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ABSTRACT

This paper provides a summary and a critique of the empirical literature on elementary teachers' mathematical knowledge. The empirical evidence addresses four major issues surrounding teachers' mathematical knowledge: 1) What is the nature of teachers' content knowledge, particularly with regard to the domain of number? 2) How does teachers' knowledge impact their instructional practice? 3) How does teachers' knowledge impact student learning? and 4) How do teachers develop appropriate mathematical knowledge? For each question, major findings are summarized, exemplars of studies that have addressed the question are provided, and the contribution of these studies to our understanding of teachers' mathematical knowledge is critiqued. The paper concludes that there is no clearly definable body of knowledge that informs teaching; rather, teachers need multiple types of knowledge, each of which is somewhat ill-defined and amorphous. (Contains 65 references.) (ASK)

An Analysis of the Research on K-8 Teachers' Mathematical Knowledge

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An Analysis of the Research on K-8 Teachers' Mathematical Knowledge

Every study or subject thus has two aspects: one for the scientist as a scientist; the other for the teacher as teacher. These two aspects are in no sense opposed or conflicting. But neither are they immediately identical.

(Dewey, 1990/1900, p. 200)

The issue of what types of knowledge are essential for teaching mathematics in the elementary school has been the subject of numerous conceptual essays and empirical studies for the last 40 years. Research upholds Dewey's claim that knowledge for teaching is different from knowledge for "doing" in a discipline. Merely "knowing" more mathematics does not ensure that one can teach it in ways that enable students to develop the mathematical power and deep conceptual understanding envisioned in current reform documents (e.g., National Council of Teachers of Mathematics, 1989).

Five major research genres are distinguishable in the literature on teachers' knowledge, and these genres follow a roughly chronological pattern. The earliest studies, conducted in the 1960s and 1970s were quantitative studies that sought to demonstrate a connection between teachers' knowledge and student achievement. These studies failed to find any statistically significant correlation between measures of teacher knowledge (such as number of mathematics courses taken, major in mathematics, grade point average) and student achievement. Although these studies have been roundly criticized for taking a naïve and simplistic view of teachers' knowledge by using such gross measures as number of courses taken, there has been little effort in the intervening 20 years to develop more appropriate research methods to answer the question about the relationship between teachers' knowledge and students' knowledge.

The 1960s, 1970s, and 1980s saw a flurry of descriptive studies that attempted to characterize the strengths and weaknesses in teachers' knowledge of particular content areas, such as fractions or geometry. Most of these studies were conducted using a combination of quantitative and qualitative research methods. Many studies administered written surveys to large numbers of teachers and conducted follow-up interviews with smaller numbers of teachers. The overwhelming majority of these studies were conducted with preservice teachers. These studies suggest that while elementary teachers generally (although not always) have a command of the facts and algorithms that comprise school mathematics, they lack a conceptual understanding of this mathematics. Their knowledge tends to be compartmentalized and fragmented and, therefore, not easily transferable from one domain to another.

The dismal results of the descriptive studies spawned comparison studies that compared the knowledge of elementary vs. secondary teachers, preservice vs. inservice teachers, and U.S. teachers vs. teachers from other countries. These studies generally employed the same quantitative and qualitative methods as the descriptive studies. The comparison studies showed that while there are some slight differences between various populations, the conceptual knowledge of all populations is uniformly low.

Over time, researchers have come to recognize that the issues surrounding teachers' knowledge, in general, and its implementation in classroom practice, in particular, are multifaceted and complex. Within the last decade, there have been a number of studies that have attempted to capture this complexity by conducting qualitative studies of small numbers of teachers engaged in teaching practice. These studies have shown that the relationship between knowledge and teaching practice is anything but straightforward. While a number of elementary teachers with weak content knowledge are predisposed to telling students rules and explaining algorithmic procedures, a number of teachers with strong content knowledge behave similarly..

Also during the last decade researchers have begun to conduct intervention studies to determine what kinds of teacher education seem to make a difference in teachers' knowledge and practice. These studies suggest that change is possible, but it is slow and tedious.

Two major research efforts undertaken during the 1980s shed considerable light on questions related to teachers' knowledge and deserve special mention here. The Teacher Education and Learning to Teach Study was conducted at Michigan State University and examined 11 different preservice, inservice, induction, and alternative route teacher education programs around the United States. The study included both case studies of programs and longitudinal studies of teachers. (See NCRTE, 1988 for a thorough description of the study.) The Knowledge Growth in a Profession Study was conducted at Stanford University and investigated secondary teachers' content knowledge in a variety of disciplines. Although focused on secondary teachers, the Stanford study illuminated a number of important issues in the area of teachers' knowledge that were relevant for the study of elementary teachers as well.

Types of Knowledge

As the genres described above progressed over time, so too did the field's conceptualization of "knowledge." Numerous scholars have articulated various types of knowledge that are essential for teaching. It is generally accepted that elementary teachers need knowledge of the subject matter, knowledge of learners, knowledge of learning theory, knowledge of teaching strategies, and knowledge of the social context of schooling in order to be effective in helping children learn. There exist extensive bodies of literature on each of these types of knowledge. The focus of this paper is on the first category, knowledge of the subject matter.

The earliest research into teachers' subject matter knowledge focused exclusively on teachers' mastery of the content of mathematics. Current research, however, takes a broader view of what constitutes subject matter knowledge. Scholars suggest that there are three particular types of subject matter knowledge that are essential for teaching mathematics–substantive knowledge, knowledge of the discipline, and pedagogical content knowledge.

Substantive knowledge is what is typically regarded as content knowledge–facts, procedures, concepts, organizing ideas, and relationships among topics (Ball, 1991). Substantive knowledge has both a quantitative and a qualitative dimension. In the quantitative dimension, there are many "things" we expect teachers to know. For example, we expect them to know that multiplication and division are inverse operations, that squares are a subset of rectangles, and that division by zero is undefined. There probably would be widespread agreement on a core body of facts, procedures and relationships that we expect elementary teachers to know, but there would probably be much disagreement at the fringes of this body of knowledge.

The qualitative dimension of substantive knowledge is usually associated with the word *understanding*, which has multifarious meanings. The issue of what it means to know or understand something in mathematics has received much attention in the literature. Understanding has been described in many ways–using adjectives, antonyms, and continua. Adjectives that have been used include correct, deep, broad, rich, robust, flexible, connected, fragmented, and compartmentalized. Terms that sometimes are used as antonyms and sometimes as ends of a continuum include procedural and conceptual (Hiebert, 1986) and instrumental and relational (Skemp, 1978). The literature on the qualitative dimension of teachers' knowledge sometimes has been interpreted to mean that procedural knowledge is bad and conceptual knowledge is good. Hiebert and Lefevre (1986) argue, however, that learners need both types of knowledge in order to be "fully competent in mathematics" (p. 9).

Knowledge of mathematics as a discipline includes an understanding of the ways in which knowledge is created and the canons of evidence that guide inquiry. Schwab (1978) labeled this type of knowledge *syntactic*. However, this term seems a bit too narrow to capture all of the knowledge about mathematics as a discipline that teachers need. Ball (1991) suggested that knowledge of the discipline includes "knowledge about the nature and discourse of mathematics" (p. 7) such as what qualifies as a solution, the role of conjectures, which ideas are arbitrary and which are not, and the role of various mathematical tools (such as proof or examples). Syntactic knowledge is rarely made explicit for teachers in their teacher preparation coursework (Ball, 1991); rather it is assumed to be obvious or, worse, unimportant. Lacking the opportunity to build their knowledge about mathematics as a discipline, teachers are left to form their own assumptions. The literature suggests that teachers think that to do mathematics is to follow procedures to get correct answers, that knowing mathematics is equivalent to being able to perform procedures, that mathematics is a collection of rules and procedures, and that mathematics has little value beyond preparing one for the next mathematics class (Ball, 1991).

Pedagogical content knowledge is a term coined by Shulman (1986) to describe "the ways of representing and formulating the subject that make it comprehensible to others," (p. 9). It includes an understanding of which representations are most appropriate for an idea, which ideas are likely to be most difficult for children, what preconceptions children might hold about an idea, and the strategies that are most likely to lead to learning on the part of the children. It is difficult to study teachers' pedagogical content knowledge because it seems to be logically connected to their substantive knowledge and perhaps their knowledge of the discipline. Thus, much of what we know about teachers' pedagogical content knowledge is intertwined with what we know about their substantive knowledge and is not particularly revealing. For example, one study showed that a teacher who had limited substantive knowledge about division of fractions was not able to provide representations and explanations that assisted children in understanding why the invert-and-multiply rule makes sense (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992).

The rest of this paper provides a summary and critique of the empirical literature on elementary teachers' mathematical knowledge. The paper reports the empirical evidence that addresses four major issues surrounding teachers' mathematical knowledge: 1) What is the nature of teachers' content knowledge, particularly about the domain of number? 2) How does teachers' knowledge impact their instructional practice? 3) How does teachers' knowledge impact student learning? 4) How do teachers develop appropriate mathematical knowledge? For each question, I summarize the major findings, provide exemplars of studies that have addressed the question, and critique the contribution of these studies to our understanding of teachers' mathematical knowledge. Please note that the exemplars are not intended to be exhaustive; rather they are intended to be illustrative.

Framing the Review

Methodological Approach

I used several methods to collect data for this analysis. Initially, I consulted existing reviews of the research on teaching, teacher education, and teachers' knowledge (e.g., Brown & Borko, 1992; Brown, Cooney, & Jones, 1990; Carter, 1990; Cooney, 1980, 1994; Fennema & Franke, 1992; Lanier & Little, 1986; Romberg & Carpenter, 1986) to gain a perspective on issues that have been addressed. Then, I used the reference lists from these reviews as well as a search of the ERIC database to locate empirical studies of teachers' knowledge. In addition, I searched the publications of the National Center for Research on Teacher Learning[1] at Michigan State University. I also consulted colleagues in mathematics education for their suggestions of literature to review.

Parameters

I defined elementary teachers as those teaching (or preparing to teach) kindergarten through eighth grade. In a few instances, I cite studies involving secondary teachers where there is a direct connection to the literature on elementary teachers. I reviewed literature pertaining to both inservice and preservice teachers and the domains of both teaching and teacher education.

I limited the review to literature published since the 1960s because this is the time period when many people indicate that mathematics education became a recognized discipline (Johnson, Romberg, & Scandura, 1994; Kilpatrick, 1992). The 1960s also seem to mark a period of increased interest in issues surrounding mathematics teachers' knowledge, most likely because of the advent of the new math era. Further, I reviewed primarily studies conducted in the United States and only pieces written in English. A few studies conducted in other countries are included where they provide useful information.

The Nature of Teachers' Content Knowledge

Substantive Knowledge

[1] These papers are available on the Internet at http://ncrtl.msu.edu. Some of the papers that I accessed did not contain page numbers while others did not print with consistent spacing. Therefore, in cases where I have used direct quotes from these papers, the page numbers I cite may not match the pages numbers on another copy of the paper printed by someone else.

Studies that document teachers' substantive knowledge of mathematics abound in every decade since the 1960s. There exists a plethora of articles and dissertations bearing titles such as "Mathematical Competencies of Preservice Elementary School Teachers," (Reys, 1968), "Mathematical Understandings and Misconceptions of Prospective Elementary School Teachers" (Eisner, 1975), "Much Ado about Nothing: Preservice Elementary School Teachers' Concept of Zero (Wheeler & Feghali, 1983), and "Preservice Elementary Teachers' Misconceptions in Interpreting and Applying Decimals" (Thipkong & Davis, 1991). The overwhelming majority of these studies paint a rather dismal picture of elementary teachers' (usually preservice teachers') knowledge of mathematical concepts found in the elementary curriculum. These studies conclude that elementary teachers have a rudimentary and procedural knowledge of mathematics, but they are considerably less competent in their ability to provide conceptual explanations for computational algorithms.

<u>Exemplars of Studies of Substantive Knowledge</u>

In most cases, this literature shows that the teachers are able to successfully perform computations. There are, however, some examples of teachers who show an alarming lack of basic knowledge of mathematics. For example, Ball (1988) found that half (5 out of 10) of the preservice elementary teachers she interviewed thought that zero divided by zero was zero, and an additional 20% of the elementary candidates (2 out of 10) stated that they could not remember the rule for division by zero and were unable to answer the question. Baturo and Nason (1996) reported that some preservice teachers in their study (conducted in Australia) were unable to produce number facts from memory and were unable to deduce facts when given related facts (e.g., given that 3 x 5 = 15, they were unable to deduce 6 x 5). However, by and large, these studies report that the majority of teachers are competent in their ability to perform procedural mathematics tasks.

But teachers are unable to provide conceptual explanations for the procedural tasks they perform. For example, a common finding of these studies is that preservice teachers lack an understanding of quotitive (measurement) division and are prone to rely only on a partitive (sharing) interpretation of division[2] (Ball, 1990; Graeber, Tirosh, Glover, 1989; Simon, 1993). This becomes particularly problematic in the case of division of fractions where it is almost impossible to make sense of the underlying ideas using a partitive interpretation of division. Many teachers are unable to generate a word problem for a whole number divided by a fraction, often providing a problem that represents a multiplication situation (Borko et al., 1992; Ma, 1999). Teachers tend to rely on their knowledge of whole numbers when working in the domain of rational numbers (Tirosh, Fischbein, Graeber, & Wilson, 1999). This overgeneralization from one number system to another leads to misconceptions and impoverished ideas about rational numbers (such as the claim that multiplying two numbers results in a product that is larger than either of the two numbers, a claim that is true for whole numbers but false for rational numbers). Further, many teachers do not know the difference between a ratio and a fraction, believing that because they can be represented with the same notation they behave in identical ways (Fuller, 1997; Leinhardt & Smith, 1985).

Another common finding from this literature is that teachers confuse the concepts of area and perimeter (Baturo & Nason, 1996; Fuller, 1997; Heaton, 1992), frequently assuming that there is a constant relationship between area and perimeter. Further, teachers often do not use

[2] See Ball (1990) for a thorough explanation of partitive and quotitive division.

appropriate units when computing area and perimeter, commonly failing to use square units when reporting measures of area (Baturo & Nason, 1996, Simon & Blume, 1994).

Analysis

There are a large number of studies in this genre, spanning four decades of research, and the results are essentially the same: elementary teachers lack a conceptual understanding of many topics in the elementary mathematics curriculum. It is tempting to conclude that the problem lies within the population of individuals choosing careers as elementary teachers. However, Lanier and Little (1986) provided a thorough analysis of the qualifications of teacher education students and demonstrated that the teaching profession gets its fair share of the best and brightest students. They found that the teaching profession also gets more than its fair share of students from the lowest quintile of achievement. Lanier and Little's analysis is now more than a decade old, and much of the data they were analyzing was even older than that. Unfortunately, we lack a similar analysis with contemporary data.

It is equally tempting to conclude that these studies suggest that prospective elementary teachers need to study more mathematics. However, similar descriptive studies have been conducted with prospective secondary teachers, and these studies show that the problem of weak conceptual knowledge of school mathematics is not confined to elementary teachers. For example, Even (1993) found that prospective secondary teachers held an equation concept of functions, expected the graphs of functions to be smooth and continuous, and were unable to provide an explanation of the univalence requirement for functions. The students knew that the vertical line test was a procedural way of determining whether a graph represented a function, but they were unable to provide a conceptual explanation for why univalence is necessary.

Further, studies comparing the mathematical knowledge of prospective elementary and secondary teachers show that secondary teachers' conceptual knowledge of elementary mathematics is not significantly stronger than that of their elementary counterparts. For example, Ball (1990, 1991) compared the mathematical knowledge of preservice elementary education majors and preservice secondary mathematics education majors on the topics of division (including division of fractions, division by zero, and division in algebraic expressions) and place value in the multiplication of large numbers. The secondary majors were more successful at obtaining correct answers than the elementary majors, but they were not adept at explaining the reasons behind the rules they invoked and their knowledge was not connected across various contexts. Thus, Ball concluded that although the secondary mathematics majors had successfully completed a number of advanced mathematics courses, this academic preparation did not provide them with "the opportunity to revisit or extend their understandings of arithmetic, algebra, or geometry, the subjects they will teaching" (p. 24). She further noted that simply requiring more mathematics of prospective teachers will not increase their substantive understanding of school mathematics. Rather, a *different kind of mathematics* is needed.

Alternative certification programs (for those already holding a bachelors degree in a content area) have gained popularity in recent years as the need for more teachers rises. Ball and Wilson (1990) compared the mathematical content knowledge of students in traditional teacher education programs and alternative route certification programs at both the entry and exit points of the programs. The mathematics content of the study dealt with the relationship between perimeter and area, proof by example, division by zero, and division of fractions. Upon entry to the teacher education programs, neither group was able to explain the mathematics underlying the problems presented, and there were no significant differences between the groups. At the conclusion of the teacher education programs, both groups showed increased evidence of mathematical understanding, but again there were no significant differences between the groups. Ball and Wilson concluded that neither group had "opportunities to unpack mathematical ideas or to make connections" (p. 7) and that neither group was prepared to teach mathematics for understanding. Their findings support Ball's (1990, 1991) claim that requiring teachers to study more traditional mathematics is not the answer as students who have pursued this course of study are not substantially better prepared to teach school mathematics.

It is striking to read the comments from prospective teachers as they are asked to solve mathematical problems or as they engage in reflecting on their teaching. In many cases, these teachers are fully aware that they lack a conceptual understanding of mathematics. For example, one student teacher noted, "I don't just like saying 'Well, this is pi. Remember it,' ... [but] where does pi come from? Well, I don't know." (Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993, p. 18). Another preservice teacher noted, "I am really worried about teaching something to kids I may not know. Like long division–I can *do* it–but I don't' know if I could really *teach* it because I don't know if I really *know* it or know how to *word* it" (Ball, 1990, p. 104). It is to their credit that these future teachers are aware of and concerned about their mathematical competence and its potential impact on their teaching.

Three weaknesses in the research on the nature of teachers' mathematical knowledge are worthy of note. First, these research studies have addressed a fairly narrow range of mathematics content areas. The topics of place value, division, rational numbers (more specifically, fractions, with considerably less attention to decimals and ratios), and geometry (focusing almost exclusively on area and perimeter) have been addressed by numerous researchers. Perhaps it has been taken for granted that teachers understand addition and subtraction of whole numbers, patterns, and counting–fundamental topics in the kindergarten and first grade curricula. A number of more contemporary mathematical topics (such as probability, data analysis, functions, transformational geometry, number theory) have been addressed by only a few researchers. Given the recent emphasis on elementary mathematics as more than arithmetic, it seems necessary to know more about teachers' knowledge in domains other than number.

For example, an area of mathematical understanding that seems to be crucial to enabling teachers to enact current reform visions but that has received limited attention in the research literature is preservice elementary teachers' understanding of mathematical justification. If teachers are to orchestrate discourse in their classrooms and encourage students to share their emerging mathematical ideas, teachers must have a sense of what constitutes a valid mathematical argument (Ball, 1994). Studies suggest that preservice teachers are prone to accept inductive evidence, such as a series of empirical examples or a pattern, as a sufficient proof (Martin & Harel, 1989; Simon & Blume, 1996). Simon and Blume (1996) paint a vivid picture of the challenges and opportunities of engaging a class of preservice teachers in mathematical arguments. Studies such as this one illuminate the nature of preservice teachers' thinking and demonstrate how their thinking impacts and can be impacted by instruction.

Second, these studies generally present "snapshots" of teachers' knowledge at a particular point in time. Few studies provide a longitudinal "videotape" of teachers' knowledge and how it changes over time. Thus, we lack data that shows us what experiences impact teachers' knowledge and how that knowledge grows and changes over time.

Third, research conducted in this genre has generally failed to provide us with rich data about teachers who *do* possess strong conceptual knowledge of mathematics. We have very few examples of the reasoning of teachers who are able to think through problems and provide suitable explanations. Most of these studies report that 50% or fewer of the teachers studied lacked conceptual understanding of mathematics. However, we rarely read any data about the other 50% of the teachers who did possess some conceptual understanding of the mathematics. It would be enlightening to see examples of teachers with strong mathematics content knowledge and an analysis of what mathematics and what reasoning processes these teachers use to solve novel problems. Further, it would be useful to know how, when, and where these teachers developed this conceptual understanding..

Do we need more studies of this type? The answer to that question depends on what is done with the results of the studies. If the studies are viewed as an end in themselves, then enough is enough. We do not need more studies that decry the impoverished nature of elementary teachers' mathematical knowledge in particular content areas; that has been established. However, if these studies are viewed as a means to an end–that end being the design of better mathematics content and methods courses–then more studies could be useful. These studies give us windows of insight into teachers' thinking, and this insight can help us design courses that address the strengths of their knowledge while challenging the weaknesses in their knowledge.

<u>Knowledge of Mathematics as a Discipline</u>

More recently, studies of teachers' knowledge have also begun to document teachers' knowledge about mathematics as a discipline. These studies suggest that teachers' views of mathematics are largely shaped by their own prior experiences with mathematics. Preservice teachers tend to have similar views of mathematics, seeing it as a fixed body of facts, rules, formulas, and procedures that is learned through memorization. Thus, when faced with a pedagogical dilemma, such as helping a child correct an error, preservice teachers generally resort to telling children how to correctly perform a procedure. Current college seniors were entering middle school when the NCTM Standards were released, and we have ample evidence (Ferrini-Mundy & Schram, 1997; McLeod, Stake, Schapelle, Mellissinos, & Gierl, 1996) that the Standards did not make an impact at the classroom level when these students were in school. Thus, it is not surprising that they have rather limited views of mathematics. By contrast, however, there is more variability in inservice teachers' knowledge of mathematics as a discipline. Some inservice teachers see mathematics in the same way as most preservice teachers, but many see mathematics as a dynamic human activity. As will be discussed in more detail later, this difference between preservice and inservice teachers' knowledge suggests that teachers' knowledge grows and changes as a result of teaching practice.

Although studies of teachers' knowledge of mathematics as a discipline address the question about the nature of teachers' knowledge, they perhaps speak more explicitly to the question of how teacher's knowledge impacts instructional practice. Therefore, exemplars of these studies are given in the next section.

The Impact of Teachers' Content Knowledge on Instructional Practice

In recent years there has been an increasing number of studies that employ qualitative methods to examine the teaching practice of a small number of teachers in detail. These studies suggest that the quality of teachers' mathematical explanations in the classroom closely mirrors the depth of their substantive knowledge. In other words, teachers whose knowledge of mathematics is only procedural are likely to provide students with algorithmic, procedural explanations, while teaches whose knowledge is also conceptual are more likely to provide explanations and pose questions that help children develop rich conceptual understanding of the content.

Teachers' classroom practices are also influenced by their knowledge of mathematics as a discipline. Teachers who see mathematics as a fixed body of knowledge that is best learned through memorization will have a different approach to teaching than teachers who see mathematics as a dynamic subject best learned through problem solving and group work.

However, these studies suggest that the relationship between teachers' knowledge and classroom practice is complicated and not at all straightforward. Teaching is a complex enterprise that is affected by a number of factors. Therefore, it is not possible to predict what a teachers' classroom practice will look like based on an assessment of his or her substantive knowledge and knowledge about mathematics as a discipline.

Exemplars

Leinhardt and Smith (1985) presented three cases of experienced mathematics teachers with differing levels of substantive knowledge of mathematics and compared their instruction in a lesson on generating equivalent fractions and simplifying fractions. The teacher with the strongest content knowledge provided the students with "a rich body of conceptual information" (p. 261) without explicitly giving the algorithms for generating equivalent fractions or simplifying fractions. She provided students with heuristics to help them in their decisions about how and when to simplify fractions. In contrast, the teacher with moderate subject matter knowledge presented algorithms based on the operations of multiplication and division with no attention to the use of the identity element or the notion of equivalence in the computations. She, in fact, talked of multiplication making larger and division making smaller, implying that the fractions generated by the algorithms were larger (in the case of equivalent fractions) or smaller (in the case of simplified fractions) than the initial fractions. The third teacher, who evidenced low content knowledge, demonstrated similar misconceptions about "reducing" fractions resulting in smaller fractions. Further, she did not present an efficient algorithm to the students. The algorithm she presented involved checking to see if numerator and denominator were divisible by two. However, the next example she presented was of a fraction that could be simplified, but not by a factor of two. Thus, the algorithm she presented was not useful to the students in the general case.

Putnam, Heaton, Prawat, and Remillard (1992) studied four fifth-grade teachers during mathematics instruction and identified aspects of the teachers' substantive knowledge that impacted their instruction. They noted that teachers who were trying to teach in ways consistent with current reform efforts often found themselves in unfamiliar mathematical territory. In these cases, "the limits of their knowledge of mathematics became apparent and their efforts fell short of providing students with powerful mathematical experiences" (p. 221). For example, one teacher's lack of familiarity with area, perimeter, and conversion of measurement units hindered her from foreseeing the complexities her students would face in attempting to compute the cost of some sand, which was priced per cubic foot. Students were allowed to erroneously multiply measurements given in feet with measurements given in yards, resulting in an incorrect and unreasonable answer. Heaton was led to ask, "Who is minding the mathematics content?" in this lesson because neither the teacher nor the students realized that faulty mathematics was being presented in the classroom.

Fernández (1997) studied nine secondary mathematics teachers who had earned a Master of Arts in Teaching degree in secondary mathematics at the University of Chicago. All of the teachers had strong credentials on paper, holding bachelors degrees in mathematics or mathematics-related fields from respected institutions. The study examined the teachers' responses to unexpected student answers in order to see how the teachers' substantive knowledge impacted their teaching practice. Fernández provided examples of where teachers' strong content knowledge enabled them to provide a counterexample to uncover an error in students' thinking, to follow through on a students comment to lead to a contradiction or a viable solution, to apply a student's method to a simpler or related problem, to understand a student's alternative method, and to incorporate a student's alternative method into instruction. Similar findings are emerging from Ball's ongoing study of her own mathematics instruction. Ball (1998) proposed that a teacher's mathematical content knowledge comes into play when she must decide whether an answer is right or wrong, decide whether an answer makes sense, decide whose ideas to showcase, anticipate how students might think, decide how to frame a lesson, determine when to close a lesson, map a curriculum, determine the relative importance of elegance and efficiency in symbolism, choose when to allow students to invent notation and when to insist that they conform to convention, and recognize the isomorphism of multiple representations.

Ball (1991) studied three experienced teachers as they taught long multiplication to fourth graders and analyzed the ways in which their knowledge about mathematics as a discipline impacted their classroom practice. Two of the teachers focused their instruction on helping children get correct answers through the proper use of procedures. They used mnemonic devices to help students remember the steps of the procedure and "placeholders" to minimize students' errors. Students were given lots of opportunities to practice procedures and get feedback from the teacher. Although one teacher did show evidence that she understood the mathematics behind the placement of partial products and the role of place value, she chose not to explain this to her students because she did not think it was important to their success in obtaining correct answers. The third teacher in the study, mathematics educator Magdalene Lampert, had a goal of developing students' mathematical and reasoning skills in order to enable them to validate their own and others' thinking. Thus, she engaged her students in the intellectual activities of inventing procedures, justifying the validity of their procedures, and explaining their procedures to peers. Lampert's students learned how to get correct answers to multiplication problems as did the students of the other two teachers, but Lampert's students also learned something about how mathematics is created, canonized, and communicated to others. These teachers' classroom practices were clearly influenced by their knowledge about mathematics as a discipline. It is less clear how these teachers' substantive knowledge of mathematics influenced their instruction because for the first two teachers, only a procedural knowledge of mathematics was needed.

Borko et al. (1992) demonstrated that when one lacks substantive knowledge of mathematics, it can be difficult to implement one's knowledge of mathematics as a discipline. They presented the case of a student teacher in a fifth-grade classroom who placed high value on helping students see mathematics as meaningful and relevant to their everyday lives. Thus, she used "real-world" examples in her teaching and tried to address students "on their level." She also thought it was important for students to understand the mathematics they were learning and not just apply procedures. However, when faced with a child's question about why the invert-and-multiply algorithm works, the student teacher attempted to place the problem in the real-world context of painting a fence. She drew a rectangle on the board to represent the fence. Despite having taken two years of calculus, a course in proof, a course in modern algebra, and four computer science courses, she lacked a conceptual understanding of division of fractions and was unable to provide a correct representation for the child. She, in fact, provided a representation–orally and pictorially–of multiplication of fractions rather than division. In a related article, Eisenhart et al. (1993) noted that, in general, the student teacher's lack of conceptual understanding of the mathematics she was teaching led her to provide students with mnemonics or memory aids to assist them in remembering algorithms. So, although the student teacher exhibited some knowledge of mathematics as a discipline and a desire to help students develop conceptual knowledge, her efforts to achieve these goals were thwarted by the limits of her own substantive knowledge of mathematics.

Heaton (1994, 1995) provided a similar example from her study of her own mathematics teaching in a fourth-grade classroom. Heaton had returned to classroom teaching during her doctoral program in order to try out some of the things she had learned about teaching mathematics in a manner consistent with current reform efforts. She was committed to fostering a community of learners in her classroom through the use of discourse and worthwhile mathematical tasks. However, she discovered that her substantive knowledge of mathematics sometimes impeded her ability to enact instruction that was consistent with her values. For example, in a lesson involving composition of functions, Heaton asked her students to identify patterns they saw in tables in hopes of helping them articulate a generalization about the composition of two functions. When her students began identifying random, interesting occurrences rather than meaningful patterns, she realized that her own knowledge of what constituted a pattern and how compositions of functions behaved was impoverished. She noted, "I was worried about the mathematics I did not understand. I was reminded of the limits of my understanding as I searched for the meaning of 'composition of functions'" (Heaton, 1995, p.

12). In reflecting on the lesson, Heaton came to realize that because she did not really understand how composition of functions was relevant to the rest of the elementary mathematics curriculum, she had no purpose or direction for her lesson or for the questions she asked during her lesson. Thus, she was unable to help children make sense of teach other's ideas or ask questions that would lead to productive discourse.

Two studies, in particular, demonstrate that teaching mathematics is a complicated enterprise, and mathematics content knowledge is only one of many factors that impacts teachers' instructional practices. Lubinski, Otto, Rich, and Jaberg (1998) studied two novice teachers, one with a strong mathematics background and one with a weak mathematics background. The one with the strong mathematics background was not inclined to use students' mathematical understandings to inform his instruction and tended to "railroad" students into doing the mathematics his way. The teacher with the weaker mathematics background was more adept at listening to students and reshaping her lessons to take advantage of their emerging understanding. However, this teacher presented incorrect mathematics to the students on a number of occasions. This study illustrates that strong content knowledge is just one of many elements that are central to good teaching. Having strong content knowledge does not guarantee that a teacher will facilitate robust student learning; neither does weak content knowledge doom a teacher to traditional methods of instruction.

Similarly, Thompson and Thompson (1994, 1996) studied a middle school teacher as he worked one-on-one with a middle school student on the concept of rate. They found that while the teacher had a reasonably robust conception of rate, he was unable to articulate that conception in a way that helped the student learn about rate conceptually. In fact, the teachers' inclination to describe rate in terms of whole numbers and whole number operations actually reinforced the student's incorrect additive (rather than multiplicative) way of thinking. The researchers noted that the teacher's strong conception of rate actually *hindered* his ability to listen to the child's mathematical thinking. He tended to automatically lay his own understanding on top of the child's explanations, which led to him assuming she understood what she was doing. Thompson and Thompson give a few examples of situations where the teacher and student seemed to be having a conversation but were actually talking past each other because they were unaware that the other did not share their mathematical understanding. The teacher, while having a strong conception of the topic, was not able to "step outside" of his own thinking to really listen to the child.

Analysis

Clearly, teachers' knowledge impacts teaching practice. Teachers who lack a conceptual understanding of mathematics are more likely to provide procedural, algorithmic explanations to children and are less able to "think on their feet" to provide conceptual explanations. Those with a stronger conceptual knowledge of mathematics are better able to plan and direct classroom activity and discourse in ways that potentially lead to conceptual understanding on the part of the students. Those who see mathematics as a search for correct answers will provide instruction that assists students in getting correct answers efficiently while those who see mathematics as a sense-making endeavor will provide instruction that engages students in the "verbs of mathematics"– conjecturing, investigating, discussing, summarizing.

However, the statements in the preceding paragraph are much too simplistic. The richness of the descriptions of teaching provided by these studies shows that teaching is a complex endeavor and that the role that the teachers' mathematical knowledge plays in shaping instruction is multi-faceted. Teachers are faced with numerous moment-to-moment, short-term, and long-term decisions every day, and they must weigh a number of competing demands in making these decisions. Factors that affect a teacher's decisions are sometimes not obvious to the researcher. For example, a teacher possesses a wealth of knowledge about the individual children in the classroom and their prior experiences as a community of learners in mathematics and other subjects, and this information is probably not easily available to the researcher. Thus, even through the conscientious application of accepted research methodologies, researchers are still only able to present a portion of the complicated puzzle that is teaching. We know very little about how various types of knowledge–substntive knowledge, knowledge of the discipline, pedagogical content knowledge, knowledge of particular learners, knowledge of learning theory, knowledge of teaching strategies–interact to inform teachers' classroom practices

The Impact of Teachers' Content Knowledge on Student Learning

The question of how teachers' knowledge affects student learning was heavily investigated in the 1960s and 1970s by researchers using quantitative methods in an attempt to find statistically significant relationships between teacher characteristics and student learning. Teacher characteristics generally were measured by such variables as number of mathematics courses taken, major in mathematics, grade point average, and other gross measures of teachers' knowledge of mathematics. Student learning was measured almost exclusively by scores on achievement tests. While it seems axiomatic that teachers should have a solid understanding of the mathematics content they are expected to teach, studies in this genre found little evidence of a connection between teachers' knowledge and student learning. Although this line of research has been virtually abandoned for the last 20 years, a few recent studies provide some insight into the possible interactions between teacher knowledge and student learning.

Exemplars

Perhaps the most widely cited example of researchers' attempts to link teachers' knowledge and student learning was done as part of the National Longitudinal Study of Mathematical Abilities (NLSMA) and summarized by Begle in his 1979 book *Critical Variables in Mathematics Education*. NLSMA found no evidence to suggest a significant positive relationship between the teacher variables of number of years of teaching, highest academic degree, academic credits beyond BA, mathematics credits beginning with calculus, credits in mathematics methods, in-service or extension courses, other preparation in the last five years, mathematics as a major or minor and student achievement in mathematics. Begle (1979) reported similar results in his review of 17 additional studies of teachers' knowledge of mathematics and student achievement.

Similarly, Fey (1969) reviewed a number of studies conducted in the 1960s and concluded that "there is almost no evidence to support or deny claims of correlation between teacher knowledge of mathematics and classroom effectiveness" (p. 80). In particular, Fey noted that the Minnesota National Laboratory Study, conducted from 1958 to 1962, "failed to find any significant correlation between students' achievement and the experience, collegiate courses or grades, and professional activity of their teachers" (p. 54).

A few studies conducted in this genre did find some evidence of a relationship between teacher characteristics and student achievement. These studies used more particular measures of teachers' knowledge of mathematics than simply number of courses. For example, Begle (1972) examined high school algebra teachers' knowledge of both abstract algebra and the algebra of the real number system and their students' computational abilities and understanding of high school algebra. All measures were taken from written tests constructed for the study. Begle found that teachers' understanding of abstract algebra was not significantly correlated with student achievement in either computation or understanding of ninth grade algebra. Neither was teachers' understanding of the algebra of the real number system significantly correlated with student achievement in computation. However, teachers' understanding of the algebra of the real number system had a significant positive correlation with students' understanding of ninth grade algebra. Begle noted that while significant, the correlation was quite small and therefore not educationally significant. Eisenberg (1977) found similar results in a replication of Begle's study.

More recently, Mullens, Murnane, and Willet (1996) conducted a study of teachers of 1043 third graders in Belize. They compared teachers' scores on the mathematics portion of a school-leaving exam (administered at the end of the teachers' own eighth grade school year) and student gain scores on a pretest/posttest covering basic concepts and operations as well as advanced concepts (e.g., applications of basic concepts). The study showed no relationship between teacher competence and student attainment in basic concepts. However, there was a statistically significant relationship between teachers' mathematical competence and student attainment of advanced mathematical concepts from the pretest to the posttest. Thus, they concluded that "third-grade students in Belize learn more mathematics when their teachers have a strong command of the subject" (Mullens et al., 1996, p. 156).

Ball (1992) and her colleagues have studied her mathematics teaching in a third-grade classroom over the course of a school year. One particular example from her work illustrates how a teachers' knowledge can impact student learning. In a lesson involving even and odd numbers, one of Ball's students, Shea, said that 6 was both an even and an odd number. He explained that 6 was even because 6 objects could be "split in half without having to use halves"

(p. 14) (the definition of evenness that the students had constructed). He explained that 6 was also odd because when 6 objects were placed in piles of 2, there were 3 piles and 3 was an odd number. This definition of odd numbers was not the one that had been agreed upon by the classroom community. However, rather than telling Shea that he was not being consistent in his application of definitions, Ball asked the other students in the class to comment on Shea's claim and engaged them in determining if other numbers had the property that Shea had noticed. Ball was able to use her own substantive knowledge of mathematics to determine that Shea's observation about the number 6 was not a random occurrence, but rather that there would be other numbers that fit Shea's description and that these numbers would have a pattern to them. Thus, she decided that engaging the students in the investigation, although not the lesson she had planned for the day, would be a worthwhile task. The students eventually developed a general rule for such numbers, and, in keeping with how things are often done in the larger mathematics community, they named these numbers Shea numbers in honor of the mathematician who first advanced the idea. Ball was using her knowledge of mathematics as a discipline to guide student learning in this lesson. As evidenced by classroom discussions and a subsequent written task completed by the children individually, the children learned the subject matter of how to determine if a number is even or odd and whether or not a number is a Shea number. Further, the children learned something about what it means to participate in a mathematical community. They learned how a conjecture is scrutinized by peers, how it is stated formally, and how it becomes an accepted part of the culture of a mathematical community.

Analysis

Much of the work early work that failed to demonstrate a connection between teachers' knowledge and student learning has been labeled surprising, counterintuitive, and depressing.

However, a thoughtful analysis of this literature demystifies the findings. In most cases, the studies used gross measures of teachers' knowledge of mathematics. Easily accessible data, such as the number of mathematics courses taken, were used as proxies for a teacher's mathematics knowledge. As noted by Mullens et al. (1996), "Variation in the quality of education and training activities...may mean that teachers with the same paper credentials possess quite different levels of content knowledge and teaching skill" (p. 139). Thus, it is not surprising that there was little evidence of a connection between teachers' knowledge, as determined by these indirect measures, and student achievement.

However, when more direct measures of teachers' knowledge were used, as in the Begle (1972) study and the Mullens et al. (1996) study, the results were more in line with what we would predict. In both studies it was found that teacher knowledge did relate to student achievement in the area of understanding but not computation. It is not surprising that teachers of varying mathematical backgrounds were equally successful in teaching students the procedures for solving computational problems and that those with stronger mathematics backgrounds were more successful in helping students understand the mathematics they were studying.

Begle's (1979) summary of the research in this genre is frequently quoted:

> It is widely believed that the more a teacher knows about his subject matter, the more effective he will be as a teacher. The empirical literature suggests that this belief needs drastic modification and in fact suggests that *once a teacher reaches a certain level of understanding of the subject matter* [italics added], then further understanding contributes nothing to student achievement. (p. 51)

I have added the italics to Begle's quote to emphasize an often overlooked aspect of his point. Begle did not contend that teachers do not need to know mathematics in order to teach it; rather he contended that there was no evidence to suggest that studying mathematics beyond *a certain level* is beneficial to student learning. Unfortunately, Begle did not elaborate on the "certain level of understanding" that he proposed, nor does the literature from the correlational genre provide much guidance.

This line of research has virtually been abandoned in the United States in the last two decades. The type of research being conducted by Ball holds possibilities for new ways of looking at the impact of teachers' knowledge on student learning. Like teaching, learning is a complicated enterprise, and studies that seek to capture evidence of the relationship between teachers' knowledge and students' learning will have to enter the messy fray of the classroom in order to do so.

The Ways in Which Teachers Acquire Appropriate Knowledge

The studies cited above lead to the conclusion that many elementary teachers do in fact lack a conceptual understanding of the mathematics they are expected to teach. However, with few exceptions the literature cited above fails to document that the participants *had the opportunity* to learn mathematics conceptually somewhere in their teacher preparation programs. There are a few studies that illuminate the possibilities for enhancing teachers' mathematical knowledge and their teaching practice.

Exemplars

Swafford, Jones, and Thornton (1997) reported that 49 middle school teachers who completed a summer course in geometry experienced significant gains in their geometry content knowledge as measured by a pretest/posttest. Seventy-two percent of the teachers increased one van Hiele level, and more than 50% of the teachers increased two van Hiele levels. Eight teachers who were selected for classroom observations and interviews during the following school year reported that they spent more time on geometry, were willing to try new ideas, were more likely to take risks to enhance student learning and were more confident in their abilities to elicit and respond to higher levels of geometric thinking as a result of their experience in the geometry course.

Studies of the Cognitively Guided Instruction program (Carpenter, Fennema, Peterson, & Carey, 1988) have shown that teachers who have opportunities to learn about problem types for addition and subtraction, the relationship between the two operations, and common strategies used by students in solving such problems are successful in engaging students in rich discourse about mathematics and in facilitating students' constructions of multiple solutions to problems (Fennema & Franke, 1992).

Educators at Michigan State University designed a three course sequence of mathematics content courses for elementary education majors and studied students during the classes, during their student teaching, and during their first year of teaching. Schram, Wilcox, Lanier, and Lappan (1988) reported some success in helping preservice teachers expand their conceptual understanding during the mathematics content courses. They found that students developed a conceptual understanding of many of the facts, formulas and rules that they had previously memorized as a result of taking a course that emphasized problem solving, reasoning, discourse, group work, and the use of multiple representations. However, when they followed these teachers into their first year of teaching, they found that some of them struggled to replicate their own learning experiences in their classrooms. Others did not appear to attempt to replicate the type of learning environment they had experienced. The researchers were led to conclude:

> Disciplinary study is necessary to develop in novice teachers a set of intellectual tools and a disposition to engage in mathematical inquiry themselves. But disciplinary study alone may be insufficient...to develop in beginning teachers the knowledge, skills, and beliefs to conceive of teaching as something other than telling or as more than a matter of technical competence. (Wilcox, Lanier, Schram, & Lappan, 1992, p. 23)

This lends further credence to the argument that a number of different types of knowledge interact when a teacher makes decisions. Although these teachers possessed some level of desirable substantive knowledge of mathematics, they lacked adequate knowledge of mathematics as a discipline and/or pedagogical content knowledge to enable them to teach mathematics in ways consistent with current reform efforts. Clearly, knowing mathematics for oneself is not the same as knowing how to teaching it.

A recent cross-national study suggests that the conditions under which one teaches have a profound influence on what one does in the classroom. Ma's (1999) book comparing the knowledge of inservice teachers in the United States and China highlights the important influence of context. Ma's data on U.S. teachers came from the Teacher Education and Learning to Teach Study (TELT) conducted at the National Center for Research on Teacher Education at Michigan State University in the late 1980s and included 23 teachers who were considered to be "better than average" because they were enrolled in graduate level inservice programs. Seventy-two Chinese teachers were selected from schools ranging from very low to very high in quality, and thus Ma concluded that these teachers were more representative of the total teaching population than were the U.S. teachers. A further point of contrast is that Chinese teachers typically complete ninth grade and attend two or three years of normal school, far less formal education than that acquired by U.S. teachers. In his review of Ma's book, Baldwin (1999) highlighted additional differences in the two populations, namely that the Chinese teachers spanned grades K-8 and that 80% of the Chinese teachers were mathematics specialists, teaching only mathematics. U.S. teachers taught only grades K-5 and are assumed to have taught all subject areas. (This is an assumption because no data is provided on the teaching loads of U.S. teachers.)

The TELT interview items that were used with U.S. teachers were administered to Chinese teachers to obtain comparable data. The interview items dealt with subtraction with regrouping, multidigit multiplication, division of fractions, and the relationship between perimeter and area.

Ma's conclusions about the U.S. teachers mirrored results reported earlier. Namely, she concluded that the U.S. teachers had an algorithmic focus to their thinking and that their knowledge was fragmented. U.S. teachers tended to be satisfied with "pseudoexplanations" and did not attempt to present multiple solutions to problems or to verify solutions using alternative methods. This phenomenon has been found outside the United States as well. Baturo and Nason (1996) reported that first-year teachers in Australia did not possess a propensity to check their answers using estimation or alternative strategies. When prompted to do so, they frequently lacked appropriate alternative strategies. In contrast, Ma found that the Chinese teachers had well-connected mathematical knowledge, particularly of the four basic operations, and they tended to provide multiple solutions to problems, often searching for more efficient and elegant solutions without external prompting. She described their thinking as more conceptual and flexible. Further, they tended to provide explanations that addressed underlying mathematical concepts rather than the procedures. For example, when explaining multidigit multiplication, U.S. teachers focused on the procedure of using placeholders in the ones place of the second partial product and moving over one place to begin the product (a pseudoexplanation). Some even suggested using nonmathematical symbols such as asterisks as placeholders. The Chinese teachers, however, focused their explanations on the distributive property and the role of place value in the partial products.

Ma characterized 10% of the Chinese teachers and none of the U.S. teachers as having a "profound understanding of fundamental mathematics," which she defined as a deep, broad, and thorough understanding of the terrain of fundamental mathematics. Thus, although the Chinese teachers appeared, on the whole, to have a better mathematical understanding than the U.S. teachers, still only 10% of them exhibited the desired level of understanding. Chinese teachers having this profound understanding had an average of 18 years of teaching experience, and Ma speculated that several aspects of the Chinese system serve to cultivate this understanding.

First, Chinese teachers teach only three or four 45-minute class period each day; the rest of their time is spent assessing student work or preparing future lessons. Second, Chinese teachers commonly "loop up," following their students through two or three successive grades. This looping perhaps gives teachers a more thorough picture of the interconnectedness of various mathematical topics because they see how the content they taught in first grade is built upon in second grade. Third, Chinese teachers spend substantial amounts of time studying the government's framework, the textbooks, and the teacher's manual to understand how topics are sequenced, why particular examples are used, and how to best make use of the materials to accomplish the stated objectives. Ma noted that this preparation time "occupies a significant status in Chinese teachers' work" (p. 135) and that much of this studying is done with colleagues. Teachers meet weekly in teaching research groups for formal discussion of and reflection on their teaching. It is during both individual and group study of teaching materials that Chinese teachers likely encounter both the opportunity and the necessity to deepen their substantive knowledge of the mathematics content they are teaching.

Echoing Baldwin (1999), I would identify a fourth aspect of the Chinese system that perhaps contributes to the differences that Ma noted. The fact that 80% of the Chinese teachers interviewed taught only mathematics means that they were able to concentrate all of their planning time and discussion with colleagues on deepening their understanding of mathematics. U.S. teachers, by contrast, are expected to understanding and teach science, social studies, reading, language arts, health, and a host of other areas.

Analysis

There are many sources for learning about mathematics and teaching mathematics–content courses, methods courses, student teaching, teaching practice, inservice programs--but these sources are often quite disjointed. Because of the multi-faceted nature of teacher learning and the multiple contexts in which that learning occurs, it is important for researchers to conduct studies that span contexts and time. As Schram et al. (1988) noted, the fact that preservice teachers evidenced conceptual understanding of some mathematics topics in their first content course was no guarantee that this understanding was solidified or that they would apply it in other contexts. The study conducted by Swafford et al. (1997) provides a possible model for intervention studies that follow teachers from an inservice program into their classrooms and document changes in both their content knowledge and teaching practice. There is evidence to suggest that teachers' knowledge does change over time and that, in particular, it is affected by teaching experience. Fuller (1997), for example, found that inservice teachers were much more likely to provide conceptual explanations of whole number operations than were preservice teachers. Ma (1999) found that Chinese teachers who had a profound understanding of elementary school mathematics had an average of 18 years of teaching experience. Future research might pay more attention to the issue of when, where, and how teachers develop their knowledge of mathematics beyond their formal teacher education programs.

Final Comments

As has been demonstrated, there is no clearly definable body of knowledge that informs teaching. Rather, teachers need multiple types of knowledge, each of which is rather ill-defined and amorphous. Because of the enormous complexity of teaching and learning, the study of teachers' knowledge is fraught with methodological difficulties. Interview protocols that present "what if" situations involving students' mathematical reasoning are useful in providing some information about how teachers *might* bring their mathematical knowledge to bear in an instructional situation, but as Thompson (1992) warns, what someone says they will do does not always match what they actually do because a number of other mitigating factors may intervene. Studying teachers while they are engaged in classroom practice is problematic because it is impossible to capture the moment-to-moment decisions a teacher is making and the influences on those decisions. Stimulated recall interviews enable teachers and researchers to partially reconstruct chains of events, but teachers often make unconscious decisions that are difficult to articulate in retrospect. This puts the researcher in the position of making inferences from teachers' words or actions. Lacking detailed information about the learners in the classroom and the norms of that community of learners, researchers may make inappropriate inferences. Studying one's own practice, as has been done by Heaton and Ball, comes closest to allowing us access into teachers' heads, but this type of research is highly impractical on a large scale.

It seems clear that we need more in-depth studies of teachers in action in various contexts–as learners of mathematics and as teachers of mathematics. However, as we begin to accumulate a substantial collection of studies that investigate in detail the knowledge and practice of individual teachers, we must guard against viewing these studies as simply a collection of stories. Researchers must return to these stories and conduct cross-case analyses in order to begin to develop a theory about teachers' knowledge, teachers' practice, and student learning. As Cooney (1994) noted, "if we are to move beyond collecting interesting stories, theoretical perspectives need to be developed that allow us to see how those stories begin to tell a larger story" (p. 627).

The study of teaching and teachers' knowledge is as important to educational reform today as it was 40 years ago. As Shulman has noted, teachers are the key ingredient in our educational system.

...the teacher must remain the key. The literature on effective schools is meaningless, debates over educational policy are moot, if the primary agents of instruction are incapable of performing their functions well. No microcomputer will replace them, no television system will clone and distribute them, no scripted lessons will direct and control them, no voucher system will bypass them. (Shulman, 1983, p. 504)

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