses recommendations fro teachers about how to teach fractions. These articles describe different pictorial models, and the degrees of success and failure that students had with one specific model. Often the recommended pictorial model is compared to one or two other pictorial models. In this category, there did not seem to be a seminal piece of research, but rather a diffuse collection of recommendations. However, many of the articles in this category refer to one of the philosophical conceptualizations mentioned in the first category.

A third category discusses different strategies for teaching addition, subtraction, multiplication, and division of fractions. Because I was interested in fraction concepts rather than computation, I read these articles to see if any fraction model, pictorial or situational, was mentioned. A large majority of articles did not include any reference to conceptual development in conjunction with computation. The few that emphasized concept development suggested a ratio situational model or related the operations with the part-whole continuous-to-discrete model for fractions. Table 1 summarizes the ways that fraction is conceptualized in the literature.

Table 1

Models of Fraction for $\frac{5}{8}$

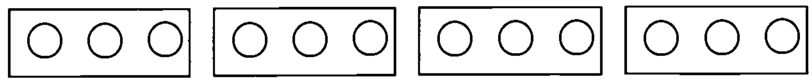
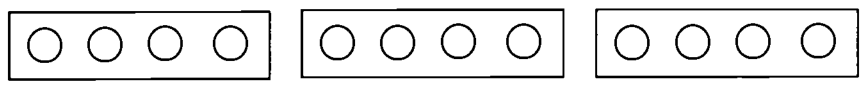
<table>
<tr><th>Model</th><th>Example of Model</th></tr>
<tr><td>Part-whole</td><td></td></tr>
<tr><td>Continuous-to-discrete</td><td> or </td></tr>
<tr><td>Discrete</td><td> $\frac{5}{8}$</td></tr>
<tr><td>Continuous</td><td> or </td></tr>
<tr><td>Ratio</td><td>5 girls to 8 boys (the ratio of girls to boys is $\frac{5}{8}$);
for Δ ABC below, the sine of an angle A, sin(A) = $\frac{5}{8}$.
</td></tr>
<tr><td>Rate</td><td>five miles in eight days ($\frac{5}{8}$ mile per day);
five pizzas for eight people ($\frac{5}{8}$ pizza per person)</td></tr>
<tr><td>Number line</td><td></td></tr>
<tr><td>Scale Change</td><td>$\frac{5}{8}$ as tall as the original chair
</td></tr>
<tr><td>Fraction Chart</td><td><table>
<tr><td colspan="8">1</td></tr>
<tr><td colspan="4">½</td><td colspan="4">½</td></tr>
<tr><td colspan="2">¼</td><td colspan="2">¼</td><td colspan="2">¼</td><td colspan="2">¼</td></tr>
<tr><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td></tr>
</table></td></tr>
<tr><td>Indicated Division</td><td>$\frac{5}{8}$ means $5 \div 8$ or $8\overline{)5}$</td></tr>
<tr><td>Performed Division</td><td>The result of a division: $25 \div 40 = 5/8 = 0.625$ (or 62.5%).</td></tr>
</table>

Division

The articles I found on division do not, in general, include a broad conceptualization on division. Two noticeable exceptions are Usiskin and Bell (1983) and Kouba and Franklin (1995). Instead, many of the articles focus primarily upon either division in terms of related facts or the inverse-operation for multiplication (see for example, Bergen, 1966 and Nesher, 1988), or the partitive and quotative structure of division problems. The work by Fischbein, Deri, Nello, and Marino (1985) is cited repeatedly. They found that the intuitive model is the partitive model, and that students are not successful with the quotative model of division until the middle-school years. Yet their findings do not confirm an earlier study by Zweng (1963), who found no significant differences between partitive and quotative models in second graders. A summary of the ways that division is conceptualized in the literature is found in Table 2.

One important note: the articles which focus upon the lack of student success with solving division problems at different age groups attribute the difficulty to the size of the number in the dividend compared to the number in the divisor (see for example: Graeber & Tirosh, 1992; Greer, 1987).

Table 2
Models of Division for 12 ÷ 4

Model	Example of Model
Partitive Division	
Rate	A child walks 12 miles in 4 hours. What is the child's average speed?
Ratio	A child walks 12 miles. A dog walks 4 miles. How many times as far as the dog has the child walked?
Quotative Division	
Rate	A child runs at 4 miles per hour. How long will it take the child to run 12 miles?
Ratio	A child rides a bicycle at 12 miles per hour, which is 4 times as fast she can run. What is her running speed?
Repeated Subtraction	12 – **4** = 8; 8 – **4** = 4; 4 – **4** = 0; 3 **4**s, so 12 ÷ 4 = 3.
Related Facts	12 ÷ 4 means that if 4 • **3** = 12, then 12 ÷ 4 = **3**.

Connections Between Fractions and Division

As described above, it would appear that fractions and division have few if any connections. This is not true. As the definition shown on page 1 illustrates, fractions and division are linked by vocabulary (two terms for the same function), by symbolism (the fraction bar and the binary symbol for division ÷), and by related uses. The *Principles and Standards* recommends that teachers specifically highlight connections in mathematics (NCTM, 2000). Thinking of a fraction as a division is found in the literature (Kieren, 1980). However, there is not a convenient pictorial or application model which supports this connection. Instead, many of the relationships between fractions and division stem from the stated definition of from one of the many applications of the definition of a fraction. Table 3 shows eight relationships that exist between fractions and division.

In many classrooms, teachers teach from textbooks in which fractions are taught separately from division. However, the idea that fractions represent division is an important one—without this understanding, expressions such as $\frac{\pi}{6}$ and equations such as $\frac{x}{4} = 9$ have no meaning. Additionally, if students do not understand the division meaning of fractions, then algorithms for rewriting improper fractions as mixed numbers and rewriting fractions as decimals (and percents) are reduced to trivial and arbitrary rules rather than applications of the definition of a fraction.

Table 3

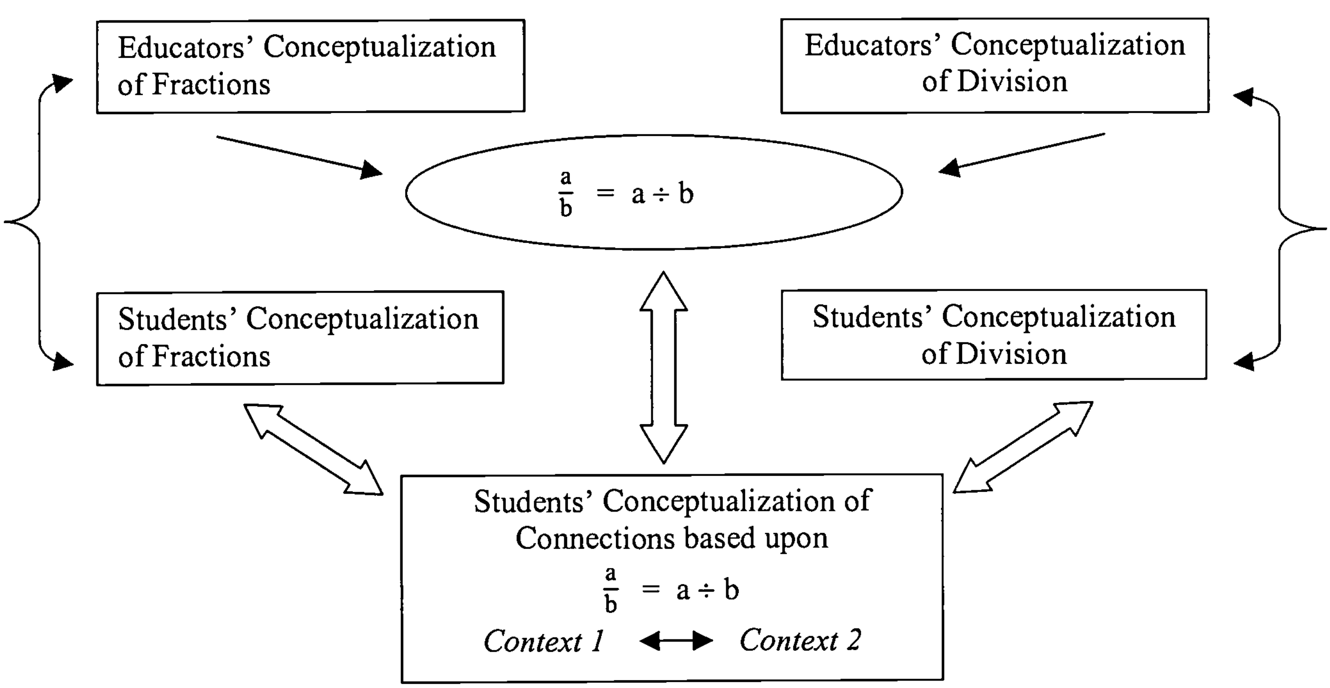
Relationships Between Fractions and Division

Relationship	Example of Relationship
A fraction can represent either indicated or performed division	$\frac{\text{2 laps}}{\text{5 minutes}}$ *indicated:* Meg swims 2 laps in 5 minutes. *performed:* Meg can swim at an average rate of $\frac{2}{5}$ laps per minute, or $\frac{2}{5}$ lap/min.
Fractions and division have different vocabulary for the same ideas	In $\frac{4}{9}$, 4 is the dividend (numerator) and 9 is the divisor (denominator)
A fraction bar is a division symbol	In $\frac{4}{9}$, the bar between the numbers means to divide.
Division can be written using fraction notation, and vice-versa	$\frac{4}{9} = 4 \div 9$; $\frac{4}{9} = 9\overline{)4}$; $4 \div 9 = \frac{4}{9}$; $9\overline{)4} = 9\overline{)4} = 4:9 = 4/9$
A quotient that is not a whole number can be written with a fraction for the remainder	$13 \div 4 = 4\overline{)13} =$ 3 Remainder 1, or $3\frac{1}{4}$
Division can be used to rewrite an improper fraction as a (whole or) mixed number	$\frac{13}{4} = 13 \div 4 = 4\overline{)13} = 3\frac{1}{4}$
Division can be used to rewrite a fraction as a decimal (or percent)	$\frac{13}{4} = 13 \div 4 = 4\overline{)13} = 3.25 = 325\%$ $\frac{1}{8} = 1 \div 8 = 8\overline{)1} = 0.125 = 12.5\%$
A fraction is a division; the definition of fraction includes the concept of division	A fraction $\frac{a}{b}$ is an indicated division $(a \div b)$ where the dividend a is the numerator and the divisor b is the denominator.

Framework for the Study

The study examined middle school students' understanding of fractions, of division, and of the relationships between fractions and division. Middle school students were selected as a population specifically because they have had exposure to fractions and division over several years—hence, the variable of opportunity to learn about fractions and division was not an issue. In other words, although the way in which fractions and division were presented to students may have differed, all of the students in the study had had a minimum of three years in which both fractions and division were significant parts of the mathematics curriculum.

To determine what students know about the connections between fractions and division, it was important to first identify what conceptual knowledge students knew about each topic. The conceptual knowledge was framed in the descriptions and applications of both fractions and division that are found in the literature and also in elementary and middle school textbooks (Levin, 1998). Then, students were asked questions about the relationships between fractions and division in several contexts. Their responses to each type of question were categorized and compared. Last, answers to the different types of questions were compared to the way that fractions and division were presented in the students' textbooks, and finally to the conceptualizations suggested by mathematics educators. The following diagram summarizes the theoretical framework.



Population

To determine what students had learned about fractions and division, all of the students in a "good" suburban middle school in the mid-west region of the country took part in a four-part assessment. The assessment was given at the end of the year. The majority of the 387 students in this study were assigned heterogeneously to mathematics classes—called Math 6, Math 7, and Math 8—all of which used a textbook from a traditional textbook.[1] Three classes were grouped homogeneously: a seventh grade pre-algebra class, and eighth grade algebra class, and an eighth grade intervention class.[2] The students enrolled in pre-algebra and algebra studied from a pre-algebra and an algebra textbook. Altogether, 6 sixth grade classes (128 students), 7 seventh grade classes (115 students in 6 grade 7 math classes and 29 students in pre-algebra), and 6 eighth grade classes (90 students in 5 grade 8 math classes and 25 students in Algebra).

The Assessment

The goal of the assessment was to determine student understanding with respect to fractions, to division, and to the relationships between fractions and division. To this end, questions were designed to elicit which models students most frequently associated with fractions and which models students most frequently associated with division. additionally, students were asked whether fractions and division were related.

The assessment consisted of four parts. (See Appendix 1 for a copy of the entire assessment.) Part I of the assessment asks students to consider four different ways to think about solving a division problem, and then to rank these four ways as the best way to teach someone

1 On the day that the assessment was given, 426 students were present. 39 of the 426 students were labeled "special education" students, and so these tests were not included in the analysis. Hence, a total of 387 students from 19 classes participated in the assessment; nine different teachers, each of whom taught between one and four classes, taught these students.

2 One of the eighth-grade classes with 12 students was labeled as an "intervention class," and was grouped homogeneously. This class was designed for "at risk" students. The students in this class used the same textbook as several of the other Math 8 classes.

how to solve $21 \div 3$. The four different ways corresponded to the partitive, quotative, repeated subtraction, and related facts division models.

Part II asks students to solve a real-world problem involving a scale change. Scale change problems involve proportional reasoning, an outgrowth of an application of division or an application of fractions. If a student has a comprehensive understanding of either fractions or division, solving a proportional reasoning problem should be meaningful, rather than a memorized routine.

Part III specifically asks students whether fractions and division are related, and if so, how? The question is designed to elicit the conceptualization that a student is able (or willing) to write down. Generating connections or relationships between fractions and division is a task that is not typically asked in textbooks (Levin, 1998); hence the description written by the student may not represent the *entire* understanding of either topic. Rather, the question is asked to determine whether a student can generate *any* connection, link, or relationship between fractions and division.

More detailed questioning on the specific relationships suggested in Table 3 appears in Part IV. The 28 questions found in this section ask students to choose *all* appropriate answers for a given fraction or division expression, pictorial model, or situation. Some of the skills include rewriting fractions and division, computation involving division, and identification of representations and uses which correspond to fractions and division expressions.

Results of the Assessment—What Students Know about Fractions and Division

The entire assessment, taken as a whole, shows that there are some concepts and skills that most students understand. In Part I, students at each grade and course level selected the related facts model for division as the way they would teach someone how to solve $21 \div 3$—a total of 144 out of 362 (39.8%) students—more often than any other model. The division model most often encountered in the textbooks that the students used was the related facts model; hence the high number of students selecting this model may reflect the presentation seen in the textbook used during the school year. That so many students are comfortable thinking about solving division in terms of related facts is promising, for it indicates that they may be ready to understand how to solve one-step equations, such as $3x = 21$, by thinking

"What number multiplied by 3 will give 21?

To solve that, I can divide 21 by 3 to get 7.

So, 7 times 3 gives me 21. $x = 7$."

Understanding the connection between multiplication and division is essential for success in later mathematics courses.

The next most selected model involved quotative division—101 of 362 (27.9%) students. This is surprising because in all of the textbooks which were used in this school, more division problems involved the partitive structure than the quotative structure. It is also interesting because it contradicts the finding by Fischbein, Deri, Nello, and Marino (1985), who found that the quotative division model was only meaningful for students by the *end* of middle school (that is, by ninth grade).

When asked for other ways to teach division, 32 students explained that repeated *addition* would be beneficial for teaching division. This model was not found in any of the textbooks used in the school. It is not clear where students encountered this meaning for division, or if they merely extended the meaning of repeated addition from multiplication on their own.

In Part III, half of the students in this school (194 of 387 students) stated that fractions and division were related and provided a reason for or example of a relationship. An additional 57 stated that fractions and division were related, but did not provide a meaningful reason why. Thus, nearly two-thirds of the students in this study acknowledge a relationship between fractions and division. This high number is encouraging, for it means that students do recognize that a connection between fractions and division exists, despite the absence of connections in their textbooks. However, the majority of the justifications for the connections focused on the statement of a connection, such as $\frac{a}{b}$ can be rewritten as $a \div b$ or $\frac{a}{b}$ can be written as a decimal by dividing $a \div b$, rather than a reason for the connection, such as $\frac{a}{b}$ is a division, or the fraction bar is a division symbol.

From Part IV, students exhibit a good understanding (75% of correct responses given) of the part-whole continuous-to-discrete (64.1% of the students) and part-whole discrete meaning for proper fractions (59.4% of the students). Additionally, a majority of the students selected correct fraction expressions for division problems (57.9% of the students for proper fractions and 54.5% of the students for improper fractions) and correct division expressions for fractions (60.0% of the students for proper fractions and 58.4% of the students for improper fractions).

Results of the Assessment—What Students Do Not Know about Fractions and Division

The results of the assessment indicate that while students are successful with some aspects of fractions and division, they have difficulty with others even after instruction. What follows below are discussions about misconceptions that the middle school students illustrated in Part II and between Parts III and IV.

In Part II, only 90 of the 387 (23.3%) students correctly answered the following question.

> On a certain map, the scale indicates that 5 centimeters represents the
> Actual distance of 9 miles.
> Suppose the distance between two cities on this map measures 2 centimeters.
> Explain how you would find the actual distance between these two cities.

This question was specifically asked because it can be solved by thinking about this as a division situation or by thinking of this as a fraction situation. The student answers below illustrate the different ways of solving this scale change problem—the solutions highlight the emphasis on either fractions or on division (Weinberg, in press).

One-fourth of the students (98 of 387) left the question blank, or merely recopied part of the problem. A majority of the 105 incorrect answers did not contain an explanation for why a particular set of steps should lead to a correct answer. It is likely that many students did not include an explanation for why their set of steps should lead to a correct answer because they did not understand the nature of scale change. Yet, all of the teachers in every grade reported that they had covered the section(s) in the textbook which dealt with scale change. Hence, it is disturbing that only 32.2% of the eighth grade students, who had been exposed to this topic in three consecutive grades, were able to solve this problem.

The students who were able to solve this problem described one of four solution strategies.

Find a Unit Rate. In this solution, the unit rate (written as a fraction) represents the number of miles equivalent to one centimeter; doubling allows one to find the number of miles equivalent to two centimeters. 43 students showed work illustrating the unit rate strategy, and 14 of these 43 included an explanation for why the work was appropriate. Two of these examples appear below.

> "I divided nine by five + came out w/ 1.8. So every centimeter is 1.8 miles. So 2 centimeters equals 3.6 miles." (grade 7)

> "divide
>
> $$\begin{array}{r} 1.8 \\ 5 \overline{)\,9} \\ 5 \\ 40 \end{array}$$
>
> that's only ½ the way so 1.8 + 1.8 equal your anser so 1.8 + 1.8 = 3.6"
> 3.6 miles " (grade 6)

Repeated Subtraction Strategy. Three students used a variation of the Find a Unit Rate Strategy, whereby once the unit rate was found, it was successively subtracted from the whole. For this particular problem, the number of miles corresponding to the unit rate for a single centimeter (1.8 miles per centimeter) is repeatedly subtracted from the initial number of miles.

"9 – 1.8 = 7.3; 7.3 – 1.8 = 5.5; 5.5 – 1.8 = 3.7" (grade 7)

The student's reasoning is mathematically valid, even though a computational error was made. Without the error, the student would have arrived at the correct answer.

1.8 miles	corresponds to	1 centimeter and
9 miles	corresponds to	5 centimeters.
So, 9 – 1.8 – 1.8 – 1.8 = 3.6 miles	corresponds to	5 – 1 – 1 – 1 = 2 centimeters.

Equivalent Fractions Strategy: Eight students solved the problem by using a string of operations which correspond to the steps used to rewrite equivalent fractions. One wrote:

> "I would divide 5 by 2, giving me 2.5, then divide 9 by 2.5. That would give me the new scale 2 cm = 3.6 m." (grade 7)

This student does not explain why the sequence of steps works. The reason this sequence words can be seen by thinking about how we teach students to make equivalent fractions. Begin with the proportion involving equivalent rates: 9 miles/5 centimeters = x miles/2 centimeters. To solve this, we need to divide both 9 miles and 5 centimeters by a form of one that yields a fraction whose denominator is 2. We teach students to think: "5 divided by what gives 2?" The answer is $5 \div 2 = 2.5$. Then, 9 divided by 2.5 gives x. $9 \div 2.5 = 3.6$ miles. This is precisely the reasoning we use to teach students to rewrite equivalent fractions when the first fraction has a larger denominator than the second—the only difference is that with equivalent fractions, we rarely multiply or divide the numerator and denominator by fractions.

Size Change Strategy: Just eight students wrote a sequence of steps which reflects the thinking that multiplying by a ratio (written as a fraction) that "stretches" or "shrinks" a given amount proportionally. Since the original number of centimeters is 5, and the desired number of centimeters is 2, the size change involves a "shrink;" hence, the ratio 2 centimetes/5 centimeters or 2/5 is used. Three students provided an explanation for a size change strategy like this one.

> "Well... $\frac{5}{5}$ of 5 cm = $\frac{9}{9}$ of 9 mi so $\frac{2}{5}$ of 5 cm = $\frac{2}{5}$ of 9 mi to 2 cm = = 3 3/5 mi" (grade 6)

It should be noted that this reasoning allows students to solve for a variable in a one-step equation of the form $x/9 = 2/5$. When a student multiplies both sides of the equation by 9, the result is $x = (2/5) \bullet 9$.

Cross Multiplication Using Equal Rates or Ratios Strategy: 27 students answered this question by setting up and solving a proportion using cross multiplication. Of these, six students provided a rationale for why using a proportion is appropriate. Two students' responses are shown below. The first answer illustrates an appropriate rationale; the second illustrates how the student set up the proportion. Both students used the proportion 5/9 = 2/n to solve this problem.

> "5 is to 9, as 2 is to ? $\frac{5}{9}$ $\frac{2}{n}$ 3.6 miles." (grade 7)
>
> "I would use cross-products. Put 5 cm on top of the first fraction and put 9 miles on the bottom. The, put 2 cm on top of the second fraction and multiply the nine miles times the two cm and divide 5 cm into the product." (grade 8)

Altogether, there are four different proportions that can be used to represent this situation. As shown below, setting up equal rates corresponds to the Find a Unit Rate Strategy, and setting up equal ratios leads quite naturally to the Size Change Strategy and Equivalent Fractions Strategy. Hence, setting up proportions to model a proportional reasoning situation can be used to reinforce the understanding of the problem as well as a means for solving the problem.

Equal Rates: $\frac{\text{cm}}{\text{miles}} = \frac{\text{cm}}{\text{miles}} \rightarrow \frac{5\text{ cm}}{9\text{ miles}} = \frac{2\text{ cm}}{x\text{ miles}}$; $\frac{\text{miles}}{\text{cm}} = \frac{\text{miles}}{\text{cm}} \rightarrow \frac{9\text{ miles}}{5\text{ cm}} = \frac{x\text{ miles}}{2\text{ cm}}$

Equal Ratios: $\frac{\text{cm}}{\text{cm}} = \frac{\text{miles}}{\text{miles}} \rightarrow \frac{5\text{ cm}}{2\text{ cm}} = \frac{9\text{ miles}}{x\text{ miles}}$; $\frac{\text{cm}}{\text{cm}} = \frac{\text{miles}}{\text{miles}} \rightarrow \frac{2\text{ cm}}{5\text{ cm}} = \frac{x\text{ miles}}{9\text{ miles}}$

For the most part, students did not include any reasonable rationale for the work (either correct or incorrect) they showed for this particular proportional situation problem. One reason might be that students are encouraged to show work, but are not encouraged to explain why the steps lead to a correct answer. In this study, of the 90 students who arrived at the correct answer, only 39 provided an explanation for why their strategy was correct. The remainder either showed their work and provided no additional explanation, or the explanation described the mathematical steps they performed. The majority of the students who supplied an incorrect answer wrote no explanation of why their strategy should be appropriate, nor did their answers reflect a recognition that this problem called for proportional reasoning.

Error Pattern #1: *Using a single step solution.* 77 students wrote a solution involving a single operation and two of the three given numbers in the problem. A sample of these types of answers appears below.

> "You take the two centimeters divide it by five." (grade 7)
>
> "By adding on 9 miles to each 5 centimeters." (grade 7)
>
> "I would subtract 5 form 9." (grade 8)
>
> "I would divid 2 by 9 and get 4.5 or 4 and a half miles." (grade 7)
>
> "times 2 by 9 = 18 m." (grade 7)

The answers indicate a limited understanding of the problem. As stated, each of the three numbers in the problem serves a role. Two of the numbers (5 centimeters and 9 miles) describe an initial situation. The third (2 centimeters) describes the outcome situation of one of the quantities. A one-step solution such as the ones above indicate that these students do not recognize the role of at least one of the numbers in the stated problem. A failure to understand the need for all three numbers in this situation will necessarily prevent students from using a proportional reasoning approach. Furthermore, some of the answers indicate the understanding that division and multiplication are involved in some way—this could indicate partial understand of the problem, for example, the notion of a rate. However, other answers involve operations which are in no way relevant to this problem. It is likely that at least some of these answers reflect students with no understanding of the problem, who wrote their best guess.

Error Pattern #2: Three numbers used, but in the wrong order. Another reason might be that students do not know why a particular method is appropriate for a given situation. This is a different type of misconception that found in the one-step solutions. Examples of student responses with this error pattern appear below.

> "Divide the five centimeters by 2, and multiply your result by 9. 22 ½ miles." (grade 8)
>
> "First I would devide 5 by 9 and get an answer then whatever the answer is I would times it by two." (grade 8)

"9 – 5 – 4   4 + 2 = 6 miles." (grade 6)

"Find 5 ÷ 9 = $0.\overline{5}$ = 0.6. From this I know that in 1 cm there is about 0.6 miles. Multiply it by 2 (from the two centimeters) and you get 1.2 the distance between the two citys is 1.2 miles." (grade 8)

These answers illustrate that some students recognize that all three numbers must be used, but may not understand the role each number plays in the solution. The error shows up when the numbers are not applied "in the right order" meaning perhaps that a student knows a solution but not why it works. Again, few of the answers which are categorized as Error Pattern #2 contained explanations as to why the sequence of steps was appropriate.

Still a third reason might be that students are "on the right track"—that is, they somewhat understand the problem, but do not know how to use mathematics to solve it. Several students wrote answers whose solutions, if they had substituted correctly, would have produced the correct answer. Other students wrote answers that are technically incorrect, but show that the student does understand the proportional nature of the problem. The answers below reflect this partial understanding.

"

cm.	1	2	3	4	5	6	7	8	9
in.	1	1 ½	2	2 ½	3	3 ½	4	4 ½	5

" (grade 6)

"Find out how many miles are in each centimeters. When you find out multiply it by two." (grade 7)

"If you look at 2 it is less then half of 5 centimeters. It is off by .5. So if you take half of 9 which is 4 ½ 2 is of by a half of half of 5. So take the half off 4 ½ and your answer is 4 miles." (grade 6)

Other Misconceptions

In Part IV, students demonstrated weaknesses in specific fraction and division skills, such as in identifying the correct number-line representation corresponding to a specific fraction, whether proper or improper. The part-whole continuous-to-discrete model was another pictorial fraction model which students only partially understood. Although 248 students (64.1%) chose the correct part-whole continuous-to-discrete model for a specific proper fraction, only 67 students (17.3%) did so with an improper fraction.

The size of the dividend in relation to the divisor appeared to affect the student success with particular skills, echoing other research (Greer, 1987). For example, students were less successful expressing the result of whole number division as a decimal when the quotient was less than one (23.3% of the students) than when the quotient was greater than one (27.7% of the students). However, when rewriting a fraction as a decimal, the students were less successful when the fraction was improper (26.4% of the students) than when the fraction was proper (33.6% of the students).

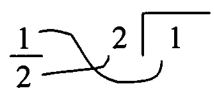
In general, students were most successful with skills and models that appeared in their textbooks with greater frequency. For example, more students selected division than fractions to mathematically express a rate. In the textbooks used by the students in this school, division situations more frequently involved rates than ratios, and fractions more frequently involved ratios than rates. Thus, for particular skills relating to fractions and division, it seems that Begle's (1973) conclusion that "if a mathematical topic is in the text, the students do learn it" is somewhat supported, but only when the topic is more than simply mentioned in the textbook. The results of the student assessment indicate that when a topic receives attention—in the explanations, worked examples, and questions—on many pages with a single textbook, then students do learn it.

Students' Inconsistent Responses

One unanticipated result was the fact that many students who described a relationship between fractions and division in part III did not consistently (75% of the possible items selected correctly) identify that relationship when it appeared in Part IV. Consider, for example, the differences in the values for rewriting fraction expressions as division, and vice versa. In sixth grade, 13 students specifically mentioned that a fraction can be rewritten as a division. Yet only 3 of these 13 students marked the correct responses that correspond with this skill in Part IV. For the school as a whole, not quite two-thirds of the students who identified that a fraction can be rewritten as a division in Part III selected correct responses in part IV, and most of these were in the 7th grade pre-algebra and 8th grade algebra classes.

Part of the reason for this inconsistency is explained directly from the students' responses. Consider the superficial understand of this sixth grade student's response to the prompt "are fractions and division related?"

> "You would say yes and explain. Well if you want to make a fraction into a decimal you must <u>divide</u>. Give an exaple like ½.
> The explain the steps as you go along. First it's like a soccer or football play.



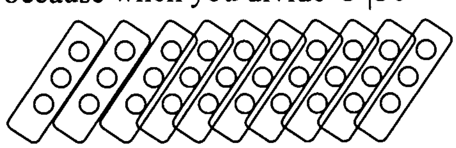
> The rewrite it. 2 $\overline{)1}$ divide
> Your answer is .5." (grade 6)

$$\begin{array}{r} 0.5 \\ 2\,\overline{)\,1.0} \\ -\ 1.0 \\ \hline 0 \end{array}$$

Although the student knows *how* to rewrite the fraction as a division, there is no description here of why this can be done. This student describes a football play; the implication is that the rewriting is just a trick.

Other answers indicating misconceptions appear below.

> "yes in a way they are because when you divide 3 $\overline{)30}$ = 10 divition
> fraction $\frac{3}{10}$ = $\begin{array}{r} 10 \\ \underline{x\ 3} \\ 30 \end{array}$



> " (grade 7)

Table 4

Relationships Identified by Students in Part III and in Part IV of the Assessment

Relationship	Grade 6 (n = 128)	Grade 7 Math (n = 115)	Grade 7 Pre-Alg. (n = 29)	Grade 8 Math (n = 90)	Grade 8 Algebra (n = 25)	Total (n = 387)
Indicated or Performed Division						
Described in Part III	2	2	2	1	2	9
Identified in Part IV	85	84	25	66	20	280
Correct in Both Parts	0	0	0	1	1	2
Vocabulary*						
Described in Part III	0	0	0	1	0	1
Fraction Bar Is a Division Symbol*						
Described in Part III	2	2	3	31	9	47
$a \div b = \frac{a}{b}$; $\frac{a}{b} = a \div b$						
Described in Part III	13	24	14	32	15	98
Identified in Part IV	43	65	24	76	24	232
Correct in Both Parts	3	12	12	18	15	60
Fraction in Remainder						
Described in Part III	3	4	0	3	2	12
Identified in Part IV	46	36	15	34	15	146
Correct in Both Parts	0	4	0	2	2	8
Improper Fraction as Mixed Number						
Described in Part III	1	8	3	1	7	20
Identified in Part IV	40	60	23	47	21	191
Correct in Both Parts	0	3	3	0	7	13
Fraction as Decimal						
Described in Part III	11	14	9	10	3	47
Identified in Part IV	21	34	23	23	18	130
Correct in Both Parts	9	8	8	8	2	35
Fraction Is Division						
Described in Part III	3	3	5	4	5	20
Identified in Part IV	70	83	19	60	17	249
Correct in Both Parts	0	0	1	0	0	1

* No corresponding items were included on Part IV of the assessment.

> "I would say all math is related. But it is like division is a way to do something, and fractions are parts of numbers. It is like compareing apples and ornges there both fruits but they are so different." (grade 6)

> "If it has a different symbol it means a different thing or a different way of finding a answer." (grade 8)

Other inconsistencies appear when comparing student responses with respect to rewriting improper fractions as mixed numbers and rewriting fractions as decimals. Although the algorithm, and the rationale for the algorithm, is nearly identical, far more students in the average track cited the decimal conversion in Part III than improper fractions. For these students, nearly all were able to select the correct responses in Part IV. However, many fewer students selected the improper fraction conversion. And, with the exception of the students in the Pre-Algebra and Algebra tracks, most were not able to select corresponding mixed number equivalents for improper fractions when presented in a multiple choice format.

Implications

Implications for Researchers

The mathematics education researcher conceptualizations of fractions and division are incomplete. With respect to fractions, these conceptualizations focus heavily on the pictorial representations of fraction. With the exception of the number-line model, pictorial models are difficult for students to apply and interpret when fractions are improper. The student assessment confirmed that none of the part-whole models proved very powerful for fractions greater than one. Moreover, three of the pictorial models—part-whole continuous-to-discrete, part-whole discrete, and number line—may encourage students to think of fractions as "double counting" rather than as a single number.

Many mathematics education researchers appear to concur that of the pictorial fraction models, the part-whole continuous-to-discrete model should be introduced to students first (see for example: Ball, 1992; Bezuk & Cramer, 1989; Dirkes, 1991; Pothier & Sawada, 1990). While it may be true that the part-whole continuous-to-discrete is an excellent first model, it should not be used as the only model. Emphasizing this model for the meaning of fraction encourages three misconceptions.

1. A fraction is defined as a part of a whole, rather than a number or an indicated division.

2. A fraction is a number less than one. This lends legitimacy to students who believe there is something illegal of inherently wrong with fractions greater than one.

3. A fraction results from "double counting," once for the numerator and once for the denominator. This leads students to conclude that a fraction is two numbers, instead of thinking of the fraction as a single number representing an amount or a measure.

Additionally, fraction conceptualizations do not focus on the connections between fractions and division. Little or no emphasis is made with respect to the indicated division meaning of fraction, which is the mathematical definition of fraction. Hence, students may not recognize that the numerator of a fraction has the same function as the dividend in binary division, and that the denominator of a fraction has the same function as the divisor. Furthermore, these conceptualizations do not call attention to the fact that the bar in a fraction is, in fact, a symbol of division. Yet it is these connections that enable us to rewrite fractions as division expressions, fractions as decimals, improper fractions as mixed numbers. The "rules" for rewriting are not arbitrary pieces of trivia to be memorized; they are grounded in the meaning of the fraction as an indicated division. Hence, the reason $\frac{22}{5}$ can be rewritten as $22 \div 5$ is that $\frac{22}{5}$ *means* 22 divided by 5.

Finally, although many of the conceptualizations do include the rate and ratio situational models, they do not include models in which fractions are measures (see Usiskin & Bell, 1983 for a more complete discussion).

The conceptualizations for division are also incomplete. The most common discussions of division focus on the structure of division situations: partitive and quotative. Other research discusses division in terms of its relation to multiplication. The articles I found in the literature focused on either partitive and quotative situations, or on multiplication as the inverse operation for division. However, division can be conceptualized as both.

Furthermore, division conceptualizations fail to discuss the various strategies that students use to solve division situations. For example, which is more effective—using manipulatives or using the long-division algorithm to solve $21 \div 3$? Which is more effective for $21 \div 4$? How are these two strategies related: applying the quotative model of division using manipulatives and repeatedly subtracting the divisor from the dividend using pencil and paper?

Certainly, conceptual development of division must include identifying similar division situations. For example, without identifying common features of partitive division situations, students may focus entirely on a situational "cover story" and miss opportunities to recognize mathematical structure. Identifying these common features helps students to understand *why* a division situation involves division. However, if the discussion in the research literature concerning division concepts ends with identifying the type of division situation, and does not include additional focus on how students determine a quotient, should we be surprised that this connection is missing in the textbooks that students use?

Many of the conceptualizations do not distinguish among (1) division of whole numbers in which there is no remainder, (2) division of whole numbers in which there is a whole number remainder, and (3) division of whole numbers in which the quotient is a decimal or fraction, and there is no whole number remainder. Yet mathematicians do make distinctions in division. When the division of numbers is described using only whole numbers, such as a whole number quotient and whole number remainder (which includes a remainder of zero), this division is called integer division.[3] When the division of whole numbers produces a quotient which is described by non-whole numbers, such as a quotient written as a decimal or mixed number, then the division is termed real-number division. Mathematically, integer division can be expressed as

3 Integer division is so named because this division is not confined to whole numbers, but also to integers: the dividend may be any integer, and the divisor may be any integer not equal to zero.

$$\text{dividend} = \text{quotient} \bullet \text{divisor} + \text{remainder}.$$

For example, in the integer division $22 \div 5$,

$$22 = 4 \bullet 5 + 2.$$

In this case, the quotient is 4, and the remainder is 2. For the real-number division $22 \div 5$, the quotient is 4.4, or $4\frac{2}{5}$.

The research is also incomplete because there is no consensus on whether quotative models of division are appropriate at all grade levels. Zweng (1963) found that it was appropriate for second grade students even before formal instruction on division was given, but Fischbein, Deri, Nello, and Marino (1985) did not, even after students had formal instruction on division. Furthermore, is it the structure of the division problem (e.g., partitive or quotative) or the size of the numbers in the dividend compared to the divisor which causes more difficulty for students? Is it more important to teach the structure of a division situational problem, or to simply expose students to many situations in which quotients are less than one and many in which the quotients are greater than one?

Last, discussions of division often ignore entirely the issue of symbolism. There are at least five possible ways of writing twenty-two divided by five, as shown below, although in the United States, the ratio symbolism is rarely used.

<u>Twenty-two divided by five:</u>

$22 \div 5$	binary division symbol
$5\overline{)22}$	housing or long division symbol
22 / 5	slash symbol
$\frac{22}{5}$	fraction symbol
22 : 5	ratio symbol

Implications for Teachers

Students have difficulty with fraction and division concepts. The textbooks do not give an adequate presentation of either of these two topics. Teachers have been counted on to fill in the gaps in the textbook curriculum. Teachers must be vigilant about explaining not just that fractions and division are related, but why they are related. They must not assume that students have been exposed to the concepts underlying fractions and division in grades three and above simply because those concepts appear in textbooks for grades kindergarten through three. For many students, third or fourth grade may be the first time that fractions are presented to students. Teachers must show that division is an operation that describes different situations regardless of the type of number involved in the problem. They must also systematically and thoughtfully eliminate *misconceptions* which are unintentionally raised within many of the textbooks, such as that division must involve a dividend which is larger than a divisor, or that improper fractions are not correct answers. The burden placed on teachers is not an easy one: either they make up for the lack found in textbooks, or the students will not, in general, pick this information up elsewhere in a meaningful way. And there is ample evidence that teacher knowledge of mathematics, especially elementary teacher knowledge of mathematics, is not strong enough to meet this challenge.[4]

Implications for Students

Because neither researchers nor textbooks have a comprehensive conceptualization which includes both fractions and division, it is not surprising that students have difficulty with both of these topics. As shown in the student assessment, students can state that fractions and division are related. This was shown to be true for students who learned from traditional textbooks and for students who learned from non-traditional textbooks (in two previously conducted pilot

4 It is *not* the case that learning about relationships between fractions and division is too complex for teachers to grasp. (Certainly, if we expect students to understand the relationships between fractions and division, then we must expect at least this much from their teachers.) In my own experience with teaching pre-service elementary school teachers, even those who were comfortable with mathematics expressed not knowing the definition of fraction or the meaning of the fraction bar. Yet, this seemed to be an exposure issue, rather than a complexity issue. However two pre-service teachers were able to explain that a fraction is a division. These students were not strong math students. When I asked them why a fraction is a division, both expressed that they had learned the "fact" in grade school—both said that they remember a teacher telling them this fact. I did not find any studies showing how much exposure to the connections is necessary for teachers to understand and integrate these connections into their own lessons. Further research on this would be instructive.

studies—see Levin, 1998 for a detailed discussion). However, the justifications they provide are the ones that show only that a relationships exists, not *why* a particular relationship exists. Again, this was true regardless of the type of textbook used. One can be optimistic that students are able to explain that a relationship exists between fractions and division even when the textbooks do not highlight this content.

Perhaps, given even more time, all students (regardless of textbook used) would be able to state that fractions and division are related because the fraction bar is a division symbol, and fractions are a form of indicated division. However, since half of the students were unable to explain why fractions and division were related after learning about these topics for at least two years, we cannot assume that the connection will be made without some intervention. The same is true for skills which are missing from the textbook curriculum, such as writing quotients to division of whole number problems as mixed numbers, or identifying the fraction bar as a division symbol.

A New Conceptualization of Fractions and Division

Therefore, to help students and teachers, and to refocus mathematics education research and textbook presentations of fractions and division, there seems to be a need for a new conceptualization which includes fractions, division, and the connections between fractions and division. I offer a new conceptualization of fractions and division together, which may be just one of the ways to address this problem.

This diagram incorporates the different representational models used for teaching fractions (part-whole continuous-to-discrete, part-whole discrete, part-whole continuous, number line, fraction chart) as well as the situations found in division problems (rate division, ratio division, rate divisor, ratio divisor). The conceptualization also includes repeated subtraction and related facts, two methods for solving division problems that can be used to develop concepts. A third piece of the diagram includes the connections between fractions and division which stem from the relationship $a \div b = \frac{a}{b}$. Within this piece are some examples where links between fractions and division can be used with specific topics and skills in the elementary school curriculum. This conceptualization is not meant to be complete, but rather the beginning of an organizational strategy for discussing the concepts of fractions and division with students, and for helping them recognize connections between the two.

Division Concepts	Concepts Relating to Fractions and Division	Fraction Concepts
o Definition of Division The result of dividing one number (the dividend) by another (the divisor) is the quotient. The quotient a/b of the two numbers a and b is that number c such that b o c = a, provided c exists and has only one possible value.	o Vocabulary in Definition of Fraction o Indicated and Performed Division	o Definition of Fraction An indicated quotient of two quantities. The dividend is the numerator and the divisor is the denominator (the numerator of $\frac{3}{4}$ is 3 and the denominator is 4.)
o Division Symbolism (÷ , / , $\overline{)\ \ }$, —)	o Symbolism in Definition of Division	o Fraction Symbolism (— , /)
o Pictorial Models for Division partitive (rate, ratio division) quotative (rate divisor, ratio divisor)	o Pictorial Models for Fractions and Division N/A	o Pictorial Models for Fractions part-whole continuous-to-discrete part-whole discrete part-whole continuous number line fraction chart
o Other Models for Division repeated subtraction repeated addition related facts rate divisor situations ratio divisor situations	o Other Models for Fractions and Division rate situations ratio situations	o Other Models for Fractions fraction as a number (measure or count)
	o Algorithms Connecting Fractions and Division write a quotient as a fraction or mixed number write a fraction as a decimal write an improper fraction as a mixed number	
	o Applications probability, scale change, slope, average (arithmetic mean), divisibility, etc.	

A New Conceptualization for Fractions and Division

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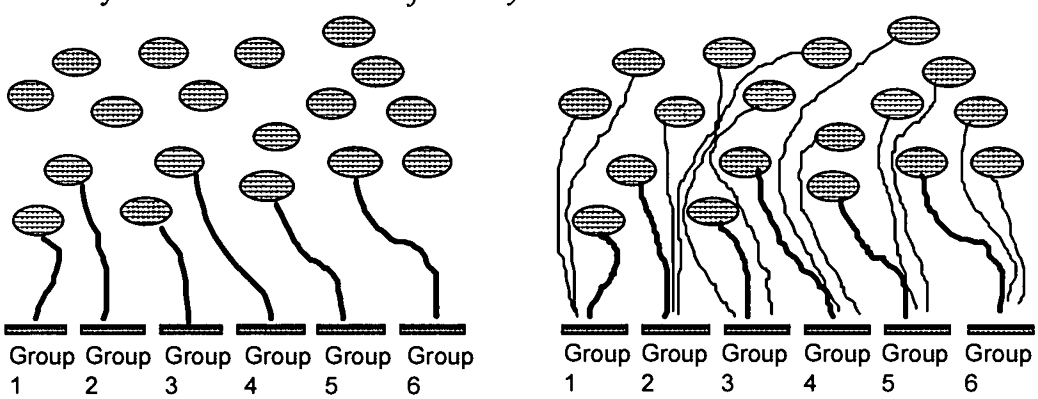
Appendix: Assessment Given to Middle School Students

Part I. (read out loud by the teacher as all students, followed along.)

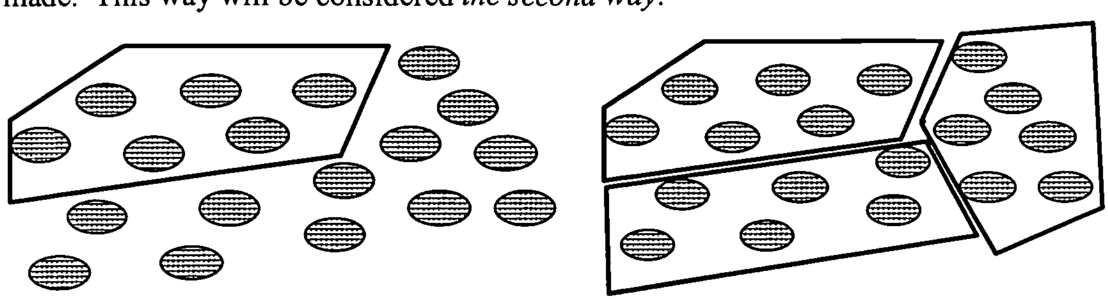
There are many ways to think about division. Some of these different ways that can be used to find the answer to 18 divided by 6 are described below. Read each of the following descriptions carefully. Then, answer the questions.

Way 1
One way is to use counters. For dividing 18 by 6, begin with 18 counters, and put these into 6 piles. Begin by putting one counter into the first pile, another counter into the second pile, and another counter into the third pile, and so on until you have six piles with one counter in each. Then, put a second counter into pile one, a second counter into pile two, and a second counter into pile three, and so on. Continue putting counters, one at a time, into the six piles until there are no more counters. The answer to 18 divided by 6 is the *number of counters in each pile.* This way will be considered *the first way.*



18 ÷ 6 = 3; 3 in each group

Way 2
Another way also involves using counters. For dividing 18 by 6, begin with 18 counters. Make one pile of six counters. Then, make a second pile of six counters. Continue making piles of six counters until there are no more counters. The answer to 18 divided by 6 is the *number of piles* made. This way will be considered *the second way.*



18 ÷ 6 = 3; 3 groups

Way 3
A third way uses pencil and paper. For dividing 18 by 6, begin by writing 18. Then subtract six. From the result, subtract six again. Continue subtracting six until you reach zero. The answer to 18 divided by 6 is *the number of times 6 was subtracted.* This way will be considered *the third way.*

$$\begin{array}{r} 18 \\ -\ 6 \\ \hline 12 \end{array} \qquad \begin{array}{r} 12 \\ -\ 6 \\ \hline 6 \end{array} \qquad \begin{array}{r} 6 \\ -\ 6 \\ \hline 0 \end{array}$$

$18 \div 6 = 3$; subtracted **6** from 12 *three* times

Way 4
A fourth way calls for thinking about multiplication and division as related operations. For dividing 18 by 6, think "what number can be multiplied by 6 to give 18?" That number is the answer to 18 divided by 6. This way will be considered *the fourth way.*

$18 \div 6 = 3$; 3 solves the equation $6 \bullet ? = 18$.

Now, suppose a student does not understand division, and asks you to explain it using the example $21 \div 3 = ?$

1. Rank the ways in the order YOU would use to teach someone what division means.

   Let Way 1 be your first choice; let Way 2 be your second choice; let Way 3 be your third choice; and let Way 4 be your fourth choice.

   _____ Way 1

   _____ Way 2

   _____ Way 3

   _____ Way 4

2. In addition to the ways that were just described, are there any other ways YOU might use to teach someone what division means? Please describe them below.

Part II.

On a certain map, the scale indicates that 5 centimeters represents the actual distance of 9 miles.

Suppose the distance between two cities on this map measures 2 centimeters. Explain how you would find the actual distance between these two cities.

Part III.

Suppose a student asks you if fractions and division are related.

How would you answer this student?

How would you justify (explain) your answer?

Write your answer in the space provided below. Be sure to include reasons for your answer.

Part IV.

For questions 1 – 28, draw a box ▭ around the letter or letters of *all* correct choices.

Sample: Which of the following are true?

a. $2+3=5$ b. $9+3=11$ c. $4+2=6$

d. $8+0=9$ e. none of these

You should put a box around the two correct sentences, a and c.

$\boxed{\text{a.}}$ $2+3=5$ b. $9+3=11$ $\boxed{\text{c.}}$ $4+2=6$

d. $8+0=9$ e. none of these

If only one choice is correct, then draw a box around the letter for that choice only.

If there is no choice that is correct, draw a box around the letter for the choice "none of these".

Follow these directions for questions 1 – 28. DO NOT USE A CALCULATOR.

1. Which of these choices equals $4 \div 9$?

a. $9\overline{)4}$ b. $9:4$ c. $4\overline{)9}$

d. $\frac{4}{9}$ e. none of these

2. Which of these choices equals as $10 \div 3$?

a. $10\overline{)3}$ b. $10:3$ c. $\frac{10}{3}$

d. $\frac{3}{10}$ e. none of these

3. Which of these choices equals $2 \div 5$?

a. $\frac{5}{2}$ b. $2\overline{)5}$ c. $\frac{2}{5}$

d. $2:5$ e. none of these

4. Which of these choices equals $7 \div 4$?

a. $7\overline{)4}$ b. $4:7$ c. $\frac{7}{4}$

d. $4\overline{)7}$ e. none of these

5. Which of these choices equals $\frac{5}{8}$?

a. $5\overline{)8}$ b. $8:5$ c. $8\overline{)5}$

d. $5 \div 8$ e. none of these

6. Which of these choices equals $\frac{16}{7}$?

a. $16:7$ b. $7\overline{)16}$ c. $16\overline{)7}$

d. $16 \div 7$ e. none of these

7. Which of these choices equals $\frac{1}{6}$?

a. $1 \div 6$ b. $1:6$ c. $6 \div 1$

d. $6\overline{)1}$ e. none of these

8. Which of these choices equals $\frac{13}{6}$?

a. $13\overline{)6}$ b. $6 \div 13$ c. $6:13$

d. $13 \div 6$ e. none of these

9. Which of these choices equals $4 \div 9$?

a. 2.25 b. 0 Remainder 4 c. $2\frac{1}{4}$

d. $44.\overline{4}\ \%$ e. none of these

10. Which of these choices equals $\frac{5}{8}$?

a. $\frac{8}{5}$ b. $1\frac{3}{8}$ c. $10/16$

d. $\frac{20}{32}$ e. none of these

11. Which of these choices equals $10 \div 3$?

a. $3.\overline{3}$ b. 30 % c. $3\frac{1}{3}$

d. 3 Remainder 1 e. none of these

12. Which of these choices equals $\frac{16}{7}$?

a. 2 Remainder 2 b. $2.\overline{285714}$ c. 0 Remainder 7

d. 0.4375 e. none of these

13. Which of these choices equals $10 \div 3$?

a. 0 Remainder 10 b. $3\frac{1}{10}$ c. $333.\overline{3}$ %

d. 0.3 e. none of these

14. Which of these choices equals $\frac{5}{8}$?

a. 0.625 b. 1 Remainder 3 c. 0 Remainder 5

d. 3.6 e. none of these

15. Which of these choices equals $4 \div 9$?

a. 2 Remainder 1 b. $0.\overline{4}$ c. 225%

d. $2\frac{1}{9}$ e. none of these

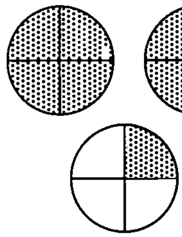
16. Which of these choices equals $\frac{16}{7}$?

a. $\frac{7}{16}$ b. 32 / 14 c. $\frac{48}{21}$

d. $2\frac{2}{7}$ e. none of these

17. Which of these choices pictures $4 \div 9$?

a.



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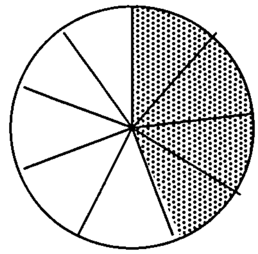

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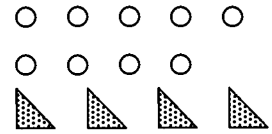
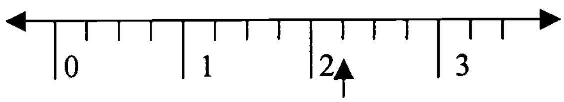


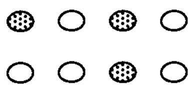
d.



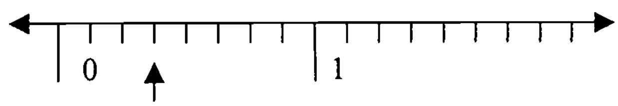
e. none of these

18. Which of these choices pictures $3 \div 8$?

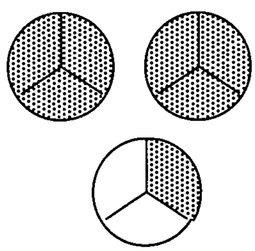
a.



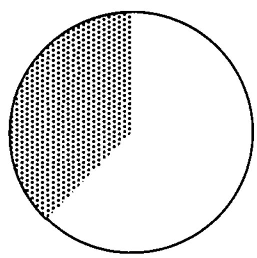
b.



c.



d.



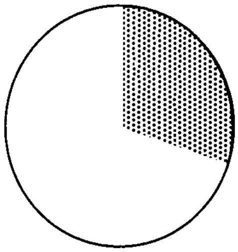
e. none of these

19. Which of these choices describes a situation which leads to $6 \div 9$?

a. six pies divided among nine people
b. 6 chances out of 9
c. What is the average speed of a plane which travels 9 miles in 6 minutes?
d. nine boys for every six girls
e. none of these

20. Which of these choices pictures $10 \div 3$?

a.



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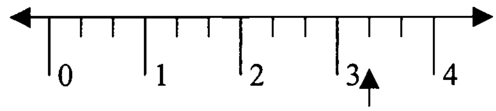
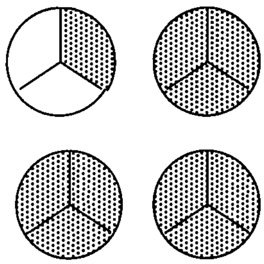
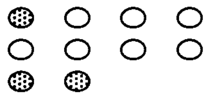


c.



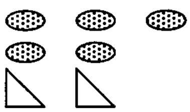
d.



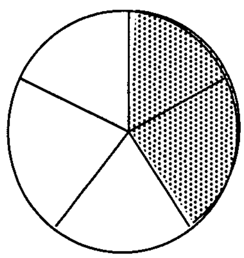
e. none of these

21. Which of these choices pictures $5 \div 2$?

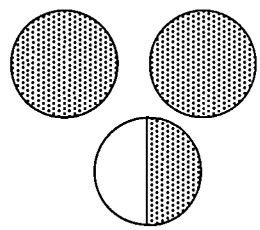
a.



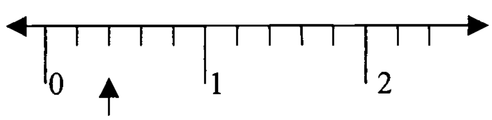
b.



c.



d.

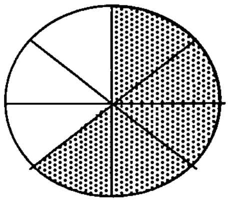


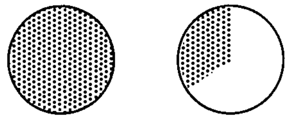
e. none of these

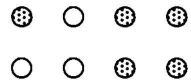
22. Which of these choices describes a situation which leads to $11 \div 4$?

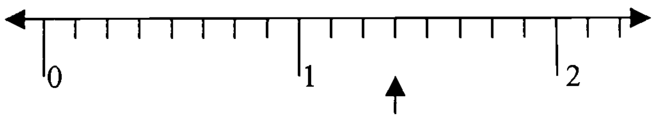
a. four pizzas divided among eleven people
b. What is the average speed of a train which travels 11 meters in 4 seconds?
c. four chances out of eleven
d. eleven boys for every four girls
e. none of these

23. Which of these choices pictures $\frac{5}{8}$?

a.


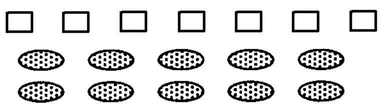
b.


c.


d.


e. none of these

24. Which of these choices $\frac{7}{10}$?

a.


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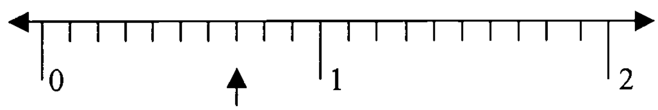
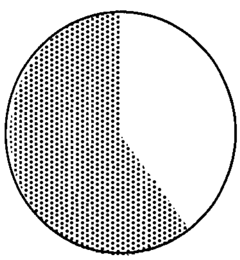
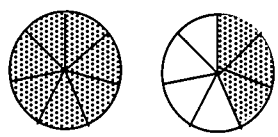


c.


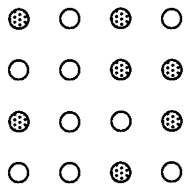
d.


e. none of these

25. Which of these choices describes a situation which leads to $\frac{3}{4}$?

a. 3 chances out of 4
b. What is the average speed of a car which travels 3 miles in 4 minutes?
c. four girls for every three boys
d. four candy bars divided among three people
e. none of these

26. Which of these choices pictures $\frac{16}{7}$?

a.


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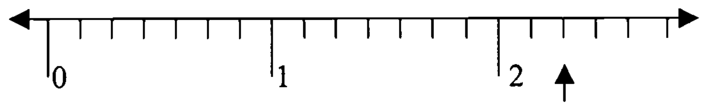
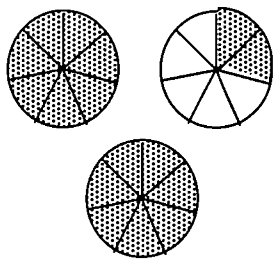
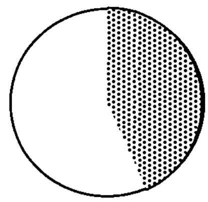
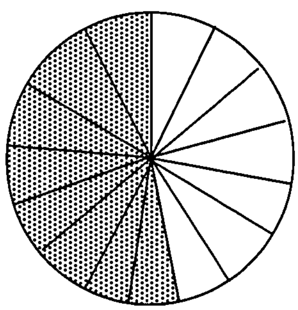


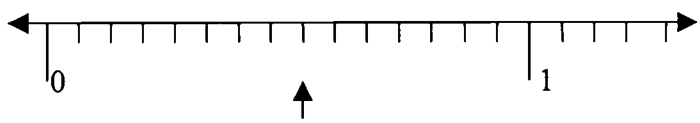
c.


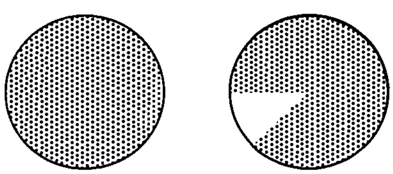
d.


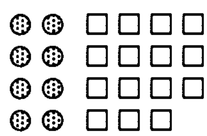
e. none of these

27. Which of these choices pictures $\frac{15}{8}$?

a.


b.


c.


d.


e. none of these

28. Which of these choices describes a situation which leads to $\frac{12}{7}$?

a. twelve cans of pop divided among seven people
b. 12 boys for every 7 girls
c. seven chances out of twelve
d. What is the average speed of a bicycle which travels 7 kilometers in 12 hours?
e. none of these

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Organization/Address: 436 Ritter Hall, Temple University 1301 Cecil B. Moore Ave. Philadelphia, PA 19122

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