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ABSTRACT

This conference proceedings contains 135 research reports, 73 short oral reports, 30 poster session reports, 4 plenary addresses, 3 research forums, 6 project groups, and 5 discussion group reports. Only the research reports, research forums, and plenary addresses are full reports; the others are generally one-page abstracts. The first volume includes: (1) "Where in shared knowledge is the individual knowledge hidden?" (R. Hershkowitz); (2) "Professional development, classroom practices, and students' mathematics learning. A cultural perspective" (G. B. Saxe); (3) "One theoretical perspective in undergraduate mathematics education research" (B. Czarnocha, E. Dubinsky, V. Prabhu, and D. Vidakovic); (4) "Cultural aspects in the learning of mathematics" (N. Presmeg and P. Clarckson); (5) "Developing skills of advanced mathematical thinking" (P. E. Kahn). The second volume includes: (1) "Teacher profile in the geometry curriculum based on the Van Hiele theory" (M. D. Afonso, M. Camacho, and M. M. Socas); (2) "Pupils' images of teachers' representations" (C. Bills and E. Gray); (3) "What kind of mathematical knowledge supports teaching for 'conceptual understanding'? Preservice teachers and the solving of equations" (D. Chazan, C. Larriva, and D. Sandow); (4) "Argumentative aspects of proving: Analysis of some undergraduate mathematics students' performances" (N. Douek); (5) "A numeracy assessment framework for the international life skills survey" (I. Gal). The third volume includes: (1) "'What Can We All Say?' Dynamic geometry in a whole-class zone of proximal development" (J. Gardiner, B. Hudson, and H. Povey); (2) "Pedagogy and the role of context in the development of an instrumental disposition towards mathematics (S. Goodchild); (3) "Alternative assessment for student teachers in a geometry and teaching of geometry course" (B-S. Ilany and N. Shmueli); (4) "Learning pre-calculus with complex calculators: Mediation and instrumental genesis" (J. B. Lagrange); (5) "This patient should be dead! Or: How can the study of mathematics in work advance our understanding of mathematical meaning-making in general?" (R. Noss, C. Hoyles, and S. Pozzi). The fourth volume includes: (1) "The research of ideas of probability in the elementary level of education" (A.-M. Ojeda); (2) "Monitoring of dynamics of students' intellectual growth in MPI-Project" (S. Rososhek); (3) "Conceptual understanding of conventional signs: A study

without manipulatives" (C. Silveira); (4) "Does the understanding of variable evolve through schooling?" (M. Trigueros and S. Ursini); (5) "Boys, mathematics and classroom interactions: The construction of masculinity in working-class mathematics classrooms" (R. Zevenbergen). (ASK)

Proceedings

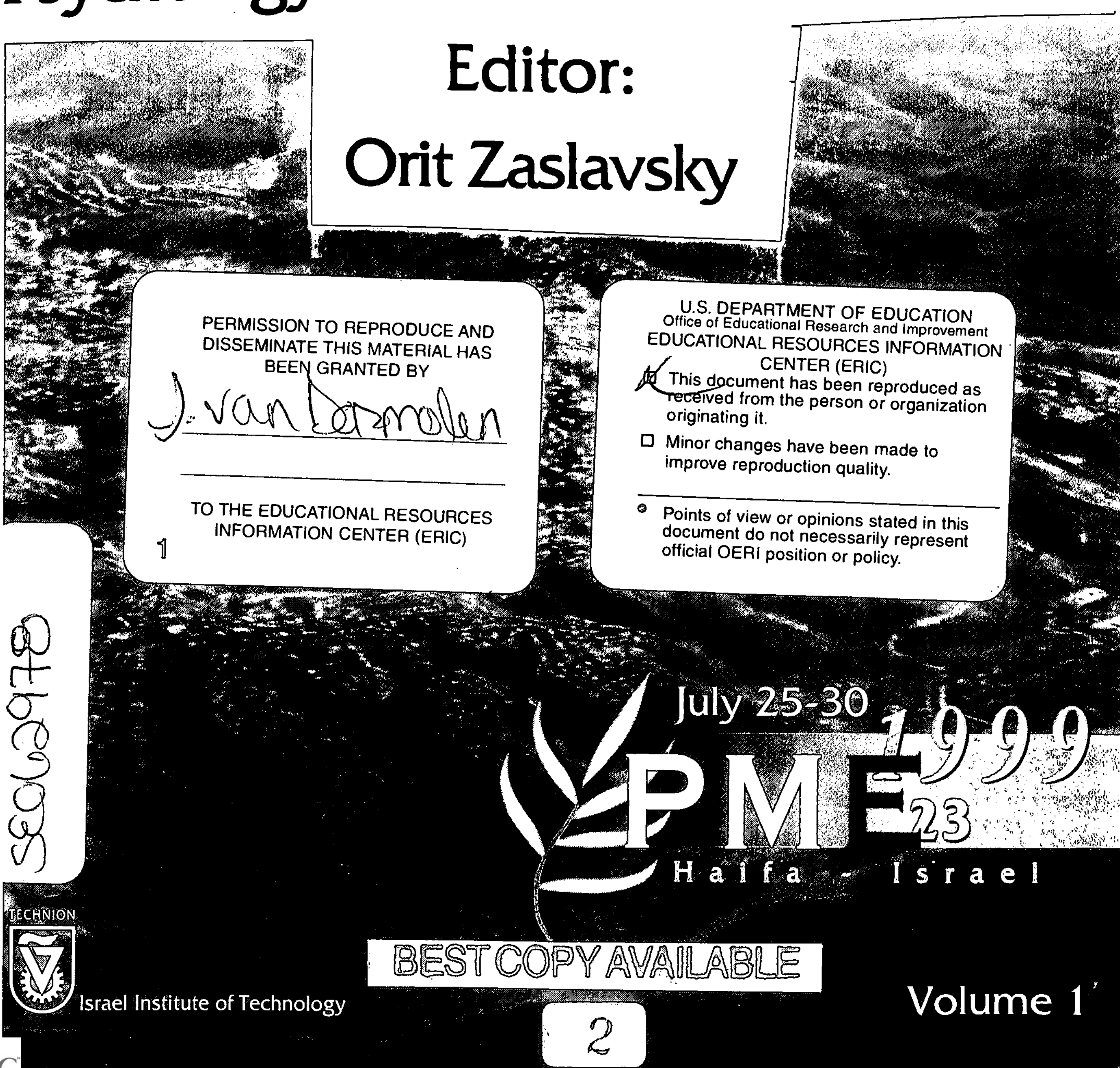
of the

23rd Conference

of the International Group for the Psychology of Mathematics Education

Editor:

Orit Zaslavsky



July 25-30 1999

PME 23

Haifa - Israel

TECHNION

Israel Institute of Technology

Volume 1

Proceedings

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23rd Conference

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Psychology of Mathematics Education

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Volume 1

Editor:
Orit Zaslavsky
Department of Education in Technology and Science
Technion - Israel Institute of Technology
Haifa 32000
Israel

Fax: +972 4 832 5445
E-mail: orit@tx.technion.ac.il

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PREFACE

It is an honor and pleasure for me to chair the 23rd PME conference in Haifa. This conference has a special meaning to all of us. It marks a year since Efraim Fischbein, the founder president of PME, left us. In his plenary address at the 20th anniversary of PME, Efraim Fischbein expressed his hope that every year we will meet and pass the *message of goodwill, of cooperation, of love for mathematics and reason, the love for our students coping with the difficulties and fascination of learning mathematics.* We share this hope and look forward to a meeting that - in addition to its scientific merit - conveys this message.

The papers in the four volumes of the proceedings are grouped according to types of presentations: Plenary Addresses, Plenary Panel, Research Forums, Project Groups, Discussion Groups, Short Oral Communications, Posters, and Research Reports. The plenary addresses and the research forum papers appear according to the order of presentation. The Groups are sequenced according to their numbers. For the other types of presentations, within each group, papers are sequenced alphabetically by the name of the first author, with the name(s) of the presenting author(s) underlined.

There are two cross-references to help readers identify papers of interest to them:
- by research domain, according to the first author (p. 1-xxvii)
- by author, in the list of authors (p. 1-369).

I wish to extend my appreciation to all the people who took part in the production of these proceedings. I am particularly indebted to Joop van Dormolen, Lea Keinan, and Doron Zur for their dedication, cooperation and endless amount of work devoted to the preparation of the proceedings.

This conference received support from many sources, without which we could not have organized it to meet PME standards. We are grateful to the sponsors, especially to the hosting institute, the Technion – Israel Institute of Technology, for the support and facilities provided to the conference organizers.

Last, but not least, many thanks to the members of the Program Committee and the Local Organizing Committee for sharing with me so willingly the responsibilities involved in this enterprise.

Orit Zaslavsky
Haifa, July 1999

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INTRODUCTION

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

History and Aims of PME

PME came into existence at the Third International Congress on Mathematics Education (ICME3) held in Karlsruhe, Germany in 1976. Its past presidents have been Efraim Fischbein (Israel), Richard R. Skemp (UK), Gérard Vergnaud (France), Kevin F. Collis (Australia), Pearla Nesher (Israel), Nicolas Balacheff (France), Kathleen Hart (UK), Carolyn Kieran (Canada) and Stephen Lerman (UK).

The major goals of the Group are:

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
- To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics educators;
- To further a deeper understanding into the psychological aspects of teaching and learning mathematics and the implications thereof.

PME Membership and other Information

Membership is open to people involved in active research consistent with the Group's goals, or professionally interested in the results of such research. Membership is on an annual basis and requires payment of the membership fees (US$30 or the equivalent in local currency) per year (January to December). For participants of PME 23 Conference, the membership fee is included in the Conference Deposit. Others are requested to contact their Regional Contact, or the Executive Secretary.
More information about PME as an association can be obtained trough its home page at: http://members.tripod.com/~IGPME (case sensitive) or through the Executive Secretary.

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PROCEEDINGS OF PREVIOUS PME CONFERENCES

Copies of some previous PME Conferences are still available for sale. For information, see the PME home page at http:/members.tripod.com/~IGPME/procee.html (case sensitive!) or contact the Executive Secretary Joop van Dormolen, Rehov Harofeh 48A, 34367 Haifa, Israel; email: joop@tx.technion.ac.il; fax: +972 4 8258071.

All proceedings, except PME 1, are included in ERIC. Below is a list of the proceedings with their corresponding ERIC codes.

PME International

No.	Year	Place	ERIC number
1	1977	Utrecht, The Netherlands	not available in ERIC
2	1978	Osnabrück, Germany	ED 226 945
3	1979	Warwick, United Kingdom	ED 226 956
4	1980	Berkeley, USA	ED 250 186
5	1981	Grenoble, France	ED 225 809
6	1982	Antwerpen, Belgium	ED 226 943
7	1983	Shoresh, Israel	ED 241 295
8	1984	Sydney, Australia	ED 306 127
9	1985	Noordwijkerhout, The Netherlands	ED 411130 (Vol. 1), ED 411131 (Vol. 2)
10	1986	London, United Kingdom	ED 287 715
11	1987	Montreal, Canada	ED 383 532
12	1988	Veszprem, Hungary	ED 411128 (Vol. 1), ED 411129 (Vol. 2)
13	1989	Paris, France	ED 411140 (Vol. 1), ED 411141 (Vol. 2), ED 411142 (Vol. 3)
14	1990	Oaxtepex, Mexico	ED 411137 (Vol. 1), ED 411138 (Vol. 2), ED 411139 (Vol. 3)
15	1991	Assisi, Italy	ED 413 162 (Vol. 1), ED 413 163 (Vol. 2), ED 413 164 (Vol. 3)
16	1992	Durham, USA	ED 383 538
17	1993	Tsukuba, Japan	ED 383 536
18	1994	Lisbon, Portugal	ED 383 537
19	1995	Recife, Brazil	ED 411134 (Vol. 1), ED 411135 (Vol. 2), ED 411136 (Vol. 3)
20	1996	Valencia, Spain	being processed
21	1997	Lahti,Finland	being processed
22	1998	Stellenbosch, South Africa	being processed

PME North American Chapter

No.	Year	Place	ERIC number
2	1980	Berkeley, California (with PME2)	ED 250 186
3	1981	Minnesota	ED 223 449
4	1982	Georgia	ED 226 957
5	1983	Montreal, Canada	ED 289 688
6	1984	Wisconsin	ED 253 432
7	1985	Ohio	SE 056 279
8	1986	Michigan	ED 301 443
9	1987	Montreal, Canada (with PME11)	ED 383 532
10	1988	Illinois	ED 411 126
11	1989	New Jersey	ED 411 132 (Vol. 1), ED 411 133 (Vol.2)
12	1990	Oaxtepex, Mexico (with PME14)	ED 411137 (Vol. 1), ED 411138 (Vol. 2), ED 411139 (Vol. 3)
13	1991	Virginia (with PME16)	ED 352 274
14	1992	Durham, New Hampshire	ED 383 538
15	1993	California	ED 372 91
16	1994	Louisiana	ED 383 533 (Vol.1), ED 383 534 (Vol. 2)
17	1995	Ohio	ED 398 534
18	1996	Panama City, Florida	ED 400 178
19	1997	Normal, Illinois	being processed
20	1998	Raleigh, North Carolina	being processed

The ERIC abstracts can be read on the Internet site of AskEric (http://www/askeric.org).
Micro fiches with the content of the proceedings may be available for inspection at university libraries.
You can also inquire about it by contacting:
ERIC/CSMEE, 1929 Kenny Road,
Columbus, OH 43210-1080
Tel: (614) 292-6717
Fax: (614) 292-0263
e-mail: ericse@osu.edu

THE REVIEW PROCESS OF PME23

Research Forum

Four themes had been suggested by the Program Committee as research forum themes for PME23: *Learning and Teaching Undergraduate Mathematics; Becoming a Mathematics Teacher educator; Visual Thinking in Mathematics;* and *Assessment, Learning and Mathematics.* The Program Committee received 12 research forum proposals for these themes (4 for the first theme, 3 for each of the second and third themes, and 2 for the latter). For each theme, all the proposals were reviewed and ranked by three reputable scholars with expertise in the respective fields. The Program Committee considered and generally accepted the research forum coordinators' evaluations of the reviews and their ranking of the proposals. Consequently, 2 proposals were selected for each of the first three themes. The two proposals of the latter theme did not seem to provide a rich and wide enough scope of the field, thus, it was decided to cancel this forum.

Research Reports

The Program Committee received 202 research report proposals. Each proposal was sent for blind review to three reviewers. As a rule, proposals with at least two recommendations for acceptance were accepted. The reviews of proposals with only one recommendation for acceptance were carefully read by at least two members of the Program Committee. When necessary, the Program Committee members read the full proposal and formally reviewed it. Proposals with 3 recommendations for rejection were not considered for presentation as research reports. Altogether, 136 research report proposals were accepted. When appropriate, authors of proposals that were not accepted as research reports were invited to re-submit their work -- some in the form of a short oral communication and some as a poster presentation.

Early Bird Proposals

The Program Committee received 21 early bird research report proposals. Each proposal was sent to 3 reviewers who were asked to suggest ways to improve the proposal for resubmission as a research report. Of the early bird proposals 18 were re-submitted as research reports and one re-submitted as a short oral. Altogether, of the 21 early bird proposals 16 were finally accepted as research reports and 3 were accepted as short oral communications.

Short Oral Communications and Poster Presentations

The Program Committee received 72 short oral communication proposals and 17 poster proposals. Each proposal was reviewed by at least two Program Committee members. Altogether, 59 short oral proposals and 12 poster proposals were accepted. There were cases in which the program committee did not accept a proposal in the form that it was intended but invited the author(s) to present it in a different form.

LIST OF PME23 REVIEWERS

The PME23 Program Committee thanks the following people for their help in the review process:

Adler, Jill (South Africa)
Aharoni, Dan (Israel)
Ainley, Janet (UK)
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INDEX OF PRESENTATIONS BY RESEARCH DOMAIN

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Affective Factors

Algebraic Thinking

Epistemology

Functions and Graphs

Gender Issues

Geometrical and Spatial Thinking

Imagery and Visualization

Probability and Combinatorics

Problem Solving

Rational Numbers and Proportion

Socio-Cultural Studies

Teacher Education and Professional Development

Theories of Learning

A TRIBUTE TO EFRAIM FISCHBEIN

Efraim Fischbein, 1920-1998, Founder President of PME
A Tribute

David Tall

The 23rd meeting of the International Group for the Psychology of Learning Mathematics in Israel is touchingly the first in which we cannot be joined by our Founder President, Professor Efraim Fischbein, who left us on July 22nd 1998. It is a time of sadness, yes, but it is also a time for celebrating the achievements of this gentle man who is responsible for the existence of our organisation. In particular, it is to him that we owe our focus on the *psychology* of learning mathematics.

Efraim Fischbein was born in Bucharest on January 20th, 1920. He was a precocious child who learned to read Hebrew from the old testament at the age of three. He spent his formative years in Romania where he had to cope with the hardship of living in a rising fascist state. When World War II broke out he was forced into hard labour with other Jewish youths. His sight was seriously damaged and at the end of the war he prepared for his university examinations by listening to the reading of friends and conversation with his fellow students. He graduated at the Bucharest University in 1947 with an MA in Psychology and the qualification to teach mathematics in high school.

His first activity was to travel to Transylvania to care for a hundred orphans who were survivors of death camps. In 1948 he returned to Bucharest as a high school teacher and then, in parallel, as a lecturer in developmental psychology at the university. His long association with the University of Bucharest culminated as head of the Department of Educational Psychology from 1959 to 1975.

He was a prolific author of articles and books during this time, including the first original Romanian text-book on Psychology (*How do we know the world*, 1958). His monograph *The Figural Concepts: the nature of geometric entities and their development in children* was published in 1963 and accepted for his PhD. Other titles published in Romanian include: *The Man, Master of His Habits* (1955), *Concept and Image in Mathematics Thinking* (1965), *The Art of Thinking* (1968), *Hazard and Probability in Children's Thinking* (1974).

He caught the eye of the international mathematics community and was invited to address the first International Congress in Mathematics Education in 1969. His outstanding presentation on "Enseignement mathémathique et développement intellectuel" and his rising eminence led to his invitation to chair the Working Group on the Psychology of Mathematics Education at the second ICME conference in 1972. This highly successful working group continued under his chairmanship at the third Congress in

Karlsruhe in 1976 where the participants voted to continue with conferences every year as "The International Group for the Psychology of Mathematics Education". Efraim Fischbein was elected its founder president and served in this role from 1976 to 1980. Meanwhile he was appointed Professor of Psychology and founder chairman of the School of Education in Tel-Aviv University in 1975. He remained here for the rest of his life, with many visiting positions abroad, at Nottingham, UK, Montréal, Canada, Pisa, Italy, Georgia, USA, Heidelberg, Germany, and Granada & Valencia in Spain.

He continued to publish prolifically throughout his working life, including books in English on *The Intuitive Source of Probabilistic Thinking* (1975) and *Intuition in Science and Mathematics* (1987). His articles are a model of carefully designed research methodology and generative theory. Although well-versed in the methods of psychology, he was critical of its limited application to mathematics and saw that the psychology of mathematics education must develop its own theoretical perspectives.

His greatest creation is surely the organisation to which we belong. Following the decision to meet annually at Karlsruhe in 1976, the first meeting occurred the following year at Utrecht, organised by Hans Freudenthal. At Osnabrüch in 1978 the organisation was formally constituted under the title "International Group for the Psychology of Mathematics Education", subsequently shortened from IGPME to PME.

I remember vividly the talk he gave at PME in 1978, for it was to change my whole professional life. He presented his empirical and theoretical ideas on individual conceptions of infinity. His slim, wiry frame resonated with vigour and emotion as he passionately advocated the theoretical implications of his empirical findings. His enthusiasm had a profound effect on me personally. My own, previously solitary studies in undergraduate thinking suddenly began to take their place in the wider picture that he painted. It inspired me to make the study of limits and infinity—and broader research in undergraduate mathematics—as the focal point of my studies at that time. By 1985 a growing interest in this area led to the formation of the Advanced Mathematical Thinking group. Thus it was that Efraim's interest in the psychology of school mathematics permeated through to mathematics education at all levels.

He had a salutary wisdom that challenged those who professed to wear Emperor's clothes. I remember explaining to him that I could "see" an infinitesimal as a graph that tended to zero. He challenged me forcefully, saying: "*Show* me an infinitesimal". I was taken aback. I could not do it. Though I could formulate the formal mathematical framework, I had never analysed what it was that made the ideas work cognitively. It took a perceptive genius to ask the right question that cause a new theory to blossom. In my own case, this question from Fischbein spurred the journey of a life-time as I struggled to understand the relationship between conceptions to *think* about mathematics and processes that allow us to *do* it.

It is a salutary thought that he continued in vigour in his sixties and seventies, producing books and research articles of great quality at a time when many others have taken a well-earned retirement. At almost every conference of PME it has been my

privilege and delight to take my turn amongst his many friends and colleagues who sought his wisdom and advice.

His research has a subtle balance between theory and empirical evidence that has always been the hallmark of his scholarship. It is these qualities which should continue to mark our present and future work in PME.

His work on primary and secondary intuitions, on children's probabilistic thinking, on the complex meaning of infinite concepts and on intuition in both mathematics and science have been seminal. They provide us with fundamental notions on which we can continue to build into the future. While we lament his passing, we therefore rightly celebrate his achievement and his legacy:—the gift of "PME" which draws us together every year to pursue our continuing quest to understand the subtleties of psychological studies in mathematics education.

Farewell dear friend, our journeys continue in your footsteps.

* * * * *

A selection of publications by Efraim Fischbein in the last decade:

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PLENARY ADDRESSES

Rina Hershkowitz
Geoffrey Saxe
Heinz Steinring
Kenneth Ruthven

WHERE IN SHARED KNOWLEDGE IS THE INDIVIDUAL KNOWLEDGE HIDDEN?

Rina Hershkowitz

The Weizmann Institute of Science, Israel

In more and more research and development projects, it is rightly assumed that "multiple interaction learning environments" (among students, between students and teacher, with the tool or with the task, etc.) are desirable for a meaningful construction of knowledge. It is then natural that social interaction within the classroom community is currently the object of intensive investigation. However, the individual as the one who uses the constructed knowledge, and shares it with others in various communities, has been neglected in these investigations. The relationships between construction of shared knowledge within a community and the individual construction of knowledge, are discussed and exemplified. Related issues concerning research paradigms and methodologies are also raised.

Background

Coming from the field of curriculum development, I view research questions from within a very comprehensive setting, which includes:

(1) Design considerations before starting the actual development and research work;

(2) Implementation of activities in a few classrooms, accompanied with observations of learning and teaching practices;

(3) Analysis of the data collected in order to redesign sequences of activities towards the creation of a complete curriculum (compatible with an official syllabus);

(4) Dissemination of the curricular aims and "spirit" on a national scale;

(5) Initiation of a new cycle of curriculum development (i.e. back to (1)).

About 6 years ago, we began a new cycle of such a curriculum development and research program at the Weizmann Institute, the CompuMath Project -- learning mathematics with computerized tools. Like others (e.g., Balacheff, & Kaput, 1996; Roth & Bowen, 1995), I believed that "multiple interaction learning environments", in the form of activities among students and/or between students and teacher through the

mediation of (computerized) tools, are desirable for a meaningful construction of individual student knowledge. Computerized tools were chosen and used because of their amplifying and reorganization capabilities, as mediators in meaningful learning of mathematics (Dörfler, 1993; Kaput, 1992; Pea, 1985). Activities were designed as multi-phased open-ended problem situations. For a detailed description of the CompuMath project, see Hershkowitz and Schwarz (1999). Experienced teachers, who usually belonged to the project team, first implemented activities in a few classrooms.

We were carried away by the overwhelming and surprising processes we observed in these classrooms, which differed from what we had observed in the last two decades of development and research. We confess that our excitement was not only due to the new activities and environment we created, but also because we decided to see the reality through new lenses. The need to describe, understand, explain (to ourselves as well as to others) and analyze what was going on in the classroom, naturally pushed us closer to the concerns of socio-cultural psychology. Like many others (e.f. Perret-Clermont, 1993; Yackel & Cobb, 1996), we felt the shortcomings of the cognitive theories, methodologies and tools we had at our disposal to describe and interpret learning and teaching processes in the classroom. We needed different kinds of units of analysis by which we would be able to describe meaningfully learning practices (Kuutti, 1996) and an appropriate "zoom of the lens" (Lerman, 1998) to observe, document, analyze and explain.

The research, which is an integral part of our work, is mostly from an interactionist perspective: the construction of knowledge is analyzed while students are investigating problem situations in different contexts. For example, we studied pairs of students in peer work during classroom activity or in an interview situation (Hadas & Hershkowitz, 1998, 1999), collaboration of a small group of students to solve a problem followed by a whole classroom discussion (Hershkowitz & Schwarz, 1997; 1999; Schwarz & Hershkowitz, 1995), or individual students interacting with a researcher during an interview at the end of a course (Dolev, 1997; Hershkowitz, Schwarz & Dreyfus, submitted).

The dilemma I would like to raise and discuss here arose and took hold during the research I have already done, and becomes clearer as I plan my further research program. My research is embedded in comprehensive socio-cultural paradigms of current cognitive research, therefore the controversies and questions I would like to discuss, are in a way derived from, and relevant to, these kinds of research as a whole.

The dilemma focuses on the individual construction of knowledge within the different "ensembles" of which he or she is part. (I prefer to use the term “ensemble” rather than the general term "community". This term, defined by Granot (1998), designates "the smallest group of individuals who directly interact with one another during developmental processes related to a specific activity context"). It raises questions about the difficulties in “zooming with our lens” on the individual’s development as

he/she participates in the collective construction of shared cognition, in an ensemble (a couple of students or a small group), or in the whole classroom community.

In the following I will attempt to detail the dilemma through considerations and questions related to theory, research and methodology. The different questions will emerge from, and be demonstrated by, the presentation and analysis of a few examples.

Theoretical Considerations

Following Vygotsky, I consider the psychological development and learning of the individual as participation in the social activity of his/her own community. For Vygotsky the relationships between the two poles -- the individual and the social -- are asymmetrical. In an article in which he argued and criticized Piaget's theory of the child's speech and thought, he expressed his principle concerning the development of thinking:

> In our conception, the true direction of the development of thinking is not from the individual to the socialized, but from the social to the individual (Vygotsky, 1986. p. 36).

Vygotsky's research concerning the ways in which social communicative speech becomes "inner speech" -- a mental function of the individual, was the basis of the above theoretical claim, as well as additional theoretical claims of his colleagues and successors. The influence of these theories on socio-cognitive research in the last decade is expressed in more symmetrical perspectives.

For example, in a chapter entitled "The zone of proximal development: Where culture and cognition create each other", Cole (1985) expresses what he calls "culturally grounded theory of cognition", and explains the incorporation of the "activity" as a unit in which the study of "both systems of social relations and of internal (cognitive) activity" can be done (p. 159). Cobb, (1998) in his plenary lecture, claimed that the relationships, between the two perspectives, are reflexive, in the sense that one does not exist without the other. This means that the psychological perspective implies that "one analyzes individual students' reasoning as they participate in the practices of the classroom community" and the social perspective implies that "communal practices are continually generated by, and do not exist apart from the activities of the participating individuals" (p. 44).

Pontecorvo (1993) in her opening paper to the special issue of Cognition and Instruction, which aimed "to identify and describe the socio-cognitive mechanisms through which thinking and learning are developed in different types of socially interactive settings" (p. 192-193), summarizes some "general presuppositions" common to research in such settings:

1. The attempt to look at the interactive situation from the *Participant's perspective* which is expressed in questioning "the meaning that participants attribute to the interactional setting".

2. The assumption that "there is a continuous interchange between the *interactional event* and the *context*, where the context is considered as the "cultural frame that surrounds the specific interactional event and provides resources for its enactment and interpretation".

3. The relevant role-played by the *Other* - who is assumed to be "an active co-participant in speech and in action, who gives support for acting, understanding, and reasoning through both agreement and opposition".

The above three points represent a modern perspective of socio-cognitive studies (see, for example, some studies in the above special issue of Cognition and Instruction, 1993, as well as some of our papers mentioned above). These studies focus mostly on the *interaction* or the *interactional event* itself. The individual student is mostly an anonymous participant (even when he/she is named) in classroom episodes, which are selected and analyzed with the intention of highlighting the social context in which some cognitive processes take place. But, as "short stories" of quite complex situations, these episodes do not have the potential to focus on one student's cognitive changes while participating in such episodes. Cobb, in a joint paper on "learning mathematics through conversation" (Sfard, Nesher, Streefland, Cobb & Mason, 1988), criticizes the kind of instruction done in order "to shape classroom discourse", where "the issue of whether students might be learning any mathematics that is worth knowing is not a focus of investigation" (p. 46).

I would like to stress that, on the one hand, I consider the interactional processes and the "shared construction of knowledge" within the classroom community, that may be grasped at least partially in such episodes, as very important, and justifies the heavy research that has been done lately. On the other hand, I cannot ignore the fact that, in the end, the individual is the smallest autonomist "unit" that can carry his/her constructed knowledge to different communities even simultaneously (the pair, the small group, the whole classroom community), and share it with other individuals in various communities during a lifetime. One way is to investigate the individual actual knowledge at the end of an interactive learning process (Perret-Clermount, 1993; Schwartz & Hershkowitz, in press). But, this will tell us mainly about "end-products", and very little about the interactive processes of constructing this knowledge. The individual's work takes place in the work of teams, so I believe that we need, in addition, to investigate the developmental processes of his/her knowledge while participating in team work. These teams tend to have a short life, meaning that the individual belongs to numerous teams in different communities in her/his productive years. In a study on collaborative problem solving, Granot (1998) showed that ensembles formed, separated, and reformed naturally among participants within the

same class activity. In this sense learning can be seen as "a changing membership of communities of practice" (Lave & Wenger, 1991, p. 54). And therefore, as noted by Pontecorvo (1993), the question of the "types of knowledge or of socio-cognitive tools that can be accessed, built, and changed through collective discourse can be subsequently used in other settings, including individual problem solving, writing tasks, and answers to an interview", becomes very important.

The Dilemma through Empirical and Methodological Eyes

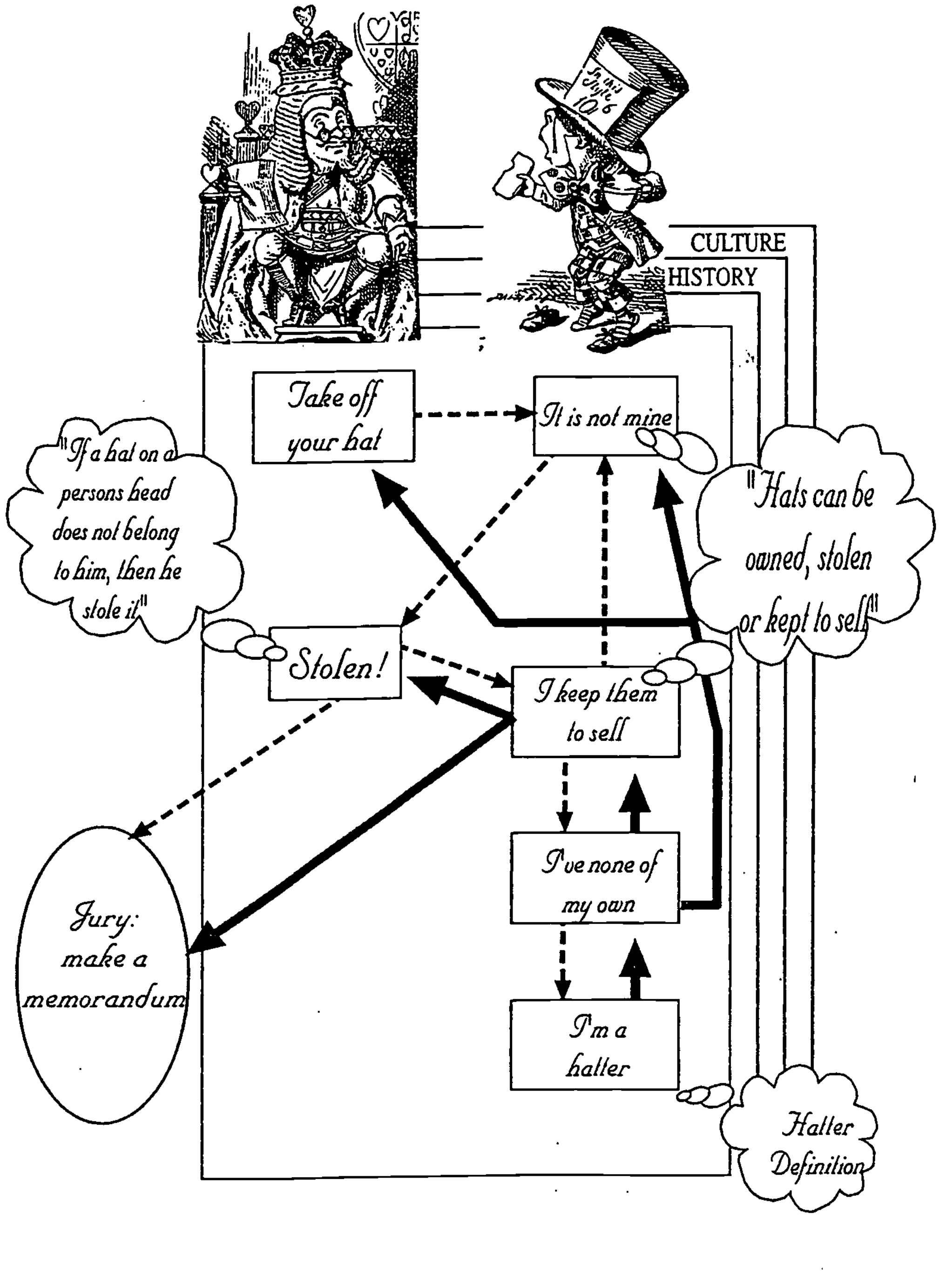
As can be concluded from the above, a central goal in empirical studies of socio-cognitive development is an empirical investigation of the individual development (learning) within the socio-cultural context of which he or she is part. I would like to use Lerman's metaphor (1998) about the zoom of a lens, and to raise the following question: If we put our lens on a socio-cognitive activity, can we, at the same time, zoom on the individual, and if yes, what we might see? I would like to discuss this issue through a presentation and analysis of some examples. The first example is borrowed from "Alice's Adventures in Wonderland" (Carroll, 1865/1965). The second stems from the investigative work done by Michal Tabach, a member of our team. The other two, which will be discussed briefly, derive from two studies that have already been completed and reported (Hershkowitz & Schwarz, 1999, and Hershkowitz, Schwarz, & Dreyfus, submitted).

The King and the Hatter Example

I have chosen a short dialogue between the king and the hatter from Lewis Carroll's book, as a parable to demonstrate the complexity of describing and interpreting individual cognitive behavior (in this case, without any evidence of learning or cognitive change) in an interactive situation. This episode is taken from the beginning of the trial in chapter 11 (see the following figure). The sharpness of Carroll's logic, the brevity of the episode, and the fact that there are only two "players", make it easy to follow and analyze the argumentation between the two figures.

I use different sets of arrows to show different kinds of links between the utterances of the king and the hatter. If we read along the arrows of the first set (dotted lines), we have the parts of the episode in their chronological order. In the second set (bold lines), I have tried to demonstrate the logical structure which underlies these utterances. For example "I keep them to sell", said the hatter *to contradict* the king's claim: "Stolen!", as well as *to explain* his utterance "It is not mine" to the public at the trial, including the jury, etc. We focus here on arguments of *contradiction, explanation,* etc., which are relations between an utterance and previous utterances or gestures. Therefore, the

arrows, in the second chain, are mostly in the opposite direction to the arrows in the first.



(From Lewis Carroll, Illustration John Tenniel)

The main frame is embedded in a sequence of other frames: First, our immediate interpretation (as notes within "clouds") of this episode and the roles of the two players in it. For example, for the king, a hat on a person's head can be interpreted as either belonging to this person, or stolen. The "hatter definition", given by the hatter himself, includes a third possibility: a hat can be kept by a person "to sell". This way of presentation and analysis of interactive episodes, which we will call "the hatter method", is quite close to other analysis methods used in various studies on interaction (e. g., Resnick, Salmon, Zeitz, Wathen, & Holowchak, 1993).

If we wanted to come closer to these two figures, and be able to grasp their different character and their different ways of behavior and thinking, we would need to know more about the contextual frames of the king and the hatter - their history, culture, etc. This could be discovered, at least partially, from other episodes in which the king and the hatter interact with other figures. Fortunately, Lewis Carroll was kind to us and left us the whole *book* on Alice's adventures.

If we leave our fable, and come back to learning situations in an interactive environment, we have to face questions such as: What kind of *book* do we need in order to be able to trace the development of an individual in interactive situations? Do we have proper methodologies with the help of which this *book* can be written and interpreted? Do we have proper psychological tools to follow and analyze the individual cognitive behavior, while observing it? It is more realistic to relate to the above questions in the context of a real classroom situation.

The example of the pocket-money activity

The pocket-money activity is part of the year-long algebra course for seventh graders, which we have developed in the CompuMath project. The course consists of a sequence of multiphase problem situations and the use of a multi-representational tool (a spreadsheet). In the "pocket-money" activity, which was given about two months after the beginning of the year, students first investigated the weekly pocket-money savings of three children governed by different linear rules. After a few days, a new rule was proposed:

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Sharon receives pocket-money each week as follows: At the end of the first week she received 2ag (0.02 shekel). Each week she receives a sum equal to what she already has in her pocket. Sharon does not spend any money.

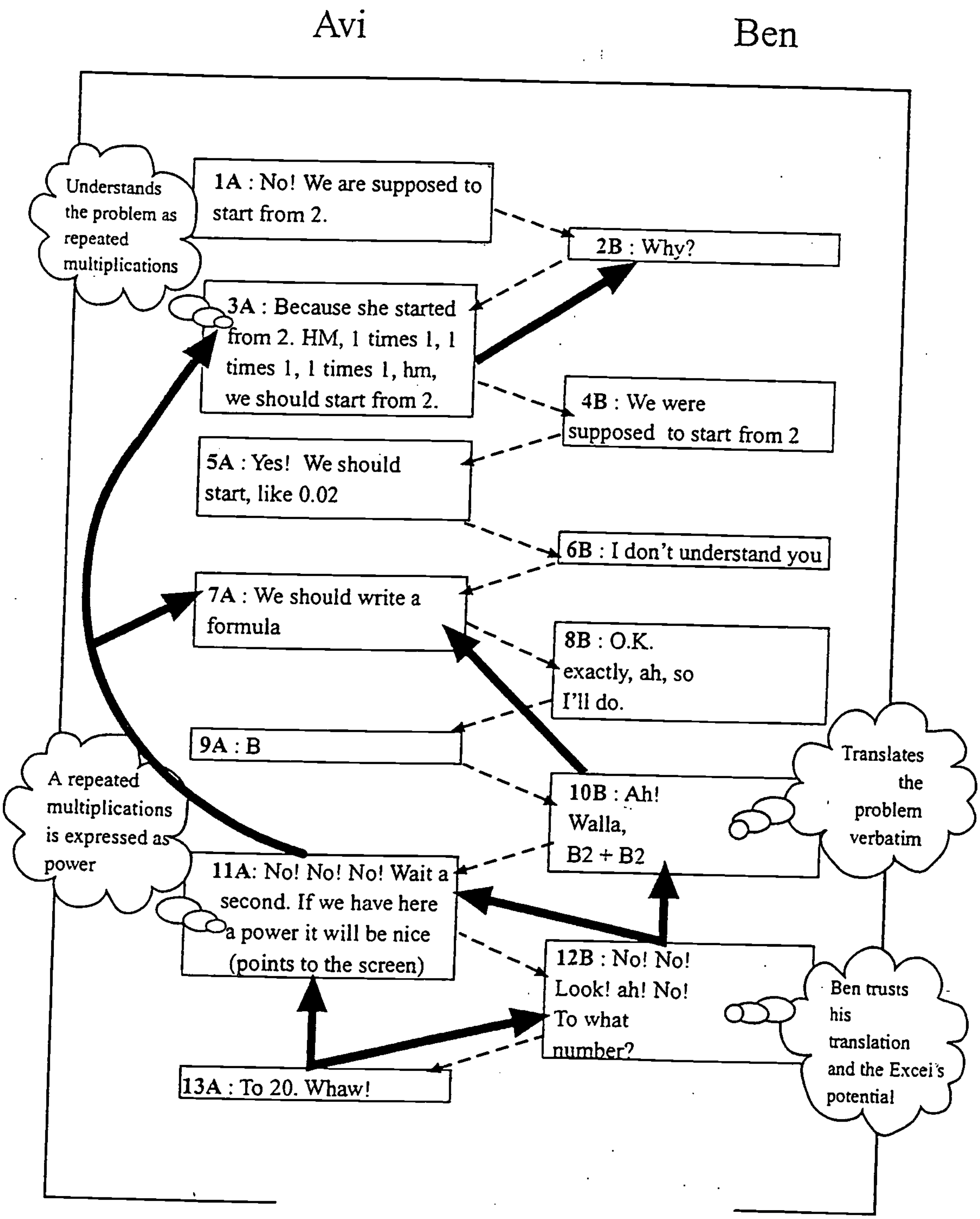
==

Students were asked to hypothesize the rate of growth of Sharon's savings as compared to other children's savings, before obtaining the exact data from the generalized rule found through the mediation of the spreadsheet (Excel). The

investigation was done in peer collaboration. It had two steps. First students were asked to hypothesize numerically the growth of Sharon's savings. After a "symbolic negotiation" with the Excel program (in order to write the savings rule in Excel symbols) and dragging down to get the weekly totals of Sharon's savings, they could compare the totals with their hypothesis. In a second step, they were asked to hypothesize the shape of the graph of Sharon's saving from the numerical data they had. Finally, after a "graphic negotiation" with the tool, they were asked to compare the graph they sketched with the graph they drew by the tool. Michal Tabach, investigated the construction of knowledge of few pairs of students working on such activities through the year. As a silent observer, she documented the work of two boys, Avi and Ben, on the above activity.

The figure on the next page is a presentation and analysis of one episode, in which the two students are involved in the "symbolic negotiation" with the Excel program to obtain the numerical data of the weekly sums of money.

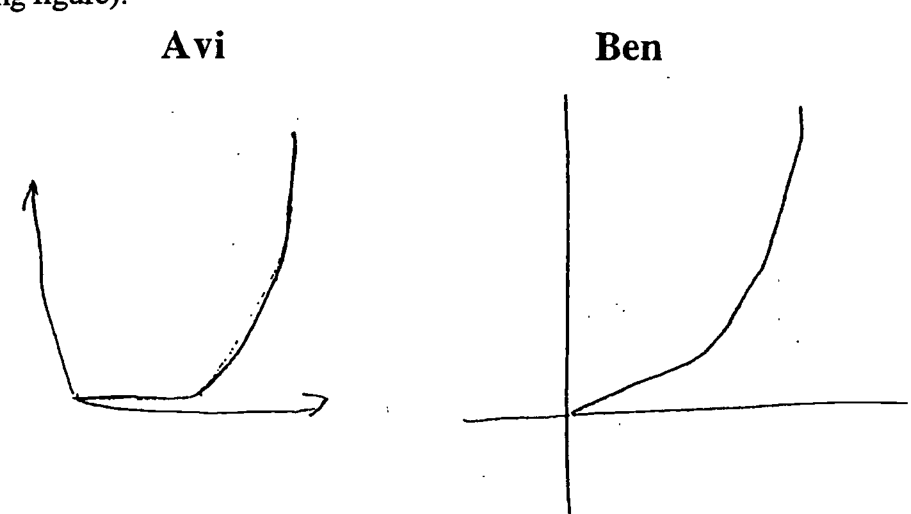
In the analysis of this episode I have used "the hatter method" — two sets of arrows in the main frame, and interpretations "around" it. The chronological set of arrows seems to indicate a symmetric interaction between the two; each utterance is immediately followed by an utterance by the interlocutor. A deeper analysis, expressed by the second set of arrows (in bold), and the interpretations inserted outside the main frame, presents a more complicated situation. The episode starts from Avi's claim: 1A: "We are supposed to start from 2". It is impossible to know whether he meant that the amount of money in the first week is 2 ag., or that the savings column has to start from B2, or something else. He then explains to Ben (who asks "*why?*" (2B)) his idea (3A): "1 times 1, 1 times 1, we should start from 2". We interpreted this (see the "balloon" interpretations) as first evidence that Avi sees the growth, at least implicitly, as "repeated doubling", and if expressed as a power with "base" 1, there is no growth. From here until the end of this episode, the discourse between the boys does not seem to evidence a significant cognitive interaction. Avi ignores Ben's need for understanding (expressed in 6B), and continues telling him what should be done (7A). The two, work on the formula to be inserted in the weekly savings column, along separate lines – as expressed by the fact that most of the bold arrows connect utterances said by the same boy. Avi (11A) looks for a formula based on a power operation, while Ben immediately translates the saving rule "verbatim" into Excel symbols — B2 + B2 (10B), drags to obtain the whole picture numerically, and shows it to Avi (12B). The two boys share the surprise caused by the numerical representation of the exponential growth.


Avi
Ben
Understands the problem as repeated multiplications
1A : No! We are supposed to start from 2.
2B : Why?
3A : Because she started from 2. HM, 1 times 1, 1 times 1, 1 times 1, hm, we should start from 2.
4B : We were supposed to start from 2
5A : Yes! We should start, like 0.02
6B : I don't understand you
7A : We should write a formula
8B : O.K. exactly, ah, so I'll do.
9A : B
10B : Ah! Walla, B2 + B2
Translates the problem verbatim
A repeated multiplications is expressed as power
11A: No! No! No! Wait a second. If we have here a power it will be nice (points to the screen)
12B : No! No! Look! ah! No! To what number?
Ben trusts his translation and the Excel's potential
13A : To 20. Whaw!

Many things can be learned and many questions can be asked on this episode:

- We have some information on the kind of interaction between the two boys, and what each of them contributed (did not contribute) to the "interactive/non-interactive" character of their work in this episode. Kieran & Dreyfus (1998), who analyzed types of interaction between two boys solving problems together, observed that moments when one boy participates in the "universe of thought" (Trognon, 1993, p. 341) of the other, were quite rare. The episode above is similar, in the sense that the universes of thought of Avi and Ben intersect only towards the end, where they realize together the rate of growth.
- It seems that both of them constructed some knowledge. Is this knowledge "shared"? NO! Because each of them has his own pace, and YES! Because both of them got some feeling of exponential growth, through the same activity in the same time.
- We can also explain the mediation role of the spreadsheet, where a combination of a "local symbolization" -- B2 + B2 -- and the dragging operation, provide Ben with a sort of a global rule of the growth, and because it is so simple and so fast, it has a convincing potential that attracts both students.

But, there is not much information on the individual construction of knowledge. Does Avi grasp a more mathematical construct of an exponential growth? If yes, how come that Ben was the first one to put his hands on the numerical data which express this growth? And does this numerical representation of the exponential growth prevent Avi from further development of an exponential construct? We can make some hypotheses and interpretations inspired by what was observed, but cannot be considered as resulting from the observations. Fortunately, in this activity, we have more data on the actions of the boys, which give us one more clue: --Students were asked to hypothesize the growth graph, and each boy proposed his own graph (see the following figure).



Ben's graph looks like a half of parabola, whereas Avi's graph is close to the exponential. While sketching his graph, Avi added verbally: "at the beginning it grows slowly and slowly and then it grows faster and faster".

But we still do not know much about whether each one of them constructed knowledge about the exponential change, and if this knowledge was consolidated, in the sense that each boy could use it as an artifact in further construction of knowledge. To be able to answer this experimentally means *to write a book* for each boy working on further activities, in different ensembles or individually. Suppose we do just that, and that we had more data, do we have a proper methodology to analyze it? Is a methodology like that used in the hatter model useful when we pass from one episode in one activity, into a sequence of episodes in sequence of activities? How can we follow the individual when we have more than two participants in small group teamwork, or in a whole class interactive discourse? I shall try to come closer to the last question in the third example.

Where in the "Overseas Activity" is the individual knowledge hidden?

In a study on reflective processes in an interactive mathematics classroom (Hershkowitz & Schwarz, 1999), we followed a group of four girls through the phases of the Overseas Activity. A class of 40 students worked first individually during a preparatory phase (phase 0). Then they collaborated to investigate and solve a problem in small groups (phase 1), and subsequently wrote group reports (phase 2). Finally they engaged in a teacher led discussion, in which different groups verbally reflected on the processes they underwent, as well as on their learning styles (phase 3). I went back to the protocols of our target group, while investigating and solving the problem (phase 1), and while reporting to the class community their investigating processes (phase 3), and to the analysis and interpretations we did of these protocols. We claimed there that "we juxtapose the cognitive-individual and the discursive-interactional perspectives, without suggesting priority for either" (p. 78). Looking back, through the dilemma eyes, it might be concluded that the cognitive-individual perspective meant for us the presentation and explanation of the girls' cognitive contribution as individuals to the "shared cognitive process" of the group. This process was expressed in actions aimed to obtain the most appropriate hypothesis about the right solution, and at the end to elaborate the solution itself. This time I tried to go in the opposite direction, from the social to the individual. I tried to re-analyze the data by zooming on one individual girl, and to see what can be said about the contribution of the group interactive discourse to her learning. I found only a few clues and identified the following difficulties:

- As in the previous example, I felt that one episode (which in this case was quite long) is too narrow. We need a kind of *book* with more chapters on this girl, before and after this episode. Can we *write* such a book for one girl? For all four?

- The documentation of this episode was meant to show the whole group investigation process, so the camera changed its focus from one dominant participant to the other, and the documentation of each individual was incomplete. In spite of our desire to see both directions within the same activity, it seems that we have to document and analyze data in two different ways within the same situation.

- I have not (yet!) found a proper methodology to analyze and interpret *one girl's cognition* within the interactive discourse of the four girls. The "hatter method", or other similar methods, become very heavy when applied to four participants, even in only one episode.

- We can learn about the cognitive development of the group through verbal utterances and gestures. The individual development is dependent or expressed in cognitive mental functions which, by their nature, are more hidden. It seems that research should be done on "translating" these mental functions to actions that can be observed, analyzed and interpreted. At this point we can rely and come back, in a way, to the research and theoretical work done in classic cognitive research. The question whether this kind of research can be incorporated in investigations done within the "interactionist perspective", and in what ways, becomes quite crucial. I discuss this briefly in the next example.

Investigating Abstraction in Context

We undertook a sequence of studies (of which only the first one has recently been completed) to investigate the mental activity of abstraction as embedded in its socio-cultural mathematical context. In this first study (Hershkowitz, Schwarz & Dreyfus, submitted), which is based on an interview of a Grade 9 student, we elaborated a special approach to abstraction, as an activity of reorganizing previously constructed structures into a new structure (see the introduction to the paper). Thus, the interview was designed to enable and to observe the emergence of new knowledge constructed out of already acquired knowledge. In the analysis of the interview we accounted for the available tools (a grapher and a calculator), the social dimension of the interaction between interviewer and student and the history of the student. This analysis led us to propose a model for the genesis of abstraction on experimental grounds. At the center of the model, we identified three epistemic actions, which are dynamically nested in each other within the flow of the student's actions. The action of *Constructing* a new mathematical construct, in which actions of *Recognizing* and *Building with* already constructed knowledge, are nested. *Constructing* is an action of reorganization of knowledge to a new construct. By *Recognizing* we meant the action in which the student makes use of a construct or structure which has been constructed earlier.

Building with is an action performed by the student to combine structural elements in order to achieve a given goal.

We also suggested a longitudinal dimension to the above model, which takes into consideration the history, as well as future activity, as part of the proposed model. This means that artifacts that result from earlier constructions mediate the three epistemic actions, and the *Constructing* action leads to artifacts available for later epistemic actions. This longitudinal part of the proposed model, especially the use of outcomes of the epistemic actions in further activities, has been hypothesized only and needs experimental confirmation. At this point, we feel that a further experimental study of abstraction depends on the existence of a *book* containing the (hi)stories of students along their engagement in activities in different groups during the year.

Concluding Remarks

Through the examples considered, I have tried to discuss our ability and maturity to trace the development of individuals while participating in interactive processes in classroom activities. I described the topic as a dilemma, because it either raises questions for which we have mostly "wishful thinking" answers, rather than experimental answers, or questions for which answers go in contradicting directions.

Research from an interactionist perspective seems to be developing dynamically, from investigations of the interactive processes of learning per-se and the contributions of the individuals participating to these processes, towards the investigation of the contribution of the interactions within the ensemble to each individual. I have tried to indicate some limitations and difficulties of the latter direction of research as illustrated by the examples, which were taken from studies I share with my colleagues in the CompuMath project.

I have tried to show that, as the number of the participants in the ensemble grows, it becomes more complicated to focus on the individual. There is a need for different kinds of documentation within the same episode, and more longitudinal methods of documentation, where the individual is observed in different contexts. The analysis of these kinds of longitudinal data raises the need for new methodology. The question whether longitudinal description and analysis may distort the analysis of the single episode has also to be taken into consideration.

In addition, the analysis of the intra-cognitive processes within the inter-cognitive processes raises the need to investigate the individual's mental cognitive functions. In such investigations the accumulated research findings, which we inherited from classic cognitive research, should be incorporated into the global frame of research and theory, enlarged, and continued.

Acknowledgement

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PROFESSIONAL DEVELOPMENT, CLASSROOM PRACTICES, AND STUDENTS' MATHEMATICS LEARNING: A CULTURAL PERSPECTIVE

Geoffrey B. Saxe

University of California, Berkeley

Abstract

I present two studies designed to illuminate ways that ongoing reform efforts in mathematics education are becoming interwoven with teachers' classroom practices and children's developing mathematics. The first study examines patterns of K-12 mathematics teachers' changing assessment practices. The second examines the classroom practices of upper elementary teachers participating in two professional development programs, each designed to support implementation of a reform-oriented curriculum. Both studies show the utility of a focus on practice for understanding teachers' professional development and students' developing mathematical understandings in the context of ongoing reforms.

My purpose in this paper is to sketch a cultural-developmental framework for the analysis of conceptual change (broadly defined), using two recent studies to illustrate the framework. In the first study, I present an analysis of the shifting assessment practices of K-12 teachers in the context of ongoing reforms in mathematics education. In the second, I present an analysis of upper elementary classroom practices in our era of reform, with a particular focus on students' developing understanding of fractions linked to classroom practices.

A CULTURAL-DEVELOPMENTAL APPROACH TO THE ANALYSIS OF LEARNING IN PRACTICES

The research approach provides a cultural-developmental frame for the analysis of learning in practices. The orientation is cultural insofar as practices are analyzed as recurrent socially organized activities that permeate daily life. A key assumption is that there is a reflexive relation between individual activities and practices – individuals' activities are constitutive of

practices and at the same time practices give form and social meaning to individuals' activities. The orientation is developmental insofar as the focus is on cognitive developments that emerge in individuals' efforts to structure and accomplish goals in practices. At core of the developmental perspective is a concern with the interplay in development between *cultural forms* –artifacts that have emerged over social history (such as forms of assessment) – and *cognitive functions*, the purposes for which forms are used (such as to gain insight into student understanding).

Study 1: Teachers' Shifting Assessment Practices[1]

The field of mathematics education has experienced waves of reform throughout its history, and each wave has been marked by challenges to teachers (Tyack & Cuban, 1995). In the recent climate of reform, particular value is placed on problem solving and conceptual understanding, a marked departure from the more traditional focus on accuracy and procedural skills (California State Department of Education, 1992; NCTM, 1989, 1993). New mathematics curriculum has been developed to engage students in problem solving, and new methods of assessment have been developed to evaluate the ways that students interpret problems and construct strategies for their solution. Mathematics teachers are pressed to implement these new approaches or to adapt their existing practices to fit the reform recommendations. We know that they are challenged, but to date we understand little of the pathways by which they develop competence with the new forms and functions of practice.

The purpose of the first study was to explore how mathematics teachers' assessment practices shift over time in relation to the presses to change. The practice of assessment is conceptualized broadly (see Figure 1). Assessment practices involve multiple stakeholder groups, including students, teachers, parents, principals who vary in their relation to students' performances. Stakeholders have different relations to the performances that are the targets of assessment – as represented by the double arrows between

[1] Maryl Gearhart, Megan Franke, Sharon Howard, and Michele Crockett were collaborators in this study

each stakeholder group and student performance. Stakeholders may communicate with one another about assessments as represented by the double arrows between groups. Further, assessment also involves the production of multiple artifacts, including primary artifacts (students' class work that may involve solutions to exercises and open ended problems), evaluations of those artifacts (number correct, scores, rubric levels), secondary artifacts, including composite scores (e.g., report cards, portfolio evaluations), and "high stakes" scores (e.g., SAT, CTBS scores). Such artifacts are often the foci of stakeholder reflection, evaluation, and communication.

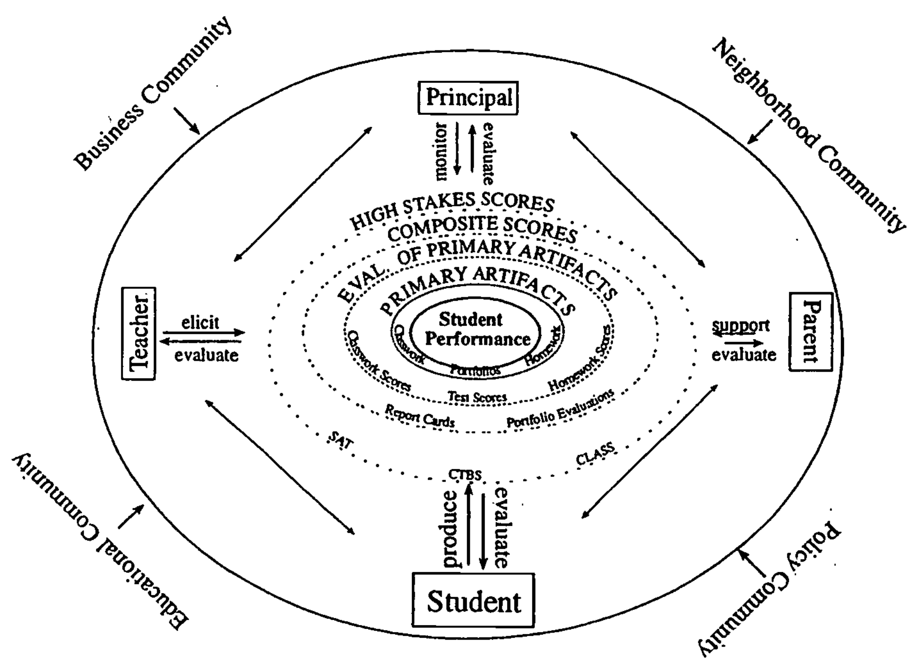

Figure 1: A sketch of stakeholder groups and principal artifacts in typical assessment practices.

I limit my focus in this talk to teachers' engagement with three forms of assessment and the functions that these forms serve in their practices. These included forms for eliciting student performances in class work such as exercises or open-ended problems as well as forms for evaluating student performances such as rubrics. While exercises are associated with traditional practices, open-ended problems and rubrics are associated with reform

practices. A focus on shifts over time in both teachers' differential frequency of use of these assessment forms and the different functions that these forms serve in assessment activities provides a window into the changing practices of teachers in this era of reform.[2]

Methods

In the first phase of the study, colleagues and I fielded surveys to two cohorts of K-12 teachers participating in a voluntary long-term professional development program (N=59). To capture the patterns of change, we asked the teachers to report on the frequency with which they were currently using the three assessment forms. In addition, to gain insight into trajectories of change. Therefore, we asked teachers to compare their current uses with their uses in the past and their anticipated uses in the future of the targeted forms.

We conducted a second phase of the study to shed light on the functions that these forms of assessment were serving in teachers' practices. We interviewed teachers, eliciting narrative descriptions of how they used targeted forms, the purposes that they served in their assessment practices, and whether and /or in what way these purposes might be changing.

Frequency of Use of Assessment Forms and Shifts in Use over Time

We analyzed the frequency with which teachers used particular assessment forms and the shifts in frequency (and projected frequency) over time. Our findings point to changing patterns of use linked to the current climate of reform and presses for change.

Though all teachers reported that they were implementing reforms, most reported using exercises frequently for purposes of assessment. Indeed, 75% of the teachers reported using exercises at least 2-3 times a week for assessment. The same was not true for open-ended problems and rubrics: Teachers reported using open-ended problems at more moderate

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[2] The way stakeholders pressures are manifest in teachers practices is also critical to understanding shifting forms and functions of assessment, but will not be discussed in this paper due to time and space constraints.

levels, the majority reporting at least weekly use. The variability in use of rubrics was quite pronounced. Indeed, 50% of the sample reported uses of rubrics in the range between rare (once or twice a year) and relatively frequently (weekly).

By comparing teachers' reported uses of assessment forms last year, this year, and next year, we were able to identify patterns of change. For exercises, most teachers reported little change in frequency of use. Indeed, more than 75% of the teachers reported stable (and high) use over past through prospective practice. In contrast to the results for exercises, most teachers showed shifts towards greater frequency of use for open-ended problems and rubrics. Between 60% and 70% of the teachers' profiles indicated increases in frequency of use either from past to current and current to projected practice.

Shifts in Functions of Assessment

Our interviews were designed to explore continuities and discontinuities in functions of assessment forms in practices. In assessment practices, continuity is manifested in a teacher's decision to continue using either an 'old' assessment form over time, or, a new form to serve an 'old' function. Discontinuity is manifested in a teacher's decision to use a new assessment form, or, to use an 'old' form for a new function. Core to our approach is the assumption that continuity and discontinuity are inherently related to one another in the process of development -- continuity preserves the coherence or integrity of practice while discontinuity allows for adjustment to presses and organizational change.

We explored shifts in the purposes for which teachers used exercises, open-ended problems, and rubrics with 12 teachers. We documented several patterns of development. None of the patterns represents a radical re-organization of practice. Rather, for each pattern, shifts over time were marked by both continuity and discontinuity.

One pattern captures the ways that teachers may implement a new form of assessment in a way that served 'old' functions. For example, some

teachers used a 'new' form of assessment, open-ended problems, in ways that served instructional function. They engaged children with the open-ended problems to provide them the opportunity to invent strategies; for this pattern, they did not examine students' responses to open-ended problems to gain insight into the character of their mathematical thinking, a function linked to student inquiry promoted by reform documents.

A second pattern captures the ways that new forms of assessment may be implemented in pro forma ways. Some teachers used rubrics developed by others that focused on the completeness of students' written explanations. Teachers exemplifying this pattern did not revise such rubrics to capture students' mathematics.

A third pattern illustrates the ways that teachers may fashion or re-fashion forms of assessment in order to assess students' mathematical thinking, the function of assessment recommended by reform. Some teachers re-purposed an 'old' form of assessment, such as an exercise, to serve a new function, supplementing the old form as necessary with new forms (written explanations) that support the new function. In addition, some appropriated a colleague's rubric for evaluating students' responses to the open-ended problems, re-designing it to suit their curriculum and their goals for her students' mathematical learning.

A fourth pattern illustrates how teachers' concerns for efficiency may work against the quality of their assessments. Some teachers were considering strategies for more frequent and more rapid rubric scoring. Some as yet had no specific strategy for increasing the speed of scoring; at least one was considering replacing her analytic rubric with a holistic approach, expressing worries about tradeoffs between frequency of scoring and quality of the evaluation.

This study provides a preliminary frame for analyzing the dynamics of change in the professional development of teachers. A key notion here is that in order to understand why 'change takes time,' we need to identify developmental patterns in the ways that teachers construct goals in their

practices, goals that interweave the presses upon them, the resources available to them, and their current knowledge and patterns of practice.

Study 2: Relations between Teachers' Shifting Classroom Practices and Student Learning in the Domain of Fractions[3]

In the second study, colleagues and I produced an analysis of teachers' changing classroom practices linked to ongoing reforms and the relation of such change to student learning. To this end, we observed 23 upper elementary teachers implementing units on fractions, and assessed the learning of their students. We sampled teachers who were committed to traditional instructional approaches as well as teachers who were committed to reform. The latter group was implementing a new unit on fractions [Seeing Fractions (Corwin, Russell, & Tierney, 1990)].[4] In analyzing the relation between classroom practices and student learning, we compared the progress of two groups of students – those who began instruction with a rudimentary understanding of fractions vs. those without a rudimentary understanding of fractions. We assumed that these two groups of children might be forming different goals related to fractions in classroom practices. Further, we assumed that children's progress would be related to the extent to which classroom practices were aligned with reform frameworks. Thus, our focus was the relation between (a) the alignment of classroom practices with reform principles (b) students' prior knowledge and (c) the developing mathematics of each of these two groups of students.

Analyzing and Rating Classroom Practices

To evaluate the extent to which classroom practices were aligned with reform principles, we developed rating scales and applied them both to videotape and fieldnote records of whole class lessons (cf. Gearhart, Saxe, Ching, Fall, Nasir, Schlackman, Bennett, Rhine, & Sloan. (in press)). The scales were used to evaluate core instructional principles espoused in reform

[3] Maryl Gearhart and Michael Seltzer were collaborators in this study

[4] Teachers implementing the reform curriculum were participating in one of two professional development programs colleagues and I organized.

documents – (a) the degree to which classroom practices elicit and build upon students' thinking (Integrated Assessment) and (b) the extent to which conceptual issues are addressed in treatments of problem solving (Conceptual Issues). To apply the Integrated Assessment scale, raters were instructed to attend to teacher questioning and public problem solving and the ways that these did or did not elicit and address students' mathematical understandings. To apply the Conceptual Issues scale, raters focused on the ways that methods for solving fractions problems were linked to core fractions concepts — part-whole relations, part-part relations, and equivalence relations. Parallel scales were developed for videotape and for fieldnotes, resulting in four scales in all. We then aggregated these measures in order to produce a single index of alignment.

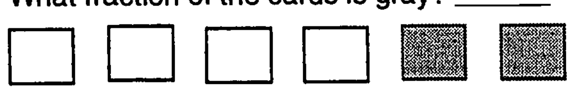
Assessing Students' Rudimentary Understanding of Fractions

To partition students into those who demonstrated a rudimentary understanding of fractions and those who did not, we coded students' performance on an additional set of elementary items that were included as part of the pretest. These additional items were elementary fractions problems depicted in Figure 2, one subset involving discontinuous quantity (items in Figure 2A) and the other subset involved a continuous quantity (items in Figure 2B). Consistency of adequate performance on at least one subset was required for children to be regarded as displaying a rudimentary understanding of fractions (see Methods section).

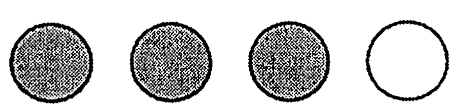
Figure 2. Elementary fractions problems.

A.

What fraction of the cards is gray? ______

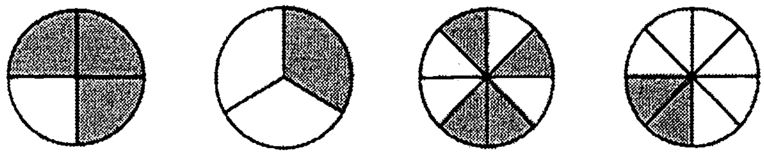


What fraction of the marbles is gray? ______



B.

For each picture below, write a fraction to show what part is gray:



Analyzing Shifts in Students' Problem Solving and Computation with Fractions as a Function of Instruction

Our assessments of student achievement in the domain of fractions were designed to measure both students' computational skills and their competence with problem solving. The distinction between computation and problem solving is captured in similar ways by other researchers using such constructs as procedural versus conceptual knowledge (Greeno, Riley, & Gelman, 1984; Hiebert & Lefevre, 1986), the syntax versus semantics of mathematics (Resnick, 1982), and skills versus principles (Gelman & Gallistel, 1978). We recognized that the distinction between computation and problem solving would become problematic when we operationalized it as distinctive sets of items. Indeed, a child might solve what we regarded as a computation task using an invented problem solving strategy, or might solve what we classify as a problem using a memorized procedure. Nonetheless, the items that we constructed provided a heuristically useful way to measure students' skills with fractions and problem solving with fractions. The computation items could be solved using routine algorithmic procedures or commonly memorized facts. The problem solving items could not easily be solved by standard computational approaches, and were more likely to require insight into the concepts underlying representations of fractions. In addition to the face validity of the distinction, we validated the distinction between computations and problem solving through confirmatory factor analytic techniques (Saxe & Gearhart, 1998).

Expected Relations between Practice and Achievement

We expected students' performances on the problem solving and computation scales to vary as a function of students' prior understandings and alignment of classroom practices with reform principles. For the problem solving scale, we expected that students without rudimentary understandings of fractions would be at risk for not learning from instruction if there were little classroom support for children's conceptual engagement

with the subject matter (as indexed by a low level of support on our classroom alignment ratings). These students should be prone to interpret classroom activities involving fractions (e.g., representational forms presented in lessons, small group work) in whole number terms. Thus, classroom activities at lower levels of support should either be very confusing or systematically misunderstood in terms of whole numbers. However, if these students participated in classroom practices involving fractions that were geared for building on their understandings (as indexed by at least a moderate level of support on our classroom alignment ratings), we might expect to see growth in students fractions concepts, and even greater growth at high levels of support.

In contrast, we expected that students with rudimentary understandings at the start of instruction would show a different profile of learning as a function of alignment. These students should be more able to make sense of representational forms presented in lessons in terms of part-whole relations even if engagement with fractions concepts was not a focus of instruction (i.e., at low levels of alignment). Further, with greater support for conceptual engagement (increasing levels of alignment on our scale), we expected that these students would show greater gains in their understandings.

For the computation scale, we did not expect to find the same pattern of relations between alignment of practices with reform principles and student performance. Indeed, there is little reason to expect that reform practices would influence directly students' developing competence with computation tasks that are often readily solved through memorization of routine facts and algorithms. Thus, we expected at best a weak relation between alignment of whole class lessons with reform principles and students' computation achievement, regardless of students' prior understanding of fractions.

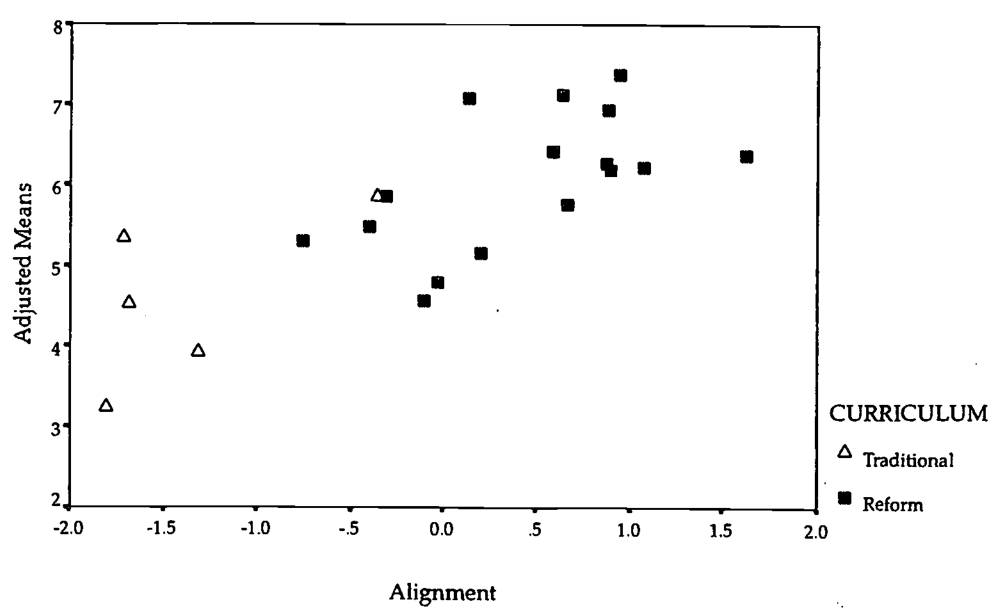
<u>Relations between Alignment and Problem Solving Scale</u>

Figures 3 and 4 contain plots of posttest performances on the problem solving scale as a function of our measure of the alignment of classroom

practices with reform principles (standard scores). The posttest means are statistically adjusted for language background and pretest performance. Visual inspection of the plots for students with and without a rudimentary understanding reveals that both slopes show a positive relation between posttest performance and classroom alignment. However, the character of the slopes differs.

For students with a rudimentary understanding, the relation between posttest performance and our measure of alignment appears linear (Figure 3). Our HLM analyses confirm this. For every unit increase in the classroom practice scale (a 1-4 point scale), there is a .87 increase in classroom posttest performance (a 13-point scale); the t statistic shows that the effect is significant ($t(19)=4.91$, $p=.0000$). We interpret this pattern as evidence that these students' rudimentary understandings of fractions allowed them to make sense of fraction problems in part-whole terms even when classroom practices were relatively inconsistent with the principles of reform.

Figure 3. Adjusted classroom posttest means on the problem solving scale for students with rudimentary understanding as a function of classroom alignment measure.



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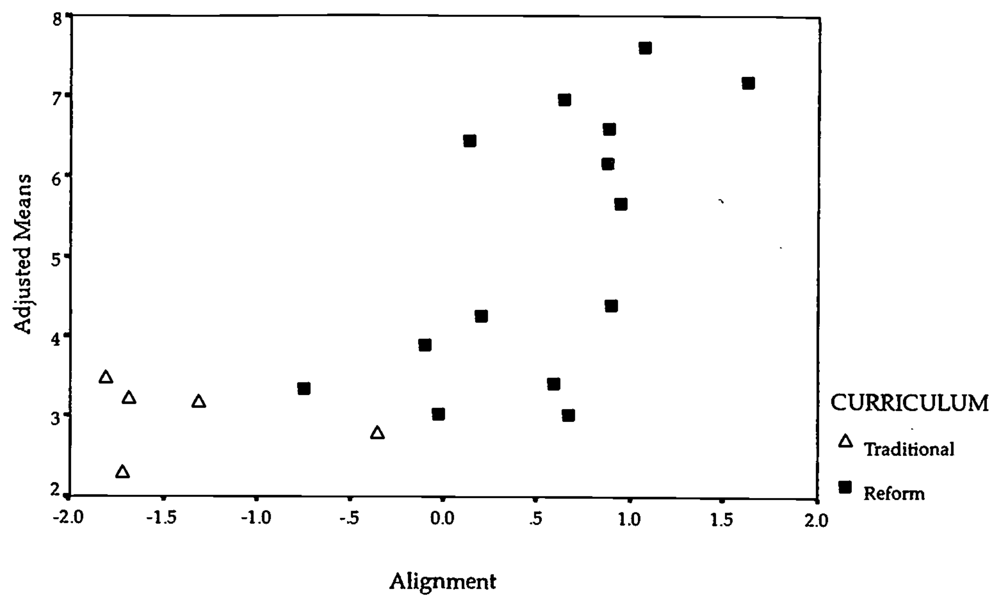
For students without a rudimentary understanding, the relation between posttest performance and our measure of alignment does not appear linear (Figure 4). Indeed, for classrooms in which alignment with reform principles was below the mean, the plot appears to show no relation between posttest performance and alignment. In contrast, for classrooms in which alignment with reform principles was above the mean, the plot appears to show a linear relation between alignment and posttest performance. To confirm the visual analysis of the plot in Figure 5, we fit a two-relation HLM to the data. Our model allows for the possibility that the relationship between class mean posttest scores and classroom practices may differ for those classes in which alignment is below average and those classes in which alignment is above average. The results of our HLM analyses supported the "two-relation" model. When alignment of classroom practices with reform principles is below average, we find no relation between posttest score and alignment ($t(16)=.81$, $p=.433$). In contrast, when alignment is above average, we find a significant effect. For every point increase in classroom alignment, there is an expected posttest score increase of 2.07 points ($t(16)=2.48$, $p=.025$). This is a significant relation that is almost five times the magnitude of the estimated effect for the relation for below average alignment classrooms. We interpret this finding as evidence that, in classrooms judged low on alignment (whether using traditional or reform curricula), students without a rudimentary understanding had little basis on which to structure mathematical goals in other than whole number or procedural terms. In contrast, at higher levels of alignment, posttest scores for students without rudimentary understandings were related to alignment, and, indeed, those scores increased sharply. We interpret this pattern as evidence of a threshold of support needed by such children. With such support, students may become engaged with mathematical goals involving fractions, leading to gains in their understanding of fractions.

Figure 4. Adjusted classroom posttest means on the problem solving scale for students without rudimentary understanding as a function of classroom alignment measure.



Together, the considerations above point to the importance of the coordinated analysis of students' rudimentary understandings, curriculum, and classroom practices in student achievement on the problem solving scale.

Relations between Alignment and Computation

In contrast to the problem solving items, the slopes for students with a rudimentary understanding and without a rudimentary understanding reveal no relation between posttest performance and alignment of classroom practice with reform principles. Our HLM analyses show that for students with a rudimentary understanding, the estimated effect was -.26 ($t(19)=-.70$, $p=.49$), and for students without a rudimentary understanding, the estimated effect was also -.26 ($t(19)=-.57$, $p=.58$).

The lack of relation between student performance on the computation scale and alignment of classroom practice with reform principles was expected. In the short term, neither support for students' conceptual engagement with mathematics nor efforts to build on student understanding are likely to enhance students' memorization of arithmetical procedures. Although some students may be able to extend their developing

understandings of fractions to computational items, it may well be that rehearsal of computational procedures under direct instructional methods is more successful in enhancing computational skills. This latter conjecture was beyond the purpose and scope of our analyses.

CONCLUDING REMARKS

Investigating relations between the shifting organization of classroom practices and student learning related to reform efforts is a complex analytic task. The cultural-developmental framework that I've sketched provides an initial foothold into this complex arena of study. Through a focus on practice and the way individuals' are structuring and accomplishing goals in classroom life, we gain insight into historical change in practice and shifts in patterns of learning and development. My hope is that greater understanding of the relations between development and practice will provide new insights about how to usefully support the professional development of teachers and mathematical understandings of students.

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Reconstructing the Mathematical in Social Discourse – Aspects of an Epistemology-Based Interaction Research

Heinz Steinbring, Universität Dortmund, Germany

Abstract. *Mathematical knowledge depends on human thinking and social interaction. Neither symbols nor contexts alone provide the objective basic substance for mathematical existence. Mathematical knowledge is created by (mental and interactive) interpretation of signs with regard to possible reference contexts. With this theoretical perspective on knowledge in mind, mathematical interaction research is faced by a complementary difficulty: The object of research – mathematical communication – as well as its observation and scientific analysis are both "sign-interpreting-processes" that are constituted in social interaction. The analysing reconstruction of mathematical discourses requires the revelation of possible interactive interpretations of communicated signs and in this way the analysis reflects its "own" understanding of mathematical knowledge as a result of a social construction processes. This article presents crucial components of the epistemology-based research of mathematical interaction by using exemplary teaching episodes from an ongoing research project on "Social and epistemological constraints of constructing new knowledge in the mathematics classroom".*

1. Introduction

The social construction of new mathematical knowledge in teaching and learning processes depends on two important conditions: The special character of instructional communication and the specific epistemological nature of mathematical knowledge. In mathematics teaching at the primary level new knowledge cannot be constructed in a formal manner by a kind of preview technique, i.e. using algebra or formulas, but this construction is linked with the children's situated contexts of learning and of experience in a characteristic way. The young students have to learn – and they are able to do so by their personal means – to see the general in the particular. To better understand this problem is an important inquiry of the research project "Social and epistemological constraints of constructing new knowledge in the mathematics classroom" (funded by the German Research Society, DFG; see Steinbring et al. 1998). How are students of elementary grades able to grasp the new, general mathematical knowledge with their own conceptions and to describe it with their own words? And what factors support or hinder this generalising interactive knowledge construction?

2. What is the Specific Nature of Mathematical Concepts?

Mathematical concepts and mathematical knowledge are not given a priori in the "external" reality, neither as concrete, material objects, nor as independently existing (platonic) ideas. For the individual cognitive agent mathematical concepts are "mental objects" (Changeux & Connes 1995; Dehaene 1997); in the course of communication mathematical concepts are constituted as "social facts" (Searle 1997) or as "cultural

objects" (Hersh 1997). From an evolutionary point of view mathematical concepts develop as cognitive and as social theoretical knowledge objects in confrontation with the material and social environment.

In contrast to objects constructed by humans as for instance a chair, a table, a knife or a screw-driver one cannot deduce the meaning of social facts, as for instance money, time or the number concept neither from their form nor from their material. There are no direct insights into the corresponding mathematical object when inspecting the "material" or the functional form of number signs as $\sqrt{2}$, -3.17 or π. The meaning of these theoretical, social respectively mental objects has to be constructed by the individual in interaction with experience based and abstract referential contexts. In a general way, mathematical concepts can be conceived as "symbolised, operational relations" between their formal codings and certain socially intended interpretation.

Mathematical knowledge can be looked at in two complementary ways: On the one side, each mathematical knowledge domain represents a consistent structural wickerwork, in which all elements are linked in an equivalent logical manner. On the other side, new concepts posing new questions and problems can be constructed in every mathematical knowledge structure, concepts that are not yet imbedded in the actual logical structure, and in this way producing new insights.

This distinction between the *logical structure* and the *mathematical objects* is in accordance with the distinction made in philosophy between a subjective ontology of reality and the subject independent structure of the world. "... the ontology of the world is created by the cognitive agent, the structure of the world depends on the mind-independent external reality. In this way, the experiential world can be seen as both created and mind-independent at once. As there cannot be a structure without an ontology, it is the cognitive agent's act of creating an ontology that endows external reality with a structure" (Indurkhya 1994, p. 106).

The "logical coherence" and consequently the "unique generativity" of mathematical knowledge often is taken as an irrefutable "proof" for the objective existence of mathematical knowledge independent of any cognitive agent (Changeux & Connes 1995, p. 12); but also this property – a specific, epistemological mechanism for the autopoietic development of mathematical knowledge – needs the cognitive as well as the social environment of the cognitive agent for its unfolding.

3. The Epistemological and Communicative Function of Signs

3.1 The Epistemological Dimension

The peculiar interrelation between "Signs / Symbols" and "Objects / Reference contexts" is central for the description and analysis of mathematical teaching as a specific culture. This relation also represents a basic item of the epistemologically based interaction analysis. All mathematical knowledge needs certain *systems of signs or symbols* for grasping and coding the knowledge in question. These signs themselves do not have an isolated meaning; their meaning must be constructed by the learning child. In a general

sense, to endow mathematical signs with meaning, one needs an adequate *reference context*. Meanings of mathematical concepts emerge in the interplay between sign/symbol systems and objects/reference contexts (Steinbring 1993; or Maier & Steinbring 1998).

The interrelation between coding signs of knowledge and reference contexts can be structured in the *epistemological triangle* (cf. Steinbring 1989; 1991; 1998). The links between the corners in this epistemological triangle are not defined explicitly and invariably, they rather form a mutually supported and balanced system. In the course of further developing knowledge the interpretation of signs systems and their accompanying reference contexts will be modified and generalised.

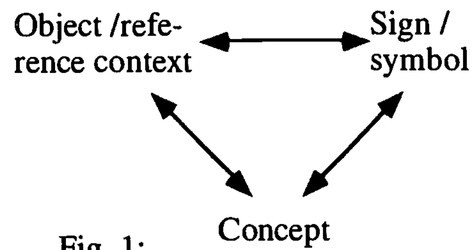


Fig. 1:
The epistemological triangle

Similar triangular schemes have been introduced in the philosophy of mathematics, in linguistics and the philosophy of language for analysing the semiotic problem of the relation between symbol and referent (Frege 1969; Ogden & Richards 1923).

Mathematical concepts are constructed as symbolic relational structures and are coded by means of *signs and symbols*, that can be combined logically in mathematical operations. With regard to the analysis of conditions for the construction of new mathematical knowledge in classroom interaction, mathematical signs and symbols are the central connecting links between the epistemological and the communicative dimension of interactive construction processes; on the one hand, signs and symbols are the *carriers of mathematical knowledge*, and on the other hand, they contain at the same time the *information of the mathematical communication*.

3.2 The Communicative Dimension

The sociologist Niklas Luhmann characterises »communication« as the constitutive concept of sociology: "... when communication shall come about, ... an autopoietic system has to be activated, that is a social system, that reproduces communications by communications and makes nothing else but this" (Luhmann 1996, p. 279).

The concept of "autopoietic system" has been introduced by Maturana and Varela (cf. i. e. 1987); it characterises self-referential systems, that exist and develop autonomously on the basis of this self-referential relation. These systems consist of components that are permanently re-produced within the system for its maintenance. With the concept of "autopoietic system" not only biologic processes are investigated but it is also applied to social and psychic processes.

What is the essential difference between a social and a psychic process? The psychic process is based on consciousness and the social system is based on communication. "A social system cannot think, a psychological system cannot communicate. Nevertheless, from a causal view there are immense, highly complex interdependencies" (Luhmann

1997, p. 28). How these interdependencies can be understood? "Communication systems and psychic systems (or consciousness) form two clearly separated autopoietic domains; ... But these two kinds of systems are linked in a special narrow relation and they form reciprocally a »portion of necessary environment«: Without the participation of consciousness systems there is no communication, and without the participation of communication there is no development of consciousness" (Baraldi, Corsi & Esposito 1997, p. 86).

Language is a central "linking mean" between communication and consciousness. Within language one has to distinguish between »sound« and »sense«; accordingly within written language one has to distinguish between »sign« (more exactly »signifier«) and »sense«. This distinction between sign and intended meaning is the starting point – the take off (Luhmann, 1997, p. 208) – for the autopoiesis of communicative systems.

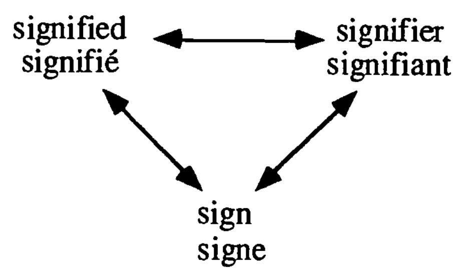


Fig. 2: The semiotic triangle

For the analysis of the conditions of the auto-poiesis Luhmann refers among others to the work of de Saussure, who made the following distinction between signifier (signifiant), signified (signifié) and sign (signe). Luhmann writes: "Signs are also forms, that means marked distinctions. They distinguish, following Saussure, the signified (signifiant) from the signifier (signifié). In the form of the sign, that means in the relation between signifier and signified, there are referents: The signifier signifies the signified. But the form itself (and only this should be named sign) has no reference; it functions only as a distinction, and that only when it is actually used as such" (Luhmann 1997, p. 208f.).

How the autopoiesis of the social, of communication, is possible? According to Luhmann, in the course of interaction or in the communication system the participants provide with their "conveyances" (or communicative actions) mutually "signifiers" which may signify certain "information" (signifieds). "Decisive might be..., that speaking (and this imitating gestures) elucidates an intention of the speaker, hence forces a distinction between information and conveyance with likewise linguistic means" (Luhmann 1997, p. 85).

The conveyor only can convey a signifier, but the signified intended by the conveyor, which alone could lead to an understandable sign, remains open and relatively uncertain; in principle, it can be constructed only by the receiver of the conveyance, in a way that he himself articulates a new signified. Luhmann explains this in the following way: "We do not start with the speech action, which will happen only when one expects, that it is expected and understood, but we start with the situation of the receiver of the conveyance, hence the person who observes the conveyor and who ascribes to him the conveyance, but not the information. The receiver of the conveyance has to observe the conveyance as the designation of an information, hence both together as a sign (as a

form of the distinction between signifier and signified) ...." (Luhmann 1997, p. 210). The receiver must not ascribe the possible signified strictly to the conveyor of the conveyance but he/she has to construct the signified himself/herself; the signified and hence the sign is constituted within the process of communication.

The possible detachment of the information belonging to the conveyance from the conveyor is the starting point of the autopoiesis of the communicative system. By this "mechanism" that describes the autopoietic functioning of the communicative system as an ongoing conveyance of signifiers which are simultaneously transformed into signs by the contrasting conveyance of other, new signifiers, general properties of the functioning of *mathematical communication* are explained, too. In a first approximation, the epistemological triangle can be seen as analogue to the semiotic triangle (according to de Saussure); in addition, the epistemological triangle contains very specific features with regard to the particularities of mathematical communication.

4. Open and Superimposed Discourses in Mathematics Teaching – Analysis of Exemplary Teaching Episodes

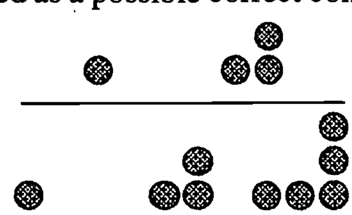
In the following, interactive patterns in two different teaching episodes are analysed in the course of constructing and justifying new mathematical knowledge. In the first episode students work within a learning environment about figurative numbers, where *geometric* reference contexts are offered for the interpretation of mathematical signs. The second episode is part of a teaching unit about special number squares, where the new mathematical signs have to be interpreted with the help of *structured arithmetical* reference contexts.

4.1 What is the "Correct" Representation of the Third Triangular Number?

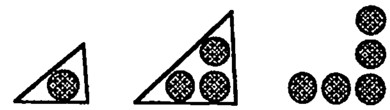
The content of the observed lesson in a 4th grade can be summarised in the following way. The teacher has placed a pattern of magnetic chips (little disks with one red and one blue side) on the black board. This pattern obviously should show the first two triangular numbers. The children are asked to construct the next pattern in this sequence. They offer different interesting proposals. The teacher guides the interactive construction process, and she asks for a justification of the last pattern she had accepted.

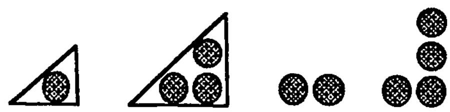
4.1.1 The Children's Proposals for Continuing the Pattern

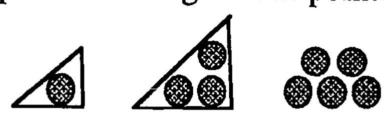
The teacher places two patterns on the blackboard. How to continue? She emphasises that special numbers are involved having to do with the chips. Dennis continues the pattern in the following way. His proposal can be seen indeed as a possible correct continuation, in which the hook is extended by placing down left and right above one chip each time. The teacher comments this proposal: "Is this already correct? ... One could have the impression, but it is not yet quite right"(5). She refers to the shape and draws rectangular triangles

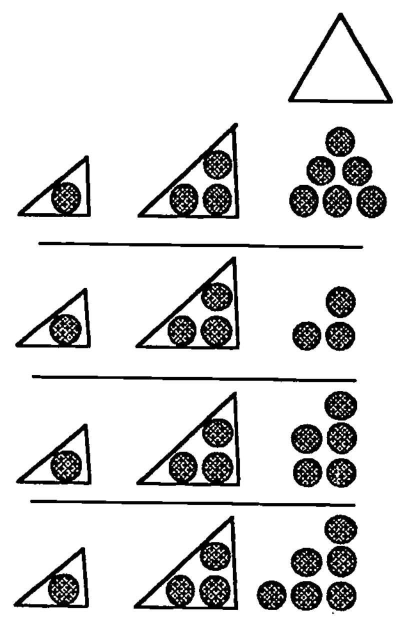


around the two first patterns. Then the teacher points to Dennis' pattern and she says that one could not draw such an triangle around it: "This could not yet be made here."(7). With her finger she goes around his pattern and in this way she outlines the shape of a hook or an angle. The teacher seems to have in mind that only one chip is still missing at the correct place and she tries to focus the children's attention to this fact. By asking the question: "Who could place this now, or use something else?" (7), the teacher expects that the one missing chip will be inserted now. But Lisa answers by making a completely different proposal and constructs the following pattern. She seems to take the initial patterns as one single figure and looks for a possible continuation. Her proposal is a plausible continuation in which to each part of the complete initial figure each time one chip is added, once in the horizontal line, and once in the vertical line. The teacher refuses Lisa's proposal by referring to the shape of the triangles. She points to the base line and to the inclined line of the triangle. Kai takes away all new chips from the blackboard and starts to place his configuration. With the help of his classmates he inserts the last missing chip. He has produced an isosceles triangle, as the teacher then accentuates: "Well, first I have to look here. We have had such a form there but now we have seemingly this [*she draws an isosceles triangle above the constructed configuration*]" (14). The teacher poses the question whether this is the same (16); in this way she refutes Kai's proposal. Kai takes his chips off from the blackboard. Once again the teacher tries to focus his attention to the drawn shape of the two first patterns and she says that the new figure should look the same as the two already existing figures, only with more chips (18). First Kai places exactly the same pattern as the second one. Then he adds two chips in the following way. The teacher confirms: "Ooh, he is very close!" (22). Also Kai's classmates make supporting comments that only one chip is still missing. But Kai is not able to finish the proposal. Tugba places the missing chip on the left side in the base row.









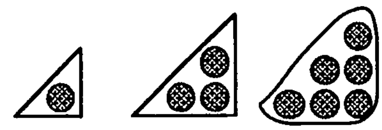
4.1.2 An Empirical Justification for the Correctness of the Third Pattern

Let us look more closely at the following short interaction phase wherein the justification of the correctness for the third pattern is negotiated.

```
24    T    … Who could now explain why this is correct now? That is
           correct, you must know. Rabea.
```

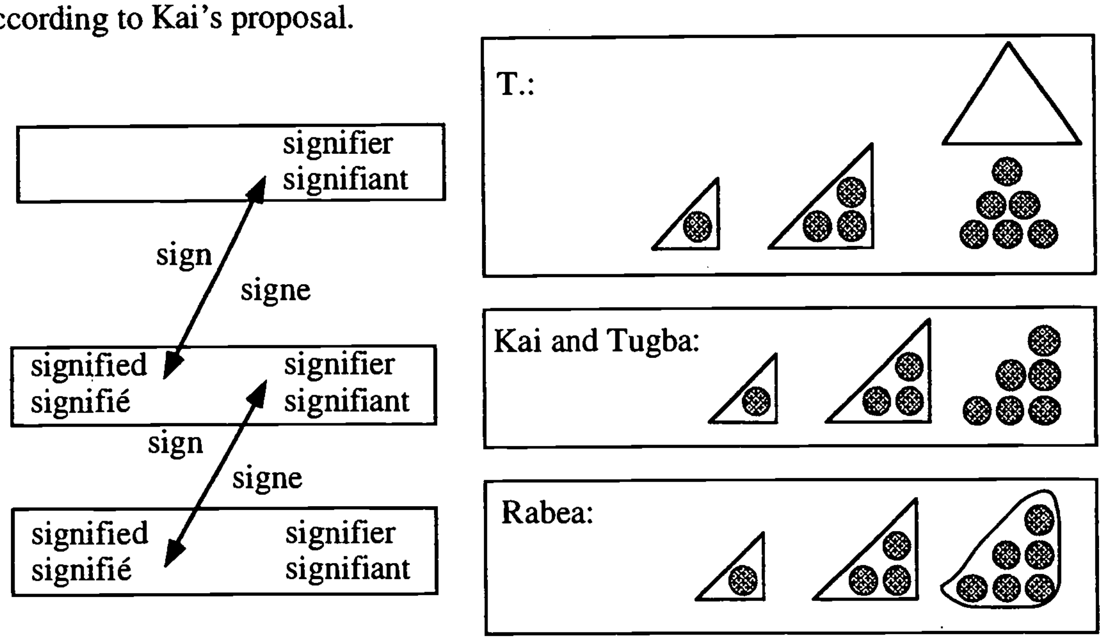
25 R Because it is again the same pattern.

26 L Mhm. Could you come and draw the pattern around it? [*Tugba goes to her place and Rabea comes to the blackboard*] … Draw first the pattern! Go once around it! [*Rabea draws with chalk a triangle around the third configuration*] Aha.



In principle, this justification consists of one statement: "Because it is again the same pattern.", which then is illustrated by Rabea as the "same pattern". How could this short statement gain the status of a justification? This justification function is only possible on the basis of the earlier interaction process. We have seen that the continuations of the teacher's two initial patterns as proposed by the students could have been possible and reasonable. But the teacher refused them one after the other and at the same time she explicated the conditions of "similarity" in the patterns. The children's proposed continuations are excluded until in the end the teacher's intended unique, correct third pattern is produced.

With the scheme of the communicative analysis the final interactive justification with Kai and Tugba can be described in the following manner. The teacher emphasises the rectangular property of the figures and as a contrast she draws the isosceles triangle according to Kai's proposal.



Kai changes his signifier, and after Tugba has completed the pattern, the teacher accepts this continuation of the series of patterns. By drawing the triangular shape of the pattern Rabea makes the "similarity" between the different shapes more explicit and in this way the "conformity" of the patterns becomes the justifying argument and this is legitimised and agreed upon interactively.

The functioning of the autopoiesis of the communication system requires that a given

signifier is not directly linked with the signified intended by the conveyor of the message. This openness is essential for the communicative process. During the analysed episode one can observe that the teacher's denials of the children's proposed signifiers aim at identifying a definite relation between the given signifier (the two first triangular patterns) and a fixed signified (the rectangular shaped pattern). The elaboration of this definite third pattern takes place by a kind of negative delimitation in the course of this interaction.

This mathematical interaction is dominated by the idea that there exists one single correct third pattern, and this idea is made explicit step by step. The teacher stresses this point at different occasions: "Is this already correct? ... One could have the impression, but it is not yet quite right"(5); "... mh, this is not yet quite correct." (10); "... he is very close!" (22); "... why this is correct now? That is correct, you must know." (24).

The communication analysis shows that the interaction is used to point out the teacher's a priori correct relation between the presented signifier (the two patterns) and the appropriate fixed signified (the shape of a rectangular triangle). To give an acceptable justification in this situation means to identify the correct relation between signifier and signified. The basis is the dogma that in mathematics there always exists one single correct relation between signifiers and signifieds. From an epistemological point of view, the new sign "third pattern" is interpreted with regard to a reference context of fixed rectangular shapes for triangles – all other possible shapes are excluded. News signs / symbols and corresponding elements in the reference context are strictly fixed with one another; the signs become names for observed empirical objects (in this case for chip configurations).

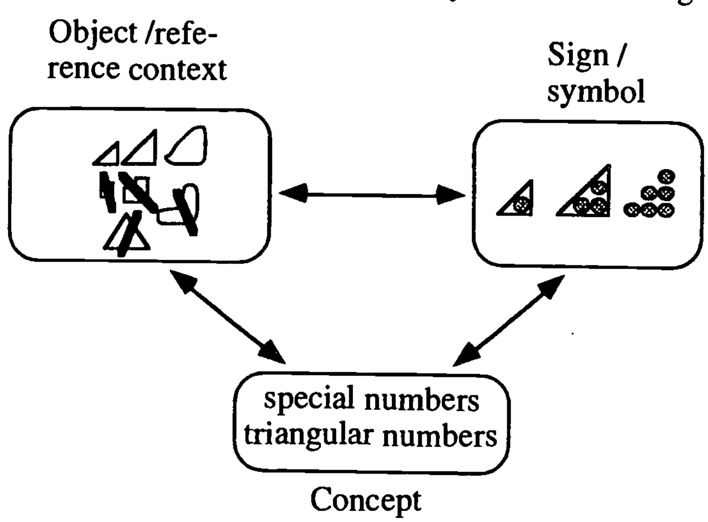


Fig. 3: The epistemological triangle

The justification of the correctness for the third pattern can be characterised in this way: The proposed patterns are compared with one invisible fixed pattern and differences or similarities are remarked until the new pattern is in agreement with the teacher's intended pattern. The last pattern is an admissible one, but it is accentuated in a special social manner as the only correct pattern. No specific reasons are provided for the choice of this pattern; the sole basis is the teacher's authority.

4.2 How is it Possible to Recover a Lost Number in the Number Square?

During this lesson in a mixed class of grade 3 and 4 the children had to work on the following problem: How could one recover a lost number in a certain number square,

in such a way that this number reproduces the former arithmetical structure? (cf. Fig. 4).The special number squares as used in this class can be constructed in the following way: First one adds some given numbers in the border row and border column of a table (cf. Fig. 5). The squares thus created have the following property: You can choose (circle) in a $(3 \cdot 3)$ number square three numbers arbitrarily such that in every row and in every column there is one and only one circled number. The sum of three numbers chosen is always constant – independent of its choice (cf. Fig. 6). Such squares are called "crossing out number squares", because when circling a certain number in the square, all

15	16	17
14	15	16
13	14	

Fig. 4

+	5	6	7
10	15	16	17
9	14	15	16
8	13	14	

Fig. 5

~~15~~	~~16~~	17
~~14~~	15	~~16~~
13	14	

Fig. 6

other numbers in the same row and in the same column have to be crossed out. The children called such a square »magical square« and the constant sum the »magical number«. In this episode, the children reproduced the lost number with three different strategies.

4.2.1 First Strategy: Using Structures in the Arithmetical Pattern

By using the arithmetical pattern of the given numbers in the square, Kevin argues that 15 is the missing number: "... because here is the fifteen, sixteen, seventeen [*points at the first row of the magical square*]. There is the fif-, fourteen, fifteen, sixteen [*points at the second row of the magical square*]. And here is the thirteen, fourteen [*points at the two numbers in the third row of the magical square*]. And then comes there the fifteen [*points at the empty field in the third row*]" (12). Kevin completes his argument by refering to the arithmetical regularities in the columns, too.

4.2.2 Second Strategy: Reconstructing the Missing Number with Numbers in the Border Lines

Some students reconstruct possible numbers in the border column and border row from which the magical square could have been built up. They start with the additive decomposition of $15 = 10 + 5$ (cf. Fig. 7) and they calculate further numbers in the border row (cf. Fig. 8) and finally in the border column (cf. Fig. 9). On this basis the children determine 15 as the missing number; this is justified by checking all calculations.

+	5		
10	15	16	17
	14	15	16
	13	14	

Fig. 7

+	5	6	7
10	15	16	17
	14	15	16
	13	14	

Fig. 8

+	5	6	7
10	15	16	17
9	14	15	16
8	13	14	

Fig. 9

4.2.3 Third Strategy: Reconstructing the Missing Number by Using the Magical Number

Already earlier in the course of this lesson Kim has sketched her idea. Later she explains her plan in detail. First she calculates the magical number 45 by adding the numbers 13, 15 and 17 in the diagonal. With this proposal she expresses that one can determine the magical number in an incomplete magical square. Then her argumentation starts.

147 K And then one could already make it this way. One circles the fifteen *[points at 15 in the first row]* and <u>this</u> fifteen *[points at 15 in the second row]* and adds it up. And then one still calculates, how much there must be up to forty-five.

The signifier "One circles the fifteen and <u>this</u> fifteen and adds it up." denotes the intention to apply the known procedure for calculating the magical number to two numbers in the diagonal. The second signifier "And then one still calculates, how much there must be up to forty-five." could be understood in this way: One has to calculate how much is left from the sum of 15 + 15 up to 45 (one has to calculate the difference); seemingly, this number has to be placed into the empty field.

At this moment, several classmates object that nothing could be really calculated here. "Well, that really leads nowhere ... Where you would like calculate up to? ... Exactly. After all, you do not at all know which number is the result here!" (152, 153).

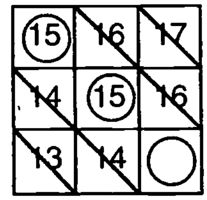
Kim formulates further explanations.

161 K First one calculates, one first calculates these numbers, that I have, which are there, what is their result. And then ..., and then one calculates ...

165 K These three, oh, yes, this, this and then afterwards one calculates fifteen *[circles 15 in the second row]*, one takes this way. Cross out that, and that. And cross out that and that *[crosses the other numbers in the same column and the same row]*. Then one also takes the fifteen *[circles 15 in the first row]*. Crosses out then the seventeen and the thirteen *[crosses out the still uncrossed numbers in the same column and the same row]*. And then one circles this here, this here *[circles the empty field]*. And then one has to calculate, fifteen and fifteen this makes thirty, how much is left up to forty-five.

The signifier "And then one circles this here, this here." indicates the application of the crossing algorithm for calculating the magical number to a missing third number – to the empty field. On the one hand, the second signifier "...one has to calculate, fifteen and fifteen this makes thirty, how much is left up to forty-five." intends the calculation of the magical number from three circled positions: 15 and 15 makes 30. But with the third circled position one cannot calculate in the known standard way. On the other

hand, one should now calculate in a "reversal manner" with the empty field: Here one has to place the number that represents the difference between thirty and forty-five (the magical number). Later, the addition task is written as a complementary task "15 + 15 + __ = 45". This task displays the form of an addition task with three terms, but in an unusual way. It reflects a generalised structure allowing to express the calculation with a partially unknown number. In this way, new mathematical signs or symbols are created. By applying the calculation scheme for the magical number to the empty field and by writing the complementary task with an unknown term, new mathematical signs are expressed and are represented as abstract icons.



13 + 15 + 17 = 45

15 + 15 + __ = 45

Fig. 10

The epistemological analysis verifies that Kim constructs genuine new knowledge when including the empty field into the mathematical operation to determine the magical number. She argues that one cannot calculate with concrete numbers only, but the algorithm for the magical number can be extended also to arbitrary fields – with or without numbers. The new mathematical knowledge constructed in Kim's argumentation can be described with the help of the epistemological triangle. The new relation (resp. "unknown number" or "variable") is symbolised in two ways; once as a "circled number" and then as a missing term in the addition task. In this domain of representation and of mathematical operation we can observe how Kim works with the "unknown number" in a specific situated manner. Kim places the unknown number into a new mathematical relation with other numbers and in this way she constructs new knowledge; the new mathematical object is created as a relation in the extended and generalised operational structure of the number square.

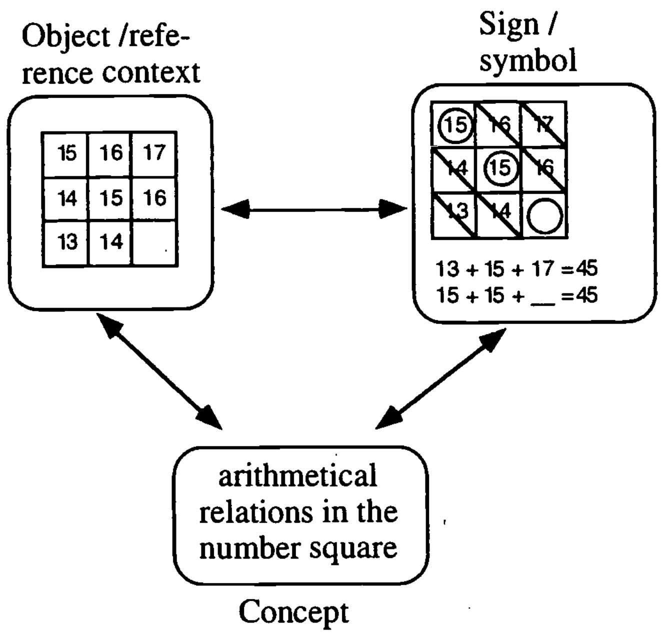


Fig. 11: The epistemological triangle

4.2.4 Eva Repeats the Strategy Using the Magical Number

During the discussion of Kim's strategy, the teacher encourages other children to explain how this "trick" works. Eva argues in the following way.

```
193   E     Well. Ohm, one has to take three numbers out of the
            magical square. Add them up. But not the empty field,
```

```
        there is not much to calculate, all right?
196  E  Yes. And then you get the magical number. Then one must,
        ohm, that, ohm take again three numbers, but now also the
        empty field must be therein. And then you have to … from
        this number, you get then … from these two, one has still
        to go to forty-five.
```

With the first signifier, Eva intends the calculation of the magical number taking three concrete arbitrary numbers from the square. She adds: "But not the empty field, there is not much to calculate, all right?". One cannot and should not use this field for the calculation of the magical number.

The second signifier indicates that the scheme of the algorithm for the magical number shall be transferred to three other numbers: "… take again three numbers, but now also the empty field must be therein." Here, the empty field is identified in a way with a number; Eva characterises a mathematical unknown in a situation specific manner. When transferring the algorithm the following calculation cannot be done in the usual direct manner, because the third term is missing; consequently Eva modifies the calculation procedure: "And then …from these two, one has still to go to forty-five".

In her description, Eva makes the distinction between the *impossibility* to calculate a sum by using the empty field as one of the three terms and the *possibility* to operate with the empty field (as a kind of pre-variable) in the same way as with a known concrete number, provided that the result of the operation is already known. Her argument expresses a fundamental epistemological dialectic between old and new mathematical knowledge. In the frame of understanding subtraction as »taking away« – an example for old knowledge – the task "5 – 7" cannot be calculated; later, with a relational understanding of the number concept – an example for new knowledge – the same task can be calculated by developing and using the new concept of "negative numbers".

5. Interrelations between the Epistemological and Communicative Dimension of Mathematical Signs

Mathematical interactions are social systems, being at the same time characterised by very specific intentions. On the one hand they are *educational* or *instructional* communications, on the other hand *mathematical knowledge* is in some special way the object of these communication processes.

• Interactions between teacher and students are pedagogically intended social communications with the aim to mediate knowledge. This implies a superposition in the autopoietic development of the social communication with an "additional sense", which is the result of the teacher's intention linked with his *teaching* and *instruction*. This intention dominates the educational communication for all participants (teacher and students). When trying to realise their instructional intentions teachers often unconsciously make the assumption that the separation between the social and the psychic system could be bridged directly and the meaning conveyed in the communicative process could be transported instantly and unchanged into the student's consciousness (as it is

tried with the well known interactive funnel patterns; cf. i. e. Bauersfeld 1978; Krummheuer & Voigt 1991, p. 18; Wood 1994, pp. 153ff.; 1998).

• Mathematical communication copes with mathematical knowledge; this implies for the (external) observer (the researcher) to analyse the knowledge which is generated in the course of communication interactively from an epistemological perspective. The analysis of the specific status of school mathematics and its interactively constituted meaning shows that it can be interpreted as a "symbolically generalised communication medium" (Luhmann 1997, p. 316ff.); in analogy to "scientific truth" (Baraldi, Corsi & Esposito 1997, p. 190) one can speak here of "school mathematical correctness".

In the course of the first episode on "triangular numbers", one can observe a kind of "compensating communicative strategy". During the interactive process of constructing new knowledge up to the short phase of the accepted justification, the teacher assists, comments and refutes the children's proposals. She uses a number of similar key words: "... trying to place <u>the next</u> ..." (1); "Is this <u>already correct</u>? ... One could have the impression, but <u>it is not yet quite right</u>" (5); "...this is <u>not yet the right</u>." (10); "Is this <u>the same</u>?" (16); "Ooh, he is very <u>close</u>!" (22); "... Who could now explain why <u>this is correct now</u>? <u>That is correct</u>, you must know." (24). With these descriptions, the teacher indicates that she starts from a very definite idea about the third pattern. In the course of the episode, the students are led to find out what, according to the teacher's opinion, is the only correct pattern. After having given their own, correct proposals for continuing the pattern, now the students do no longer interpret the conveyed signifiers with reference to other *mathematical* signifiers for constructing in this way new epistemological mathematical relations, but they start to seek for the intended signified belonging to the teacher's signifier for attaining in this way the correct solution.

This episode shows in an exemplary manner that the necessary condition for starting the autopoietic behaviour of the *mathematical* communication is "destroyed". First of all, the receiver of the conveyance (a student) can ascribe the given conveyance only to the conveyor (the teacher or another student). The possibility to detach the information of the conveyance that the conveyor "attached" to his conveyance implies the possibility of the autopoiesis of the social system. The students are more and more urged to deduce directly from the teacher's signifiers the intended signified. Such a type of communication takes place with the tacit assumption that mathematical conveyances (the signifiers; or the mathematical signs) possess *unequivocal* information (definite referential signifieds), which can be derived in a communication *about* the conveyances. In this way, the referential links of the signifiers are shifted. The new signifiers no longer refer to mathematical referential contexts immediately but they refer to the interpretation that is postulated by the teacher in the existing reference context. The conveyances become the new, proper object of communication. The signifiers are no longer conveyed in communication for relating them to other *mathematical* signifiers or *mathematical* signs mediated in interaction and thus constituting an interpretation.

During the second episode, the communication has a different character. The attempts to reconstruct the missing number are not dominated by explicit aims of the teacher. Different proposed strategies are allowed: Kevin uses the number pattern of the given number square; then several children reconstruct the missing number by reproducing possible numbers in the border row and border column. Subsequently, Kim presents her ideas to calculate the missing number with the help of the magical number; moreover, three children explain this strategy with their own descriptions. The teacher does not confront these proposals with her own fixed ideas about the correct solution; she moderates and supports the children's construction of new knowledge that develops in the course of interaction. Instead of pushing ahead her own ideas, the teacher places the students' solution strategies into the foreground of the communication process, and this makes possible to maintain a "true" mathematical discourse in this classroom.

The autopoiesis of the communication system is possible only if the signifiers are not connected with the intention of the conveyor strictly but always have to be interpreted in the course of communication in a new manner. For a given signifier there is no definite, fixed signified in communication, and therefore there is no unique universal sign, but different interactively evolving interpretations. Accordingly, a successful authentic mathematical interaction requires that a communicated (new) mathematical sign is not fixed a priori by a given referential object, but the participants have to develop their own different and multiple readings of the communicated sign. Such multiple, evolving interpretations of mathematical signs are possible only if these signs are not explained by linking them with concrete properties of pre given empirical objects but if the referents of mathematical signs are seen as relational structures.

An open referential interpretation of communicated signifiers or of mathematical signs is indispensable for the progression of a *mathematical* communication process. A successful mathematical discourse requires not to fix the mathematical signs definitely and once for all, but to respect a rather open relation between mathematical signs and referential contexts the learner permanently has to establish in new ways in interaction. The realization of an open interpretation of mathematical signs strongly depends on the acceptance of mathematical objects as »symbolic relational structures« in interaction. When mathematical knowledge is reduced to its formal terminology and its logical consistency with reference to fixed referents then the mathematical discourse is in danger to turn into a communication about the definte "correct" interpretation of mathematical signs what in the end is decided by the teacher's authority.

The successful functioning of authentic mathematical interaction requires "open" mathematical objects. A central implication is that every theoretical analysis of possible reasons for the success or the failure of everyday mathematical interaction has to take into account the very specific epistemological nature of mathematical knowledge as "symbolised relations" and it has to reconstruct the social epistemology of mathematics in interaction, i.e. the characteristic forms and situated descriptions for this knowledge. The particularity of the mathematical in classroom interaction – and also in any

reconstructing analysis of such interaction – is basically established by the symbolic, relational character of mathematical concepts; these concepts represent open interrelations between formal sign systems and situated referential structures that have to be negotiated in mathematical interaction.

Consequently, the manner of interpreting mathematical knowledge influences the mathematical communication process essentially: Strictly fixed readings of mathematical signs may cause a paralysis of mathematical communication and they also may lead to a transformed, a ritualised communication. Open readings of mathematical signs with regard to multi relational, structural reference contexts are preconditions for any authentic mathematical interaction. The development of a successful mathematical communication requires to take into account the epistemological particularities of mathematical knowledge and at the same time the specificity of instructional interactions between teacher and students. Direct, intentional teaching and interactive construction of new mathematical knowledge often constitute a fundamental dilemma that cannot be dissolved easily in mathematical discourse.

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Heinz Steinbring
Universität Dortmund
Vogelpothsweg 87
D–44221 Dortmund
Germany
e mail: heinz.steinbring@math.uni-dortmund.de

Constructing a calculator-aware number curriculum: the challenges of systematic design and systemic reform

Kenneth Ruthven, University of Cambridge School of Education

Abstract: *In England, the idea of developing a 'calculator-aware' number curriculum for the primary school has been pursued over a lengthy period. Initial explorations within a government-supported curriculum development project proved a significant influence on the design of a national curriculum for mathematics. Hence, the English experience provides an unusual case in which the attempt to frame and implement a calculator-aware number curriculum has moved beyond localised innovation to an attempt at national institutionalisation.*

This paper will examine this experience, and some important emergent issues of wide interest. First, it will outline the approach of the pioneering Calculator-Aware Number (CAN) project, and its influence on the design of the national curriculum for mathematics. Second, it will investigate the continuing experience of CAN schools after the end of the project, coinciding with the introduction of the national curriculum. Third, it will examine the progress of the cohort of pupils who entered these schools at the start of this period. Fourth, it will relate pupils' mathematical strategies to aspects of the curriculum framework. Finally, it will suggest some lessons to be learned from this experience.

The development of calculator use in English primary education

The Calculator Aware Number project

The Calculator-Aware Number (CAN) project (Shuard et al., 1991) was a component of the Primary Initiatives in Mathematics Education (PrIME) programme, sponsored by the government curriculum agency. Before recruiting schools and teachers, the project team formulated a set of basic working principles:

- classroom activities would be practical and investigational, emphasising language and ranging across the whole curriculum;
- exploring and investigating 'how numbers work' would be encouraged;
- children would always have a calculator available; the choice as to whether to use it would be the child's not the teacher's;
- the importance of mental calculation would be emphasised; children would be encouraged to share their methods with others;
- traditional pencil-and-paper methods of column addition, subtraction, multiplication and division would not be taught; children would use a calculator for those calculations which they could not do mentally.

Between 1986 and 1989, the project team worked collaboratively with several clusters of primary schools and teachers. The tangible outcomes of the project were these curriculum principles, illustrated through a range of classroom activities and accounts, rather than a structured curriculum plan. This reflected the pedagogical approach adopted by the project teachers:

The teachers began to develop an exploratory and investigative style of

working, which allowed the children freedom to take responsibility for their own learning. Topics for exploration took the place of practice exercises as the prevailing classroom activities. Because the number sections of the mathematics schemes used in the schools had been discarded, the teachers were able to move towards a different style of working. No longer did they have to 'cover' set topics in a set order. They began to notice that children's mathematics learning did not seem to progress in the ordered linear way in which it was traditionally structured. Individual children seemed to be putting together the network of mathematical concepts in their own individual ways (Shuard et al., 1991, 44).

The project team reported favourable findings from a participating local authority in which the mathematical achievements of first cohort of project pupils were compared with those of peers in other schools (Shuard et al., 1991). The following year, the second cohort was involved in a similar comparison. The comparison was still favourable, but less strongly so (Foxman, 1996).

Framing the National Curriculum

When the decision was made to introduce a national curriculum, it was not surprising that the CAN project and its personnel should influence the proposals of the working group charged with devising the programme of study in mathematics:

> The universal availability of electronic calculators is changing our views about the kinds of facility in computation which are needed of pupils ... Along with the ability to use and interpret the results obtained from calculators there is general agreement that a greater facility in mental arithmetic should be encouraged (National Curriculum Mathematics Working Group, 1997: 8).

The rejoinder of the then minister of education was that:

> it must be important that pupils themselves understand and are proficient in the various mathematical operations that can now be done electronically (Department of Education and Science, 1988: 100).

Later, he asked still more more pointedly:

> Is it justifiable to exclude the pencil and paper methods for long division and long multiplication from the attainment targets for mathematics, as the mathematics working group have recommended? (National Curriculum Council, 1988: 92).

The final programme of study achieved a superficial resolution of this conflict through the deliberate ambiguity of its references to 'non-calculator' methods of computation, with a more explicit account of the approach favoured by the working group confined to the accompanying pedagogical guidance which suggested that:

> For most practical purposes, pupils will use mental methods or a calculator to tackle problems involving calculations. Thus the heavy emphasis placed on teaching standard written methods for calculations in the past needs to be re-examined. Mental methods have assumed a greater importance through the introduction of calculators, and the use of mental methods as a first resort in tackling calculations should be encouraged. Work should be based in a firm understanding of number operations, applied to problems in a variety of

contexts, and encourage pupils to select from different methods with confidence depending on the nature of the problem and the suitability of the method. (National Curriculum Council, 1989, E6).

Discussing the new national curriculum, the CAN project team wrote:

The CAN project has been very fortunate that National Curriculum mathematics is much in line with the thinking the project has developed. Teachers who work in the project have welcomed the emphasis in the National Curriculum on a broad curriculum in mathematics, on using and applying mathematics, on the encouragement for children to use their own methods of calculation, and on the possibility of using calculators for much of the work. Teachers in project schools have commented on a number of occasions that they need to make many fewer changes to their curriculum than other teachers (Shuard et al., 1991, 71).

Calculators in the National Curriculum

The programme of study was designed within a levelled framework intended to prescribe progression within teaching programmes and to describe the resulting development of pupil capabilities. The expectation was that pupils in the lower primary (infant) phase (known as Key Stage 1) would cover the material of levels 1 to 3, and that by the end of that phase (at age 7), the great majority would be assessed as attaining a level between 1 and 3, with the average pupil at level 2. Similarly, during the upper primary (junior) phase (known as Key Stage 2), it was anticipated that, by the close of the phase (at age 11), the great majority of pupils would achieve a level between 3 and 5, with the typical pupil at level 4, having covered the programme of study to level 5; with level 6 'intended for only the most able children performing significantly above the normal range' (School Curriculum and Assessment Authority, 1995).

Table 1 abstracts, from the mathematics programme of study, those references to calculators relevant to the primary school (excluding levels 1 and 6 which make no references of this type). This is certainly a curriculum which acknowledges the calculator; but is it a calculator-aware curriculum? References to 'using a calculator' start warily. At levels 2 and 3, this is to be for checking or 'where necessary'. Half of the examples concern money calculations. At levels 4 and 5, cautiously affirmative references to 'using a calculator where necessary', working 'with the aid of a calculator' and 'check[ing] using a calculator' are easily outnumbered by prohibitive references to working 'without a calculator' or 'using non-calculator methods'. A product of the fudged compromise over the relative emphasis to be given to mental and written, standard and nonstandard methods, this negative phraseology comes over as subversive of calculator use.

While there is an emphasis on interpreting results 'on a calculator' and 'reading calculator displays', there is only one reference to the need to 'translate the problem...in order to use a calculator'. However, there is one important innovation, in the form of 'trial and improvement' as a solution strategy. Clearly dependent on calculator availability, although avoiding reference to it, this is the sole recognition of the possibility of distinctive calculator methods.

Table 1: Calculator references in National Curriculum mathematics programme of study (1989 & 1991)

Level	Programme of Study	Example
2	Describe current work, record findings and check results.	Devise stories for adding and subtracting numbers up to 10 and check with calculator or apparatus. [1989]
3	Solving problems involving multiplication or division of whole numbers or money, using a calculator where necessary.	Find the cost of four calculators at £2.45 each. [1989, 1991]
	Using decimal notation in recording money.	Know that three £1 coins plus six 1p coins is written as £3.06, and that 3.6 on a calculator means £3.60 in the context of money. [1989, 1991]
	Appreciating the meaning of negative whole numbers in familiar contexts.	Understand a negative output on a calculator. [1989, 1991]
4	Adding and subtracting two 3-digit numbers, without a calculator.	Work out without a calculator how much longer 834 mm is than 688 mm. [1989]
	Multiplying and dividing 2-digit numbers by a single-digit number, without a calculator.	
	Solving addition and subtraction problems using numbers with no more than two decimal places, and multiplication and division problems starting with whole numbers.	Work out how many chocolate bars can be bought for £5 if each costs 19p, and how much change there will be. [1989, 1991 without the aid of a calculator] Find out how many 47-seater coaches will be needed for a school trip for a party of 352. [1991 with the aid of a calculator, interpreting the display]
	Reading calculator displays to the nearest whole number and knowing how to interpret results which have rounding errors.	Interpret 7 ÷ 3 x 3 = 6.9999999 if it occurs on a calculator. [1989]
	Recording findings and presenting them in oral, written or visual form.	Translate the problem of finding the number of 28p packets of crisps that can be bought for £5 into 500 ÷ 28 = in order to use a calculator; record the result as 17.857142 and thus decide that the result is 17. [1991]
5	Understanding and using non-calculator methods by which a 3-digit number is multiplied/divided by a 2-digit number	Use any pencil-and-paper method to find the number of coaches needed to take 165 Year 7 pupils on an outing if each coach has 42 seats
	Calculating fractions and % of quantities using a calculator where necessary.	Calculate 15% of 320; $^{3}/_{5}$ of 170 m; 37% of £234; $^{1}/_{10}$ of 2 m. [1989]
	Using 'trial and improvement' methods and refining.	Estimate the square root of 10 and refine to 3 decimal places. [1991]
	Approximating, using significant figures or decimal places.	Read a calculator display, approximating to 3 significant figures. [1989]
	Make and test simple statements.	Explore the results of multiplying together the house numbers of adjacent houses, make a statement about the results, and check using a calculator. [1989]

Calculators and national testing

As the first pupils completed each Key Stage, a national programme of assessment was introduced, incorporating external testing. Again, at first sight, the place of calculators is acknowledged. To take the example of the 1995 Key Stage 2 mathematics tests, the opening instructions explain the use of icons to indicate where the use of a calculator is stipulated or prohibited, and state that its use is permitted on items without an icon. Of the 40 items across the two test papers, 14 prohibit use of a calculator. A number of these items took the form of missing digit problems presented in the vertical format of standard written methods, rather than in a more methodically neutral, horizontal format. Such presentational features, combined with concern about the acceptability of nonstandard approaches when meeting requirements that pupils should show their working, led many teachers to conclude that the testing process entailed a preference towards standard written methods. Use of a calculator is stipulated on only one item within the two test papers. Only one further item might be regarded as calculator affirmative both uses the machine as its setting, and permits its use.

Beyond the optimistic speculation, then, and beneath the veneer of calculator recognition, both national curriculum and national testing emerge as more 'calculator-beware' in spirit than 'calculator-aware'.

Impact of the reforms on professional practice

The official evaluation of the implementation of the new curriculum found that teachers made little reference to the non-statutory guidance, and were already confident of their teaching of number (Brown et al., 1993). Combine these factors with the tone of the curriculum document and national testing, and it is not startling to find school inspections reporting a largely unchanged pattern of professional practice:

> In all the schools visited the teachers placed a strong emphasis on the written practice of the basic operations of addition, subtraction, multiplication and division. This dominated the work in half the schools ... For many schools there was an imbalance between the written practice of basic number skills and mental, oral and practical work involving number...The skills of using a calculator were neglected in a high percentage of the schools; in only a tenth of the lessons seen were calculators used ... Only in the most successful schools was a policy of calculator use thought out in relation to the acquisition of mental and 'pencil-and-paper' skills (Office for Standards in Education, 1993: 9-11).

However, let us now turn to examine a group of schools which had been involved in the CAN project.

The evolution and long-term impact of CAN in project schools

As the CAN project drew to a close in the summer of 1989, the National Curriculum came into force. A Cambridge research study examined the mathematical experience of a cohort of pupils who entered reception class during the 1989/90 school year, progressing to the final year of primary education in

1995/96. Data was gathered in six neighbouring primary schools, all covering the full primary phase. Three of these schools participated in the CAN project between 1986 and 1989, and in the much smaller-scale continuation project to 1992.

The influence of national reforms

The accounts of teachers in the post-project schools indicate how their practice was influenced by the national reforms. The general tenor was of seeking to retain valued principles and activities from CAN; to establish the legitimacy of these principles and activities within the new order; and to tighten aspects of their implementation.

> It's made very little difference in the way I taught maths personally. Very little. The other thing it had done in this school - it would have happened without the National Curriculum anyway - it refocused people's ideas on the structure that was needed right through the year groups. [Richard]
>
> I don't think the delivery has done anything except become a little cleaner. It's forced me to sharpen up my act. I think if we want to hold on to this we've got to really be able to...justify it. We've got to show that it can be done in this way...It's not that we're being subversive. It's there in the National Curriculum. It's that it's not terribly common what we do. I feel we have to justify it. [Stephanie]

The substantive influence of these external pressures had been threefold. First, some of the expansiveness of investigative work had gone, with a stronger tendency to structure and foreclose an activity than in the past.

> I think what's altered is that pre National Curriculum I would have had a much broader, less clear picture of what I wanted to get out of a session, and therefore I would have been more open to other things that came up, and would have been able to pick up on those other things and delve a bit deeper with the children. I think what its forced me to do is to keep on a much narrower pathway. [Stephanie]
>
> I would probably [have done] more longer investigations. Now, I do a lot more shorter activities, to get the coverage in a year. [Tracy]

Second, although calculators continued to be readily available in the classroom, there were occasions when their use was challenged or proscribed.

> There might be some negative reference: 'How are you going to do that, to use the calculator?'... So there might be some kind of sideways swipe at it. [Stephanie]
>
> The policy is that they are there all the time, but [sometimes we say] 'Actually for this activity I don't want you to use a calculator'. [Tracy]

Third, standard written methods of recording and calculating were now taught. At Key Stage 1, teachers felt obliged to introduce pupils to vertical methods of recording, and to 'sums' presented in this way.

> [Tests] brought back more formal work. Having been through the first maths test - it was formal sums set out, which the children were not used to seeing. That's when we decided that we were going to have to introduce formal sums set out ready for them. They'd set them out any way they wanted. [Rachel]

The first [tests] affected [the Key Stage 1 teachers]. After being used to CAN and recording in their own way, after the first [tests] they were suddenly putting everything in columns. Worried children wouldn't understand. [Tracy]

At Key Stage 2, standard written methods were more prominent, although the expectations of secondary schools were often cited as the direct reason for this.

Conceptually they had a pretty good insight of number, but on paper, when they were presented with any kind of traditional sum on paper, which they would be at the high school, the children were worried and the parents were worried. We just felt we had to teach it them. [Richard]

[Emphasis on pupils' own strategies] has caused some difficulty on transfer. If they don't use a traditional method, it is considered that they can't do it. So even if it is not their preferred option, they have been taught a traditional method so they know how to use it, for that reason. [Stephanie]

Awareness of tensions within the CAN model

Teachers had developed a more subtle view of the complexities of supporting pupils' development of methods of calculation. They were conscious of having to manage an important tension between personal insight and authenticity on the one hand, and accuracy and efficiency on the other.

We've built on what the children have actually used... try out the different methods and encourage them to find the one they feel most happy with...There is one child I did change ... because he was not accurate, and he was slow. His methods were so long-winded ... It is important that children do have quick accurate methods. One of the things which is really important is ... that the children have conceptual awareness of what's happening with the numbers. If they know that then they are secure. But some of the children are going through the motions with methods they don't understand. [Richard]

We put [pupils' strategies] very high up [but] the older a child is the more likely I am to say 'That's fine but it takes twice as long as this one'...There is kind of a seductiveness in working investigatively...and they forget that there can be a directness that is important as well. [Stephanie]

Another issue which emerged from teachers' accounts related to the systematisation of CAN within schools. Salient themes here were of the uncertainty and effort arising from the abandonment of a conventional mathematics scheme, with limited alternative means of support.

I came to this school having a fairly sketchy knowledge of CAN, having seen it in operation, but having a sketchy knowledge about how to proceed, and finding no resources. The resources there were were photocopiable resources and packs. There would be one copy so you had to have copies made. It was incredibly hard work preparing lessons each day. [Richard]

We more or less abandoned schemes and went in at the deep end with CAN. Two members of staff in particular were heavily involved with it and went to meetings and then fed back to staff. But, as I remember, you were left floating about a bit and not knowing what was right or what was wrong to do. I remember thinking if I just give them investigations and problems and help

them to solve them, that's how I'll survive this. You felt as thought there was nothing to support you...When you have a scheme, you don't use it rigidly, but you know it is there as a support for you if you need it...The two who went to the meetings seemed to be more capable at it. You needed to go to the meetings. They got the ideas from the meetings. We just got the 'trickle down'. [Tricia]

In these circumstances, it was difficult to plan for continuity and progression in children's learning, both from lesson to lesson and from year to year.

In CAN it was difficult to know how to progress. After an exciting lesson you thought 'Where do I go now? Where do I take them next?' You'd be rooting around for ideas. [Tricia]

There was no structure through the school...I noticed in my first year that teachers were photocopying an investigation for Year 3 children and the same one was being used for Year 6 children, and nobody knowing what the children had covered at all. [Richard]

One important effect of the national curriculum and assessment reforms had been to press schools to develop a more systematic approach to number, building on the national frameworks. Indeed, this was an important strand of the 'focusing' and 'sharpening up' attributed to the reforms in the opening quotations of the previous subsection.

The long-term influence of CAN on pupil attitude and attainment

The Cambridge study compared the progress of pupils in the post-project schools with that of their peers in the non-project schools (Ruthven, Rousham & Chaplin, 1997). National assessment levels awarded at the end of Key Stage 1 (aged 7) and of Key Stage 2 (aged 11) were analysed, to determine whether the odds of high or low attainment in mathematics differed between schools, after taking account of the general scholastic attainment of pupils.

At Key Stage 1, the odds of high mathematics attainment (level 3) were found to be significantly greater in the post-project schools, as were the odds of low mathematics attainment (level 1), with no individual school in either group diverging from this pattern. In the post-project schools, then, pupils were more likely to be found at either extreme of the attainment distribution. Comments from the teachers of the CAN cohort suggest that a plausible explanation is that the emphasis on investigative and problem-solving tasks within CAN produced a greater differentiation of experience between pupils, creating higher expectations of, and greater challenges for, successful pupils, but providing less systematic structure and support for the learning of pupils who were making poor progress.

One of the things that keeps me working in this way is that low ability children don't get so complexed about it...I think the weak ones do benefit from a lot of talk and being involved in things. They are not excluded because they didn't manage to get quite as much done. And for the high flyers, I think it is a brilliant way of working because they can go as far as they want; there is no ceiling on them. They can take off and go a long way with things and the talk is good for them at that end. [Stella]

You always thought 'Do children really understand - particularly the less able children? Do they really understand what it is they are doing? I think it showed up with more able children, if they got an answer which was clearly wrong, they knew it was wrong. But that estimating thing was not there with less able. You'd have outrageous answers and they wouldn't have a clue it was not right...I didn't ensure that, like I do now, that children could add up quickly, mentally in their head...Looking back I think I should have done that. That would have helped the less able with their estimating...Some children struggled, but the children who had a gift for maths did very well. If they had a good understanding of the structure of numbers and estimating skills, then they went quite far. [Tricia]

However, this differential pattern did not persist through to the Key Stage 2 results, where no significant group differences were found between non-project and post-project schools, although in one non-project school the odds of low mathematics attainment (level 2 or 3) were significantly greater.

Pupils were also tested on a range of concepts related to place-value, using five blocks of items involving larger whole numbers and numbers with decimal parts. Relative to overall mathematical attainment (as assessed by national test levels), patterns of achievement on these blocks of items did not produce any significant effects interpretable as differences between the groups of schools.

Pupils also completed an attitude questionnaire. No significant differences between schools were found on the constructs *Enjoyment of number work, Reluctance to use a calculator* and *Calculator use as a support for number learning.* However, on the construct *Preference for mental over machine calculation,* there was a discernible, but non-significant, trend for pupils in the post-project schools to be more positive. And on the construct *Mental calculation as a support for number learning,* pupils in the post-project schools were significantly more positive.

A further study strengthened these findings (Ruthven, 1998). It examined the strategies used by a structured subsample of pupils -excluding those at the extremes of attainment- in tackling a set of number problems. Whereas 38% of pupils from post-project schools tackled all problems mentally, without any use of written or calculator computation, and only 24% had recourse to these supports on more than one occasion, the corresponding proportions for non-project schools were 19% and 52% respectively. Pupils from post-project schools were not only more prone to calculate mentally, but also more liable to adopt relatively powerful and efficient strategies for doing so.

These outcomes seem to reflect contrasting numeracy cultures in the two groups of schools. In the post-project schools, pupils had been encouraged to develop and refine informal methods of mental calculation from an early age; they had been explicitly taught mental methods based on 'smashing up' or 'breaking down' numbers; and they had been expected to behave responsibly in regulating their use of calculators to complement these mental methods. In the non-project schools, daily experience of 'quickfire calculation' had offered pupils a model of mental calculation as something to be done quickly or abandoned; explicit teaching of calculation had emphasised approved written methods; and pupils had little experience of regulating their own use of calculators.

Pupils' computational strategies and the curriculum framework

We now return to consider the programme of study which guided the primary schooling of these pupils, in the light of further evidence from the Cambridge study. At level 4, pupils are expected to be able to 'Find out how many 47-seater coaches will be needed for a school trip for a party of 352', 'with the aid of a calculator, interpreting the display'; and at level 5, to 'Use any pencil-and-paper method to find the number of coaches needed to take 165 Year 7 pupils on an outing if each coach has 42 seats'.

In the Cambridge study, then, pupils in the structured subsample were presented with a version of the 'coach problem': *313 people are going on a coach trip. Each coach can carry up to 42 passengers. How many coaches will be needed? How many spare places will be left on the coaches?* (Ruthven & Chaplin, 1997) Pupils had been told that they could work out the problem however they liked; using their head, pen and paper, or calculator, or a mixture of these. Drawn from across the six schools, their attainment distribution corresponded very closely to that which would have been found in a similarly truncated national sample.

Direct strategies for the coach problem

The first strategic idea used by most pupils was some form of direct division. Some attempted to use a non-calculator method of computation (typically a standard written method). None was able to accomplish the computation successfully. Around a third proposed, nonetheless, an incorrect solution; around a third switched to use a calculator to carry out the division; and around a third abandoned the problem, or changed approach.

Many pupils appeared surprised by the result of the calculator division.

Karen keys [313][÷][42][=]7.452380952
Karen: Whoopsee!
Interviewer: What have you got?
Karen: I've got loads of numbers.
Interviewer: Are they any good to you?
Karen: No
Interviewer: Why?
Karen: I don't know
Interviewer: Can you understand what they say?
Karen shakes her head
Interviewer: Okay.
[pause]
Rekeys [313][÷][42][=]7.452380952
[pause]
Keys [42][÷][313][=]0.134185303

Karen's initial interpretation of the string of digits on the calculator display was that she has miskeyed; and when rekeying produced the same result, she then supposed that she must have reversed numbers within the calculation. Behind such responses lay an expectation -or perhaps an aspiration- that the result of a division should be a whole number. Certainly, the 'commonsense' of this problem points in this direction, as in Tom's initial reaction: 'You can't split a coach up'. But other

factors are also in play. For these pupils, from their experience of mental and written calculation, division was a process yielding whole numbers as quotient and remainder. By contrast, the calculator provides a (not necessarily whole) ratio. The results of the two processes are the same only for 'exact' division. Karen did not recognise the string of digits as incorporating a decimal resulting from a division. Yet -on the evidence of testing a few weeks earlier- she was capable of working successfully with the one- and two-digit decimals specified by the curriculum.

Although Damon did recognise the result as that of a division, his response illustrates a common difficulty: that of confusing decimal part with remainder.

Damon keys [313][÷][42][=]7.452380952
Interviewer: What have you got? Any good?
Damon: About seven coaches.
Interviewer: About seven coaches.
[pause]
Damon: I think it's four.
Interviewer: Four.
Damon: Yeah.
Interviewer: Spare places?
Damon: Yeah.
Interviewer: How did you work that bit out?
Damon: Because it's seven point four.

None of the pupils attempting this calculator division proceeded successfully to interpret the calculated result as implying 8 coaches. Around a quarter carried forward an estimate of 7 into a further strategy.

Vera shows how the calculator itself could be deployed to make sense of the result.

Vera keys [313][÷][42][=]7.452380952
Interviewer: What does it say?
Vera: I don't know
Interviewer: Okay, so that's not helpful. What else could you do?
[pause]
Keys [313][÷][14][=]22.35714285
Interviewer: Oh there's another one.
[pause]
Interviewer: What do you think you could do?
Keys [42][÷][10][=]4.2
Vera: Four point two.
[pause]
Interviewer: Any ideas?
Keys [42][x][7][=]294
Interviewer: Why did you try that?
Vera: I don't know really

Vera gave little away, and any interpretation of her intermediate moves must be speculative. One conjecture is that they enabled Vera to build a bridge between incongruous digits and familiar decimals. The division of 313 by another number resembling 42 confirmed the 'appropriateness' of the original result. By then dividing 42 by 10, Vera produced a simple decimal, completing the process of

anchoring the unexpected result in a familiar category, and supporting the idea of carrying forward 7 to the next move. In this interpretation, then, these moves become transitional ones of sense-making.

Kylie, too, carried forward 7, to the written multiplication 7 x 42 = 294, followed by the written subtraction, 313 - 294 = 19, in effect translating the ratio result of the calculator division into quotient and remainder form, although she then fell for the tempting interpretation that this implied 7 coaches and 19 spare places.

These episodes highlight the special character of calculator division and the demands that it makes on pupils' mathematical understanding. Carrying out this apparently simple computation proved to be anything but the mindless, automatic process that using a calculator is commonly reputed to be.

Indirect strategies for the coach problem

Less direct strategies were also used to tackle the problem. Two pupils adopted forms of trial and improvement as their opening strategy, using a calculator to carry out the trial computations. Both solved the problem successfully in this way.

Joanne employed trial multiplication.

Joanne keys [42][x][12][=]504

Interviewer: Why did you do that?

Joanne: Forty two times any number, but it was a bit too high.

Keys [42][x][10][=]420

Joanne: Forty two times ten, that's too high so..

Keys [42][x][8][=]336

[pause]

Joanne: They'd need eight coaches, and they'd have..

[pause]

Joanne: Twenty three places left over.

Note Joanne's use of the calculator to multiply 42 by 10. Using the machine to carry out computations in a predictably routinised way, Joanne freed her attention to monitor her strategy and interpret results. She was very capable of doing such a calculation mentally; a few minutes earlier she had successfully multiplied 24 by 10 in her head, answering within one second.

Around 30% of pupils employed some form of repeated addition as their opening strategy, and a further 16% took up this type of strategy at a later stage. Around a quarter of such attempts used the calculator for computation. Liam's experience was typical.

Liam: So you need to add up how many forty twos go into. I'll do that. I'm sure you could do it a quicker way but, well.

Keys [42][+] [42][+] [42][+] [42][+] [42][+] [42][+] monitoring intermediate totals

Keys [252][+]

Liam: Oh no!

Interviewer: Where have you got to? What's happened?

Liam: Hmmm. Don't know.

The calculator leaves no trace of intermediate results, making any extended calculation incorporating a parallel mental computation extremely vulnerable to

failure through miskeying or losing track of where the calculation has reached. All calculator attempts of this type broke down in this way. Pupils who tried to compute mentally without recording had similar difficulties.

Reviewing the curriculum framework

These pupil responses to the coach problem can be related to characteristics of the National Curriculum under which their primary schooling had been conducted; notably, to features of the progression implied within the levelled framework.

First, pupils such as Karen and Damon had difficulties in making sense of the result of the calculator division. Interpretation of division calculations certainly features explicitly in the curriculum framework in the form of 'understanding remainders in the context of calculation and knowing whether to round up or down' at level 3; and 'reading calculator displays to the nearest whole number' at level 4. For this problem, however, pupils needed not so much to read the display to the nearest whole number, but to recognise it as a number lying between 7 and 8. Equally, they needed not so much to understand remainders, but to distinguish them from decimal parts. This highlights the importance of seeing curricular objectives as embedded in a wider conceptual system.

Decimals appear in the curriculum framework for the first time at level 3: explicitly in 'using decimal notation in recording money', and (hence) implicitly in 'solving problems involving multiplication or division of whole numbers or money, using a calculator where necessary'; then at level 4 in the form of 'using, with understanding, decimal notation to two decimal places in the context of measurement' exemplified as 'read scales marked in hundredths and numbered in tenths (1.89m)', and 'solving addition and subtraction problems using numbers with no more than two decimal places'. Gaining familiarity with these monetary and measurement contexts and the corresponding calculation schemes is undoubtedly important, but too literal a treatment risks encouraging a view of the decimal point as a 'separator' within a system of super- and sub-ordinate units such as pounds-and-pence or metres-and-centimetres. Not until level 6, is there explicit reference to underlying relationships between division, fractions and decimals in the form of 'understanding and using equivalent fractions and equivalent ratios and relating these to decimals'.

A similar issue arises in relation to checking. Repeating the original computation, as illustrated by Karen, appeared to be the major strategy employed by pupils to check their calculations and solutions. There was no evidence, in particular, of pupils mentally calculating an approximate value for 313÷42, either as a rough check on a non-calculator or calculator division, or as the basis of some further strategy. The pedagogical guidance certainly emphasises this issue.

> Whether using mental, pencil and paper or calculator methods, pupils must be able to estimate, approximate, interpret answers and check for reasonableness. The development of these skills is crucial to pupils becoming effective and confident in performing calculations, and should match the development of methods and techniques for calculating. (National Curriculum Council, 1989).

However, closer analysis suggests that the curriculum framework does not actually make provision for suitable methods of checking. The type of approximate mental

calculation appropriate for estimating the result of this level 4 calculator division or level 5 written division does not feature until level 6, in the form of 'using estimation and approximation to check that answers to multiplication and division problems involving whole numbers are of the right order', exemplified by 'Estimate that 278 ÷ 39 is about 7'. And no explicit reference is made within the framework to the more viable alternative of checking the solution through working back to the original data.

A third issue is the assumption that such checks should be mental; related to the wider assertion -prominent within the pedagogical guidance- that pupils should be encouraged to view mental methods as a first resort. Clearly, developing pupils' expertise in mental calculation is an important curricular goal, not least because components of this expertise underpin estimation through approximate calculation as well as written methods of calculation. But an overgeneralised insistence on prioritising mental calculation can impede pupils' thinking and inhibit their learning of other aspects of mathematics. We have already seen how, while focusing on a higher level solution strategy, Joanne employed a calculator to execute a computation which, under other circumstances, she had shown herself perfectly capable of carrying out mentally. Equally, the reluctance of some pupils to make use of a calculator to implement a direct division led them to adopt alternative strategies based on addition, which they felt better able to compute mentally, but which often proved unreliable.

Again, the programme of study largely ignores such issues. In particular, there is no explicit reference to the importance of developing effective use of the calculator constant. This reflects a more general lack of vision. With the exception of trial-and-improvement, there is no recognition of the possibility of distinctive calculator methods to parallel those of written computation.

Here, Kylie's approach points the way to a distinctive calculator-based method of 'quotient and remainder' division. Figure 1 shows a systematic version, capable of incorporating a range of checks. The scheme incorporates two parallel number lines, spatially encoding the relationships between different elements of the record. Might such a calculator method serve to cap the calculator-aware number curriculum, drawing together and integrating important strands, in much the same way as the proponents of the written long division method see it as capping the traditional number curriculum?

Lessons of the English experience

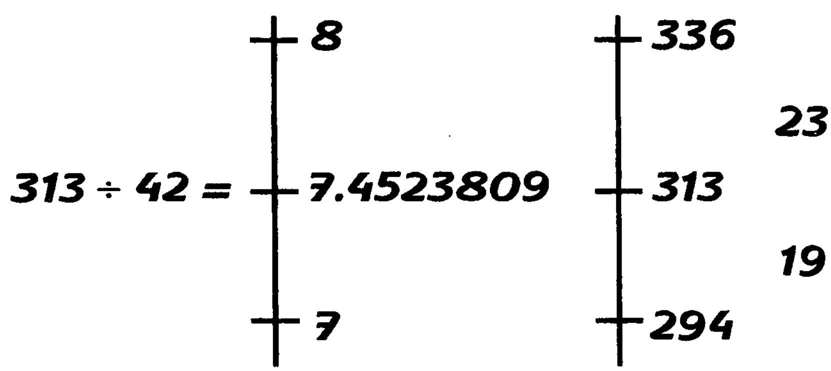
There are two important lessons to be learned from the English experience. The first is that the design of a calculator-aware number curriculum calls for more thoroughgoing analysis -both of content and progression- than it has received to date. This emerges both from the accounts of the teachers involved in CAN, and from the analysis of pupils' performance in terms of the National Curriculum framework. This is an important issue which deserves attention from researchers; they should aim to develop and evaluate both a systematic design for a calculator-aware curriculum and an appropriate pedagogy of calculator use (Ruthven, forthcoming).

The second lesson is that, however well designed such a curriculum, its successful implementation is likely to depend on it being treated as part of a coherent and committed process of school development -and ultimately of systemic reform- rather than as the isolated responsibility of individual teachers. This can be seen both in the problems of coordination which emerged within the CAN schools, and in the limited impact of the National Curriculum reforms on the teaching of number across the system. These organisational issues are often neglected by mathematics educators, but they emerge as critical to the successful institutionalisation of change.

One and a half of these lessons have now been learned in England. A National Numeracy Project has piloted and refined a more systematic programme for the primary number curriculum; and its implementation has been treated as an aspect of school improvement, calling for sustained commitment and support from school managers, as well as the involvement and professional development of all teachers. This will form the basis of a forthcoming national programme (Department for Education and Employment, 1998).

And the missing half? The new programme of study will be systematic, but far from calculator-aware. In the moral panic over standards of numeracy in English primary schools, the calculator has been cast as scapegoat, despite evidence that it was little used; and that, where it was used, this was not to the detriment of pupils' achievement (School Curriculum and Assessment Authority, 1997a; 1997b). The political compromise that has emerged confines use of calculators to the last two years of primary education, treated as a relatively isolated element of the number curriculum, concerned with teaching children 'when it is, and is not, appropriate to use a calculator' and 'the technical skills needed to use it constructively and efficiently' (Department for Education and Employment, 1998: 53).

Figure 1:Calculator-based method for 'quotient and remainder' division



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PLENARY PANEL

Theme:
Doing Research in Mathematics Education in Time of Paradigm Wars

Coordinator: Anna Sfard

Panelists: Ellice Forman
Stephen Lerman
Pearla Nesher

Plenary panel:
Doing research in mathematics education in time of paradigm wars

Coordinator: Anna Sfard, The University of Haifa, Israel

Panelists: Pearla Nesher, The University of Haifa, Israel
Stephen Lerman, South Bank University, London, UK
Ellice Forman, University of Pittsburgh, USA

1. Panel overview

These days we are witnessing an unprecedented proliferation of research frameworks in human studies at large, and in the field of mathematics education in particular. Along with the wide range of traditional cognitive approaches and their updated current versions, there is a steadily intensifying socio-cultural trend including a whole spectrum of research frameworks, from situated cognition through distributed cognition to a number of schools that include the term 'discourse' in their name. This unusual situation provides researchers with many exciting possibilities, but at the same time creates communication problems and leads to partitioning of the community into a growing number of 'camps', only too likely to argue against each other (see, e.g. Anderson et al., 1996; Greeno, 1997). The aim of the panel is to raise and discuss some questions about research, which in these circumstances must urgently be answered. This will be done by considering a research proposal of a Ph.D. candidate who wants to devote her dissertation to mapping student's difficulties with negative numbers. The panel members will act as a Ph.D. committee, which has to decide whether to accept the proposal. Each of the panelists will review and evaluate the proposal from the vantage point of a certain well-defined research framework: Pearla Nesher will take a *cognitive approach,* with its roots in Piagetian theory, Steve Lerman will speak from a strong *sociological/postmodern position*, and Ellice Forman will try to present a more moderate vision by taking a *developmental social constructivist* stance. Discussion will then be opened to the audience who, by the end of the session, will be required to decide whether the proposal should be accepted, substantially modified, or simply rejected.

2. Research proposal by Ph.D. candidate Angelica L. Pabst: Investigating and informing middle school students' conceptions of the negative numbers

1. The theme of the study and its rationale. Negative number is one of the most fundamental, and at the same time most problematic mathematical concepts taught today in our middle schools. Its central role in modern mathematics can hardly be overestimated. The appearance of negative numbers in Medieval Indian mathematics and then through Arab texts in Renaissance Europe (see e.g. Boyer, 1985; Kline, 1980) was as a groundbreaking event, which led to important developments in algebra and beyond. One can hardly envision the impressive mathematical advances of the last three centuries without it. Neither can one imagine an educated citizen of the third millenium society who is not familiar with the notion.

In spite of the retroactive obviousness of their usefulness and indispensability, the negative numbers had a difficult and prolonged birth. The doubts and mistrust with which Renaissance mathematicians greeted this important notion had been obstructing their progress for centuries. Today these difficulties should be of special interest for the mathematics education researcher. The historical phenomena may be indicative of an inherent difficulty of the concept – a difficulty that is only too likely to hinder its learning in our schools today. There is much evidence that, indeed, the idea of negative number is a major challenge to many learners. To illustrate the nature of the most common difficulties, let me use the personal testimony of the French writer Stendhal who, after stating his inability to cope with the concept on his own, issues the following complaint about his teachers: "Imagine how I felt when I realized that no one could explain to me why minus times minus yields plus!" (Stendhal, quoted in Hefendehl- Hebeker, 1991). That Stendhal's case is not exceptional is evidenced, among others, by a number of studies (see e.g. Hart, 1981).

In spite of the importance of the concept, the issue of its difficulty and, more generally, the question of how it is being acquired by the learner has not been given the due attention by the mathematics education community. When surveying the work done during the last few decades one cannot help being surprised by the striking difference between the abundance of studies devoted to basic arithmetic and algebraic concepts, to the concept of function and to rational numbers on the one hand, and the scarcity of research on negative numbers, on the other hand. Thus, considering the fact that our knowledge and understanding of the topic are still fragmentary, there a study such as the one to be presented in this proposal seems to be urgently needed.

2. The purpose of the study. The study has a double goal. First, it aims at enhancing our knowledge of the processes underlying the acquisition of the concept of negative number and our understanding of the difficulties which obstruct this process. Second, it is expected to lead to operative implications about the ways in which teaching and learning of the subject may be improved.

With respect to the first, more theoretical, goal, the intention is to get to know types of misconceptions about negative numbers and kinds of repetitive procedural errors that commonly appear in the middle school population. The study will suggest a categorization of students' mistakes and will attempt an explanation of their deeper cognitive roots.

The second goal is to formulate and to test some operative didactic ideas based on the findings on students' misconceptions and procedural errors. In the most extreme case, if the results of our investigations show a definite prevalence of faulty conceptions and procedural difficulties in the middle school population, we may conclude that the children of this age are not yet developmentally ready for this intricate topic. In this case, we shall recommend delaying it to later grades. With the intention to improve the learning of the subject we will then develop a new teaching unit on negative numbers aimed at students of the proposed age.

3. Theoretical framework. Thanks to a few generations of cognitive psychologists, but mainly to Piaget, the human mind is no longer conceptualized as a mere reservoir of accidental concepts, facts, and procedures but rather as a well organized structure, composed of ever growing mental schemata. Thus, once a person learns a mathematical concept, this concept becomes a part of a web-like cognitive system that connects it to other mathematical and non-mathematical notions. This is what keeps this new concept conveniently in the student's long term memory, and this is what makes it meaningful in his or her eyes. Cognitive schemata, and thus individual understanding of concepts, may change from person to person, and in any given person all these cognitive entities may vary considerably over time; nevertheless, at any given moment any given notion is conceived by a person in a specific, rather well defined way. It is this private conception which is responsible for the way its owner acts whenever application of the given notion becomes appropriate. Thus, the researcher or teacher endowed with the knowledge of a learner's understanding of negative number has a reliable basis for predicting this student's behavior in situations involving negative numbers.

Within this theoretical framework, learning is defined as an ongoing process of construction and modification of mental schemata. It is the basic tenet of this theory that rather than being a passive recipient of knowledge, the student is the builder of his or her own cognitive structures. Sometimes, the individual constructions resulting from school learning are incorrect and must therefore be termed *misconceptions*.

Misconceptions are faulty understandings resulting from inadequate cognitive schemata. These inadequacies may be an outcome of an inappropriate former learning that failed to prepare the necessary foundations. They may also stem from the lack of developmental readiness on the part of the learner. For example, there is little doubt that such an advanced and overly abstract mathematical idea as negative number requires fully fledged formal thinking, whereas there is no guarantee that the target population of twelve and thirteen year olds has, indeed, the required cognitive maturity.

According to another basic assumption of this study, an appropriate instructional strategy may greatly reduce the occurrence of misconceptions and procedural difficulties related to such concepts as negative number. Documentation of common mistakes and underlying cognitive schemata may be expected to help in finding an appropriate teaching approach. It is also believed that computers, with their power of turning abstract mathematical ideas into perceptually accessible objects, are a highly desirable ingredient of the learning environment. Finally, it is assumed that team learning should be promoted as an instructional approach, which greatly increase the chances for effective learning and diminishes the danger of misconceptions.

4. Design and method of the study. The study will be performed in two steps. First, an investigation of students' conceptions of negative numbers will be conducted among seventh and eighth graders in a number of middle school classes. At the second stage, the implications of this investigation will be translated into a teaching unit on negative numbers. The unit will subsequently be experimentally taught in one or two classrooms.

Part 1: Study of students' conceptions of the negative number. The population in this part of the study will be 12 to 14 year old middle school students, all of whom have learned the concept of negative numbers according to the obligatory curriculum and in the ways implied by the regular curriculum. This means that their first formal encounter with the subject was at the beginning of the seventh grade, just before an introduction to algebra, and that ever since they are supposed to use the concept of negative number and to be able to perform computations involving these numbers. The study will include at least 300 students of varying age and ability.

Within this population a comprehensive survey will be conducted with the help of a specially designed questionnaire. The written responses will be followed by in-depths interviews with a representative sample of the respondents. The questionnaire will be built according to a prognosis on possible learning difficulties and resulting misconceptions designed on the basis of historical investigations, subject matter analysis, and findings of the few former studies reported in the literature. A few examples of questions to be included in such questionnaire are shown in the box below (cf. Hart, 1981).

QUESTION 1: Calculate:

-8+4= 2+(-2)= -6-(-8)= -5+(-6)= (-3)•2= 5•(-7)= (-11) •(-3)=

QUESTION 2: Order the following numbers in a line, according to their magnitude (from small to large): 45.3, -11, 0.2, -1/2, -18.1, 18, 3, 2, -2, -3

QUESTION 3: For each of the following calculations, write a word problem which would lead to this calculation:

a. 13+(-3)
b. 18.5 – (-5)
c. (-6) •12
d. (-5) •(-4) (-5) •(-4)

Data analysis will be carried out in a number of steps. First, students' written responses will be scrutinized for errors and inaccuracies, which will subsequently be systematized and divided into categories. With the help of the in-depth interviews we will then try to formulate some conjectures as to the nature of the cognitive schemata underlying different types of students' mistakes.

Part 2: Curriculum development and formative evaluation. The curriculum development part of the study will begin with an examination of the existing textbooks with an intention to identify those aspects of the current teaching methods that may be held responsible for the students' misconceptions and common computational errors found in the first part of the study. On the basis of our findings we will then propose an alternative approach, in which a special care will be taken of the problematic aspect of the traditional instruction. Eventually, we will produce new technology intensive teaching materials, aimed at students learning in pairs in a computer laboratory. While developing the teaching unit, we will try to remain within the boundaries of the obligatory curriculum. The proposed teaching sequence will subsequently be run experimentally in two or three classes. To assess the effectiveness of the method, the students will undergo a series of interviews and will eventually be administered the same questionnaire which had been used in the survey part of the study.

5. The importance of the study. We believe that the proposed study has the potential to make significant contribution both to educational theory and teaching practice. Its theoretical importance expresses itself in the way the powerful conception of cognitive schema is applied to the investigation of the learning of a particular topic. This investigation promises to bring new knowledge on the hitherto unstudied subject and, at the same time, to put to the test the theory of cognitive schemata itself. The practical importance of the study is in the explanatory and predictive power of the theoretical model that will be built, and in the improvements of teaching it is expected to bring.

3. Reviews of the proposal

Review 1
by Pearla Nesher, The University of Haifa, Israel

In the review that follows I would like to draw a distinction among the *historical*, *cognitive* (psychological), and *didactical* paradigms for research, each of them mentioned in Ms. Pabst's proposal. Each one of the above paradigms formulates differently the research questions and employs different methodologies. Each one of them is a legitimate research framework, but they should not be confused since this can lead to confounded results.

I see serious problems with the Ph.D. proposal. I could not find the research question. Among the five sections of the proposal I miss even the title of a section named "the research question". There is a section "Themes and rationale". There are sections dealing with the "Purpose of the study", "Theoretical framework"; "Design and method" and the "Importance of the study". But nowhere do I find a clear research question based on the theoretical framework, from which methodology and design are derived. It could be framed within the section of "purpose". Unfortunately I missed it even there.

Since I should be constructive with my review I will try to help the candidate in this most difficult part of planning Ph. D. research: formulating and selecting a research question. If I read carefully the content of the various chapters of the proposal we have here more than one possible proposal.

First, there is an introduction where students' difficulties in learning negative numbers are related to the historical perspective, that is to the question how negative numbers have evolved through ages. The second deals with the cognitive aspects of the issue and connects cognitive theories (in particular about cognitive schemata) to students' misconceptions. The third is of a completely different nature. It deals with pedagogy. Though Ms. Pabst speaks about "translation", there is no direct (and maybe even non-direct) translation from answering a cognitive theoretical question and writing a learning unit. Preparing a learning unit calls for much more than cognitive theoretical implications. It calls for other theories about classroom interactions, the role of the teacher, etc. Understanding the underlying cognitive schema is just a part of the story.

Each of the above three main parts (or perspectives) calls for a completely different methodology. The historical approach will require making comparisons and finding parallels between certain difficulties occurring now in learning negative numbers and

their historical evolution. Actually, the relation between the historical dilemmas and present learning difficulties is a very interesting question. We should note that in dealing with practical instances of debt, temperature, etc. one can operate merely with natural numbers (or positive integers), avoiding negative numbers. When one speaks about a debt of $100, he speaks in terms of positive numbers: "a debt of $100" and not "a debt of -$100". One speaks about "5 degrees below zero", and not "–5 below zero", which is redundant. It is only when we move to a uniform scale of money at the bank or temperature that we need the zero point of departure and the negative numbers.

Those who point to the usefulness of negative numbers in the context of the simplistic everyday applications, as it is done usually in schools to justify their introduction, do not tell the whole story. The real historical reasons for the introduction of negative numbers as objects in mathematics were different. These numbers were recognized as proper numbers only when they became legitimate members of the number system within an algebraic framework. The algebraic notions of groups as mathematical objects helped to define rigorously the operations with negative numbers. The everyday applications could not serve this purpose. Whether all this is related to the difficulties encountered by students of middle school is an interesting question. In order to answer it, one has to delve into historical research, asking about stages in the evolution of negative numbers into the mathematical structure and investigating the reasons for their gradual acceptance by the mathematical community. Following that, the researcher might ask whether his findings bring any support to what can be found in the textbooks used today to teach negative numbers. However, none of these were mentioned in the proposal. The nomenclature is also changing along the proposal: "students' difficulties" become "misconceptions" in the second part, even though the notion of misconceptions is taken from a completely different paradigm, one that belongs to a psychological perspective rather than historical.

The second part of the proposal suggests a cognitive perspective. According to my reading of the candidate's proposal, the purpose of the study is "*... understanding difficulties which obstruct the process of acquisition of the concept of negative numbers...* (I omit purposely the didactic goal). Since I am supporting the cognitivist approach, I will invest most of my comments here.

As Ms. Pabst already pointed out, the cognitivist research that started with Piaget has demonstrated the fruitfulness of this paradigm. Indeed, during the last three decades it spurred much insightful studies on learning of mathematics. Findings about early counting, addition and subtraction at the primary level, multiplication, estimation, algebra concepts, etc. have changed completely our notions about the process of learning. The constructivist approach dominating now in many schools have emerged

as a pedagogical outcome of the findings of the research from the cognitivist point of view, although not by a direct translation.

Ms. Pabst is right to point out that cognitive research on negative numbers is scarce. However, what is suggested by the candidate, cannot qualify as a typical cognitive study.

First, there is some confusion about the notion of schema. Referring to this notion is important. Lets remember Rumelhart's very useful formulation: "..*schemata truly are the building blocks of cognition*" (Rumelhart, 1980). The emphasis in the candidate's proposal, however, is on a schema as an individual understanding. This is a partial and narrow interpretation of Piaget's and others notion of schema. Piaget himself wrote that:

> *A schema of an action consists in those aspects which are repeatable, transposable or generelisable, i.e. the structure of the form in distinction to the object which represent its variable content. "* (Piaget, 1950).

Fischbein (1997) in his last year worked on the notion of schema. He writes:

> *The term schema covers a large variety of psychological phenomena and for that reason it is difficult to express its meaning in commonly accepted definition.*

According to him:

> *Schemata play in the adaptive process of human beings, a role which is similar to that of instincts at lower level....Schemata which develop after the child is born may also be based on some innate structures. They also are complex programs for interpreting information and adequate reactivity... they develop with age by maturation and exercise.*

True, every aspect of development leaves much room for individual differences. Not every child starts to walk or speak at the same time. Every individual behaves according to his available schemata at any given time. Yet, there is also a general notion of schema typical of mature (or expert, if you wish) mental structures that enables the shared cognition of human beings. The notion of schema includes the tension between the individual development and the shared cognition. Schema is very important in the process of assimilation and accommodation (which are the mechanism of development). These are schemata, rather than isolated pieces of events, which are taking part in that process.

I do not want to continue this already too long list of characteristics of the notion of schema, or speak of its categorization either according to Piaget (into Presentative, Procedural and Operative), or according to Fischbein (into Specific and Structural). I would like to see how all this is connected to the issue of mathematical learning, in general, and to the learning of negative numbers, in particular.

First, the question is what constitutes the presentative, procedural and operative schemata for the domain of negative numbers. Unfortunately, whilst there are abundance of studies describing procedural errors students tend to commit while working with negative numbers, and equally many studies describing linguistic difficulty connected to the pejorative connotations of the word "negative", I cannot recall, in the field of mathematical education or psychology, a study of negative numbers that rely on schemata and structures. This is surprising because, as mentioned before, the essence of mathematical operations with negative numbers was born within the structure of a group.

Past research shows that examining procedural and algorithmic errors proved to be very illuminating only when done from the "schematic" point of view. All the difficulties in learning decimal numbers could be explained via the existence of the whole number schema and the fraction schema that could not be structurally integrated into the decimal schema. A program like "Buggy", that detailed all possible errors existing in students' performance, cannot help us. It is the presentative and procedural schemata the student already constructed, and the ones that he did not yet construct that can help us understand his behavior and predict his future cognitive growth.

While proposing to employ the cognitive notion of schema Ms. Pabst should be more specific and say explicitly what *kinds* of schemata related to negative numbers she is going to deal with. It is our understanding of the particular schemata underlying the notion of negative numbers that will allow us to understand better what is going on in the process of developing and building up these schemata.

Here are the possible research questions she could focus on in her research:

a) Where in the hierarchy of schemata of the number system can one expect to build up a schema for negative numbers? What are the necessary sub-ordinate schemata? How is the newly built schema different from previous number system schemata (already constructed by the student)?

b) Where can we expect the learner of negative numbers to be misguided by his old numeric schemata? Does the hypothetical schema model suggested in (a) enable the interpretation of observed behaviors, erroneous as well as correct ones?

Such questions are derived from the theoretical review Ms. Pabst has written. Note, that I do not include direct implications for pedagogy among the above research questions. Possible implications could be mentioned at the end of the dissertation, but should better be left for a separate research. The proposal intends to enrich our knowledge about schemata underlying negative numbers. Better understanding, which will be attained in this way, is important for teachers. Yet, none of the methods used in this research are directly transferable to schools. The empirical part of the study will include individual open-ended interviews with students, the aim of which is to reveal various aspects of the negative number schemata and their sub-ordinate schemata, as they occur within individuals at a certain level. These interviews are not units of learning, despite the fact that students might learn a lot from them.

I come now to the third point in the proposal, the pedagogical one. I think it would be too hasty to conclude on the basis of one cognitive study about the proper ways to teach the subject matter. First, before running with the results to the classroom, I would expect to see more studies replicating or elaborating on the finding of the first study. I would like to remind us how many studies were performed on elementary addition and subtraction word problems, before 'combine', 'change' and 'compare' could be introduced to the classroom teachers.

I also want to mention, in brief, that the methodology of research which aims at the question what works at school is completely different from the methods one employs when trying to understand a cognitive schema. Research that aims at testing a teaching approach must include many classes rather than a single group and one teacher; it must involve a long term teaching, because we cannot expect the student to construct a schema in just two or three weeks; and, of course, it must include a comparison to other approaches. If we do not do it in the proper way, we end up with systems of beliefs rather than with scientifically sound findings.

To sum up, the issue of learning negative numbers is important. Yet, the proposal, as it is, includes too many questions and I cannot support it. I suggest that Ms. Pabst defines better the paradigm within which she would like to investigate the learning of negative numbers.

3. Reviews of the proposal (cont.)

Review 2

by Stephen Lerman, Centre for Mathematics Education, South Bank University, London, UK.

The candidate has chosen a most important topic for her study. There are certainly a great many confusions and difficulties around the teaching and learning of negative numbers. However, I fear that the candidate will not resolve any of these problems if she pursues the perspective outlined in the proposal. I will indicate some concerns and then I will make some suggestions below for what I consider a more appropriate study.

Understanding. First, it is high time we abandoned words and phrases such as 'understanding', 'misconceptions', and 'acquisition of concepts' in mathematics education. They are useless from a teacher's and a researcher's point of view, since they are in essence totally unobservable, and are effectively tools of regulation, since we take it upon ourselves to be the only ones qualified to identify when understanding has taken place. They are predicated upon a notion of the gaining of knowledge as an individual cognitive process. 'Real understanding', with the associated ideas of its non-occurrence, is interpreted as the construction of context-free mental schemata. These schemata, when properly formed, ought to be available for operationalisation and application whenever the 'situation', which is interpreted unambiguously, requires it. We tend to be very isolationist in mathematics education research, relying on psychology as an explanatory domain long after socio-linguistics and cultural studies, to name just two domains, have altered all other fields of social and intellectual inquiry.

A far more useful notion than 'understanding' is that of the forming of identity in the mathematics classroom (Lave, 1996; Winbourne and Watson, 1998). To know something is to use that something in an appropriate way, at appropriate times, these being judged by those who are already initiated. Identity is constituted in discursive practices, which carry what constitutes knowledge in that practice. That approach directs us to study what it is to become a person who functions knowingly in particular social, historical and cultural situations, and how one will be apprenticed into that practice (following Bernstein's (1996) use of the term apprentice, rather than Lave's (1988), although there are similarities between them). One will need to focus on conscious desires and goals but identity will also be conceptualised through the unconscious (Evans & Tsatsaroni, 1993; Pimm, 1994). One needs to be acutely aware of the regulative practices of initiation and to view the task of teaching mathematics as initiation into a hierarchical knowledge structure, with boundaries between school mathematics and everyday knowledge which must be recognised if

they are to be traversed in any way.

In relation to the classroom, and to school mathematics in particular, the research focus must be on: the regulating structures of society and the classroom through the pedagogic mode; the changing identities, goals and needs of the pupils; and the modification and adaptation of their language and behaviour, and the development of meanings, towards those of the teacher. There is a danger in using 'behaviour' as contrasted against 'cognition', since these are usually presented as bipolar alternatives, the former being 'bad' and the latter 'good' in popular discourse of course. What I am referring to here is neither, it is rather a social epistemology, which is materialist whilst not merely behaviourist. Knowledge is what the various social formations in which we live deem to be knowledgeable ways of being, with the multiplicity and fragmentation of identity that this implies.

It will be clear that I am proposing a deconstruction of 'individual' in the often used duality of social/individual but I want also to deconstruct the notion of 'social'. Any situation, including particular activities in the mathematics classroom, is experienced differently by the pupils, and the teacher as a matter of fact, according to their positioning in overlapping social and cultural histories, such as gender, ethnicity, class.

Mathematical meanings. The proposed study falls into following the nineteenth century reductionist theory of ontogeny replicating phylogeny, suggesting the need to find, through historical studies, the obstacles to the development of the currently accepted concept of negative numbers, as these are "only too likely to hinder its learning in our schools today". Such an approach has been a self-fulfilling prophecy in mathematics education in recent decades. We teach to a curriculum constructed around proposed epistemological obstacles, thus ensuring that they will become learning obstacles. I would suggest that we must see mathematical concepts as cultural tools, through which thinking in society is transformed, for all time. The number line, infinitely extensible in both directions, has transformed the discourse of number, but whilst its emergence in history is important, there is no reason to assume that pupils must proceed through the same steps to construct the number line. Today, we can recognise the integers as forming a group under addition, and that can offer us ways of teaching negative numbers, even to young children (Lerman, 1996), by fore-grounding sign-sign connections.

Models of negative numbers and their operations are just that: bank debt or credit, sub-zero temperatures or whatever, are not the 'meaning' of negative numbers. All such metaphors bring with them inevitable limitations (Lins, 1994). That is not to deny the role of models in mathematics, nor their usefulness in teaching, but to question the ubiquity of such approaches which only contribute to what the candidate calls misconceptions. It makes the mistake of confusing metaphor, the link into other systems of meaning, with metonym, where meanings are internal to the system. I was asked, some years ago, by a group of primary teachers on an inservice course, to

explain why $(-1)\times(-1) = +1$. They expected an explanation based on such metaphors. My reply was "What else could it be?"

Readiness. I am convinced that ages and stages, or its consequent notion of readiness, is not a useful way to think about children's development. Indeed the notion of 'development' retains the modernist themes of universality and naturalness and obscures the ideological intent. Notions of normality, which enable teachers to speak of a particular child being ready, pathologises other children as not ready and therefore somehow deficient. The alternative I am proposing, namely a focus on the ways that children form their identities in the practices of school mathematics, admits to the regulating function of teaching. At the same time the teacher and researcher are forced to acknowledge the multiple practices in which each child is already situated, and enable them to bring those identities, gendered, ethnic, class-based etc., into the learning situation (Ellsworth, 1989). The notion of the zone of proximal development, seen as a product of the moment, of the teacher, the pupils, the texts and all the other available resources (Meira & Lerman, forthcoming), is the focus I would recommend for thinking about the problem of when to teach anything. Thus, in the paper I mentioned earlier (Lerman, 1996) the need for negative numbers arose in a problem on which I was working with a group of 9/10 year olds, adding two numbers to make a third, and in that context it presented no difficulties for them.

The study. I would suggest that you abandon the programme of pre-test, investigating existing texts, designing new ones, and testing their efficacy. In its place, you and your teacher(s), should immerse yourselves in the body of literature to which I have referred above, in order to theorise the notion of pupils becoming mathematical actors, in the specific context of negative numbers. You should then design activities, both with and without technological tools, which elicit the need for negative numbers in the solution of those problems. As with all tools, the various forms of IT have specific features that structure and transform the world for the student. I am sure the number line extended infinitely in both directions from zero is analogous. You should pay attention to the role of metaphor and its limitations, enabling a shift of signification towards sign-sign foregrounding (Vile & Lerman, 1996), or metonymic links, in the tasks you devise. You should then undertake ethnographic studies of the students' discussions, actions, reactions to questions, and their other productions, as well as those of the teacher(s) and yourself. You should design tasks that require writing, both to elicit descriptions of meanings and to encourage the development of those meanings. The aim of the study would be to investigate the internalisation of the cultural tools that mediate the appropriate actions of operations with the integers.

Following this programme,. you should have something to say about this important problem, through a focus on the ways that pupils' ideas and actions become orientated to working with negative numbers, by the teacher, other students, the tasks and the texts.

Review 3

by Ellice Forman, Department of Psychology in Education, University of Pittsburgh, Pittsburgh, USA

Teaching and learning about negative numbers is clearly an important and under-investigated topic. Unfortunately, Ms. Pabst has not explicitly chosen a consistent theoretical approach to this topic. In her review, Pearla finds at least three paradigms in the proposal (historical, psychological, pedagogical) and each requires a different methodology. Using three different paradigms in a study can also create purely theoretical problems. Each paradigm may depend upon different (and perhaps contradictory) assumptions about learning (such as an active versus a passive learner). For example, it seems that Ms. Pabst assumes an active learner when she refers to the psychological paradigm but implies a passive learner when she discusses instruction.

Both Pearla and Steve propose different theoretical frameworks, each of which are internally consistent, instead of the inconsistent paradigm of the original proposal. It seems to me that another way to think about paradigms is to consider the two broad theoretical frameworks currently in use in mathematics education. These frameworks have been called a variety of names (e.g., cognitive science vs. situated cognition; radical constructivism vs. social constructivism; acquisition metaphor vs. participation metaphor; cognitive vs. situated social practice) and their strengths and weaknesses are currently being debated in the education literature (Ernest, 1996; Greeno et al., 1998; Sfard, 1998). In the interests of consistency, I suggest that Ms. Pabst consider selecting <u>one</u> of the above frameworks to guide her research study.

Supporters of either framework are unlikely to admit that their theory is incapable of addressing important practical questions in mathematics education. To the contrary, adherents of both positions claim to provide the most comprehensive and useful set of concepts needed to address any meaningful basic or applied question. Each perspective, however, begins its analysis with a set of assumptions and concepts that take us only so far. For example, Greeno and his colleagues note that the cognitive science approach begins with the analysis of individual cognition and gradually works outward to incorporate individuals learning in social contexts whereas situated cognition begins with complex activity settings and works inward to individuals. Yet, it is unlikely that either one position will be able to realize its ambition to be everything for everyone (Sfard, 1998). In education, we can't afford to be doctrinaire in our research or practices. Nevertheless, eclecticism can be confusing. One way out of this relativist dilemma is to let our practical concerns guide our theoretical choices. I suggest that Ms. Pabst clarify the practical aims of her research and then select one of the two theoretical frameworks that is most likely to help her attain her goals.

The first choice: A study of the acquisition of negative numbers. As Pearla has argued in her review, a cognitive approach would be the best way to investigate the acquisition of negative numbers by middle school students. The term, acquisition, as a synonym for learning, implies that the individual student is expected to gain some mental entity (concepts, representations, schemas, rules, etc.) as a result of prior knowledge and particular experiences (Sfard, 1998). Moreover, as Pearla has noted, schemes do not exist in isolation but in hierarchies or complex structures. These mental entities, in Steve's view, are "context-free", in the sense that they would be available for use in situations other than those in which they were first encountered (i.e., transfer of knowledge from one context to another would be possible, at least in principle). The unit of analysis for this type of investigation would be the individual learner who is presumed to change (at least mentally) as a result of experiences with material and symbolic objects. The nature of this change can be quantitative (e.g., enriched domain-specific knowledge) or qualitative (e.g., stage-like transformations of schemes as in Piagetian theory).

If Ms. Pabst's goals are to expand our notions about the nature of students' knowledge base in the area of negative numbers, then she should go with the cognitive science position. From this perspective, she would be able to focus on the contents and organization of an individual's mental structures. Although this position is closest to the one in her original proposal, I want her to rid herself of this deficit model of students' understanding that is inherent in constructs such as misconceptions and developmental readiness. We have characterized students' thinking for too long in terms of what it lacks (as a result of no instruction or poor instruction). The notion of "misconception", for example, is theoretically misguided, from a cognitive developmental point of view. As Siegler (1976) has shown us in his investigations of children's scientific concepts, many mature conceptions, such as torque, arise when children are able to revise and integrate their previous notions of distance and weight in increasingly sophisticated ways. That is, children initially predict that a balance beam will tip in the direction of the greater amount of weight (and ignore the relative distances of the weights from the fulcrum). Older children take distance as well as weight into account but in an unsystematic fashion. Adolescents and adults use a mathematical formula to compute the relative contributions of both weight and distance. Thus, development does not involve the replacement of misconceptions with correct conceptions but, instead, involves the improved ability to encode or understand more relevant features and to quantify the relationships between those features. A similar issue may occur in the domain of negative numbers.

The second choice: A study of situated social practice in a mathematics classroom. I agree with Steve that if one is primarily interested in understanding the complex social, institutional, personal, and discursive dynamics of the classroom, then one should abandon notions of learning as acquisition of decontextualized mental schema. This Platonic view of mathematics learning is inconsistent with a focus on situated

social practice. If one were primarily interested in understanding how mathematics is practiced by mathematicians and by students of mathematics, then one needs to change one's views of both mathematics and learning. In this alternative approach, mathematics is defined by what mathematicians do—communicate with other members of the professional community, explicate their arguments through the use of cultural tools (graphs, symbols and operators), make conjectures, pose puzzles, etc. (Rotman, 1993). Likewise, learning mathematics should resemble, to some extent, an apprenticeship in thinking, speaking, and acting mathematically (Lampert, 1990; Rogoff, 1990).

Returning to Ms. Pabst's research proposal, if she is truly interested in understanding teaching and learning activities in a classroom community, then she would need to reconceptualize her study from a situated social practice perspective. Like Steve, I feel that Vygotsky's writings about the zone of proximal development would help her articulate a dynamic perspective to understanding teaching and learning. Vygotsky argued that we must study not just the fruits of intellectual development but the process by which the fruits are created (1978). Thus, he proposed the integration of teaching and learning in his famous concept of the zone of proximal development—the difference between assisted and unassisted performance on a task. The notion of developmental readiness has no real validity from this perspective. As Bruner noted, "we begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (1960, p. 33).

The unit of analysis for this investigation would have to be larger than the individual—it could be a dyad, small group, classroom, etc. This is because learning is no longer viewed as an individual accomplishment or possession. In contrast, indices of learning would involve analyses of group participation patterns. Also, the design of the study should include evidence of change over time: the genetic analysis that is a key component of a Vygotskian approach. For example, changes over time in teachers' and students' involvement in posing problems, arguing solution strategies, orchestrating discussions, integrating alternative explanations, and inscribing justifications would indicate how expertise was socially created in this community. In addition to examining expertise as an emergent phenomenon, this study could focus on affective issues. How do students' and teachers' motives for participating in mathematical activities change over time? Who is seen as resistant to learning about negative numbers? Who is viewed as eager to master the cultural tools necessary for effectively communicating about negative numbers? How are those notions of resistance or eagerness to learn socially constructed?

Summary. Ms. Pabst's topic of interest allows one to conduct at least two quite different types of studies, employing two distinctly different theoretical frameworks. The choice of one approach over the other should be done for pragmatic reasons. Once a choice is made, however, it is essential that the concepts employed and the methods used to investigate those concepts be convincing and coherent (Sfard, 1998).

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RESEARCH FORUM

Theme 1:
Learning and Teaching Undergraduate Mathematics

Coordinators:	Annie Selden John Selden
Presentation 1:	*One Theoretical Perspective in Undergraduate Mathematics Education Research* Broni Czarnocha, Ed Dubinsky, Vrunda Prabhu & Draga Vidakovic
Reactor:	David Tall
Presentation 2:	*Teaching and Learning Linear Algebra with Cabri* Anna Sierpinska, Jana Trgalová, Joel Hillel & Tommy Dreyfus
Reactor:	Uri Leron

ONE THEORETICAL PERSPECTIVE IN UNDERGRADUATE MATHEMATICS EDUCATION RESEARCH

Broni Czarnocha, Hostos Community College
Ed Dubinsky, Georgia State University
Vrunda Prabhu, William Woods University,
Draga Vidakovic, Georgia State University

This presentation is based on three principles. First, research in undergraduate mathematics education (RUME) should be closely connected with, if not embedded in, and at least potentially applicable to, curriculum development and teaching practice. Second, a theoretical perspective involving one or more theories of learning, should play a major role in any research project; this role should be explicit and chosen at the beginning of the research so as to provide direction for the investigation and analysis of results. Finally, empirical data is also important and the best research synthesizes theoretical analysis, pedagogical applications, and the gathering and analysis of data.

We begin, in Section 1, with a discussion of how we go about conducting research that is both theoretical and empirical and relates to teaching practice. In Section 2 we give a very brief discussion (with references to more extensive descriptions) of APOS Theory which is the main theoretical perspective used in the program. This is followed in Section 3 by examples in which our approach is used --- both by its developers and others and for both postsecondary mathematics for which it was designed and for K-12 levels from which it came. In Section 4 we describe the pedagogical component of some projects that use our approach. Next, in Section 5 we describe our methods of analyzing and gathering data with some
indications (together with references) of the results. Finally, in Section 6, we consider approaches that are alternatives to our use of APOS Theory.

1 An approach to research and curriculum development in undergraduate mathematics education

As Alan Schoenfeld argues (Schoenfeld, 1998), models and theories should support prediction, have explanatory power, and be applicable to broad ranges of phenomena. To this we have added (Dubinsky & McDonald, 1999) that a theory should help organize one's thinking about complex, interrelated phenomena, serve as a tool for analyzing data, and provide a language for communication of ideas about learning that go beyond superficial descriptions.

We have tried to develop APOS Theory so as to meet these criteria and combine with teaching practice and empirical data to form a paradigm consisting in a repeated traversal of a circle of the following activities: theoretical analysis, empirical data, and design and implement instruction.

In this paradigm, investigation of a particular topic begins with a theoretical analysis based on APOS Theory, the researchers' own knowledge of the mathematics involved, and any informal observations of students, for example, in teaching the material in a traditional way. The purpose of this analysis is to propose, in a preliminary and tentative manner, what we call a *genetic decomposition* of the concept in question. That amounts to suggesting specific mental constructions which a student can make in order to learn the concept. The next step is to design and implement instruction aimed at getting students to make the proposed mental constructions. As the students are experiencing this instruction, data is collected in several different ways, using both quantitative and qualitative methods of gathering information.

The final step in the traversal is to coordinate the empirical data obtained with the theoretical analysis. This means, on one hand, that the theoretical analysis suggests questions to ask of the data: to wit, does it appear that the proposed mental constructions were made by students? Focusing the analysis of data in this way can help the researcher deal with a huge amount of information meaningfully, but far from exhaustively. Indeed we have very often used the same data more than once in studies differentiated by the questions to which a theoretical analysis points us. On the other hand, it sometimes occurs that the mental constructions students appear to be making are different from what has been proposed. One possible response to this is to reconsider one or more aspects of the particular theoretical analysis of the concepts in question. Another response which we have been forced to make, albeit rarely, is to reconsider the theory itself and make appropriate revisions.

It is important to note that it is here that APOS theory exhibits yet another important feature. There is contained within its use the potential for completely rejecting it, which would happen if it had to be revised too often in order to fit with the data.

To summarize, the theoretical analysis drives the instruction which creates the data. The theoretical analysis directs the analysis of data and is simultaneously subject to revision as a result of this data analysis. This circle of activity is then repeated with the (possibly new) theoretical analysis. It is repeated as often as appears necessary to understand the epistemology of the particular topic.

If things work properly, then learning should improve in a natural way as a result of instruction that relates to how the students can learn the concept or concepts. For a more detailed discussion of this approach, see Asiala et al. (1996).

2 APOS Theory

In this section we will sketch the general theory in its present stage of development and describe the nature of mental constructions that the theory proposes along with some examples.

The original source of APOS theory was Piaget's epistemology of mathematics learned from infancy through adolescence (see, for example, Beth & Piaget, (1966)) and an attempt to apply his mechanism of reflective abstraction to learning post secondary mathematics. For more details on the relation between Piaget's ideas and APOS Theory, see Dubinsky (1991). The fact that the development of a theory for advanced mathematical thinking is based on a theory for thinking about elementary mathematics raises the question of the applicability of APOS theory to elementary concepts. We will return to this question in Section 3.

APOS theory begins with a statement of what it means to learn and know something in mathematics.

> An individual's mathematical knowledge is her or his tendency to respond to mathematical problem situations by reflecting on them in a social context and constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations.

There are, in this statement, references to a number of aspects of learning and knowing. For one thing, the statement acknowledges that what a person knows and is capable of doing is not necessarily available to her or him at a given moment and in a given situation. All of us who have taught (or studied) are familiar with the phenomenon of a student missing a question completely on an exam and then really knowing the answer right after, without looking it up. A related phenomenon is to be unable to deal with a mathematical situation but, after the slightest suggestion from a colleague or teacher, it all comes running back to your consciousness. Thus, in the problem of knowing, there are two issues: learning a concept and accessing it when needed.

Reflection is an important part of both learning and knowing. Mathematics in particular is full of techniques and algorithms to use in dealing with situations. Many people can learn these quite well and use them to do things in mathematics. But understanding mathematics goes beyond the ability to perform calculations, no matter how sophisticated. It is necessary to be aware of how these procedures go, to get a feel for the result without actually performing all the calculations, to be able to work with variations of a single algorithm and to understand relationships among algorithms.

It is a controversial point, but this theory takes the position that reflection is best performed in a social context. There is evidence in the literature (see Vidakovic (1993), for example) of the value to students of social interaction and there is also the cultural fact that almost all research mathematicians feel very strongly the need for interactions with colleagues before, during, and after creative work in mathematics.

APOS theory asserts that "possessing" knowledge consists in a tendency to make mental constructions that are used in dealing with a problem situation. Often the construction amounts to reconstructing (or remembering) something previously built so as to repeat a previous method. But progress in the development of
mathematical knowledge comes from making a reconstruction in a situation similar to, but different in important ways from, a problem previously dealt with. Then the reconstruction is not exactly the same as what existed previously, and may in fact contain one or more advances to a more sophisticated level. This whole notion is
related to the well known Piagetian dichotomy of assimilation and accommodation (Piaget, (1972)).

Finally, the question arises of what is constructed, or what is the nature of the constructions and the ways in which they are made? It is when we talk about this that our theoretical perspective, which may appear applicable to any subject whatsoever, becomes specific to mathematics. We will deal with this question in the next paragraph.

2.1 Mental constructions for learning mathematics

Understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, processes and objects can be organized in schemas.

Actually, there is a potentially misleading aspect of this description in that there may be two much of a suggestion of a linear progression from action through process to object and, ultimately, schema. In fact, although something like a procession can be discerned, it often appears more like a dialectic in which not only is there a partial development at one level, passage to the next level, returning to the previous and going back forth, but also the development of each level influences both developments at higher and lower levels.

In describing briefly the main mental constructions, we will use the example of cosets of a subgroup, Lagrange's Theorem and quotient groups of a

group and carry this example through the remainder of the paper to illustrate other points.

Action. A transformation is considered to be an action when it is a reaction to stimuli which the subject perceives as external. This means that the individual requires complete and understandable instructions giving precise details on steps to take in connection with the concept.

One example of an action conception comes from the notion of a (left or right) coset of a group in abstract algebra. Consider, for example, the modular group $[Z_{20}, +_{20}]$ --- that is, the integers $\{0,1,2,\ldots,19\}$ with the operation of addition mod 20 --- and the subgroup $H = \{0,4,8,12,16\}$ of multiples of 4. It is not very difficult for students to work with a coset such as $2+H = \{2,6,10,14,18\}$ because it is formed either by a listing of the elements according to some rule ("begin with 2 and add 4") or an explicit condition such as, "the remainder on division by 4 is 2". This is an action conception. Something more is required to work with cosets in a group such as S_n, the group of all permutations on n objects where simple formulas are not available. Even in the more elementary situation of Z_n, students will have difficulty in reasoning about cosets (such as counting them, comparing them, etc.)

According to APOS theory, all of these difficulties are related to students' inability to interiorize these actions to processes, or encapsulate the processes to objects.

Although an action conception is very limited, it is an important part of the beginning of understanding a concept. Therefore, instruction should begin with activities designed to help students construct actions.

Process. When an individual reflects on an action scheme and interiorizes it then the action can become perceived as a part of the individual who can establish control over it.

For cosets, a process understanding consists of thinking about the formation of a set by operating a fixed element with every element in the subgroup. It is not necessary to perform the operations, but only to think about them being performed.

Object. When an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that transformations (whether they be actions or processes) can act on it, and is able to actually construct such transformations, then he or she is thinking of this process as an object.

In the course of performing an action or process on an object, it is often necessary to de-encapsulate the object back to the process from which it came in order to use its properties in manipulating it.

In the case of cosets, given an element x and a subgroup H of a group G, if an individual thinks generally of the (left) coset of x modulo H as a process of operating with x on each element of H, then this process can be encapsulated to an object xH. Actions on cosets of H, such as equipping the set of cosets with a binary operation (quotient group), or counting their number, comparing their cardinality, and checking their intersections (Lagrange's Theorem) can make sense to the individual. Thinking about the problem of investigating such properties involves the interpretation of cosets as objects whereas the actual finding out requires that these objects be de-encapsulated in the individual's mind so as to make use

of the properties of the processes from which these objects came (certain kinds of set formation in this case.)

In general, encapsulating processes to become objects is considered to be extremely difficult (Sfard, (1991)) and not very many pedagogical strategies have been effective in helping students do this in situations such as functions or cosets. A part of the reason is that there is very little in our experience that corresponds to performing actions on what are interpreted as processes.

Schema. Once constructed, objects and processes can be interconnected in various ways: for example, two or more processes may be coordinated by linking them through composition or in other ways; processes and objects are related by virtue of the fact that the former acts on the latter. A collection of actions, processes, and objects can be organized in a structured manner to form a schema which may also include previously constructed schemas. The structure of a schema has coherence in the sense that an individual understands, implicitly or explicitly, which phenomena the schema can be used to deal with.

Schemas themselves can be treated as objects and included in the organization of "higher level" schemas.

For example, sets can be formed as objects and linked with binary operations are linked to form pairs which may or may not satisfy certain properties. All of this can be organized to construct the schema for coset. Groups and rings and other such mathematical objects might be organized in a schema called algebraic structures.

The idea of schema is very important for our story in that it is one of the few cases in which difficulties in analyzing data have led to significant revision of the theory. In research studies connected with the chain rule (Clark et al, 1997), using properties of the derivative to draw the graph of a function (Baker et al, in preparation) and understanding sequences of numbers (Mathews et al, in review), researchers found the mechanisms of actions, objects and processes inadequate. Moreover, the idea of schema had only a superficial description as given in Dubinsky (1991) where the development of a schema is not explained. The notion of the *triad* developed by Garcia & Piaget (1983) could be seen to be very closely related to that of schema and it turned out that it could be used to provide better explanations of what a schema is and how it is constructed.

In the triad notion an individual's understanding of a concept develops through three stages: the *intra* stage in which there is a focus on single objects (which might be encapsulated processes or thematized schemas); the *inter* stage in which there is a construction and understanding of transformations between these objects (such as the relations of Skemp (1976) or interiorized actions); and finally *trans* in which there develops a coherence among the transformations in that the individual understands and is ready for, explicitly or implicitly, an organized system of transformations including both those which have been experienced and those that are only potential. It is this coherence that gives substance to our notion of schema and connects the various actions, processes, objects and other schemas.

Thus, for example, in the concept of the chain rule, and individual might begin at the intra level by being able to take the derivative of individual expressions obtained by replacing the variable in a given expression with another expression. Then at the inter level there are certain "rules" such as the "power rule" or the "inside-outside" rule; finally at the trans level, all of these individual rules are encompassed by a single operation of taking the derivative of the composition of two functions. The result is a schema in which the individual understands that if a function can be written as a composition of functions whose derivatives are known, then the derivative of the original function can be obtained.

In this way, APOS Theory connects with the Triad Theory through the notions of coherence and schema, to form a more powerful mechanism for studying learning advanced mathematical concepts.

The Triad Theory is not the only case in which APOS Theory relates to other theories of learning found in the research literature. Other examples include the operational/structural characterization of Sfard (Sfard, 1991) which is very similar to our process/object analysis, the concept image/concept definition dichotomy of Tall and Vinner (Tall and Vinner, 1981), The procept notion of Gray & Tall (1991, 1994), and the didactical engineering of the French school (see, for example, Farfan (1997) and references therein.) In some cases such as the use of the Triad to develop the schema concept discussed above and the work of Arnon, to be discussed below, in combining APOS Theory with Nesher's Theory of Knowledge and Exemplification Components a synthesis of two different but related theories leads to advances in our understanding of the learning process and improvements in our students' learning.

3 Examples of research projects using APOS Theory

There are three categories of examples of research projects using APOS Theory that we wish to consider: studies of advanced mathematical concepts by those who are engaged in developing the theory; studies of advanced mathematical concepts by those other than the developers; and studies of elementary mathematical concepts that use APOS Theory.

3.1 **Developers of APOS**

After a period of several years in which the approach described here was used by only a few people who were the original developers, an organization, called a Research in Undergraduate Mathematics Education (RUMEC) was formed for the purpose of bringing together experienced and novice researchers to work in teams to conduct specific research studies. This group, which consists of 30-35 members, decided to focus on the development and use of APOS Theory and engaged in a large number of projects, following the paradigm described in

Section 1, most of which have already resulted in published research reports and applications to teaching practice.

Completed and ongoing RUMEC studies include investigations of understanding the behavior of functions, infinity, existential and universal quantification, permutations, symmetries and cooperative learning. Studies of calculus are about the chain rule, slope, limits, graphical understanding of the derivative, using the derivative to draw graphs, intuitive conceptions of the definite integral, sequences and series. There is a long-range study of the effect on student learning and involvement with mathematics of pedagogy based on our paradigm in comparison with pedagogy in traditional courses. In statistics, studies have considered means, medians and the central limit theorem. In abstract algebra there are studies of binary operations, groups, subgroups, cosets, normality and quotient groups. There is a study of the attitudes towards abstract algebra in particular and mathematical abstraction in general as a result of pedagogy based on our paradigm, again in comparison with pedagogy in traditional courses.

Details of these studies and the specific results are summarized in Clark et al (In preparation). The results presented in this paper represent a vast amount of information, both quantitative and qualitative in nature, about learning the topics. It is possible to draw some general conclusions for which justification can be found in the references. First of all, it seems that APOS Theory (including the more sophisticated version of schema) and the genetic decompositions it provides are effective as tools for analyzing and describing what might be the mental processes involved in learning a wide variety of topics. Second, in a fairly weak, but not totally meaningless, sense it can be argued that the theory is predictive in that one can postulate that if students make the mental constructions in a genetic decomposition that an APOS-theoretical analysis of a concept provides, then they are likely to learn the concept. Third, the comparisons, taken as a whole, seem to point to the effectiveness of pedagogy based on APOS Theory in comparison with traditional pedagogy. There is a large number of comparisons in which the former leads to substantially better learning (both in the sense of statistical significance and in the sense of differences that are so large that statistics are unnecessary). There are even more cases of differences that are small, often not statistically significant, but all in the same direction, favoring APOS-based pedagogy over traditional. There are almost no examples in which students taking courses with traditional pedagogy appear to learn better than the students in the APOS-based courses. Finally, in addition to these results in terms of learning, APOS students end up with better attitudes about mathematics and abstraction, they are more likely to take more math courses after, for example, calculus, and they are not adversely affected in their other university courses by what appears to be more time spent outside of class on working on APOS-based courses.

One of the major RUMEC studies was in abstract algebra and concerned the concepts of cosests, Lagrange's theorem and quotient groups. The design of this study very closely followed the paradigm and is taking place in several phases. The first phase considered a group of high school teachers taking a professional development course in abstract algebra that made use of an early version of what would become our pedagogical approach (the ACE Teaching cycle, which is described below in Section 4.) The result was a preliminary genetic decomposition describing mental constructions that might help someone understand cosets and quotients. The second made use of this decomposition, designed a course following very closely the ACE cycle (it was necessary to produce a textbook), and then designed a number of instruments to try to find out about what the students had constructed and what they had learned. The results of this study can be found in Clark et al (In preparation). Finally, a follow-up study is in progress to see if the (very encouraging) results could be obtained by teachers other than those who had developed the course.

3.2 Users of APOS

Use of APOS Theory has not been restricted to those who have been involved in its development. For example, Carlson (1998) has used it to help categorize students' understanding of the concept function; in their doctoral theses, Tostado (1995) and Carmona (1996) applied APOS Theory to analyze student responses to a questionnaire and transcripts of interviews about their understanding of the idea of a tangent to a curve and its relationship to the derivative of a function; Wahlberg (Submitted) used the theory to analyze both interview transcripts and writing assignments in an experimental approach to helping students in a calculus course understand the limit concept; Zazkis & Gunn (1997) worked out action-process-object developments of pre-service elementary school teachers' understanding of sets, elements, cardinality, subset and the empty set.

In all of these (and other) studies, the theory invariably proved to be a useful tool for explaining the development of understanding and the difficulties students had with mathematical concepts at the post-secondary level. In some cases, APOS Theory also served a predictive and applications role in that it pointed to pedagogy that could help students learn better and when such pedagogy was developed and employed, the results were very encouraging.

3.3 Elementary mathematical concepts

It is both interesting and satisfying that a theory based on an epistemology of elementary mathematical concepts (that of Piaget) but developed explicitly in relation to advanced concepts, turns out to also be useful in studies of elementary concepts. The circle is completed.

For example, an interesting approach of I. Arnon is to combine APOS Theory with one or another theory developed in the elementary context, such as Perla Nesher's Theory of Knowledge and Exemplification Components, to obtain a powerful tool for studying children's construction of their conceptions of fractions. Using a synthesis of these theories, Arnon is able to establish a stage in the development of concrete actions into abstract objects: children use mathematical language to describe imaginary concrete activity. The results of pedagogy based on these ideas compare favorably with that of other approaches. Her results are reported in several papers, for example Arnon (1998).

Another example is the work of Zazkis who, in collaboration with Gunn, Khoury and Campbell, used APOS Theory in studying pre-service teachers' understanding of several elementary concepts: the positional number system using bases other than 10 and focusing on numbers between 0 and 1 (Zazkis and Khoury (1994); sets, elements, subsets, cardinality and the empty set (Zazkis & Gunn (1997); and divisibility (Zazkis & Campbell (1996).)

Finally, in a study in progress A. Brown and G. Tolias are using APOS Theory to explain difficulties students are having with the idea of factoring a positive integer into primes and representing this with a "factor tree".

These investigations indicate that APOS Theory has grown both beyond the community in which it was born and raised and beyond the content area for which it was intended.

4 Teaching practice

A second component of our framework is the design and implementation of instruction. What sort of pedagogical approach can be used in our paradigm to support the theoretical analysis, that is, to induce students to make the mental constructions proposed by the theory and to help them move from these constructions to understanding and knowledge of mathematics per se? Our response to these questions has been to design a pedagogical approach called the *ACE Teaching Cycle*. It is a methodology that makes use of cooperative learning and students writing computer programs. In this design, the course is broken up into sections, each of which runs for one week. During the week, the class meets on some days in the computer lab and on other days in a regular classroom in which there are no computers. Homework is completed outside of class. We have the students working in cooperative groups in all of these activities.

Following is a description of the three components of this structure with some indications of the pedagogical goals of each component and examples for the topic of coset.

Activities:

The course meets in a computer lab where students work in teams on computer tasks designed to foster the specific mental constructions proposed in the genetic decompositions of the course topics. The lab assignments are generally too long to finish during the scheduled lab and students are expected to come to the lab when it is open or work on their personal computers, or use other labs to complete the assignment.

In the abstract algebra course there were a large number of computer activities through which students could construct, on the computer, many of the basic processes and objects of elementary group theory such as: examples of binary operations, examples of groups; tests for closure, associativity, existence of an identity and inverses, and commutativity.

After some work with such constructs, the students were given a rather difficult task. They were asked to write a computer program that would accept a set and a binary operation that formed a group and return a new binary operation that could accept any of the four combinations of inputs that were either elements of the set, or subsets. This binary operation would then determine what the inputs were and, as they dictated, return the product (in the group) of the two elements, a left coset, a right coset or the set of all products of elements of the two sets.

This is an extremely difficult task for the students, but the program they must write is actually very simple. As a result, all of their struggles to get this program working correctly lead them to confront important mathematical issues and construct deep meanings for concepts related to cosets. Here is a program that solves this problem.

```
PR := func(G,o);
            return =func(x,y);
                  if x in G and y in G
               then return x .o y;
            elseif x in G and y subset G
               then return {x .o b : b in y};
            elseif x subset G and y in G
               then return { a .o y : a in x};
            elseif x subset G and y subset G
               then return { a .o b : a in x, b in y};
            end;
      end;
   end;

oo := PR(S5, comp);
GH := {{g .oo h : h in H } : g in G };
K .oo L;
```

In the last three lines, the program defines the new operation, computes the set of all cosets of the group G by the subgroup H and computes the coset product of the two cosets K and L.

Class:

The course meets in a classroom where students again work in teams to perform paper and pencil tasks based on the computer activities in the lab. The instructor leads inter-group discussions designed to give students an opportunity to reflect on the calculations they have been working on and further construct their own meaning for their mathematical

experiences. On occasion, the instructor will provide definitions, explanations and overviews to tie together what the students have been thinking about.

In the case of cosets, constructing the lines of the program that form left and right cosets leads, according to APOS Theory to students constructing in their minds a process conception of cosets. Returning the two cosets as the result of the operation fosters development of an object conception. Finally, writing code to form the product of all pair of elements from two cosets relates to de-encapsulating these objects back to the processes from which they came. There are also many other examples of constructions in this short program that relate to process and object conceptions of other mathematical entities such as group as object and binary operation as both process and object.

After writing and working with the above program **PR** students can find it meaningful when they are asked in class to talk in their groups about the possibility of the intersection of two cosets, the number of elements in a coset, and set of all elements that are in at least one coset. Each of these questions has a complete answer for which the formal proof is rather easy for students who have constructed meanings for all these ideas. It is not difficult for the instructor to lead a discussion that brings out, not only the facts, but reasonable proofs of what amount to the steps in a proof of Lagrange's theorem. Furthermore, the computer experiences lead many students to think about cosets as objects and the result of PR as a binary operation on them. In class, they are asked to consider if the group axioms are satisfied and the notions of normality and quotient groups emerge.

Exercises:

Relatively traditional exercises are assigned for students to work on in teams. These are expected to be completed outside of class and lab and they represent homework that is in addition to the lab assignments. The purpose of the exercises is for students to reinforce the ideas they have constructed, to use the mathematics they have learned and, on occasion, to begin thinking about situations that will be studied later.

In the above-mentioned papers by the RUMEC teams, the results are based on pedagogy that uses the ACE Teaching Cycle. In other studies (some of which we have mentioned here), other pedagogies supporting this theoretical perspective have been used by a variety of authors.

5 Gathering and analyzing data

Gathering and analyzing data in our approach is a complex matter. This is for several reasons. One reason is that as we indicated in Section 1, our analysis of data is both driven by and drives our theoretical analyses and possibly even the development of the theory itself. Another is that we do not feel that a choice between gathering quantitative or qualitative data is appropriate. In our view, the best research will combine the two forms of information. This introduces complexity both in the interaction of two different types of results and the special needs of qualitative research which can be less straightforward than quantitative investigations.

We have already discussed the interaction between data and theory in Section 1. Regarding the qualitative/quantitative approaches, we don't just use

both, but try to develop a synthesis of the two. For example, one can administer a written instrument to a large number of subjects. We do this at times and analyze the results statistically. But we also use such data in the following way. Often it is possible to partition a large number of subjects into a relatively small number of categories by putting together people in a category whose responses on the written instrument are similar. Then we can apply the more time-consuming and laborious
qualitative methods, such as in-depth interviews, to one or two representatives in each category. Our assumption is that the interview transcripts we obtain in this manner both tell us a great deal about the student being interviewed and are also, to a greater or lesser degree, representative of all of the students in the same group.

Another purpose of combining the two kinds of data is triangulation. An in-depth interview can bring out a perhaps unexpected phenomena that needs to be explained. One can use written instruments to prepare for an explanation for example by considering questions such as: With how many subjects does this phenomena occur? What are its variations? What relationships does it have to other features of the subjects? Triangulation is of course, of critical importance in qualitative research. In addition to using combinations of different forms of data, we also take advantage of the fact that research in this approach is generally conducted by a team. This allows us to have different people on the team interpret qualitative data independently. Some of our best insights have been obtained from discussions designed to resolve the inevitable differences in interpretations. Triangulation consist in that when several researchers can agree on an interpretation, its validity and reliability is enhanced.

Again returning to Section 1, we recall that our data, both quantitative and qualitative tries to answer two kinds of questions: Have the students made the mental representations that the instruction was designed to foster? and To what extent do the students appear to understand the mathematics? Of course, the critical issue for our approach is whether there seems to be a causal relation from the former to the latter. Without giving details, to complete our discussion of the example of cosets, Lagrange's Theorem and quotient groups, we can report that a very high percentage of the students who experienced the pedagogy described here appeared to construct strong conceptions of cosets as processes and objects. As a result, they performed very well on examinations, for example, most could prove
Lagrange's Theorem, and indicated in interviews a good understanding of it and related notions. The results for quotient groups were not so strong, but this is an extremely difficult topic at the undergraduate level and what we saw in our data was much better than one gets from traditional courses in abstract algebra. The interested reader can pursue these questions by looking at the large amounts of

information about all of the RUMEC studies that appears in Clark et al (In preparation).

6 Alternative approaches

We have taken the position in this paper that good research in undergraduate mathematics education *must* be based on one or another kind of theoretical perspective. There is value in making assertions of universality in full awareness of the existence of counter-examples. This is the situation here. We believe our assertion is true almost all the time and that is important. We also believe that there are, and should be, a small number of counterexamples and they are important as well. Between the two extremes of *always* using a theory and use one or not as seems most reasonable, we believe there is a middle ground. We suggest that a research program should always expect to use a theory and one should proceed without one only if there is compelling reason to do so. Moreover, one should be quite concerned if the latter occurs very frequently.

Here is one example of research conducted by a RUMEC team that did not involve a theory. In the work reported in Dubinsky & Yiparaki (in review), we were interested in understanding student difficulties in interpreting mathematical statements involving two quantifications, one existential and one universal. Such statements are at once among the most difficult for students to grasp and absolutely essential to understand modern mathematics. Our basic premise was that people are successful in interpreting such statements when the context has to do with phenomena, not from the world of mathematics, but from the world(s) that most people in our society can think about. We thought that if we could understand how people did this, we could use that knowledge to get them to apply the same thinking to mathematical statements. This was a purely exploratory study and there did not seem to be much sense to involving theoretical considerations, at least at first.

All of this worked and we feel that we learned a great deal from the study without using any theoretical analyses. What we learned, incidentally, was that in fact, people *do not* tend to understand statements about the standard world if they include two quantifiers. Moreover, when they do, this understanding does not appear to transfer very well to mathematical contexts.

7 Conclusions

We do not have any definitive conclusions to which the considerations of this paper force us, or should force the reader. We do feel that we have made a strong case supporting a critical role, in research in mathematics education, for theoretical perspectives in general and APOS Theory in particular. Regarding the latter, we can say that its use seems to be increasing internationally. Indeed, there are at least 40-50 researchers throughout the world who are engaged in active research projects that make explicit use of APOS Theory. It is hard to say how

this compares with other theoretical perspectives. We can say, however, that we are obtaining results that may have some importance both for a basic understanding of learning advanced mathematical concepts, and for developing a basis for pedagogical reform that has the potential for making significant improvements in student learning. For this reason, we feel that APOS Theory and its applications deserve a place of some importance in the mathematics education enterprise.

Finally, we want to say that if the reader is at all convinced of the last point, or even curious about it, then he or she might be pleased to know that, in addition to the topics mentioned here, it seems that our approach can be applied to all topics in mathematics at the undergraduate level and at least some topics at the K-12 level. We have only scratched the surface and there is much to be done and plenty of room for more people to come in and see if this kind of approach makes sense to them and if the work might be something they would like to do.

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Reflections on APOS theory in Elementary and Advanced Mathematical Thinking

David Tall

Mathematics Education Research Centre
University of Warwick, UK
e-mail: David.Tall@warwick.ac.uk

What are the processes by which we construct mathematical concepts? What is the nature of the cognitive entities constructed in this process? Based on the theories of cognitive construction developed by Piaget for younger children, Dubinsky proposed ***APOS*** *theory to describe how* ***actions*** *become interiorized into* ***processes*** *and then encapsulated as mental* ***objects****, which take their place in more sophisticated cognitive* ***schemas****. He thus takes a method of construction hypothesised in (elementary) school mathematics and extends it to (advanced) college/university mathematics. In this paper I respond to Dubinsky's theory by noting the need for cognitive action to produce cognitive structure, yet questioning the primacy of action before object throughout the whole of mathematics. Biological underpinnings reveal cognitive structures for object recognition and analysis. I use this to suggest that APOS theory has already shown its strength in designing undergraduate mathematical curricula but question its universal applicability, in particular in geometry, and, more interestingly, in the formal construction of knowledge from definitions to deductions in advanced mathematical thinking.*

Introduction

The purpose of this paper is to respond to the research forum presentation of Ed Dubinsky (Czarnocha et al, 1999) on APOS theory as "one theoretical perspective in mathematics education research". It is clearly more than this, offering a major contribution to mathematics education at the undergraduate level. Indeed, in the Calculus Reform in the United States in the late 80s, it formed the basis of the *only* curriculum project that had a coherent cognitive perspective.

My response will analyse APOS theory within wider realms of mathematical learning and thinking, in particular a comparison of its roles in various contexts in elementary mathematical thinking (EMT) and its extension to advanced mathematical thinking (AMT). (These acronyms were introduced by Gontran Ervynck who was responsible for the formation of the AMT Working Group at the Psychology of Mathematics Education Conference in 1985.) AMT referred initially to "mathematics learning and teaching at 16+" including the activities of mathematicians in research. The work of Dubinsky and his colleagues has focused on undergraduate mathematics (RUME), in particularly in developing suitable working practices (eg co-operative learning) and learning sequences (genetic decompositions) in a wide range of specific mathematical areas, including discrete mathematics, logic, calculus, linear algebra, group theory. In this paper I shall concentrate on the role of APOS theory.

The biological foundation of action-process-object-schema

I wish to begin by showing that the broad brush-strokes of APOS theory seem to have a deep underlying biological structure. Figure 1 shows a simplified model of three stages of brain development as a result of successive stimuli (which could be perceptual or reflective). Stage represents an external stimulus to neuronal group 1, which is sufficiently strong to fire neuronal group 2 but not group 3. The firing causes the link between 1 and 2 to become more sensitive for a period of hours or days (so that we are more likely to recall recent events). If the connection is reactivated, it becomes more easily fired until it reaches as stage where any excitation of 1 also fires 2. This *long-term potentiation* of the neuronal connections builds new structures. The combined strength of 1 and 2 may now cause group 3 to be excited, and so on. In this way an external stimulus can cause a firing between two states perceived initially as separate, then joined together, then part of more complex neuronal groupings that can fire in more complex situations. The broad action-process-object-schema therefore has a natural biological underpinning.

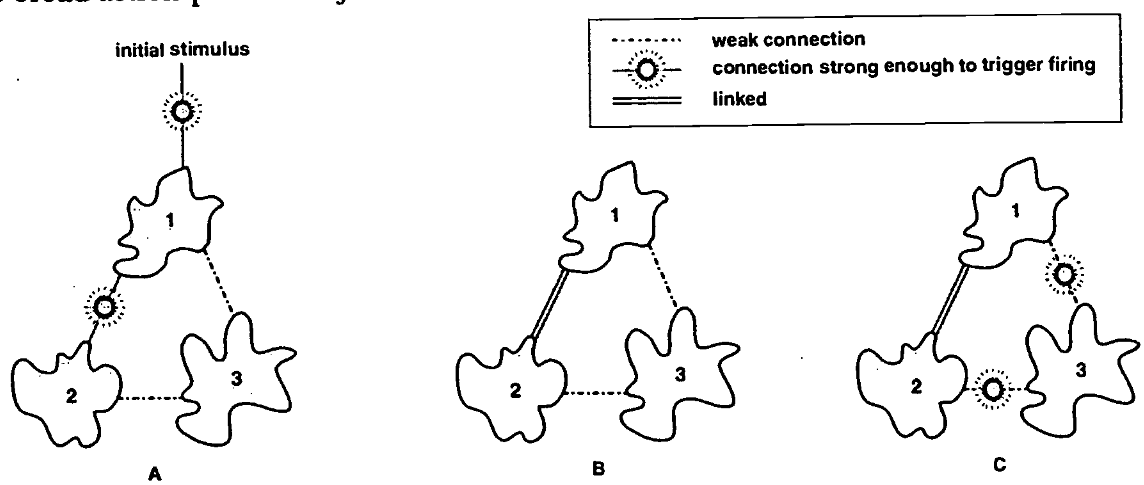

Figure 1: building memories in the brain by long-term potentiation (Carter, 1999, p. 160)

However, suppose that the first stimulus is a perception of an *object*. Then the same sequence of diagrams could represent a growing relationship with the properties of that object and the building of a more complex system of properties and connections. Both of these theories occur within the description of APOS in Czarnocha et al, (1999), There is a sub-sequence beginning with actions and moving to objects, and a further sub-sequence beginning with objects and moving to a broader schema.

The primacy of actions and objects

APOS theory begins with actions and moves through processes to encapsulated objects. These are then integrated into schemas—consisting of actions, processes and objects—which can themselves be encapsulated as objects. This suggests a primacy of action over object. At a fundamental level it absolutely clear that cognitive actions are required to construct cognitive objects. This I refer to as an application of "strong

APOS", in which the actions and processes are *any* cognitive actions or processes (conscious or unconscious), not just (conscious) mathematical ones.

However, even with this strong interpretation of the theory, the primacy of actions needs to be questioned. Dubinsky and his co-workers have made an impressive effort to formulate everything in action-process-object language. However, the urge to place this sequence to the fore leads to a description that, to me, soon becomes over-prescriptive.

The part of development that uses the triad theory of Piaget and Garcia (1983) moving from object to schema describes the initial object as "an encapsulated process" or a "thematized object" to maintain the primacy of the APOS sequence. The first stage of the triad, denoted *intra*, is simply described as "focus on a single object", followed by *inter* (study of transformations between objects) and *trans* (schema development connecting actions, processes and objects). Such a description (based on the language of action, process and object) seems to be at pains to avoid other more widely used terms such as "inter" including the study of *properties* of objects, or "intra" being concerned with *relationships* between them. The term "transformation" is one that I sometimes find impenetrable. Sometimes it has a mathematical meaning, but at others it seems obscure. In comparing the size of one object with another, is there a transformation of objects in some sense, or does the child just declare one is bigger because it pokes out beyond another?

APOS theory even formulates the notion of "permanent object" as arising through "encapsulating the process of performing transformations in space which do not destroy the physical object" (Dubinsky *et al.*, 1988, p.45). Thus the permanent mental object in the mind is created by a physical or perceptual action on an external object, to maintain the primacy of process over object.

Looking closer at the structure of the brain suggests a distinctly different scenario. The research of Hubel & Wiesel (1959) revealed single neurons in a cat's brains that respond to orientation of an edge. Similar experiments with other animals revealed the same phenomenon. In the brain of Homo sapiens, in addition to a specific area of the cortex that builds a point by point copy of the visual field, other brain modules specialize in a variety of analytic activities, including the perception of edges, orientation, movement, colour, binocular vision, and so on. Homo sapiens and many other creatures therefore have complex systems for the visual perception and analysis of *objects*. One may attempt to state that such objects only arise as the result of cognitive actions, and this use of *strong* APOS cannot be denied at a theoretical level. However, the practicalities of consciousness tell a different story. When an individual is looking at an object, the *conscious* experience is that of the object being seen, not of the multi-faceted unconscious processes by which the internal brain processes information.

Furthermore the brain has a highly subtle collection of modules to detect object properties. Recent research has shown that animals and very small babies have primitive brain modules that distinguish between one object, two objects, and perhaps more. This distinction occurs in babies at a far earlier age than Piaget's theories predict. (See "Piaget's errors, p.44 *et seq.* in Dehaene, 1997).

The primitive brain also operates in other ways that impinge on high mathematical and philosophical constructions. Lakoff & Johnson (1999, p.16) hypothesize that the primitive "embodied mind" plays a role in *all* our thought, including what may be perceived as logical deduction when brain modules sense such things as "inside inside" is "inside" (ibid, p.32). This occurs not through logical deduction but because the brain models just observe that "it is". Furthermore, intuitive deduction occurs using "embodied arguments" such as the use of "prototypical" exemplars rather than strict quantification (ibid. chapter 7). The objects of the world and the embodied structures in the brain have a full role to play in learning and thought.

The previous discussion, whilst failing to deny the primacy of *cognitive* action to construct *cognitive* concept, indicates that the brain observes objects, and what seem to be primitive mathematical and logical concepts in ready-made brain modules. This seriously questions a rigid Action-Process-Object-Schema strategy in every curriculum. Even the APOS curriculum has sub-sequences building on objects.

APOS Theory in Elementary Mathematical Thinking

To see the relevance of APOS in mathematics, I begin with its source in EMT. Here Piaget spoke of three modes of abstraction: *empirical abstraction* from objects of the environment, *pseudo-empirical abstraction* from actions on objects in the environment and then *reflective abstraction* from mental objects.

Geometry includes many acts of empirical abstraction focusing on objects, beginning with an intra stage coming to terms with the nature of objects themselves. I contend therefore that geometry starts as an *object-based theory*. This is not to say that there are no processes—of course there are (drawing, measuring, constructing etc). However the focus of these processes is to gain knowledge about the objects themselves.

Concepts in geometry occur with many parallel activities involving physical interaction with the real world, but also, in a very real sense, they depend on the growing sophistication of language. Rosch's theory of prototypes (Rosch et al, 1976) shows that children first recognize "basic categories" such as 'dog', or 'car', only later moving to super-ordinate or sub-ordinate categories, such as 'poodle-dog-animal' or 'Ford-car-transport'. Such basic categories have properties that relate naturally to immediate perception. A basic category can be represented by a prototypical mental image; it is the highest level at which category members have similarly perceived overall shapes and the highest level at which a person uses similar motor actions for interaction with the members. The focus on a category of basic *objects* occurs naturally through a coherent combination of perceptions. I would contend here that it is the *object* that is the focus of attention, with the actions being the agents of that perception. It is only later, in a Van Hiele type growth (using a dialectic back and forth rather than a strict sequence of stages) that language used for description enables the conscious mind to build platonic objects such as lines with "no width" and "infinite extensibility".

On the other hand, the growth of knowledge in arithmetic and algebra begins with *pseudo-empirical* abstraction and hence more closely follows an APOS sequence. In counting, there is the action of repeating the number words and beginning to accompany

this by pointing at objects in turn. Later various learning sequences set up neuronal connections in the brain, routinizing the procedure, seeing it as a process when it is realised that different orders of counting the same set give the same number, and then "encapsulating" the process into the concept of number. In fact the encapsulation follows a sequence of counting "one, two three, four", then silently counting all but the last number, then saying just the last number without counting at all. Number names and number symbols play an essential role in this development.

I was utterly flabbergasted to see that *nowhere* in Czarnocha et al, (1999) is there a single mention of the word *symbol.* Symbols are at the heart of cognitive development of arithmetic (and later in algebra). They can be spoken, heard, written, read, used in action games and songs. They are the stuff that children work with. Indeed, those who focus more on the objects being counted than on the symbols for the counting prove to have much greater difficulties in later development (Pitta & Gray, 1997).

To develop a more elaborated theory to describe these phenomena, Gray & Tall (1994) formulated the notion of procept as follows:

> An *elementary procept* is the amalgam of three components: a process which produces a mathematical object, and a symbol which is used to represent either process or object.
>
> A *procept* consists of a collection of elementary procepts which have the same object.

The procept notion has strong links with APOS theory, but there are significant differences. We have always insisted on focusing on the cognitive structure itself and do not imply that the mathematical process involved must first be given and "encapsulated" before any understanding of the concept can be derived. For instance, in introducing the notion of solving a (first-order) differential equation, I have designed software to show a small line whose gradient is defined by the equation, encouraging the learner to stick the pieces end to end to construct a visual solution through sensori-motor activity. This builds an embodied notion of the existence of a unique solution through every point, with difficulties only occurring at singularities where the differential equation does not give the value of the gradient. It provides a skeletal cognitive schema for the solution process before it need be filled out with the specific methods of constructing solutions through numeric and symbolic processes. It uses the available power of the brain to construct the whole theory at a schema level rather than follow through a rigid sequence of strictly mathematical action-process-object.

Figure 2 shows a succession of uses of processes and concepts in symbolic mathematics (Tall, 1998). Arithmetic has computational processes, algebra has potential evaluation processes but manipulable concepts, the dynamic limit concept at the beginning of calculus involves potentially infinite computational processes that lead to the mental imagination of "arbitrarily small", "arbitrarily close" and "arbitrarily large" quantities. It is no wonder that so many students cling to the comfort of rote-learned finite rules of the calculus.

In Advanced Mathematical Thinking, students meet an entirely new construction: the axiomatic object in which the *properties* (expressed as axioms) are the starting point and the *concepts* must be constructed by logical deduction. Although (strict) APOS again can

Formal definitions & proof	*defined* properties *logical* processes *formally constructed* concepts
calculus	*computational* processes (rules), *manipulable* concepts (formulas)
(dynamic limit concept)	*potentially infinite* processes, *arbitrarily small, close,* or *large* concepts (variable quantities)
algebra	*potential* processes (evaluating expressions) *manipulable* concepts (expressions)
arithmetic	*computational* processes *computational* concepts (numbers)

Figure 2: Development of process/concept in symbolic mathematics (Tall, 1998)

describe the learning processes (as it always will), there is a far more serious area of study in the relationship between embodied knowledge and formal deduction (Alcock and Simpson, 1999). Procepts are only of value here in certain aspects (for instance, in the element of a transformation group can be both process and an object), but the notion of group is not as it does not have a symbol dually representing process and concept. Is this a failure of the notion of procept compared with the broader application of APOS theory? Superficially, of course. However, the very fact that there is a serious cognitive reconstruction using symbols in formal mathematics in an entirely different way from the procepts of elementary mathematics suggests a chasm that many students have difficulty in crossing.

Dubinsky and his colleagues have a *brilliant* way of looking at the group axioms: formulate it as a function which outputs whether a set and its operation is a group or not. This to me is a fantastic solution (both in terms of computers and cognition). However, I still have serious concerns about the cognitive constructions made in such a sequence.

Different styles of Advanced Mathematical Thinking

One of my favourite quotations, which I have used often before, is the following:

> It is impossible to study the works of the great mathematicians, or even those of the lesser, without noticing and distinguishing two opposite tendencies, or rather two entirely different kinds of minds. The one sort are above all preoccupied with logic; to read their works, one is tempted to believe they have advanced only step by step, after the manner of a Vauban who pushes on his trenches against the place besieged, leaving nothing to chance. The other sort are guided by intuition and at the first stroke make quick but sometimes precarious conquests, like bold cavalrymen of the advanced guard. (Poincaré, 1913, p. 210)

The approach to the development of APOS is closer to Vauban than the bold cavalryman. As implemented it often begins with symbolic procedures or programming activities with visual activities coming later, if at all. Such a curriculum clearly can be of great value but I cannot help sensing we need to develop bold cavalrymen with insight, even if it is flawed for these leaps can be the stuff of future solid conquest.

There is such a difference between analysts who fear the fallibility of pictures and geometers or topologists that live by them. But this does not mean that either style is necessarily always superior:

> In the fall of 1982, Riyadh, Saudi Arabia ... we all mounted to the roof ... to sit at ease in the starlight. Atiyah and MacLane fell into a discussion, as suited the occasion, about how mathematical research is done. For MacLane it meant getting and understanding the needed definitions, working with them to see what could be calculated and what might be true, to finally come up with new "structure" theorems. For Atiyah, it meant thinking hard about a somewhat vague and uncertain situation, trying to guess what might be found out, and only then finally reaching definitions and the definitive theorems and proofs. This story indicates the ways of doing mathematics can vary sharply, as in this case between the fields of algebra and geometry, while at the end there was full agreement on the final goal: theorems with proofs. Thus differently oriented mathematicians have sharply different ways of thought, but also common standards as to the result. (Maclane, 1994, p. 190–191.)

Even though different approaches to research both end up with formal proof, the mathematical insights gained are very different. In a time of fast technological development, it is clear that we need our cavalrymen to make precarious advances as well as those who carefully operate safely step-by-step.

Pinto and Tall (1999) reveal a wide spectrum of thinking processes in undergraduate mathematics students including some who build meaning for definitions from their own experiences and others who take the definitions given by others and build meaning mainly by deducing theorems. The latter students seem more amenable to an action-based APOS course than the former, who build on a whole range of embodied cognitive constructs. Whilst cognitive actions are always necessary to construct cognitive concepts, is it providing a service to necessary diversity in human thought by restricting the learning sequence to one format of building mathematical actions, mathematical processes and mathematical objects?

Summary

Strict APOS can be used to formulate the idea that cognitive concepts must be preceded by cognitive operations. In this sense APOS theory is a "ToE" (Theory Of Everything). However, given that the learner has a wide range of embodied constructs in need of reflection and reconstruction, I contend that the accent on sequences solely built on action-process-object-schema distorts the wider enterprise. Figure 3 shows my own vision of the development of major themes in mathematics. APOS theory has many applications in the elementary mathematics of arithmetic, algebra, and calculus, but is of less relevance in the study of space and shape. In RUME the evidence shows its power in designing certain types of highly successful curricula. However, it is not the whole world. The varieties of thinking in professional mathematicians need an expression beyond that of the measured action-process-object-schema development. For instance, in my own approach to calculus, I begin not with the new process of programming functions in a computer language, but with the embodied visuo-spatial notion of "local straightness" which is then explored in parallel with the symbolic operations of differentiation. I welcome APOS as a major contribution to the understanding of mathematical cognition, but as a valued tool, not a global template.

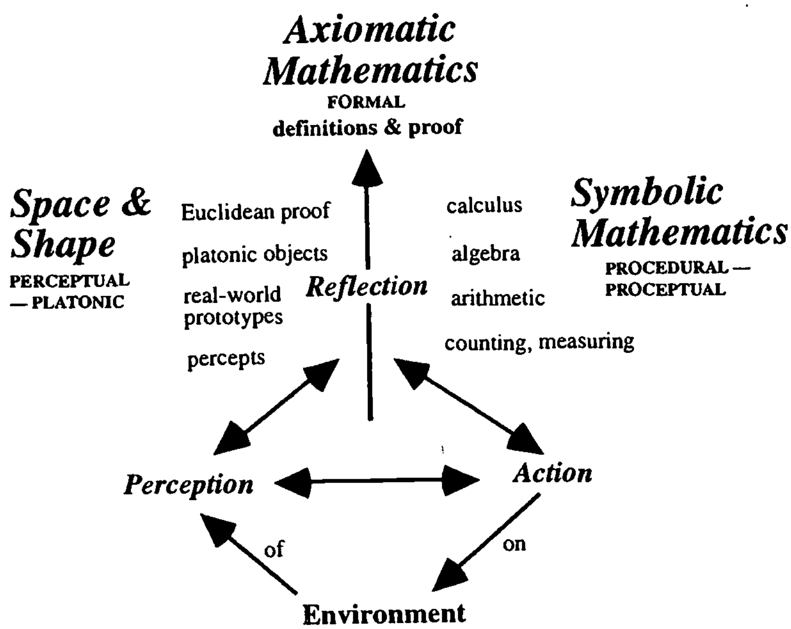

Figure 3: cognitive themes in the development of mathematics (Tall, 1995, 1998)

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TEACHING AND LEARNING LINEAR ALGEBRA WITH CABRI

Anna Sierpinska, Jana Trgalová, Joel Hillel, Concordia University, Montréal, Canada

Tommy Dreyfus, Center for Technological Education, Holon, Israel

Abstract: *The paper gives an account of a research program concerned with the study of computer environments in the teaching and learning of linear algebra. In an attempt to help students overcome the so called 'obstacle of formalism', we designed an entry to linear algebra based on a geometric model of the two-dimensional vector space within the dynamic Cabri-geometry II environment. We present (i) the theoretical framework of the research whose main ideas are the necessity of the use of 'multiple representations' in the learning of mathematics and the notion of quasi-fundamental situation, (ii) the methodology which is close to didactic engineering, and (iii) the design and its evaluation in terms of an analysis of discrepancies between the objects constructed by the students and the intended ones.*

1. Introduction

Our research program inscribes itself within the large domain of mathematics education which is concerned with the *design and study of the computer environments for human learning* ('environnements informatiques de l'apprentissage humain', Balacheff, 1998), whose main questions regarding mathematics learning are:

- How are the computer representations of mathematical objects related to the meanings of these mathematical objects in the mathematical theory?
- What should be the conceptual furnishings of the student's mind in order for him or her to interpret these representations not in phenomenological terms of what is happening at the interface between the user and the computer program but in terms of mathematical theory?
- Can the computer learning environment be designed so as to promote the development in the student, of such conceptual, rather than phenomenological, interpretations?

There is already an awareness in the mathematical community of the general 'disparity between mathematical conceptual structures and their software manifestations' (Burton & Jaworski, 1995, p. 8) and this phenomenon has been thematized in the concept of 'computational transposition' (Balacheff, 1993, p. 147). According to Balacheff (1998), the study of the transformations of meaning and of the epistemological status of objects in the computational environments should become an important field of research in mathematics education.

At the source of our research there lies the practical didactic problem of overcoming, in students of linear algebra, the quite widespread 'obstacle of formalism' (Dorier, Robert, Robinet & Rogalski, 1997). We have considered this obstacle as responsible for, among others, the notorious confusion of linear transformations with their matrix representations and lack of awareness of the

relativity of the matrix representations with respect to a basis (Hillel & Sierpinska, 1994). In order to overcome the obstacle of formalism, our idea was to design an entry into the topic of linear transformations using a definition of this concept condition, without, however, using only the discourse of the axiomatic theory in which vectors and linear transformations are nothing more than variables (e.g. 'let $\{v_1, v_2\}$ be a basis of a two-dimensional vector space'; 'let T be a linear transformation on V', etc.). We needed an informal way of speaking about vectors without immediately speaking about their coordinates.

The Cabri-geometry II environment appeared to provide us with some means to reach this goal: We have created a geometric model of the two-dimensional vector space where vectors are positions of points with respect to a fixed point called the origin and where to describe the position of a point one needs only two independent oriented directions and units in each direction. The arbitrary element of the vector space was represented, in Cabri, by an arrow - the Cabri 'vector' - whose tail was attached to a fixed point O and whose head could be dragged freely on the screen. Linear transformations were introduced as transformations preserving vector addition and scalar multiplication. Transformations in general were given, in Cabri, by representations of dependencies between two vectors: One 'free' vector would be put on the Cabri screen, labeled v, its image under a transformation would be constructed, labeled T(v), and all traces of the construction would be hidden. The construction would be recorded in Cabri as a macro and given a name. It would be possible to obtain images of vectors under the transformation by using this name. If a transformation was given this way, the only way to know whether the dependence between v and T(v) is linear was by checking the conditions of the definition.

Thus the Cabri environment appeared to make it possible for the students to achieve a more direct contact with the objects of the abstract theory without too quickly replacing these objects by computational procedures.

From the point of view of the idea of the relativity of the matrix representation of a linear transformation with respect to a basis, a very useful feature of Cabri is the possibility of creating systems of axes on an arbitrarily chosen pair of non-collinear vectors ('New Axes'). Moreover, the dynamic features (the dragging mode) of Cabri and the possibility of obtaining simultaneous geometric and arithmetic representations of vectors and linear transformations, supported our theoretical perspective on the role of multiple representations in grasping abstract mathematical objects.

An experimentation, with several groups of students, of a series of activities based on the above ideas, has shown, however, that our expectations are not easily realized. This problematized our approach and our use of the Cabri technology for the teaching of linear algebra. Since then our research efforts have been directed towards explaining the sources of the perceived discrepancies between our expectations and students' reactions to the designed activities.

2. Epistemological position

The theoretical framework underlying our research program is constructed on a set of *epistemological assumptions* (rather than psychological ones). A detailed presentation of this framework can be found in Sierpinska, Dreyfus and Hillel (1999).

Our main assumption is that there is no direct access to objects of scientific knowledge; scientific knowledge, by definition, is semiotically mediated. The development of scientific knowledge was triggered by the need to compare, order, and predict the behavior of objects whose identity, magnitude, appearance in time, etc. could not be evaluated with the use of the senses alone. Direct observation had to be replaced by the construction of technical instruments, and of systems of representation, notation and computation, as well as of theories containing means of the validation of inferences made on this basis. In the absence of direct access to an object, one of the main questions is: how can we tell whether two given representations are, in fact, representations of the same object? Each representation captures certain aspects of an object and ignores other aspects, and, therefore, the answer to this question is not obvious (for a thorough presentation and justification of this epistemological position, see Duval, 1998).

Much of the scientific effort is invested in the development and study of instruments and systems of representations. This is especially (but not solely) the case of mathematics. The objects of mathematics being theoretical objects, the question of their identity is a particularly delicate one.

We take mathematical objects to be the invariants in the reference of several semiotic representations (Duval, ibid.). In other words, we do not take them to be the contents of mental representations in the psychological subject, nor do we consider them as mind independent entities. One consequence of this epistemological position is the necessity of the existence of several representations of an object of knowledge. If there was only one, there would be no reason to distinguish between the object and its representation. A learner of mathematics can grasp an object of this knowledge only through working with several representations of this object, processing them and converting from one to another.

Some caveats are in order with respect to the above presentation of our theoretical perspective. We assume an independent existence of mathematical objects (independent of individuals' minds and social conventions), but we do not award mathematical objects any kind of absolute character. What is the object and what is its representation may well depend upon the context of the mathematical situation at hand. For example, in Analysis, an equation like y=2x+3 is usually the object and a straight line drawn in a system of coordinates is its geometric representation; but in Geometry the line will be regarded as the object and an equation like y=2x+3 as one of its possible representations relative to one of the many possible systems of coordinates. What accounts for the 'objectivity' of

mathematical concepts is their theoretical and systemic character (cf. Steinbring, 1998). The existence of mathematical objects is of a logical, not psychological or sociological order.

But, in taking this epistemological position, we are not denying that the processes of construction of mathematical knowledge do indeed have a psychological and social character. We are only saying that what we shall call 'knowledge' will be the *products* of these processes in the form of systems of signs and that we are interested in studying the internal logical coherence of these systems, their field of reference (i.e. the objects of the signs), and their relations and relevance with respect to other sign systems. As mathematics educators looking at groups of undergraduate students learning linear algebra under the guidance of a teacher, we study what is being said and written and drawn on paper or on a computer screen, in the aim of creating theoretical models of the sign systems used by the groups. Our analysis and evaluation of this model is done not in terms of the cognitive characteristics of individual students and the teacher as psychological subjects, nor of the nature of the social interactions in the group, but in terms of the mathematical and logical relations between the knowledge produced by the group and the 'intended knowledge' assumed by the didacticians who planned the teaching/learning activities. This is not to say that the impact of didactic phenomena such as 'contract didactique' or 'effet Topaze' (Brousseau, 1986a) on the knowledge produced by a group of students and a teacher is not taken into account. It is, but we require that this impact be described in terms of a mathematical model of the difference between the intended objects and concepts and the actually constructed objects and concepts.

3. Quasi-fundamental situations

The methodology of our research is close to (but not identical with) *didactic engineering* (Artigue, 1990; Artigue and Perrin-Glorian, 1991) in that (a) both the *a priori* decisions and the analysis following a realization of the design are founded on epistemological, cognitive, and pedagogical considerations, and (b) its methods of validation are internal, based on an interaction between the *a priori* and the *a posteriori* analysis, and not external, based on a comparison between experimental and control groups.

Didactic engineering methodology has been originally based on Brousseau's theory of didactic situations (Brousseau, 1997; Perrin-Glorian, 1994). Our design deviates from the classical applications, usually structured along the situations of action, formulation and validation. In Brousseau's theory, one of the basic distinctions is between didactic and a-didactic situations, and it is assumed that, for some important elements of mathematical knowledge, it is possible to construct a-didactic situations in which this target mathematical knowledge will emerge as an optimal solution to a problem for the students ('*fundamental situations*').

Considering that the specific character of linear algebra concepts (generalizing, unifying existing theory rather than tools for solving specific problems, see Dorier, 1995) does not allow for the construction of a-didactic, 'fundamental situations' in this sense, and acknowledging the importance of the 'didactic contract', we propose a somewhat different categorization.

We consider only didactic situations and we divide them into 'social' and 'cognitive' ones. 'Social didactic situations' are those based on a social contract; 'cognitive' ones are those based on a cognitive contract. In a social contract, the object of the negotiations between the students and the teacher are the social and institutional rules governing the process of following and passing a course within the given institution. The students' most frequent questions are concerned with the presentation of their solutions, e.g. 'Do we have to write the solution with these curly brackets, or is it enough to write x equals something?', or 'Do we have to justify our answer or is it OK if we just write the numbers?'. If a teacher 'teaches to the test', the contract is of this 'social' nature. In a cognitive contract, the object of the negotiations between the students and the teacher are the interpretations of the representations used in the communication of knowledge and the nature of the studied [mathematical] objects.

We shall call a didactic situation a '*quasi-fundamental situation*' (QFS) with respect to a given mathematical object if it has a high potential to evolve into a 'cognitive didactic situation' in which the nature of this mathematical object will not fail to be discussed. This does not mean that it has to evolve into such a situation for all students and in all circumstances. We do not assume any kind of 'epistemological necessity' or determinism in the development of knowledge. We admit that much depends on the students' attitude towards learning mathematics, their previous mathematical experience, as well as on the teacher's perception of the meaning of the subject matter. (For the presentations and discussions of the concept of fundamental situation, see Brousseau, 1986b; Berthelot et Salin, 1995; Legrand, 1996; Sierpinska, to appear).

4. The evolution of the research program

Our research program has been evolving within the framework described above since 1996. It is also then that the idea of using the Cabri-geometry software arose. From the outset, our work concentrated on the development, within this environment, of (a) a coherent system of representations of objects such as two-dimensional vector space, linear transformation, eigenvector, and (b) a set of activities and problems for the students using this system of representations. A sequence of experimental sessions with students was designed aiming at testing whether, in this linear algebra enriched Cabri environment students develop a better grasp of the mentioned mathematical objects than is usually the case in ordinary lecture style, paper and pencil setting.

The first experimentation took place early in 1997: There were 7 highly controlled sessions with only a pair of students, both after a college level vectors and matrices course. The students were working together, partly with a teacher, partly without his intervention. A computer with the enriched Cabri environment was always there and the students could use it whenever they wanted. Most, but not all, activities were to be done using the computer. The students' interpretations of the Cabri representations and the activities and problems turned out to be rather far from those that we had intended and expected, yet logically quite justifiable. Our *a posteriori* analyses concentrated on modeling the mathematical objects that appeared to emerge in the interactions of the students with the learning environment, and justifying them on the basis of, mainly, the characteristics of the representations and the activities and problems proposed to the students. The results of these analyses can be found in Sierpinska et al. (1999).

A second experimentation (spring 1998) was then prepared, based on a revised version of the previous design. Only five sessions were designed, focusing on those concepts about which there was the most confusion in the first experimentation: vector, basis, linear transformation and especially the notion that the values of a linear transformation on a basis determine the transformation completely. Three pairs of students worked each with a different teacher. The experiment was still very tightly controlled: The students were not given any materials to study outside of the sessions, and all their conversations were recorded. The *a posteriori* analyses of this experimentation concentrated, as before, on understanding the mathematical objects that have emerged in the interactions within the environment, but not so much on ways to improve the design of the representations, activities and problems. It appeared to us that some unintended interpretations are inevitable, whether due to the specificities of the Cabri environment or to the axiomatic nature of the linear algebra concepts, or to other reasons. We started thinking that, instead of trying to avoid them, we might capitalize on the didactic situations that bring them to light, and create follow up activities that would allow the students to become aware of these interpretations, and see the possibility of alternative ones.

This led us to the notion of the 'quasi-fundamental situations' and the preparation of the third experimentation focused on the development of these for the notions of vector, linear transformation and eigenvector. These situations had to be tested in an ordinary course context, not an experimental context, because in the latter, the risk of the evolution of a 'social type of contract' is rather unlikely (the situation may not even be perceived by the participants as a didactic situation). We chose a course in linear algebra for students preparing for a master's degree in mathematics education. The course, taught by A. Sierpinska, was titled 'Linear Algebra with Cabri and Maple' and lasted 13 weeks, at the rate of one 2 hour meeting per week. It took place in the fall term of 1998. It was only partly controlled (materials to study between classes and assignments were given, but the students' activities between classes were not recorded nor monitored). In the class,

or rather computer lab, the students were given worksheets in which they had to write the results of their activities with Cabri and solutions to problems. These worksheets were collected each week. The students were also writing diaries. At the end of the course, the students wrote a test. Nine students started the course, seven completed it. Two students dropped out, feeling unable to cope with the material and the pace of the course.

All these students had taken undergraduate linear algebra courses prior to entering the master's program. The course was not planned as a re-take of an undergraduate course; rather the students were supposed to experience this course both as students and as potential teachers of linear algebra courses at the college level. The aim was to help them see the presumably known concepts from a different perspective and judge of the potential or pitfalls of this technological approach for the teaching of college students. The course started with the notions of vector, operations on vectors and linear transformations like the previous experiments but then continued with the arithmetization of these notions: coordinates, change of basis, matrix representations of linear transformations, 'eigentheory' and its applications to dynamical systems.

The analysis of the outcomes of this last experiment is ongoing at the time of writing this text. So far, the analysis has focused on the potential of some of the designed situations to evolve into quasi-fundamental situations.

5. Samples of results

The results of our research comprise

- descriptions and justification of the didactic situations as designed;
- description of the students' behavior during the experimentation of the designed didactic situations;
- models of the mathematical objects constructed by the students; their analysis in terms of their inner consistency and possible differences between them and the intended mathematical objects; and possible reasons why the students constructed these objects;
- evaluation of the potential of some of the designed didactic situations to evolve into quasi-fundamental didactic situations.

The results reflect the foci of our experiments: the *a priori* analysis of the linear algebra-extended Cabri environment in the first experiment; models of the objects constructed by the students in the second, and the identification and study of the candidates for quasi-fundamental situations relative to the basic objects of linear algebra in the third.

In the sequel we present some samples of such results of our research. We omit an example from the first experiment, whose thorough description can be found in Sierpinska et al. (1999).

5.1 The second experiment: models of mathematical objects created by the students

The results of our research in relation to the second experiment are a product of a confrontation of an analysis of the students' behavior with our theoretical assumptions and expected behavior. We give here a sample of these results, focused on the concept of linear combination of vectors.

5.1.1 Overview of the design with respect to the notions of vector and linear combination

These concepts were addressed in the first two sessions. In the first session, the students became acquainted with a representation of the two-dimensional vector space, constructed in the dynamic geometry environment Cabri-geometry II. Vectors were introduced as models of translations in geometry and of forces in physics, and were represented by an arrow produced by the Cabri command 'Vector'. A scalar k was represented in Cabri by a variable point attached to a number line; its coordinate on this number line was the value of the scalar k. The first activities were aimed at focusing the students' attention on two characteristic features of a vector: length and direction, and at introducing the notion of equality of vectors. Then the convention of representing all vectors by arrows starting from the same point called the 'origin' was introduced. The second part of the session was devoted to the operations of addition and scalar multiplication defined on vectors. The operations were introduced by defining new vectors produced by Cabri macros 'Vector addition' and 'Scalar multiplication' respectively, and the students were asked to explore the properties of these new vectors. They were expected to focus on those properties that appeared as invariants of the representation when the given vectors and scalars were dragged around the screen. The notions of zero vector and opposite vectors were introduced at this stage.

The second session was devoted to the notion of linear combination. The linear combination of two vectors v_1 and v_2 was introduced algebraically as a vector $w = k_1v_1+k_2v_2$, where k_1 and k_2 are scalars. The students were asked to translate this definition into a Cabri representation of a linear combination and construct w on the screen. For this purpose, two number lines with variable scalars k_1 and k_2 were created on the screen and two vectors v_1 and v_2 starting from the origin were given. The students were then asked to explore the properties of a linear combination and examine special cases when one of the scalars is set to zero and when both scalars are equal to 1. In the next activity, the students were asked to put any vector u on the screen and express it as a linear combination of the vectors v_1 and v_2. This could be done by adjusting the scalars k_1 and k_2 so that the vectors u and $k_1v_1+k_2v_2$ overlapped. This activity aimed at creating an intuition that any vector in a two-dimensional vector space can be decomposed into a linear combination of two given non-collinear vectors. In the next activity, the students were composing and decomposing vectors on the Cabri screen as well as on paper. At the end of the

session, the students were asked to determine which vectors in the plane can be obtained as linear combinations of (a) one vector and (b) two non-collinear vectors and which vectors cannot be obtained. This question aimed at an intuition of the notion of basis in two dimensions.

5.1.2 Description and analysis of students' behavior with respect to the notion of linear combination

The design described above was experimented with three pairs of students who had not taken any linear algebra course yet, but some of them, one pair in particular, had encountered vectors in physics classes.

As we said above, a linear combination of two vectors v_1 and v_2 was defined algebraically as a new vector w given by the formula $w = k_1v_1+k_2v_2$. The invariant of this representation was intended to be *the sum of scalar multiples of v_1 and v_2.* The students were asked to construct a representation on the computer screen of a linear combination of two given vectors v_1 and v_2 stemming from the same point, using two scalars already on the screen. Following the formula, the students constructed first the scalar multiples k_1v_1 and k_2v_2, and then added these two together to obtain the vector w. To explore the properties of w, they were invited to move the scalars along the number lines and observe what happens to w. To help them identify the invariants of this dynamic representation, they were asked to connect by segments the endpoint of w with the endpoints of k_1v_1 and k_2v_2 respectively, thus obtaining a parallelogram with w as a diagonal. The intended invariant of the dynamic representation on the screen was: Linear combination of v_1 and v_2 as *a diagonal of the parallelogram built on vectors k_1v_1 and k_2v_2.*

As intended, the students were focusing on the parallelogram and its diagonal, however they seemed first to be overlooking its sides. The object 'linear combination of v_1 and v_2' became for them the sum v_1+v_2 with a specific property: *The lengths of v_1 and v_2 were variable and adjustable by the scalars.* This was the difference between linear combination and vector addition, the latter being defined for *vectors with fixed lengths.* The object that for the students was referred to by the term 'linear combination of v_1 and v_2' was '*a set of all vectors v_1+v_2, with v_1 and v_2 having fixed direction, but variable length*'. Such an object could only exist in the dynamic environment; in the paper and pencil context it was identified with the vector addition. In the activities to be done on the computer, such as expressing a given vector as a linear combination of two given vectors, the students were successful and it was impossible for a teacher to notice that the concept being developed in the students was not exactly the intended one. It was only revealed when the students were asked to find scalars k_1 and k_2 such that the linear combination $k_1v_1+k_2v_2$ is a zero vector. The task was to be done away from the computer. It is clear that, within the students' understanding of linear combination as being the sum, the only possible answer, when v_1 and v_2 are non-collinear, is that both vectors are zero, and when v_1 and v_2 are collinear, they are opposite to each

other. These were exactly the students' responses. This concept became even more transparent in the activity of decomposing a given vector w into a linear combination of two given non collinear vectors u and v. The students were able to solve this problem on the computer: They first constructed the linear combination of the given vectors using the values of scalars that were already on the screen, and then adjusted the scalars so that the linear combination and the given vector w overlapped. When they were presented the same problem on paper, they claimed that it was impossible to solve it because w could not be expressed as u+v and they said they '*don't have a scalar*' to adjust the lengths of u and v so as to make w the diagonal of the parallelogram on u and v.

We now analyze in more detail the activities of composing and decomposing vectors into a linear combination of two given vectors. These activities were to be done both on paper and with the computer. The composition of two given vectors into a linear combination did not present any difficulty for our students: all necessary information (vectors and scalars) was explicitly given in both environments, and the students knew 'what to do' with the givens. The students needed only to draw scalar multiples of the given vectors, using the given scalars, and then draw the sum of those multiples. On the other hand, the decomposition of a vector into a linear combination of two given vectors v_1 and v_2 is, in general, mathematically more difficult. This task cannot be done without thinking about the linear combination as an object described by its definition (rather than thinking about it as a sequence of things to do). One needs to, in fact, create the parallelogram whose diagonal is the vector to be decomposed, and express the sides of the parallelogram in terms of the given vectors, i.e. find the corresponding scalars. However, when working on the decomposition with the computer, one can easily bypass this difficulty and solve the problem via composition, by first constructing a linear combination w with scalars that are on the screen, and then adjusting the scalars to make w overlap with the vector to be decomposed. This approach had even been encouraged by the teachers in order to avoid the technical problems related to the accurate geometric construction of the parallelogram with Cabri. For this reason, only the decomposition on paper, where there was no other way of solving the problem, presented real difficulties for the students and thus allowed to reveal the students' understanding of linear combination.

As the students were working on generating vectors from a set of given ones, they developed other objects that were for them referred to by the term linear combination. For example, when asked which vectors can be obtained from two non-collinear vectors by way of linear combinations, the students first drew two vectors v_1 and v_2 stemming from the same point, perpendicular to each other, one horizontal, the other vertical, and they started generating the new vectors as follows: The vector v_1 being given, one can obtain all vectors on the line through v_1 using scalar multiplication. One can obtain the line through v_2 in the same way. One can also obtain the vector v_1+v_2 and the whole line through this vector. By symmetry of the figure that the students were drawing simultaneously, one can obtain the

vector v_1-v_2 and the whole line through it. One can then obtain the vector v_1+(v_1+v_2) and the whole line through this vector. After that, the students concluded that all directions can be obtained, a vector in a new direction being obtained as a sum of any two given or previously obtained vectors, and any other vector in this direction being obtained by scalar multiplication. This object developed by the students and standing, for them, for a linear combination can be expressed by the formula '$k(mv_1 \pm nv_2)$, *where m, n are integers and k is a real number (scalar)*'.In this formula, mv_1 and nv_2 represent the additions of m times the vector v_1 and n times the vector v_2 respectively, and only $k(mv_1 \pm nv_2)$ represents the multiplication of the sum by the scalar k.

5.1.3 Possible reasons for the discrepancies between the expected and observed behavior

Reasons related to the graphical representation of linear combination. The processing of the graphical representation of a linear combination w of two given vectors on the screen (by changing the values of the scalars) made the students focus on the parallelogram whose diagonal was w, and the diagonal was for them associated with vector addition. Moreover, the way of decomposing a given vector w into a linear combination of vectors v_1 and v_2 in Cabri by first constructing any linear combination and then adjusting the scalars in an appropriate way, seems to have conveyed the idea that scalar multiplication occurring in the definition of linear combination is just a tool to control the length of the component vectors v_1 and v_2 to obtain the desired diagonal (sum).

Reasons related to some students' background in physics. As mentioned, one pair of our students were already familiar with vectors and operations on vectors, in particular vector addition, which they encountered in their physics classes. Indeed, they were often referring to vectors, even in an abstract context, as representing forces (e.g., one of these students described the sum of two vectors as a vector having '*the same force with the same effect with these two vectors*'). In physics, vectors usually represent forces applied to objects. Typical problems involving forces concern combining forces acting on an object to find the resultant force (which is in fact vector addition), or the converse, decomposing a given force into forces acting in given directions (decomposing a given vector into a sum of vectors). The fact that the forces are multiples of a unit force remains implicit in the sense that the unit force is never drawn in the diagram representing a composition or a decomposition of forces. In other words, the notion of linear combination is never explicitly encountered in physics. This fact might have been at the origin of the students focusing on the vector sum exclusively.

Reasons related to the student's 'elastic' view of vectors. From the students' behavior when faced with various problems involving the operations on vectors, it seems that while, for them, vector addition produces a new vector (in a new direction between the two components), this is not the case for scalar multiplication,

which just makes a given vector longer or shorter (elastic view). Such a conception of scalar multiplication goes hand in hand with the concept of linear combination as '*a set of all vectors* v_1+v_2, *with* v_1 *and* v_2 *having fixed direction, but variable length*'. In addition, it allows to explain the emergence of the concept of linear combination as a vector given by '$k(mv_1 \pm nv_2)$': in fact, the students were first generating new directions using the operation of vector addition, and then were making the new vector longer or shorter by scalar multiplication, thus obtaining the whole line.

5.2 The third experiment: the search for quasi-fundamental situations

Several of our candidates for QFS were designed so as to fully take into account the indirect access to scientific knowledge. We give two examples: A situation based on a game of communication ('jeu de message') and a situation in which the students are asked to find some information about an object in two cases: (a) the information can be almost directly read off a graphical representation; (b) the information can only be derived using analytical means (a definition, an equation, etc.).

Example 1. *A game of communication*

One of this type of situations was aimed at bringing to the students' awareness the need for a reference system in communicating the position of a point in a plane. This situation was meant to address the students' interpretation of vector as 'an elastic arrow' with no definite length nor direction, which we observed in the first and second experiment.

The students worked in pairs. Each member of the pair had a circle cut out of paper with its center marked and (a) nothing else marked, (b) one other point marked, (c) two more points marked. One of the students had to put a point in the circle and communicate its position to the other student in writing, without using drawings. The other student had to follow the instructions to reproduce the position of the point in his or her circle. In case of ambiguity, he or she would ask the first student additional questions. They had then to compare their circles. The rationale behind this activity was the following: A coordinate system is necessary only when the position of an object cannot be given by ostension. This is why, historically, the first coordinate systems were constructed in the areas of cadastral surveys, navigation and astronomy (Boyer, 1994). This situation did, indeed, bring an awareness, in all the students in the class, of the need for a reference system. Pairs did, however, differ in the number of attempts needed to obtain an unambiguous message, and in the degree of mathematical sophistication of the reference system they came up with. An interesting observation is that students used polar coordinate systems rather than Cartesian ones.

Example 2: *From reading an information off the screen to calculating it*

The concept targeted by this situation was that of the coordinates of a vector in a basis. The students were asked to put, on the Cabri screen, two non-collinear

vectors v_1 and v_2 stemming from the origin, and draw NEW AXES on them. They were asked to construct the vectors $w_1=.5\ v_1+.3\ v_2$ and $w_2=.8\ v_1-2.3\ v_2$ and put NEW AXES on these vectors, as well. Next, the students were asked to put any vector u from the origin on the screen and find its coordinates in both bases $\{v_1, v_2\}$ and $\{w_1, w_2\}$. This task did not require any use of algebra. The coordinates could be read directly off the screen with the Cabri command EQUATION and COORDINATES. It was not even necessary to use the notion of coordinates of a vector in a basis. The notion of coordinates of a point in a system of axes was sufficient: In the latter the notion of linear combination is not necessary; the notion of unit of length is implicit and the coordinates are found by dropping parallels (and not by linear decomposition). However, an awareness of the process of dropping parallels is needed because the systems of axes created by the students were not orthonormal and were not in a horizontal/vertical position.

The next question required a very different approach: the students were asked to find the coordinates, in the basis $\{w_1, w_2\}$, of a vector u whose coordinates in the basis $\{v_1, v_2\}$ are (100, 85). A vector with coordinates of this magnitude could not be constructed could not be constructed on the Cabri screen and, consequently, its coordinates in the other basis could not simply be read off the screen. It was necessary to think in terms of linear combinations of vectors, and to use analysis: the vector u had to be written in two ways, as a linear combination of the vectors v_1 and v_2 with coefficients 100 and 80, and as a linear combination of the vectors w_1 and w_2 with unknown coefficients, x and y. Then these two representations would be equated. Using the relation between the vectors v_1 and v_2 and w_1 and w_2, and the independence of the vectors v_1 and v_2, a system of linear equations in two unknowns x and y would be obtained.

$$u = 100v_1 + 85v_2\ ,\ u = x\ w_1 + y\ w_2,\ w_1 = .5\ v_1 + .3\ v_2\ ,\ w_2 = .8\ v_1 - 2.3\ v_2 \qquad (*)$$

$$(.5x + .8y - 100)\ v_1 + (.3x - 2.3y - 85)\ v_2 = 0 \qquad (**)$$

$$\begin{cases} .5x + .8y = 100 \\ .3x - 2.3y = 85 \end{cases} \qquad (***)$$

whence $x \approx 214.39$ and $y \approx -8.99$

The leap from the first to the second task was conceptually quite big, and all students experienced some difficulties with the latter. Some students solved the problem with the help of a hint from the teacher. The main topic of discussion with these students was why would one have to refer to the linear independence of the vectors v_1 and v_2 in solving the problem. For the students, the uniqueness of the representation in a basis was something obvious. They would naturally conclude from (**) that the coefficients should be zero, on the grounds that v_1 and v_2 are non-zero. But a few other students were directing their conversations with the teacher so that she tells them 'how to start', 'what to do'. One of these students found the correct coordinates for u, writing equations (*), (**), and (***) and

solving the last one, without any justifications between them. He then substituted these coordinates into the equation (**) obtaining '$v_1(0) + v_2(0)=0$'. His concluding statement was: 'v_1 and v_2 are linearly independent'. The reason why the student wrote this 'conclusion' could be that he overheard the teacher discuss the importance of the condition of linear independence of v_1 and v_2 in solving the problem with other students. He thus knew that this fact must be included in the solution; it was not important for him that putting it where he did, did not make sense from the point of view of the question in the problem. 'Making sense' was not his main concern; 'surviving the course' was.

In spite of the fact that three out of the nine students in the class were perceiving the situation as based on a social rather than on a cognitive contract when working on this task, we are willing to claim that the situation is a candidate for a QFS; not a QFS for the awareness of the importance of the condition of linear independence for the idea of basis, however, but a QFS for understanding the notion of coordinates of a vector in a basis. Some of the students' diary entries for this class give support to our conviction.

There were also other types of situations in our design with a strong potential to evolve into QFSs. One of them was based on the principle of sustained intellectual engagement in not just a simple task, but a series of related problems, whose solution requires the overcoming of some deeply entrenched obstacle. The series of problems which required students to extend a transformation of a basis to a linear transformation of the whole plane, belonged to this kind. Students had serious difficulties in solving these problems because they tended not to think about linear transformations analytically, in terms of the axiomatic definition, but synthetically, as a term referring to a kind of proportional change, or a name for transformations such as reflections, rotations, dilations etc. A description of the evolution of students' thinking in confrontation with one such problem can be found in Sierpinska et al. (1999). In the third experiment, not all students engaged in the intellectual challenge that these problems presented. But, those who did, achieved an understanding rarely encountered in the ordinary linear algebra courses.

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The third experiment confirmed the inevitable character of some interpretations of the Cabri representations in students, but it has also pointed to the limitations of observations about the learning process that can be done in tightly controlled teaching experiments, limited in time and based on a small number of tasks. In particular, the time spent in the lab with the teacher could account for only a small part of the process of understanding the defining condition of linear transformations in the whole class experiment. Students were given a large variety of problems to work on between the sessions. The students reported being engrossed in some problems for the whole week between the sessions, working alone and with their peers, and even dreaming the solutions. Three weeks after the problem of extending a transformation of a basis to a linear transformation of the

whole plane was given in class, one student suddenly exclaimed: "It's been three weeks now that we have been working on it, and it's only today that it clicked into place. Now I don't even understand why I didn't see it before. Now it's so clear. And so beautiful". But the whole class experiment has also brought forth the fragility of the cognitive contract and the difficulty of maintaining it in an ordinary course, where the question of 'survival' in an academic program (as one of the students put it) may easily become more important than that of understanding.

We find the chosen direction of research promising because it leads to results which are relevant at the level of theory and methodology of research while making some products of the research (adapted descriptions of the didactic situations, with caveats concerning the possible unintended interpretations of the representations) accessible to those instructors of linear algebra who are not engaged in research themselves.

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Finding the Student's Voice vs. Meeting the Instructor's Expectations

Uri Leron

Technion – Israel Institute of Technology

Reaction Paper to:

"Teaching and Learning Linear Algebra with Cabri"

by Anna Sierpinska, Jana Trgalová, Joel Hillel and Tommy Dreyfus

The topic under discussion can be decomposed into four levels, which I list from the bottom up:

A. Linear Algebra (LA);

B. The teaching and learning of (A);

C. The use of technology (specifically, dynamic geometry software; more specifically, Cabri II) in (B);

D. Research on (C).

In my reaction I will say something on each level. The remarks on the first two levels build up towards the last two levels, where the teaching and research issues raised by authors are explicitly discussed. It is convenient to arrange my observations around pairs of dual views that exist at each level.

Note: I use the abbreviation STHD to refer to both the paper and the authors.

A. Linear Algebra: Global-Geometric View vs. Local-Arithmetic View.

LA consists of a tight and powerful synthesis between two ubiquitous parallel threads: The Global/Geometric View (GGV) and the Local/Arithmetic View (LAV). GGV is a generalization and an abstraction from geometry of the 2-dimensional plane and the 3-dimensional space, especially in its vector representation. LAV is a generalization and an abstraction from the theory of linear equations in 2 and 3 unknowns. The LA synthesis is a generalization and an abstraction from analytic geometry, that is, the synthesis of geometry and algebra (of linear equations) developed by René Descartes in the 17^{th} Century. The objects of GGV are usually given abstractly by postulating their defining properties; they include scalars, vectors, vector spaces, bases, dimension, linear transformations and the like. The objects of LAV are given concretely by specifying their form; they

include real numbers, linear equations, n-tuples and matrices over the real numbers (the restriction to real numbers is for convenience only). The synthesis is achieved by choosing a basis (analogous to coordinate system), and representing vectors by n-tuples and linear transformation by matrices relative to that basis. The power of the synthesis stems from the fact that this representation is an isomorphism between the two systems, preserving the corresponding operations (n-tuple or vector addition, multiplication by a number or scalar, and multiplication of matrices or linear transformations); thus all 'abstract' statements that are true in one system are automatically true in the other. [1]

B. Teaching and Learning Linear Algebra: Conceptual Understanding vs. Computational Facility.

Understanding linear algebra involves conceptual understanding (especially manifest in GGV), computational facility (especially manifest in LAV) and facility with the synthesis – the ability to move smoothly between the dual views, utilizing their complementary strengths and weaknesses: Some problems or theorems are much easier to solve or prove in one of the views than in the other, and some require clever combination of methods from both. Unfortunately, it is not easy to teach both views at the same time. Moreover, computational facility has traditionally been much easier to teach -- and test! -- than conceptual understanding; consequently, traditional lecture-based LA teaching has been much more successful with LAV than with GGV, even in classes where the instructor did aim at conceptual understanding. One reason for this unfortunate state of affairs is that students find computations with numbers and their derivatives (such as n-tuples and matrices) much easier than reasoning with abstract entities (such as vectors and linear transformations). Since in the early stages of learning LA students must work with some more-or-less concrete representations for the abstract entities, the traditional lecture-room instructor didn't really have much of an alternative.

C. Technology in Teaching and Learning of Linear Algebra: Finding the Student's Voice vs. Meeting the Instructor's Expectations

Modern technology has changed all this, since it has introduced new intermediate (computational) objects, new representations and new *interactive* ways to manipulate these objects. STHD give a beautiful example of these new possibilities, in using Dynamic Geometry software to enable students to manipulate 'pure' geometrical vectors, thus bypassing their 'arithmetical' representation via

[1] The last two sentences require some modification in case several vector spaces are involved.

bases and coordinates.[2] Thus technology in principle enables STHD, in their role as teachers, to promote conceptual understanding, relative to what can be achieved in traditional LA teaching. The benefit of such a change is obvious – we all want our students to have a better and deeper understanding of mathematical concepts. But, as I explain below, this advantage does come with a price, and the instructor who aims for such understanding must be willing to pay the price.

First, learning for conceptual understanding is more time consuming than the usual more 'instrumental' learning, since students must go through a lot of *interaction*, socially and with the computational-mathematical environment, to allow for the necessary mental constructions to take place. Secondly, and for the same 'constructivist' reasons, conceptual understanding must come gradually by *successive refinement* [Leron & Hazzan, 1998] and, in intermediate stages along the way, students will inevitably form 'misconceptions' and 'mispractices' when judged by standard mathematical criteria. In particular, their computational facility and their performance on standard tasks and tests may be expected to temporarily lag behind. This should be no cause for alarm; on the contrary, students' urge to 'debug' their computational products can be a powerful engine for ascending to the next stage in the sequence of successive refinements. This urge can come about spontaneously or by appropriately designed activities, as seems to be the direction STHD have aimed at in the third phase of their research. Indeed, the authors' *Quasi-Fundamental Situations* may be just that – a situation which affords students feedback on their computational constructions in such a way that they can find and eliminate bugs in their software products and in their thinking (signaled by a feeling of surprise which results from a clash between their expectations and the feedback from the computational environment). The need to pay this price forms in my experience a major obstacle in practical implementations of such computational environments and the attendants teaching methods, since most undergraduate math faculty, who do not usually subscribe to 'constructivist' beliefs on learning mathematics, will not be willing to pay it.

With this background, it seems to me that STHD in their second experiment, having clearly aimed at conceptual understanding of their LA students, and having designed a beautiful computational environment for achieving it, were not fully ready to pay the necessary price. My (admittedly tentative) reconstruction of the educational situation, based on the description in section 5.1 of the STHD paper, consists of the following observations.

[2] Other examples of using of technology in teaching undergraduate algebra, utilize student activities in a Computer Algebra System such as Maple [Dreyfus and Hillel, 98; Hazzan *et al*, 98], or a programming language such as ISETL [Meunch, 93; Leron & Dubinsky, 95].

- By emphasizing GGV via the Cabri environment, STHD have in effect chosen to promote more conceptual understanding of their students (relative to computational facility) than conventional LA instructors do.
- The total duration of the teaching episodes in the second experiment was too short to expect a solid conceptual understanding to emerge, except for very bright students.
- In the intermediate stages, the students should have been *expected* to temporarily be less facile with computational procedures and to form 'misconceptions' concerning the mathematical objects and operations involved.
- STHD's research agenda has got the better of their teaching agenda, leading them to an over-pessimistic view of their students' achievements and preventing them from exploiting the full potential of their innovative teaching design. This claim is based on their research results and is explained more fully in the next section.

D. Research on Technology in Teaching and Learning Linear Algebra: Clean Research vs. Messy Reality

The educational researcher, in trying to describe the messy reality of teaching, learning and the classroom, is pulled in opposite directions by two ideals: producing clean, solid, 'scientific' research on the one hand, and giving a *faithful* (or *viable*) description of the educational situation under study on the other. Since clean research cannot faithfully account for messy reality, there really *is* a conflict between the two ideals, and striking an appropriate compromise is basically a *value* choice, not a scientific one. Since this is a matter of values, I cannot really argue with the authors, only state my own preference, which leans a bit more in the 'messy' direction.

I believe that the authors by choosing an epistemological, not psychological or social definition of mathematical objects, and by choosing to compare the intended objects with those constructed by the students, rather than to look at the *process* of construction, have obtained a relatively clean and solid research, but one that gives (to my taste) a not quite satisfactory view of the educational situation. Instead, I'd like to offer an alternative interpretation of their data.

The different analysis stems from three sources. First, I take a psychological and social view of mathematical objects (similar though not identical to Hersh's [97] *humanist* philosophy of mathematics), rather than the authors' 'purely' epistemological one. Secondly, I focus on the *process* involved in students' constructions, rather than comparing endpoints (the researchers' intended objects vs. the student-constructed ones at a fixed point in time). Thirdly, and most importantly, I bring in an interpretative approach to students' productions as

described in [Confrey, 94]: "It recognized that students' views are not simply inadequate or incomplete adult views, and it allows for the reconceptualization of the mathematical content of the expert in light of student invention." Indeed, this approach enables me to see in the students' constructions a source for *new mathematical insight*, rather than a cause for disappointment. I will now go into some detail in order to explain and substantiate my claim.

STHD present a very clear and insightful analysis of the kind of understanding (enabled by the Cabri environment) that the students *did* form. They describe the students' 'elastic' view of scalar multiplication, which has led to the "concept of linear combination as '*a set of all vectors* v_1+v_2, *with* v_1 *and* v_2 *having fixed direction, but variable length*'"(italics in the original). I claim that the students' intuitions in fact were very much in line with a mature mathematician's intuition: the elastic view and the resulting interpretation of linear combinations is pretty much the intuition behind the powerful notion of *decomposing a vector space as a direct sum of two subspaces*. In this view each vector in the 2-dimensional space is represented (uniquely) as a sum of two vectors, each taken from one of the 1-dimensional subspaces. It is quite natural then, and quite powerful too, to view the first vector as a variable vector, with variable components ranging over the respective subspaces. This is the GGV version of the LAV version of representing a vector as a linear combination of two vectors. Since the teaching design used Cabri to promote GGV, it is quite natural -- and welcome! -- that this view indeed emerged in student's intuitions. A mature mathematician can of course easily see the equivalence of the two representations and shift from one to the other as needed. Given the limited time of the experiment, the students simply didn't get there *yet*. But with proper planning, enough time, and a different set of expectations, I see the STHD approach and the result of their experiment as very promising in ways which are very hard to achieve with traditional tools and methods.

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RESEARCH FORUM

Theme 2:
Becoming a Mathematics Teacher-Educator

Coordinator:	Tom Cooney
Presentation 1:	*Interweaving the Training of Mathematics Teacher-Educators and the Professional Development of Mathematics Teachers* Orit Zaslavsky & Roza Leikin
Reactor:	Konrad Krainer
Presentation 2:	*Becoming a Mathematics Teacher-Educator: Conceptualizing the Terrain through Self-Reflective Analysis* Ron Tzur
Reactor:	Barbara Jaworski

Interweaving the Training of Mathematics Teacher-Educators and the Professional Development of Mathematics Teachers

Orit Zaslavsky and Roza Leikin

Technion – Israel Institute of Technology

In this paper we present a study conducted within the framework of an inservice professional development program for junior high and secondary high school mathematics teachers. The focus of the study is the analysis of powerful processes encountered by the staff members, which contributed to their growth as teacher-educators. We offer a conceptual framework to think about becoming a mathematics teacher-educator.

The Setting

In the past decade there have been several calls for reform in mathematics education that are based on the assumption that well prepared mathematics teacher-educators are available to furnish opportunities for teachers to develop in ways that will enable them to enhance the recommended changes. Unfortunately, there are hardly any formal programs that provide adequate training for potential mathematics teacher-educators, let alone research on becoming a mathematics teacher-educator.

The work described in this paper was conducted within the framework of a five-year reform-oriented inservice professional development project ("Tomorrow 98" in the Upper Galilee) for junior and senior high school mathematics teachers. The goals and design of the project were very much along what Cooney (1994), Cooney and Krainer (1996), Comiti and Ball (1996), and Borasi, (1999) suggest as essential components for teacher education programs. They were inspired by *(a)* Constructivist perspectives of learning (von Glaserfeld, 1991; Kilpatrick, 1987; Davis, Maher, & Noddings, 1990). *(b)* Jaworski's (1992, 1994) teaching triad. *(c)* theories of reflective practice, which are addressed by Jaowrski (1994, 1998), Calderhead (1989), Gilbert (1994), Krainer (1998), Simon, (1994) and Borasi (1999); and *(d)* theories on social aspects of teaching and learning (Adler, 1998; Lave and Wegner, 1991; Rogoff, 1990; Leikin and Zaslavsky, 1997; Winbourne & Watson, 1998).

Thus, the project goals included the following:

- Facilitating teachers' knowledge (both mathematical and pedagogical) in ways that support a constructivist perspective to teaching;
- Offering teachers opportunities to experience alternative ways of learning (challenging) mathematics;
- Preparing teachers for innovative and reform oriented approaches to management of learning mathematics (particularly, the kinds with which they have had very limited experience);
- Fostering teachers' sensitivity to students and their ability to assess students' mathematical understanding;

- Promoting teachers' ability to reflect on their learning and teaching experiences as well as on their personal and social development;
- Enhancing teachers' and teacher-educators' socialization and developing a supportive professional community to which they belong.

The teachers who participated in the full program took part for four consecutive years in weekly professional development meetings, six hours per week, throughout each school year. In the first year there were two groups of about 20 teachers each, one group of junior high school teachers and the other group of senior high school teachers. In the second year most of the teachers continued and another two new groups began. In the third year again another two groups began. By the fourth year some teachers switched from full participation to monthly meetings. Altogether, about 120 teachers participated to some extent in the program. The meetings consisted of a wide range of activities led by the project team. Some of the teachers gradually became more involved in the program and towards their third year assumed responsibility for many of these activities. As the program progressed, the location of the activities shifted from a central regional location into the schools in the region.

The project team consisted mainly of experienced and highly reputable secondary mathematics teachers. The team members varied with respect to their expertise and experience. None of them had any formal training (such as the Manor Program reported by Even (1999)). Some did not have any previous experience in mentoring or teaching other teachers. Consequently, the project was designed to enhance the development of the project team hand in hand with the development of the inservice teachers who participated in the program. The design of both the staff enhancement component of the project and the research that focused on the staff members' professional growth stemmed from the project's goals, and was based on two main assumptions: *(a)* Similar to the ways in which teachers learn through their own (teaching) practice (Mason, 1998; Steinbring, 1998; Brown & Borko, 1992; Leikin, Berman, & Zaslavsky, forthcoming), teacher-educators learn through their practice. *(b)* There are learning aspects that are fundamentally inherent to the structure and nature of the community of practice which the project team constitutes (Adler, 1998; Lave, 1996; Roth, 1998; Winbourne & Watson, 1998).

This paper focuses on the professional development of the staff members as teacher-educators within the context of the project. We analyze some powerful processes in which the project team engaged as they became more proficient, and the conditions that contributed to their training and professional growth as a community of teacher-educators.

Conceptual Framework

The design of the program of the project as well as the research that accompanied it, were driven by constructivist views of learning and teaching. According to this perspective, learning is regarded as an ongoing process of an individual or a group trying to make sense and construct meaning based on their experiences and

interactions with the environment in which they are engaged (von Glaserfeld, 1991; Kilpatrick, 1987; Davis, Maher, & Noddings, 1990). The constructivist view applies to any learner regardless of age and context. It follows that teachers and teacher-educators can be seen as learners who continuously make sense of their histories, their practices, and other experiences.

In order to prepare teachers to teach in reform-oriented ways (such as suggested by NCTM, 1989, 1991, 1995), there is much they need to "unlearn" (Ball, 1997). The reform of mathematics teaching requires that teachers play an active role in their own professional development (Grant, Hiebert, & Wearne, 1998). Grant et al. raise the issue that until now teachers were only told about the new ways in which they are expected to teach, but did not experience these ways of learning and teaching. Thus, "teachers must make sense of proposed changes in the context of their own prior knowledge and beliefs about teaching, learning, and the nature of the content being taught" (p. 218, ibid.). In order to help teachers see new possibilities for their own practice they must be offered opportunities to *(a)* learn challenging mathematics in ways that they are expected to teach; *(b)* engage in alternative models of teaching (Cooney & Krainer, 1996; Ball, 1997; Grant et al., 1998; Malone & Taylor, 1992; Brown & Borko, 1992). Thus, the main task of the project staff members was to offer such opportunities for the participating teachers.

Providing the above kinds of experiences for teachers is not sufficient. It is also necessary to emphasize "a reflective component of inservice programs in which teachers explicitly consider the implications of their own learning experiences for their teaching and for creating contexts in which pedagogy and content are intertwined in a reform minded way" (Cooney and Krainer, 1996, p. 1162). Taking the stand that reflection is a key issue in teachers' and teacher-educators' learning, many elements of the program were incorporated in order to enhance reflection and self-analysis of both the participating teachers and the project staff. Similar to aspects of reflection such as those that Simon (1994), Krainer (1998), Borasi (1999), Tzur (1999) discuss, and depending on the situation, reflection in our study involved various aspects, e.g., the mathematics in which the teachers or teacher-educators were engaged, pedagogical considerations, implications for students' learning, or accounts concerning one owns practice and growth.

Jaworski (1992, 1994) offers a teaching triad, which is consistent with constructivist perspectives of learning and teaching. The triad synthesizes three elements, which are involved in the creation of opportunities for students to learn mathematics: The management of learning, sensitivity to students, and the mathematical challenge. Although quite distinct, these elements are often inseparable. Jaworski claims that "this triad forms a powerful tool for making sense of the practice of teaching mathematics" (1992, p. 8). We borrow the idea of this teaching triad for describing in a general way the tasks of the different groups of mathematics educators who were involved in the study, as well as for analyzing and discussing our findings.

In the current study, there were three different groups of mathematics educators: The teachers who participated in the program, the project staff members who were the teacher educators, and the project-director/leading-researcher (the first author of this paper), who can be seen as a teacher-educators' educator. We describe some similarities and differences between mathematics teachers (MT), mathematics teacher-educators (MTE), and mathematics teacher-educators' educators (MTEE), with respect to two aspects of their work: ***Task*** and the ***Context*** within which they carry out their Task (see Table 1). By Task we refer to the teaching triad relevant to each group.

Group of Mathematics Educators	**Task**			**Context**
	Manage Learning of:	**Be Sensitive to:**	**Provide Challenging Content in:**	
Mathematics Teachers (MT)	Students		Mathematics	Classroom Settings ...
Mathematics Teacher-Educators (MTE)	Teachers (MT)		Mathematical Challenge Management of Students' Learning ⟷ Sensitivity to Students	Classroom Settings **Workshops for MT** Individual Mentoring Staff Meetings of MT University Courses ...
Mathematics Teacher-Educators' Educator (MTEE)	Teacher-Educators (MTE)		Mathematical Challenge Management of Teachers' Learning ⟷ Sensitivity to Teachers	**Workshops for MT** **Workshops for MTE** **Staff Meetings of MTE** **Individual Mentoring** University Courses ...

Table 1: Tasks and Contexts of Different Groups of Mathematics Educators

The main task of all three groups has to do with managing learning, being sensitive to the learner and providing challenging content. The intended learners are different for each group. For a MT's task the learners are students, for a MTE's task the learners are teachers (or prospective teachers), and for a MTEE's task the learners are teacher-educators (or prospective teacher-educators).

The target learning content for which challenges should be provided also varies from one group to another. The task of MTs requires challenging mathematics. The task of MTEs requires challenges regarding the three components in Jaworski's teaching triad (see second row in Table 1). However, with respect to challenging mathematics, they need to address the needs and interests of both MTs and students. The task of MTEEs

requires challenges regarding the contents that MTs and MTEs should challenge (first and second row in Table 1), and in addition, they need to provide challenges concerning a similar triad. This triad consists of challenging mathematics for MTEs, sensitivity to MTs, and management of MTs' learning (third row in Table 1). This conveys the hierarchical nature of the content that plays a role for each group. It should be noted that the content relevant for a particular group is accumulative and includes the contents of the groups in the previous rows.

All tasks are influenced by the context in which the learning is managed. Table 1 presents the most commonly recognized contexts for each group. Those that were central to the current study are in bold.

Similarly to the ways in which teachers gain expertise in teaching as a result of their own teaching practice, an important element in becoming a mathematics teacher-educator entails experiences in facilitating the learning of mathematics teachers. Table 1 conveys this learning-through-teaching process that teachers often encounter (Mason, 1998; Steinbring, 1998; Brown & Borko, 1992; Leikin et al., forthcoming). A teacher (MT), by teaching (students) mathematics and reflecting on it, may develop his or her awareness and knowledge with respect to the teaching triad. Similarly, a teacher-educator (MTE), by gaining experience in providing opportunities for teachers to develop their knowledge in the three domains comprising the teaching triad (in the second row), is likely to become more knowledgeable of the three domains with respect to teacher learning (in the third row). Thus, learning-through-teaching is reflected in Table 1 by moving from one row to the content in the subsequent row.

Steinbring (1998) suggests a model of teaching and learning mathematics as autonomous systems. According to this model, the teacher offers a learning environment for his or her students in which the students operate and construct knowledge of school mathematics in a rather autonomous way, by subjective interpretations of the tasks in which they engage and reflection on their work. The teacher, by observing the students' work and reflecting on their learning processes constructs an understanding, which enables him or her to vary the learning environment in ways that are more appropriate for the students. Although each system is autonomous, the students' learning processes and the interactive teaching process, the two systems are interdependent. This interdependence can explain how teachers learn from their teaching.

We adapt our interpretation of Steinbring's model to help us think about and offer explanations to some ways in which mathematics teacher-educators may learn through the learning offers which they provide for teachers. It should be noted that mathematics teacher-educators interact with mathematics teachers in many different ways and settings. Thus, this model does not equally apply to all settings. It applies mostly to those that are structured as workshops for teachers and resemble to a certain extent the learning settings in school. According to the modified model, mathematics

teacher-educators offer learning experiences for mathematics teachers. As shown in Table 1, the content of these learning experiences includes many aspects of teachers' professional knowledge, in which mathematics is one of the components. The teachers work on the learning tasks, make sense of them and construct meaning in subjective ways. The problems that they solve may be mathematical, pedagogical or both. By reflecting on and communicating their ideas, teachers develop their professional knowledge. The teacher-educators, by observing the teachers and reflecting on the teachers' learning processes become aware of subtle aspects of their own knowledge, assumptions and practices and of the teachers' interpretations and needs. Consequently, they modify the learning offers for the teachers.

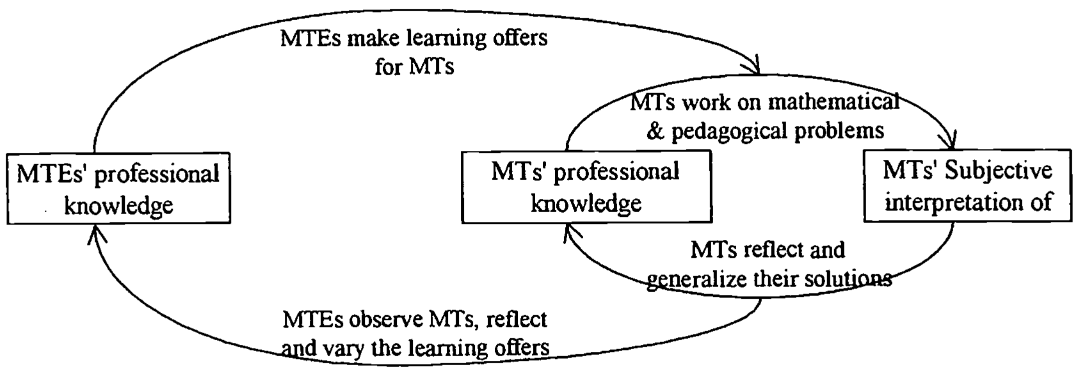

Figure 1: A Modification of Steinbring's Teaching and Learning Model (1998, p. 159)

As mentioned above, the model presented in Figure 1 may account for some ways and aspects of teacher-educators' growth. There are other aspects of the development of teacher-educators that can be better explained in terms of their participation in a community of practice (Lave & Wenger, 1991; Lave, 1996; Adler, 1998; Roth, 1998), where members learn from each other's expertise in an apprenticeship manner (Rogoff, 1990): "Vygotsky's model for the mechanism through which social interaction facilitates cognitive development resembles apprenticeship in which a novice works closely with an expert in joint problem solving in the zone of proximal development. ... Development builds on the internalization by the novice of the shared cognitive process appropriating what was carried out in collaboration to extend existing knowledge and skills." (ibid., p. 141). As mentioned earlier, the community of teacher-educators in the project was heterogeneous with respect to their prior experience and expertise. Thus, similarly to the mechanism, through which cognitive development is facilitated in social contexts, there are aspects of professional development of (teachers and) teacher-educators which resemble apprenticeship. Consistent with this perspective, the design of the project incorporated, for example, many opportunities for staff members to observe, participate in, and reflect on learning activities offered to teachers by more experienced staff members including the project director.

Methodology, Data Collection and Analysis

The methodology employed in the study followed a qualitative research paradigm in which the researcher is part of the community under investigation. It borrows from Glaser and Strauss's (1967) *Grounded Theory,* according to which the researcher's perspective crystallizes as the evidence, documents, and pieces of information accumulate in a dynamic process from which a theory emerges. The methods, data collection and analysis grow continuously throughout the progressing study. The researcher acts as a reflective practitioner (Schön, 1983) whose ongoing reflectiveness and interpretativeness are essential components of this type of research (Erickson, 1986).

The project was carefully documented. In order to investigate the development processes of the team members, and to be able to draw connections to the development of the participating teachers, a number of means and sources of documentation were employed. It should be noted that some of the means contributed in themselves to these processes (e.g., responding to the written questionnaires and being interviewed stimulated reflection and influenced the awareness to many aspects of their professional and private lives).

The documentation included:

- Interviews with the team members and with the participating teachers fostering their reflection on their personal professional growth and its connection to the project's goals and to the various activities in which they engaged;
- Written questionnaires for the staff members addressing the influence of their work in the project on various aspects of their lives, and the characteristics of the environment to which they attribute such influence;
- Self and peer reports on and analyses of the activities with the teachers;
- Videotapes of workshops with teachers;
- Videotapes and protocols of the project team's meetings and mini-conferences;
- A documentary film about the project;
- Resource materials developed by the project members.

Based on Bell & Gilbert's (1994) findings indicating that professional growth (i.e., developing ideas and actions) is interconnected to personal growth (i.e., attending to feelings) and social growth (i.e., developing collaborative ways of relating to colleagues) we looked for indications of these three aspects of growth. Other researchers have found similar evidence (Even, 1999; Halai, 1998). In addition to the three aspects of growth which Bell and Gilbert identify, they also point to three main stages in teachers' growth, which are consistent with more detailed stages that Jones et

al. (1994) describe. These stages were helpful in interpreting the data, particularly with respect to the differences between the group of teachers and the staff members.

Our conceptual framework guided our analysis of the professional growth of the staff members. We looked for evidence of growth associated with mathematics challenges, sensitivity to teachers, and management of teachers' learning. In addition, we looked for indicators of professionalization and professionalism in Nodding's (1992) terms.

The Project Team

As mentioned earlier, the project team consisted of people with diverse expertise, which is one of the characteristics that Roth (1998) considers essential to a community. Although there were altogether over 20 team members, only 14 were involved in the project from its early stages until its completion. The latter formed the focal group for our study. All members had experienced teaching mathematics for a number of years (they varied between 4 to 30 years of experience). Ten members continued teaching junior high or senior high school mathematics in addition to their work in the project. Of the above 14 members, at the beginning of the project there were only 6 members who had had some previous experience in managing inservice professional development activities for mathematics teachers. Three of the members had had experience in incorporating technology in innovative ways in the mathematics classroom. One member was competent in implementing cooperative learning methods in mathematics classrooms and inservice workshops for teachers. One member was particularly knowledgeable about new trends of assessment in mathematics education. Two members had had experience in curriculum development. One member was a practicing mathematician. Thus, it was expected that as a team, many elements underlying the calls for reform would eventually be addressed, although at the initial stage only 3 members were familiar with the NCTM Standards (1989, 1991) or similar calls for reforms. It should be noted that at the early stages of the project most of the project team members seamed to hold a transmission metaphor of the teaching-learning process. In addition, they found it hard to reflect on their work, let alone elicit reflection of the teachers.

In the first three years of the project, the tasks of the staff members were mostly directed toward designing and carrying out inservice workshops (see the second row in Table 1). Some were in charge of the inservice activities with the teachers, and others facilitated the activities by participating in the weekly inservice meetings or by assisting in the preparation of resources that were required for the inservice meetings.

It should be noted that at the beginning of the project there were many staff members who were not very confident of their qualifications as teacher-educators, and expressed a need for guidance by the project director or other relatively experienced members. From the very beginning it was clear that it would be a long term project. Thus, the staff members expected to gain expertise as teacher-educators within the framework of the project, in order to become more capable in their work. It was only

towards the end of the project that several staff members considered assuming the role of mathematics teacher-educators within other frameworks.

As mentioned earlier, the project director assumed the role of teacher-educators' educator (MTEE) in various ways (in later stages, an expert mathematics teacher-educator shared this role with the project director).

Components Designed to Contribute to the Development of the Team Members

As mentioned earlier, the main task of the staff members was to design and carry out weekly workshops and related activities with the inservice teachers. Contrary to the task of a mathematics teacher, who teaches according to a rather specific mathematics curriculum, there was no readily available and agreed upon curriculum for inservice programs. From the start, the stated goals of the project called for considering the teaching triad (Jaworski, 1992, 1994). The decisions regarding the mathematics, the teaching strategies, and the aspects of students' learning that should be addressed required analysis and self-reflection of the staff members with respect to their teaching experience in school. Translating these (content-oriented) decisions into powerful activities for teachers entailed considerations with respect to: *(a)* teachers' backgrounds and interests; (2) appropriate ways to manage teachers' learning and foster their development. All workshops were accompanied by resource material, which was developed or adapted by the staff members.

Although there was an overall plan for each year, the staff members had to attend to the constant need for (unexpected) adjustments and changes in plan and to be flexible in carrying them out. This approach was often seen as an obstacle for the staff members, however, it conveyed the need to listen - in the broad sense - to the teachers and involve them in the plan.

The teachers were grouped according to the grade levels they taught (junior/senior high grades) and the year in which they enrolled in the program. Thus, in the third year of the project there were 6 groups of 15-20 teachers in each group. There was a staff member in charge of each group. Some activities were conducted separately within each group, and some were conducted in different groupings (e.g., with all junior high school teachers, or all teachers who had been participating for a given number of years). For each workshop there was at least one staff member in charge. As part of their working load, staff members were required to observe their colleagues' workshops, to take part in the workshops' activities when they felt comfortable to do so, and to provide written and oral reactions to their colleagues. In the first stages of the project the project director would suggest for each member which colleague to join. Her considerations were based on her knowledge of the different domains of expertise of the staff members. The intention was to create situations in which less experienced members would learn from more experienced ones in an apprenticeship like manner. Along the same lines, when the project director was in charge of a workshop for the teachers, she invited all the staff members who were not

simultaneously engaged in other responsibilities to take part. The hope was that through such workshops a constructivist perspective to teaching would be conveyed and some forms of reform-oriented learning environments would be modeled.

In order to foster reflection and self-analysis of the staff members they were required to give written accounts of the workshops for which they were in charge (Borasi, 1998, reports the significance of writing for enhancing reflection). In addition, mutual peer (written and oral) evaluation and teachers' (written and oral) feedback were provided on a regular basis.

There were regularly scheduled staff meetings (Even, 1998, stresses the importance of such meetings), in which staff members could reflect on their work, share their experiences, consult with their colleagues, and negotiate meaning with respect to the goals and actions of the project. Small groups of staff members, who had common tasks, initiated additional meetings according to their specific needs and interests.

Staff members were continuously encouraged to initiate ideas and suggest new directions and actions within the project. Members' personal interests and enthusiasm drove many of the activities of the project. The underlying assumption was that ownership and responsibility, which are indicators of professionalism (Nodding, 1992), would contribute to their position in their community of practice.

All staff members (and teachers) had access to an electronic network (in their native language). The network served for informative communications, as well as for sharing, reflecting, and debating.

Special professional meetings and mini-conferences for the project team (2-3 times a year) were scheduled. These meetings were designed to address the evolving needs and interests of the staff members (this exemplifies one aspect of the MTEE's triad, as presented in the third row of Table 1). In these meetings staff members had the opportunity to conduct workshops and various activities for their colleagues. In addition, experts who were not part of the project team were invited to elaborate on their expertise.

To summarize this section, it seems worthwhile to note that the above components address all four dimensions which Krainer (1998, 1999) considers as describing teachers' professional practice: Action, Reflection, Autonomy, and Networking.

Professional Growth through Cooperative Learning and Learning to Cooperate

Overall, the findings indicate that the staff members encountered powerful experiences, which led to their professional, social, and personal growth (Bell & Gilbert, 1994). As we reflected on our experience we noticed a few patterns of development that seemed to emerge. The analysis of the accumulating data kept supporting these patterns and provided numerous pieces of evidence confirming them.

In order to convey the nature of the growth of the project members along the professional development of the participating teachers, we begin with a story (as

Krainer (1999) advocates). This is a story of a theme -- *cooperative learning* -- that evolved within the context of the project in many different layers by individual and groups of participants.

At the initial stage, there was only one staff member, Tami, who was an expert in managing cooperative learning experiences for students and teachers. Tami had strong beliefs regarding the significance of fostering cooperative learning settings in mathematics, based on her prior experience as a secondary mathematics teacher and a research program which she conducted (Leikin & Zaslavsky, 1997, 1999). Thus, she felt the need to incorporate in the teachers' workshops activities that would facilitate their appreciation of the potential of this approach for their students. In addition, Tami was very eager to convince her (project staff) colleagues to adopt this approach in their workshops. In staff meetings when planning the workshops, Tami always offered to conduct "cooperative learning" workshops. These workshops were very structured and were managed in one of five cooperative learning methods, which she had developed for her students, prior to her participation in the project. Several staff members observed her workshops and often took part in them as learners. Since the main purpose Tami set forth was "to convince the teachers to use *these* methods in their classrooms" she treated the challenging mathematics in the workshops mainly as a vehicle for promoting her goal.

In the first year of the project Tami conducted several cooperative learning workshops with the teachers, who were asked to adjust appropriate mathematical challenges for their students and to apply cooperative learning methods in their teaching. The teachers made some adjustments in the mathematics, however, used the cooperative methods exactly as they had experienced in the workshops. When sharing their experiences with their colleagues in the project some claimed that it was not realistic to expect a teacher to apply this approach regularly because of many constraints in school. Towards the end of the year, Tami encouraged a small group of teachers to adapt the material adjusted by their colleagues and themselves and to produce a booklet including a collection of resource materials for cooperative learning of school mathematics. This collection was restricted to one particular method. Following the teachers in their second year, Tami found out that those who applied cooperative learning methods in their classrooms used mostly the specific activities in the booklet without any modifications or additions. One of the reasons they gave for this was that in order to fully apply a cooperative approach to learning they needed to collaborate with other teachers in their schools. In fact, in schools from which a number of teachers participated in the project, Tami realized that collaboration began and more flexible ways were employed to facilitate students' cooperative learning. Of the small group of teachers that collaborated on the production of the booklet, some became rather competent and began preparing workshops for other teachers in cooperative approaches. Eventually, two of these teachers became teacher mentors in schools.

Parallel to the teachers' increasing appreciation of cooperative learning, staff members also started seeing its potential for teachers. They gradually began incorporating in their workshops elements that facilitate cooperative learning. However, unlike the teachers, they did not stick to the structured methods that Tami had developed. On the contrary, they

objected to the given constraints of each method. Each member adapted his or her own interpretation of what constitutes conditions for cooperative learning. Yet, some staff members felt they needed more theoretical background with respect to the principles underlying and research findings supporting cooperative learning in mathematics. Thus, Tami was invited by the project director to give a presentation to the staff members on cooperative learning in mathematics. Following the rather theoretical presentation, the staff members were asked to design cooperative learning experiences for the teachers to be included in the program of the following year. As a result of this request, a group of four staff members decided to collaborate on this task. This began a whole new story.

This group of four staff members encountered a unique experience in collaborating on this task. They devoted enormous time developing their ideas in ways that neither Tami nor the project director anticipated. Instead of planning a workshop for a period of an hour and a half, they expanded it to twice the time. They tried to incorporate many of the aspects of learning, for which they developed an appreciation, such as, connecting the mathematics to real life situations, addressing subtle issues of graphing and scaling, designing tasks that are open and engaging, in a way that inherently encourages full cooperation among learners. They applied all the principles they had learned on ways to facilitate cooperative learning of challenging mathematics. As they became more and more engaged in their task, they wanted to share their ideas with the staff members before conducting the workshop with the teachers. Thus, a special meeting for the staff was scheduled, in which the group of four staff members tried out their ideas with all the rest (including the project director). During this workshop, their colleagues were truly acting as learners, which turned out somewhat painful for the four, since they were confronted with some unforeseen problematic mathematical and pedagogical issues. They then returned to revise their plan and the resource material, and in the following year conducted one of the most exciting workshops with the teachers.

This story illustrates some of the dynamics, which the project fostered, as portrayed in Figure 2. In many other cases a similar pattern was identified. Staff members with an area of expertise provided learning experiences for teachers associated with their expertise. Consequently, the teachers started to appreciate the potential of this kind of learning experience for themselves as learners, and began -- often as a given assignment -- to try out very similar approaches with their students in rather limited ways. At the same time, staff members who did not have any expertise in this area, by observing and taking part in the offers for teachers, gradually began -- by their own initiative -- to incorporate various aspects of this approach in the activities they provided for teachers. Although both groups, teachers and staff members seemed to be empowered by these interactions, we suggest that there are differences in the extent and nature of these empowerments.

These differences can be explained mainly by the differences in the time and opportunities staff members had, in comparison to the teachers, to reflect and share their experiences, which are key elements in professional development (e.g., Borasi, 1998; Krainer, 1998; Jaworski, 1994). Another explanation can be seen in the

differences in the nature of the task for which each group was responsible (see Table 1).

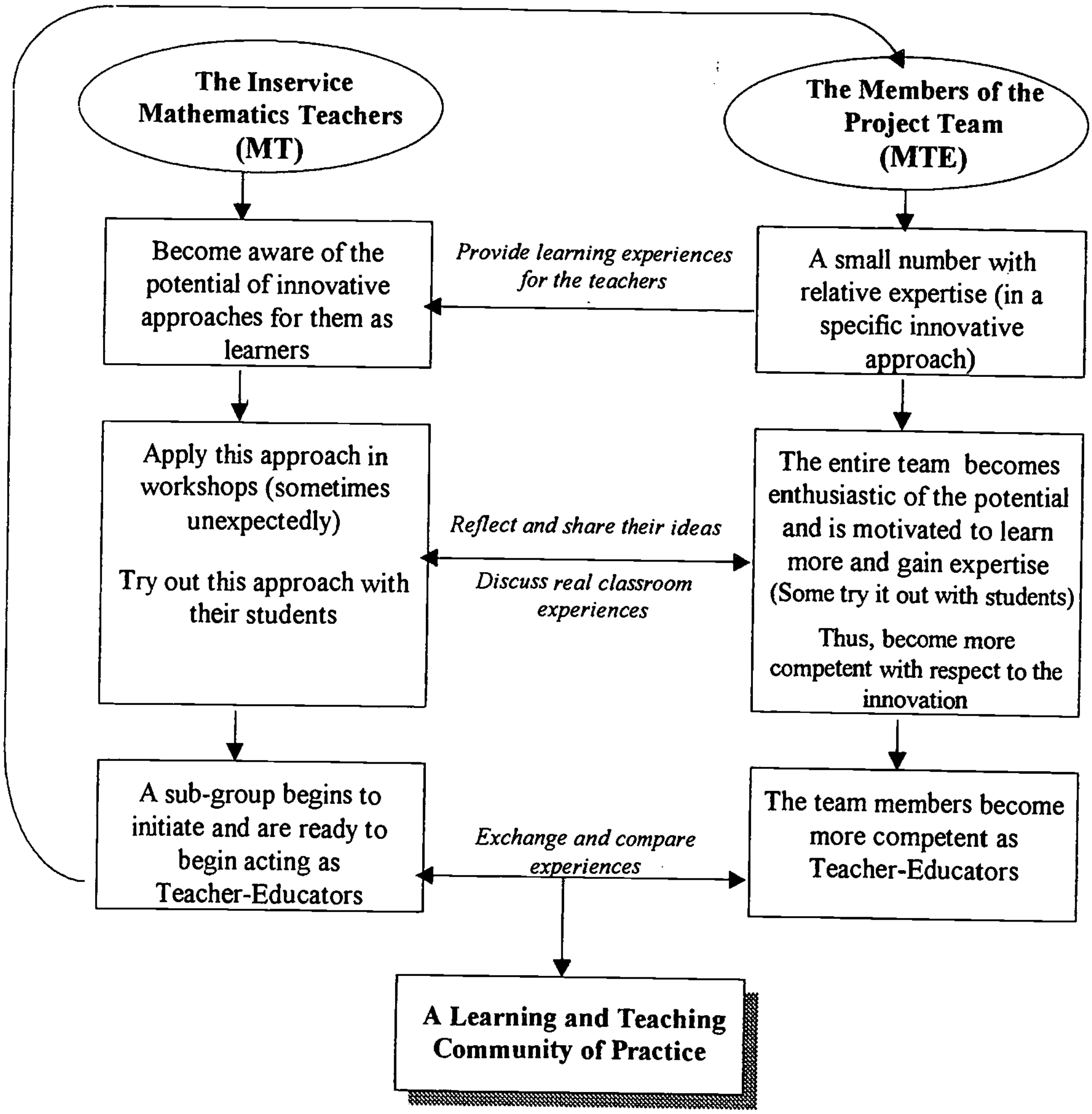

Figure 2: The Learning Dynamics within and between the Two Groups (MTs & MTEs)

The mere planning of the workshops for teachers entailed considerations of a teacher-educator triad in addition to a teacher's triad (see Table 1). There were also different kinds of constraints. The staff members had much more freedom and flexibility in terms of the "curriculum" they had to address and time allocated for the different activities. Finally, for the staff members, the project was their working place, thus their commitment to carrying out their task was inherently connected to their positions. However, teachers enrolled in the program for various reasons, and for the most, felt less committed and accountable for their role in the project. It should be noted that some of the above differences can be seen as reflecting a different phase in

the development of the teachers and teacher-educator (Bell & Gilbert, 1994; Jones et al., 1994).

To better understand the processes, which the staff members encountered, we bring excerpts from an interview with Ronit, one of the members of the "group of four". In this interview she reflects on the group's work:

"I think we all felt that after the first two hours, in which we sat and worked on the task and decided what to do, we thought that this is it, that we finished the work and there is not much more work. It was clear to us, that we have the idea and we know what to do. So there were a few doubts, maybe from the mathematics aspect it isn't so new, but it was clear to us that we have the idea and only have to sit and write and it would take another three hours and that would be it. But, like many things, it turned out that there was much more to it and needed more investment. But since each of us already invested a lot, and I, I don't know, I liked what we were doing, I found myself, not having a problem to invest the time. I found myself constantly thinking about it, even, you know, when I did other things. It kept bothering me, what can be done and how can it be improved. I enjoyed the group and the collaboration, and I think I really learned a lot".

"I thought we produced something good and that we would benefit from it. I found myself dealing with questions that interested me, because I asked the questions, I wanted to see if they could be answered and how they could be answered. So I sat and solved and thought: How can they be answered, then maybe in different ways, and I tried to see what could be done with it and I hoped it came out good. I still want it to stand a test, others judgement, because, maybe, I am too involved and not objective, like a mother to her child ... Yes, this was really a process of giving birth".

The above excerpts express the professional, personal, and social aspects of Ronit's development, which are interconnected.

We argue that the dynamics portrayed in Figure 2 influenced the growth of the more expert staff members as well. We conclude with some of Tami's reflections on her own development. Tami, who was considered by herself and her colleagues as a rather competent teacher-educator to begin with, reflected on her own development at the end of the project: "Today, I am much more open to many different ways of facilitating teachers' and students' cooperative learning in mathematics. I now accept and use methods that are not very structured, that seem to give way for many different kinds of cooperation to different extents."

In the beginning stages Tami was reluctant to conduct reflective discussions with the teachers. She had many debates with the project director on the need and ways to foster such reflections. Today she reflects: "When I first observed the project director conduct a reflective discussion with the teachers I told myself that I would never be able to do so myself. I wouldn't know what to ask, and would be insecure if the teachers would not react right away. Silence was a threat to me. Having observed over and over again in many different situations ways in which the project director conducted such sessions, with the teachers and with us -- the staff members, I began to try it myself. Slowly, I began to listen.

I no longer felt uncomfortable waiting for responses. I reached the extent that my colleagues often accuse me of waiting too long for replies".

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Promoting reflection and networking as an intervention strategy in professional development programs for mathematics teachers and mathematics teacher educators

A reaction to the paper "Interweaving the Training of Mathematics Teacher-Educators and the Professional Development of Mathematics Teachers" by Orit Zaslavsky and Roza Leikin

Konrad Krainer

University of Klagenfurt, IFF, Austria

0 Introduction

"I found myself dealing with questions that interested me, because I asked the questions ... ". This quotation stems from Ronit, a mathematics teacher educator engaged in the five-year professional development program *"Tomorrow 98"* in Israel, led by the first author of the above mentioned paper. Ronit's words give a first impression of the *freedom* enjoyed by the teacher educators in this project to define their own ways to grow professionally when working with teachers and when reflecting on their work as teacher educators, i.e. using their teaching as a basis for their own learning. In addition, the quotation indicates that the program fostered teacher educators' *investigative attitudes* in order to achieve a better understanding of their role in the interaction process with the teachers.

Following this first impression, it seems appropriate to sketch a broad picture of the value of the paper of Orit Zaslavsky and Roza Leikin before going into closer detail. One essential strength of the paper is the description and reflection on an apparently really successful project that creatively combines the professional growth of mathematics teachers and mathematics teacher educators within the framework of a *"supportive professional community"*. It demonstrates the dynamics and power that heterogeneous learning groups achieve through *building on the specific strengths* of individual learners (in this project teachers and teacher educators). The paper sometimes would gain additional value through more deeply linking the described processes in the project with the different theoretical considerations indicated in the text. However, the paper represents a variety of powerful ideas and interpretations that need to be highlighted and appreciated. It yields a good learning opportunity for our scientific community to reflect on the process of becoming a mathematics teacher educator, an issue that needs far more consideration in the future.

The paper of Zaslavsky & Leikin mainly considers the interaction between *mathematics teachers* (MTs), *mathematics teacher educators* (MTEs), and the *project director* who – among others – has the role of a *mathematics teacher educators' educator* (MTEE). In a broader sense all the MTEs can also be seen as having the role of MTEEs when providing workshops for their MTE colleagues. Figure 1 indicates the authors' explicit interest in their analysis of the paper, namely the professional growth of the MTEs.

The data collected mainly come alive to the reader through some brief insights into one teacher educator's, namely *Tami's* (individual, social, and) professional growth, and partially through reflections of her colleague *Ronit*.

MTEs as participants in teacher education: MTEEs ⟷ **MTE**

↕

MTEs as organisers of teacher education (eventually also for their colleagues): (MTEs ⟷) **MTE** ⟷ MTs

Figure 1

The methodology used in the study follows the paradigm of *"Grounded Theory"* (Glaser and Strauss, 1967) and refers to several theoretical considerations by other researchers, primarily to Jaworski's *teaching triad* (1992), Krainer's *four dimensions of professional practice* (1998), and Steinbring's *model of teaching and learning mathematics* (1998), for which links to the program are made.

The following analysis firstly aims at finding *common ground* between Jaworski's, Krainer's and Steinbring's considerations. Next, the *story of Tami*, an MTE in the program, is reflected upon in greater detail through the lenses of the four dimensions of teachers' professional practice, namely *action, reflection, autonomy, and networking*. Finally, this paper sketches some *future challenges* for similar professional development programs.

1 Searching for common ground between the main theoretical perspectives

Let us start with Jaworski's *teaching triad*, which Zaslavsky & Leikin use for *"describing in a general way the tasks of the different groups of mathematics educators [MTs, MTEs, MTEE] who were involved in the study, as well as for analyzing and discussing our findings"*.

The teaching triad used by Jaworski (1992, p. 8) *"as a powerful tool for making sense of the practice of teaching mathematics"* can be regarded as a possible concrete form of the classic *"didactic triangle: teacher – student – content"*, insofar as it puts three major *tasks of the teacher* in the foreground (see Figure 2).

providing challenging

content (= mathematics)

↙↗ ↖↘

management of learning by the **teacher** ⟷ being sensitive to the **students**

Figure 2

One benefit of this triangle – as used by Zaslavsky & Leikin – is its variable use in different contexts, for example, in the versions *MTE – MTs – mathematics teaching* or *MTEE – MTEs – mathematics teacher education*.

We must always bear in mind that the *didactic triangle* can be realised in a variety of different models, expressing different norms, values, beliefs, philosophies, ... behind learning and teaching, influenced by the teacher's explicit or implicit views on mathematics, education, society, world, sense of life, etc. and the explicit and implicit socially constructed views etc. in the context he or she lives in.

Zaslavsky & Leikin refer in their paper to two perspectives of learning and teaching, namely the *"constructivist view"* and the *"transmission metaphor"*, as representing opposite perspectives. The authors highlight the notion that the project is driven by a constructivist view where *"learning is regarded as an ongoing process of an individual or a group trying to make sense and construct meaning based on their experiences and interactions with the environment in which they are engaged"*, while at the same time also noting that *"at the early stages of the project most of the project team members seem to hold a transmission metaphor"*, finding it *"hard to reflect on their work"*.

This difference between the philosophy of the project and the starting perspectives of most team members (the MTEs), and, in addition, the explicitly stated project goal *"Facilitating teachers' knowledge ... in ways that support a constructivist perspective to teaching"*, leads us to assume that the project goal mentioned above was extended, tacitly at least, to the team members. All in all, this apparently indicates that the project intended to promote teachers' and teacher educators' change from the transmission metaphor to the constructivist view.

Referring to the didactic triangle, we can visualise the intended change as a transfer from a *Model T* to a *Model S* as illustrated below (Figure 3). The use of such opposites as *"top vs. bottom"* and *"indirect vs. direct"* is intentional and aims to amplify – and to some extent to exaggerate – the differences between the two models:

Model T: content → teacher → student

Model S: student ↔ content; teacher ↑ (student ↔ content)

Figure 3

Whereas in *Model T* the teacher is transmitting the top-positioned content down to the students, thus providing them with only an indirect access to the content, *Model S* sees the students' direct and partially self-organised confrontation with the content as the major goal, with the teacher supporting and facilitating this process.

The *"transmission metaphor"* versus *"constructivist view"* distinction is one concrete form of the difference between Model T and Model S. Other authors, for example, use the difference *"teacher-oriented"* (or *"content-oriented"*) versus *"student-oriented"* (or *"process-oriented"*). We would need more discussion of what constructivist views, student-orientation, process-orientation, and other perspectives have in common or how they differ. We should also take into consideration *other versions* (than models S and T) of the didactic triangle in order to avoid the trap of uniformly painting everything black & white. In addition, we need to consider the question of to what extent the everyday practice of teaching students, student teachers, teachers, teacher educators, ..., influenced by general conditions like curriculum, assessment system, class sizes, etc., fosters or hinders these perspectives. Furthermore, we should ask how we could describe, for example, a teacher educator's position with regard to these models (whereby the position might change from context to context), or describe (aspects of) change in a teacher educator with regard to these models.

When Zaslavsky & Leikin bring Steinbring's *model of teaching and learning mathematics* (1998) into play in order to look more deeply into the *MTE-MT base line* of the *didactic triangle*, their bigger cycle (Figure 1 in their paper) has – slightly modified – the following structure (Figure 4):

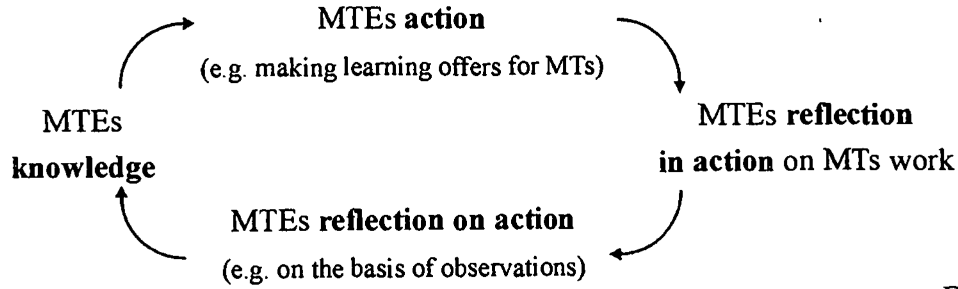

Figure 4

With reference to Figure 4, we can highlight *two additional features* of the "Tomorrow 98" program. Firstly, the program not only puts an emphasis on MTEs actions and reflections when working with MTs, but also strongly initiates and promotes *joint actions and reflections* among MTEs, for example, through internal workshops where MTEs (or the project director) work as MTEEs for their colleagues, through workshops where MTEs take part in workshops that experienced colleagues conduct for MTs, or through joint reflections among the MTEs. Secondly, experience in this program (as well as in general) shows that in addition to MTEs knowledge, their *beliefs*, *world views*, *etc.* also play an important role. Taking into account these two extensions, Figure 4 might be extended as follows (Figure 5):

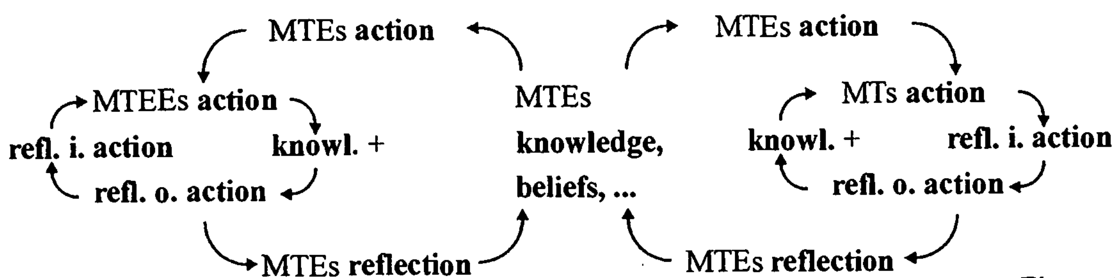

Figure 5

Figure 5 makes the connection to Krainer's *four dimensions of teachers' professional practice* (1998), to which Zaslavsky & Leikin refer in their paper. It is apparent that in the "Tomorrow 98" program the dimensions of *action* and *reflection* play an important role for the MTEs (as well as for the MTs). In addition, if we examine the difference between the right and the left circle, the dimensions of *autonomy* and *networking* also come to the fore: whereas the right circle mainly refers to MTEs autonomous inservice work with the MTs, the left one mainly refers to situations where the MTEs share their ideas and experiences with other MTEs. The two circles, the autonomy and the networking circles, meet in MTEs professional knowledge, beliefs, etc.

From this point of view, *MTEs professional knowledge, beliefs, etc.* can be seen as influenced by the four dimensions and visualised by the following schema (Figure 6):

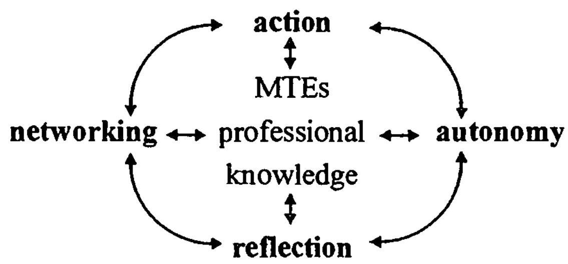

Figure 6

All in all, looking at Zaslavski's & Leikin's use of the models of Jaworski, Steinbring and Krainer, we can see *common ground.* Jaworski's teaching triad is used in order show the different *tasks* the MTs, MTEs and MTEEs (the project director and MTEs when leading workshops for MTEs) have in the project. Steinbrings' model of teaching and learning is brought into play in order to allow a closer look at the *interaction process* between MTEs and MTs. Finally, Krainer's dimensions of action, reflection, autonomy and networking can be used as lenses to look at *promoting and hindering factors* that influence MTEs professional practice, highlighting that a crucial part of MTEs growth is sharing experiences within the group of MTEs.

Before using these *four dimensions* in the next section in order to reflect on the story of Tami, I shall briefly describe how action, reflection, autonomy, and networking are interpreted in this model:

Action: The attitude towards, and competence in, experimental, constructive and goal-directed work;

Reflection: The attitude towards, and competence in, (self-)critical work that reflects on one's own actions systematically;

Autonomy: The attitude towards, and competence in, self-initiating, self-organised and self-determined work;

Networking: The attitude towards, and competence in, communicative and co-operative work with increasingly public relevance.

Each of the pairs, "action and reflection" and "autonomy and networking", expresses both *contrast and unity*, and can be seen as complementary dimensions which have to be kept in a certain balance, depending on the context. The interplay between these dimensions seems to be of great importance: in general, more reflection contributes to a higher quality of actions, and the sharing of experiences enriches one's own view; furthermore, a higher quality of action and autonomy promotes the quality of reflection and networking, etc. Experience shows that teachers' practice is usually characterised by *a lot of action and autonomy* but *less reflection and networking*, in the sense of critical dialogue about one's teaching with colleagues, mathematics educators, the school authority, the public, etc., and, linked with that, by putting an emphasis mainly on the dimensions of action and autonomy in their classrooms (see e.g. Krainer, 1999). Therefore, it is a great challenge for teacher education programs to foster particularly the dimensions of *reflection* and *networking* in order to support teachers' flexibility to generate and to keep an adequate balance between the four dimensions in their practice.

2 Looking at the story of Tami through the lenses of the four dimensions action, reflection, autonomy, and networking

The description of Tami, *"who was considered by herself and her colleagues as a rather competent teacher-educator to begin with"*, by Zaslavsky & Leikin in the early stages of the "Tomorrow 98" program can briefly be interpreted with regard to the four dimensions as follows (I can only refer to the data presented in their paper).

Tami in the early stages:

Tami is an *"expert in managing cooperative learning experiences"* (CLE), and she has a *"strong belief regarding the significance"* of CLE. Tami is *"eager to convince"* other colleagues to adopt her CLE approach and to use the same methods she uses. Within the project she offers to conduct CLE workshops for MTs and MTEs. She even encourages *"a small group of teachers to adapt the material ... and to produce a booklet"*. Tami's workshops are *"very structured"*, she is *"reluctant to conduct reflective discussions with the teachers"*, *"silence"* is *"a threat"* to her.

Concerning the four dimensions, Tami has clear strengths with regard to *action and autonomy*. She is an expert in CLE and practices it with conviction. It seems that around CLE she constructs a considerable domain of autonomous action with strict rules where she feels competent and secure. It seems that Tami has some weak points in *reflection*. She avoids reflective discussions with teachers, and the use of very structured designs decreases the chance of such reflections. She is very convinced by her CLE approach and seems to feel not much need to reflect on modifications. Concerning *networking* the situation seems to be more complex. On the one hand, she is interested in spreading her knowledge to other teachers. She practices CLE in her workshops, initiating cooperative learning among her inservice teachers, thus promoting networking among them. On the other hand, she is reluctant to conduct reflective discussions with the teachers and avoids departures from her given workshop structure (which decreases the chance that the teachers themselves generate new ideas and ways), thus hindering networking among teachers.

Tami at the end of the project:

By the end of the "Tomorrow 98" program, Tami seems to have changed a lot. Again, we refer to the four dimensions of professional practice. Tami made great progress with regard to *reflection* and *networking*, as her reflection on the program shows: *"Today, I am much more open to many different ways of facilitating teachers' and students' cooperative learning in mathematics. I now accept and use methods that are not very structured, that seem to give way for many different kinds of cooperation to different extents."* Tami now no longer feels *"uncomfortable waiting for responses"*, having learned *"to listen"* to her participants' voices. She succeeded in supporting a small group of teachers writing a booklet about CLE, and in convincing other MTEs of the potential of CLE, in particular through motivating a small group of team members to collaborate on designing CLEs for workshops with teachers. This progress in reflection and networking also improved the dimensions of *action* and *autonomy*. She gained further expertise in CLE and in leading CLE workshops.

She increased both her flexibility and her self-confidence in taking autonomous decisions: *"I reached the extent that my colleagues often accuse me of waiting too long for replies."*

What made this *progress*, in particular with regard to *reflection* and *networking*, possible? Let us look at the program's effort to foster these two dimensions.

Zaslavsky & Leikin regard *reflection* as a *"key issue"* in their program and incorporated several elements of it *"in order to enhance reflection and self-analysis of both the participating teachers and the project staff"*. A considerable part of the reflection was organised as *joint activities* by MTEs, thus promoting exchanges about their experiences at different levels, for example, experiences from workshops they led for MTs or their MTE colleagues, or from participating in workshops led by the project director or other MTEs.
Examples of other essential elements that strengthened the *combination of reflection and networking* were:

- The MTEs were *"required to give written accounts of the workshops for which they were in charge"*
- *"mutual peer (written and oral) evaluation and teachers' (written and oral) feedback were provided on a regular basis"*
- *"regularly scheduled staff meetings, in which staff members could reflect on their work, share their experiences, consult with their colleagues, and negotiate meaning with respect to the goals and actions of the project"*
- team members were *"continuously encouraged to initiate ideas and suggest new directions and actions within the project"* (thus promoting participants' ownership and responsibility for the project)
- *"special professional meetings and mini-conferences for the project team ... were scheduled"* where also *"external experts"* were invited to share their expertise
- the project was accompanied by a systematic documentation, for example including interviews, questionnaires, self and peer reports, written analyses, videotapes and protocols, thus enriching the reflection and networking processes
- all staff members (and teachers) had *"access to an electronic network"* that *"served for informative communications, as well as for sharing, reflecting, and debating"*
- a project director who apparently was really successful in coordinating this complex task and in taking a considerable amount of responsibility for the whole project, being involved intensively, professionally and socially.

All these factors (and more) together seem to have generated a fruitful and dynamic *project culture*, as it had been intended by those responsible for the project, as articulated in the project goals, for example: *"Enhancing teachers' and teacher-educators' socialization and developing a supportive professional community to which they belong."* Nevertheless, this supportive professional community cannot be the full explanation of Tami's growth, although it surely promoted her efforts extraordinarily. There was an interplay between the project's challenging *network* of actions and reflections and her *autonomous* struggle for growth. In particular one situation seems to have played a very *critical role* in her development, namely the one where a group of four MTEs, who had been motivated by Tami to collaborate on designing CLEs, conducted a workshop for the project team and were *"confronted with some unexpectedly problematic mathematical and pedagogical issues"*.

However, in this network of "critical friends" it apparently was possible for the group to appreciate the criticism as a learning chance and they *"returned to revise their plan and the resource material, and in the following year conducted one of the most exciting workshops with the teachers"*. This challenging situation seemed to have greatly influenced Tami's way of making her CLE approach more open and flexible. We can assume that the openness and flexibility of the whole project, for example, with regard to MTEs freedom to define their own ways to grow professionally within the project was a supporting factor. This is, for example, expressed by Ronit who was one of the group of four MTEs that conducted the CLE workshop for the project team: *"I found myself dealing with questions that interested me, because I asked the questions ..."*. Such a motivation for developing oneself further, accompanied by colleagues who have similar goals and ambitions, and based on a project culture that provides challenging learning environments, is a good starting point for professional growth.

The "Tomorrow 98" project reveals some crucial features of promoting professional growth of teachers and teacher educators, with the particular strength that it combines their learning into a whole framework. It might be interesting to get an even closer view on Tami's development, possibly indicating how she dealt with other (critical) situations that challenged her. It would also be valuable to look more deeply at the growth of the four MTEs that collaborated on CLE, a group, which became a *professional mini-community* of its own. And, in addition, we could benefit from hearing a story about an MTE who was not so successful in his or her further development, even though he or she was involved in the project's rich learning environment.

3 Some future challenges for similar professional development projects

The paper of Zaslavsky & Leikin shows a *unique project* that combines the professional growth of mathematics teachers with mathematics teacher educators. The short story about the professional growth of Tami gives a brief insight into the processes that influenced her development. It is worth writing *data-rich stories* about mathematics teacher educators' and participants' development which allow us also to compare their growth.

One main feature of the uniqueness of the "Tomorrow 98" program is its *complexity*, as the following aspects demonstrate:

- The challenge of *interweaving* the professional development of MTs and MTEs.
- The *size* and the *structure* of the project with regard to the people involved: 1 project director (only supported by an MTE in the latter stages of the project), (a big group of) 20 team members, and about 120 participants.
- The *heterogeneity* of the *team members'* competence and experience (only 6 out of 14 *"had some previous experience in managing inservice professional development activities"* and *"were not very confident of their qualifications as teacher-educators"*, which means that many team members apparently were really busy with their own learning process, and thus did not have all their energy for taking some responsibility off the project director, or for looking more closely at mathematics teachers' growth).
- The *heterogeneity* of the *participants' background* (junior high school and high school teachers, the different length and intensity of their involvement).

- The *variety* of *activities* and *meetings* in the program.
- The *variety* of *data gathered* (observations, interviews, questionnaires, videos, ...).

It is astonishing that the complexity of the project – given its low personnel resources – seemed to have been mastered so successfully. On the one hand, this is due to a high degree to the work of a *project director*, who clearly is a highly skilled in leading projects and is prepared to take more responsibility than usual. On the other hand, we can imagine that *more resources* (people, time, money) for this project could provide the responsible people with more freedom to concentrate on specific development and research questions.

In the following I confine myself to some challenges for *similar professional development programs*, in particular looking at the *differences in MTs and MTEs growth* as reported in the paper by Zaslavsky & Leikin.

Regarding the MTs and the MTEs further development in the "Tomorrow 98" project the authors highlight *differences* in the *extent* and *nature* of their growth. Whereas the progress of the MTEs is stressed through expressions like *"entire team becomes enthusiastic"*, *"is motivated"*, *"become more competent with respect to innovation"* or *"become more competent as Teacher-Educators"*, the progress of the MTs is described far less spectacularly, for example, through expressions like *"become aware of the potential of innovative approaches"*, *"apply this approach"* or *"a sub-group begins to initiate and is ready to begin acting as Teacher-Educators"*. Zaslavsky & Leikin also point out that the MTs felt *"less committed and accountable for their role in the project"* than the MTEs. Besides the fact that, of course, MTs and MTEs had different starting points and tasks within the project, it is interesting that the relative effects, professionally and atmospherically, seem to be stronger in the case of the MTEs. Zaslavsky & Leikin explain it mainly by the *"differences in the time and opportunities staff members had, in comparison to the teachers, to reflect and share their experiences"*. This is certainly a very important point. However, I would like to bring into play some *additional hypotheses* that might explain the difference:

- The MTEs had more *freedom* to define their own ways to grow professionally, whereas the MTs had more the role of *applying* the approaches introduced by the MTEs. Connected with that, the MTEs were able to build their workshops on their individual strengths (as e.g. CLE in the case of Tami), whereas for the MTs these approaches often apparently meant the need to cope with situations that were really new for them (e.g. alternative ways of learning). Both factors have an impact on one's energy to reflect on one's own practice and on one's commitment with the project.
- The authors' assessment that at *"the early stages of the project most of the project team members seemed to hold a transmission metaphor of the teaching-learning process"* demonstrates the deep learning process of the MTEs. According to that, a lot of the *project's energy* was covered by promoting the growth of the MTEs who really were challenged in reflecting on their own learning process.
- The *research* mainly built on investigations into MTEs and MTs growth from the perspective of MTEEs involved in the project. To some extent, the MTEs systematically investigated their own practice (mainly following a research question that arose from

their own interest), wrote down their experiences and shared it with their colleagues, and thus practised a form of *action research* (see e.g. Jaworski, in press). The MTs, too, were invited to reflect on their practice, however, it seems that their investigations were not mainly aimed at giving answers to their self-defined questions, but more towards informing the MTEs about how successful of their – the MTEs – interventions in promoting a constructivist view of learning and teaching had been.

- The *project director* had a complex task and took a considerable amount of responsibility for the whole project. She was very active in different roles, for example, through leading the project, defining and reinforcing further its philosophy and its conceptual framework, being the chief researcher, conducting several workshops, giving feedback and guidance to individual people, writing reports and research papers, and through keeping all the things together (being supported by a colleague only in the latter stages of the project). She seems to have been a focal point in the further development of many MTs and MTEs. This is shown, for example, in Tami's reflection *"Having observed over and over again in many different situations ways in which the project director conducted such sessions, with the teachers and with us - the staff members, I began to try it myself."* It can be assumed that the project director's competence and support reached the MTEs more intensively than the MTs.

Finally, some *brief suggestions* with regard to *future professional development programs* that might have similar goals and general conditions like the "Tomorrow 98" project:

- Given a complex project like above, it is worth considering getting enough resources in order to enable a team to share a *fairly balanced responsibility* from the very beginning, i.e. to differentiate between *different roles* (research, management, coaching, workshops, evaluation, ...), and to initiate other elements of a project structure that allow the project director to concentrate on important strategic and research-related activities.

- With regard to the challenge of *interweaving teacher educators' and teachers' professional growth*, it is worth considering putting more emphasis on teachers' investigations into their own practice, thus also providing the teacher educators with additional authentic reflections by the teachers.

- Referring to Zaslavsky's & Leikin's interesting statement, *"In fact, in schools from which a number of teachers participated in the project, Tami realized that collaboration began and more flexible ways were employed to facilitate students' cooperative learning."*, it is worth indicating that the organisational aspect of teachers' and teacher educators' professional growth needs more consideration, both with regard to teacher education programs and research on teacher education.

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Other references see paper of Zaslavsky & Leikin.

BECOMING A MATHEMATICS TEACHER-EDUCATOR: CONCEPTUALIZING THE TERRAIN THROUGH SELF-REFLECTIVE ANALYSIS

Ron Tzur

The Pennsylvania State University

Abstract: *My purpose in this paper is to contribute to the conceptualization of the complex terrain that often is indiscriminately termed mathematics teacher educator development. Because this terrain is largely unresearched, I interweave experience fragments of my own development as a mathematics teacher educator and reflective analysis of those fragments as a tool to abstract notions of general implication. In particular, I postulate a framework consisting of four stages of development that are distinguished on the basis of the domain of activities one's reflections may focus on and the nature of her or his reflections. Drawing on this framework, I present four themes that contribute to thinking about and conducting research in the terrain of mathematics teacher educator development.*

During the third year of my work as an assistant professor I encountered a problem that is not uncommon. A prospective doctoral student canceled her participation in the program, thus we needed an instructor for a mathematics education methods course for preservice teachers. I asked Kelly, a graduate student who has been working for two years as a research assistant in the mathematics teacher development (MTD) project if she would take on the responsibility. On the *intuitive* (unformulated) basis of knowing Kelly as a developing researcher and as a student in courses I taught, I expected that she would be able to take on the responsibility, and that this would promote her development. Yet, while talking with Kelly and my colleagues it soon occurred to me that I had never before articulated the perspective underlying my practice as a mentor of developing mathematics teacher educators, nor did I articulate my own development as a novice teacher educator. To my surprise, in searching for research literature I could find many generic studies on development of teacher educators (Denemark & Espinoza, 1974; Diamond, 1988; Ross & Bondy, 1996; Wayson, 1974) or on such development in other disciplines like English teaching (Farrell, 1985), but only one (Onslow & Gadanidis, 1997) that focused on development of mathematics teacher educators. Therefore, in this paper I attempt to begin conceptualizing the terrain of mathematics teacher educator development through self-reflective analysis.

My interest in mathematics teacher educator development is rooted in my work of studying mathematics teacher development (Simon, Tzur, Heinz, Kinzel, & Smith, 1998b; Tzur, Simon, Heinz, & Kinzel, 1998). I became convinced that mathematics teacher educators can, and should, construct clearer (a) conceptions of desired teaching practices that can serve in setting goals for the development of preservice and inservice mathematics teachers (Simon, Ball, Dekker, & Russel, 1998a), (b) explanations of mathematics teachers' development toward such practices (Tzur & Timmerman, 1997), and (c) ways in which teacher educators might think about, and promote, such development (Cooney, 1994). It seems as important that mathematics

teacher educators construct clearer (a) conceptions of desired teacher-educating practices that can serve in setting goals for the development of mathematics teacher educators, (b) explanations of mathematics teacher educators' development toward such practices, and (c) ways in which mentors of teacher educators might think about, and promote, such learning processes.

I organized the paper as follows. Following Guilfoyle, Hamilton, Pinnegar, & Placier (1996) writing, I first interweave narratives and analyses of experience fragments of my own development. Then, I present my way of thinking (conceptual framework) about learning and teaching processes, stressing the critical role that reflection plays in learning. This is followed by my conception of mathematics teacher educator development in terms of four interconnected foci of reflection—learning mathematics (student), learning to teach mathematics (teacher), learning to teach mathematics teachers (teacher educator), and learning to teach mathematics teacher educators (mentor). In this context I present four themes for further research and conversation.

Reflecting on Experience Fragments

In this section I interweave narratives and analyses of experience fragments that greatly impacted my current praxis[1] as a teacher educator and a mentor of teacher educators. The experiences are ordered according to my understanding of the developmental sequence in which the four foci mentioned above are interwoven. I will elaborate on relationships among those foci later. However, it is critical to note that I view the four foci as highly interconnected, and distinguish among them on the basis of what issues (activities) serve as "material" on which the learner reflects and on differences in the nature of the reflective process (awareness, rigor, depth).

Learning Mathematics

I learned mathematics at my kibbutz elementary and high schools. The kibbutz is a small community operated by its members on the basis of socialist ideas such as equality, cooperation, and mutual aid. To educate the young toward realization of those ideas, the kibbutz school is operated as a mini-community in which adult educators and students negotiate social norms and practices (notions borrowed from Yackel & Cobb, 1996) to guide students toward gradually taking responsibility for academic, social, and physical aspects of their life. In particular, capable students are expected and encouraged to tutor their peers or younger students. Below, I present a fragment pertaining to my learning through peer tutoring.

> My peers knew of my affection for mathematics, and frequently asked me to help them in doing their homework. My usual technique was that I worked one or two problems step-by-step, then asked my peers to try the next problems, which for me seemed "just the same." However, my peers usually got stuck. To help them, I explained what I saw as the reasons behind what we were told or asked in class, and how all the pieces

[1] I use the term praxis in Dewey's (1938) sense of the combined conglomerate of theory and practice that characterizes the teacher's work.

"clicked together" in the process of solving the problem. Often, the struggle to figure out how to explain the matter brought about new insights *in me*. Sometimes, my peers "got it" and I experienced a feeling of accomplishment; other times they did not and I felt dull and disturbed. In both cases, I was curious about their understanding. These experiences were related to my decision to take several courses in education during my junior and senior years in high school, where I was first introduced to the educational and psychological thought of Dewey and Piaget. However, in spite of the courses, at that time I continued to use the same tutoring techniques.

In reflecting on this fragment, I notice three significant aspects—the social context in which I learned, the advancement of *my* mathematical understanding, and the lack of change in my ways of tutoring. Regarding the social context, it seems that my tutoring was encouraged and affected not only by the generic social norm of helping others, but also by our classroom sociomathematical norms (Yackel & Cobb, 1996) of being expected to justify a solution and to make connections between results of different problems. Regarding my learning, it seems that my continual reflection on activities I used to explain my solutions *to others* played a key role in advancing my mathematical understanding.. While I repeated the process of interpreting the information provided in the problem, setting some image (goal) as a solution, carrying out activities to work to the solution, and relating the effects of my activities to the solution image, I noticed new regularities in activity-effect relationships. Regarding the lack of change in my ways of tutoring, I did not understand that many of my peers could not see what I saw because their mathematical conceptions were different than mine (Cobb, Yackel, & Wood, 1992). Thus, my reflection was focused on my own understanding of the (obvious to me) mathematics, not on others' thinking and how it might be changed via *their* activities and reflective processes. Simply put, I used activities in an attempt to promote others' mathematics, but I did not reflect on the relationship between those activities and the effects in terms of my peers' learning.

<u>Learning to Teach</u>

I started my formal education as a mathematics teacher in 1982, at the kibbutz branch of Haifa University School of Education ("Oranim"). My development continued while I taught mathematics at the kibbutz high school and while I studied (1990-92) toward a master's in mathematics education at the Technion - Israel Institute of Technology. While at "Oranim," I became friends with a small group of students who not only prepared themselves as teachers in different disciplines, but who also participated in the creation of a new program focusing on social education issues. At the kibbutz high school, although I taught all grade levels (7-12) and tracks, I worked mainly with the older, failure-experienced (non-college bound) students who struggled to understand basic mathematical ideas and to not drop out of mathematics courses. At the Technion, I focused on the role of assessment in students' mathematics learning. I created an alternative assessment method in an attempt to facilitate failure-experienced students' learning of mathematics and

studied the impact of using this method on students and teachers (Tzur & Movshovitz-Hadar, 1998). Below, I present fragments pertaining to my learning to teach mathematics through teaching students and teachers, and researching.

> A few years after high school I decided to realize my dream of being a mathematics teacher at the kibbutz high school. I vividly remember how my teacher educators encouraged me to give voice to my thoughts about how my (often painfully) growing understanding of higher level mathematics would help (or impede?) my abilities to make a difference in my future students' understanding of mathematics. They also encouraged me to conduct several mini-research projects on diverse issues (e.g., the development of the Cartesian coordinate system, gender differences in the kibbutz, drug abuse impact on the brain). Most important, they supported my decision to begin teaching full-time while completing my studies as a part-time student. Teaching failure-experienced students was challenging and rewarding. I was engaged in taking apart mathematical ideas that an expert might consider rather simple and unproblematic, and continually searched for a variety of activities that might promote the students' learning. I was pleased when their work indicated progress and felt somber when they were frustrated and stuck.
>
> The master's program was a breeze of fresh air, but it was not because I stopped teaching—I deliberately continued full-time teaching (cf. Halai, 1998) while joining a research and curriculum development project (Movshovitz-Hadar, 1992). The project team focused on developing learning materials to provide non-college bound students with enjoyable, meaningful, and successful learning experiences. Besides using those materials in my teaching, I also traveled every week to one of 6 high schools and met with teachers to promote their *understanding and use* of the materials. During those visits, I collected data (observe, interview, survey) for my thesis. The teachers' use of the curriculum indicated (to me) misunderstandings on their part. By communicating how I thought of the materials and demonstrating ways I would use them, I tried to encourage the teachers to not only use the materials as given, but also to understand the underlying educational reasoning and to adjust teaching activities to students' work. In working with the teachers, I had many insights into teaching my students, but rarely did the teachers' indicate the intended understandings. In writing my thesis, I struggled to formulate commonalties I noticed in the teachers' use of the alternative method and students' cognitive and social-emotional development. Consequently, I reconstructed my conception of Piaget's and Vygotsky's theories, emphasizing the key theoretical construct of interaction and applying it to the case of assessment.

In reflecting on these fragments, I notice three significant aspects besides the obvious continuation of my learning mathematics—the institutional norm to conduct research; my learning to teach as a result of integrating activities of teaching, research, and course work; and the lack of significant changes in my attempts to promote other teachers' reasoning about teaching-learning processes. Regarding the first two aspects, it seems that both my undergraduate and master's programs stressed a view of learning to teach that is compatible with Schön's (1983, 1987)

notion of the reflective practitioner, and with recent trends of teacher action research (Cooney & Krainer, 1996; Crawford & Adler, 1996; Jaworski, 1998; McEwan, Field, Kawamoto, & Among, 1997). My teacher educators expected me to actively engage in doing research, to integrate my diverse learning experiences, and to figure out how to make my research available to others. Being simultaneously engaged in the activities of teaching mathematics, conducting research, taking courses, and guiding teachers provided me with many situations in which to reflect on my own teaching. For example, in teaching the high school classes, I actively experimented with activities in an attempt to promote my students' understanding. That is, I looked for tasks and problems in the learning materials that had been developed in the project or that were in research articles, or I created new tasks on the basis of my interpretation of learning theories. When the activities did not bring about the intended learning I felt responsible for adjusting them, and the adjustment process focused my reflection on relationships between my teaching activities and their effects (students' learning). A similar focus was also evident in my work with other teachers. I shared with these teachers activities that promoted my students' learning, which again engendered my own thinking about why the activities were useful. Simply put, I was actively exploring and transforming my own mathematical and educational praxis (my students would say "improving as a teacher").

An important reason for the lack of change in my work with teachers seems to be that I did not understand the teachers' praxis—the conceptions of mathematical knowledge and learning underlying their teaching. In retrospect, I think that both my own mathematics teaching and my attempts to promote teacher development were rooted in a perception-based perspective (Simon et al., 1998b). That is, I *implicitly* viewed mathematical (or pedagogical) knowledge as a well connected web of concepts that exist independent of the knower, and thought of learning as the essentially unproblematic process in which a learner comes to see a growing portion of the web by connecting pieces he or she can already "see" with new pieces, connections that take place through learners' active, first-hand experience. What I lacked was an elaborated epistemology (Steinbring, 1998) that would provide an explanatory mechanism as to how one can "see" a mathematical or pedagogical concept in particular situations, let alone how one might develop new concepts.

Learning to Educate Teachers

In 1992, three years after the publication of the first National Council of Teachers of Mathematics (1989) *Standards* document, I started my doctoral program at the University of Georgia. Although my previous work as a high school teacher and as a guide of teachers included influencing others' mathematics teaching, it was then that I began to seriously consider issues of teacher education. Besides doing course work in mathematics, mathematics education, psychology, and philosophy, I participated in two research projects: a teaching experiment focusing on children's construction of fraction knowledge and a teacher enhancement program focusing on teachers' use of alternative assessment. After two years of working mainly as a research assistant,

I added a teaching role to my program. Moreover, I was asked to let a new doctoral student (Lori) who had little teaching experience join me in teaching a mathematics course and two methods courses for elementary preservice teachers as a means of assisting *her* learning to be a teacher and a teacher educator. Upon graduation (1995), I assumed an assistant professor position in the School of Education at the Pennsylvania State University. Here, I am currently developing as a teacher educator through: reforming and teaching methods courses for teachers, creating and teaching new graduate courses, promoting and researching mathematics teacher development, and mentoring novice teacher educators. Below, I present fragments pertaining to my learning to teach teachers and teacher educators.

> Entering a doctoral program in the USA was a major cultural transition (e.g., language difficulties, organization of teaching). My mentors encouraged my desire to articulate relationships between several theoretical approaches and the real-life experiences of students and teachers. I clearly remember how I was stuck when Les Steffe asked me: "When interacting with children, how do you know that you met a scheme?" Such questions sparked periods of intensive thinking, reading, and conversations with my peers and mentors that led to focusing my dissertation on both children's construction of fraction schemes and the teacher-learner interactions that might promote this process.
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> From day one, Lori and I were excited to team-teach. We shared full responsibility in planning, implementing, and reflecting on our teaching reform-oriented mathematics and pedagogy to the prospective teachers. Having excitedly worked with the students throughout the course, I was startled when I received their low rating of my teaching. Their feedback threw me into an intensive period of reflection. I painfully recognized my failure to understand *the teachers'* experience (conceptions) as a critical first step to teaching them anything. When discussing those issues with Lori, I also realized that team-teaching with her enhanced my understanding of teaching and helped her to experience activities of teaching, but it did not seem to be as useful in terms of Lori's conceptualization of teacher education.
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> At Penn State, my learning about teachers' experiences and how I might build on them to promote their development is taking a new form while I am integrating teaching teachers and researching teacher development. As I supervise the graduate students who teach methods courses, I try to inspire them with the kind of cooperative, inquisitive, and creative work I enjoy in my research team. I cherish and contribute to that kind of culture as a deliberate effort to confront the pressures of academia. I try to translate the "aha" experiences of my research into new or revised methods courses for teachers. In the last four years, I have been struggling with Marty Simon's question, "Do you have a way of thinking, a theory, about graduate students' learning processes?" For example, half way into a graduate seminar I realized that just reading and conversing about children's mathematics resulted in superficial understandings. Consequently, I engaged the graduate students in interviewing a child as a means to make sense of the child's mathematical conceptions. Such adjustments in my teaching intensified the question I came to view as preliminary to Marty's question: "What are

graduate students' perspectives of mathematics education and, in particular, of mathematics teacher development and the ways one might promote such development?" It is this question that provoked my work on this paper.

In reflecting on the last three fragments, I notice three significant aspects: how contextual features impact one's learning; how one's explicit ways of thinking about the development of teachers and teacher educators empowers her or his acting as a teacher educator or a mentor; and how it is difficult to translate such explicit ways of thinking into daily planning for or reflecting on teaching, let alone adjusting (on the spot) one's interactions with teachers and teacher educators. Next, I elaborate on each of these three aspects.

The first aspect seems to be a case of the general notion of situated learning (Lave & Wenger, 1991). Three examples in my personal experience seem to highlight this notion. First, culture change intensified my learning by pushing me to see "old" experiences in new light, hence to abstract new ideas about teaching and learning to teach. Second, team-teaching also intensified my learning because of the need to constantly reflect on my teaching in the context of observing Lori's teaching and receiving her feedback on my teaching. Third, there is a mutual impact between my research and my teaching (Jaworski, 1994) while I continually struggle with the inherent tension between expectations and reward structures in academia (e.g., prioritizing writing or teaching, cf., Collins, 1997; Guilfoyle, 1995; Guilfyole et al., 1996). It seems reasonable to expect that my development as a teacher educator would differ from the learning of leader-teachers (Zaslavsky, this volume) whose tenure and promotion are not determined by conducting and publishing systematic research. As my experience indicates, the activity of writing serves as a main vehicle for constant, rigorous, and deep reflection on and reorganization of thinking about teaching and teacher educating.

The empowering impact that constructing explicit conceptions of teaching had on my work was indicated in two shifts: from teacher to teacher educator and from teacher educator to mentor. I now realize how the lack of such conceptions limited the repertoire of plans and interactions that I, or other teacher educators and mentors, could generate and use. For example, Lori could carry out most of the teaching activities we planned for our students. However, her contributions indicated that she had not yet constructed a global perspective about mathematics teaching that allows one to generate or adjust teaching activities, let alone translate learning from research with children into opportunities for teachers. Or, in teaching doctoral students, I frequently found myself adjusting plans and activities after noticing new regularities in their participation. Those adjustments were rooted in my perspectives of the roles that students' conceptions play in their learning and indicate the limited nature of my understanding of their conceptions. In all, it seems that formulating explicit ideas about teaching-learning processes is a necessary (though not sufficient) condition if one is to promote others' teaching or teacher educating.

With respect to the translation of theory into practice, I came to distinguish between reflective and anticipatory states of knowing with respect to my understanding of the learning of students (children, teacher, or teacher educators). Being able to notice students' learning while I reflected on concrete experiences with them is different from being able to form anticipatory ideas as to how students might develop. The latter allows for flexibility and creativity in planning or for adjusting one's interactions on the spot. Moreover, I propose that developing one's ability to construct reflective and anticipatory understandings at a specific enough level is a long and difficult process. My experience (e.g., the negative feedback from prospective teachers in my early methods courses) indicates that the process of reconstructing for oneself a theoretical stance and applying it first to teaching children, then to teaching teachers, then to teaching teacher educators, requires reflective work on various "local" issues before abstracting and connecting the local pieces into an integrated, well articulated praxis. Such a process involves the development of new understandings of mathematics, its learning, and teaching.

Conceptual Framework

The previous section highlights the following seven main areas of potential development of a mathematics teacher educator and a mentor: mathematical knowledge, perspectives on the nature of mathematical knowing, how mathematics is learned, how mathematics is taught, perspectives on the nature of mathematics teaching, how mathematics teaching is learned, and how mathematics teaching is taught. In this section I briefly describe my conceptual framework with respect to these areas. This framework affects what I notice, what I consider important, and what challenges my current understanding.

A foundational component of my conceptual framework is the emergent perspective (Cobb & Yackel, 1996). Using this perspective, I regard the relationship between social and individual aspects of human experience as reflexive (inseparable) in nature. To account for human experiences such as knowing mathematics or educating mathematics teachers, I try to understand both social and psychological aspects of human beings' (inter)activities in their social-cultural-physical milieu. I assume that people learn as they participate in, and *reflect on*, the continual negotiation of norms, practices, and taken-as-shared meanings concerning their (inter)activities in various communities of practice. I also assume a constructivist epistemological stance and theory of learning (Piaget, 1980; von Glasersfeld, 1991) that stress the inseparability of a person's history of (inter)activities in her or his milieu and what she or he knows and feel.

I conceive of knowledge (e.g., mathematical, pedagogical) as a person's conceptions that are used to make sense of (organize) her or his experiential world. Accordingly, I view learning as the adaptive process in which people, while participating in their communities of practice (e.g., a mathematics classroom), transform their current conceptions via *reflecting* on the relationships between their activities and the effects of those activities. In this context, I emphasize two key

to teach mathematics—differentiating plausible explanations of learning processes that afford mathematics teachers' advancement from lower to higher stages, and (*3c*) how someone's activities promote others' learning of mathematics teaching—differentiating plausible relationships between teacher educators' activities and teachers' learning. In this sense, being teacher educators themselves, mentors of teacher educators can develop their mathematical and pedagogical knowledge through such experiences as teaching mathematics to students, teaching pedagogy to teachers, or teaching epistemology to teacher educators.

Fourth Focus: Mentors of mathematics teacher educators develop ways of thinking about and intentionally participating in others' learning to educate mathematics teachers. Through reflection (indicated by arrows 1, 2, 3, 4, 5, & 6 in Figure 4),

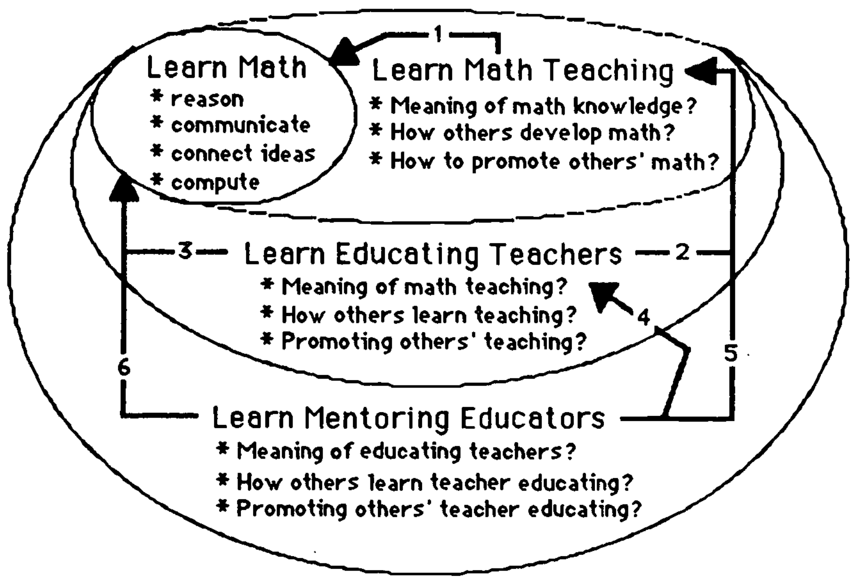

Figure 4: A four-Foci model of Teacher Education.

mentors may become aware of the perspectives that underlie their practices in terms of: (*4a*) what it means to educate mathematics teachers—elaborating one's own perspective of *3a* (hence *1a*, *1b*, *2a*, *2b*, and *2c*), *3b*, and *3c* above and differentiating levels and stages in mathematics teacher educators' practices, (*4b*) how someone comes to know how to educate mathematics teachers—differentiating plausible explanations of learning processes that afford teacher educators' advancement from lower to higher stages, and (*4c*) how someone's activities promote others' learning of how to educate mathematics teachers—differentiating plausible relationships between mentors' activities and teacher educators' learning.

A key to understanding this four-foci model is that development from a lower to a higher level is not a simple extension (i.e., doing more and better of the same thing). On the contrary, development entails a "conceptual leap" resulting from making one's *and others'* activities and ways of thinking at a lower level the *explicit focus of reflection* (cf., Cooney & Krainer, 1996; Edwards, 1996). Through such

reflection, the developing teacher educator (or mentor) may construct *anticipatory conceptions about learners and learning at the lower level(s)*, conceptions that become the theoretical ground for one's praxis. One may question this distinction by noting that all four foci include mathematical thinking. However, I emphasize that even in the case of mathematical thinking, let alone pedagogical and epistemological thinking, each focus is qualitatively different from the previous foci in that it embodies the lower level as an explicit way of thinking abstracted via reflecting on lower level ways of interacting in communities of practice (cf., Schifter, 1998). This is why, for example, proficient mathematics teachers often feel limited in their ability to teach pedagogy to other mathematics teachers on the basis of what they came to experience as empowered understandings of mathematics or students' conceptions and learning (i.e., being a good teacher does not necessarily imply being a good teacher educator).

Themes for Conversation and Research

As the four-foci model indicates, we learn through reflecting on our experience regardless of whether we are pupils, teachers, teacher educators, or mentors of teacher educators. In this context, I delineate 4 themes significant to the mathematics education community that can spark conversations and research.

1. Research Sites and Subjects: Graduate programs seem to be a good site to study teacher educator development, because such programs may promote critical shifts of foci and understandings. For example, one may study the interconnected nature of doctoral students' development of anticipatory perspectives of mathematics, mathematics learning, and mathematics teaching, as they teach mathematics teachers. In doing so, researchers will have to consider the impact of graduate students' motivations and backgrounds (e.g., beliefs about mathematics) and the political-cultural contexts in which they learned (e.g., collaboration vs. competition) on continual changes in their understandings (Acker, 1997; Diamond, 1988; Ryan, 1987). Moreover, it seems important to study development of novice teacher educators after their graduation, and to study groups of teacher educators who were not doctoral students, such as teacher-leaders or principals (Clemson-Ingram & Fessler, 1997; Collins, 1997; Hord, 1988; LeBlanc & Shelton, 1997; Levine, 1997; Rowley, 1988; Simon & Schifter, 1993; Talmage & Monroe, 1970; Zaslavsky, this volume). I anticipate that this last point may become an issue of debate, because mathematics educators may very greatly in their idea about "Who is considered a mathematics teacher educator?" (Cooney & Krainer, 1996).

2. Research Focus/Scope: It seems useful to study both general trends and personal shifts. Obviously, individuals in different communities of practice reflect on very different experiences as they construct (anticipatory?) perspectives of mathematics teaching (third focus) that underlie their shift from teacher to teacher educator. Thus, researchers may be interested in identifying regularities in teacher educator development that are unique to a specific sub-group (or political-cultural context), or regularities that characterize the majority. Additionally, as my story and the

four-foci model imply, teacher educator development is a very long and complex process. Thus, longitudinal studies seem most appropriate, but I think that short-term studies can contribute to conceptualizing specific aspects of new foci and how they are developed on the basis of previous foci.

3. Research Questions: A few fruitful questions highlighted by my story as possible research problems are: "How can models be developed for conceptualizing professional development and what would they look like?" (cf. Cooney, Grouws, & Jones, 1988) "What is important mathematics and mathematics pedagogy that teachers need to learn and how do mathematics teacher educators develop anticipatory perspectives of such learning in teachers?" "What specific shifts of focus occur in teacher educators and how do political-cultural contexts and the educators' perspectives about mathematics, mathematics learning, and mathematics teaching impact such shifts?" (Kinzer, 1972) "How can teacher educators promote bridging the gap between knowledge about students' thinking and teachers' learning to teach, or how does teaching/researching impact focus change in teacher educators?" (Cooney & Krainer, 1996; Denemark & Espinoza, 1974; Hudson-Ross & McWhorter, 1995) "How can mentors be involved in teacher educators' development and what models of teacher-learner interactions can inform the mentors' activities?" (Rahal & Melvin, 1998; Simon & Schifter, 1991) "How do teacher educators assimilate professional experiences into their praxis; what dilemmas do they confront?" (Cherland, 1989).

4. Explicating Teacher-Educating Models: Along with research on teacher educator development, it seems important to clarify the community's understandings of what constitutes teacher education consistent with principles of reform (Wilson & Ball, 1996). This is analogous to Simon et al.'s (1998a) plea to clarify models of teaching as a central means to explicate goals for teacher education. In the last 4 years, my struggle to conceptualize models of mathematics teaching through focusing my reflection on other teachers' development proved very *difficult and useful* to my understanding of and involvement in teacher development. I therefore anticipate that mathematics teacher educators will find it very difficult and useful to study developing teacher educators as a means to construct models of teacher-educating.

Closing Comments

The four-foci model presented in this paper is a work in progress and it is certainly far from being formulated and used in an anticipatory way. However, it highlights the long process required to reflect upon components at one level and develop, first locally and then globally, a new, higher level, integrated praxis. In particular, the model captures a primary goal for teacher educators and their mentors—to promote teacher educators' appreciation of the different foci of reflection. Key to this appreciation is that moving from the teacher's classroom to the teacher educator's classroom requires much more than a shift in the curriculum; it requires a shift in the kind of reflective analysis in which both parties engage. In both cases, such a shift calls into question the context of teaching and its impact on students' learning.

This paper is but an incomplete, self analysis of one teacher educator's story. Thus, it is limited to an individual's evolving perspective of teacher educator development, including recent learning resulting from reflecting on literature and reviewers' feedback while preparing the paper. Guilfoyle et al. (1996) suggested that self-reflective analyses should not be underestimated as a way to research teacher educator development. Although they stressed the need to extend studies of the meaning that teacher educators give to their development beyond this method, they saw this kind of research as appropriate because it provides access to important aspects of the teacher educator's past and current experiences (Ayers, 1988; Kaufman, 1996). In this sense, the hope is that the paper provides stimulating and relevant material for further collective and individual reflection among mathematics educators about the phenomenon loosely called teacher educator development.

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WHAT DOES IT MEAN TO PROMOTE DEVELOPMENT IN TEACHING?

A response to Ron Tzur's paper: *Becoming a mathematics teacher-educator: conceptualising the terrain through self-reflective analysis*

Barbara Jaworski – University of Oxford

Through a process of 'self-reflective-analysis' Ron Tzur sets out towards conceptualising mathematics teacher-educator development. He presents a deep and searching account of his own growth of knowledge as a teacher-educator on which he bases a four-focus model to 'conceptualise the terrain'.

In this response I shall focus firstly, and briefly, on two aspects of Ron's theory and methodology that are of significance to his substantive focus: (i) his use of fragments as a methodological tool for self-reflective analysis; and (ii) his juxtapositioning of the individual and the social as a foundation for discussing growth of knowledge in becoming a teacher-educator.

Secondly, and at slightly greater length, I will then respond to the substantive material of Ron's paper to discuss the main elements of his analysis and the resulting model, and to raise an issue which seems implicit in the paper but not extracted explicitly or subjected to critical scrutiny. This, I believe, raises questions central to much mathematics teacher education as it exists currently.

Fragments as a tool for self-reflective analysis

Ron begins by reflecting on 'fragments' from his own experience, anecdotes, narratives or stories which carry in the them the essence of the concepts he seeks to extract. John Mason (1988a) has suggested a *fragment* is

> a temporarily continuous recallable incident whose content can be negotiated and agreed. It is generally of short duration because it must be recallable … and since its content can be agreed, it must be detailed enough not to require extensive interpretation.

Mason uses the phrase "brief-but-vivid" to capture the two essential ingredients of a fragment. A working group of the Mathematical Association in the UK expounded a theory of the use of *stories* or *anecdotes* to develop professional practice - the Anecdoting Process (Mathematical Association, 1991). The notion of *narrative* as a source of theoretical grounding is now well developed in the educational literature through the work of, for example, Connolly and Clandinin (1995), acknowledged by Bruner (1986, 1996) as taking its place alongside nomothetic and ideographic forms of evidence and demonstrated in the mathematics education literature in the work of Deborah Schifter (1996) and Leone Burton (in press). Ron himself refers to the work of Guilfoyle et al (1996) in the use of narratives and analyses of experience to present their work.

As a methodology leading to emergent theory, however, fragments must be seen as more than unsubstantiated personal reflections from which generalisation is dubious. The importance of such stories, narratives or fragments lies in their offering particularities which resonate so strongly with the experiences of others in a community that general principles can be extracted. The notion of 'essence' is a key idea, since the fragment is chosen to carry

this essence, to be generic rather than particular. It is not that we can generalise from the one example, but rather that the one example points paradigmatically towards the principles we wish to expound or issues to extract. The *essential* nature of the story is resonated in experiences of its audience, who are able to recall similar instances of their own, and thus enter into the general principles involved.

Ron's choice of fragments is done selectively and succinctly to point to the key stages in his development and bring his readers, effectively, into the terrain he seeks to analyse. The levels of generality or particularity in the fragment seem central to its effectiveness in convincing its audience of the ideas or issues to be extracted. I should have liked to see rather more *specific* instances of Ron's experience in some cases. For example, in his first fragment, where he talks of his mathematical learning from peer tutoring, he says,

> "Often the struggle to figure out how to explain the matter brought about new insights *in me*. Sometimes my peers "got it" and I experienced a feeling of accomplishment; other times they did not and I felt dull and disturbed."

I can enter into this experience, knowing myself of occasions when I as teacher learned more from a situation than the students I was trying to teach. Thus the fragment has important face value reflected in my experience. How much more powerful this might have been had we been presented with a specific occasion in which something mathematical suddenly was crystalised for the teacher while seeming not to be appreciated by the peer. Similarly, in subsequent fragments, it could have been valuable to be given a glimpse of a particular issue discussed with Lori, and its differences for Ron's and her development. While a general description might resonate with a reader's own experience, such generic particularity has an intrinsic power to convince, thus providing powerful evidence for a theoretical conjecture. Mason (ibid) writes, "It is essential that a fragment be known by what people experience, and not by some interpretation or generalisation of their experience".

As a research tool, the use of fragments can be extremely valuable in analysing incidents or situations leading to general articulations or characterisations. A brief-but-vivid account of an incident which manages to capture its essential nature becomes data for analysis. Analysis looks critically at the issues raised by the fragment and the particular circumstances in which it occurred. For example, in a *particular* case of team teaching with Lori, in which Lori's learning appeared not to be what Ron might have hoped, what were the key circumstances and issues? How do these relate to circumstances and issues in other fragments? Such identification across a range of fragments could start to provide insights into the practices described and to a theoretical account of their problematic nature. In this way, reflection shifts into research, and research provides clearer evidence on which further practice can be more knowledgeably designed.

Reflection and Reflexivity: the individual and the social

Ron's position on knowing and the growth of knowledge influences overtly his conceptions of himself as a learner. He acknowledges this position as a *constructivist* one in which "people, while participating in their communities of practice (e.g., a mathematics classroom), transform their current conceptions via *reflecting* on the relationships between

their activities and the effects of those activities"(emphasis in original). He sees a reflexivity between individual and social aspects of human experience, assuming that "people learn as they participate in, and *reflect on*, the continual negotiation of norms and practices, and taken-as-shared meanings concerning their (inter)activities in various communities of practice (emphasis in original).

This reflexivity between the individual and the social is no simple matter. It involves a complexity of theory which depends on assumptions about the growth and status of knowledge and is lengthily articulated by theorists in psychological and sociocultural domains in the mathematics education literature and beyond. It is understandable that Ron did not include a critique of such theoretical positions in the short space allowed for his paper in this forum. However, such critique is now well-known in the public domain. See for example the excellent juxtapositioning by Jere Confrey of elements of Piagetian and Vygotskian theory in Confrey, 1995. She, along with Jerome Bruner (1996, in his key address in Geneva, bridging the conferences celebrating the centenary of the births of Piaget and Vygotsky) points to the incommensurability of these theories where the origins and growth of knowledge is concerned. The debate in mathematics education, between constructivism and socio-cultural theory, continues currently between Les Steffe and Stephen Lerman in JRME (forthcoming).

My point in mentioning this debate is to emphasise the complexity of the theoretical arena in which learning is conceptualised, and the problematic nature of 'giving the nod' to both constructivist and socio-cultural positions as supporting a conceptualisation of learning. Ron speaks of "the social and psychological aspects of human beings' (inter)activities in their social-cultural-physical milieu", and "the negotiation of norms and practices, and of taken-as-shared meanings concerning [people's] (inter)activities in various communities of practice". He views learning as an adaptive process in which current conceptions are transformed via people "reflecting on relationships between activities and the effect of those activities". As he indicates, this is undoubtedly a *constructivist* position. However, it uses terminology from a number of paradigms (for example, 'communities of practice' from social practice theory (e.g. Lave and Wenger, 1991) and 'taken-as-shared meanings' from social interactionist theory, (e.g. Bauersfeld, 1988) to explain communication of knowledge.

There have been many attempts to explain, from a constructivist position, apparently 'correct' understandings by one person of the thoughts or actions of another, or apparently 'mutual' understandings between one or more people – what might be called 'common knowledge', or intersubjectivity. Les Steffe and colleagues' *second-order models* offer one persuasive example (e.g. Steffe, in press; another is Cobb , Wood and Yackel's focus on developing *taken-as-shared meanings.* in classroom interactivity (e.g., Wood et. al., 1993). It is nevertheless hard to explain intersubjectivity in constructivist terms. The communication and 'sharing' of knowledge is central to consideration of social norms within classrooms, yet, often, the problematic nature of such sharing, from a constructivist theoretical perspective, is not addressed.

Socio-cultural theory, on the other hand, is *based* on a position of understandings being in the community *first*, and individual learning being derivative of social practices, or even

embedded in those practices. Reflexivity could be interpreted, in Vygotskian terms, as the shifting between social and individual planes, with the social plane having pre-eminence. Where reflection is concerned, it seems unclear whether its transformative nature is a construct relating to the individual, or whether (and how) it leads to transformations within a community. From a sociocultural view, learning is an enculturative process in which learners grow into the intellectual life of the community in which they engage. What the community espouses and practices (for example, critical reflective practice in teaching), it is likely that participants in the community will develop as a result of participation (Lave and Wenger, 1991). However, if such practices are not central to the community, it is unlikely that they will be developed through a process of enculturation. In terms of the Vygotskian construct of Zone of Proximal Development, by engaging with a learner in the learning process, the more experienced other can interact to support learning within the ZPD. Whether the outcomes would be those desired is far from certain.

These various theoretical positions (treated very sketchily here) depend on epistemological assumptions regarding the nature and status of knowledge. It is dangerous to interpret constructive intersubjectivity, participative enculturation, or supported learning as promoting *desired* learning without examining critically such assumptions. Thus, in trying to *promote* learning, we find ourselves in ambiguous positions theoretically. It is perhaps not surprising therefore, that promoting *desired learning* in practice is so difficult, as Ron's paper shows.

Another issue concerns the nature of the community in the mathematics classroom, where negotiation of norms and practices may be central to achieving goals, for example goals of mathematical described by Wood et al.,(1993) where it was clear that a *changing* of norms was central to the success of their programme. When we set out to negotiate, or change, the social norms of the classroom community, we have to look beyond mathematics and the immediate institutional environment to the wider sociocultural influences on relationships within the classroom: whether the practices being nurtured are compatible with *societal* norms – issues of equity for example, and the implications of changes in the classroom power structure. Does reflection and negotiation include addressing these wider (and potentially stronger) influences? If so, is this an individual or a collective act; constructive or enculturative? How is it achieved and what are its consequences for participants? Ron sidesteps these issues by acknowledging what he calls the "non-academic issues", but deciding for this paper to leave them "in the background". I believe they are too central to all levels of development, including the academic, for this to be a real option, and this raises issues for teacher educators as well as theoreticians and researchers.

The Teacher-Educator's Dilemma

Teacher-educator Development

The main theme of Ron's paper is the drawing of a conceptual model of teacher-educator development based on deep reflections on, and critique of, his own development over a lengthy time period. Ron describes his learning through reflection on his own activities, starting with early experiences of peer tutoring leading to his own enhancement of mathematical understanding, and continuing through a critique of teaching practices leading

to adaptations to his teaching. The key notion here is that the reflective activity needs to relate to a learner's own experiential domain, so that the learner tackles and finds solutions to his own problems.

Ron has taken the questions from his own mentors (Steffe and Simon, for example) and, through working on those questions relative to his own previous knowledge and experience, has adapted his approaches to teaching, educating teachers and mentoring teacher educators. He points to the importance of cultural changes , of the need to reflect on his experiences, and of the mutual impact between his research and teaching leading to his own enhanced awarenesses of growth of knowledge in teaching and educating teachers. In the terms of Jaworski and Watson, 1993, Ron has an extremely well-developed 'inner mentor'.

The developmental path that Ron traces for himself resonates strongly with my own experiences of developing as a mathematics educator: for example, the recognition of key stages in one's own learning of mathematics; the realisation that, in trying to teach another person, the teacher's own mathematical knowledge grows; the value of reflecting on experiences in order to be more consciously aware, and critical, of aims and outcomes. I believe that most educators will be able to produce their own fragments of experience to speak to the stages of development described and conceptualised. It is quite clear that the reflective process so clearly articulated is a compelling way to characterise effective growth of knowledge both of mathematics and of the complexities of the learning and teaching of mathematics for the critically conscious[1] teacher educator.

Ron outlines a four-focus developmental model based on his own experiences. This model offers a clearly articulated and convincing conceptual account of the inter-related stages of teacher-educator development, taking in learning of mathematics, learning of teaching mathematics, learning of educating teachers of mathematics, and finally learning of educating mathematics teacher-educators. It is an account which characterises the development of an aware teacher educator. It also fits with other articulations in the same terrain (e.g. Jaworski, 1999). However, it is an account in which it is not clear how any student, teacher or educator comes to engage in this process. It is also not a model of how educators promote the engagement of others in the process. A critical question, given that the process is clear and convincing and well evidenced in the experience of many educators, is how can educators use this conceptualisation to promote the learning of others at any level within it? Or does it just *describe* a developmental process?

Educator centrism

In his clarity of articulation, what I found especially outstanding, and poignant, were Ron's references to occasions in which, despite his own evident growth and development, his teaching ambitions had not been realised in the learning he observed. For example, Ron says

> Often the struggle to figure out how to explain the matter brought about new insights *in me*. Sometimes my peers "got it" [something mathematical] and I experienced a feeling of accomplishment; other times they did not and I felt dull and disturbed.

[1] In using such terminology I draw on a substantial literature relating to critical reflective practice, most seminally represented for me in the works of Dewey, Kemmis, and Schōn. See for example Jaworski, 1994

> In working with the teachers, I had many insights into teaching my students, but rarely did the teachers indicate the intended understandings.
>
> When discussing those issues with Lori, I also realized that team teaching enhanced my understanding of teaching and helped her to experience activities of teaching but it did not seem to be as useful in terms of Lori's conceptualization of teacher education.

As Ron acknowledges, in many of his teaching interactions the chief learner is himself. What the other participant learns is not clear, but, often, it seems not to be what Ron would ideally like to be learned, or at least there is no evidence that it is. Such situations are clearly recognisable to me, and I could find many similar ones in my own experience. I suggest this might be true for other educators. How does one build from such recognition to a clearer understanding of the problematics involved? Research consisting of a systematic analysis of fragments (mentioned above) might be one way to achieve this.

In particular, such research lifts reflective activity to a more objective plane. As teacher educators individually reflect on their development and ask questions about its problematic aspects, it is often the "I" which impedes progress in perceiving the nature of the problems - a sort of educator centrism. A caricature of the situation is, '*I* have learned this, or achieved this – how can *I* promote this in others?' The focus on the "I" both illuminates and obscures the essential nature of the developmental exercise in which the "I" has participated.

'Promoting' the learning of others

Ron makes it very clear that his own learning relates strongly to his cycle of raising his own questions and setting his own goals as a result of reflecting on outcomes of his own research activity. This recognition leads to questions of how he, or other educators, might *promote* such learning in others. I want to argue for some dichotomy between these aspirations. The poignant nature of the quotations above lies in the frustration of the experienced, aware and knowledgeable educator in trying to promote "intended understandings" in others. This might be typified as "the teacher-educator's dilemma", and it relates closely to dilemmas of the teacher, recognised for example as the *teacher's dilemma* (Edwards & Mercer, 1987) and the *didactic tension* (Mason, 1988b). Perhaps a key phrase is "intended understandings".

When Ron was faced with Les Steffe's question "When interacting with children, how do you know that you met a scheme?" his learning was enhanced. He himself knows how this question impinged on his experience at that time, and the subsequent effect of it. Could Les Steffe have envisaged this? Did he see, in Ron, his own intended understandings? Was he, perhaps, disappointed because of Ron's apparent understandings did not coincide with what he had intended in posing the question? Were there parallels between his perceptions of Ron's awareness and Ron's subsequent perceptions of Lori's awareness? Like Ron, Lori can be seen to have a personal learning trajectory, one part of which relates to her team teaching with Ron. Her learning relative to the team teaching is also relative to other parts of her experience in communities of practice to which, perhaps, Ron has little access. According to Ron's model of development, based on critical reflective practice, each person promotes

their own development relative to the issues they address. What is problematic is for some other person to try to direct that development.

A constructivist view is that the only person who can 'know' what has been learned – through reflection on experience – is one's self. Through attempts at reaching intersubjectivity, another person can gain insights to this learning, but can never be sure what has been learned. More importantly, the other person can only try to approximate to the desired learning, based on intersubjective experience. These issues are subject to the all the theoretical problems mentioned earlier.

Towards resolving the dichotomy

Ron's team teaching with Lori seems to have been especially powerful in Ron's learning. Here he was working *with* Lori, rather than (or as well as?) trying to promote her development. Of course we know nothing from Lori about this development, only Ron's perception of it. Lori's development can only be *her* development. It can never be Ron's. Also Ron cannot be expected to have insights into the cultural origins of Lori's experiential development. However, as two people conduct classes together, and work together on ideas and issues, their intersubjective understanding would grow. This would allow critical questions to be recognised and addressed jointly, albeit from different experiences and developmental levels. A form of co-learning, or co-mentoring would result, which would support the inner mentor of each of them. Thus reflexivity is interpreted as acting between the social plane and the individual plane for the individuals in a co-learning partnership. The idea of such a partnership is based on a typology of researcher-practitioner co-operation offered by Jon Wagner who writes:

> In a co-learning agreement, researchers and practitioners are both participants in processes of education and systems of schooling. Both are engaged in action and reflection. By working together, each might learn something about the world of the other. Of equal importance, however, each may learn something more about his or her own world and its connections to institutions and schooling. (Wagner, 1997)

I have proposed that Wagner's concept of co-learning agreements might be extended to work between teachers and educators in an enterprise of mutual development (Jaworski, 1999b). For any participant in the developmental enterprise this involves growth of the individual in practice through reflexivity between co-mentoring and the inner mentor. A co-learning agreement requires both participants to reflect on activities, jointly and separately, and to critique outcomes of activity. The aim of a co-learning agreement is to create learning situations in which all participants can be reflective partners, and through which ideas can be offered and issues and tensions addressed overtly. Such a partnership would remove the necessity for one person to 'promote' learning in others. Partners take responsibility for their own learning and development, but levels of mutual responsibility within the partnership offer and support development of ideas. There is no intention that partners should be equal, but more responsibility rests with the more experienced partners to ensure minimisation of power differentials and ease collaborative communication and critical reflection.

Of course there are many issues to address, not least the epistemological groundings of such a theoretical perspective, and the nature of relationships in an evolving praxis. These are addressed to some extent in Jaworski 1999a and b, but further research and more rigorous attention to theory is still required.

In Conclusion

In my response I have valued greatly Ron's articulate account of his own development, and his consequent four-focus model to describe development in the teaching and education process more generally. I have recognised what might be seen as a dichotomy in the aspirations of educators in valuing development through self-reflective analysis, while seeking simultaneously to promote particular development in others. In making a critique I have touched on the methodology of using fragments as an analytical tool, and the theoretical commensurability of constructivism with versions of sociocultural theory. It must be recognised the enormous issues in epistemology, cognition and pedagogy which are raised when we start to offer conceptualisations for teacher-educator development. Ron's four-focus model offers an important starting point in addressing these issues.

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RESEARCH FORUM

Theme 3:
Visual Thinking in Mathematics Education

Coordinator:	Norma Presmeg
Presentation 1:	*Visualization as a Vehicle for Meaningful Problem Solving in Algebra* Michal Yerushalmy, Beba Shternberg & Shoshana Gilead
Reactor:	Bernard Parzysz
Presentation 2:	*The Role of Visualization in Young Students' Learning* Kay Owens
Reactor:	Eddie Gray

Visualization as a Vehicle for Meaningful Problem Solving in Algebra

Michal Yerushalmy, Beba Shternberg, Shoshana Gilead

University of Haifa and Centre for Educational Technology - Tel-Aviv.

Our experience with a new algebra curriculum has challenged us to identify both strengths and complexities of the variation function approach we take. Among a few noticeable effects of the approach, we learned that functions form an important basis for solving algebra word problems. The approach described here maintains that graphing is not necessarily an outcome of measurement and plotting points. This approach is supported by the use of a software environment. Already in the very first stages of the course, a natural language text turns into a script of events and processes, and then into qualitative graphs sketched using iconic notations; then it turns into a subject for qualitative analysis of the rate of change. This sequence forms a visual basis for modeling. In this article we discuss the relevancy of this approach to problem solving in algebra. We also do an exhaustive analysis of the major types of algebra word problems in order to point out the structural differences between various algebra rate problems and the possible interactions between the visual mental models and the symbolic equations involved in the solution process.

The use of various representations of functions offers new ways for supporting strategies of solving algebra problems (Chazan 1993, Heid 1995, Yerushalmy 1997). Most attempts offer the use of multiple representations as a means of <u>bridging and deepening connections</u> between areas assumed to be separated (e.g., symbolic manipulations in algebra and knowledge of functions). Other proposals call for democratizing the access to calculus using MBL and simulations that can help describe complex phenomena (Kaput 1994)<u>without relying on formal algebraic symbolization</u>. As a result of studying our students who were using the "*Visual Mathematics*" curricular materials and software that offer syntax and semantics for qualitative visual description of processes[1], we became interested in the link between the two; We wondered how we could use modeling attempts based on the concept of variation and on visual descriptions and analysis of rates, that naturally lead towards symbolization in calculus (e.g. simple differential equations

[1] "*Visual Mathematics*" is an algebra curriculum focused on the concept of function and organized around major ideas and representations rather than technical manipulations (Centre for Educational Technology CET 1995).

such as $f'(x)=Af(x)$), in order to introduce and establish an understanding of algebra.

To demonstrate a step within this exploration, we will look first at strategies and representations used by calculus students and 7^{th} graders solving the same non standard modeling problem. Both groups were equipped with tools for solving the given problem, which requires a comparison of rates of two processes: the calculus students were familiar with derivatives and algebra, and the 7^{th} graders have learned how to describe and analyze models using a basic set of iconic symbols. The analysis of these solution attempts offers some ideas about how to treat this gap between graphing and meaningful algebraic symbolization. From the non traditional modeling we will move on to look at the possible effects of the visual foundation on studying traditional algebra word problems. We will start by looking at attempts of solving a non trivial algebra problem prior to any teaching intervention; continue by identifying problems that were harder for students to solve, and finally suggest exhaustive classification of all linear-models algebra problems by the deep structure of their graphical models.

Part I: Coping with the Ovens problem

The Ovens problem requires skills that are not usually taught in algebra; it deals qualitatively with a phenomenon related to the lows of heat. It does not provide numerical data. Therefore, students who are not familiar with the physics involved nor provided with a simulation tool, can hardly benefit from the use of graphing technology while solving this problem. Thus, this was an appropriate activity for studying modeling and solving skills of both symbolically mature and immature students.

> The Ovens problem: A cook has to cook a large lamb as quickly as possible. The meat is at room temperature. The cook has a conventional oven and a microwave oven. In a cooking test it was found that during the cooking time, the temperature of the lamb in the conventional oven is always higher than the temperature in the microwave and that the cooking time required in both ovens is exactly 2 hours. In the microwave oven, the heat of the lamb increases at a constant rate, and in the conventional oven kept in constant temperature, the heat of the lamb increases at a changing rate. Could the cook use the two ovens to reduce the two-hours cooking time ? (adapted from Taylor, 1992, p. 20)

A symbolical approach

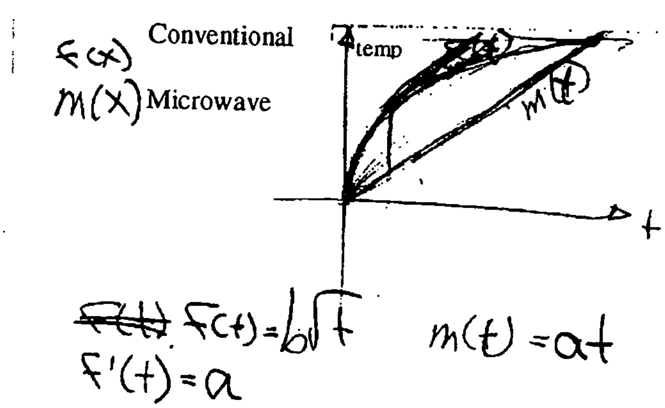
In order to learn about the contribution of formal symbolic knowledge to the construction of models using rate of change, we interviewed a group of 11^{th} or 12^{th} graders. The students took algebra, pre calculus and calculus courses. However, none of their courses was centered on modeling or realistic situations. We will demonstrate typical attempts taken throughout the interviews.

Ella started her solution by looking for specific information about the processes:

Ella: I think that some givens are missing...
Interviewer: Which one do you miss?
Ella: I think I need the temperatures...
Interviewer: So you'd like to have the measurements from the experiment?
Ella: Yes!
Interviewer: Why don't you make them up yourself?...

It was not easy for Ella to give examples of numbers which could fit the processes. She used constant differences to represent the constant rate condition for the microwave sequence. But she could not form any model of the collection of numbers she fabricated to represent the conventional oven. Nothing looked as mathematics to her in this sequence, and as a result she gave up trying to solve the problem. Ella was one of many students (12 out of 34 interviewees) who tried but failed to express their intuitions symbolically or in any other general way.

Yoni, on the other hand, immediately started by graphing and assigning symbolic descriptions:
Let's say that m(x) is the microwave and f(x) is the conventional oven.
He sketched the graphs and inferred that the cook should start by using the conventional oven:



.... it should stay there to a certain point, something like that (he pointed on the graph) and then... Intuitively, I would say that it should be at a point where the slope here (conventional oven) will be equal to the slope here (microwave), since from this point onward, the first slope begins to be smaller than the second one.... m(t) = at, f(t).... it doesn't matter for a moment what it is. We have to find the derivatives of m(t) and f(t).

Still, Yoni did not consider his answer to form a solution and continued his efforts:

I have to find the value of the t...

Here Yoni began to look for an expression that would fit the description of the curve. He offered the square root function but commented with great disappointment that there is no way to know:

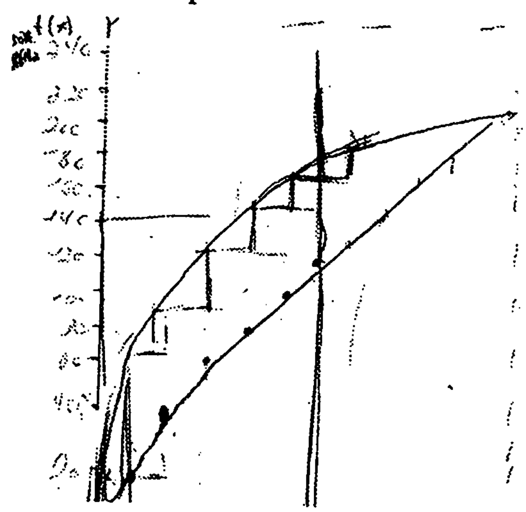
I have a problem with curves which aren't parabolas and here I am not sure that it is a parabola...

So how do high achiever calculus students deal with a modeling problem that does not provide information to form an algebraic equation? Yoni belongs to the group (20 out of 34 students) that made an attempt to use known symbolic conventions of functions to formulate a model. Yoni's intention to provide a parametric description rather than a specific expression was exceptional. Only 6 interviewees out of the 34 reached a solution for the Ovens problem. Yoni, who was one of them, was unhappy with his answer. His disappointment stemmed from his inability to use rate as a tool for explaining the behavior of a curve (for Yoni, any non linear curve is a parabola) and was probably linked to the traditional sequence of an algebra course where the visual situation models are being established as an outcome of an algebraic equation rather than being a basic tool for description and analysis.

A qualitative approach

While technology usually serves for plotting data or performing computations symbolically (Heid 1995), we were interested in using technology to support the two qualitative stages: the <u>recognition</u> of typical graphical models (as demonstrated in curricular materials of the Netherlands and the UK [Swan 1982]), and the visual non-quantitative <u>analysis methods</u>. To do this, we developed a software environment, "The Algebra Sketchbook" (Yerushalmy , Shternberg 1992/4). We were interested in understanding how this environment supports evolution of visual and linguistic modeling capabilities and how these capabilities support the solution of problems of the Ovens problem type that the students were unfamiliar with. The interview of 16 7th grade pairs took

place right after the completion of this qualitative modeling unit. The students were intuitively using symbols as generalized numbers but did not yet deal formally with manipulating or solving any algebraic relations. The following is an attempt (made by Erez and Ilanit) to solve the problem by offering a graphical method to evaluate the rates of change of each of the processes.



The graph above that these students sketched represents the general approach of students in this sample: they made a sketch, analyzed the two rates, and compared the stairs size of the two processes as a way of deciding if and how to use the two ovens. The modeling capability demonstrated by 7th grade students was not only convincing; we found, again and again, that it was a strong mathematical foundation which they were able to build their further knowledge.

Part II: Visualizing traditional algebra word problems

For decades, algebra students encountered word problem of the following kind :

> Problem 1: A biker left Haifa for Tel-Aviv riding at an average speed of 10 km/h. His friend left Haifa 4 hours later, took the same road to Tel-Aviv, and rode at an average speed of 20 km/h. How long after the first biker left Haifa did they meet?

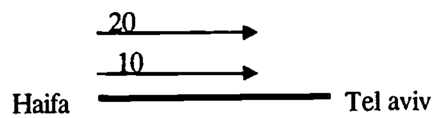
The expectation has always been that students would assign symbols to variables such as distance or speed, arrange them in an algebraic relation, and solve the equation to discover the answer to the problem.

Organizing table for Problem 1

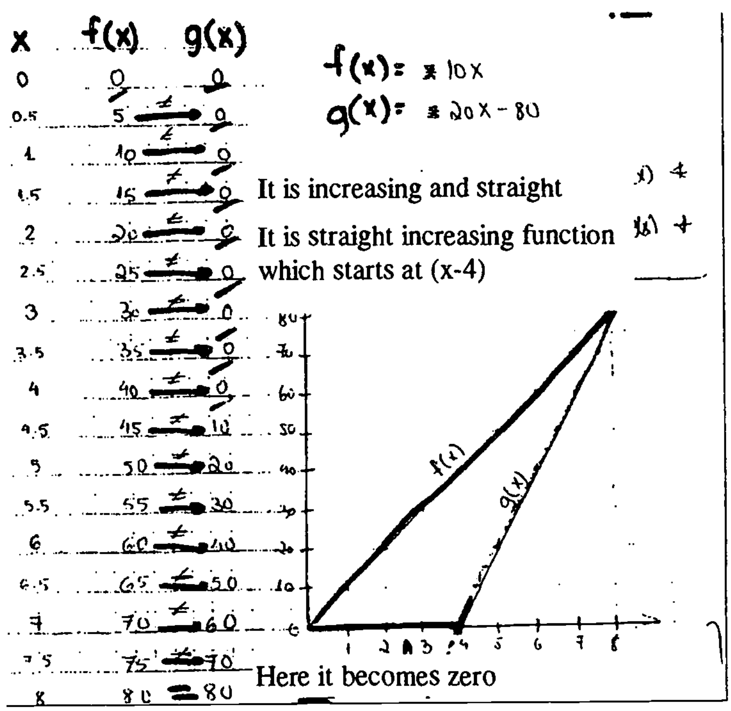
	time	speed	distance
Biker	t	10	$10t$
friend	t-4	20	$20(t$-4)

$10t = 20(t-4)$

Traveling path in Problem 1



This translation from situation to symbols proved to be difficult for many students. Previous studies suggest that the re-representation of a situation by a symbolic expression requires the evolution of what is called "*the situational model*" and the use of graphs, tables, and diagrams of various sorts. With the function approach to algebra, this translation process takes a different direction. The following is a solution of the problem offered by Tal, an 8th grader of the "*Visual Mathematics*" curriculum prior to any formal instruction of word problems:



Tal was mostly interested in the connections between the formal functions expressions and the situation structure. At first, Tal described the functions properties. Then, he found the value of the intersection point. However, this numerical information from the graph and the table did not coincide with the information retrieved from the functions expression: "At $x = 2$ the function *g(x)* assumes the value -40, but a negative distance does not exist, so there the function is zero but I still use the expression $g(x) = 20x - 80$ since I don't know how to find a better procedure". The negative values, which resulted from substitution in the

function expression, were inconsistent with his perception of the situation. But at that time, Tal did not know how to write symbolic descriptions of functions in multi intervals, and therefor gave up. A way to settle this conflict, one that we did not take at that time, was to follow Tal's thought about the meaning of the "negative" distance. One could do that by considering another situation having the same symbolic description and the same solution . For example, if the two bikers would have left at the same time from two locations 80 kms apart, then Tal's conflict would be set. Thus, a problem that we assumed to be a routine exercise turned to be a rich modeling activity for students who seek meaning in connecting the situation and its representations and are equipped to do so.

Do graphs always simplify problem solving?

Here is an example of another motion problem taken from the same set of practice problems as problem 1:

> Problem 2: A biker traveled from town1 to at an average speed of 10 km/h. Arriving at town2, she immediately turned back and traveled from town2 to town1 at an average speed of 20 km/h. The return trip was 4 hour shorter than the outward trip. How long took the trip in each direction?

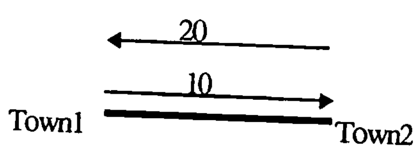
In the traditional approach this problem is almost identical to problem 1.

Organizing table for Problem 2

	time	speed	distance
out	t	10	$10t$
back	t-4	20	$20\,(t\text{-}4)$

$$10t = 20(t-4)$$

Traveling path in Problem 2



In the function approach, there are noticeable differences between the two problems.

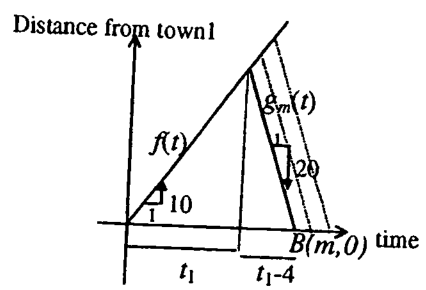
$f(t) = 10t$

$g(t) = -20(t-m)$

$f(t) \overset{?}{=} g(t)$

$10t \overset{?}{=} 20(t-m)$

Problem2



The distance in the second problem cannot be graphed as a function of time but must be sketched. Distance can be evaluated only for the first

segment of the round trip, and the return point cannot be numerically determined by a comparison of the two functions. The function rule g is dependent on a parameter m to specify the x-coordinate of point B, and it represents a family of parallel lines with a slope of -20. The equation $10t = -20(t - m)$ represents a comparison between the function f and a family of functions g_m. Only one member of the family, for which $m = t_1 + (t_1 - 4)$, satisfies all the constraints defined by the given story problem. The solution requires substitutions that transform the original equation, that describes processes as a function of t, into a function of m. This function does not describe the situation structure as initially graphed by the distance-time graph.

Thus, two similar problems pose different challenges regarding the construction of symbolic semantics and the possible information that can be evaluated from the visual model. What might be the source of the different levels of complexity encountered while formulating functional expressions? Since the problems have the same givens and constraints (see table below), one may suggest the difference can probably be explained by their different situational context.

The givens in problem 1	The givens in problem 2
v_1= 10 km/hour, v_2= 20 km/hour	v_1= 10 km/hour, v_2= 20 km/hour
$d_1 = d_2$ (same distance until they met)	$d_1 = d_2$ (same distance in both directions)
$t_1 - t_2$= 4 hours	$t_1 - t_2$= 4 hours

To contradict this assumption we will re-examine problem 2, but this time we will compare it to problem 3. Problem 3 describes a similar situational context as problem 2 but defines a different time constraint.

> Problem 3: A biker traveled from town1 to town2 at an average speed of 10 km/hour. Arriving at town2, she immediately turned back and traveled from town2 to town1 at an average speed of 20 km/h. The whole trip took 7 hours. How long was the trip in each direction?

The givens in problem 2	The givens in problem 3
v_1= 10 km/hour, v_2= 20 km/hour	v_1= 10 km/hour, v_2= 20 km/hour
$d_1 = d_2$ (same distance in both directions)	$d_1 = d_2$ (same distance in both directions)
$t_1 - t_2$= 4 hours	$t_1 + t_2$= 7 hours

Problem 3 in contrast to problem 2 can be graphed (using a point and a slope for each of the two functions) and the expressions, the equation

$10t=20(7-t)$ (or its equivalent equation $10t=140-20t$) and the solution values can be directly related to the visual situatuational model.

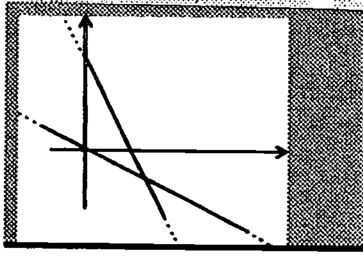
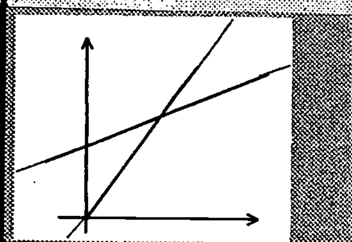
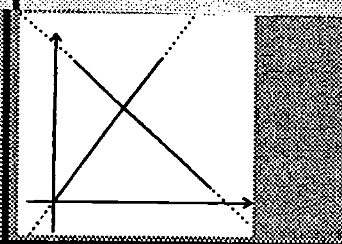
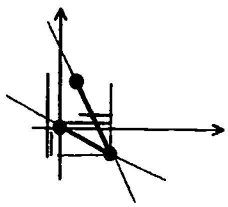
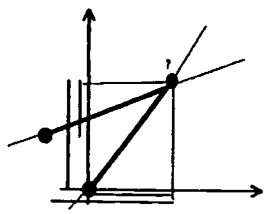
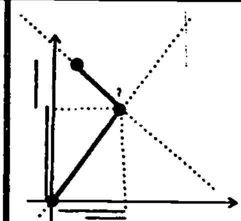
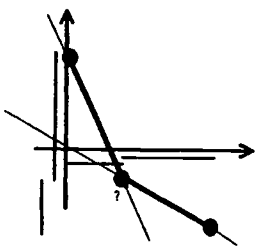
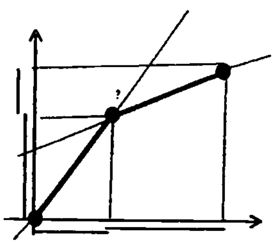
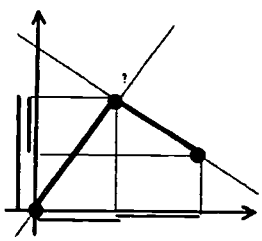
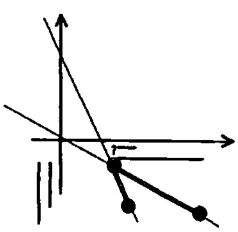
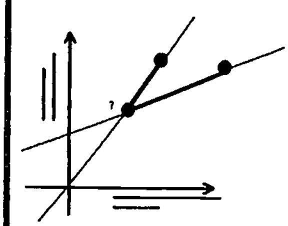
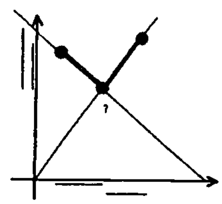
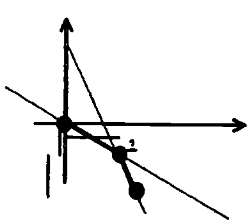
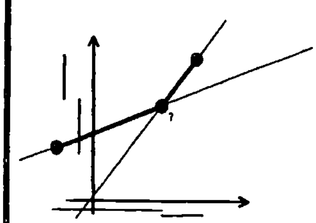
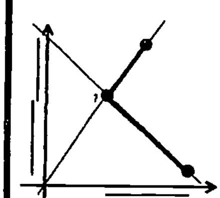
Use of visual models to classify algebra word problems

The different challenge each of problems 1 ,2 and 3 pose, suggests that neither the situational context (which problems 2 and 3 share) nor the set of givens and constraints (which problems 1 and 2 share) can each of them alone explain the differences between problems when taking the function solution approach. We conjecture that the interaction between the two is the crucial component and therefore can serve to analyze the deep structure of most algebra word problems.. To systematically test our conjecture, we have chosen terminology drawn from cognitive research on arithmetic and algebraic word problems. Problem structure can be examined at two levels of abstraction: the *quantitative structure* describes arithmetic operations and relations among symbolic or numerical entities, and the *situational structure* describes relations among physical properties of the entities within a story problem. Both the quantitative and the situational structure of a problem can be described in different forms and representations. (Nesher and Hershkovitz 1994, Nathan, Kintch and Young 1992, Hall 1989). In this work, we describe the quantitative structure of constant-rate problems by the number and type of given quantities, the number and type of unknowns, and the constraints formed by arithmetic operations between pairs of unknowns (e.g. $t_1+t_2=5$ or $t_1-t_2=5$ or $t_1/t_2=5$). We describe the situational structure of constant-rate problems through sketches of linear functions that describe the change of the output (e.g., signed distance) in time where the rate is described by the slope of the line.

At this stage of the study, our main goal was to examine different structures of "algebra rate problems" as an outcome of all the possible interactions between the situational structure and the quantitative structure. The problems in this study consist of two processes that for some input share the same output, and the question in the problem asks about the value of this shared input, its shared output or both.

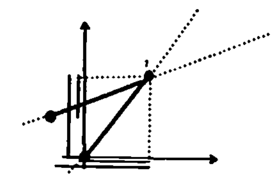
Defining a space of the situational structures: The basic elements of the functional language of rate problems in our work are two intersecting linear functions from R to R, the intersection point, and two other points (one on each line). There are three different possible combinations of the two intersecting lines (see table below): 1. slopes of negative signs (left column), 2. slopes of positive signs (middle column), or 3. slopes of positive and negative signs (right column). For any of the three combinations there are 4 different ways to create the set of the two

points on each line. The graphic representation provides a visual global perspective on the space of algebra temporal-constant rate problems.

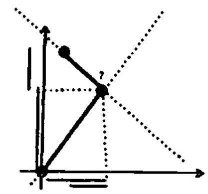
Two negative slopes	Two positive slopes	Two opposite slopes
		
 Shared domain – Shared range	 Shared domain – Shared range	 Shared domain- Adjacent range
 Adjacent domain - Adjacent range	 Adjacent domain- Adjacent range	 Adjacent domain- Shared range
 Shared domain – Shared range	 Shared domain – Shared range	 Adjacent domain- Shared range
 Adjacent domain- Adjacent range	 Adjacent domain- Adjacent range	 Shared domain- Adjacent range

Looking at the twelve situational structures in the table, four different types of relations between and within domain and range can be observed:

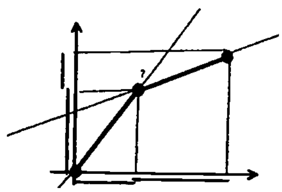
1. shared domain, shared range, 2. adjacent domain, adjacent range (both obtained by two lines of the same sign slope), 3. shared domain, adjacent range and 4. adjacent domain, shared range (both obtained by two lines of opposite sign slope). This organization allows the mapping of most algebra word problems onto one of the four types of situational structures.



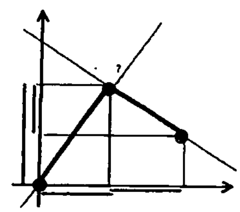
Shared domain – Shared range



Shared domain- Adjacent range



Adjacent domain- Adjacent range



Adjacent domain- Shared range

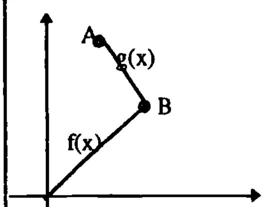
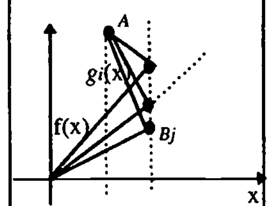
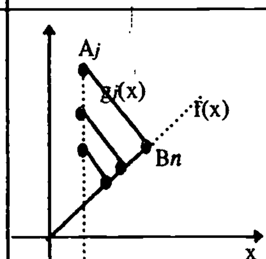
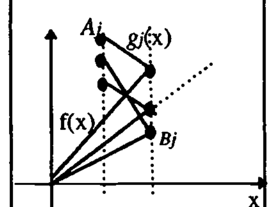
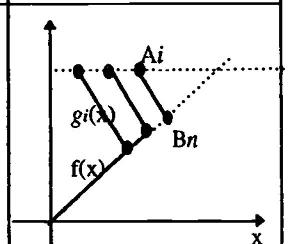
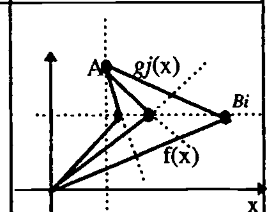
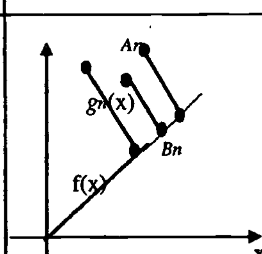
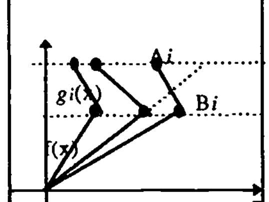
Definition of the space of the quantitative structures: Each of the situation problems classified above can also be determined by the following quantitative elements and relations:

1. Each problem deals with two triads of quantities, d, v, t, connected by a multiplicative constraint $d = v \bullet t$. Thus, there are exactly six elements involved in each problem.[2]
2. Only magnitudes of quantities are dealt with.

The problems which are normally called "algebra" problems describe additional two constraints, each of them linking two quantities of the same type by a binary operation (e.g., $t_1 + t_2 = c_1$ $v_1 + v_2 = c_2$, "One left 5 hours before the other", "His speed is 4 times larger", " They started at the same station and reached the same station"). There are always two known quantities and the unknowns are of a third quantity type (e.g. for given constraints about t and v, d is unknown). All possible combinations of two givens and two constraints produce a space of 24 different quantitative structures. The 4 situaitonal structure types and the 24 quantitative types produce 96 types of algebra problems. Our next step was to investigate the 96 interactions in order.to reduce the number of different classes. To do so,

[2] The quantities referred to here, time, distance and speed, may be replaced by other quantities for non temporal situations or for work problems.

we considered again the differences between problems 2 and 3. The structure of the given situation and quantities in problem 2 allow to draw a unique graph. The structure of problem 3 allows to graph a family of possible graphs; the family is created by the parameter *m* which provides "freedom" to shift function *g* horizontally. Here is a demonstration of one of the four situation structures:

Information about slopes and points	Constraints and "freedom" of the situation graph	Information about slopes and points	Constraints and "freedom" of the situation graph
Class 1 v1(✓) v2(✓) B(? ,?) A(✓,✓)		**Class5** v1(?) v2(?) B(✓,?) A(✓,✓)	
Class 2 v1(✓) v2(✓) B(? ,?) A(✓,?)		**Class 6** v1(?) v2(?) B(✓,?) A(✓,?)	
Class 3 v1(✓) v2(✓) B(? ,?) A(?,✓)		**Class7** v1(?) v2(?) B(?,✓) A(✓,✓)	
Class 4 v1(✓) v2(✓) B(? ,?) A(?,?)		**Class 8** v1(?) v2(?) B(?,✓) A(?,✓)	

Analysis of all 96 structures according to "types and degrees of freedom" reduced the number of different classes to 32; eight types of "freedom" are repeated in each of the 4 situaitonal structures.

Part III: Directions for further research

Using images of rate to construct symbolic models

A covariational approach in analyzing a problem situation has been already suggested as more powerful than a correspondence approach (Confrey & Smith, 1994, Nemirovsky 1996). Even young children use rate of change as a way to explore functional relations (Nemirovsky and Rubin, 1991). A central question in our studies is whether the consideration of rate of change is a vehicle towards formulating models in algebra and calculus and whether the description of phenomena by their rate of change (rather than by dealing with values of specific events) can support symbolic understanding. We found that consideration of rate of change as a vehicle towards formulating and explaining symbolic models is absent in traditional algebra and calculus strategies. We conjecture that this absence could be a major obstacle of the 'modeling based' algebra curriculum designed to motivate the understanding of the mutual relations between phenomena and it symbolic models. It is even less clear how the qualitative visual covariation description of phenomena (and not of numerical values of discrete events) can support symbolic understanding or even create symbolic awareness.

Mathematics offers methods allowing the construction of symbolic expressions from given data; here is an example: Imagine a "stairs" graph that models a motion of a car that during an hour constantly accelerates from 50 to 60 miles/hour (adapted from Thompson 1994). The visual image of the rate can be translated and manipulated by the following procedure:

$$a(t) = 10\ mi/hr^2 \qquad\qquad v(t) = 50 + 10{*}t\ mi/hr$$

$$d(t) = 50 + (50+10) + (50+10+10) + \ldots + (50 + 10{*}t) =$$

$$= 50{*}t + (10 + 20 + 30 + \ldots + 10{*}(t-1)) = 50{*}t + \frac{10{*}(t-1)}{2}{*}t\ mi$$

In order to reach a symbolic model that describes the distance traveled at any instant along this hour, one should employ calculus methods:

$$a(t) = 10\ mi/hr^2 \qquad\qquad v(t) = 50 + \int_0^t 10du = 50 + 10t\ mi/hr$$

$$d(t) = \int_0^t v(u)du = \int_0^t (50 + 10u)du = 50t + \frac{10t^2}{2}\ mi$$

Although the first procedure yields information only about instants and suffers from inaccuracy, the sum expression is to a certain extent similar to the expression resulting from the continuous model computation.

The visual based covariation approach is successful in supporting complex non quantitative modeling. It also supports moving from graphs to symbols in the case of linear expressions. But for other algebra activities that involve quadratic (for example the acceleration -distance model described above) or exponential models it is yet unclear how to support the transition from graphs to the construction of symbolic expressions. Although the mathematics of sequences can be considered as a possible pedagogical bridge, it might be a too difficult task for algebra beginners; it cannot be easily generalized along various models and even more important, it must be accompanied by an intervention that would justify the swap between discrete and continuous views of the process. Acknowledging the benefits and the difficulties, we are seeking ways to study further the visual foundation for supporting symbolization.

The impact of the classification

Data obtained from problem solving studies indicate that novices tend to classify problems according to the surface structure of the problem and experts according to the deep structure. (Schoenfeld 1985). The analysis begun here identifies the deep structures of algebra rate problems. Clearly, other dimensions of the problem structure should be further explored by means of an exhaustive analysis of algebra word problem. Our underlying assumption is that explicit representation of the deep structure of a problem facilitates student's ability to recognize the properties of the problem and choose appropriate strategies. We propose that once problem structures are accurately defined, a tool of significant predictive power would be formed that could be examined empirically. The pedagogical implications of such an analysis will help us become more sensitive to the sequence of instruction and the types of strategies that should be adopted, and will also allow better understanding of the difficulties encountered by algebra students.

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Visualization and Modeling in Problem Solving: From Algebra to Geometry and Back

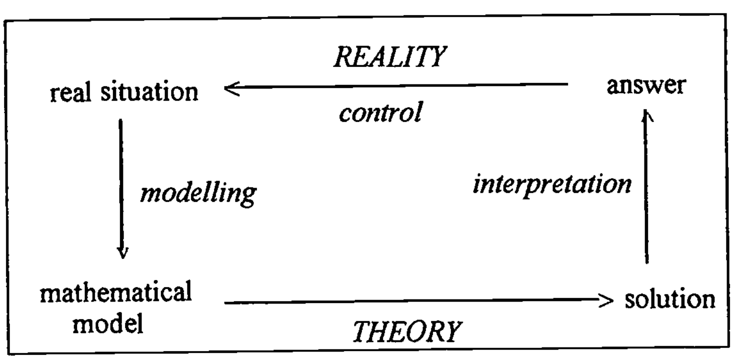
A Response to Michal Yerushalmy, Beba Shternberg, and Shoshana Gilead's paper: *Visualization as a Vehicle for Meaningful Problem Solving in Algebra*

Bernard Parzysz
Equipe DIDIREM, Université Paris-7

When I was asked to react on M. Yerushalmy, B. Shternberg and S. Gilead's paper *Visualization as a vehicle for meaningful problem solving in algebra*, I was rather puzzled since I am not a specialist in algebra problem solving, my area being mostly geometry. But, as I came to reading it, I was brought to link some ideas developed in the paper with some I had met in my own field of research. I wrote them down as they appeared, as well as some questions, and then tried to organize the whole of them. So, you will not be surprised to see some references to geometrical situations.

1- The Modeling Process

The first question, which came to me, was: *What is a realistic situation?* I know that a teaching of mathematics based on 'realistic situations' has been developed recently in some countries [Hershkowitz *et al.*, 1996]. I believe it a sound idea, training students to start from a problem they might be confronted with in their everyday life, and undertaking to solve it with the help of a mathematical model that they elaborate themselves, or with one that they already know and think well fitted to the problem. This process -well known- can be described by the following diagram:



It appears clearly in 'double' statistics (regression) problems. This came to my mind from Yoni's behaviour in the 'ovens problem': he started by graphing; then, basing his choice on his graph, he got to a mathematical model (functions), looked 'for an expression that would fit the description of the curve' and finally chose a particular type (square root). Even if he could not go any further, he had undertaken a genuine mathematical process. On the contrary, Ella and other students were not able to

proceed because they clung to the wording of the problem (written in terms of changing rate) and were unable to associate it with a mathematical model.

2- Real or Realistic?

Let us go back to the above diagram. The starting point is a 'real' situation, that is to say a situation as it presents itself (or may present itself) to somebody in 'real' life. But the authors are in fact talking of 'realistic' situations, and we can at first take this term as meaning 'looking real'. It is the case with the 'ovens problem' as well as with the 'bikers problem'. For instance, in the microwave oven 'the heat of the lamb increases at a constant rate'. In fact, there is little chance of this actually happening: there would probably be some variations of the rate, due to various factors. But 1° it is impossible to determine these variations accurately and 2° it can be assumed *grosso modo* that they will be of small range. Hence, at the cost of a small 'twist' of the real facts, we shall suppose that 'the heat of the lamb increases at a constant rate'. This situation is 'realistic', although not real; one can say that it is a 'pseudo concrete' situation. It does not belong to a mathematical model, but it is does not belong to reality either; it belongs in fact to a kind of 'idealized reality', in Plato's sense:

REALITY		*THEORY*		
real situation	——>	pseudo-concrete model ('realistic' situation)	——>	mathematical model
	idealization		*mathematization*	

This 'idealized reality' is in fact a model, in the following sense: '*A model is an abstract, simplified and idealized interpretation of an object of the real world, or of a system of relations, or of an evolutive process, stemming from a description of reality*' [Henry 1997 p. 78, tr. B.P.]; hence, it belongs to theory. The difference between a 'pseudo-concrete' model and a mathematical model being that the latter is clearly included in a mathematical theory.

Many 'word problems' are of the 'pseudo-concrete' type, a type which can also be found in geometry and probability. Here are two examples:

1- *A box contains 5 red balls and 3 white ones, indiscernible by touch.*
Three balls are drawn from this box simultaneously, at random.
What is the probability for getting at least two red balls?

2- *A wooden stick is broken into three pieces.*
Is it possible to make a triangle from these pieces?

In the first problem, the situation is already idealized: the balls are 'indiscernible by touch' and 'drawn at random'; in fact, the function of such features is to indicate which mathematical model is to be used to solve the problem. In the second problem, the situation is described as a 'real' one ('wooden stick'), and the student has to imagine the pseudo-concrete model by him/herself, that is a 'straight' stick with 'no thickness'.

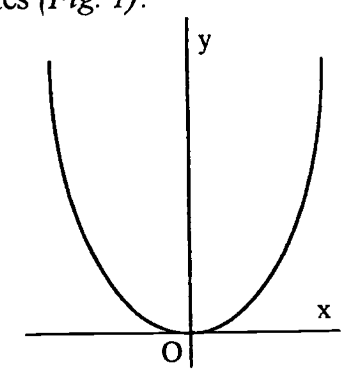
Now, if we look from this point of view at the two problems dealt with in the paper, we can say that the 'ovens problem' is a pseudo-concrete model of a real

situation, both from the qualitative data (changing rates) and the numerical datum (which, anyway, is of no use apart from giving the problem a 'realistic' touch). What's more, the data have already been selected; for instance, nothing is said of what 'cooking a lamb' consists of: is it reaching a given temperature? or being kept at a given temperature during a given time? As this piece of information is not useful, it is not mentioned. Nevertheless the authors choose not to give sufficient information to solve the problem completely; the wording says that in the conventional oven 'the heat of the lamb increases at a changing rate', but nothing is said about how it changes. So the mathematical model is to be built by the student him/herself. This explains, to some extent, the difficulty which the students were confronted with: they usually have nothing to choose in a mathematics problem (cf Yoni's disappointment); and in fact, when it is the case, they have the impression that some data are missing, and consequently ask their teacher for them (cf Ella).

In the 'bikers problem' the given situation is somewhat idealized (by the choice of the numerical data), but, as it is worded, it cannot be solved, since knowing the average speed of a biker is not enough to determine his position at any time. As a result, in order to solve the problem you have to make an extra hypothesis *(model hypothesis)*, the simpler one supposing that the speed of the bikers is constant during the whole ride. Whatever the hypothesis you choose, it is only at that stage that you get to a pseudo-concrete model.

3- Visualization and Modeling

Visualization can play an essential role in the modeling process, and especially by helping to choose an appropriate model. This is the case for Yoni, who refers the curve he has drawn to a parabola, and hence to the square root function. In that respect, it seems important to provide students with what can be called a 'herbarium' of functions, *i.e.* a collection of visual images of curves, each one of these being associated with a set of functions and a set of properties such as symmetry, tangents, asymptotes *(Fig. 1)*.



$y = kx^n \quad k \in R \quad n \in N$

$k > 0 \quad n$ *even*

$x = 0$ *tangent at point O*

$y = 0$ *axis of symmetry*

$y = 0$ *asymptotic direction*

Fig. 1

In such a 'herbarium' the connection *curve × functions × properties* has to be precise in order to be useful; the mental image of a curve, though essential (it acts as a starting point enabling one to choose a possible model) is not sufficient by itself, and neither is the symbolic relation. Fischbein's theory of figural concept seems quite

suitable to take this connection into account [Fischbein 1993]. The process is quite similar in geometry problem solving; here, the diagram constructed from the wording may help to find a 'key figure', *i.e.* a subdiagram related to a known and referenced situation, as in the following problem *(Fig. 2)*.

ABCD is a parallelogram.
E is the middle of [AB].
F is the middle of [CD].
(DE) intersects (AC) at M.
(BF) intersects (AC) at N.
Demonstrate that AM = MN = NC.

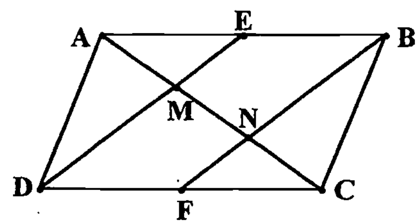


Fig. 2

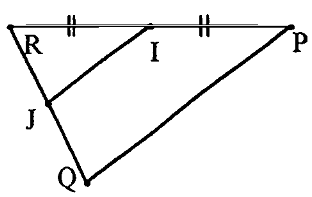


Fig. 3

The key of this classical problem lies in the spotting of the subdiagram shown on *Fig. 3*, which appears twice on Fig. 2. Since (IJ) is parallel to (PQ), one can assert that J is the middle of [RQ].

Once these two subdiagrams are identified, the solution becomes simple, <u>on the condition that one can remember the associated property</u>.

4- How to Solve It?

Tal's solution of the 'bikers problem', as it is described in the paper, appears to be most interesting, for two main reasons. The first one is that it illustrates, in a brilliant way, the control process which intervenes in modeling: having found that the distance, as a function of time, can be expressed by $g(x) = 20x - 80$, Tal undertakes to check if this makes sense, by testing some values of x. This is worth to be noted, for such a behaviour is not common among students, for whom 'mathematics is not reality', and hence it does not matter whether the result is plausible or not [Girard & Parzysz 1998]. However, such a behaviour as Tal's should be developed among students, since it is an essential component of the modeling process: indeed, what use can a model be if it does not fit with the real situation it is supposed to represent ? By the way, we can here get an insight into Tal's conception of a function (a most frequent one), which is mainly that of a functional relation between two variables; and a function defined by $g(x) = 0$ if $0 \leq x \leq 4$ and $g(x) = 20x - 80$ if $4 \leq x \leq 8$ is perceived as a juxtaposition of <u>two</u> functions, not as a single function. This narrow conception prevents Tal from overcoming his problem.

The second interest of Tal's behaviour is that he is able to establish relations between several 'frames' [Douady, 1986] and 'representation registers' [Duval, 1995]; in this case: numbers, graphs and symbols *(Fig. 4)*.

x	f(x)	g(x)
0	0	0
0,5	5	0
....	....	....
4	40	0
4,5	45	10
....	....	....
8	80	80

$f(x) = 10x$

$g(x) = 20x - 80$

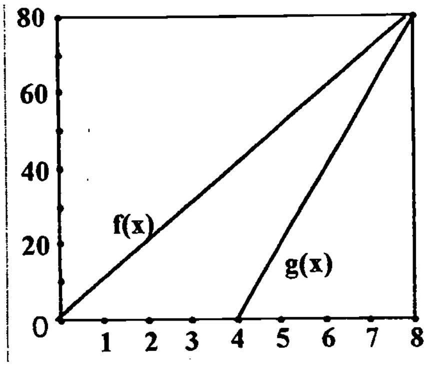


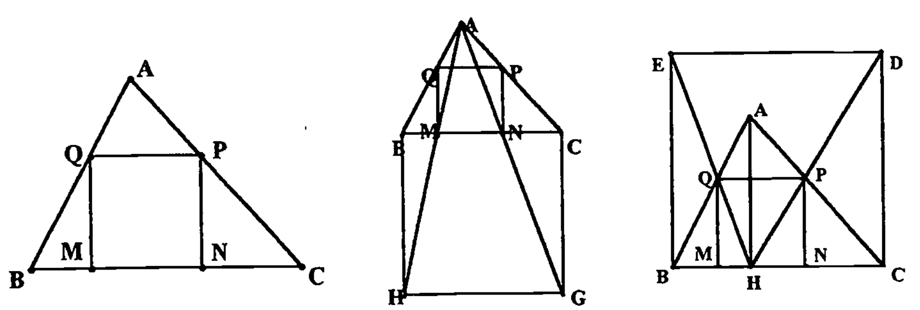
Fig. 4

This ability to connect several areas of mathematics and/or several ways of expressing mathematical situations is most important in problem solving, because it enables the student moving to an area in which he/she will be more at ease, being thus in better conditions for solving the problem. Here, Tal was able to solve the problem with numbers (left column) and with graphs (right column), and he would have solved it with algebra (center column), had he not been upset by the 'negative distance'.

5- Towards Dynamic Problem Solving

The second 'bikers problem', although looking similar to the other, confronts the students with quite a different approach, since one has to introduce a parameter (for instance the time when the biker is back to town 1, or the distance between the two towns). A consequence is that it becomes impossible to draw an accurate graph (one can only draw a sketch). The presence of the parameter defines, not a single function, but a whole family, and 'only one member of the family (...) satisfies all the constraints'. This problem reminded me another (classical) one *(Fig. 5, A)*:

A triangle ABC is given. Construct a square MNPQ so as M and N belong to [BC], P belongs to [AC] and Q belongs to [AB].



A B C

Fig. 5

In textbooks this problem is usually solved with the help of one of two diagrams *(fig 5, B and C)*. But I have experimented that this is not how students proceed. In fact, they begin with a 'trial and error' process, drawing several squares successively

(Fig. 6, A). This process leads some of them to an accurate solution, by using homothecy, *(Fig. 6, B)*.

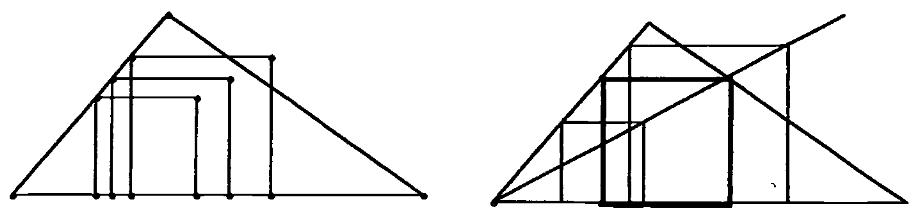


Fig. 6

In fact, the various squares drawn in the first stage can be considered as 'snapshots' of a same square MNPQ growing from point O, the points M and N remaining on (CB) and Q remaining on (BA). This same 'dynamic' solution, in which visualization plays a essential role by suggesting the suitable geometrical transformation, may be used in the second 'bikers problem': supposing that the distance between the two towns is, say, 100 km, you can draw a first graph *(Fig. 7, A)*. By so doing, you can see that the return trip is 5 hours shorter than the outward trip (instead of the 4 hours requested). And then you can finally adjust the second line by using a homothecy, the center of which is O and the ratio 4/5 *(Fig. 7, B)*.

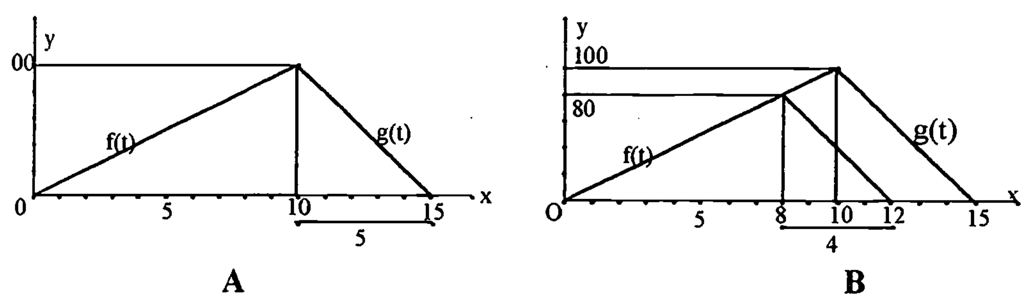


Fig. 7

This type of solution can easily be put into play by using graphical software. In the above problem, for instance, the line representing the distance covered during the outward journey (y = 10x) can be graphed. Assuming now that the distance between the two towns is 100 km, the point A(10; 100) will also belong to the line representing the distance covered during the return journey. And, if the software makes it possible to move this line parallel to itself (by changing the coordinates of A with the constraint that it remains on the first line), a graphical solution of the problem can be found. Here, students do not have mere 'snapshots' of the situation, but they can see it moving, they have a <u>continuous</u> view of it. This gives sense to resorting to a homothecy and can also lead to the idea of replacing point A by <u>any point</u> of the first line: B(t; 10t), the question being then to determine the value of t for which the

difference between the lengths of time of the outward journey and the return journey is 4 hours.

The efficiency of such a move from a discrete to a continuous point of view can also be observed in phenomena such as radiocarbon dating. Knowing the value of the 'half-life' of C^{14}, and provided some model hypotheses are defined, one can study it by using a geometrical sequence, obtaining an interval of time, or one can solve a differential equation, which gives a -theoretical- date [Parzysz 1999].

Similarly, in the 'ovens problem' Erez and Ilanit -after having drawn a continuous curve- use a discrete graphical method *(Fig. 8)*: They compare the changing rates of the two ovens in each time interval ('stair') and choose, for this interval, the oven with the highest rate. By so doing, they can conclude that, in the beginning, the conventional oven is better, but towards the end it is the contrary. But this process can only give them an interval during which the move from the conventional oven to the microwave one has to be operated; one cannot get a precise time.

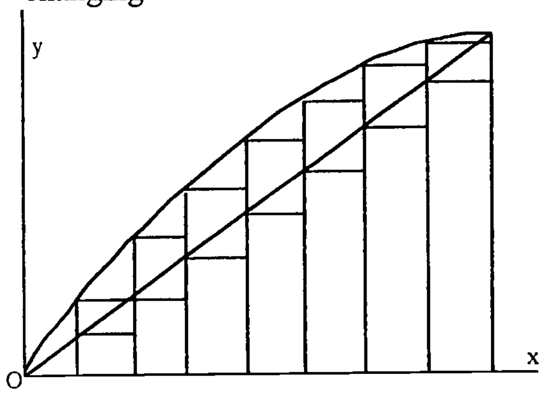


Fig. 8

On the contrary, Yoni's process, which is a continuous one based on the slopes of the curves, enables him -at least theoretically- to find this time: on the graph, it is the point where the slopes are equal. This 'superiority' of the second process over the first one, as regards the 'cost' and accuracy of the answer they can give, may be a motivation for the students to shift towards the concept of derivative.

6- Classifying the rate problems

The idea of classifying the rate problems (in which two lines intersect on the associated graph) according to 1° the signs of the slopes and 2° the unknown quantities seems interesting. The authors distinguish between the situations which have no degree of freedom, allowing to draw a unique graph (I should call them *static* situations) and those which have one degree of freedom or more, allowing to draw a family of possible graphs (I should call them *dynamic* situations). As we have just seen, this leads to different processes, from the point of view of problem solving.

Such a mapping of the 'deep structure' of the problems seems promising, because it can help teachers find problems in a more rational way, thus leading their students to explore the different types. It can also help researchers study the relation between the structure of a problem and the difficulties encountered by students, in order to have 'a tool of significant predictive power', as the authors say. Some questions could perhaps be answered, as for instance:

How can the surface structure of a rate problem obliterate the deep one?
Are there realistic situations, which prove to make the solving easier?
Can rate problems help students move from graphs to symbolic understanding?

How can students be helped to grasp the deep structure of a rate problem?
Can graphical software help students become aware of the similarity of structure between such algebra problems and some geometry problems?
Can the move from 'static' to 'dynamic' situations help students with the use of parameters for solving problems?
Can dynamic situations help students deal with qualitative problems? etc.

This list, far from being exhaustive, shows that there is still some work for the authors and other educational researchers to be done. I wish them good luck.

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THE ROLE OF VISUALISATION IN YOUNG STUDENTS' LEARNING

Kay Owens

University of Western Sydney, Macarthur

The study draws together a number of research studies carried out by others and the author into a framework of early spatial mathematics learning. The emphasis is not only on conceptual knowledge but also on the role that visualisation plays in learning spatial mathematics. Several tasks were developed to assess students in primary school and the types of responses collated in terms of the framework.

Related Research Studies with Adults

There are several aspects of visualisation that I have explored with adults although my focus is on children's visualisation. Adult participants can articulate their thinking as we explore new ideas, and comparisons between adults and children have highlighted certain aspects of thinking. In one study (Owens, 1990), a comparison of the *order* in which adults and children made the different pentomino shapes (shapes made from five squares with sides exactly aligned) indicated that conceptualisations and associated images influenced decisions. For example, children were more likely to consider that shapes had to have a name or be symmetrical when making their first few pentomino shapes. For both adults and children, responding in the problem-solving situation was reliant on a range of cognitive processes including imagery, concepts, affect, and heuristics such as investigative tactics and self-monitoring. This responsiveness was idiosyncratic but an important part of problem solving (Owens, 1996b).

Another study (Owens, 1998) was an experimental study of adults asked to recognise equal angles in complex designs presented on computer screens. The adults responded to the training tasks in four different ways. During the training those who were in the *Doing* group could use acetate and pens to trace over angles to check size; those in the *Speaking* group were asked to explain how they were deciding; those who were in the *Listening* group listened to the researcher on earphones explain why angles were equal; while those in the *Looking-only* group could only read the explanation which was presented to all the participants. Learners were presented with both analytic reasons such as "vertically opposite angles are equal" and reasons such as "they just look the same and (another) angle is smaller". The computer was programmed to show the designated angle move to be positioned on the equal angle. Some of the adults were later asked what they were attending to, and how they were thinking during the tasks. There was significant variance between the scores of members of the groups who were speaking or doing. The lower variance in the Listening group suggested that this group had their attention more focussed. Overall there was no clearcut difference between responding groups nor support for the effect of cognitive load due to training responses. However, participants did refer to their use of visual imagery, their search for equal angles and how they used geometric knowledge to reduce their searches. A

significant effect of affective processes was noted in their retrospective comments. Responsiveness was also a key to learning through investigating when the children participated in the series of tasks that linked to angle conceptualisation (Owens, 1996b).

Studies on Children's Spatial Strategies

Relevant Findings from Earlier Studies

I have previously reported on a matched-group experimental study that showed that primary-school students engaging in spatial problem-solving activities improved in spatial thinking (Owens, 1992b). Based on the positive outcome of this experimental study, it seemed important to explore *how* spatial problem solving improved learning in classrooms. Over 180 problem-solving episodes were analysed. The role and diversity of visual imagery in problem solving was one key aspect that emerged in this grounded-theory investigation (Owens, 1994; Owens & Clements, 1998). The work of Presmeg (1986) was useful in classifying this diversity.

Students' apparent visualising was classified as concrete pictorial imagery, imagery associated with patterning, dynamic imagery associated with movement within the image structure, action imagery involving movement of body parts, and imagery that involved following a series of procedures. The examples of dynamic imagery in such young children were noticeable. For example, one child imaged a square becoming a rectangle and then becoming thinner as he developed his rectangle concept. Another dynamically changed a trapezium into a parallelogram. Action imagery was exemplified by one child who explained that she could see her hand moving around the line of four squares as she investigated new pentomino shapes. The fact that action imagery was associated with investigative tactics provided a further focus for the present study (Owens, 1994; Owens & Clements, 1998).

Pattern imagery was associated with students' developments in tiling areas. For example, one child explained how a triangle could be covered by smaller, similar triangles using the zig-zag pattern of one-up, one-down (Owens & Clements, 1998). Some students on the test, *Thinking about 2D Shapes* (Owens, 1992a), used in the experimental study, spontaneously drew tiles and grids to answer the questions on tiling. A comparison of their drawings and those of children in a study by Outhred (Owens & Outhred, 1997, 1998), and the earlier studies of Mansfield and Scott (1990) on young children covering shapes, suggested that young students can develop a sense that a particular shape could tile a given shape. Students were likely to do this by recognising certain attributes of the tile such as type of angles, size of tile, relevant pattern for filling with the tile, and recognition of gaps. The early ideas on tiling are part of the current study.

Students' responses to the items of *Thinking about 2D Shapes* on angles and the activities related to angles, showed that young children can recognise angles but relevant experiences are needed as well as an ability to "see" the angle (Owens, 1996a). Rosser, Lane, and Mazzeo's (1988) study indicated the importance of rotation

and visual memory in early spatial thinking. These two ideas also needed to be considered in exploring young children's spatial thinking.

The ideas of reseeing, seeing parts that fit, and completing images have been discussed extensively in the spatial abilities literature (see Eliot, 1988). Several tasks were developed by Del Grande (1988) and a succinct summary made by Tartre (1990). These seem to be important features of early spatial thinking. Students also increase their visual memory, and sequential use of perceptions. For example, young students can mentally fold a net of an open cube (Owens, 1992a).

Theoretical Framework Development

Reviewing the earlier studies resulted in the development of a theoretical framework that could be used to inform teachers of young students' early spatio-mathematical development. The framework was also designed to build on ideas developed by *The Count Me in Too* project (NSW Department of Education and Training, 1998) through which teachers became familiar with such terms as *emergent*, *perceptual*, and *figurative* (imagery) stages.

Orientation and Motion

Students need to recognise shapes in different orientations and to develop the skill of appreciating what an object or group of objects might look like from another perspective. These changes in perspective and orientation are related to motion. Motions with manipulatives (e.g. card cutouts and tiles) that represent two-dimensional shapes include flips, slides, turns, and folds. These motions assist students to develop concepts such as (a) reflection symmetry (flips in horizontal, vertical, and diagonal lines or folding), (b) area (slide repetitions associated with covering of areas), and (c) rotational symmetries.

Movement is imaged by students as they make associations between shapes. For example, they can image one triangular shape moving to become another triangular shape as a point slides along a taut string. They might see how a triangular shape becomes a quadrilateral by bending one side into two or how a square can be pushed over to make a rhombus.

Movement necessarily involves position concepts. Actions are described in conjunction with directions such as left, right, or straight ahead. A particularly important change of direction or turn is associated with the concept of angle. Early learning is often stimulated by action and this turning of one arm of an angle away from the other does seem to be one of the ways that children first begin to learn about angles. However, they also begin to notice angles on shapes.

Part-Whole Recognition

All shapes are made up of parts. When students notice the parts, then they develop their concepts about shapes. For example, a student might notice three corners on a triangle and decide that this is a defining feature of a triangle. The corners are at first

just considered as the pointy parts. Gradually, the parts become specific in their features and students notice right angles or equal sides.

Students might only notice dominant features of shapes such as a pointy part or they might only see the overall shape. It is important that students develop both skills. Noticing the parts requires students to see the part within the context of a shape or configuration of lines. This is the skill of *disembedding* and it has a counter skill of embedding by which a student is able to complete shapes imagined in the mind.

The ability to see angles on shapes, to see differences in angles, to see shapes within other shapes, and to complete shapes using imagery are necessary skills for students if they are to develop a repertoire of properties of shapes or to apply geometry in real examples.

Classification and Language

Students will realise that a variety of examples of a shape can be categorised as one particular shape. Students will begin to associate more and more properties of parts as necessary or not necessary for a shape to belong to a particular category. Verbal expressions are associated with visual imagery and help define it. As students group and regroup, they develop relationships between shape categories and properties of shapes and lines.

Students will associate particular words consistently with particular actions, shapes, and other spatial relationships. They can use words to represent their imagery. Students need to identify spatial features such as parallel lines, perpendicular lines, spatial patterns, slopes, shapes, and corners in their environment, and discuss what they see.

Through discussions, students begin to abstract concepts such as shapes, to describe comparisons between parts, and to recognise why certain shapes make patterns and tessellate.

Imagery Strategies

Each of the above aspects of spatial knowledge becomes evident in the way that students behave and respond to tasks. Their imagery strategies can be inferred from their actions and words. It is through student's language and the selection and use of objects (including recognition of spatial features of card cut-outs, pictures, and shapes in their environments) that decisions on children's learning will be made. By saying and pointing, students indicate that they notice parts and visualise their relationships.

Certain experiences may encourage the development of dynamic changes and patterning in their imagery. These can challenge and modify limited concept images. For example, a full range of triangles can be included for the concept of a triangle.

Nevertheless, students' abilities to represent shapes by drawing or using materials like sticks can have different influences on their thinking. For example, some think that their drawing of a square is not a square because it no longer matches their mental

imagery. They may then decide a problem cannot be solved because the square does not look right but other students will accept that they just had trouble with the drawing and their mental imagery dominates. In a similar way, when students are making shapes, they make decisions about whether gaps and overlaps are important or not (Owens & Outhred, 1997). It is important, therefore, for teachers to wait and probe during assessment tasks in order to recognise how the student is thinking.

There are five groupings of strategies:

- *Emergent strategies*. Students using emergent strategies are *beginning to* attend purposefully to aspects of spatial experiences, to manipulate and explore shapes and space, to select shapes like ones shown or named, and to associate words with shapes and positions.
- *Perceptual strategies*. Students using perceptual strategies are attending to spatial features and beginning to make comparisons, relying on what they can see or do.
- *Pictorial imagery strategies*. Students using strategies involving pictorial imagery are developing mental images associated with concepts, with increasing use of standard language.
- *Pattern and dynamic imagery strategies*. Students using strategies involving pattern and dynamic imagery are using pattern and movement in their mental imagery and developing conceptual relationships.
- *Efficient strategies*. Students using efficient strategies, are beginning to solve spatial problems and constructions successfully by using imagery, classification, part-whole recognition, and orientation.

These strategies are more or less likely to emerge and be used by children in the above order. Intuitive and incidental learning can influence these strategies in unexpected ways. The casual use of a spatial term can be picked up by a young child in such a way that a well developed understanding of the concept is formed earlier than expected. This might occur if a child realises that the term *triangle* is used over a number of different occasions to refer to shapes that are not all exactly the same.

Orientation and motion, part-whole recognition, and classification and language aspects of spatial thinking are described for each strategy group in Table 1.

Table 1
Framework of Imagery for Space Mathematics

	Orientation and Motion	*Part-Whole Relationships*	*Classification and Language*
	The student:	The student:	The student:
Emergent Strategies	recognises shapes that match the student's fixed image(s)	attempts to put pieces together to see what is obtained	matches shapes with everyday words e.g. ball for a circle
Perceptual Strategies	recognises shapes in different orientations and proportions, checking by physical manipulation	recognises whole shapes used to build a shape or picture	describes similarities and differences and processes of change as they use materials
Pictorial Imagery Strategies	generates images of shapes in a variety of orientations and with different features	disembeds parts of shapes from the whole shape, matches parts of different shapes completes a partially represented shape or simple design	discusses shapes, their parts, and actions when the shape is not present
Pattern and Dynamic Imagery Strategies	predicts changes by mentally modifying shapes and their attributes using motion or pattern analysis; represents relationships created by change and patterns by modelling or drawing	develops and uses a pattern of shapes or relationship between parts of shapes; plans and dynamically modifies a shape to illustrate similarities between different representations of the same concept	discusses patterns and movements associated with combinations of shapes and relationships between shapes
Efficient Strategies	selects effective strategies to make changes needed to achieve a planned product	assesses images and plans the effective use of properties of shapes and composite units to generate shapes	describes relevant use of properties of shapes to generate new shapes

Assessment Tasks

Associated tasks were specifically designed to assess students' thinking in terms of the framework and whether, in particular, students were developing pictorial imagery, pattern imagery and dynamic imagery, and associated concept images. Several tasks (e.g. Tasks 2 and 7) were developed from the concrete introductions and items used in the test *Thinking about 2D Shapes* used in the initial experimental study with children (Owens, 1992a) and from the above references (e.g. Task 2B is based on Mansfield & Scott, 1990; Task 5 from Rosser, Lane, & Mazzeo, 1988). The revealing shape task (Task 3) is an extension of a common task used in studies based on the van Hiele theory.

The development of the tasks has involved the assessment of over 50 students in the first three years of schooling (ages 5 to 7 years) by the researcher, curriculum consultants, and several teachers. Some students were video-recorded during the tasks, and discussions on viewing the tapes assisted in checking the validity of the framework and in improving the tasks.

Table 2 gives the details of some tasks and brief descriptions of the remainder. Task 1 emphasises the idea of context for learning. It also seeks to see what properties the student might know. Subtle changes were made to wording, (e.g., "tiles to *make* (not cover) a given shape" in Task 2), and tasks were modified so they were manageable by teachers and students (e.g. Task 6). The tasks were also devised to allow students to show they were visualising before they were given the opportunity to try with concrete materials (e.g. Tasks 2, 6 & 7). Several probes are suggested to help when students are having difficulties with the harder question or if they are not familiar with the language (e.g. Tasks 2C, 8).

The tasks require some equipment but it is kept to a minimum. The card cut-outs, are shown in Figure 1. Spare small rectangles and small squares and another large square, drawings of the square at a 30° slope (plus photocopies) and of an inverted equilateral triangle double the size of the right-angled triangle are needed. Nets for Task 7, 3 circles with tabs for Task 5, string, and sticks are used.

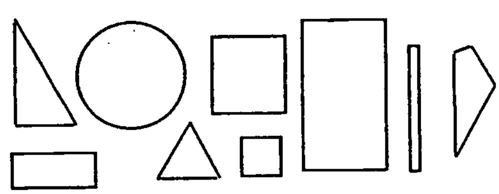


Figure 1. Card cut-out shapes required for the tasks.

Table 2

Selection of Tasks for Assessing Students Early Space Learning

Task 1A: Recognising shapes in the environment

Use card cut-out shapes, one of each type except the quadrilateral, (i.e. large square, circle, large rectangle, and both types of triangles).

Place the card cutout shapes on the table.

Ensure the layout of the shapes is not aligned horizontally.

- *Look around the room. Can you see a shape like one of these shapes?*

(Indicate the cut-out shapes on the table. Allow the student to nominate the shapes.)

- *How are they like each other?*

  Probe question

  If a student is unable to complete the task, point to the rectangle. (Push it away from the other shapes towards the student, turn it into horizontal alignment)

  - *Can you see a shape like this one?*

Task 1B: Sorting shapes and identifying properties

All the shapes shown in Figure 1 and the drawn shapes are to be sorted. Probes are used, for example, if a picture is made.

Task 2A: Recognising double tiling

The square is used and students asked to image another joined to it. They are asked if they can choose the shape it makes. If not, two squares are used to try. The squares are used to explain that they join without gaps or overlaps in preparation for the next two parts.

Task 2B: Imagining triangle tiles

The student is asked how many and how the right-angled triangle might cover the drawn equilateral triangle, and then given the tile if needed.

Task 2C: Imagining tiling of areas

Use card cut-out shapes, drawn square, photocopies of square, pencil, more square and rectangular tiles (if needed)

Display the card shapes and the drawn square on the table.

- *Which one of these cardboard shapes could you use to start making this drawn shape?*
- *How many of those would you need to cover this drawn shape?*
- *Show me how you would make it.*

Provide the student with the photocopy of the square.

- *Draw what it would look like.*

  Probe. If the student cannot show how to move the tile and cannot draw it on the diagram, give the student more of the same tiles to make the square. Cover it. Then ask

  - *Draw what they have made.*

Show me another shape to use to make this square.

If the student chooses an appropriate shape ask:

- *How many of these would you need?*

Provide a second photocopy of the square.

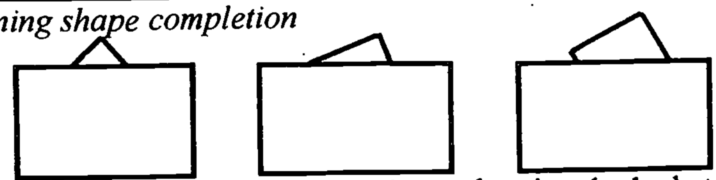
- *Draw what it will look like.*

  Probe. If the student cannot draw it, give the student the tile (rectangle or square) and say:

  - *Show me how you would make it.*

- *Please draw it.*

Task 3: Imagining shape completion



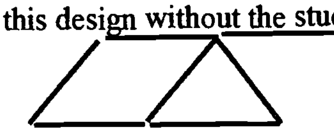
A square is gradually revealed. Each time, the student is asked what it might be and to trace where it might be. They are encouraged to give more than one answer.

Task 4A: Seeing shapes within shapes

Students use sticks to form 2 squares joined together and then 2 triangles. They are asked to draw the 2 triangles, preferably while covered.

Task 4B: Seeing shapes within shapes

Use diagram of a trapezium, 7 equal-length sticks, sheet A4 paper.
Arrange the sticks to make this design without the student seeing.



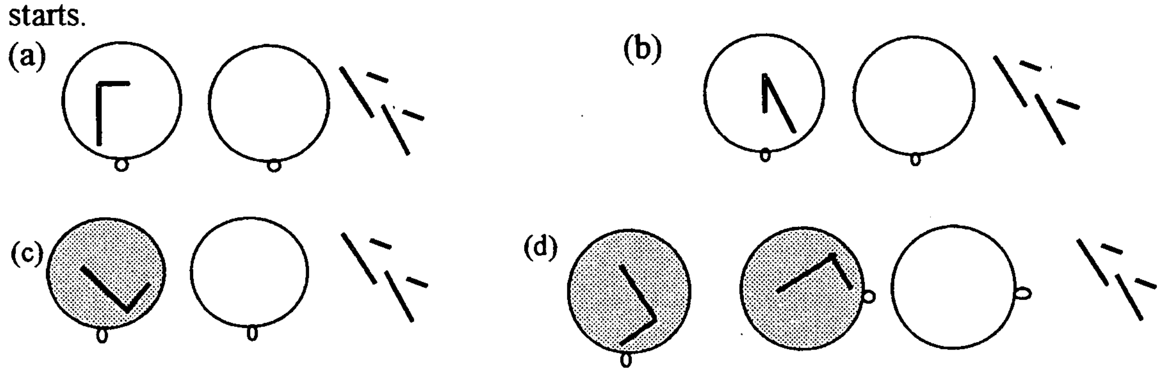
Show the outline drawing of the trapezium to the student. As you talk run your finger around the perimeter of the trapezium.

- *This is a drawing of a trapezium. Look at the design I made with the sticks.*
- *Do not touch the sticks but point to the sticks you would need to take away so that it is the same as the drawing of the trapezium.*

  Probe: If the student is unable to identify the correct sticks to be taken away, allow them to manipulate the sticks.

Task 5: Angle recognition, visual memory, and rotation skills

Make the following diagrams on a circle using long and short sticks or pipecleaners, point out the tab, let the student make the same diagram on their circle with tab mark aligned with yours. The first two are uncovered, the third is covered before the student starts, and the fourth is shown to the student, covered, and turned before the student starts.



Task 6: Dynamic imagery

Use 40 cm string, joined to form a loop; a firm stick.
Place the loop of string on the table and hold two points firm, about 15 cm apart (the string needs to form a triangle when one side is shortened). Provide the student with the stick.

- *Use this stick to pull the string tight and make a triangle.*

- *How would you describe the triangle you have made?*
- *Could you make other triangles?*
- *How would they change?*
  Probe: If the student cannot explain, let them use the stick to demonstrate and tell about the triangles they are making.

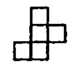
Point to one of the sides of the triangle.

- *Tell me what you would have to do to make this side shorter.*

Point to the other side.

- *As the first side is made shorter, what will happen to this side?*

Task 7: Imagining, folding and turning nets to make three dimensional shapes

Use nets of the open triangular prism and open cube. 

Show the net of the open triangular prism and fold it up to make an open triangular prism. Talk about how you folded up and turned the sides. Talk about how the 2D shape became a 3D shape. Let the student try to fold it. The word "box" may be used as well as prism.

Show the net of the open cube but do not let the student actually fold the sides

- *Tell me how you might fold it up and where each square will end up.*
- *What will it make?*
  Probe if the child cannot explain.
  - *Try to fold it up, and as you do, tell me what you are doing.*
  - *As soon as you know what you are making tell me.*

Task 8: Visualising turning three dimensional shapes

Place a square pyramid on its base in front of the student.

- *If I push this over and the point lands away from you, what shape will you see?*

After the student responds, push it over so the point lands away from the student and the square is facing the student.

- *If I stand it on its point, where will the square be?*
  Probe: If the student is unable to say the square is on the top, tell them and indicate with your hand how the pyramid is turned up onto its point.
- Draw how the triangle facing you will look.
  Probe: If the student cannot draw it, hold the pyramid up on its point and indicate the triangle to be drawn.

Results

Our preliminary use of these tasks has indicated students portraying different strategies. Table 3 indicates the kinds of responses given by students and how they indicate different strategies within the framework. The efficient strategies were not well established by any of the assessed students in Kindergarten to Year 2 at school. It was possible to distinguish responses for the different categories of Orientation and Motion, Part-Whole Relationships, and Classification and Language. Clearly some tasks addressed one area more than another. More importantly, the tasks did provide a range of strategies to be observed by different students.

While students did not necessarily show the same type of strategy across all questions, there was a tendency for this to happen. That is, if a student tended to use perceptual strategies in one task, the student was likely to show these same strategies across all tasks. Some students were only showing the beginning of Perceptual

Strategies; for example, they seemed inexperienced with diverse shapes. It is also interesting that in some cases, one area such as Language and Classification assisted a higher strategy for another area such as Part-Whole Relationships. For example, students who learnt about quadrilaterals and that they had four sides were more likely to use pictorial imagery strategies for Part-Whole Relationships in Task 1B.

Results of some tasks are presented in Table 3. Task 1A on shapes in the environment and their properties shows how the students' responses may be best considered in terms of one of the main conceptual areas. Task 2B on tiling with the right-angled triangle shows the importance of the investigative tactic of flipping. Task 2C on tiling shows how the tasks can show a range of strategies. The idea of tracing and encouraging different responses in Task 3 was successful in drawing out responses of different kinds of strategies.

Table 3
Typical Responses for Space Tasks for Early Space Framework

Strategy	*Orientation and Motion*	*Part-Whole Relationships*	*Classification and Language*
1A: Recognising Shapes in the Environment			
Emergent		* Tries to match the rectangle	
Perceptual	◊ Selects much larger, differently oriented shapes		◊ Selects and names ○ □ ▭ △ ◺
Pictorial Imagery			⇒ Readily selects and gives properties
2B: Imagining Tilings of Equilateral Triangle with Right-Angled Triangles			
Emergent		* Can't guess 2 and places triangle in centre of equilateral triangle	
Perceptual	◊ Eventually flips or turns shape into position	◊ Covers shape with triangles when both given	
Pictorial Imagery		⇒ Quickly says 2, places and flips one triangle	
2C: Imagining Tiling of Square with Smaller Square or Rectangle Tiles			
Emergent			* Selects any shape and places inside square
Perceptual	◊ Turns tiles to be aligned with sides and corners	◊ Says 3 to 5 tiles, selects □ ▭ , covers square when given extra tiles	
Pictorial		⇒ Says 4□, 3▭ rough	

Imagery		drawing, shows with 1	
Pattern, Dynamic Imagery		• Quickly says correctly, Uses grids in drawing	
3: Imagining Shape Completion by Tracing Possible Hidden Shapes			
Emergent			* Says any shape name
Perceptual	◊ Says square		◊ Says triangle or square but cannot trace where it might be
Pictorial Imagery		⇒ Traces for a triangle or square or rectangle	
Pattern, Dynamic Imagery		• Traces out several possible shapes	
Efficient Imagery		▪ Indicates tracings and various changes	▪ Readily explains how different shapes could be underneath
4B: Seeing Trapezium within Shape			
Emergent		* Tries to place sticks to make a shape	
Perceptual		◊ Can select a stick when allowed to try it	
Pictorial Imagery		⇒ Can select at least one stick to take away	
Pattern, Dynamic Imagery		• Can quickly point to all sticks to take away	
6: Dynamic Imagery Using a Stick to Move a Loop of String			
Emergent	* Moves stick but does not make or recognise a triangle		
Perceptual			◊ Knows a triangle will be made
Pictorial Imagery	⇒ Makes different triangles	⇒ Tells other side will get longer	⇒ Knows names and properties of different types of triangles
Pattern, Dynamic Imagery	• Automatically slides stick to make different triangles, commenting on them		• Explains why continuous range of triangles can be made in general and by type
Efficient Imagery	▪ Shows an arc of points to shorten side		

Task 4B shows how the skill of disembedding or re-seeing parts is manifested while responses to Task 5 indicated the development of orientation skills and noticing angles. Results for Task 6 on making triangles show how a carefully designed task can illustrate a full range of strategies. It was a particularly novel task for consultants and teachers. The turning of a face for the 3D shape in Task 7 was a critical step for pattern and dynamic imagery strategies under Orientation and Motion. Task 8 was an important task for 3D concepts, and responses to the task could be placed in most of the cells of the first four strategies.

Discussion

Task 1B was difficult to use if the idea of sorting and grouping was new to the students. Both Tasks 1A and 6, as well as Task 1B, encourage students to express their concept images. However, Task 1B may be very fruitful in ascertaining students' knowledge, and in helping teachers realise just how limited students' concepts can be due to the lack of mathematical ideas in the common language of the classroom. For example, the lack of language such as "a quadrilateral has four sides" often meant that students tended to group the quadrilateral with a triangle or turned square.

The tasks provide information about the students in the different areas of the framework. One task can be multifaceted and used for a range of outcomes. In this sense, the tasks are rich in informing the teacher about the students' learning. Not all possible responses or cells of the table have been completed. It is possible that further testing will assist in showing other common responses for different cells. Nevertheless, this is not expected.

The tasks allow teachers to know the strengths and weaknesses of different students. For example, Natalie in Year 2, mostly showed emergent strategies and seemed to have most difficulty in the classifying and language area. She was able to show perceptual or pictorial imagery strategies in Task 6, 7, and 8. These were novel questions for her and may have been less associated with her general struggle with learning. The assessment provided the basis to plan suitable activities for her. For example, she needed experiences in sorting and grouping many different kinds of triangles, squares and rectangles (two kinds at a time); talking about the reasons for grouping e.g. four sides or four corners; seeing shapes within shapes in matchstick type puzzles; doing more jigsaws and making geometric shape like squares with pieces.

On the other hand, Jack in the same class, generally showed pattern and dynamic imagery strategies and an ability to see shapes within shapes assisted by good general language. However, his recognition of diversity when referring to a shape like "a triangle" still needed extension. He needed activities like matching parts of different shapes in order to notice similarities and differences, and to develop properties. He also needed more language to describe the parts and types of shapes. Interestingly, he showed some hesitation in placing the position to shorten the side of the triangle in

Task 6, trying to indicate that it would be further away than the line of the string (tending towards efficient strategies). He was ready to use properties to establish that squares are rectangles and that the same names apply when the shapes are in turned positions (a problem that was exacerbated by the use of words like diamonds), and to use words like rhombus, trapezium, quadrilateral or four-sided shape.

The tasks can be used for individual assessment or for the basis of activities for the class. The questions and probes can be used by the teacher to assist in students' learning and assessment during class experiences. The technology may be as simple as card cut-outs but computer-generated tasks could extend learning from previous activities with concrete materials.

Further research is needed to assess how consistently students are showing the same type of strategies across each task. It was clear in modifying the tasks that slight changes could make a task more difficult so that students were not using the same strategy as on other tasks. Research is also needed to see how classroom experiences can encourage students to use more advanced strategies. Such longitudinal studies would give further evidence of the validity of the framework, and its value for classroom teachers.

Conclusion

The previous studies of the researcher and those of numerous other people working in the area of spatial mathematics have been drawn upon to develop a framework that is useful for teachers. Synthesising so much research into a framework is an important step forward in developing a theoretical base for early childhood education in Space mathematics. In particular, the framework shows how visualisation strategies are a key to development of spatial thinking (knowledge and skills).

The framework provides a basis for teachers to assess students' current learning and to plan learning experiences that will extend their students' knowledge and visual skills. The facets of the framework interlink to provide holistic learning experiences and assessment tasks for students. The tasks provide a richer forum for teachers' understanding of the framework.

Students' responses to the tasks have provided evidence that the various facets of the framework can be assessed. In particular, the tasks can assess more than one facet, and each can generate responses showing different types of strategies. For this reasons, some tasks are constructed to encourage students to visualise first, and probes are suggested to elicit an emergent or perceptual response.

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SPATIAL STRATEGIES AND VISUALISATION:

Eddie Gray
Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL, UK

Abstract

This paper is an invited response to the Imagery and Visualisation Research Forum of PME XX111. It takes as its starting point the keynote *paper The Role of Visualisation in Young Students Learning* (Owens,1999). The review attempts to consider the paper in the wider context of mathematical thinking. It addresses difficulties associated with the investigation of imagery and draws the conclusion that a reappraisal of the data could suggest the possible existence of a divergence in spatio-visual thinking similar to that in arithmetical/algebraic thinking.

Introducing a Perspective

Writing in 1996, Owens suggested that:

> Student's responsiveness during active engagement in problem-solving activities is precipitated by their own thinking and feelings … responsiveness implies a degree of understanding as well as involvement and interest in the activity. (Owens, 1996, p. 101)

With this notion in mind, I present a personal review of the paper *The Role of Visualisation in Young Students Learning* (Owens, 1999). In doing so, I am also aware that each comes to the task of considering the paper with different personal repertoires of knowledge. Inevitably, we see the paper in our own personal way.

Within her paper, Owens considers two features designed to inform teachers about young children's early spatio-mathematical development. The first is a 'framework' that provides a basis for teachers to assess children's thinking and build a teaching programme. The second is a mechanism for assessing the children against the framework. It is claimed that an important aspect of the two is the relationship between spatial understanding and visualisation. Indeed, some of the tasks are 'specifically designed to encourage visualisation' and the framework itself is associated with a 'hierarchical' list of imagery strategies.

My personal reading of Owens has been influenced by two questions:

- first, how should we interpret the fact that some pupils seem to find the study of mathematics relatively easy whilst others find it virtually impossible?
- second, what can it add to our efforts to establish some kind of underlying theoretical structure that may help us further our understanding of mathematical thinking?

Perception, action and reflection

Geometry builds from the fundamental perception of figures and their shape, supported by action and reflection to move from practical measurement to theoretical deduction and Euclidean proof. This was a theme which Tall (1995) alluded to in his plenary address to PME in Brazil. He spoke of the way in which elementary mathematics begins with '*perceptions of*' and '*actions on*' objects in the external world.

> The perceived objects are first seen as visio-spatial gestalts, but then, as they are analysed and their properties are teased out, they are described verbally, leading in turn to classification (first into collections and then into hierarchy's...Tall, 1995, p 61.

To establish the distinction between the perceptive and manipulative aspects of early spatial development and the verbal/symbolic development of arithmetic and algebra Tall suggested that a different kind of development stemmed from *actions on* objects. Here the process of counting is developed using number words that become conceptualised as number concepts.

Early mathematical concepts are strongly associated with preliminary activities involving perception and action within the physical world and reflection on both perception and action (Gray, Pitta, Tall & Pinto, in press). Such a development requires the ability to concentrate the mind and give careful thought to an act or idea and then to filter out irrelevancies and separate notions from their context. It involves the construction of relationships between and amongst objects and of the inter-relationships of the actions on them. It may be that such a process works to the advantage of the more successful. An emphasis on one or more of the activities of perception, action or reflection leads not only to different kinds of mathematics, but also to a spectrum of success and failure depending on the nature of the focus in the individual activity.

In any context involving an action with objects, the individual has the possibility of attending to different aspects of the situation. Indeed, this is an issue that Cobb, Yackel and Wood (1992) see as one of the great problems in learning mathematics, particularly if learning and teaching are approached in a representational context. In their search for substance and meaning, some children may be distinctly disadvantaged right at the start of their mathematical development, but it is a disadvantage that may not make itself apparent in the earlier stages of cognitive development.

Any considered attempt to establish the way children think as opposed to simply measuring their level of achievement has to be welcomed. I see the Owens paper in this context. But there is more — it hints at a divergence in spatial thinking that may match the one identified in arithmetic/algebraic thinking. This is not to say that I applaud the paper in all of its aspects. As a research paper it has many technical weaknesses, one of the major ones being the extensive inclusion of the mechanisms of assessment (Table 1) at the expense of empirical evidence from which the reader may make judgements. What we see is the researcher's

interpretation of the evidence. In a sense, we have a description of the researcher's image of the child's image.

The framework and possible links with van Hiele

The earlier work of Owens (see for example 1992, 1996, 1998) led to the development of a framework to inform teachers of young students' early spatial mathematical development. The 1992 test was developed at the recognition level of spatial development (Owens, 1992,) and was therefore associated with Van Hiele Level 1 (van Hiele 1986).

Dominated by perception, van Hiele Level 1 suggests that appearance becomes the mechanism through which learners operate on shape and other geometric configurations. However, though perception is the foundation of geometry, it takes the power of language to make hierarchical classifications. Figures are initially perceived as gestalts but then may be described and classified through verbalising their properties, to give the notions of points, lines, planes, triangles, squares, rectangles, circles, spheres, etc. Initially these words may operate at a single generic level, so that a square (with four *equal* sides and every angle a right angle) is not considered as a rectangle (with only *opposite* sides equal). Again, through verbal discussion, instruction and construction, the child may begin to see hierarchies with one idea classified within another, so that "a square is a rectangle is a quadrilateral", or "a square is a rhombus is a parallelogram is a quadrilateral".

Owens' contribution to this development has been to tease out the coarser framework and provide a finer grained analysis that may be beneficial to teachers. Focussing on orientation and motion, part whole recognition and the use of language and classification, she presents what may be a useable set of criteria to establish children's thinking. However, I wonder how these criteria differ from the van Hiele level 1, that of establishing a visual gestalt, and van Hiele level 2, that of being able to characterize shapes by their properties. It seems self evident though that these two need embedding in an associated language context in recognition that:

> The criterion for having a concepts is not that of being able to say its name, but that of behaving in a way indicative of classifying new data according to the similarities which go to form this concept. (Skemp, 1986. P. 26)

Evidence of divergent thinking

Of course, it is important to place these notions into a context which has meaning for the teachers of young children. However, it is equally important not to over-generalise from the outcomes — a factor that does seem to dominate this part of the paper, particularly since we are considering a "theoretical framework". We may accept that students 'need to recognise shapes in different orientations' but it is far less easy to accept (in the absence of any empirical data from which to make a personal analysis) that:

- movement is imagined by students as they make associations between shapes

- that the need to complete shapes using imagery are necessary skills for students.

Equally, it is difficult to accept that *all* students will "realise" and/or "develop relationships" or "associate particular words" or use "words to represent their imagery". These expressions project the hopes of the pedagogue not the realism of the learning experience.

In the cognitive context, we see the framework presented as a model which informs us about children's thinking. In the pedagogic context, we see the model as a framework from which teaching could be developed. Such a suggestion reminds me of efforts to turn Piagetian theories into a mathematics curriculum. Indeed, we see similarities with the Piagetian notion of a stage theory and Owens claims that the 'imagery strategies' through which the different aspects of spatial knowledge become evident 'are more or less likely to emerge and be used by children in an order'. Any divergence from this hierarchy of development is simply explained in terms of 'intuition or incidental learning'. No evidence in the paper supports either of these arguments. Most appears to be based upon teaching experiments (for example Owens, 1992, 1996) and/or through the analysis of children's spatial problem solving abilities (see for example Owens & Clements, 1998). Any claims about a hierarchical process of growing strategy sophistication would seem to be best addressed by a longer-term developmental study with individual students.

The notion that there are different strategies associated with children's spatial thinking is important and perhaps we are seeing the first stages in an attempt to mirror for spatial development the research in elementary arithmetic which has proven to be so beneficial. The evidence presented within typical examples suggests that there are differences in the way that children think. Though the study refers to a sample of 50 children, 'typical responses' presented for the reader to make judgements are only drawn from 6. For four of these children it is possible to consider their responses over the 6 items which stimulated responses. An analysis of the responses of these four suggests distinct behaviours not an ordered development. The data within Table 3 may be re-interpreted; not to simply provide 'typical responses' but to indicate qualitatively different thinking representing a spectrum of varying between perceptual and figurative extremes. We see, for example, that "*" always uses "emergent" strategies, "perceptual" strategies dominate the thinking of "◊", pictorial imagery that of "⇒" and the more sophisticated strategies are used by "●" and "■". Responses associated with the emergent strategy, suggest that the child identified as "*" is at a stage of "pre-recognition" (Clements and Battista, 1992) based upon a deficiency in perceptual activity whilst child "◊" appears to be firmly embedded in perceptual strategy use. My conclusion from the evidence would be that some operate on a perceptual level, whilst others operate at a figurative level which itself may reflect differing degrees of sophistication.

A focus on imagery and visualisation

Piaget and Inhelder drew our attention to the relationship between imagery and perception:

> Perception is the knowledge of objects resulting from direct contact with them. As against this, representation or imagination involves the evocation of objects in their absence or, when it runs parallel to perception, in their presence. It completes perceptual knowledge by reference to objects not actually perceived... (Piaget

As more progress is made on the research front the concept of image has become less clear (Cooper, 1995). In a sense, Owens adds to this lack of clarity. Through her implicit recognition that human cognition requires different representational constructs, she adds another dimension to our perceptions of image and visualisation. Images are significant components of cognition and their interpretation has particular relevance to the study of mathematics. However, our interpretation of the notions, and the evidence we use to describe and classify them, is often somewhat speculative and open to interpretation.

It may be correct to make the assumption that there is a functional equivalence between images and processes formed on images, and the corresponding external objects and the perceptual operations that the images and the imagined operations were thought to stimulate. To do so, however, suggests that images, like visual perceptions, have depictive or picture-like qualities. Here we are faced with an elementary problem — whether or not the notion of imagery can be synonymous with the notion of visualisation.

Guttierez (1996), in his excellent discussion of the various ways visualisation, image and mental image have been used in mathematics, suggests that a

> "mental image is a mental representation of a mathematical concept or property containing information based upon pictorial, graphical or diagrammatic elements [whilst] visualisation or visual thinking is a kind of reasoning based on the use of mental images".
>
> (Gutierrez, 1996, p. 6)

He considered that mental images were a basic element in visualization. Using the notions of concept image (the student's mental picture of a geometrical figure) and concept definition (the student's verbal definition to define a geometrical figure) Matsuo (1993) suggested that the more these converge. the more likely it is that the student moves from level 1 to level 2. Such a hypothesis would seem to be consistent with Tall's (1995) view and Skemp's (1986) criteria for the possession of a concept. Equally, the evidence suggested by Owens' responses from "●" and "■" would seem to support this view. Interestingly, Matsuo then suggested that there might be a difference between a state of understanding that is specified by the visual mode and a state of understanding which is clarified by the visual mode. I take this to mean a difference between the visual mode being necessary for understanding and the visual mode being a generator of understanding. In an entirely different study that looks at imagery in the context of elementary

arithmetic, Gray & Pitta (1997) draw a distinction between the use of imagery that is essential to thought and the use of imagery that generates thought.

It is possible that such a distinction may be interpreted from the evidence presented by Owens, where we see notions of perception and imagery being jointly classified to interpret imagery strategies. The fact that we see distinctions between children who rely extensively on perception and others who evoke imagery of a visual form may not be the whole of the story.

Though it would seem that the ability to visualise provides strength in the context of developing spatial awareness, the labels 'visualiser' and 'non-visualiser' may not be indicative of level of mathematical achievement. Pitta (1998) has shown that children at extremes of achievement may be strong visualisers. Equally, children at the extremes could be identified as non-visualisers. To focus solely on the incidence of visual imagery may not provide an insight into the qualitative differences between children, whereas a focus on imagery in both visual and non-visual form may.

The framework devised by Owens may lead us some way towards our understanding of children's understanding. It may also be suggesting that there is a diverging approach to elementary geometry which matches diverging approaches to elementary arithmetic (see Gray and Tall, 1994). Such a divergence, based upon qualitatively different interpretations that children place on spatial activity, may be manifested in the formation of qualitatively different forms of imagery and/or qualitatively different kinds of imagery. Visual imagery may be one of these forms but a broader view of the notion of image would not disregard the product of imaging in any modality.

I started this paper by asking two questions, one associated with the way in which children deal with the study of mathematics; the other associated with the formation of an underlying theoretical structure. The evidence from the framework developed by Owens would suggest that more sophisticated thinking reflects the ability to disregard details and focus on the generalizations that support choice. This was not to say that the detail was unavailable to them — it was and could be incorporated and used if needed.

The ability to create mental imagery is inherent in us all and it would seem that a visual stimulus would evoke visual imagery to a greater extent that verbal stimulus. Pitta (1998) has suggested that it is quite possible for children to posses the same visual image but to then attach different meanings to it. Therefore, whereas the visual image of the more successful child may be used to refresh memory or as a skeleton that ideas, equivalencies and relationships may be attached to, that same visual image may be used in an active mental episode for the less successful.

It is implicit from the Piagetian and the constructivist perspective that the knowledge and the beliefs that learners bring to a situation can influence the meanings they construct from that situation. Kay Owens uses the word

responsiveness to mean a similar thing. The responsiveness of the sample in the survey illustrates that the differences in strategy use may not only be apparent when children's responses to the problems are considered but, also a manifestation of the use of memory. A tendency to focus on surface characteristics of the stimuli may be more strongly related to the use of short-term memory. On the other hand, emphasis on long term memory use may mean retrieval of mental imagery which could be both meaning related information or carry surface characteristics. The truth of the matter is we really do not know. The study of imagery in any context is fraught with difficulty. We make an assumption that report, description and external representation in the form of words, drawings and actions provide an indication of the nature of the mental image. Any efforts to explain or provide a description that relates to reality can be very different from the truth. However, I would suggest that the strength of this paper lies in the possible insights it provides to another divergence in mathematical thinking.

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PROJECT GROUPS

Project Group PG1: Algebra: Epistemology, Cognition and New Technologies

Epistemological and Cognitive Issues about *Equations* and *Inequalities* keeping into account the aspects related to the use of Computers

Coordinators: Jean-Philippe Drouhard, IREM & IUFM de Nice, France

Alan Bell, Shell Centre for Mathematical Education, University of Nottingham, UK

Sonia Ursini, Departamento de Matemática Educativa, Instituto Politénico Nacional, México

Aims of the group:

The members of the 1998 PME Working Group on "Algebra: Epistemology, Cognition and New Technologies" wished to continue their work as a Project Group. They will be involved, until April 1999, in the elaboration (through e-mail exchanges) of a "Discussion Document" which will be circulated before the Haifa Conference, with the aim of involving all people willing to attend the PME-XXIII Project Group.

This Discussion Document will concern "Epistemological and Cognitive Issues about Equations and Inequalities, keeping into account the aspects related to the use of Computers" as a "case study". This document will be available to the participants in the Project Group website:

http://math.unice.fr/~iremnice/pme_wg_aecnt/index.html

The PME-XXIII Project Group will be devoted to discuss prepared reactions to the Discussion Document, and to elaborate the guidelines for the chapters of an electronic (or paper) volume which might be written during the following year and discussed at the PME-XXIV Conference.

Project Group PG2:
Classroom Research Project Group

Coordinators: Simon Goodchild, College of St Mark and St John, Plymouth, UK
Ruth Shane, Kaye College of Education, Beer Sheva, Israel

This Project Group has evolved from the Classroom Research Working Group that has met at PME conferences for a number of years. The focus of the group is the methodology of classroom research. In the past the group has discussed a range of issues crucial to all who are engaged in classroom research, from the consideration of the types of questions that are explored, through the methods and technologies that can be applied, to the approaches taken in the analysis of data that arises from classroom research.

When the group met in Stellenbosch last year it was decided to collaborate in the production of a book that collected together accounts of a variety of methodologies used by members of the group in their own classroom research activity. This project is progressing and one intention, as planned in 1998, is to share a draft version of the resulting book with the Project Group in Haifa.

The other intention of the Group is to move on to consider in more detail approaches taken in the analysis of qualitative data arising from classroom research. This issue was raised in discussion in 1998 and it became clear then that a useful outcome from the Project Group would be the publication of a resource that could be used to both illustrate and provide experience of different approaches to qualitative data analysis. The Group will consider the most suitable media for publishing this work; this could be a CD ROM, the development of a web site or a collection of printed papers.

The goal is to produce a useful companion to the book about methods for gathering data. The sessions will incorporate opportunities for members to share their own approaches to data analysis.

Project Group PG3: Cultural Aspects in the Learning of Mathematics

Coordinators: Norma Presmeg, Florida State University, USA
Marta Civil, University of Arizona, USA
Phil Clarkson, Australian Catholic University, Australia

The four subgroups of this Project Group are working towards functioning websites that will enable continued collaboration between the members of each subgroup between meetings. Suggestions for a published product will be considered at the meetings in Haifa. The specific topics of the subgroups are as follows:

1. Theoretical perspectives on cultural aspects of learning mathematics.
2. Classroom culture and social practices.
3. Power relations (political, socioeconomic, etc.).
4. Language and culture.

As in Lahti and Stellenbosch, the twin aims of the meetings will be to welcome newcomers to the group as well as to continue the exploration of issues introduced in the interactions of the specific topic groups. This exploration will have the purpose of consolidating the website constructed by Phil Clarkson after the meeting in Stellenbosch. A presentation on a separate page of the website for each specific topic will highlight the results of the Project Group's work in each of these areas, and enable further collaboration.

Project Group PG4: Research on Mathematics Teacher Development

Coordinators: Andrea Peter-Koop, University of Münster, Germany

Regina D. Möller, University of Landau, Germany

Vania M. Santos-Wagner, Federal University of Rio de Janeiro, Brazil

This Project Group has emerged from a Discussion Group (1986-1989) which was continued as a Working Group between 1990 and 1998. One major asset of this group has been its cohesiveness and its wide representation across many countries.

Goal of the Project Group

During recent years the group has been concerned with developing, communicating and examining paradigms and frameworks for research in mathematics teacher development. From previous group discussions and individual presentations emerged the need to explore the implications of collaboration and collaborative research in mathematics teacher education in a variety of different settings across the world. Therefore, in 1998 the group decided to collaboratively prepare a book on this issue. The working title of the book in progress is „Collaboration and Co-operation in Mathematics Teacher Education: Chances and Implications".

Since the meetings in Stellenbosch last year, altogether 18 abstracts were submitted, of which 13 have been accepted as potential chapters. At this stage additional chapters are still possible and most welcome. The authors of the accepted proposals agreed to submit draft versions of their manuscripts to the co-ordinators by May 99 in order for them to be distributed prior to and during the 1999 sessions.

Plans for the Project Group Activities at PME 23

During the two sessions in 1999 we will need to (1) clarify the theoretical foundations and concepts of the book, (2) develop its structure and (3) begin the peer review process with respect to the manuscripts submitted so far.

Central to the first session will be the discussion and clarification of the concepts of collaboration and co-operation as guiding elements of the joint product. In order to facilitate intense discussions, we will work in small groups each focussing on different aspects with the aim to jointly develop a 'concept map'.

At the end of the session, copies of all submitted manuscripts will be distributed in order to allow the participants to be prepared for the discussion of the papers.

The main focus of the second session will be to reflect on the levels, modes and purposes of collaboration and co-operation addressed in the individual chapters in order to prepare the structure of the book. At this stage it will become clear which additional chapter(s) would complement the publication. Our final task will be to develop a working plan for the time period until PME 24 in Japan.

Project Group PG5: Social Aspects of Mathematics Education

Coordinators: Jo Boaler, Stanford University, USA
Paola Valero, Royal Danish School of Educational Studies, Denmark

The recent evolution of mathematics education as a field of research has brought growing interest in the social dimension of mathematics education practices. The existence and permanence over the years of the PME working group "Social Aspects of Mathematics Education" is one of the multiple evidences of such an interest. Recognition of the importance of social perspectives on learning poses questions and challenges for researchers working within mathematics education. What does it mean to adopt a research approach that considers the social aspects of mathematics education? Which theories and conceptual frameworks are useful and appropriate for such an approach? Which questions may be addressed by such a perspective? Which research methodologies are suitable for this endeavor? Which criteria for research quality are relevant? What are the social and economic implications of this approach?

Mathematics education is facing the new challenge of taking and understanding a social perspective on the teaching and learning of mathematics, aperspective that has been shown to illuminate significant research questions, processes and results. Such a perspective requires deliberation about the range of research methods and methodologies that are consonant with social dimensions of mathematics education. Such deliberations should broaden our understanding about the practices of mathematics education and possibilities for action.

Therefore, this project group intends collecting a series of papers and publish them in the form of a book whose working tittle is "The Social Dimension of MathematicsEducation: Theoretical, Methodological and Practical Issues". The book addresses the issue of research on the social dimension of mathematics education, and it will include one or more of the following topics:

1. The meaning of the "social" for research in mathematics education
2. Theoretical approaches encompassed within such a dimension
3. Appropriate methodologies and methods
4. The quality of research and research findings
5. The connection between research and practice

Given that the process of the call for papers and submission of proposals has already advanced, the sessions of the project group will be devoted to presenting some of the papers that will make part of the book and receiving critical comments from the group members.

Project Group PG6: The Teaching and Learning of Stochastics

Coordinators: John Truran, University of Adelaide, South Australia
Kathleen Truran, University of South Australia
Dani Ben-Zvi, Weizmann Institute of Science, Israel
Carmen Batanero, University of Granada, Spain

This project group exists as a focus for members interested in the psychology of the teaching and learning of probability, statistics and combinatorics. It maintains an informal network between conferences by means of an electronically distributed newsletter. It particularly seeks to bring together interested people from all language groups, and does its best to provide translation facilities as appropriate.

Part of our Project Group meetings in the PME 23 Conference will also be devoted to ensuring that all of us have an opportunity to talk about our work. People who wish to be involved in this Project Group are invited to make a 10 minute presentation on their interests. This might be supported by two or three overhead transparencies and perhaps some handouts of work, which they think will be of interest for others. We shall also discuss Projects being done jointly with the International Study Group for the Teaching and Learning of Probability and Statistics.

Work is continuing on the preparation of a book on *Teaching Statistics – Teaching and Learning Statistics: Implications for Research* – which has its basis in psychological and educational research. The basic outlines of this work were decided at our meeting in 1998. Some material is also being prepared for a book on Advanced Mathematical Thinking. Some material will be available for group comment at this PME meeting.

DISCUSSION GROUPS

Discussion Group DG1:

Exploring Different Ways of Working with Video in Research and Inservice Work

Coordinators: Judy Mousley, Deakin University, Australia
Chris Breen, University of Cape Town, South Africa
Harold Frederick, Arizona State University, USA

This discussion group is concerned with the use of video as a means for obtaining data for research purposes and in teacher education.

During the 1998 PME Conference, the group centered its work around the ways in which video data could be re-presented when endeavouring to communicate research to others and to enrich our research stories. Issues around the 'truth' of classroom data obtained from a videotape was explored against a background of conflicting realities. Discussions around these themes elicited energetic debate.

We would like to invite people who are interested in this area to contact Judy Mousley directly and to submit questions or issues that may stimulate discussion and practical activities. Similarly, anyone who would like to offer a short activity that will maximise the participation of those attending the sessions should send a brief proposal for it to Judy (who can also give details about data projection and VCR facilities that will be available, as well as about videotape format).

Questions that will be used to shape the 1999 sessions include:

- How does the viewer's experience influence what they get from a video lesson?
- What impact do extras (commentary, dialogue, transcription) have on interpretation?
- Are there clinical versus descriptive/analytical orientations to viewing lessons?
- Does video data enable researchers to come closer to classroom reality?
- Can experts identify and isolate examples of their theoretical perspectives in videos of mathematics classrooms?
- What are various "points of dialogue" in videos of mathematics classrooms?

The coordinators wish to signal their interest in this becoming a Project Group in 2000, so part of one session will be used for planning future work.

Discussion Group DG2: Learning and Teaching Elementary Number Theory

Coordinators: Rina Zazkis, Simon Fraser University, Canada

Stephen Campbell, University of California, Irvine, USA

The discussion group on learning and teaching elementary number theory convened for the first time at PME 21 in Lahti, Finland and continued its work at PME 22 in Stellenbosch, South Africa. Referring to "elementary number theory" we think of concepts associated with multiplicative structure of natural numbers, such as factors, divisors, multiples, prime and composite numbers, GCD and LCM, divisibility and divisibility "rules", prime factorization and the Fundamental Theorem of Arithmetic, among others.

A central mandate and guiding motivation for this group is to explore the extent to which elementary number theory can serve as a gateway leading from concrete arithmetic understanding to more abstract levels of algebraic understanding. During the last two years the discussion group covered a broad spectrum of issues, including

- the nature of number theory, its influence on the historical development of mathematics, and its philosophical implications regarding the nature of mathematics more generally
- the role of number theory in the K-16 curriculum
- difficulties teachers and students alike may encounter in understanding introductory concepts of number theory
- the utility of number theory as a vehicle for teaching and learning mathematical concepts such as uniqueness, variables, and proof
- theories and methods for conducting research into these areas

Participants enthusiastically recognized the need to encourage, conduct, and disseminate research into the learning and teaching of elementary number theory. This year, we propose to discuss in more detail some of the issues, especially with respect to the utility of number theory in learning and appreciating mathematical structure and patterns. We also wish to share several insights from research in this area that have influenced or may influence our teaching practice.

Some of the participants from last year's discussion group have offered to give short 5-15 minute presentations of work that they have been conducting. These presentations will serve as a focus for discussion and as an opportunity for feedback from other participants. Another objective is to identify and pursue specific items for potential collaboration in the areas of learning and teaching introductory number theory, further research and its dissemination.

Discussion Group DG3: Mathematics in Working Practice

Coordinators: Richard Noss, University of London, UK
Celia Hoyles, University of London, UK
Rudolf Straesser, University of Klagenfurt, Austria

The Discussion Group on "Mathematics in Working Practice" starts from the well-known dilemma:

- Apart from basic arithmetic, Mathematics is not used at the workplace (a statement always offered by workers when asked to describe the Mathematics they use at work).

- The use of Mathematics is growing in production, reproduction and other societal domains and is penetrating more and more domains of practice.

Starting from these contradictory statements, the Discussion Group will look into:

- the current use of Mathematics at the workplace,

- research methodologies to explore the uses of Mathematics in the workplace,

- current and future ways to teach/learn Mathematics so that it connects more strongly with the workplace,

- the role of "new technologies" and other artifacts in using, teaching and learning work-related Mathematics,

- the foundation of a research network on Mathematics in vocational contexts.

Discussion Group DG4: Teachers' and Pupils' Mathematics-Related Beliefs

Coordinators: Erkki Pehkonen, University of Helsinki, Finland
Fulvia Furinghetti, University of Genoa, Italy

The importance of beliefs/conceptions in the study of cognitive and metacognitive phenomena, as well as teachers' behavior and attitudes in the classroom are widely recognized. The following passage[1] expresses very significantly this importance with respect to the study of students' performances:

> "purely cognitive" behavior is extremely rare, and that what is often taken for pure cognition is actually shaped - if not distorted - by a variety of factors. [...] The thesis advanced here is that the cognitive behaviors we customarily study in experimental fashion take place within, and are shaped by, a broad social-cognitive and metacognitive matrix. That is, the tangible cognitive actions produced by our experimental subjects are often the result of consciously or unconsciously held beliefs about (a) the task at hand, (b) the social environment within which the task takes place, and (c) the individual problem solver's perception of self and his or her relation to the task and the environment.

As for teachers, many studies have shown that beliefs are behind teachers' behavior in their classroom and act as a filter to indications of curriculum developers[2].

The discussion group will focus on the following three main issues, in addition to issues which will emerge from participants' contributions:

- Terminology: Some points need a careful discussion in order to agree on terminology and description of terms. Even the words 'belief' and 'conception' have different meanings for different researchers.
- Methodology: How do we detect and analyze beliefs, which in their nature are entities hidden and elusive?
- Effect: How may studies on beliefs/conceptions affect the classroom practice and strategies for teacher education?

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[1] Schoenfeld, A. H. 1983, Beyond the purely cognitive: beliefs systems, social cognitions, and metacognitions as driving forces in intellectual performance, *Cognitive Science*, v.7, n.4, 329-363.

[2] Thompson, A. 1992. Teachers' beliefs and conceptions: A synthesis of the research. In A.D. Grows (Ed.), *Handbook of research on mathematics learning and teaching*, 127-146. New York: Macmillan.

Discussion Group DG5: Understanding of Multiplicative Concepts

Coordinators: Tad Watanabe, Towson University, USA
Julia Anghileri, Homerton College, University of Cambridge, UK
Angela Pesci, University of Pavia, Italy

Mathematical concepts within the 'multiplicative conceptual fields" include multiplication and division operations, fractions, ratio and proportion, and linear functions (Vergnaud, 1988). Understanding of these concepts mark significant points in students' development as mathematics learners. Moreover, although these concepts are taught at different grade levels, their developments are closely connected (Harel & Confrey, 1994).

This dicussion group is a continuation of the Working Group on the same topic which had been in existence for 3 years. There will be opportunities for participants to informally share insights they have gained in their previous studies, discuss ideas they are planning to pursue and develop potential collaborative research opportunities. Although we will not limit our discussion to any specific topic, we have agreed to spend some time discussing two ideas: (1) proportional reasoning, and (2) development of algorithms for multiplicative operations.

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Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (eds.), *Number concepts and operations in the middle grades* (pp. 141-161). Reston, VA: National Council of Teachers of Mathematics.

SHORT ORAL COMMUNICATIONS

BECOMING A MATHEMATICS TEACHER EDUCATOR: REFLECTIONS AND ANALYSIS

Bridget Arvold
University of Illinois

If findings from research on becoming a mathematics teacher extend to becoming a mathematics teacher educator, then investigation into becoming a mathematics teacher educator is crucial. Based on the assumption that mathematics teacher educators impact not only their students but ultimately their students' students, studies of becoming a mathematics teacher educator may illuminate not only our understanding of a scaffolding of impact but also the complex process of becoming. Unfortunately, research related to teacher educators is minimal and focuses primarily on problems of identity (See Reynolds, 1995).

To initiate research in mathematics teacher educator's becoming, I scoured recent PME-NA conference proceedings for author interest in teacher education. A diverse group of seventeen of the 72 mathematics educators contacted, responding electronically to an open-ended request to share their thoughts on the meaning of becoming a mathematics teacher educator and on significant experiences that had influenced or were influencing them in becoming mathematics teacher educators. Attribution of sense making to the constructions of individuals and the introduction of only "sensitizing concepts" to research participants supported a research framework of interactionist theory as promoted by Blumer (1969).

Although interpretations of becoming a mathematics teacher educator were diverse and exposed distinct perspectives such as personal growth, upward movement in a hierarchy of authority, moral development, and service, a common focus on communication permeated the data. Expressions of personal "naivete" as a teacher as well as struggles to understand the "complexity of teaching" and of one's "impact on teachers" contrasted with expressions of exploring "mirrors to our souls" and "compromising an intellectual side." This small selective study suggests that a wide variety of interpretations of becoming a mathematics teacher educator and related attributions are prevalent. Follow-up research into how various interpretations of one's own becoming interacts with one's practice and impacts students and students' students warrants continued research in this area.

Blumer, H. (1969). *Symbolic interactionism: Perspective and method.* Englewood Cliffs, NJ: Prentice Hall.

Reynolds, R. J. (1995). The professional self-esteem of teacher educators. *Journal of Teacher Education, 46*, 216-221.

TWO ASPECTS OF THE MATHEMATICAL RATIONALITY. FUNCTIONS OF THE COUNTEREXAMPLE

Gustavo Barallobres, Mabel Panizza
Universidad de Buenos Aires, Argentina

This work deals with two identically constituent aspects of mathematical rationality, involved when dealing with conjecture, and which depend whether the aim is to determine the conjecture's truth or its validity domain.

According to this context, two different marks are particularly worth reading a counterexample when dealing with a refutation: either to inform us whether the conjecture is true or false (logical status) or to inform us the of conditions under which it would be true (mathematical status).

Both sides, neither opposed to nor subordinated to each other, have to do with the didactic responsibility domain.

After a cognitive and epistemological analysis, we have identified a number of variables that could be considered to be relevant for a parallel and complementary development of both aspects of mathematical rationality as they could command different focusing.

In doing so, we have mainly considered Balacheff' contributions (1991) in relation with the student's treatment of a counterexample and Duval's contributions(1995) in reference to the components of the sense of a proposition outside a theoretic context of enunciation.

The **variables** which have been identified are:

As regards the conjecture:

- Validity domain (finite, infinite)
- Author of the conjecture(student, partner, teacher)
- Knowledge nature (intuitive, learned)
- Interaction between the "epistemic" value and the logical true value

As regards the counterexamples:

- Quantity (one, few, a lot, infinite)
- Characterizable ensemble (yes, no)

During the oral communication, we will ground our choices of these categories and present a few problems belonging to the algebra domain which have been chosen on account of them. We will also report some results of an empirical study which aims to observe the cognitive functioning of the students (ages 15-18) when facing such problems.

References

Balacheff; N. (1991). Treatment of refutations: aspects of the complexity of a constructivist approach to mathematics learning. In E. Von Glasersfeld (ed.), *Radical Constructivism in Mathematics Education*, 89-110. Kluwer Academics Publishers.

Duval, R. (1995). *Sémiosis et pensée humaine*. Peter Lang.

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FROM NATURAL LANGUAGE TO SYMBOLIC EXPRESSION: STUDENTS' DIFFICULTIES IN THE PROCESS OF NAMING.

Luciana Bazzini, Department of Mathematics, University of Padova, Italy.

The use of symbolic expressions seems to be a relevant cause of difficulties in students at different school levels. Most difficulties emerge because of the incapability to relate the algebraic code to the semantics of natural language. The student able to express the relationships among the elements of a given problem correctly, by means of the natural language, can be unable to express the same relationships through the algebraic code.

The passage from natural to symbolic language is a key point in the development of algebraic thinking and asks for special attention in teaching.

Starting from the analysis of students' difficulties when they have to coach their stream of thinking in order to condense the most relevant aspects of a given problem into a formula, we have deserved special attention to the initial approach in the construction of algebraic expressions. Our focus is on the *process of naming*, i.e. the process of assigning names to the elements of a problem in order to construct a symbolic expression functional to the problem's solution.

In this process the role of letters is crucial. Letters can be used to give a name to the elements involved in the problem and to emphasize the relationships among the elements within an algebraic expression. The choice of names to designate objects is strictly linked to the control of the variables introduced: since algebraic formulas are usually not a linear stenography of what is expressed by means of natural language, difficulties emerge, especially for novices (Arzarello, Bazzini and Chiappini, 1994)

An additional contribution is given by a study carried out with a sample of 244 students (aged 15-16 years) attending the first two grades of upper secondary school in Italy. The study was set when the students have attended at least two years of algebra. A questionnaire was submitted to the pupils: all items required a translation of statements expressed in natural language into symbolic expressions involving the use of letters. In particular we are concerned in studying the students' responses in the case that the mathematical elements are already named by a letter, compared with the case when they are not.

Two main features emerge from the data analysis: a difficulty in the items not simply requiring a step by step translation from natural to symbolic language and a drop down of correct answers in the items requiring the process of naming for the elements of the problem. There is evidence that the semantic control of letters should be systematically carried out by means of adequate coordination of different semiotic registers (according to Duval, 1995).

Arzarello F., Bazzini L., Chiappini G.,1994, The process of naming in algebraic problem solving, *Proceedings of PME XVIII*,Lisboa, Portugal, Vol.II, (40-47).

Duval R., 1995, *Semiosis et pensèe humaine*, Peter Lang, Bern.

A RELATIONAL ANALYSIS OF A MATHEMATICAL LEARNING EPISODE

Margot Berger

University of the Witwatersrand

Most members of the mathematics education community would agree that learning is a complex and interrelated phenomenon. In this paper I demonstrate how the framework of Activity Theory (Leont'ev) offers a tool for the analysis of mathematical learning episodes which is commensurate with that perspective.

Certain mathematics educational researchers (such as Crawford, Meira) use Activity Theory or aspects of it in their interpretations, but detailed and multi–leveled analyses in terms of this theory are not common in the literature of mathematics education research.

In line with Activity Theory, which proposes purposeful human activity as the unit of analysis, I explicate mathematical learning in the classroom in terms of the **relationships** between the pupil's activities and the different elements of the learning system: the pupil with his/her own history and motives, the goal of the mathematical task as subjectively perceived by the pupil, the activities of the pupil (purposeful or automated), the actual mathematical task, the social interactions (which include the teacher's mediations), the tools and the setting.

Furthermore I argue that although it is necessary to regard these components as analytically distinct, they are, in fact, interrelated and mutually constitute the learning episode.

The episode which I analyse and interpret, derives from a case study of a lesson in which two Grade 8 girls, working as a pair, are videod as they embark on a mathematical task. The task involves moving from a graphical representation of the changing water levels in a bath tub to a real–world description of a sequence of events which could generate such a graph. The task is given in a lesson published on the internet and as such, is presented to the girls on a computer screen.

In this particular episode, the relationship between goals and activities is foregrounded. The analysis shows how the pupils struggle to articulate the goal of this task; how teacher mediation is required to explicate the goal; how the motives and cultural background of each girl serve to re–orientate the goals of the other and how the maths activities change as the goal does.

This small example illustrates the usefulness of Activity Theory as a multi–focal lens through which to view the mathematical learning that may take place in the classroom.

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THE EFFECT OF DRAGGING IN A DYNAMIC GEOMETRY ENVIRONMENT ON STUDENTS' STRATEGIES SOLVING LOCUS PROBLEMS

Irina Bershadsky and Orit Zaslavsky

Technion, Haifa

The concept of locus of points can serve as a unifying idea in geometry, reflecting a basic and general approach in modern mathematics. In school, traditionally, the approach to the notion of locus puts more emphasis on deductive inference and computational methods for finding the algebraic equations of a limited class of geometric loci, rather than pays attention to intuitive, geometric, visual aspects, which are at the heart of this notion (Hershkowitz, Friedlander, Dreyfus, 1991).

The special visual features of Dynamic Geometry Environments open new perspectives and powerful opportunities for mathematics education. However, these opportunities can raise new difficulties. To make optimal use of computerized environment, one must understand how students interact with the environment and how students' mathematical thinking is reflected and affected by their use of the environment (Hazzan & Goldenberg, 1997).

The goals of this study are to explore the way in which dynamic features are used by students while solving locus problems and the effect that this use might have on the understanding of the concept of locus.

We report the main results of one case-study investigating the knowledge constructed by a Grade 11 student while solving locus problems in a Dynamic Geometry Environment during 12 sessions of about 2 hours each. A qualitative data analysis was based on everything that the student said, constructed on the computer screen and wrote on paper during these sessions.

Data analysis points to several stages through which the student went: At first, the student was inclined to implement a *segment-based approach* to construct loci defined by their properties. Then, by dragging points and marking their trace, the student was able to find and reflect on the geometrical relations and properties that remained unchanged under the dragging transformation. This process led to the student's insight of "seeing" in his mind the locus as a global entity. The dragged points on the computer screen played a significant role in the student's strategies. Finally, the student developed an awareness of the limitations of his own *segment-based* strategy and moved to a more global perception of locus and its construction.

Hazzan & Goldenberg, (1997). Students Understanding of the Notion of Function in Dynamic Geometry Environments. *International Journal of Computers for Mathematical Learning*. Vol.1, pp. 263-291.

Hershkowitz, Friedlander, Dreyfus, (1991). Loci and Visual Thinking. In *Proceedings 15th PME Conference, Assisi (Italy)*. Vol. 2, pp. 181-188.

USING THE UNDERGRADUATE MATHEMATICS CLASSROOM TO CHALLENGE PROSPECTIVE SECONDARY TEACHERS' NOTIONS OF MATHEMATICAL DISCOURSE

Maria L. Blanton

University of Massachusetts Dartmouth

The way that teacher and students interact with mathematical ideas in the social context of the classroom structures students' thinking about mathematics (Wertsch & Toma, 1995) and so ultimately, prospective teachers' [PTs] thinking about teaching mathematics (Lortie, 1975).The purpose of this proposal is to discuss results of using a one-semester, undergraduate geometry course to challenge PTs' notions of mathematical discourse [MD]. The course was structured to engage PTs (9 total) as student participants in MD, as students in the pedagogy of MD, and as architects of MD. As geometry students, PTs were expected to establish mathematical ideas through conjecture, justification, and peer argumentation. The resulting MD, which PTs reported as differing from their previous experiences in undergraduate mathematics, became a springboard for in-class analyses on the nature of our dialoguing. The characterization of discourse as univocal or dialogic (Wertsch & Toma, 1995; see also Lotman, 1988) was used to frame discussions about MD. To engage PTs as architects of MD, each PT taught a one-hour lesson as part of the class. The lesson was videotaped, transcribed, and analyzed with respect to the nature of MD by the PT. Data for this study consisted of video recordings of the geometry class, PTs' discourse analyses of their teaching, clinical interviews with each PT about his or her discourse analysis, PTs' pre/post reflections about their notions of MD, and PTs' reflections about the pedagogical and mathematical structure of the geometry course. Results indicate that PTs matured throughout the semester in their ability to engage in and reflect on MD. Moreover, by examining MD in the mathematics classroom, PTs were able to dissect a mathematically authentic discursive event *as they created it*, allowing them to apprentice more powerful techniques of MD than might be possible in other contexts. Additionally, PTs' teaching episodes and subsequent discourse analyses revealed to PTs the complexity of MD and deepened class discussions about MD. In conclusion, the model used in this class seemed to give PTs a concrete, experiential approach to understanding the MD needed to support students' mathematical thinking.

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MATHEMATICS AND LITERATURE: LEARNING MATHEMATICS IN A FOLK-TALE CONTEXT

Malka Brender, Nomy Dickman, Rachel Heller, Larisa Marku, Alla Shmukler,

Clara Ziskin, and Orit Zaslavsky

Technion, Haifa

In the past decade there have been many calls for connecting mathematics and other domains (e.g., NCTM, 1989). The reasons for such calls are cognitive as well as affective. The current study is based on the assumption that connecting literature and mathematics can help decrease students' math anxiety, motivate them to learn mathematics, and consequently, improve their attitudes towards mathematics

For the purpose of the study, a special booklet was developed called "On the Fairy Tale Wings". This booklet continues a long lasting tradition of combining mathematics and literature, which began in the 19th century. It includes the original text in addition to a Hebrew translation of a Russian folk-tale called *Princess the Frog*, new episodes interlacing the main plot and its characters, and mathematical and logical problems associated with these episodes. The booklet and its activities are intended for middle and high school students. For example, one type of problem deals with plotting pictures of the various folk-tale characters by computer generated graphs of functions.

In order to study possible effects of learning mathematics in such a context, an experiment was conducted in which students had to read the folk-tale and engage in the accompanying assignments. At the end of the experiment the students were asked to pose new problems based on this folk-tale as well as other tales that they know.

Three 8th grade classes of students participated in the study. An attitude questionnaire was administered prior to the experiment and after its completion. In addition, classroom observations and student and teacher journals were documented and analyzed. The findings point to students' increase in positive attitudes towards mathematics and their inclinations and desires to combine mathematics and literature in school. Interestingly, students' motivation to pose worthwhile mathematical problems was also enhanced.

ACQUISITION OF THE MODEL OF PROPORTIONALITY SUPPORTED BY A HYPERMEDIA DOCUMENT

Isabel Cabrita and Armando Alves de Oliveira
Universidade de Aveiro – Portugal

Our study was based on three fundamental pillars — problem-solving, the model of direct proportionality and the 'new' information and communication technologies. The main objective of the study, which was carried out with 7th Grade students, was to develop and validate a hypermedia document which, designed according to a problem-solving methodology and a constructivist view of learning, might intervene in the acquisition of the model of Direct Proportionality (Alves de Oliveira & Cabrita, 1997 e Cabrita 1999).

We opted for an extra-curricular experimentation of our hypermedia programme for two main reasons: firstly, the use of computers in the classroom situation is not (yet) a systematic reality in Portuguese schools; and secondly, for several reasons which have to do essentially with the democratization of schooling, the classroom space is not the most suitable to cope with the individual rhythm of each student, or, cause and consequence of the above, to respect different browsing behaviours. Although we believe that this strategy could bring added advantages, we were conscious of the fact that it would certainly bring with it new responsibilities for teachers who will increasingly have to know how to negotiate the knowledge acquire in spaces other than the microcosm of the classroom, leading to a broadening of the horizons of their professional development

In this context, and following the conclusion of the research (Cabrita, 1998), which we propose to divulgue in this paper, it was possible to offer indications in three directions: about how interactive teaching programmes should be designed and put into effect; about how, where and in what conditions they should be exploited and about the implications of this study for teacher education.

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PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' CONCEPTIONS OF FUNCTION: MATHEMATICAL AND PEDAGOGICAL UNDERSTANDINGS

Insook Cha, University of Michigan, Ann Arbor, USA
Melvin (Skip) Wilson, Virginia Polytechnic Institute and State University, USA

This study investigates the three research questions: (1) What are the common definitions of function given by preservice teachers?, (2) What are the common preferred definitions of function for teaching given by preservice teachers?, (3) What conceptions tend to impede or allow the selection of certain definitions of function for teaching? Twenty-one preservice teachers enrolled in a required mathematics methods course in the Fall Semester 1997 at a large mid-western university in the US were subjects of the study. Preservice teachers' conceptions were investigated both before and after they completed an instructional unit about the teaching of functions. Open-ended questionnaires, observations, and written documents of 21 subjects were collected and three subjects were each interviewed twice.

Most subjects could not give more than one definitional description nor discuss features of definitions. Analogical (machine/equations) definitions were the most popular. Logical (set-theoretical) and genetical (dependence) definitions were not common. Before the instructional unit, when selecting definitions to use in teaching, most subjects were concerned with pedagogical reasons (whether they thought students could easily understand/memorize the definition) rather than mathematical reasons (whether the definition allows students to learn the nature of the function concept). In contrast, after the unit most subjects were concerned with mathematical reasons. This study also illustrates the importance of integrating different domains of knowledge for a good pedagogical content preparation. An understanding of functional relationships and their relation to the genetical definitions as well as an appreciation for the usefulness of functional relationships and the importance of multiple representations within and outside mathematics were all significant factors allowing the selection of genetical definitions. Preservice teachers' conceptions of students' understandings and potential misunderstandings of the function concept were important factors impeding logical definitions (particularly, correspondence and set definitions). Some preservice teachers were reluctant to select a correspondence definition for students because they believed that students might not readily grasp the ideas of correspondence or sets.

MATHEMATICS TEACHERS' USE OF THE NEW TECHNOLOGY

Cosette Crisan
South Bank University

Interest in teachers' subject matter knowledge and pedagogical content knowledge has risen in recent years (Cochran *et al*, 1993, Meredith, 1993, Ruhama *et al*, 1996). With the increased availability of the new technology (graphics calculators and/or computers together with different items of educational software) in our schools, it is important to examine how these components of teachers' professional knowledge base affect and are affected by its use in teachers' instruction (Zehavi, 1997, Simmt, 1997).

The focus of this session is to report on a pilot study that involved a number of secondary mathematics teachers who regard themselves as confident and competent users of IT in their mathematics lessons. Three main factors affecting teachers' use of the new technology emerged from the analysis of the interviews: types of hardware available in schools and their accessibility, software & resources on the use of IT and departmental policy. The presentation will also give an account of teachers' reasons for starting to use IT, reasons for using IT in their mathematics lessons as well as reporting on teachers' views regarding the value of using IT with respect to todays' school mathematics, teaching of mathematics and their knowledge of mathematics.

In the analysis of the data there is evidence to suggest that the use of the new technology enriches and challenges teachers' knowledge of mathematics as well as their instruction. The findings point out to the importance of teachers' own learning experiences when doing mathematics with the new technology.

The longer term intention is to develop a theoretical model which accounts for teachers' learning about IT and teachers' incorporation of IT into their planning for teaching mathematics. It is hoped that this will have useful implications for pre-service and in-service teacher education area.

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An Activity Approach to Dynamic Assessment

Chris Day

South Bank University

Ferrara (1987) investigated static tests which provide an index of a child's developmental level. She discovered that the rate of adaptation to similar, but more complex problems can provide a far more comprehensive index. This is indicated inversely by the number of hints needed in the second training phase of a test-train-retest procedure. In applying her assessment procedure, Ferrara (1987 p 106) found that children who scored highly on the post-test were not simply the ones who did well on the pre-test. Multiple regression indicated that background knowledge accounted for 22% of the variance in residual gain scores and neither IQ nor 'Learning Efficiency' (total hints to reach mastery in simple addition problems) produced any increase in this. However 'transfer efficiency' (total hints to reach mastery in 'transfer and far transfer' problems) explained an additional 32% of the variance in residual gain scores.

I have adapted Ferrara's basic method for use in the evaluation of a mathematics teaching program that I have conducted in English middle schools. This mathematics teaching program was based on principles of Activity Theory described by N Talyzina (1981). Although my intentions were rather different, my own assessment results were consistent with Ferrara's. I have, however, taken the view that Dynamic Assessment is essentially an assessment of social interaction and cannot provide an index of individual potential. I will suggest some theoretical considerations concerning the genesis of the generalisation and fluency of mental actions that I consider to be of central importance for the development of Dynamic Assessment procedures. I will also briefly present some of my results and I will outline some modifications to Ferrara's method which I have used to develop the Dynamic Assessment procedure within the theoretical framework of Activity Theory.

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THE " CESAME " PROJECT: MATHEMATICAL DISCUSSIONS AND ASPECTS OF KNOWLEDGE

Jean-Philippe DROUHARD
DIERF (IUFM de Nice)
drouhard@unice.fr

Catherine SACKUR
GECO (IREM de Nice)
sackur@unice.fr

Within the research field of Social Constructivism (Ernest, 1995), we are working on a project called CESAME: in Nice (France): J-Ph. Drouhard, M. Maurel & C. Sackur; in Paris (France): T. Assude & N. Douek; in Madrid (Spain): Y. Paquelier; in Buenos Aires (Argentina): G. Barallobres & M. Panizza. The word, "CESAME", is an acronym of the following French words: "Construction Expériencielle du Savoir" et "Autrui" dans les Mathématiques Enseignées ", which could be more or less well translated as: " the Subject's Experience of Dialogues with Interlocutors ("Others") in the Social Construction of Taught Mathematical Knowledge ".

Our starting point was the following. What is the teacher supposed to do, at the end of a *Scientific Debate* (Legrand, 1988) or more generally of a *Mathematical Discussion* (Bartolini-Bussi, 1991), in order to turn the many statements yielded by the discussion into "official" mathematical statements, which is "Institutionalisation" in the terms of the Theory of Didactic Situations (Brousseau, 1997, Margolinas 1992, 1993)? And what is s/he actually doing? On the one hand, in a strict Social Constructivist point of view, the group is supposed to " construct " the whole mathematical knowledge by itself, the teacher's role being just to lead the discussion. On the other hand, the teacher actually says something about the new common knowledge. Well, what does s/he says?

Obviously, the teacher institutionalises something more than the strict 'content' of the mathematical text. In order to give a framework to this idea, we propose to consider that knowledge present three aspects. The *first aspect* is made of the mathematical *content* of the knowledge, the *semantics* of the related statements (definitions, theorems...). The *second aspect* contains the rules of the mathematical game ('game' in the sense of Wittgenstein), the principles that make the definitions define as they are supposed to do, the theorems prove as they are supposed to do etc. The *third aspect* contains the most general believes about mathematics, as for instance "mathematics is a matter of understanding" or in the contrary (as many students say), "algebra is just a matter of formal rules".

Within this framework, to learn mathematics is to learn mathematical contents (aspect I), is also to learn *how to do* mathematics the way mathematics are supposed to be done (aspect II) and is also to learn *what* are mathematics (III). On the other hand, to teach that a statement is mathematical, is also to teach that mathematics are made (III) of statements like the one which is taught (I)!

After a first theoretical study we are at present collecting more empirical data.

WHAT DO STUDENTS DO WITH CONJECTURES? PRE-SERVICE TEACHERS' RESPONSES TO A GENERALIZATION TASK

Laurie D. Edwards

St. Mary's College

Rina Zazkis

Simon Fraser University

Problem-solving and generalization are central activities in mathematics, with the solution to difficult problems often requiring generalization across specific cases. In order to support students' learning of generalization , teachers of mathematics require experience with this process themselves. This study examines the work of undergraduate students enrolled in a mathematics course for prospective elementary school teachers, as they grappled with a challenging problem in number theory (determining the number of squares crossed by a diagonal in a rectangular grid). The central research question examined how students responded to evidence that did not confirm their conjectures. Although in such a case, it is mathematically normative to reject or revise the conjecture, not all students had internalized this norm. An analysis of 27 written problem solving journals revealed that although few students were successful at finding a complete and correct general solution to the problem, most responded appropriately to disconfirming evidence. Variants from the normative response are described and implications for instruction discussed.

Literacy-in-mathematics perception of mathematics educators: From Literacy-in-mathematics to mathematics literacy

Hanna Ezer, Levinsky College
Dorit Patkin, Hakkibutzim College of Education
Shosh Millet, Achva Academic College

In this paper we present the relations between literacy and mathematics as perceived by mathematics educators such as pre-service teachers, in-service teachers and teacher trainers. The relations between mathematics and literacy develop on the backdrop of the perception of using literate activities in the disciplines in order to develop learning and thinking (Fulwiler and Young, 1990; Ackerman, 1993).

The purpose of this study is to investigate through questionnaires and interviews the participants' perceptions of the notion of literacy in general, the relations between literacy and mathematics in particular and the contribution of literacy to the teaching and learning of mathematics. All participants had been exposed to the notion of literacy in mathematics during a one year in-service and pre-service courses.

Findings indicate that students and teachers of mathematics hold medium to low perception of literacy in general and literacy in mathematics in particular, though teacher trainers hold high perception of both. Yet, pre-service teachers reveal higher perception of literacy in general, probably due to exposure to literacy learning in other courses in a teacher training college.

Following the study results we recommend the use of literacy activities such as oral discussion, reflective journal, writing process (Waywood, 1992; Scheibelhut, 1994) etc. in mathematics in-service and pre-service courses. The training period should be longer, using reflective tools and coaching to strengthen the perception of **literacy-in-mathematics** in order to develop **mathematics literacy** in teachers and student teachers as well as in school students.

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TEACHER DEVELOPMENT: A SELF-INQUIRY APPROACH

Razia Fakir Mohammad, University of Oxford, Department of Educational Studies, Oxford, UK

In this presentation I offer and analyse accounts of how teachers and a mentor learned about their practice through a process of self-inquiry in a research study in Pakistan. Action research was the method for analysing and learning from the significant accounts of the teachers' practice. The purpose of the study was not to examine the meaning of being a teacher, nor to provide teaching strategies and underlying teaching practice, but to explore the ways in which action research contributed to the teachers' process of development by helping them to create their own theories of teaching and learning. The investigation was carried out in a Karachi school as a partnership between a mentor and two mathematics secondary school teachers in 1996. The agenda for study had two parallel strands: in one, teachers learned to search their own classroom practice on their own as insider and perceived themselves as an agent of change of their own practice rather than received knowers; in the second the mentor was involved in researching the teachers' learning as a result of teachers' research. It was a two-level inquiry with a co-operative learning agreement. As a participant observer the mentor-researcher performed the role of an observer, who observed the lessons and took notes; a listener, who listened to the teachers very carefully; a stimulus, who encouraged them to reflect on the significant aspects of teaching and learning; a task-keeper, who focused their thinking; and a guide, who helped them to improve their practice and develop their own theories of learning. On the other hand the teachers were involved in planning, implementing and reflecting through the period of study. They had three roles: teachers - planning and implementing their teaching; researchers - researching their practice by examining and reflecting; learners - developing new understanding of children's learning mathematics. The data was collected in one academic year. The mentor-researcher's data included field notes, transcription of audio recording of dialogue and notes from reflective journals. Teacher-researchers maintained reflective journals as a record of significant accounts of teaching and meeting with mentors. All the details of data were shared among the participants(teacher-researchers and mentor-researcher) on an on-going basis. There was on-going data analysis, which helped the participants to plan and decide further actions. The findings show action research as an evolving experience, because of the nature and significance of development of self-reflection and a sense of ownership of the participants concerning their actions and development of theories in collaboration. This presentation will share some insights, which show how reflection and dialogue on the classroom events helped the teachers to learn and understand the different aspects of mathematics and teaching mathematics and an understanding of action research as a result of practical experience of conducting the research.

MEANING PRODUCTION FOR FUNCTION: PARALLEL AXES

Authors: Janete B. Frant and Monica Rabello de Castro

Affiliation Universidade Santa Úrsula - Instituto de Educação Matemática

The purpose of this study is to investigate how high school students produce meaning for linear function while working in an unusual environment. This environment, called P.A.R—Parallel Axes Representation, was proposed by Arcavi and Bruckheimer (1996). In this presentation we will discuss the process followed by two students while developing a formula to relate the cartesian representation to PAR representation.

The theoretical framework is based on the notion of meaning production proposed by Lins (1997). We agree with Lins that meaning production is related to enunciation, actually he defined meaning production as everything that is effectively said about an object. In this way, we are analyzing the dialogues that arise during students' argumentation. In order to analyze data we develop a argumentation strategy model based on Perelman's Argumentation Treatment (1996).

The study took place in a public high school in 1997. Two 10th grade students were videotaped during 6 one-hour-and-forty-minute meetings.

Three points will be discussed: 1) How to find the image of a point in PAR, 2) the Crucial Point and 3) the Formula to enable relating cartesian and PAR representation.

Fragments of Crucial Point Dialogue

1. P- I told you to not connect
2. F- So it's all wrong
3. P- No. Why is it all wrong?
4. F- Because if it doesn't work for one [point], it can't work for the others.
5. P- Cool
6. F- Uaaauuu, it always connect in the same point. I discovered it the Crucial Point.
7. F- Is everything already invented or we can invent things too?
8. P- I don't know
9. F- Because I'll call it crucial point because I think it's cool, but I do think they already invented it.

Partial Findings: We found that working in this unusual environment students were able to be flexible in using algebraic and graphic representation, to elaborate a formula to pass from cartesian to PAR representation and vice-versa. Moreover, they believed they were able to invent mathematics.

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Computers and the teachers' role in mathematics learning environment. Some episodes from the classroom.

Anne Berit Fuglestad

Agder College, Kristiansand, Norway

In a teaching experiment, with students of age 10 - 14, computers were used in the teaching of decimal numbers with some spreadsheet tasks (Fuglestad, 1996b; Fuglestad, 1996a). Based on a constructivist view of learning (Davis, Maher, & Noddings, 1990), the aim was to implement a diagnostic teaching approach (Bell, 1993) to stimulate the students' construction of knowledge.

In order to help the class teachers combine the use of computers with a diagnostic teaching approach, some spreadsheet tasks were provided on student worksheets. The teachers were given an introduction to computers in the classroom including spreadsheets, diagnostic teaching and the worksheets over three days.

The teachers' role appeared to be crucial in utilising the potential of the computer and spreadsheet tasks. Clearly, this can be seen from episodes where the students' misconceptions were provoked and discussed, and from other episodes which revealed less. The teacher's or observer's intervention, asking questions or giving suggestions for further trial, was of major importance. The teacher should not give the answers too quickly, but give the students time to reflect on their experience and discuss in small groups and in class. There was also a need to give clear introductions, to follow up, to summarise findings and to provide further discussion in the class.

The teachers' awareness of diagnostic teaching was vaguer than expected at the end of the research. However, some elements were implemented and in particular some spreadsheet tasks helped them to achieve this. The computer apparently has the potential to be a useful tool in diagnostic teaching, but there is a need to strengthen the teachers awareness of their role in implementing this.

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What do Mathematics Senior Phase Teachers Understand about the new Outcomes Based Curriculum 2005?

Ms. Mellony Graven
RADMASTE Centre, University of the Witwatersrand, South Africa

The Programme for Leader Educators in Secondary Mathematics Education (PLESME) began workshops with teachers from schools in Soweto and Eldorado Park (both urban townships outside Johannesburg) in January 1999. The primary goal of this programme is to:

* create leader teachers in mathematics with the capacity to interpret, critique and implement current curriculum innovations in mathematics education in South Africa and to support other teachers to do the same.

South Africa is currently embarking on radical curriculum change which aims to implement an outcomes approach to education. This curriculum places a significant emphasis on the contextualisation of mathematics, socially, politically, economically and historically. It also places a significant emphasis on mathematising particular mathematical processes such as mathematics communication, interpretation and justification.

Previous research work, conducted by myself, indicated that very little information about curriculum development was being disseminated in these areas. The interim curriculum, which was to be implemented at the Senior Phase Level in 1995, had never reached the schools and little or no information had reached them about the new outcomes based education curriculum which was initially scheduled to be implemented at the Senior Phase level in 1997.

The research work presented here is part of a broader two year research plan which aims to investigate mathematics teachers' learning especially as it relates to implementing 'new' aspects of mathematics emphasised within the current South African context of curriculum change.

The PLESME Programme and its related research began in October 1998. This research paper will look at some preliminary findings of baseline questionnaires and interviews conducted with ten teachers from Soweto and Eldorado Park. The interviews and questionnaires covered a range of topics such as teachers' views on how mathematics should be taught, their understanding of the new curriculum and its related mathematics specific outcomes, who teachers talk to about their mathematics teaching etc.

In this presentation I will discuss teachers' understanding of the new curriculum and the related mathematics outcomes as revealed by analysis of the transcriptions of recorded interviews with teachers and their written responses to the questionnaires.

Some preliminary findings indicate that:

- teachers have not received any documentation relating to the new curriculum. Most of their opinions have been informed by what they have read in newspapers and what they have heard from teachers who have implemented it in grade 1.
- there is a mixture of positive and negative attitudes towards the value of the new curriculum and variation in their understanding
- teachers have difficulty understanding the meaning of the maths specific outcomes

WHEN OBSTACLES SEEM TOO BIG

Michael D. Hardy

University of Southern Mississippi Gulf Coast

This paper is based on a case study of Sharon, a middle school mathematics teacher who endorsed a constructivist epistemology and strove to use a variety of instructional strategies to teach for understanding as well as computational skill. The goal of the study was to gain insight into how Sharon made sense of her teaching experiences by investigating both her constructions of such experiences and her beliefs about teaching and learning.

Over the course of the study, it became apparent that a discrepancy existed between Sharon's teaching and her ideal for teaching. Sharon was aware of this discrepancy but was able to justify it. Hence, it did not perturb her. Sharon's efforts to use a variety of instructional strategies were further impeded by student resistance, her conception of her role as an educator, and limited engagement in critical reflection. However, participation in communicative discourse (Habermas, 1987) could have perturbed Sharon with respect to her beliefs and practice, helped her envision alternatives for her practice, and provided support for her efforts to alter her pedagogy. This is significant because all of these potential outcomes are components of the process of teacher change (Shaw & Jakubowski, 1991). Thus, participating in communicative discourse could help teachers alter their practice and overcome pedagogical constraints.

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ATTITUDES OF PROSPECTIVE HIGHSCHOOL MATHEMATICS TEACHERS TOWARDS INTEGRATING INFORMATION TECHNOLOGIES IN THEIR FUTURE TEACHING

Orit Hazzan

Technion – Israel Institute of Technology, Haifa

& Oranim – College for Education, Tivon

ISRAEL

Hundreds of papers are published nowadays arguing that computers become an integral part of our lives and, as such, should be integrated into educational systems as well (Cf. Edelson, Pea and Gomez, 1996; Flake, 1996). Since such integration requires a change in teaching methods, teachers play a central role in such a transition. Of course, this is also true in regard to teaching of mathematics.

The talk presents attitudes of prospective mathematics teachers towards integrating computers in their classes in the future. A course took place, which focused on didactical and cognitive aspects of learning mathematics with computers. At the end of a course, ninety-four prospective teachers (from 4 classes) were asked to present reasons, pro and con, that would influence their use of computers in their future mathematics teaching. Based on written questionnaires and class discussions, the reasons given by the prospective teachers were grouped into the following two-dimensional theoretical framework:

	learner	teacher	mathematical content	learning environment	class atmosphere
cognitive factors					
affective factors					
social factors					

In the presentation I explain what each category means, present reasons given by the prospective mathematics teachers, and suggest some plausible implications of these attitudes on mathematics education.

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STUDENT'S OWN MATHEMATICS AS AN EXPLANATION FOR ERRORS IN DRUG CALCULATIONS

Sinikka Huhtala

Department of Teacher Education, University of Helsinki, Finland

Accurate mathematical calculations are a critical skill that nurses must demonstrate in order to safely administer medications. However a review of published research shows that many nursing students are unable to accurately calculate medication dosages because they are deficient in basic mathematical skills: they have deficiencies in decimals, fractions and metric conversion. They also fail to recognize incorrect answers, they have negative attitudes toward mathematics and little confidence in their ability to solve problems. (e.g. Cartwright 1996, Huhtala 1996, Pozehl 1996)

This study explains the errors in drug calculations with student's own mathematics. Data for this study has been collected by tape recording small group instruction of practical nurse students in a mathematics clinic (Case studies).

The student's own mathematics consists of experiences as a learner in mathematics, emotions towards mathematics and strategies and mini-theories which students use while studying mathematics. When a student has very negative experiences and math anxiety, she/he tries to avoid mathematics and chooses superficial strategies while working with mathematical problems instead of trying to understand. These strategies may change to permanent mini-theories like "in division you must always divide the bigger number by the smaller" or that "multiplication always makes bigger and division always makes smaller" or " when you convert grams to milligrams the answer always has four numbers". As a result this own mathematics leads to errors like this:

> Mrs. Malmi (aged 64) is to take 40 IU of Insulin. The strength of Insulin is 100 IU/ml; how much should the patient be taking?
>
> 100 IU/ml : 40 IU = 2,5 ml

The short oral presentation will further describe the student's own mathematics and how this should be taken into consideration in the instruction.

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INSTITUTIONAL RELATION TO A MATHEMATICAL CONCEPT: THE CASE OF LIMITS OF FUNCTIONS IN MOZAMBIQUE

Danielle Huillet and **Balbina Mutemba**
(Eduardo Mondlane University, Maputo, Mozambique)

Changes of settings introduced by the teacher could help the students to understand different aspects of a concept and make their conceptions evolve (Douady, 1986). The concept of limit may be defined and limits determined in a purely analytical setting, but concept may also be illustrated and studied in numerical and graphical settings. Students may learn to work in various settings, but experience indicates that they often see no link between them.

This paper presents the first results of an on-going research which aims to construct a didactical engineering (Artigue, 1992), using changes of settings, in order to improve the teaching and learning of limits of functions in the Mozambican Secondary School.

In the very first place, and using the anthropological approach of Chevallard (1992), the institutional relation of the secondary school to this concept has been analyzed through the contents of curricula, textbooks and final exams. This study suggested that limits of functions are shown as algebraic transformations and procedures that the students have to learn in order to calculate limits. Usually the results of the calculation are not interpreted.

The only book produced in Mozambique on this topic at this level has a very formal approach (formal definitions and demonstrations). However it only requires the students to apply procedures. In the final secondary school exams, there are always two exercises to calculate complicated limits and very few exercises linking limits with graphs. It seems that the secondary school institution has an "algebraic conception" of limits of function as an opportunity of creating algebraic transformations for the learners to apply.

In that moment the relationship of some secondary school's teachers to this concept is being studied through a questionnaire and interviews.

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THE PROCESS OF ACCEPTANCE OF THE REALISTIC MATHEMATICS CURRICULUM BY TEACHERS IN HARARE

GODWIN HUNGWE

UNIVERSITY OF ZIMBABWE

The study will investigate the process of acceptance of the realistic mathematics curriculum by secondary school mathematics teachers in Harare, Zimbabwe.

According to the realistic mathematics curriculum students should be guided to re-discover mathematical concepts and skills from contextual situations and from class discussions. Teachers should stimulate pupils to find their own solutions instead of teaching by demonstration (Freudenthal, 1973; Gravemeijer, 1994; van Galen & Feijs, 1991). Research in The Netherlands showed that initially teachers followed the realistic mathematics curriculum mechanically but after familiarisation, they implemented it more flexibly (Gravemeijer, 1994).

The current mathematics curricula in Zimbabwe were investigated for evidence of the presence of the realistic mathematics curriculum. A sample of mathematics teachers in Harare, will complete questionnaires, be interviewed and observed while teaching. Only the teachers who are unaware of the realistic mathematics curriculum will be in-serviced in the processes of this curriculum and be observed while teaching. They then will complete questionnaires and be interviewed again.

Documentary evidence revealed that the current mathematics curricula in Zimbabwe do not promote the realistic mathematics curriculum. The pre-treatment questionnaires and interviews will hopefully reveal the teachers' current beliefs and practice in mathematics instruction, while on the other hand the post-treatment ones may reveal acceptance or non-acceptance of the curriculum by the teachers.

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The Effects of the Exposure to the "Math is Next" Database on Teachers' Methods in Teaching Mathematics to Young Children

Bat-Sheva Ilany, Ilana Binyamin Paul, Miriam Ben Yehuda, Rina Gafny, Nehama Horin
Beit Berl College, Israel

"Math Is Next" is a computerized database intended to help the teacher to construct a work program for developing mathematical thinking in young children. The database assists the teacher in planning the work and in constructing a rich, experiential and authentic mathematical learning environment, that combines creativity and use of language and is appropriate for the individual needs of every learner. At the foundation of the database is a matrix that lists the academic skills, on one axis (mathematical concepts and development of thinking skills), and the cognitive processes through which we transmit information (processing of visual, audio and sensomotor information and memory), on the other axis. These information processing options play an important role in everyday life and in acquiring the foundations of academic skills, reading and arithmetic. Junctions of coordinates form "cells" and we inserted the appropriate activities in each cell. Attached to the activity card there are, frequently, related cards with additional suggestions and games for similar materials that can be prepared for the children.

The program was tested in 21 kindergartens and lower elementary school grades. The study findings indicate improvement in the teachers' self-confidence and sense of competence to teach mathematics, as well as changes in their methods of working in the kindergarten and classroom. The study method was qualitative, employing the following tools: Semi - structured interviews, Observations, Supervision and reporting. The study findings indicate that following the use of the database, changes took place in two principal areas: A. Changes in the teacher's sense of competence and degree of confidence and readiness to deal with the subject of the development of mathematical thinking in young children. B. Changes in the teacher's teaching methods came along with the developing sense of competence and acquired self-confidence:

The findings show that the "Math Is Next" database is user friendly and easy to operate. The teachers did not view the database as "just another tool", but as a pivotal tool that can assist and can be "relied" upon in constructing the curriculum for young children. Other teachers pointed out the substantial contribution of the database to the teaching of mathematics to young children and the changes that took place in their teaching. According to them, the principal contribution of the database is its structured organization (search table) that enables the teachers to teach mathematics by employing a variety of teaching methods, with which they were unfamiliar prior to encountering the computerized database.

The conclusions of this study are that when teachers are offered a tool that presents content in a well organized and well defined form that fulfills their teachers' needs, they feel greater confidence and competence in dealing with teaching mathematics. As a result, these teachers engage the children in mathematical activities more frequently and are more inclined to attempt a variety of new teaching methods.

*Demo of "Math Is Next": **http://www.beitberl.ac.il/~intmat/eng.htm***

DEVELOPING SKILLS OF ADVANCED MATHEMATICAL THINKING

Kahn, P. E., Department of Mathematics, Liverpool Hope University College

It is becoming apparent that many students studying mathematics in higher education do not possess the necessary skills of advanced mathematical thinking, especially in the UK [1]. This paper describes an action research project that sought to help 40 first year undergraduates at the author's institution develop a variety these skills, going beyond a focus on one skill.

The project initially identified several skills as critical to the study of advanced mathematics. These were the ability to write mathematical text, think logically, solve problems, improve own learning of mathematics, and make connections between concepts [2]. A metacognitive approach to skill acquisition was followed, in which the aim was to direct students towards active and deep, rather than surface, approaches to learning. The students were alternatively introduced to content from abstract algebra and to the skills themselves, with learning in each of these areas reinforcing learning in the other area.

A variety of methods were employed to consider the effectiveness of the innovation. Student evaluation schedules and examination results indicated that the innovation was positively received and led to good outcomes. Coursework and the teaching experience indicated that the skills were unfamiliar. They further indicated that too many elements of the first skill, writing mathematical text, had been covered in the time available. Work on developing the remaining skills was thus scaled down, with outcomes on coursework improving. The need for an appropriate balance of familiar and unfamiliar material was also apparent. A standard instrument [3] indicated increases in students' confidence for their study and that motivation had remained good. Qualitative questionnaires further linked increases in confidence to good outcomes and suggested this had been aided by the integrated nature of the work on skills and content. Responses also indicated a shift towards more cohesive views of mathematics, which is of importance to deep learning. Promoting skills as relevant to every area of mathematics may have helped lead to this shift. Responses finally indicated that some students had begun to take greater account of the nature of mathematics in their learning.

The project indicates that a need exists to help students develop the skills of advanced mathematical thinking and provides a variety of lessons for programmes that make dedicated teaching and learning available to meet this need. Finally, it is interesting to note that such programmes may wish to consider ways in which these skills can be applied in a wider context than just the study of mathematics.

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COMMUNICATIVE INTERACTIVE PROCESSES IN PRIMARY VERSUS SECONDARY MATHEMATICS CLASSROOM

M. Kaldrimidou[1], H. Sakonidis[2], M. Tzekaki[3]
[1]University of Ioannina, [2]Democritus University of Thrace, [3]Aristotle University of Thessaloniki, Greece

A number of studies have put emphasis on the importance of the interactive patterns of teaching and learning in the acquisition and the development of mathematical knowledge. An essential aspect of this view is that the way the teacher defines the frame of mathematical knowledge, poses questions, refuses and reinforces students' answers and propositions provides decisive orientation for the children with regard to what the legitimate conceptions of mathematical knowledge are.

However, as many researchers argue, there should not be a total shift of analytical attention from subject matter-structure to social-interactional structure because there is then «a risk of destroying theoretical mathematical meaning by a reduction and an hypostasis of mathematical relations instead of inducing an enrichment of meaning by the interactive construction of new and more general relations» (Steinbring, 1998).

An analysis of the interaction in the mathematics classroom that takes into serious consideration the epistemological as well as the social-interactional conditions helps to provide a better understanding of how the communicative patterns and routines emerge and «makes it possible to re-establish a sound interactive mathematical reasoning that has been destroyed by these communicative patterns and routines» (Steinbring, 1997).

The results of an earlier study (Ikonomou, Kaldrimidou, Sakonidis and Tzekaki, 1998) showed that the teaching approach adopted by Greek primary teachers, possibly because of their poor mathematical pre-service training, does not often allow pupils to conceive the epistemological features of mathematics. In the present study, a comparison between primary and secondary mathematics lessons is carried out with respect to the following questions: What kind of ideas and meanings regarding mathematical knowledge are constituted during the course of the mathematics teaching? How do the communicative patterns and the epistemological constraints of the mathematical knowledge influence each other? What hinders or favours the development of mathematical meaning during the course of classroom interaction?

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Diagnose and Treat –Pupils with mathematics difficulties in middle schools

YOUSEF KHOURY & NASIF FRANCIS - Wiezmann Insititute

Arab pupils in middle schools in Israel learn mathematics according to the Rehovot program. The textbooks and the supplementary material of the program have been created and implemented by the mathematics group at the science teaching department of Weizmann Institute of Rehovot. Among this group there is a special team whose members are Arab educators, their work is devoted to the translation of the textbooks and the supplementary material in the Arab junior high school. The present project is one of the special and important activities of the Arab team of the mathematics group. The clear objective of the project is to identify the specific difficulties that face the low capacity student in regular classes as well as in classes of lowest level where from the beginning the material was written for these pupils, and trying to create remedial materials.

This project is distinguished by:

- The approach that adopt from the beginning is the implementation of the research to create the materials.
- The involvement of the teachers in diagnostic stage of the process and his commitment to implement the remedial material.
- The originality and the accordance for needs of the Arab pupils and teachers.
- The development of the tools and the suggestions for treatment in the project are done on the basic of academic research. Those tools contain a lot of attractive worksheets, games and computer activities.

The project process:

First stage: numbers and number problems: (From the elementary school curriculum)
Second stage: algebraic expressions (early algebra learning)
Third stage: algebraic expressions (in process)

Diagnose and treat algebraic expressions

The importance of the subject for studying mathematics in later grades of the high school, and Algebra "is a source of considerable confusion and negative attitude among pupils"[1], motivated us to diagnose the difficulties faced by Arab pupils concerning this topic and to find how to treat them

The diagnosis

We choose to check three topics:

1) Substitution in different algebraic expressions;
2) Operations on algebraic expressions in order to get similar expressions;
3) Translation of problems in mathematical language;

Findings:

We find most difficulties in distributive law and its applications to get similar expressions, and translation of word problems with two variables.

The following topics were found difficult for part of the pupils

* Substitution of big numbers in expressions,
* Number substitution in expressions with multiplication or subtraction.
* Negative number substitution in expression with subtraction and multiplication.
* Similar expression especially applying the distributive law
* Associatively and distributive laws and the application in expressions with multiplication with the variables out of the parentheses.
* Translation of problems in mathematical language.

The project has been implemented in some of the Arab middle schools in Israel and the results were satisfied. This was obtained from the assessment that was carried out at the end in classes and replies of the pupils, teachers, directors and parents.

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RELATIONSHIP BETWEEN PCK AND LESSON PLANS: DOES IMPROVEMENT IN TEACHERS' PCK EFFECT THEIR LESSON PLANS?

Ronith Klein and Dina Tirosh

Kibbutzim College of Education, Tel Aviv

School of Education, Tel Aviv University, Tel Aviv

Recent journals and PME papers discussed the importance of teachers' familiarity with students' ways of thinking (e.g. Even & Tirosh, 1995; Jaworski, 1998). Studies have shown that participation in programs focusing on children's thinking could increase teachers' awareness of students' incorrect responses and their possible sources (e.g., klein, Barkai, Tirosh and Tsamir, 1998). Several programs (CGI, for example) explore the impact of teachers' knowledge of students' thinking on actual teaching practices. The main aim of this study was to explore the effect of a workshop specifically designed for enhancing inservice teachers' knowledge of students' ways of thinking about rational numbers on lesson plans on Multiplication and Division word problems with Rational Numbers (MDRN).

Fourteen experienced elementary teachers participated in the course. Participants were asked to plan a teaching unit on MDRN, at the beginning and again at the end of the workshop. Our main aim was to explore the extent to which the participants took account of students' ways of thinking in this planning.

We hypothesized that teachers would adjust their lesson plans, taking account of common, systematic students' conceptions and misconceptions when planning instruction. Our data did not fully confirm this assumption. Still, in the lesson plans submitted at the beginning of the course, only few written references were made to possible, common incorrect students' responses whereas at the end of the course some building on students' common incorrect responses occurred .

During our presentation we shall describe the course and raise some alternative explanations as to why more substantial improvements in teachers' lesson plans were not found.

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Mistaken conjectures as a trigger to develop basic probabilistic reasoning

Lilya Kot, Sara Kiro, Abraham Arcavi
Department of Science Teaching
Weizmann Institute of Science - Israel

In a comprehensive review of the research on understanding and learning probability and statistics, Shaughnessy (1992, p. 465) claims that most people's intuitive and immediate responses to probability problems make no use of elementary concepts to estimate the likelihood of events and thus produce wrong answers. Moreover, he points out that "nonmathematical ways of estimating likelihoods" can be deep rooted and good teaching may not always change them. Nevertheless, we propose that an appropriate learning trajectory with the characteristics described below may be successful with non-academically oriented high school students.

The learning trajectory proposed: a) presents engaging situations close to students' experiences, b) relies on students' common sense, c) respects all conjectures, d) supports and encourages the use of visual representations to make sense of situations, and e) promotes the discussion of opposing conjectures which can be empirically tested and analyzed.

The following is an example of a series of problems used in our curriculum project (Arcavi, Hadas and Dreyfus, 1994). "Efrat and Donna play the 'fingers' game: each of them shows simultaneously a number of fingers on their right hand. If the sum of what both show is even, Efrat wins, if it is odd, Donna wins. If you think the game is fair, explain, if not, who has greater chances to win, and why?"

This is a common game, well known to the students. A great majority of them do not hesitate, and without further analysis, claim that the game is fair, possibly because of its popularity. Students are encouraged to build a table representation to display all possible situations, in the light of which they realize that the game is slightly unfair. After a series of such problems, students learn to develop the habit of checking their initial conjectures against a visual representation of the space of all possibilities.

In the short oral presentation, we will show data from pairs of students solving a series of problems (similar to the above) in which they face opposing conjectures to check. Some pairs had previous instruction in probability and others had not, but all of them raised initial conjectures which were wrong. We will show how they slowly changed their conjectures by using table representations to make sense of the problem, and how they started to develop a different approach towards similar problems.

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RESEARCH ON THE VALIDITY OF "TWO-AXES PROCESS MODEL" OF UNDERSTANDING MATHEMATICS

Masataka Koyama
Hiroshima University, Japan

The problem of understanding mathematics has been a main issue buckled down by some researchers in PME. Koyama (1992) discussed basic components that are substantially common to the process models and presented the so-called "two-axes process model" of understanding mathematics as a useful and effective framework for mathematics teachers. The model consists of two axes, i.e. the vertical axis implying levels of understanding such as mathematical entities, relations of them, and general relations, and the horizontal axis implying three learning stages of intuitive, reflective, and analytic at each level. There are two prominent characteristics in the "two-axes process model". First, it might be noted that the model reflects upon the complementarity of intuition and logical thinking, and that the role of reflective thinking in understanding mathematics is explicitly set up in the model. Second, the model could be a useful and effective one because it has both descriptive and prescriptive characteristics.

It is a significant task for us to examine both validity and effectiveness of the model in terms of practices of the teaching and learning of mathematics. Focussing on the validity of "two-axes process model" of understanding mathematics, Koyama (1996) demonstrated the validity of three stages at a certain level of understanding mathematics by analyzing a fifth grade elementary school mathematics class in Japan. The purpose of this research is to closely examine the validity by analyzing data collected in three different mathematics teachers' classrooms at the national elementary school attached to Hiroshima University.

As a result of this research, we find out the followings. First, the "two-axes process model" of understanding mathematics is valid in such a sense that it could describe children's development of understanding mathematics in their classroom. Second, we could characterize such a teaching and learning of mathematics as the dialectic process of children's individual and social constructions that enables them to understand mathematics deeply and in their meaningful way. In order to realize such mathematics classroom, it is suggested that a teacher should make a teaching and learning plan in the light of "two-axes process model" of understanding mathematics and that she/he should play a role as a facilitator for the dialectic process. There are two important features of teacher's role: The one, related to children's individual construction, is to set a problematic situation in which they are able to be conscious of their own learning tasks and encourage them to have various mathematical ideas and ways. The other, related to children's social construction, is to encourage and allow them to make, explain, and discuss their various representations.

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INTERACTION BETWEEN KNOWLEDGE AND CONTEXTS ON ABILITY TO SOLVE PROBLEMS: THE ROLE OF DIFFERENT LEARNING CONDITIONS

Bracha Kramarski

The Institute for the Advancement of Social Integration
School of Education, Bar-Ilan University

How do children solve mathematical problems, which are the same in mathematical content, but different in the contexts presented in them? Are mathematical problems, which embedded in concrete contexts easier to solve than those, which appear in abstract contexts? To what extent do contexts exert different effects on children's understanding of abstract mathematical concepts? How do different learning conditions influence the development of the ability to solve problems presented in different contexts?

The current research relates to these issues and focuses on problems which deal with the linear function graph. 384 students participated in the study. Boys and girls from the eighth grade who were studying the subject: "the linear function graph" under different learning conditions: cooperative learning with metacognitive training, the whole class with metacognitive training, cooperative learning and the whole class without metacognitive training. The metacognitive training is based on a model of self-addressed questions: comprehension questions, strategic questions, and connection questions. (Mevarech & Kramarski, 1997; Schoenfeld, 1985). All the students were tested both at the beginning of the experiment and afterwards with problems presented in concrete and abstract contexts, which tested skills of graph reading and graph drawing (transfer). It was found that students who were exposed to metacognitive training whether in cooperative learning or in whole class improved by the end of the experiment the ability to read and draw graphs (transfer) as opposed to the students who were not exposed to this training. The greatest improvement was found among the students who were exposed to this training in cooperative learning. Similarly, the students exposed to metacognitive training revealed an equal ability in solving problems presented in different contexts: concrete/abstract, as against students who were not exposed to metacognitive training, who showed a greater ability to solve problems presented in concrete contexts. The findings are discussed from a theoretical and practical aspect.

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CONSTRUCTIONS OF NEW MATHEMATICAL KNOWLEDGE IN DIFFERENT LEARNING ENVIRONMENTS

Christian Kratzin

Institut für Entwicklung & Erforschung des Mathematikunterrichts, Universität Dortmund, Germany

In this paper I would like to report some findings and results of our ongoing research-project "Epistemological and social-interactive constraints for the construction of new mathematical knowledge (in primary mathematics teaching)"[1].

During autumn 1997 we observed and video taped about 30 mathematics lessons in grades 3-4 of primary school. In this lessons the teachers used two different kinds of learning environments, which beforehand were developed by our research-team. One typ of learning environment can be characterized as arithmetic-structural and the other one as geometric-visual.

The epistemological and interpretative analysis of the interactions in some selected transcribed teaching episodes from the two kinds of learning environments shows, that the constraints and the chances for the children to develop generalized arguments (which are necessary and undispensable for the construction of new mathematical knowledge) are depending on some special features of the respective learning environment.

When preparing the learning environments we assumed that it would be easier for the children to develop the view for the generalities within the scope of a geometric-visual reference context, because there one can construct relations and connections not only on an arithmetical level but also on a geometrical one. Our observations and analyses have set us right: In a geometric-visual learning environment the children mainly produce concret interpretations and construction that are merely related to individual cases; whereas in an arithmetic-structural context the children more often reach the point where they are able to see the general in the particular.

In a geometric-visual context it is much more difficult for the children to get a certain distance from the concrete (geometrical) objects, that sometimes even can be touched by the children (for instance in forms of the counters used to build up movable dott patterns) and which perhaps chain the view too strongly to the particularities.

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1 This project is conducted by Heinz Steinbring and it is financially supported by the DFG (Deutsche Forschungsgemeinschaft).

MATHEMATICS ASSESSMENT IN A NEW CURRICULUM MODEL IN SOUTH AFRICA

Daniel Krupanandan, Springfield College Of Education South Africa.

South Africa has seen the introduction of " Curriculum 2005 " or commonly known as " Outcomes Based Education(OBE) " , a new vision of curriculum transformation in our country since 1998. This renaissance in the teaching and learning of mathematics has been initiated through readings of many policy documents and attendance of many curriculum workshops.

Despite the stirling efforts of the education departments, the speed and urgency with which the training and informing process has taken place has left many committed mathematics teachers lagging on the road to the successful introduction of " Curriculum 2005 ".

This paper will provide the results of a research conducted amongst 400 primary mathematics teachers, who completed a questionnaire based on their understandings of the new curriculum model, with particular reference to their views on mathematics assessment in the OBE model. 40 primary mathematics teachers were also interviewed.

Since assessment in mathematics is a central component to any curriculum model, it was not encouraging to find that almost 97% of the teachers involved in the research had conflicting views about assessment in mathematics or they were implementing assessment strategies that were not entirely consistent with an OBE curriculum model. Mogens Niss(1998) comments that this is an international phenomenon," During the last couple of decades, the field of mathematics education has developed considerably in the area of ideals and goals, and the theory and practice, whereas assessment concepts and practices have not developed so much ".

The results of the research has challenged teacher educators to undertake intense in-servive programs to assist primary mathematics teachers make the paradigm shifts in implementing assessment strategies that measure not only knowledge, but attitudes and values as well.

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THE LANGUAGE OF MATHEMATICS AS THE OBJECT FOR SPECIAL STUDY

Raissa Lozinskaia, Tomsk Polytechnical University, Russia

Mathematics is a definite world outlook. "Mathematics is of great interest in itself, first of all as the totality of objective truths. Besides, mathematics gives convenient and fruitful ways of describing various phenomena of real world and in this sense really performs the function of a language (Kudrjavtsev, 1980).

One of the aims of "MPI-project" (the leader — Prof. E. Gelfman) is working out the system of methods and ways of successful acquisition of mathematics language. Our basic task was to compare psychological - pedagogic fundamentals of mathematics education as well as of a foreign language.

L.S. Vigotsky (Vigotsky, 1982) pointed out that the main role in concept formation is played by "a functional usage of a word" as a means of advancement of characteristics of an object, their synthesis and generalization. That's why our system of exercises includes such tasks as to teach students work with different values of one and the same sign, to be able to single out essential, general and particular for the solution of a concrete problem. It was pointed out that students fail in mathematics very often due to mistakes in translating mathematical expressions into natural language (Bell & Malone, 1992; Ferrari, 1996). We have worked out special tasks, which form the skills of performing verbal-figurative translation, teach students to draw schemes, to pass over from real situations to mathematical models.

Mathematics language, like natural language is metaphorical (Sfard, 1996). We look for metaphors of mathematics language. Perfect acquisition of any foreign as well as of mathematics language is a long process and presupposes operating units of both the languages without translation. That's why we specially initiate situations where students work within the frames of some conventional agreement of mathematics language, when students themselves have to construct the system of this language. Teaching mathematics language should include study of semantics (values of signs, symbols, concepts) as well as of syntax (rules according to which signs are united in sentences or formulas); that's why a student should understand how a new notion or a symbol is connected with a system of concepts, i.e. it is necessary to establish the system of different links between notions, including genetic one.

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AVERAGE, TEACHERS AND STUDENTS

Zvia Markovits
Oranim School of Education, Israel

Mathematics-Classroom–Situation (MCS) Cases are real or hypothetical classroom situations involving mathematics, in which the teacher has to respond to a student's question or idea. Previous research studies (e.g. Even and Markovits, 1997) indicate the potential of MCS-Cases in raising mathematical as well as pedagogical issues.

The Average Situation deals with the dilemma of accepting an answer which seems to be unreasonable in everyday life situations as an answer to a mathematical problem.

The Average Situation:

> "A student was given the following problem:
> *At Narkisim School there are 3 fifth grade classes. In the first class there are 31 students, in the second 24 students and in the third 28 students. Find the average number of students in the fifth grade classes. The student answered: There is no average here. You* [teacher] *told us that there is not such an answer 27 2/3 students."*
> How would you respond?

The situation was given to about 60 teachers who teach mathematics in the upper grades of elementary school. Most of the teachers agreed that there is an average in this situation, but many of them suggested that the average should be 27 or 28, since the answer is people. Many of fifth and sixth graders, given this problem, were able to correctly use the averaging algorithm, but unable to accept 27 2/3 as the answer to the given problem. It seems that students and teachers developed an uncompleted understanding of the average concept.

It is interesting to point out that while many research studies suggest that students do not use "out of school" experience when dealing with mathematics problems in school, in this case they did apply everyday life considerations.

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PROBLEMS OF CONCEPTUAL CHANGE ON THE ENLARGEMENTS OF THE NUMBER CONCEPT

Kaarina Merenluoto, Department of Education, University of Turku, Finland

Erno Lehtinen, Department of Teacher Education, University of Turku, Finland

The cognitive processes of concept acquisition do not follow the logical hierarchy of mathematics (Tall 1991) of a typical curriculum. The structure of mathematics may appear fragmentary and discontinuous for the student The crucial idea in conceptual change is the radical reconstruction of prior knowledge. This complicated process leads however often systematically to misconceptions because the prior knowledge may not be transformable into the new subject matter (Vosniadou 1994).

The dual nature of mathematical concepts (Sfard 1991), the long development period between the operational use and structural definition of the concept (Boyer 1959), the high level of abstraction of advanced mathematics and the low nature of abstraction in the every day mathematics all seem to refer to the problems of conceptual change in the learning of mathematics.

We collected an extensive data from students (n=640) in high school calculus courses. The results showed that the vast majority continued to use the logic of natural numbers in tasks on the domain of rational and real numbers and their concept of more advanced numbers was confused. These findings suggest important considerations for planning conceptual change supporting learning environments.

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TUTORIAL INTERACTIONS AND DIDACTICS OF MATHEMATICS : RECOGNITION OF FRACTIONS IN PRIMARY SCHOOL AND VOCATIONAL EDUCATION

Maryvonne MERRI, Assistant Professor of learning psychology
Ecole Nationale de Formation Agronomique, Toulouse (France)
Research Team "Cognition and didactics", University of Paris VIII
Marie-Paule Vannier
Institut Universitaire de Formation des Maîtres de Melun (France)
(Dissertation director : Gérard Vergnaud)

When working as a mathematics teacher trainer, every cognitive psychologist has to face a recurring epistemological problem : initially, the most important psychology concepts did not refer to didactical situations. This short oral communication intends to discuss the pertinence of Piaget's and Vergnaud's concepts of *schème* in describing tutorial interactions in student problem-solving activity.

The same fraction recognition task was proposed to primary school teachers (for 10 year-olds) and to vocational school mathematics teachers (for 15 year-olds). Every teacher could adjust this task to his own particular teaching methods. Tutoring sessions are defined as interaction periods for problem-solving achievement. They are analysed and then related to the larger didactical process.

A few videotaped examples will illustrate one of the most important characteristics of action on problem-solving *schèmes* in didactical contexts : tutorial interactions must therefore achieve much more than simple completion of the task.

When interacting, teachers and students do not just consider problem-solving as a "private" cognitive activity. They can (explicitly or implicitly) refer individual knowledge to mathematical norms inside and outside the classroom. How can tutorial interactions improve students *schème* and prepare them to share this skill with the community ? The authors will analyse the theoretical consequences of these two dialectics (private vs public and individual vs collective knowledge) on Gérard Vergnaud's concept of *schème*.

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ON DEVELOPING TRIDIMENSIONAL SPACE AT SCHOOL*

A.L. Mesquita (U. de Lille/IUFMNPdC)

Geometry appears nowadays as "one of the most universal and useful tools in all parts of mathematics" (J. Dieudonné, 1981, quoted by C. Mammana & V. Villani, ICMI Discussion Document, 1998). However, there is "a gap between the increasing importance of geometry [...], as well in research and in society, and the decline of its role in school curricula" (*op. cit.*, p.338). It is the case in France, where school programs give a reduced place to geometry. The aim of the project we are developing at present has as a central aim the valorization of geometry and space at school. It is a longitudinal case-study, in course of implementation in a primary school of the northern of France, since 1997/98, and it concerns the two groups of pupils (i.e., about fifty pupils) during their school attendance. The main assumptioms of the project are:

1) The first step to introduction of geometry at school concerns the space: for us, the beginning of geometry at school is centered on tridimensional space, and this in articulation with geography, the other subject-matter concerning the study of space at school.

2) A didactical progression is clearly assumed and developed along all the school-attendance: from space to plane; from plane to line; from line to point. Interactions between these entities are strongly stimulated; in particular, transitions between them are considered decisive steps in the learning of geometry .

3) A special attention is given to the different registers of representation (in the sense of R. Duval, 1995, i.e., the semiotic systems of presentation of knowledge) used in geometry, and to their articulation.

In the beginning of school-attendance (ages 6 to 8), our attention is centered on tridimensional space and on the transition from space to the plane. Activities of problematization appears to be decisive to this transition, in general neglected by teaching (A.L.Mesquita *et al.*, 1998). These activities are associated with the construction of objects and other activities of material manipulation, which are privileged, at this phase. The presentation will enable a discussion about the main assumptions of this study, as well as some initial results concerning its implementation.

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* Funding from IUFMNPdC, project # R/RIU/98/079. Other participants in the project are: Anna Abbes, Francis Delboë, Nathalie Owsinski, Annie Régnier, Sabine Rossini, Jean Vandenbossche, Nathalie Vasseur.

COOPERATIVE GROUPWORK A VEHICLE FOR DEMOCRACY IN BLACK SOUTH AFRICAN MATHEMATICS CLASSROOMS?

Duduzile Mkhize, RADMASTE, Witwatersrand University, SA

The behaviour and attitudes of learners in mathematics classrooms is by and large determined by the way the teacher interacts with them. On the other hand teachers are driven by the curriculum to act the way they do. The new curriculum , Curriculum 2005 (C005) that was launched in 1997 in South Africa has to change mathematics classroom practices, especially in black schools. The EduSource Survey (1996) revealed that authoritarian approaches were prevalent in the teaching of mathematics in South African black schools. Hindle(1997) contends teachers in the past were turned into factory workers, thus uncritical compliance was expected from them. This was all in line with the non-democratic curriculum of the day! C005 is viewed as a curriculum that should promote democracy in mathematics classrooms and cooperative groupwork could be an ideal vehicle for this.

C005's view of mathematics, "It is a human activity that deals with patterns, problem-solving, logical thinking, etc. in an attempt to understand the world and make use of that understanding. This understanding is expressed, developed and contested through language, symbols and social interaction", implies the need to create social interaction to express and debate the understanding in the classroom.

This study aimed to improve cooperative groupwork practices in mathematics classrooms. Learners were encouraged to develop rules for working in groups. Linear programming lessons in a cooperative groupwork mode were prepared and conducted with grade 11 learners. Each session took two hours. After five sessions learners were asked to complete the self assessment forms. The rules and self assessment comments are summarised below.
Rules: "No parasites, give others a chance to talk, respect other members' opinion, we help each other, listen and pay attention to one another, patience."
What went well: "Everyone came with an idea, everyone was thinking, we understood very well most of the things we did, at last we gained something from a maths lesson and I enjoyed the lessons."

The rules indicate what learners value in cooperative groupwork. Clearly, the inculcation of the democratic principles such as listening to one another, a chance for all to give their opinions is inculcated through cooperative groupwork.

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TEACHING MATH WITH TECHNOLOGIES: A NATIONAL PROJECT IN MEXICO

Simon Mochon and Teresa Rojano

Center for Research and Advanced Studies, IPN, Mexico

Since 1997, the Ministry of Education of Mexico has being sponsoring a national program to teach mathematics with technologies at the secondary level. The tools used in the classroom are a combination of calculators (TI-92) and computer 'open' software: "Spreadsheets" (all purposes), "Stella" (Modeling package), "MathWorlds" (Mathematics of Change package) and "Cabri" (Dynamical Geometry package). Parallel to this program, there is an ongoing research project that has as its main purpose to investigate the impact of this technological implementation in students' learning, teaching practices and curricular transformation.

The Mexican Math Curriculum calls for a close connection with real life situations. The actual practice however is far from this. In addition, topics are introduced from general principles down to particular examples. This new educational program stresses the opposite, bottom to top approach. Another important element is collaborative learning. The interaction student-computer and student-student are being studied within the framework of Vygotsky's perspective on mediational means and the zone of proximal development. Cultural influences are also being considered.

Within the classroom, we have used very effectively a teaching strategy that in part consists of coupling a math modeling approach with worksheet guidance. A recent antecedent of this method is a collaborative Mexico/UK research project (Sutherland et al, 1996) aimed at investigating the role of modeling with spreadsheets across a range of subject areas (physics, chemistry, and biology). In this research, the spreadsheet was introduced into the students' science classrooms to construct models, "artificial worlds" (Mellar et al, 1994), that were explored and analyzed and which enhanced students' understanding of the scientific ideas related to the model.

This presentation will concentrate on two aspects of the project: the phase of teachers' training and the design, implementation and evaluation of activities for the classroom. The teachers received seminars on 1) the use of the software (each teacher learned and used only one software package), 2) the methodology related to teaching with computers and 3) how to structure the activities. We found that teachers still followed a traditional model to design the activities but have a substantial advance on their role in the classroom. We will also describe briefly some preliminary results on the students' cognitive and affective development. This data was obtained from classroom observations and tests throughout the school year (the tests contain questions about the most relevant concepts introduced by the software packages).

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MODELS OF MATHEMATICS UNDERSTANDING

Judith A. Mousley

Deakin University

The problem

Meanings that teachers, teacher educators and curriculum documents hold for the term *mathematical understanding* shape curriculum planning, teaching, and assessment in schools and in teacher education. However, these are rarely articulated.

Some models from the literature

_structural progression (e.g. Piaget; Sinclair; von Glasersfeld)
_zones of achieved and potential (e.g. Vygotsky)
_different forms of knowledge (e.g. Maslow; Skemp; Gray & Tall)
_levels of understanding (Herscovics & Bergeron)
_lattice: acts interwoven with situations (Sierpinski)
_recursive process of organising knowledge structures (Pirie & Kieren)
_socio-linguistic activity (Wittgenstein)

Results from a (pilot) survey of mathematics educators

Ideas that best matched the respondents':

_a shift from "unable to explain" to "able to explain" _grasping meaning _a mental organisation or structuring of experience _moving from one zone of knowledge to another _appreciation of what lies beneath a statement (or solution, etc.)

Physical models that best suited their ideas:

_a three-dimensional web	_a lattice or woven fabric
_a tree	_a spiral

Different types of mathematical understanding most commonly mentioned:

_instrumental relational (4 people)	_iconic symbolic relational factual conceptual analytical	_visual/spatial logical numerical inter-relation al	_rote concrete conceptual abstract	_instrumental relational logical symbolic

As well as outlining the key characteristics of the results of this pilot study and their relationships to the literature, the presentation will identify some of the methodological problems experienced when analysing responses to forced-choice items on the questionnaire.

MULTI-WORLD PARADIGM IN MATHEMATICAL LEARNING

Tadao NAKAHARA
Hiroshima University JAPAN

Lately,the mathematical education is under significant influence of three perspectives including radical constructivism, social interactionism, and social culturism, and a lot of studies are being conducted on the mathematical education based on each of such paradigms or combinations thereof. Table 1 shows comparisons among such thoughts from different view points.

View point	Radical const.	Social interactionism	Social culturism
Nature of cognition	Construction by individuals	Construction by community	Enculturation
View of language	Means of thought expression	Containing a process of interpretation	Medium of cultural transmission
View of learning	Cognitive self-organization	Social interactions	Participation to cultural practices

The above comparisons reveal a common aspect among three perspectives: all of three perspectives view that children's activities play very important roles in mathematical learning, and they position social interaction as an important means in learning. However, on the other hand,they have distinct differences in any of the views of cognition, language,and learning. In particular, difference in the view of the nature of cognition is a matter of principle that is mutually incompatible, and the integration of those perspectives is considered theoretically impossible.

However,in the observations and analyses of children's practical activities in mathematical learning, the three perspectives are frequently seen at the same class. While those three perspectives are mutually conflicting theoretically,it is considered unavoidable to integrate and coordinate the three perspectives in order to explain the realities of learning in fair manner.The studies of mathematics learning by Cobb and Bauersfeld should be considered as have been suggested based on such standpoint.

The author calls the above-shown view the multi-world paradigm in mathematical learning.It contains the three worlds of the radical constructivism, social interactionism, and social culturism. He shows theoretical as well as practical study of mathematical learning based on the multi-world paradigm.

HELPING TO DEVELOP THE ABILITY OF ARGUMENTATION IN MATHEMATICS[1]

L. Nasser and L. Tinoco
IM – UFRJ- Brazil

This research was motivated by the project carried out by C. Hoyles (1997) in the U.K., in which students are led to evaluate and write proofs to basic mathematics statements, in Algebra and Geometry, under the light of the British National Curriculum. The functions of proof and its role in the teaching of mathematics have been the focus of several papers (Hanna, 1990; de Villiers, 1991; Hanna & Jahnke, 1996). Concerning the functions of poof, we agree that:

In the educational domain it is natural to view proof first and foremost as explanation, and in consequence to value those proofs which best help to explain.(Hanna & Jahnke, 1996, p.903)

The kinds of argumentation and the analysis of the difficulties considered in this work are based on Balacheff (1987), Rezende & Nasser (1994) and van Hiele (1986). The main aims are to:

- identify some kinds of argumentation used by secondary mathematics students in Rio de Janeiro;
- develop and test activities to promote the progress in their levels of argumentation;
- suggest trends to the enhancement of the levels of argumentation of students, as well as of prospective and in-service mathematics teachers.

After the first trials with 14-16 year old students, it became clear that the majority of the Brazilian mathematics teachers do not require their students to justify their answers, mainly because this was not stressed in the curriculum (it appears in the new curriculum proposed). As a consequence, students show difficulties in justifying a statement or in explaining how they reason when solving a problem. Several strategies to improve these abilities have been tried by the teachers and undergraduate students involved in our project, such as:

- after answering a task individually and listening to the teacher's explanation, students work in groups, discussing a joint solution to the same task;
- students have to evaluate justifications given by other students;
- items requiring logical reasoning are often proposed, despite the topic being studied;
- the same task is proposed both to students that have learned or not the correspondent mathematical content, in order to investigate the differences in the kinds of argumentation used;
- students use dynamic geometry softwares to verify the veracity (or not) of a statement and after convinced of its truth (or not), are led to justify or prove it (or to seek for a counter-example);
- activities requiring the recognition of a theorem and its reciprocal have been used with pre-service and in-service teachers.

This research is still in development, but we can already conclude that the lack of experience in self-assessment stresses the difficulties to evaluate someone else's justifications and that the symbolism of the mathematics teaching may be an obstacle for students to write in mathematics. One strategy to improve the students' level of argumentation is requiring justifications to all their answers, and proposing challenging problems requiring logical reasoning. This must be done all the time, all over the year.

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[1] Research sponsored by CNPq (Brazil)

A STUDY ON THE FUNCTION OF TRANSACTIONAL WRITING: A FUNCTION MODEL OF SOCIO-MATHEMATICAL SKILL IN MATHEMATICS EDUCATION

Hiroyuki Ninomiya

Hiroshima University Graduate School, Japan

Writing activities in mathematics education has been focused on, and many writing activities have reported, e.g. practices in NCTM's Standards based curricula in America, whereas there are some reports about Mathematical Writing in Japan. Most of the writing activities can be classified into the following three types; Journal writing, Expository writing, and Creative writing. Journal writing is about mathematics classes; Expository writing is about the topics of mathematics with problem solving activities; and Creative writing is literatures with mathematical terms.

Britton et al.(1975) classified the functions of journal writing into the following two types, expressive function and transactional function, and Ninomiya(1998) defined two types of Journal Writing: Expressive writing, and Transactional writing. Expressive writing expresses students' emotional affairs, and Transactional writing refers to the students' understanding or the development of mathematics abilities.

Examining the precede studies, we can find two aspects in Transactional writings; as a method of mathematics learning, and as an aim of mathematics learning. These two aspects are two faces of a coin, and any Transactional writing activities have both of them. Moreover, Transactional writing has two functions. When the aspect as a method is highlighted, writing works in students' interaction as Social Skill. On the other hand, the aspect as an aim is highlighted, writing works as Mathematical Skill when students think and do mathematics. Because of the duality of the aspects in Transactional writing, two functions are related, and a Function Model of Socio-Mathematical Skill in Mathematics Education can be established as Figure 1.

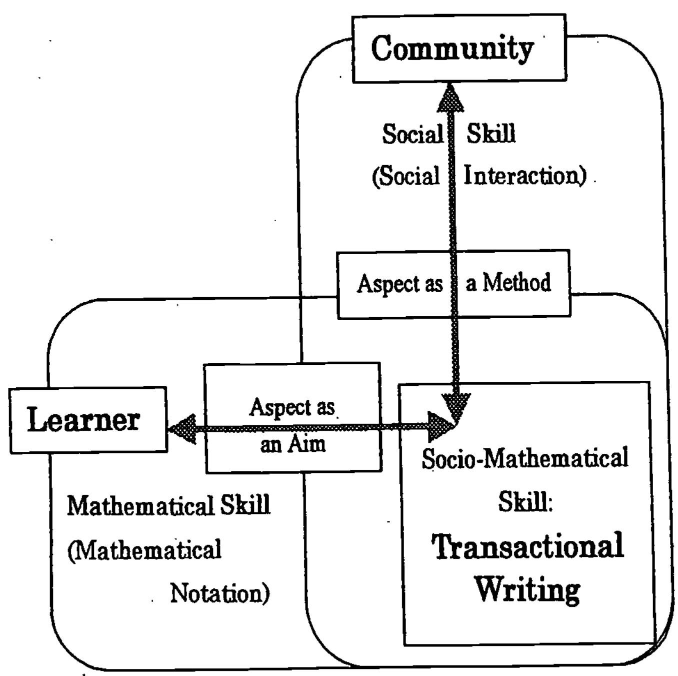

Fig. 1 A Function Model of Socio-Mathematical Skill

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Difficulties in calculating the volume of three-dimensional arrays of cubes

Guri A. Nortvedt, Department of Teacher Education and School Development, Faculty of Education, University of Oslo, Norway.

On a national survey of students understanding of measurements and units for grades 6 and 9 in Norwegian schools students were asked to write texts explaining how they solve specific volume problems. These texts have been investigated qualitatively in order to identify patterns in the children's approaches and to seek information about their conceptual understanding of volume. As a result of this investigation several categories for different text stereotypes have been developed in an emergent manner, making it possible to investigate the texts quantitatively as well.

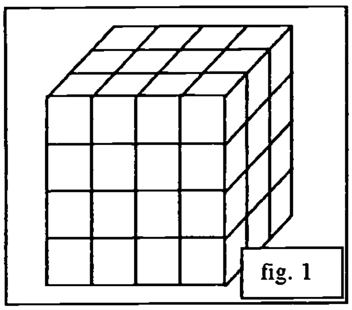

fig. 1

When calculating the number of cubes needed to build a three-dimensional array, the two most common approaches for successful children was to calculate the volume by a formulae alike approach or building the volume in terms of layers (Nortvedt, 1998). Another group of children gave answers like 30, 40, 80, or 96 when asked how many cubes are needed to build the box in fig. 1. (19,2 % of grade 6 students, 16,4 % of grade 9 students).

These students wrote short texts and did not always explain in a sufficient manner their approach to the problem. From the analysis of the texts it seems likely that some of these students did not read the illustration as three-dimensional. From other explanations an emerging possibility is that some students viewed the side of the cube as a representative for the volume of the cube, concluding that when they could count 40 "sides", the number of cubes needed is also 40. Battista and Clement (1996) suggest that many students are unable to numerate the cubes due to lack of ability to co-ordinate separate views of the array and thereby fail in constructing a mental model of the array.

The presentation at the PME-conference will focus on presenting stereotype texts written by students calculating the number of cubes on the surface of the array, and possible consequences for the teaching of this topic will be outlined.

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'EVERYDAY' CONTEXTS IN SCHOOL MATHEMATICS

Thabiso Nyabanyaba
University of the Witwatersrand, South Africa

It is now widely accepted among educationists that school mathematics is not an abstract, immutable and monolithic body of knowledge that applies across contexts. To that end, the use of contexts, especially 'real' life contexts, in order to generate school mathematics knowledge has become widely recommended. My study of how teachers understand 'relevance' as it refers to relating school mathematics to the everyday experiences of students adds to the growing debate on the use of contexts in school mathematics (Boaler, 1993). In my study, teachers were asked to discuss, in focus group interviews, how they would make school mathematics more 'relevant' to the students' everyday experiences and what difficulties they anticipated in this practice. Practical scenarios were used in order to provoke discussion. Among the many interesting issues that emerged from this study was the ease with which teachers were distracted by the very contexts or scenarios that were supposed to generate their mathematical discussions.

The new *Curriculum 2005* for South Africa, advocates 'relevant', 'integrated' learning, rather than the memorisation of discrete facts (DoE, 1997). The discussions of the teachers in this study highlight the challenges that teachers are likely to face in the implementation of a 'relevant' curriculum. The study also suggests the prospect of such contexts to produce highly complex understanding.

In this presentation I will use two critical incidents to illustrate the nature of the discussion and what issues emerged around the use of contexts in school mathematics practice. In one of the incidents, teachers were to discuss how they would make the teaching of 'parallel lines' more relevant to the students' everyday experiences. Several very interesting suggestions were made, including the use of electricity power lines to illustrate parallel lines. The discussion centred on precise mathematical terminology and what was proper to do pedagogically. In another scenario teachers were to discuss the usefulness and limitation of using a football log table to assess students' mathematics ability. The discussion centred explicitly on the scenario. The usefulness and limitations were only referred to implicitly.

These scenarios illustrate what can happen in school as students are immersed in contexts, especially contexts that they know about. Therefore, potent as 'real' life contexts are in the teaching of school mathematics, many teachers are going to find it a difficult exercise to draw the students' discussions towards the mathematical experiences and away from the novelty of the 'everyday' experiences.

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MATHEMATICAL ACTIVITY IN THE CLASSROOM: GEOMETRIC DYNAMIC ENVIRONMENT AS A WINDOW FOR LEARNING

Gisélia Correia Piteira

Centro Competência Nónio

Faculdade de Ciências da Universidade de Lisboa

The following principles are assumed as a basis for the ongoing research: (1) the mathematics classroom is a system of activity, including all activity in and among its elements' interactions; (2) students don't interact only with the environment but all their action is mediated by artefacts; (3) knowledge is located in the community, being shared by the persons involved through social interaction and mediated by artefacts; (4) mathematical objects make sense within the social interactions, (5) technology is relevant as a window for mediation of knowledge (Kuutti, 1996). It is also accepted that the environment that supports our exploration of geometry influences in different ways the appropriation of concepts and skills. Learning geometry with paper, pencil, ruler and compass is quite different from having access to dynamic software such as *Geometer's Sketchpad* or *Cabri-Géomètre*. Putting away mechanic tasks of construction, measure and calculation, time is used to a dynamic and active work in geometry (Laborde, 1993).

With the support of this assumptions, this communication presents a research project which aim is to study mathematical activity in the classroom, occurring among its elements and mediated by geometric dynamic environments identifying how mathematical meanings emerge from that activity (Voigt, 1994; Wertsch, 1991). Drawing on a qualitative approach my analysis focus on the mathematical activity of an ordinary 9th grade class, working in geometry with Sketchpad. Eight lessons were observed and video-recorded. Students work was saved and copied to be observed and analysed. An overview of the outcomes and results of the preliminary analysis are described and analysed in this paper.

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A SOCIAL REPRESENTATION APPROACH TO INVESTIGATE LEARNING

Authors: Monica Rabello de Castro and Janete B. Frant

Affiliation Universidade Santa Úrsula - Instituto de Educação Matemática

Abstract

The purpose of this study is to better understand how students learn mathematics. Specifically, our aim is to investigate and analyze how high school students deal with the idea of function and, how can a teacher tell that a student acquire the function concept. In order to talk about learning, we reviewed the notions of concept and representation in mathematics education. A glimpse on the works of Winner, Herschkowitz, Fischbeim and others lead us to raise the hypothesis that concept acquisition is extremely related to the notion of representation. We argue that an approach from the field of social representation can help us to build a theoretical model to analyze mathematical learning.

This study has been developed in a federal high school in Brazil, students from 9th grade and 11th grade are being observed while working in activities about the concept of function. Some of these activities involve using computer software.

The relationship between concept and representation is found mainly in research findings about different representation for functions using multirepresentational software. The work of Hershcowitz and Schwartz was very relevant. (Herschkowitz and Schwartz 1997, Villareal and Borba 1998). To build a framework for this study we review the literature about concepts, definitions, and representation. (Fainguelernt 1999, Fischbein 1994, Vinner 1994, Varela 1991, Nuñez et all 1997). It is accepted that a concept is different from its definition, being the concept broader. The notion of representation is regarded as a medium between the external world and the internal one. The concept is situated at the thinking level, the representation is regarded as an expression of this concept and it is distinct from the concept. However the nature of concept was not touched. A different perspective come from the works of Vinner, Hershcowitz and Schwartz, they add a component to the concept-relation paradigma that is the concept image. Varela and Nuñez bring different perspectives to this discussion establishing that it is not a pregiven outer world and an internal one. A new model to look for the concept acquisition based on the theory of social representations (Moscovici) and the argumentation strategy theory (Frant and Rabello in press) gave us a strong theoretical support for analyzing data and will be shown.

Partial Findings: Students used linear and quadratic functions as prototypes, confirming Hershcowitz.

The concept of function seems to be part only of the school environment, the students did not give any example about everyday life problems.

Students use strategies of daily life while learning mathematics concepts.

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DEFINITIONS FOR THE CONCEPT OF MAXIMUM / MINIMUM OF A FUNCTION

Shakre Rasslan, Oranim School of Education. Israel

Definitions of a relative (local) maximum / minimum of a function in a certain domain were examined in 204 Israeli Arab high school students. A questionnaire was designed to explore some aspects of the concept. One of the research questions aimed to check whether the students knew how to define the concept of a local maximum / minimum of a function. Another question was whether the students knew how to apply the techniques of calculating extremum points for specific functions. A third question examined the misconception that a maximum / minimum of a function is the largest / smallest value of the function. The results show that 56% of our sample knew the definition, but the entire picture was not encouraging.

This study examines several aspects of the definitions that junior high school students have regarding maximum / minimum of a function. Concept images and concept definitions have been discussed in detail in several papers (Vinner and Hershkowits, 1980). We will therefore introduce them here very briefly. All mathematical concepts except the primitive ones have formal definitions. Many of these definitions have been introduced to high school or college students at one time or another. The student, on the other hand, does not necessarily use the definition when deciding whether a given mathematical object is an example or a non-example of the concept. In most cases, he or she decides on the basis of a concept image, that is, the set of all the mental pictures associated in his / her mind with the *name* of the concept, together with all the properties characterizing them.

The concepts of the maximum as well as the minimum of a function are central in the chapter about derivative of functions. In many countries, including Israel, the chapter on derivative of functions is taught in the tenth grade. The topic is mentioned time and again in high school courses and elementary college courses (pre-calculus and calculus). In most mathematical textbooks one can find definitions such as the following: *We say that f has a* ***relative (or local) maximum*** *at x_o iff there exists a neighborhood V of x_o such that $f(x) \leq f(x_o)$ for all $x \in V$.* (Kitchen, 1968). The definition of a relative minimum can be obtained simply by reversing the inequality in the above definition.

Sometimes, in order to present a new concept, authors of mathematics textbooks limit themselves first to a "special case" which is supposed to illustrate the rigorous definition. The "special case" in our instance was the *continuous and polynomial functions*. The "special case" approach frequently causes serious difficulties in the formulation and the application of concept definitions (Rasslan and Vinner, 1998).

Taking into account the difficulties mentioned in this study, at least some doubts should be raised whether the "special case" approach to the maximum / minimum concept is the most effective way of teaching such a concept. The pool of examples introduced to the students should include many different examples. Only this may increase the chance that one or two examples will not become a prototype and, as such, also a concept substitute.

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Spatial ability and declarative knowledge in a geometry problem solving context

Kristina Reiss

Department of Mathematics, Carl von Ossietzky University Oldenburg, Germany

Research question

Is there a correlation between declarative knowledge about geometrical concepts and spatial ability as it is assessed by standardized tests?

Method

The sample comprised 60 students of grade 7. They took part in individual interviews. In the first part of the interview the students were asked to solve spatial geometry problems (compare Pospeschill & Reiss, 1999, for a description of the tasks presented). In the second part of the interview the students constructed a concept map using concepts related to the problem solving context. Furthermore aspects of spatial ability and general intelligence were assessed using appropriate subtests of the German non-verbal intelligence test PSB (Horn, 1969). The subjects were incidentally assigned to one of two groups. The problems were presented to these groups either in an environment making use of real solids or in a computer environment

Results

The quality of a concept map does not depend on the environment in which the students worked whether it was the computer or the real cube environment ($r_t = .140$; $p = .1149$). The interpretation for this can be twofold. On one hand it may mean that the two groups do not differ essentially in their verbal abilities. On the other hand it may mean that neither the computer environment nor the cube environment induces a certain cognitive style. Analyzing the correlation between the concepts maps and the PSB scores one has to distinguish between the two groups of. Items which present aspects of spatial ability are not indicators for the students' achievement in concept mapping ($r_t = -.009$; $p = .9923$ rsp. $r_t = .094$; $p = .2896$ for the subtests). In contrast, the correlations between concept mapping and general interlligence ($r_t = .217$; $p = .0142$ rsp. $r_t = .212$; $p = .0165$ for these subtests) are siginificant. Thus, general intelligence is a better predictor for good results in concept mapping on geometry concepts than spatial ability.

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ON APPLICATION OF ELEMENTS OF COMMUNICATION THEORY IN CLASSROOM PRACTICE

Kaarin Riives, University of Tartu

The process of teaching and learning is one of many communication processes [2,3]. This note refers to a test undertaken to check the effects of certain theoretical considerations and possibilities in practical work with first-year university students..

The **aim** of a teacher is creation of a positive attitude, motivation to study and interest in the material to be handled. The aim of a student is, in general, to gain a pass in the compulsory subject of the chosen university course. The active party in the **alignment** of these **aims,** at last during the initial stage, is the teacher, whose knowledge and experience as well as **emotional intelligence** will determine the success of the joint undertaking. In **classroom**, at the first meeting, I explained my vision of the forthcoming work to all groups, emphasizing the need for sustained independent effort and feedback, from the student's as well as the teacher's poits of view. Aims and the means for their achievement were formulated. It was interesting to observe the process of alignment of aims within the separate groups. The specialized groups showed a greater initial interest in what was presented than the groups who took mathematics as a general subject. The level of previous knowledge was also essentially different. During the course of the study both categories displayed a noticeable increase in appreciation, but for different reasons. The former began to see their subject material in the light of greater generality than previously comprehended while the latter perceived a new ability in a subject that had formerly seemed difficult and uninteresting. Ongoing work as a **contact preserving** phase requires a variety of means to disseminate information. Parallel use was made of analytic-axiomatic treatment and visualization. Essential results were drawn together and tabulated for systematization and revision of the subject. Reception turned out positive beyond expectations in all groups as the material was made intelligible to all students willing to cooperate regardless of **their cognitive and learning style** [4] as well as differences in initial interest and level of knowledge. This became evident through **feedback** already quite early after commencement of work when the first assignments were lodged and discussions took place on the topics dealt with. Informal discussions gave a picture of the student's attitudes towards the subject studied [1]. The **effectiveness** of the joint work can be assessed by how much the results justified the aims. The formal assessment turned out to be positive for all students who were prepared to cooperate. The moral assessment ought not to be overlooked. Our resultative strenuous work gave joy and satisfaction to all participants.

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"WRITTEN CONVERSATION FORM" AS PROMOTER OF TEACHERS' CHANGE

Naomi Robinson
Department of Science Teaching
Weizmann Institute of Science

Na'ama Adin
Department of Science Teaching
Weizmann Institute of Science

There is a growing recognition of the importance of the social context in which teaching and learning occurs. It becomes clear that any attempt to dissociate the cognitive from the metacognitive, and social aspects of learning, denies the reality of the learning situation (Clarke, 1987). Journal writing, on a regular basis, in the mathematics classroom, has become a tool for promoting reflection. Researchers indicate that teacher reflection on their teaching has two important results: (1) if reflection on instruction becomes a habit, it improves teaching and (2) reflective teachers will also encourage their students to reflect on learning (Cobb et al, 1997). The present study is part of a project aimed at teaching mathematics classes with a heterogeneous student population. The purpose of the project is to promote the learning of mathematics within groups of students of various mathematical abilities, by basing the learning processes on complex, and open-ended mathematical tasks. In order to work according to this philosophy, teachers must change their views and practice in many aspects. During in-service courses, the "Written Conversation Form" (WCF) was chosen as the tool for teacher reflection. The idea of the WCF was based on Clarke's (1987) report about IMPACT – a program aimed to promote student reflection. The WCF was intended to be a means of communication between each participant teacher and the course's teaching staff. The WCF contains questions about both cognitive and affective issues and also allows for other remarks. At the end of each session, the participants were asked to fill in a WCF and received written feedback during the next meeting. Participants' responses to WCF were collected and were photocopied before returning to the participants. The use of WCF during the course had several results. It strengthened and deepened the teachers' understanding of the course's main ideas. It enhanced teachers' reflection on teaching, on students' learning, and on learning materials. It also raised teachers' awareness to the WCF's potential to provide information about their students, and their own instruction.

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IS A COUNTER EXAMPLE ALWAYS ENOUGH TO REFUTE A MATHEMATICAL STATEMENT?

Gila Ron and Orit Zaslavsky

Technion, Haifa

The present study examines students' understanding of the role of counter examples in mathematics. This is another part of a larger study, which was reported in Zaslavsky and Ron (1998). The focus of this presentation was on the validity students attribute to the use of a counter-example in establishing the truth of a mathematical statement.

Two hundred and four students participated in the study. The students were top level 9^{th} and 10^{th} grade students from four different schools.

Two parallel questionnaires were constructed. Both questionnaires describe an imaginary debate between 3 students who try to establish the truth of a false mathematical statement and to justify their conclusion. The first imaginary student claims that the statement is true and justifies his decision by an example. The second imaginary student claims the statement is false and justifies this claim by using a counter example. The third student claims, like the second, that the statement is false, but he criticizes the use of an example for refutation and suggests another way to refute it. The two questionnaires differ in the context. The first deals with a geometric statement, while the second deals with a question taken from a pre-calculus context. The students were requested to provide written responses expressing their opinion about the standpoint of each of the imaginary students.

The analysis of the written responses led to the following findings:

Most of the students did not accept the verification of a statement based on an example as sufficient evidence that a statement is true.

Many students did not accept a counter example as sufficient evidence that a statement is false. The extent to which students were willing to accept a refutation based on a counter-example varied according to the different tasks: About 80% of the students accepted the use of a counter-example to refute a geometric statement, while only 23% of the students accepted it for the pre-calculus context

There were also students whose responses reflected confusion and inconsistency. These students agreed both with the imaginary student who used a counter-example to refute the false statement, and with the student who criticized him for the use of a counter-example.

Zaslavsky, O., & Ron, G. (1998). Students' Understanding of the Role of Counter-Examples. In A. Olivier and K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group of the Psychology of Mathematics Education, v. 4,* 225-232.

TEACHERS' VIEWS ON MATHEMATICS, MATHEMATICS TEACHING AND THEIR PRACTICES

Lynn Rossouw & Eddie Smith
University of the Western Cape, South Africa

This research project describes teachers current views on school mathematics and classroom teaching in relation to the new curriculum requirements.

Purposes: We address three main questions: (a) *What views on mathematics and mathematical activity appear to be prevalent among teachers?* (b) *What views of teaching mathematics that would facilitate learning are used in classrooms?* and (c) *What teaching strategies are employed by these teachers in their classroom?* To address and synthesise these questions we constructed a theoretical framework around teacher's views on mathematics and that of teaching, in relation to an Outcomes-Based curriculum, using data from the eight grade 3 teachers'.

Methodology: An ethnographic research design is used (Hammersley & Atkinson, 1995), as its qualitative methods enabled the researchers' sufficient flexibility for describing, interpreting, exploring and explaining the views teachers have of mathematics and their teaching. The research data was gathered through (a) direct observation and (b) indepth interviews. The research analysis draws on the twenty-four classroom observations and sixteen pre- and post-interviews.

Summary Findings: The views on mathematics and teaching held by the teachers can be categorized into three groups: (a) transmission, (b) empirical and (c) connected. They are by no means water tight categories, as there is some overlap in teachers' views however, it helps us to identify the dominant views held by a specific teacher.

The views of the teachers involved in this study about mathematics and mathematical activities are in direct conflict with a pedagogical practice articulated in an Outcomes-based approach, which offers learners opportunities to engage in problem-solving, logical thinking, recognising patterns, and implementing a pedagogy that focuses on conjecture, conceptual exploration and reflective, critical discussion. The predominant views of mathematics and mathematics teaching among the subjects of this study is that, of a system of algorithm transmitted by teachers to be committed to memory by their students.

Through a process of systematic observation of classroom interactions and interview it was possible to identify teaching styles that do not accord with the expectation of the Outcomes-based approach.

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TRIANGULAR RELATIONSHIP INSTITUTION-TEACHERS- SECONDARY SCHOOL PUPILS: THE CASE OF INEQUALITIES

(Reseach in Progress)

By Luis Weng San and Ribas Gwambe

Brijlall (1996) contends that the forever burdening task of solving inequalities by senior secondary higher grade pupils is consistently an issue debated by mathematics educators. This contention is also a concern in Mozambique. A significant number of new comers at the Eduardo Mondlane University pursuing Physics majors, Chemistry majors and Engineering seem unable to relate certain mathematical concepts to each other. Particularly, in solving inequalities students seem not to see and explore the connection between algebraic and graphical approaches. Besides, it seems that a lot of students epistemologically consider equations and inequalities the same mathematical entities.

Although, there has been a great deal of research on issues regarding inequalities, little was done in this area in Mozambique (e.g. San, 1996) and almost nothing was investigated related to the research question of this study.

The purpose of this study is to improve mathematics teaching at secondary school level, especially the syllabi and teaching methods. In order to approach this objective, we intend to study how the current way of teaching affects the pupils' skills and understanding of mathematical concepts, particularly in the teaching of inequalities.

Therefore, out of several activities it will be studied the contents of curricula, textbooks and final exams of secondary school in order to find out if and how the link between algebraic and graphical approaches is presented. For this short paper it is just aimed to present some preliminary findings of the study of curricula and final exams of secondary school focusing on link between algebraic and graphical approaches.

BRIJLALL, D. (1996). Relook: Graphical Solution to algebraic Inequalities. Paper presented at the Second Nacional Congress of AMESA..
SAN, W. L. (1996). Eight first-year University Students' Solution Strategies in Quadratic Inequalities. Unpublished M. Sc. Research Report, University of the Witwatersrand, Johannesburg.

TASKS FOR ASSESSING RELATIONAL UNDERSTANDING OF FUNCTIONS BASED ON THE OPERATIONAL/STRUCTURAL DISTINCTION

Thomas L. Schroeder, John E. Donovan II, Corinne M. Schaeffer, &
Christopher P. Reisch
State University of New York at Buffalo, USA

We are interested in gaining insight into the teaching and learning of mathematics by developing and administering assessment tasks that probe students' understanding. In particular, we are interested in students' relational understanding (Skemp, 1978) of functions in precalculus and calculus classes. Our work in this area has been influenced by Sfard's (1991) views on the duality of operational conceptions and structural conceptions in mathematics, and by the algebra tasks discussed by Sfard and Linchevski (1994) in terms of the distinction between operational and structural conceptions.

We have developed a number of non-routine problems which we think demand structural conceptions of functions. Unlike routine tasks which may appear to require structural thinking but which students can handle successfully with an instrumentally learned and understood procedure, we believe that these tasks, when used in interviews, can provide evidence concerning the versatility and adaptability of students' knowledge and the nature of their conceptions -- operational, structural, or pseudostructrual.

In this short oral communication we will describe some of the tasks we have developed, and we will discuss the results of the interviews in terms of the qualities of the students' understanding of functions.

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RE-THINKING THE NOTION OF SLOPE UNDER CHANGE OF SCALE

Hagit Sela, Orit Zaslavsky, and Uri Leron

Technion, Haifa

The current study addresses the mathematical, cognitive and pedagogical implications of changing the scale in a coordinate system. It was stimulated by the increasing use of graphical technologies for learning mathematics in the past decade, in which students are often told that they can zoom in or out as they like, since the "behavior" of the graphs of the functions under investigation does not change. They observe very different pictures, and are expected to "see through" these pictures a common behavior. A number of studies indicate the illusions and pitfalls with which such technologies present students (e.g., Goldenberg, 1988; Hillel et. al., 1992). Some of these illusions and pitfalls have to do with the connection between geometry and algebra, and particularly, with a number of basic mathematical concepts that may be interpreted differently from a geometric perspective than from an algebraic one. Our study focuses on the notion of slope of a straight line, and the interplay between its geometric and algebraic meanings.

For the purpose of the study we constructed a set of tasks, in which the change of scale played a role in eliciting how people interpret the connection between slope as a geometric entity related to angle, and slope as an analytic/algebraic property of a function related to the difference-quotient. The tasks were given to a diverse population of people: High school students, secondary mathematics teachers, mathematics educators and mathematicians. In addition, interviews and classroom observations were conducted and documented.

Two main results will be discussed:

- Generally, three approaches were identified: A dominating geometric approach to slope, a dominating analytic approach to slope, and a combined approach that takes into account the conditions under which the geometric system is isomorphic to the algebraic system.
- Similar approaches and conflicts were found across mathematicians, mathematics teachers, and high school students.

In our attempt to offer explanations to these findings, we revisit the notion of slope of a linear function, the assumptions for which it is invariant under different representations, and the connections between the "worlds" of algebra and geometry.

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Hillel, J., Lee, L., Laborde, C. & Linchevski, L. (1992). Basic Functions Through the Lens of Computer Algebra Systems. Journal of Mathematical Behavior, 11. 119-158.

MATHEMATICAL MODELING BY PRE-SERVICE TEACHERS IN A PHYSICS COURSE[1]

Gilli Shama[2] and John Layman, University of Maryland

In the last two decades there is a growing call for making connections between mathematics and science. Scientific situations include stories and experiments. Experiments are sensorial, they include detailed raw information, and they are bounded by physical constraints. Stories are verbal and abstract, they serve as a summary and they are not necessarily bounded. Shama and Layman (1997) found that modeling processes carried by students are affected by the type of the situation.

This paper describes a study of the effect of a physics course on pre-service teachers' (N=22) ability to model a scientific situation (as a story and as an experiment). In the physics course small groups of students designed experiments, carried them out, and presented their results with supporting evidence. Graphs of linear relationships from experiments involving a microcomputer based laboratory were mainly examined.

The course affect was examined by a pen and paper pre and post tests. Each test is composed of two parts, asking to obtain an equation for scientific situation. In Part I, given a story and observing only an experimental setting, individual students are asked to design an experiment, to predict data and to give a representing equation. In Part II the same situation was given to pairs of students to be designed and carried out with the actual experiment.

The students' responses to both tests revealed that in the experimental part they used more graphs and tables, constructed algebraic representations with a better format, used better methods to obtain the slope, and described better the connection between the equation and the data, than were the situation was described only verbally.

From the beginning of the course to its end, the students' responses improved in both parts of the tests. The responses to the part II improved more than the responses to part I. No improvement was found only in checking of the equation back with the data, as Hodgson and Harpster (1997) found "students often proceed from problem to solution without looking back at their efforts or revising their models" (p. 260).

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[1] The research reported in this paper was supported by the National Science Foundation under grant No. DUE-9255745

[2] Currently at The Israeli Open University

MODELS FOR TEACHER ENHANCEMENT IN MATHEMATICS EDUCATION: A TWO-YEAR PROGRAM IN THE BEDUIN SCHOOLS

Ruth Shane
Kaye College of Education Beersheva, Israel

Both standardized tests and informal evaluation showed a low level of mathematics achievement in the Beduin elementary schools of the Southern District in Israel. Recent articles have confirmed the conclusion that children's achievement is affected by teacher expertise which can be improved by teacher education. In particular, curriculum-centered professional development can improve the teaching of mathematics (Cohen & Hill,1997). Therefore, math educators at Kaye College proposed an intervention focusing on teacher education, merging two interrelated components: on-site school supervision and a weekly workshop for the teachers. This research examines the results of the program for the years, 1997-1999.

Traditionally, there have been two separate components to teacher education: subject matter and "methods." Shulman's (1986) definition of pedagogic content knowledge is, however, a reorganization of this knowledge base for teacher development, such that the mathematics be transparently relevant for the elementary teacher and the didactics be integrally faithful to the mathematics. This perspective is particularly relevant in designing a teacher enhancement program.

While cultural issues have been raised in how children learn mathematics, little has been studied about culture and teacher development programs. The awareness that everyone – senior staff, teachers, and pupils - is bringing to the project their own knowledge base for teaching and learning mathematics and that this base must be acknowledged, respected, shared, and developed, is a another key feature of the program.

In the fall of 1997, forty teachers from thirteen Beduin elementary schools were chosen for the program by their principals, 3 from each school, at least one from grades 1-3 and one from grades 4-6. In 1998-99, the program moved into a second phase with the same participants. Two of the Beduin teachers became part of the on-site supervision team.

The findings of this study are based on an analysis of the audiotapes of staff meetings and interviews with participants. Issues of cultural reference are considered - the borders of mathematics, language, politics, and cooperation. The focus is on identifying and categorizing evidence of the professional development of the teachers, of their developing knowledge for teaching mathematics (Shane, 1998).

THE TEACHER – THE DECISIVE AGENT IN THE QUALITY OF TEACHING

Nadja Stehlíková and Milan Hejný [1]
Faculty of Education, Charles University, Prague

The contribution is based on the research and experiences of the authors related to the teaching of mathematics in the Czech and Slovak Republics. From the idea expressed in the title, which is the authors' strong belief, it follows that the improvement of teaching mathematics is only possible via *teachers' pedagogical work*. About twenty years ago we hoped that this could be done by giving the teacher new effective teaching methods, and that he/she would implement them and thus improve his/her teaching. However, this way yielded little improvement, and failed completely to change the instructive way of teaching. The similar experience has been confirmed by many authors[2].

Our understanding of the problem of *influencing teachers* was substantially changed by the following experience.

Dana was a secondary school teacher, who had taught in an instructive way for twenty years. She was satisfied with her work. She enthusiastically took part in our research. Then came conflict. Dana's opposition to our opinions was very strong. She was not willing to accept that the results showed that her students knew algorithms but their knowledge suffered from formalism (= parrot-like knowledge). She prepared her own experiment to persuade us that her students understood mathematics well.

The results of her experiment were not as she expected. This influenced her considerably. When she told us about her experience, she was crying. She was sorry: "for the years when I taught in an ineffective instructive way". In the next years, she systematically and creatively changed the way of teaching different topics and looked for new ways of working with students. She was excited by this work.

The most important result of Dana's case is that the change in her pedagogical beliefs was not brought about by our advice or instruction, but by her independent work. The didactic knowledge cannot be transmitted from researchers to teachers but rather it must be *reconstructed and made meaningful by teachers themselves*.

We believe that a teacher who does independent experimental and research work changes his/her pedagogical consciousness and attitude in a positive direction. This belief has brought about a profound change in our approach to research collaboration both with practising teachers and students - future teachers. In this collaboration, we create a similar climate for them as we, the researchers, have when solving different research problems. Rather than tell the teachers the basic results and principles of the research methods we encouraged them to do their own experiments and to analyse the students' outcomes.

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[1] The contribution was aided by the grants GACR No. 406/97/P132 and GACR No. 406/99/1696.

[2] For more detail see the full version of this paper on http://www.pedf.cuni.cz/k_mdm/pracovnici/stehliko/stehliko.htm.

TEACHERS IN A PROCESS OF CHANGE: REFORMING MATHEMATICS BY BUILDING ON CHILDREN'S THINKING

Steinberg Ruti

Hakibbutzim State Teacher College and
Ministry of Education, Elementary School Dept.,Israel

This research describes case studies of six teachers (grades 1 to 3) who implemented an innovative approach to teaching mathematics, Cognitively Guided Instruction. The teachers learned about children's thinking in solving mathematics problems from current research and from listening to the children. This knowledge served as the basis for instruction in a constructivist approach. Reflectiveness and teacher-research were encouraged. This paper describes the teachers' changes and provides insight into the process of change.

Research methods included about 100 in-class observations conducted in a full school year, interviews with the teachers at the beginning and the end of the year, conversations with them and short reflective writing by the teachers.

Results and discussion. Five of the teachers changed their beliefs markedly during the year. They came to believe that children can learn mathematics concepts without direct instruction and actively construct mathematical knowledge. The teachers also moved toward a new view of their role in the classroom, typified by coaching, listening, creating dialog, directing discussion and giving children opportunities to solve problems, reflect on their solutions and present them orally and in writing. The teachers made substantial changes in their teaching: they gave many more challenging problems, they altered classroom organization, created a rich mathematics environment and learned to listen to and learn from the children. Teachers worked with heterogeneous small groups and encouraged the acceptance of a variety of solution strategies.

The five-level model of Fennema, Carpenter, Franke, et al. (1996) was used to assess the teachers' changes in beliefs and instruction. All 6 teachers started at level 1. One new teacher stayed in level 1. One teacher changed to level 3, two teachers moved to level 4 and two teachers moved to level 5 in only one year. Teachers who moved towards the higher levels also developed more in their ability to conduct discussions. I present here quotations of the teachers' perceptions of their changing roles in the classroom:

- *In the past I used to stand in front of the class, teach, repeat, drill. Today I let the students experiment. I am more of a coach, standing next to them, looking from the side, observing, giving advice, helping if someone gets stuck.*
- *To give challenges that encourage children to think, act, search, and generalize. To present conflicts, dilemmas, to let the children find varied ways to solve problems. To monitor, to be present. When a child needs help, to be supportive. I see my role a lot less as transmitting knowledge.*

EMPHASIZING MULTIPLE REPRESENTATIONS IN ALGEBRAIC ACTIVITIES

TABACH MICHAL

Department of Science Teaching, Weizmann Institute of Science

Computers facilitate, and can thus be used to promote the use of multiple representations. Conventional algebra teaching focused mainly on the symbolic (algebraic) representation. This can pose serious obstacles in the process of effective and meaningful learning (Kieran 1992). Hence, the use of various representations is recommended from the very beginning of learning algebra (NCTM Standards 1989).

Each representation (*algebraic, verbal, numerical* and *graphical*) has advantages and disadvantages. The need to encourage students' individual styles of thinking make the importance of working with various representations obvious. In the case of algebra, spreadsheets and graph plotters allow for quick and well-organized sequences and tables of numbers and a wide variety of graphs.

Flexibility in work with a variety of representations cannot be expected to occur spontaneously. Appropriate tasks are needed, designed to raise student awareness and ability to use several representations simultaneously, whether learning algebra is technologically based or in a more conventional environment.

Savings is an activity designed for beginning algebra learning in the 7th grade, and was developed as part of a comprehensive curricula project – *Compu-math*. In the presentation, I will use the *Savings* activity to demonstrate both guidelines for the design of such activities and episodes from students' activity.

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GENERALIZING WITH EXCEL
AT THE BEGINNING OF LEARNING ALGEBRA.

Naomi Taizi

Weizmann Institute of Science, Israel

Generalizations in early algebra

At the beginning of learning algebra, generalizations are made and applied in a variety of ways. The use of variables is an advanced stage in generalizing or applying a pattern. The use of large numbers, or the "reversed" use of a pattern are two possible ways to require students to generalize without variables (Friedlander et al., 1989).

We observed that students tend to avoid generalizations by performing a step-by-step development of a sequence. Therefore, the requirement to apply a pattern for large numbers does not allow students to follow this path.

Generalizing with Excel

Over the last few years, we experimented with Excel as a tool for generalizing patterns, at the beginning algebra stage. The students were exposed to three main methods of developing number sequences: (a) "dragging" two adjacent cells to produce a linear sequence with the corresponding rate of variation, (b) "dragging" a formula that uses the previous cell to construct the next one (i.e., regression) and (c) "dragging" a formula that uses the place indicator (situated in another column) to construct the sequence.

Several difficulties arise in students' attempts to generalize patterns, using Excel in this way:

* Students prefer to follow their natural tendencies and use regression, rather than employ a formula with the place indicator as a variable.
* Students may get confused by the need to cope with the Excel syntax of writing formulas, in addition to the traditional algebraic form.
* The power of Excel allows students to perform a direct search of numbers in very long sequences, rather than encourage them to use more sophisticated ways which rely on the general rule of the sequence.
* Excel's ability to produce large quantities of numbers de-emphasizes students' need to justify an observed pattern.

Examples of these difficulties and of attempts to overcome them will be reported in our presentation.

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MODELING FIBONACCI: TWO UNIFIX CUBE PROBLEMS

Lynn Tarlow and Emily Dann
Rutgers University

Theoretical Framework Research at Rutgers University carried out over the last decade has centered on the use of unifix cube towers to build models for constructing various mathematical ideas. Chief among these have been those related to combinatorics and proof (Maher & Martino, 1996; Maher & Speiser, 1997). Papert (1980) encourages the use of models for mathematics that provide learners rich concrete environments through which they can explore mathematical ideas and relationships. This gives them references for future problem modeling. This paper presents an analysis of two problems, illustrating the power of immersion in model understanding for extension into other areas of mathematics.

Motivating Question When presented with the challenge of finding a non-recursive formula for the nth term of the Fibonacci sequence, one of the authors created a tower model representation. This representation answered her question: "How could I make this task understandable to my students?" Problem situations were developed that resulted in two different ways of building towers that provide a concrete representation of the Fibonacci structure. These models lead to a non-recursive formula based on combinatorics.

The Models The original tower problem requires one to find all possible towers of height n when allowed to select from two colors. This results in a concrete model of the task of finding how many combinations exist. The first model created to produce the Fibonacci structure uses individual white unifix cubes and blue unifix cubes paired in twos. The second uses individual unifix cubes of white and blue and requires that no two blue cubes will be adjacent.

Presentation The oral presentation will invite the audience to participate in forming each inductive step as images of the towers are created on an overhead projector. Participants will be able to see the Fibonacci sequence develop in a unique setting.

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Children's number concepts: Implications for teacher education.

Noel Thomas
Charles Sturt University

Results of research involving the clinical interviewing of 132 children (Grades K to 6) are used to draw implications for the training of primary school teachers in understanding the needs of young children when learning mathematics. In the study, it was found that children have difficulties with using numeration as a number system and this could be a result of limited instructional experiences. Common teaching practice focusses on the numbers 1 to 1000 (the limits of the usual representations with Dienes blocks) and algorithm-related techniques using place values separately. Also, multiplication and division need to be more closely linked, and more experiences bringing out the recursive nature of repeated groupings need to be provided.

In particular, the research highlights that the teaching of numeration as compartmentalised knowledge restricts the construction of relationships. It is argued that the teaching of numeration requires a more holistic perspective of what children need to develop in their learning of number. Curriculum and teaching need to reflect the goal of achieving understanding of the structure of the number system through key processes of: counting, grouping, partitioning and regrouping and the formation of multiunit values. Moreover, children must be helped to build their own mental connections between their intuitive knowledge, various models that might be used and the formal rules of numeration. Finally, children need to build their own models as a means of constructing meaning for the number system.

In this report, the implications of a reliance on the modelling of number with Dienes base ten blocks is explored through the use of explanations given by teacher education students to a question requiring an extension to common use of the blocks. It is demonstrated that some student teachers do not understand the multiplicative structure of the number system. The students use numbers efficiently but do not see a relationship between the model they are using and number as a base ten system.

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UNIVERSAL THEOREMS: A PARADIGMATIC MODEL OF MATHEMATICAL THEOREMS

Chaim Tirosh
Kibbutzim College of Education, Tel Aviv

A paradigmatic model is a particular instance or a subclass of objects, accepted as a representative of the whole class (Fischbein, 1987). This particular representation of the concept has a strong impact on students' cognitive decisions, and may exert a coercive effect on their ways of reasoning. Examples of paradigmatic models of various mathematical concepts are described in the mathematics education research literature (e.g., Hershkowitz & Vinner, 1982).

Mathematics teaching and learning attends to different types of mathematical theorems, including universal (quantifier) theorems, equivalence theorems, existence theorems and uniqueness theorems. This study aims at determining whether one of these types of theorems acts as a paradigmatic model for prospective elementary teachers.

Three classes of mathematics majors, prospective elementary teachers, women in their second or third out of four years' teacher education program in two Israeli State Teachers' Colleges, participated in this study. At the beginning of the academic year, these prospective teachers answered two questionnaires. The first included nine mathematical propositions. Participants were asked to consider each proposition, to decide whether it was true or false and to prove their assertions. The second questionnaire consisted of six propositions, each accompanied by four to seven prepared arguments in favor of or against the proposition. The prospective teachers were asked to judge the truth of each statement as well as to determine if each presented argument was a valid mathematical truth. Analysis of prospective teachers' responses to both questionnaires suggests that universal theorems serve as paradigmatic models of mathematical theorems, for this population. In the presentation I shall provide various examples of items from both questionnaires, typical prospective teachers' responses and some, educational implications.

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WHAT COUNTS AS EVIDENCE: UNCOVERING THE FOUNDATIONS OF FAILURE IN MATHEMATICS LEARNING

Dianne Tomazos and Carrol Hall
Education Department of Western Australia

The Education Department of Western Australia is implementing a major reform of the curriculum using an outcomes based approach. Rather than prescribing what must be taught, the department has made explicit the learning outcomes which all students should achieve, and assigned to each school the responsibility and flexibility to develop "their own teaching and learning programs according to their circumstance, ethos and the needs their students" (Curriculum Council 1998 p.6). The role of the classroom teacher in making ongoing professional judgements about student learning has become central to this process. Determining what counts as evidence of having achieved outcomes, along with the early identification of students 'at risk' of failing to achieve outcomes, are therefore essential research foci.

In this presentation the authors will report on a collaborative action research project in progress* which is designed to assist teachers develop strategies for identifying, accelerating and monitoring students who, by the end of primary school (approx 12 years old), have not achieved a sufficient level of mathematics learning to enable them to make adequate progress in secondary school. A feature of the project has been the opportunity for year seven primary generalist teachers and year eight secondary mathematics teachers to work together to investigate and analyse the features of children's developing mathematical ideas which are likely to put them 'at risk' of future failure to achieve. Although the project is still in the early stages of development, some important issues are emerging.

There are distinctive differences in the way primary and secondary schools in this state operate and this has implications for the ways in which the respective teachers are able to interact with students, and consequently the types of evidence they use to make judgements about students' learning. Developing a common unbiased means by which evidence of achievement in mathematics is determined by teachers in their day to day classroom practise is a major challenge, as Watson (1997) has also noted.

What children cannot do in mathematics is usually the initial focus when teachers are asked to identify students 'at risk', and this is relatively easy to ascertain. Why they cannot do it then becomes the central question as teachers explore strategies for accelerating students' learning. However, questions about why children CAN do mathematics are also emerging as important issues in the project - there are children who can do the expected mathematics by the end of their primary schooling, but for the wrong reasons, thereby laying the foundations for later failure.

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*Transition Numeracy Project - an initiative of the Education Department of Western Australia funded by the Australian Government Department of Education, Training and Youth Affairs (DETYA).

"NO ANSWER" AS A PROBLEMATIC RESPONSE: THE CASE OF INEQUALITIES

Pessia Tsamir* Nava Almog**

Kibutzim College of Education* Tel Aviv University* Beit Berl College**

There is a growing interest in the last couple of years, in learning and teaching algebraic inequalities. Traditionally, the related mathematics education literature consisted mainly on papers suggesting preferable teaching methods, and only few papers either describe students' ways of thinking about inequalities or examine the impacts of given teaching methods on students' conceptions (e.g. Linchevski & Sfard, 1991, Piez & Voxman, 1997, Tsamir, Almog & Tirosh, 1998).

Our study investigated 355 high school, mathematics majoring students' solutions to various types of inequalities, including linear, quadratic, absolute value, rational and irrational ones. A 30 items questionnaire was administered to the participants, and they were given two hours to solve these inequalities and explain their solutions. Those who presented unusual (correct or incorrect) solutions were orally interviewed; in order to get a better insight into their ideas.

The first phase of the analysis of data related to each type of an inequality, focusing on preferred solving methods, common mistakes, and possible reasons for these mistakes. In the second phase, a more general perspective was taken. Incorrect solutions, common to several types of inequalities were detached and possible sources were suggested. This analysis revealed that in all inequalities, cases in which the solution was either "there is no value to satisfy the inequality", or "no numbers satisfy this inequality" were found to be extremely problematic for a substantial number of the participants. Most outstanding were students difficulties with '≥' inequalities, where students tended to claim, even in purely numeric cases such as $3 \geq 3$ or $7 \geq 3$, that no numbers can both be smaller than and equal to another number, and thus there is 'no answer'. In the oral presentation examples will be presented, and possible implications to instruction will be described and discussed.

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MATHEMATICAL BECOMING: THE PLACE OF MATHEMATICS IN THE UNFOLDING STORIES OF LEARNERS' IDENTITIES.

Peter Winbourne
South Bank University

The research to be reported in this session is framed by theories of situated cognition. The tools for analysing apprenticeship models of learning provided by this perspective (Lave 1988,1996, Lave and Wenger, 1991) are used to describe mathematics classrooms in terms of multiple intersections of practices and trajectories. (Winbourne and Watson, 1998.)

A central focus of this session will be the exposition and justification of a methodology which enables the learner's developing identity (or, more appropriately, identities) to be characterised. This methodology, essentially biographical, draws upon the work of writers such as Ricoeur (1984, 1985, 1988), and Brown (1997). It is used to claim the legitimacy and validity of adding a narrative dimension to the researcher's description of learning. In this research, the processes of mathematics learning observed in the classroom are placed in the wider contexts of the lives of the people who happen to come together in that classroom (in that school.)

Data drawn from observations of a small group of learners - 'inside' and 'outside' of the classroom - will be presented. These data, woven into narratives, will allow particular classroom events, associated with learning of specific pieces of mathematics, to be presented as strands in possible stories of the learners' lives. The discussion will focus on aspects of those stories which describe the learners' developing identity within the practice of school mathematics seen as but one of the many practices in which learners participate. These stories provide a powerful account not just of what, but of how and why people learn.

The research suggests that learning happens as much as a result of the complex identities in practice that teachers and learners bring with them when they step into the classroom as anything that happens to them once they are there.

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VISUALIZATION OF SOLUTIONS OF A QUADRATIC EQUATION

Kiyosi Yamaguti
Kyushu Sangyo University, Fukuoka 813-8503, Japan

To solve a quadratic equation is one of important topics of high school mathematics. If the numbers considering are restricted to the real numbers, then the equation has no solution when the discriminant of the equation is negative. By introducing the complex number, any equation has (complex) solution. A real solution can see as an intersection of the graph of quadratic function and the x-axis. How visualize elementary an imaginary solution on the complex plane would be important for teaching of high school mathematics.

We first recall three visualizations of the products of complex numbers in the complex plane in the sense of Hamilton. Secondary, we consider the geometric constructions of solutions of quadratic equations. To construct imaginary solutions is essentially equivalent to construct two real solutions. In fact, given a quadratic equation (1): $ax^2+ bx+ c=0$, then consider a quadratic equation (2): $ax^2+bx+c^*=0$, where $c^*:=(b^2/2a)-c$, associated with (1), then (1) has two imaginary solutions if and only if (2) has two real solutions. We shall give some geometric constructions by using parallelism and drawing of circle in the orthogonal coordinate plane. Example: Assume the discriminant D of the equation (1) is negative. Then, the solution of (1) is a solution of the system of equations (3) $x=-b/2a$, (4) $x^2+y^2=c/a$. Hence, the imaginary solution is obtained as intersection of the axis (3) of parabola and the circle (4). Assume $a>0$, $D>0$ in (1). Then, (2) has imaginary solution points P^*, Q^*, that are obtained by above. The real solutions of (1) are the intersection of the x-axis and a circle with center $(-b/2a, 0)$ and P^*Q^* as a diameter.

The above geometric constructions are elementary. However, these constructions need that student understands the theoretical reasoning on solutions of quadratic equations and the properties of the graphs of quadratic functions.

A graphing calculator can also use for these constructions. Therefore, such a construction of the solution could assist students for visual and theoretical understanding for a solution of quadratic equation.

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UNDERSTANDING THE CONNECTIONS BETWEEN THE GRAPH OF A FUNCTION AND THE GRAPH OF ITS DERIVATIVE

Smadar Zamir and Orit Zaslavsky

Technion, Haifa

The notion of derivative is fundamental in pre-calculus and calculus courses. There are many calls recommending that emphasis be drawn to connections between properties of the graph of a function and the graph of its derivative (Hallet, 1991; Tall, 1991; NCTM, 1989). Yet, in spite of the qualitative and visual aspects the notion of derivative, students often treat it technically without understanding its meaning. One way to create opportunities for the recommended kind of learning experiences dealing with derivatives can be done with a *Gradient Measurer* (Tall, 1991). This tool provides ways to pointwise construct the graph of the slopes of a given graph. Thus, it enables the student to construct the graph of a derivative function without having to know its analytic expression.

The purpose of the present study was: (1) to explore ways in which students understand and solve problems associated with the connections between the graph of a function and the graph of its derivative; and (2) to examine the influence of learning experiences based on the use of a *Gradient Measurer* on students' interpretation and application of these connections.

Nineteen 11-grade students participated in the study. The students encountered a series of learning activities based on the use of a *Gradient Measurer* to construct graphs of derivatives. They were given a written questionnaire prior to this intervention and again after its completion. Both questionnaires were similar in structure and content, and included 13 problems dealing with visual connections between the graph of a function and the graph of its derivative. All the problems could be solved without the use of any analytic expression of the function or its derivative.

Analysis of students written responses to both questionnaires pointed to five main approaches students used: 1. Expressing the need for an explicit analytic expression of a function or of its derivative; 2. Attributing similarity between the graph of a function and the graph of its derivative; 3. Turning to familiar families of functions; 4. Applying familiar theorems; 5. Examining or constructing the slopes of the graph at various points. Some approaches were more helpful than others for solving the above kind of problems. The findings indicate an increase in use of the more helpful approaches.

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CONFRONTING AND MODIFYING STUDENTS' INTUITIVE RULES IN NUMBER THEORY

Rina Zazkis

Simon Fraser University.

This study adds another example to a rich collection of mathematical and scientific situations presented by Stavy and Tirosh in which the intuitive rule "the more of A, the more of B" has been applied. It is students' intuitive belief that a larger number has more factors.

This study is situated within the research on learning mathematics at the undergraduate level in general and learning introductory Number Theory by preservice elementary school teachers in particular. I will introduce "the rule" by demonstrating excerpts from interviews with preservice elementary school teachers. I will discuss the robustness of this rule by considering how students assigned truth value to the following two statements:

(1) "If a natural number *a* is bigger than a natural number *b*, then the number of factors of *a* is bigger than the number of factors of *b*."

(2) "If a natural composite number *a* is bigger than a natural composite number *b*, then the number of factors of *a* is bigger than the number of factors of *b*."

I will analyze the differences in students responses to the two statements, pointing out to a pattern emerging in students' struggle to reconcile their intuition with conflicting evidence. Further, I will show students' tendency to consider "conflicting evidence" as exceptions, their willingness to amend the rule, but not to give it up. In conclusion, I will argue the importance of instructor's awareness of students' potential misconceptions. Pedagogical approaches that confront students' popular beliefs (without giving explicit reference to those), and attempt to deepen students' understanding of the relationship between natural numbers and their factors will be suggested.

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IMPROVING STUDENTS' MATHEMATICAL THINKING: THE ROLE OF DIFFERENT INTERACTIONS IN A COMPUTER ENVIRONMENT

ORIT ZEICHNER, BRACHA KRAMARSKI, ZEMIRA MEVARECH
SCHOOL OF EDUCATION
BAR-ILAN UNIVERSITY, ISRAEL

The goals of mathematics teaching published by the National Council of Teachers of Mathematics (NCTM, 1989) placed special emphasis on doing mathematics in a manner that encouraged students to develop their ability to solve problems, think and give reasons for the solution process. The present study examined the effect of learning in a computerized environment with interactions based on different types of feedback, on achievements in mathematics and on developing mathematical communication. The study is based on the Self-Regulation Learning model and two types of feedback provided by the computer when the student made a mistake: metacognitive feedback (SRL) (Mevarech, & Kramarski 1997) and knowledge of results (KR) feedback. To test mathematical thinking and communication, subjects (186) were given an achievement test in arithmetic series that examined general term formula, rule of recursion, verbal problems, and overall score. To test mathematical communication, students' explanations were analyzed, first as right/wrong explanations, and then in terms of type of rationale according to the categories: algebraic rationales, verbal rationales, and algebraic and verbal rationales. The main conclusion of this study is that students working in a computerized environment with metacognitive feedback attained higher achievements and demonstrated better ability to explain their solutions than students who worked with knowledge of results feedback. The authors also bring theoretical and practical applications of the research findings.

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POSTER PRESENTATIONS

MOTIVATING TEACHERS TO USE ALTERNATIVE ASSESSMENT

Jeanne Albert
Weizmann Institute of Science - Israel

This poster describes a national project which uses mathematical investigative tasks for assessment in elementary schools. During the first three years of the project, the researcher found that elementary school teachers believe that understanding and thinking are their most important goals in teaching mathematics. Nevertheless they are still creating mostly traditional tests.

In 1994 the Israeli Ministry of Education funded a project to create test item banks for use in school-based assessment. The mathematics investigative tasks which we created were guided by the following principles:

- to present mathematics as a subject which requires dealing with authentic, worthwhile problems.
- to emphasize the process of solving instead of final solutions.
- to emphasize processes of higher level thinking - e.g., generalization, justification, estimation.
- to downplay the use of calculation techniques and encourage the use of calculators.
- to introduce the students to unknown problems without algorithmic solutions.
- to introduce the students to problems with many solutions.
- to integrate various branches of mathematics.
- to enable each student to reach his highest possible mathematical ability.
- to enable even students with lower mathematical abilities to experience mathematical investigations.

In order to encourage teachers to use investigative mathematics tasks as assessment tools workshops were given. The general plan of a workshop was adapted from a model Clarke (1995) developed for familiarizing students with newly introduced criteria of assessment. There is evidence that teachers' participation in these workshops has an impact on their instruction and assessment.

In the poster will be displayed:

- Teachers' beliefs.
- Examples of typical elementary school mathematics tests.
- Mathematical investigative tasks from the test bank with students' work.
- Examples of tests which have been influenced by workshop participation.

Clarke, D. (1995). Quality Mathematics: How Can We Tell? *Mathematics Teacher*, 88(4), 326-328.

Mathematical sense-making strategies of non-academically oriented students

Yoni Amir and Orly Gottlib
Department of Science Teaching
Weizmann Institute of Science - Israel

"Shay" is a curriculum development project for non academically oriented students (ages 15-18) in Israel, who undertake a national matriculation examination at the end of high school. The curriculum includes the basics of analytic geometry, calculus, statistics, probability, linear programming, trigonometry and stereometry. The approach of the learning materials stresses graphical and visual aspects, uses of and connections among multiple representations, informal intuitive reasoning, and estimation. In the records collected, we found instances of flexibility, control mechanisms and resourcefulness in student thinking which may exceed those usually found in more mathematically able students.

In the poster, we will display:

- Background information of the "Shay" program, and its pedagogical rationale.
- Examples of student work in which:

a) a problem clearly identified with one mathematical subdomain is solved using tools from another subdomain
b) problems which require formal tools, are solved informally,
c) the search for alternative perspectives on a solution approach reveals a mistake in the original approach,

The examples will be presented in its original written form, translated into English, and analyzed. The analysis will suggest the ways in which the learning materials support students' sense making attempts and strengthen their confidence and self-esteem.

TEACHERS' BELIEFS AND USE OF NON-ROUTINE PROBLEMS IN MATHEMATICS TEACHING

Dalia Asman Gordon College, Haifa , Israel **Zvia Markovits** Oranim School of Education, Israel

This study focuses on teachers' beliefs regarding mathematics non-routine problems and ways in which teachers cope with this kind of problem.

Interviews were conducted with thirty elementary school teachers from three different groups.

a. Ten in-service mathematics teachers. (T)

b. Ten in-service teachers who participated for at least two years in teachers' development programs. (TT)

d. Ten pre-service teachers specializing in math, in their last year of study (PST)

Teachers were first asked about their general beliefs concerning mathematics problem solving. Then, each teacher solved eleven non-routine problems. Each problem was followed by a series of questions exploring their beliefs about the problem. Questions included: Would you pose this problem to your students, to the whole class or part of it? Would you give this problem in an examination?

One of the problems was the Bell Problem.

At the top of a high tower, there are three bells. One Rings every 10 minutes, the second every 8 minutes, and the third every 5 minutes. They all rang together at 12:00. When is the next time they would ring altogether?

The following table indicates the number of teachers who solved this problem and distribution of their answers concerning some of the follow up questions.

	T	TT	PST	Total
Solved correctly	5	8	3	16
Would pose problem to the whole class	9	9	10	28
Would give the problem in examination	1	2	0	3
Encountered the problem in textbooks or math courses	0	0	0	0

Findings indicated that almost half of the teachers solved the problems incorrectly. Nevertheless, most of them after realizing the correct solution were willing to pose the problem to their entire class. However most teachers would definitely not pose the problem in an examination, in order to avoid unnecessary difficulties for their students during exams. None of the teachers had encountered such a problem in any of elementary schools' textbook.

The analysis of the other ten research problems draws much a similar picture. The TT teachers solved more problems correctly than the other teachers did. In addition during the interviews, TT teachers expressed much more the kind of beliefs which we would like and expect math teachers to hold. This might indicate the importance of in-service teacher development programs.

LEARNING MATHEMATICS BY PROJECTS USING WEB PAGE

SEP – SSEDF – DGENAMDF
Yolanda Campos – Centro Siglo XXI
Teresa Navarro de Mendicuti – BENM

Objective and description: In this experience we have elaborated one prototype project which represents an interinstitucional effort to produce some kind of material that encourage the critical and creative incorporation of computer technologies in the area of teachers training as part of an Academic Programme of Computer for Education in Mexico D.F.

The objective of this prototype is to promote the learning of Mathematics in elementary school based in projects, to stimulate teachers and students to use computer technologies critically and creatively as a didactic assistance and elaborate a Web page as an integrative element within the multimedia elements generated during the projects process. Concerning the level of teachers training we intend to promote the active participation of teachers and students the use of computer technologies as a support for educational projects that can be applied to elementary education after being introduced to the prototype.

Among the computer technologies used we have: productivity tools and structured educational software elaborated specially for the prototype, Logo, software for exploration, Internet, and the elaboration of a Web page. This Web page integrates the project description, the software, book and notebook exercises, participation in learning circles and discussion forum. Didactic suggestions are offered as well as a teachers manual, parental advice and lines of investigation. It is our wish to prepare the teachers trainers together with the students so that they can apply the project working model in elementary school and generate new projects with the assistance of the prototype.

Some Results: Until now in the experimental phase of the prototype, teachers in training involved have acknowledged the agreement between the project and the focus of the new plans and study programs within Primary School Study Licence. For the moment we have not yet gotten personal project production but we hope to achieve this in 99. Working with the children in open participation within the pilot phase we have observed the way they have advance not only in Math but also in their attitude towards school according to their parents. We were able to capacitate teachers from the Valle Arispe Primary School on the project focus and collaborative learning around the prototype; each teacher managed to produce a personal project that was included in the Web page. We successfully organised a direct workshop of Web page project elaboration with the National Association of Teachers of Mathematics.

We have considered pedagogic follow up for the learning circles for children, teachers, parents and researchers as well as discussion groups with other interested teachers. It is our intention to document every action realised and generate lines of investigation to sustain the thesis proposed in the corresponding section. Developmental courses for training and actualising, as well as other higher studies have been organised for teachers in service to be innovated and reflective in this new system.

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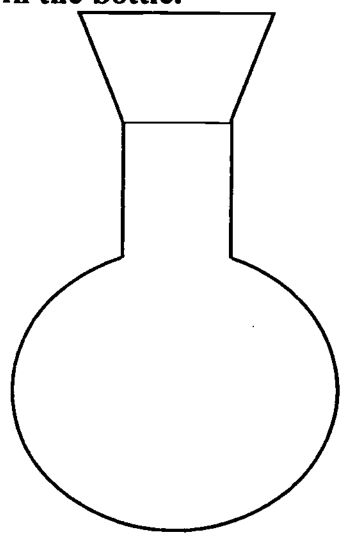
A Study of Second Semester Calculus Students' Notion of Covariation

Marilyn P. Carlson
Arizona State University

Research has shown that calculus students possess weak understanding of impor aspects of the function concept, with particular difficulty understanding covariant aspects function relationship (Carlson, 1998; Thompson, 1994). The present study investigate calculus II students' understanding of covariation.

Imagine this bottle filling with water. Sketch a graph of the height as a function of amount of water that's in the bottle.



On the above problem, 14 of the 20 students constructed a graph that was str concave up or concave down, conveying that "as more water is added, the graph should r However, when asked to explain why, most appeared to be responding with a memor phrase. When prompted to discuss the changing shape of the graph, fewer than half mention of the changing rate, and when further probed, did not appear to understand varying magnitude of the variables represented in this situation. Few students attended to "rate change" from positive to negative. Although these students had successfully complet course that emphasized rate and covariation, most did not appear to possess deep-se understandings for applying this knowledge in fairly routine settings.

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LEARNING CONSTRUCTIVIST TEACHING BY DOING: A COURSE FOR IN-SERVICE TEACHERS

Chang, Ching-Kuch

National Changhua University of Education, Taiwan

E-mail: macck@cc.ncue.edu.tw

A problem-centered and investigation-based course on teaching for in-service mathematics and science teachers has been developed in a summer program. The purpose of this course was to learn how to teach mathematics or science from the constructivist perspective. The course development was based on constructivism, especially social constructivism (Ernest, 1991). Major adjustments concerning the structure, contents, and the ways of teaching have been made on this course. The course not only introduced constructivism, but also taught according to its principles, letting the teachers construct their teaching knowledge by doing, talking, presenting, and writing. Furthermore, the course also had the teachers investigate their own teaching problems. During investigation the teachers have processes experiences such as exploring, searching, formulating, planning, testing, justifying, conjectures, reflecting, and generalizing. It was found that the new course was more effective than traditional course—by lecturing. Teachers' learning was illustrated by excerpts from their journals.

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THE COMPLEMENTARY ROLES OF IDEA INITIATOR AND INQUIRER IN MATHEMATICS CONJECTURING ACTIVITY

Ing-Er Chen
Kaohsiung Normal University

Fou-Lai Lin
Taiwan Normal University

We had developed a thinking model of mathematics conjecturing. Evidence in our research had shown that two different thinking paths in this model conforming to the functions of conjecturing and refuting respectively could keep the fluency of one's thinking. Thus, we applied this model to design mathematical conjecturing activities to develop students conjecturing abilities.

This paper reports the discourse between two students in the activity conducted in a Taiwan eighth-grade mathematics class. All students were encouraged to guess a conclusion and judge the correctness of a proposition. Their performance in this activity could be divided into four stages: making conjectures, judging reasonably, refuting arguments, and comment and reflection. Some learning materials were quoted or rewritten from Mason (1989) and Polya (1954). Data was obtained from classroom observations, interviews, transcription of videotapes and audiotapes of the lessons, and students' booklets.

According to the episodes of their discourse, one student can be identified as an idea initiator, and the other as an inquirer. The idea initiator was an active conjecturer. Her contribution in the class is to provide some vague but good ideas followed by others. The contribution of the relative role, idea inquirer, is to arouse everybody's thinking. She got benefit from the activity by absorbing the initiator's ideas and her query could help the initiator to put her thinking in order.

Our evidences show that successful conjecturing needs two roles, idea initiator and idea inquirer. One student may play these two roles spontaneously. But most students could only play one of these two roles. Thus, the interaction between initiator and inquirer, who both offered complementary scaffolding by each other and conformed to the functions of conjecturing and refuting respectively, made the conjecturing activity successful. Based on this phenomenon, we suggested that cooperative learning and social construction might be the appropriate way of developing conjecturing activitiy.

THE HETEROGENEOUS CHARACTER OF THE STUDENT TEACHERS THOUGHT

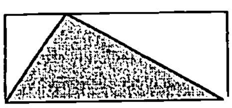
Climent, N.; Contreras, L.C. & Carrillo, J.
(University of Huelva, Spain)

Teachers usually consider their students as a *group* (*homogeneity* tendency), but every student builds their own *individual* knowledge in their personal way, based on their previous knowledge and their beliefs on mathematics and its teaching and learning. Their mathematical abilities and their own logic come also into play in every situation they approach.

Problem solving emerges as a challenge for students. They, when tackling problems, apply their mathematical abilities and views. Hence, problem solving is a very appropriate realm in which one can analyse knowledge as well as beliefs.

We have posed to related statements (C1 and C2) of the Dickson et al. Problem to two groups (A, B) of student primary teachers (C1 to group A and C2 to group B).

C1	C2
Find the area of the triangle of the picture, given 20 cm^2 as the area of the rectangle.	What part of the rectangle is the triangle of the picture, given 20 cm^2 as the area of the rectangle?



Inspired in Dickson et al. (1984) *Children Learning Mathematics: A Teachers Guide to Recent Research.* London: Cassell.

The students were asked to solve it at home and to write the process in as detailed a way as possible. After that, we analysed their protocols and we chose one representative of each different way of solving. Each student was interviewed twice (individually, to light some aspects of their processes and, in groups, to promote sharing and further discussion, comparison and revision of the solutions). Our analysis focused on main global processes and perspectives, as well as major errors, to find some differences related to students' beliefs and knowledge organisation.

We conjectured that there would be different approaches depending on the numerical or geometrical character of the statement. It happened, nevertheless almost all the students seemed to be keen to give numerical support, making explicit some aspects of their mathematical beliefs. In addition, we found an absence of reflection on what is being done in each moment of the process.

With these comments we want to highlight the importance of metacognitive knowledge and beliefs in teacher education. The metacognitive features go beyond the specific mathematical knowledge towards the students' current and future professional knowledge (metacognitive knowledge lies in the basis of professional knowledge, supporting it).

AUTHENTIC LEARNING IN MATHEMATICS: A REAL POSSIBILITY OR AN ACADEMIC'S FANTASY ?

Anne D. Cockburn,
University of East Anglia, U.K.

Kindergartens and elementary classrooms tend to be full of brightly coloured tools and toys which are rapidly produced when the children are introduced to the early stages of mathematics. Unfortunately, as the students grow older and the work becomes more formal, there is less and less evidence of apparatus - let alone everyday objects - to develop skills and promote understanding. Thus, in effect, the quality of 'authentic learning' (Desforges, 1995) is diminished leaving mathematics as an abstract series of equations and problems which appear to have little, or no, relevance to the real world.

I am certainly not the first to advocate that learning from everyday experience is effective (see Froebel, 1887; James, 1899; Dewey, 1933; Kamii, 1985 and so on) but it may be that (1) it is an unrealistic aspiration when one is endeavouring to teach thirty children or (2) only fairly simple mathematics lends itself to the use of real examples which may be familiar to the children.

This poster focuses on the second of these possibilities. It will include some examples of mathematical concepts which have been presented in an everyday manner in order to aid pupil insights and understanding. It will also encourage delegates to explore the possibility that more complex mathematics might be presented in such an accessible way and invite them to express their opinions on such an approach.

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DOES THE AUTHENTICITY OF THE CONTEXT AFFECTS THE TENDENCY TOWARDS IMPROPER PROPORTIONAL REASONING?

Dirk De Bock* **, Lieven Verschaffel* and Karen Claes*
University of Leuven* and EHSAL, Brussels**; Belgium

Two recent studies by De Bock, Verschaffel and Janssens (1998) revealed a very strong tendency among 12 to 16-year old pupils to apply the linear model in non-linear scaling problems (such as "Farmer Carl needs 8 hours to manure a square piece of land with a side of 200 m. How many hours would he approximately need to manure a square piece of land with a side of 600 m?"). However, these studies did not explain why so many pupils improperly made use of this model. One possible explanation for pupils' misuse of the linear model is the inauthentic or unrealistic nature of the problem situations. Some evidence for this hypothetical explanation can be found in Treffers (1987), who realized a design experiment in sixth graders on the influence of linear enlargement on area and volume that was built around the context of "Gulliver's travels", and claimed that, in this realistic mathematics education approach, pupils have no difficulty with a problem like "How many Lilliputian handkerchiefs make one for Gulliver?". However, the facilitating power of realistic and attractive contexts on pupils' ability to solve non-linear scaling problems has – as far as we know – never been investigated in a systematic way.

To investigate the influence of authentic contexts on pupils' well-documented tendency towards improper proportional reasoning, we executed a new study. In this study, 152 13 to 14- and 161 15 to 16-year olds were matched in two equivalent groups. A paper-and-pencil test, consisting of two proportional and four non-proportional scaling problems was administered to all pupils. In the first group, the test was preceded by an assembly of well-chosen fragments of a film version of Gulliver's visit to the isle of the Lilliputians, and all experimental items were linked to these film fragments. In the second group, an equal number of mathematically isomorphic problems was presented in the form of a series of non-related traditional school problems, without any contextual support.

In this poster, we present a selection of the materials and the main results of this research. In addition, we exemplify the different correct and incorrect strategies pupils applied to represent and solve the non-proportional problems in both experimental groups.

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VISUALISATION AND THE INFLUENCE OF GRAPHICAL CALCULATORS

Sally Elliott, Brian Hudson. Sheffield Hallam University, Sheffield, UK.
Declan O'Reilly. The University of Sheffield, Sheffield, UK.

Visualisation is increasingly being accepted as a fundamental aspect of mathematical reasoning. Indeed many researchers stress the importance of mental imagery in the construction of meaningful mathematics (Presmeg, 1995; Wheatley and Brown, 1994). Zimmerman and Cunningham (1991) further argue that visual thinking needs to be linked to other modes of representation in order for students to learn optimally.

The potential of utilising graphical calculators to promote and encourage visualisation skills has been recognised in numerous studies. In particular, graphical calculators can be used to enable students to develop a deeper insight into functions and their graphs (Carulla and Gomez, 1997, Ruthven, 1990). Borba (1996) suggests that the use of graphical calculators mediates both teacher-student relationships and interactions between students.

This study investigated ways in which the graphical calculator mediated students' powers of visualising functions. The findings indicate that this occurred in three distinct ways. Firstly, it appeared that the graphical calculator enabled the students to access graphical images of functions quickly and easily, when perhaps they may have had difficulty otherwise. This in turn allowed them to see the problem more clearly and proceed towards a solution. Secondly, it seemed that the graphical calculator influenced students' perceptions in a positive way towards the validity of visual methods in mathematics. Thirdly, the observations suggested that the graphical calculator was instrumental in improving levels of student confidence surrounding functions. It did so by providing scaffolding for student-student interactions, which enabled students to make connections between visual and symbolic modes of representation more easily.

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"*MANOR PROJECT*": PREPARATION OF TEACHER-LEADERS AND IN-SERVICE TEACHER EDUCATORS

Ruhama Even, Hasida Bar-Zohar, Orly Gottlib, Nily Hirshfeld, Naomi Robinson, Josephine Shamash, Weizmann Institute of Science, Rehovot, Israel

The *Manor Project* is used as a vehicle for the preparation of teacher-leaders and inservice teacher educators whose role is to promote teacher-learning relevant to mathematics teaching as part of effecting changes in school mathematics. The poster presents the central aspects of the *Project* in an attempt to expand and enrich understanding about professional preparation of teacher-leaders and teacher educators.

The aims of the *Manor Project*: 1) to prepare promising mathematics educators to serve as teachers, leaders and guides for secondary school teachers and provide support in the process of studying mathematics teaching and introducing changes in school mathematics, 2) to prepare resource materials for project participants and other mathematics teacher educators relevant to their work with teachers.

The preparation program is in the framework of a two-year intensive course whose central aims are: 1) to develop an understanding about current views in mathematics teaching and learning; 2) to develop leadership, mentoring knowledge, skills and work methods with teachers; 3) to create a professional reference group.

The program focuses on cognitive, curricular, technological and social aspects of teaching different mathematical topics, such as: algebra, analysis, geometry, the real numbers, probability and statistics. It also examines critical educational issues, such as assessment and teaching in heterogeneous classes, enhances mathematical knowledge, emphasizes the development of leadership skills and methods for working with teachers, encourages discussion of practical difficulties and dilemmas. The program also focuses on educational initiatives whose purpose is to effect changes in school mathematics teaching and learning.

The resource materials focus on central subjects in mathematics teaching and learning and are field-tested. Three resource files have so far been developed: Algebra, Functions and "π". The major themes in these files are: historical aspects of the central subject, selected mathematical topics relevant to the subject matter of the file, students' conceptions and ways of learning and thinking, aspects of mathematics teaching relevant to the central subject.

All resource files contain detailed suggestions of activities for teacher-development meetings and provide examples of different models for such meetings, intended to serve as a guide to teacher-educators and teacher-leaders.

The Use of Images in Primary mathematics texts

Tony Harries, Bath Spa University College
Rosamund Sutherland, Graduate School of Education, Bristol University

This presentation represents part of the work from a larger study which compared mathematics text books used in primary schools in 5 different countries – France, Hungary, Singapore, United Kingdom, United States of America. Text books play an important role in influencing the ways in which primary teachers think about teaching and learning mathematics. We take the view that what appears in a mathematics text book does not appear by chance. It is influenced by the multifaceted aspects of an educational culture. In this way mathematics text books provide a window onto the mathematics education world of a particular country. A full consideration of the results of the study, which included a comparative analysis of the use and consistency of use of multiple representations, are presented in the final report (Harries, Sutherland 1999). A number of ways have been developed for classifying illustrative representations in texts. But one of the most useful for our purposes is developed from Botsmanova (1972). Three categories are used: **Objective-illustrative, Object-analytical** and **Abstract spatial diagrams and sketches**.

Using these categories the presentation will contain illustrations from a range of text books showing in particular the way in which images are used to introduce pupils to the concepts which underlie multiplication and division. We suggest that it is important to make more explicit the principles influencing text book design and development so that consistency in the way in which concepts are represented and developed can be pursued and theory and practice can develop in an iterative way.

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SOLVING ALGORITHMIC PROBLEMS ASSISTED BY THE COMPUTER

Ronit Hoffmann,
Kibbutzim College of Education, Israel

The main consideration leading to the development of the above mentioned course was the **need** to use the computer as a tool for solving mathematical problems. In the current situation, most college students in Israel do not know how to program. We believe that the future mathematics teacher should know how to build algorithms and computer programs. The concept of the algorithm is essential both to computer science and mathematics. The Harari Committee (1992) and the NCTM standards (1989) recommended it to be among those topics that should be expanded and emphasized in the mathematics curriculum. Therefore we decided to develop an educational unit about algorithms, in which the emphasis is on writing algorithms for a given mathematics problem, and 'running' them on a computer.

While solving problems in the suggested way, the students are exposed to new mathematical thinking (for them) - **algorithmic thinking**. They are exposed to another facet of mathematics, and different aspect of integrating computers in its teaching.

Mastery in the course, which deals with **discrete algorithmic mathematics,** invites further opportunities of integrating the computer in mathematics classes in all the learning stages, and gives the opportunity to solve complex problems, assisted by the computer, even in more advanced mathematics courses. During our study, this course served us as an introduction to that of Computer Oriented Numerical Mathematics. The fact that the students (176 students during the four years of the experiment) who had not known any programming language, were successful in tackling the numerical problems in the later course, demonstrates that the aims of the introductory course were achieved, and within a relatively short time.

The poster will demonstrate a sample problem, one of several problems that are dealt within the above-mentioned course. The problem is followed by a variety of different solutions that are presented in a gradual way. It will emphasize how the students, while solving the problem, are exposed to several kinds of algorithms - due to different problem solving strategies, and how they learn to perform each of them in the computer, using the spreadsheet.

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A Study on Students' and Teachers' Conception of the Effects of Dynamic Geometry Software

Kyoko Kakihana *, Katsuhiko Shimizu **, Nobuhiko Nohda ***
*Tokyo Kasei Gakuin Tsukuba Women's University,
**National Institute for Educational Research, ***University of Tsukuba

Dynamic geometry software (DGS) has been used in schools for about ten years in Japan. DGS such as Cabri Geometry, Geometric Constructor and etc. are commonly used. Workshops/ seminars for the use of DGS have been held for teachers for its implementation. As a result of these works, DGS started to be disseminated in many classes. On the other hand, the possibility that researchers identified as merits of the use of DGS seems to be realized in a limited extent, in these classrooms [(1)]. The reasons why this limited realization might due to the purposes for what teachers use DGS and the recognition of actual effect of DGS by teachers and students.

Purpose: The purpose of this research is to identify students' and teachers' conception of effects of DGS and to find similarities and differences between these conceptions of effects and research findings.

Method: The opinions/ impressions from 23 reports of Cabri classes were categorized and analyzed: # of students: 164,sentences labeled: 270, # of teachers: 15, sentences labeled: 106

Result

(1) By using DGS, students recognized they could learn and enjoy geometry more actively. There were 44% of students' sentences and 26% of teachers' sentences on affective effects. These affective effects are coincident with the research finding of the effects of using Cabri[(2)].

(2) Some teachers' expectations were recognized also by students. Some teachers expect explanatory works/motivating to proof in DGS as researchers' finding [(3)]. And teachers recognized that students were motivated to make a proof through exploration/discovery.

(3) The problems on situation/ timing when to use DGS, on development of drawing ability, on the task designation for DGS and on teachers' over expectation were pointed by teachers themselves. These obstacles are also pointed out by researchers like Laborde C. [(4)] and others. These obstacles explain the limited realization of merits of DGS mentioned above.

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THE FUNCTION AND THE STUDENTS OF ASETEM/SELETE: A CASE STUDY

Athina Katalifou, University of Athens, School of Philosophy,
E-MAIL: akatal@cti.gr

This study is an exploratory activity (teaching experiment) for the teaching of functions involving a multi-representational software «Function Probe» (Confrey, 1992) in ASETEM/SELETE, a teacher's college.

The aim of this study was the observation of the way students understand functions and especially transformations, using one more tool: the computer (Kynigos, 1995).The main questions of the study concerned: the attitude of the students towards the software, the way they deal with mathematical problems and the impact of technology in the problem solving stragedies they used.

The teaching of transformations, that mainly concerned the parabola $y = x^2$, started with the visual and graphical forms and was extended to data table and algebraic equations. It should be noted that the experiment consisted of four parts:

1. **phase 1**: the students had to match different parabolas starting from the prototype $y = x^2$ and using the transformation icons of Function Probe.
2. **phase 2**: using the graph and the data table of some parabolas the students were asked to try to think out the algebraic equations.
3. **phase 3**: the students studied the relationship between changes in different graphs and changes in coefficients of their algebraic equations.
4. **the task**: it concerned the possibility to use the tool demanding an exploratory behaviour from the students.

The data, and especially the observed talk, were analysed qualitatively (Cohen and Manion, 1997). According to recent research emphasis is placed on the joint construction of knowledge (teacher-learner, learner-learner).Though the time available was insufficient for differentiating the already established concepts of the students, formed through a traditional environment, the results pointed our that the utilisation of Function Probe leads probably to a better understanding and suggests more research on the function concept.

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A GEOMETRIC APPROACH TO BUILD INEQUATION MEANING

Estela Kaufman Fainguelernt — IEM, Santa Úrsula University,Brazil

Franca Cohen Gottlieb — IEM, Santa Úrsula University, Brazil

The data has been collected since 97, students from a public high-school, a low income group working with a teacher-researcher, Alzir Fourny Marinhos, in Rio de Janeiro, Brazil. He is working in his master thesis advised by the authors.

Our main question: What is the role of graphical representation in the process of build inequation meaning?

To answer this question we suppose that through the visualization process students could have a better understanding of the algebraic solution for inequations.

For example, the majority of our students solve inequations following rules:

$x-1 > 0 \qquad -x > 1 \qquad x > -1$

They use a single technique, without connection with the relation order nor with functions. It is a kind of ritual and before, when they used the same technique to solve equations, "it works".

Using a geometrical approach they begin represent intervals, after that they plote functions as, $f(x) = ax + b$.

Using the example above, $f(x) = -x-1$, the function is ploted and students observe when the function is positive and when it is negative.

Comparing the graphic solution with algebraic solution they easily understand that the first solution is not the right one.

Seems to us a certain unhappy circularity, which states that inequation is badly taught because is misunderstood and it is misunderstood because it is badly taught.

Several kinds of mathematical concepts are tied to similar situation. The conceptual field provides a framework for this research, Vergnaud (1991).

> ***"A conceptual field is defined as a set of situations, the mastering of which requires mastery of several concepts of different natures"***

The research mastery historical evolution of relation order, algebraic aproach through daily-life situation, and a geometric aproach.

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Complementary Strategies To Cognitive Analysis:
<u>The Case Of Mathematically Successful Populations</u>

Evgeny Kopelman
Hebrew University of Jerusalem, Israel

It is not a surprise that the less researched school populations in mathematics education are those at more advanced levels of studies and who generally had no problems in passing standard evaluations. This doesn't say that they may not have any difficulties with basic mathematical notions. In fact, they do - as has been shown elsewhere (Vinner & Kopelman, 1998; Kopelman, 1996). While understanding their difficulties is very instructive for mathematics education, it poses a real challenge before a researcher, since standard methods seem to be not easily applied in this case.

First, those who do such research run a risk to fall out of the research tradition, since they can't draw on the known theoretical frameworks and results, which, as a rule, tend to speak about an average student. For a researcher studying mathematically successful populations it is useful to ask instead, whether he or she may imagine a chosen model being applied also to oneself. Secondly, a researcher analyzing responses to questionnaires delivered to these populations or transcripts of conducted interviews will rarely find there flaws in logic, inability to think in abstract terms or to appreciate the idea of the mathematical proof. On the contrary, the explanations will bear the unmistakable qualities of experienced learners of mathematics and the researcher will have to look for the extra-cognitive reasons which had brought the respondents - often unanimously - to their certain line of thinking and, eventually, to wrong answers. Whereas traditional cognitive studies focus on student's cognition grappling with a certain mathematical notion, for this type of research neither of them is certain; the responses of the students often make sense only in the context of the didactic tensions which accompany the teaching of that notion.

The poster presents examples from research which revealed difficulties of mathematically advanced students when they had to apply some notions of school algebra and geometry. The examples are accompanied by historico-didactical deconstruction of the corresponding notions, which may be also used in teacher education.

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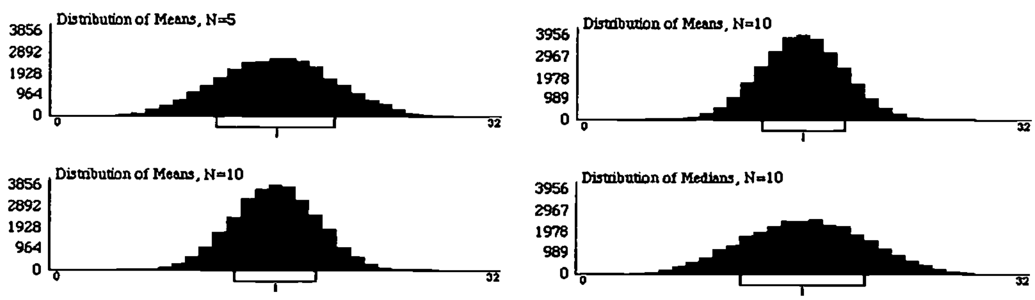
THE RICE VIRTUAL LAB IN STATISTICS: A WEB RESOURCE FOR TEACHING STATISTICS

David M. Lane, Rice University

The Rice Virtual Laboratory in Statistics is an integrated combination of a set of simulations/demonstrations, an electronic textbook, case studies, and a data analysis program. The lab currently contains 16 simulations and demonstrations designed to make abstract concepts concrete and allow students to investigate various aspects of statistical tests and distributions. Topics such as how the choice of bin width affects a histogram, sampling distributions, restricted range, and repeated measures designs are illustrated. Example results of the sampling distribution simulation are shown below. The left portion of the figure illustrates the effect of sample size on the sampling distribution of the mean; the right portion shows how the sampling distribution of the mean differs from the sampling distribution of the median when the sample size is 10 and the parent population has a uniform distribution.



The electronic textbook covers topics typically included in an introductory course in statistics and contains over 2,000 links among related topics. The case studies demonstrate the real-world applicability of statistics and illustrate many methods of descriptive and inferential statistics. They include the raw data so students can perform their own analyses if they wish. The data analysis program can create boxplots, histograms, stem and leaf displays, and scatterplots as well as basic statistical analyses such as correlation, regression, and simple analysis of variance.

The four components of the lab are closely integrated. The textbook has links to simulations that illustrate concepts in the text and the simulations have many links to the textbook. When data from case studies violate an assumption of the inferential test performed, simulations are used to assess the practical effect of these violations. Many of the graphs and analyses presented in the case studies were produced by the statistical analysis program.

The URL is http://www.ruf.rice.edu/~lane/rvls.html. This work was supported in part by the NSF Division of Undergraduate Education grant DUE# 9751307.

REINFORCING TEACHERS UNDERSTANDING OF LIMITING PROCESSES BY CONSIDERING SEQUENCES OF PLANE FIGURES

J. Mamona Downs & Martin Downs
University of Macedonia

In our presentation we will indicate what sequences of plane figures and their limits are. The topic is not commonly familiar and, partly because of this, has not been "institutionalised". This helps in raising the following issues of educational interest:

(1) It is well known that many students have cognitive problems with limits of real sequences (e.g. [1]). It would be of interest to know how these problems may change with limits of figures.

(2) It has been remarked often in the literature (e.g. [2],[3]) how difficult it is for teachers who have assimilated a complicated concept (say, that of the limit of a real sequence) to understand the problems a student has when (s)he first faces it. In teachers training, the introduction of a "parallel" concept (but new to the teachers) may prompt better understandings towards these problems of students in the original concept.

(3) The exercise of forming definitions may provide a way of partially dissipating the "Platonic" bias found in many people towards Mathematics. (Whereas the importance in regarding Mathematics as a product of the human mind is widely acknowledged in research). As with limits of figures we have several different approaches to take for our definition, we have much room in our paradigm for debate and criticism, activities rarely found in pedagogical practices in maths.

With these issues in mind, we designed a fieldwork to use in a pilot study. This involved 16 maths. secondary school teachers with varying teaching experience. The fieldwork consisted of one three - hours session, where at different stages the teachers were asked to examine given tasks concerning limits of sequences of plane figures and to address their thoughts publicly to the whole audience and hence to invite open discussions.

The study revealed many phenomena of diffuse character and its results suggest some particular avenues worthy of further research. The poster presentation will largely consist of diagrams of sequences of plane figures and (when appropriate) their proposed limits; these will be carefully selected to illustrate some of what we feel are our more significant findings.

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ON DYNAMIC SOLUTIONS OF THE QUADRATIC EQUATIONS USING A COMPUTER

Matumoto Yosifumi
Nishinippon Institute of Technology,kandamati,Fukuoka,Japan

A solution of a quadratic equation is obtained by factoring decomposition or the quadratic formula algebraically.The solution is explained as the intersections of the parabola and the x-axis.The purrpose of this communicationis to construct the solutions on the computer display by using parallel lines and circles dynamically.

We consider a quadratic equation with real coefficients

$$\text{(A)} \qquad ax^2 + bx + c = 0 \quad , \quad a \neq 0$$

such that the graph of quadratic function $y = ax^2 + bx + c$ with vertex (r , s) and passing through a point (p , q). Then $b = -2ar$,$c = ar^2 + s$ and $q = a(p - r)^2 + s$.

We regard the display as the orthogonal coodinate plane . The solution points are constructed step by step by using parallel lines and circles .In order to the constructions can be seen dynamically , we present a program with the following properties.

(i) Through the process of constructions, when a figure is constructed in a step,data need for later remain and others are wipe out and go to next step.

(ii) Drawing of parallel lines : Drawing of a parallel line m through the point P which is parallel to the given line L. Starting from L, we construct the family of lines $m(t)$ which is parallel to L such that when a line $m(t)$ is drawn , then next line is drawn and $m(t)$ is wiped out . Thus, we can imagine a parallel line $m(t)$ approch to m from L dynamically.

(iii) Drawing a circle : A dotted point moves continuously and draw a circle.

Following our program ,a student can easily construct the solutions of a quadratic equation on the display.The construction is visual and dynamic, which means that the student can imagine the solutions geometrically.And we believe he also understand that a real solution and complex solution has the same character essentially.

Exposure of "Self Knowledge" in Solid Geometry among Mathematics Teachers Through Reflective Process

Shosh Millet – Achva Academic College
Dorit Patkin –Kibbutzim College of Education

The teacher's "self knowledge" is an element in his or her pedagogic-practical knowledge (Zuzovski 1998, Patkin & Millet 1997, Clandinin 1987). This study exposes the "self knowledge" of mathematics teachers in primary schools regarding solid geometry, through reflection. Children are exposed to solid geometry on various levels, from kindergarten age up. Previous studies have testified to the fact that pupils encounter difficulties - aversion and fear engendered by geometry. A good number of teachers have aversions to solid geometry, as well (Ben-Haim 1987). Therefore, those engaged in teaching the subject must address the problem and try to overcome these difficulties. In this poster we have introduced the reflective process among teachers, including application of this process to the Van-Hiele Theory in solid geometry.

In order to expose the "self Knowledge" of a group of teachers enrolled in the enrichment courses in solid geometry, a two-stage reflective questionnaire and a post-intervention questionnaire regarding application were employed. The intervention plan involved an encounter with the Van-Hiele's theory in plane geometry and adaptation and adoption of this theory to spatial concepts.

Sample: 18 primary school teachers certified to teach math, with at least five years teaching experience, enrolled in the enrichment course

Results indicate that 50% of the teachers evaluated themselves to be at the third level, 6 teachers rated themselves at the final level, and the rest rated themselves at the second level.

The rationale and examples of findings and evaluations expressed by course participants will be presented in the poster.

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MAGIC CIRCLES: AN INVITATION TO EXPERIENCE CAS TECHNOLOGY FROM A DIDACTICAL POINT OF VIEW

Esther Openheim and Nurit Zehavi

Weizmann Institute of Science, Israel

In order to prepare teachers to teach with modern technology, Balacheff (1993) suggested that teachers be provided with opportunities to experience mathematics from a mathematician's point of view, and to experience technology from a didactical point of view. We have provided such opportunities by introducing a relatively new technology, Computer Algebra Systems (CAS). The teachers learned to use the CAS software *Derive* in rich problem-solving situations and explored the didactic strategies that make effective use of this technology. One of the problems that the teachers encountered in the course was *Magic Circles.*

Magic circles involve a rich variety of mathematical structures, concepts and ideas, such as composing linear functions, inverse functions, fixed points, group theory, complex numbers, and vectors. Given a closed circle of 10 linear functions, we begin at the top of the circle and substitute a number of our choice. The outcome is substituted in the next expression and so on, proceeding clockwise, until we complete the circle, and the final output turns out to be the same number with which we started. We posed several problems. For example:

1. If we break the circle at any other function, substitute a number, proceed clockwise and complete the circle at the new point of entry, will we always finish with the same number with which we started?
2. Is it possible to construct *magic circles* such that if we traverse them in either direction, the output is the same as the input?

To answer the first question, "the breaking of the circle", it should be realized that the whole circle reduces to the identity function, function composition is associative, and a linear function has an inverse. Thus,

$$(F_6 \circ F_5 \circ F_4 \circ F_3 \circ F_2 \circ F_1) \circ (F_{10} \circ F_9 \circ F_8 \circ F_7) = I(x) = x$$

In using a CAS, the input requires a particular forced way of viewing things and expressing relationships and the output needs to be interpreted similarly. The poster will include a variety of approaches to a series of problems related to magic circles, demonstrating the expressive power of the technology in integrating mathematical concepts, methods, and visual representations.

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ELEMENTARY CHILDREN AS REAL-WORLD PROBLEM SOLVERS: THE IMPLICATIONS OF GROUP WORK

Andrea Peter-Koop
University of Münster, Germany

The study-in-progress reported in this poster was initiated with the concern that little is known about elementary students' real-world problem solving competencies and strategies as related to the dynamics of group problem solving. Although we are reasonably well aware of the individual capabilities of single students, the complexity of the 'normal' classroom in terms of mixed abilities, and the variety of socio-cultural (out-of-school) experiences of children as well as their quantitative and qualitative participation in classroom interaction have been almost completely neglected (Pehkonen 1991). The objective of the research is to identify the collaborative and individual mathematical problem solving strategies and competencies that groups of third and fourth graders (8- to 10-year-old children) apply with respect to open real-world related tasks.

The following criteria guided the development of the open problems used in this classroom based study: the problems should be 'open-beginning' as well as 'open-ended' real-world tasks that provide 'reference contexts' for elementary students; the wording of the problems should not contain numbers in order to avoid that the children immediately start calculating without first analysing the context of the given situation and in order to challenge the students to engage in estimation and rough calculation and/or the collection of relevant data.

To date, four such problems have been posed in both grade 3 and grade 4 classes which were subsequently divided into working groups of 4-5 children. Each group was videotaped while solving the problem.

The methodological framework of the project is based on an 'Interpretative Classroom Research' approach (Bauersfeld et al. 1988) following a strict analytical procedure for the interpretation of the video data.

As the analysis of the data is not yet completed, this poster will show examples of students' solutions and excerpts from their discussions which illustrate some of their strategies and difficulties, focussing on their estimation and/or rough calculation strategies and the forms of visualisation they developed in order to represent sub-problems/aspects which arose during group discussion.

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What is the relationship between the teaching and learning of early addition in the primary classroom?

Alison J. Price

Oxford Brookes University, OX33 1HX, UK

Background

Young children's development in the learning of addition has been extensively researched and minutely documented (see for example Nunes and Bryant 1996) so that we know that there is a development through 'counting all', 'counting on', 'known facts' and 'derived facts'. However it would seem that not all children progress through these stages, as Gray and others (Gray and Tall 1994) have identified older children still using elementary counting strategies to attempt to solve more complex addition. The effect of teaching on the children's development has been little researched. In England children begin full time schooling at the age of four or five and encounter formal arithmetic very early. What effect does this early teaching have on their understanding?

The Study

This study therefore investigates teaching and learning, in terms of how the teacher presents and represents mathematics to the children and the sense that the children are making of their work. Participant observation carried out in four classes (four teachers and 121 children aged four to six years) during one lesson a week over a period of six months, provided 60 teaching episodes for analysis. The teacher's representation of mathematics was analysed using an adaptation of a model from Lesh, Post and Behr (Lesh, et al. 1987) which identifies the inter-relationships between manipulative materials, real world scripts, pictures, spoken language and symbols. Analysis of the children's responses defines the way in which their previous social, linguistic and mathematical understanding affects their learning. Analysis of this data is still continuing.

The poster

The poster display will further explain the process of data analysis. Details of initial findings from this research will be presented, with emphasis on the role of written recording and the use of symbols in identifying the children's thinking. This will be illustrated with examples of children's work.

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CONSTRUCTING INQUIRY QUESTIONS BY STUDENTS

T. Resnick M.Tabach

Weizmann Institute of Science.

Compu-Math project is a development, implementation and research project for middle school and high-school mathematics. The learning environment includes tasks organized around large-scale problem situations, which can be investigated using various approaches. The students, as investigators, are involved in mathematical thinking processes, such as looking for appropriate strategies, posing hypothesis and accepting or rejecting them, monitoring, asking questions, etc. Students have at their disposal computerized tools such as multi-representational software. They are encouraged to decide which representations to use, when to switch from one to another, when and how to link them, and in which medium to work (e.g., paper and pencil, technological tool, discussion in teams).

One type of activity, *The Problem of the month,* is open-ended, and includes most of the topics that the students have learned recently.

Our poster will illustrate an example of *The Problem of the Month* called *The Animal Park,* in which we present a situation, without any questions. Each part of the situation is presented in a different representation (words, table, graph or algebra). The students are required to construct inquiry questions. We give a collection of words, concepts and expressions, connected with the situation, to help and guide the students. In their responses, most students refer to more then one part of the situation. Hence, they have to deal with more than one representation, and to perform transitions between them. They does these transitions deliberately, and the construction of the questions involves all the mathematical thinking processes that were mentioned above.

Teachers and Computers: Teachers Cultures

Elvira Santos - Escola Básica de Álvaro-Velho - Portugal

This research intends to contribute to a better understanding of teachers cultures, related with the use of computers in the classroom.

The study's framework draws from "teaching cultures are embodied in the work-related beliefs and knowledge, teachers shared - beliefs about appropriate ways of acting on the job and rewarding aspects of teaching, and knowledge that enables teachers to do their work" (Feiman-Nemser and Floden, 1986, p.508). The content of teacher cultures consists "of the substantive attitudes, values, beliefs, habitats, assumptions and ways of doing things that are shared within a particular teacher group, or among the wider teacher community. The content of teacher culture can be seen in the way teachers think, say and do." (Hargreaves,1995,p.166)

In this research we studied how teachers use their knowledge about computers and their practice. The conceptual framework this study is shown in Figure 1.

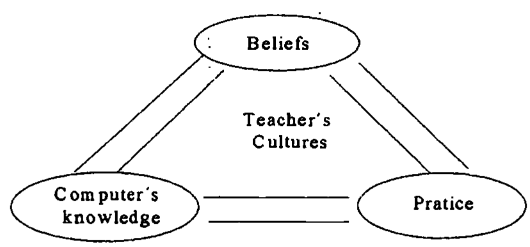

Figure 1

In this poster we will show and compare teacher's cultures of three mathematic teachers: i) one teacher who use the computer in classroom; ii) another that had abandoned the computer's use with pupils; iii) and, finnaly, one teacher who never used it.

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CAN YOU TELL ME WHETHER THIS IS A PROOF?

Annie Selden
Mathematics Department
Tennessee Technological University
Cookeville, TN 38505 U.S.A.

John Selden
Math Ed Resources Co.
P.O. Box 2781
Cookeville, TN 38502 U.S.A.

Eight mid-level university mathematics majors, the entire cohort in a "transition to abstract mathematics" course, were interviewed individually and asked to think-aloud as they determined whether four "proofs" of a theorem, generated by similar students, were proofs. All eight students considered the "proofs" carefully line-by-line, finding some notational and computational anomalies; however, they apparently often did not notice or check structural difficulties, such as the proof of the converse being offered.

First, the students were asked to read the theorem, give examples, and attempt a proof on their own. Then each of the four "proofs" was presented separately. At this stage, the students were only able to judge correctly about half the time, with some saying they were "unsure." Next, students were given all four "proofs" and asked whether they would like to rethink their judgments. As the interviewer continued to probe over the hour to hour-and-a-half interview, they gradually came to make more correct judgments, eventually being correct in about 80% of the instances. At the close of each interview, students were asked several questions about how they read proofs. All said they read proofs very carefully and checked all steps to make sure everything followed logically. However, their actual ability to tell whether the four "proofs" presented in the interview were correct was initially no better than chance.

Since all but one were preservice secondary mathematics teachers, this calls into question their ability to judge the correctness of secondary pupils' proofs. Perhaps validation of proofs, that is, the process of checking them, which is now only a part of the implicit curriculum of most mathematics departments should be made part of the explicit curriculum.

The Statement and One of the Four "Proofs"

For any positive integer n, if n^2 is a multiple of 3, then n is a multiple of 3.

(a). Proof: Assume that n^2 is an odd positive integer that is divisible by 3. That is $n^2 = (3n + 1)^2 = 9n^2 + 6n + 1 = 3n(n + 2) + 1$. Therefore, n^2 is divisible by 3. Assume that n^2 is even and a multiple of 3. That is $n^2 = (3n)^2 = 9n^2 = 3n(3n)$. Therefore, n^2 is a multiple of 3. If we factor $n^2 = 9n^2$, we get $3n(3n)$; which means that n is a multiple of 3.?

ANGLES IN TRIANGLES AND PRALLEL LINES - WHAT ARE THE DIFFICULTIES AND MISCONCEPTIONS?

Behiye Ubuz
Middle East Technical University, Ankara, TR

The concept of angle is one of the more basic ideas in understanding all geometrical concepts and further mathematics such as calculus. This study investigated 10^{th} and 11^{th} grade Turkish students' learning of the concept of angle given in triangles and parallel lines and the difficulties in finding its measure. In addition to that this study also dealt with whether and how the gender issue could influence learning. A diagnostic test including 11 open-ended questions was administered to 67 students from a private college at the end of the second semester of 1998. 34 of these students were 10^{th} grade and the rest were 11^{th} grade students. There were 23 female and 11 male students for the 10^{th} grade, and 11 female and 22 male students for the 11^{th} grade. The test was administered in the usual classroom conditions. The time allowed was about 50 minutes.

Geometry as a separate course is given starting from 10^{th} grade. While geometry I in grade 10 includes Lines, Angles and Triangles, geometry II in grade 11 includes Polygons, Circles, Vectors, and Solids. Time allowed for Geometry I and Geometry II is two and four hours a week respectively.

The analysis of the written responses given to the diagnostic test questions revealed that: (i) the percentage of students answering correctly increased with class grade level (maturity, experience, or both). On the contrary, misconception or errors had the same pattern of overall incidence from one grade level to another; (ii) questions including angles in parallel lines are much easy than questions including angles in triangles for both sexes; (iii) the comparison of "no answer" and "incorrect answer" categories for both gender, for almost every questions, showed some differences. It seems that the female students, because of their initiative, made much more errors and the male students.

Also it was noticed that the errors made were due to: 1) incorrect choice of angles as the base angles of a isosceles triangle; 2) thinking that the base angles of each isosceles triangle should be the same; 3) incorrect application of exterior angle theorem in a triangle: incorrect choice of angles; 4) not knowing the meaning of traversal line cut two parallel lines; 5) assuming something which is not given.

During the presentation the results mentioned above will be shown on the examples of questions.

An in-depth analysis of Japanese elementary school mathematics teachers' manuals: A preliminary report

Tad Watanabe, Towson University, USA

The recent results from the Third International Mathematics and Science Study (TIMSS), specially the videotape study, showed that the Japanese mathematics classrooms are organized very differently from those of the US and Germany. Although the TIMSS videotape study focused on the 8th grade level classrooms, other researchers' reports indicate that these differences are also present in elementary school classrooms.

On the other hand, some researchers have reported that many elementary school teachers, both in Japan and in the United States, rely heavily on their textbooks (for example, Shimahara & Sakai, 1995). Although textbooks have been analyzed previously, most of the time, the focus was to identify the contents of mathematics curricula. However, if teachers are to 'rely' on textbooks, the books they will use is most likely the teachers' manuals.

In this session, I will present the results from a study that is analyzing the contents and organization of Japanese elementary school mathematics teachers' manuals. The questions the study addresses are: What information is provided in the teachers' manuals? How are the teachers' manuals organized? How consistent are the contents and organization of the teachers' manuals with the nature of mathematics teaching in Japan reported in the existing literature.

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LIST OF AUTHORS

LIST OF AUTHORS

Adin, Na'ama 1-312
Weizmann Institute of Science
Dept. of Science Teaching
Rehovot 76100
ISRAEL

Afonso, Maria Candelaria 2-1
Univ. of La Laguna
Dept. of Mathematical Analysis
C/Astrofisico Francisco Sanchez
La Laguna 38257
SPAIN
mcafonso@ull.es

Ainley, Janet 2-9
University of Warwick
Institute of Education
Coventry CV4 7AL
UNITED KINGDOM
janet.ainley@warwick.ac.uk

Albert, Jeanne 1-337
Weizmann Institute
Dept. of Science & Technology
Rehovot 76100
ISRAEL
geanne@netvision.net.il

Alcock, Lara 2-17
Univsersit of Warwick
MERC
Coventry CV4 7AL
UNITED KINGDOM
lja@maths.warwick.ac.uk

Almog, Nava 1-328
Dov-Hoz 24/2
Kfar-Saba 44356
ISRAEL
nava@beitberl.ac.il

Alston, Alice S. 4-169
35 Edgehill St.
Princeton, NJ 08540
USA

Alves de Oliveira, Armando 1-268

Amir, Gilead 2-25
Bosem St. 7, Gilo
Jerusalem 93903
ISRAEL

Amir, Yoni 1-338
Weizmann Institute of Science
Dept. of Science Teaching
Rehovot 76100
ISRAEL

Arcavi, Abraham 1-289
Weizman Inst. of Science
Dept. of Science Teaching
Rehovot 76100
ISRAEL
ntarcavi@wiccmail.weizmann.ac.il

Argyris, Michael 3-177
1st Karistou Street
Athens 16233, GREECE
margiris@cti.gr

Arnon, Ilana 2-33
Center for Educational Technology
Klausner St. 16
P.O.Box 39513
Tel Aviv 61394
ISRAEL
ilana_a@cet.ac.il

Arvold, Bridget 1-261
University of Illinois
388 Education Building
1310 South Sixth Street
Champaign, IL 61820
USA
arvold@uiuc.edu

Asman, Dalia 1-339
Albert Schweizer St. 40
Haifa 34995
ISRAEL
asman@mofet.macam98.ac.il

Ayres, Paul 2-41
School of Teaching & Ed Studies
UWS Nepean
POBox 10
Kingswood, NSW 2747
AUSTRALIA
p.ayres@usw.edu.au

Baba, Takuya 2-49
Hiroshima University
Graduate School for Intern. Developm. and Coop.
1-5-1 Kagamiyama
Higashi-Hiroshima 739-0025
JAPAN
takuba@jpc.hiroshima-u.ac.jp

Baker, Penelope 4-25
University of New England
School of Curriculum Studies
Armidale 2351
AUSTRALIA
pcurrie@metz.une.edu.au

Baldino, Roberto Ribeiro 2-57, 2-65, 2-329
Cx.P. 474
Rio Claro, SP 13500-970
BRAZIL
baldino@linkway.com.br

Barallobres, Gustavo 1-262

Bar-Zohar, H. 1-348

Batanero, Carmen 2-313
Facultad de Educacion
Campus De Cartuja
Granada 18071
SPAIN
batanero@goliat.ugr.es

Baturo, Annette R. 2-73, 2-81
Queensland University of Technology
Victoria Park Road
Kelvin Grove Q 4059
AUSTRALIA
a.baturo@qut.edu.au

Bazzini, Luciana 1-263
Via San Zeno 2
Pavia 27100
ITALY
bazzini@dimat.unipv.it

Becker, Joanne Rossi 2-89
San Jose State University
Dept. of Math. & Computer Science
San Jose, CA 95192-0103
USA
becker@mathcs.sjsu.edu.us

Ben-Yehuda, Miriam 1-284

Ben-Zvi, Dani 2-97
Weizmann Inst. of Science
Dept. of Science Teaching
P.O.Box 26
Rehovot 76100
ISRAEL

Berger, Margot 1-264
University of the Witwatersrand
Dept. of Maths
Private Bag 3
Wits 2050
SOUTH AFRICA
036mab@cosmos.wits.ac.za

Berry, John 2-105

Bershadsky, Irina 1-265
Segel Zutar Dormitory, 4/8
Technion City
Haifa 32000
ISRAEL
irisha@tx.technion.ac.il

Bills, Christopher 2-113
University of Warwick,
Institute of Education,
Coventry CV4 7AL
UNITED KINGDOM
chris.bills@warwick.ac.uk

Binyamin-Paul, Ilana 1-284

Bjuland, Raymond 2-121
Konvallvegen 18
Naerbo 4350, NORWAY
raymond.bjuland@c2i.net

Blanton, Maria L. 1-266
285 Old Westport Rd.
No. Dartmouth, MA 02747-2300
USA
mblanton@umassd.edu

Boaler, Jo 2-129
1177 Middle Avenue
Menlo Park CA 94025
USA
joboaler@standford.edu

Boero, Paolo 2-137, 3-9
Universita Genova
Dept. Matematica
Via Dodecaneso 35
Genova 16146
ITALY
boero@dima.unige.it

Brender, Malka 1-267
Meshek 36
Moshav Zur Moshe 42810
ISRAEL

Brodie, Karin 2-145
Wits University
Po Wits,
2050 Johannesburg
SOUTH AFRICA
022brod@mentor.edcm.wits.ac.za

Brown, Laurinda 2-153
University of Bristol
35 Berkeley Square
Bristol BS8 1JA
UNITED KINGDOM
laurinda.brown@bris.ac.uk

Buzeika, Anne 2-161
Auckland College of Education
Privat Bag 92601
Symonds St.
Auckland,
AUSTRALIA
a.buzeika@ace.ac.nz

Byers, Bill 2-169
7141 Sherbrooke St. West
Montreal H4B IR6
CANADA
wpbyers@vax2.concordia.ca

Cabral, Tania C. Babtista 2-57
UNESP, Faculdade de Ciências
Departamento de Matemática
Av. Engenheiro L. E. Carrijo Coube, s/n
Bauru, SP 17033-360
BRAZIL
tcabral@bauru.unesp.br

Cabrita, Isabel 1-268
Universidade de Aveiro
D.D.T.E.
Aveiro 3810
PORTUGAL
icabrita@dte.ua.pt

Camacho, Matias 2-1
University of La Laguna
Dept. of Mathematical Analysis
c/ Astrofisico Francisco Sánchez
La Laguna 38257,
SPAIN
mcamacho@ull.es.

Campos, Yolanda C. 1-340
Fresno 15
Mexico DF 06400
MEXICO

Carlson, Marilyn P. 1-341
Arizona State University
Mathematics Department
Box 871804
Tempe, AZ 85287-1804
USA
carlson@math.la.asu.edu

Carrera de Souza, Antonio Carlos 2-65
Av 1 – A, 899
Rio Claro 13506-785
BRAZIL
carrera@wconect.com.br

Carrillo, Jose 1-344
University of Huelva
Dept. of Science Education
Avda.Fuerzas Armadas, s/n
Huelva 21007
SPAIN
carrillo@uhu.es

Carroll, Jean 2-177
14 Ebden St.
Brighton, Vic. 3186
AUSTRALIA
carroll@rmit.edu.au

Cha, Insook 1-269
2-1002 Seoul
Apt. Yeidodoney Young Deung Po Gu
Seoul
KOREA
chainsuk@umich.edu

Chang, Ching-Kuch 1-342
National Changhua
Yniversity of Education
6F-3, 13 Taiyi Street
Taipei
TAIWAN ROC
macck@cc.ncue.edu.tw

Chapman, Olive 2-185
University of Calgary
Fac. of Education
2500 Univ. Drive
Calgary, Alberta T2N IN4
CANADA
chapman@ucalgary.ca

Chazan, Daniel 2-193
Michigan State University
Dept. of Teacher Education
116L Erickson Hall
East Lansing, Michigan 48824
USA
dchazan@msu.edu

Chen, Ing-Er 1-343
National Koahsiung Normal University
Department of Mathematics
116 Ho-Ping Road
Kaohsiung
TAIWAN ROC
r2208@ms3.url.com.tw

Chiappini, Giampaolo 3-9
Instituto Per LaMatematica Applicata
National Research Council
Via De Marini, 6
Genova 16149
ITALY
chiappini@ima.ge.enr.it

Christou, Constantinos 2-201, 4-57
University of Cyprus
Dept. of Education
75 Kallipoleos, POBox 20537
Nicosia 1678
CYPRUS
edchrist@ucy.ac.cy

Chronaki, Anna 2-209
9 Cavendish Place
Bath BA1 2UB
UNITED KINGDOM
a.chronaki@open.ac.uk

Cifarelli, Victor 2-217
University of North Carolina
Mathematics Dept.
9201 University City Boulevard
Charlotte, NC 28223-0001
USA
vvcifare@email.uncc.edu

Claes, K. 1-346

Climent, Nuria 1-344
University of Huelva, Dept. of Science Education
Avda. Fuerzas Armadas, S/N
Huelva 21007
SPAIN
climent@uhu.es

Cockburn, Anne D. 1-345
Univ. of East Anglia,
School of Educ. & PD
Norwich NR4 7T3
UNITED KINGDOM
a.cockburn@uea.ac.uk

Cohen Gottlieb, Franca 1-353
Rua Cinco de Julho 323/1001
Rio de Janeiro 22051-030
BRAZIL
ogottlieb@abc.org.br

Coles, Alf 2-153
Kingsfield South
Dept. of Math.
Brook Road
Bristol BS15 4JT
UNITED KINGDOM
alfcoles@compuserve.com

Contreras, Luis C. 1-344
University of Huelva
Science Education
Avda. Fuerzas Armadas S/N
Huelva 21071
SPAIN
lcarlos@uhu.es

Cooper, Tom J. 2-73, 2-81, 3-89
Queensland University of Technology
Victoria Park Rd.
Kelvin Grove Q 4059
AUSTRALIA
tj.cooper@qut.edu.au

Crisan, Cosette 1-270
12 Causewayside
Pen Causeway
Cambridge CB3 9HD
UNITED KINGDOM
crisancd@vax.sbu.ac.uk

Crowley, Lillie 2-225
Lexington Community College
Cooper Drive, 138 Moloney Bldg
Lexington KY 405060235
USA
lillie@pop.uky.edu

Csíkos, Csaba A. 2-233
Attila Jozsef University
Dept. of Education
Petifi sgt. 30-34
Szeged 6722
HUNGARY
csikoscs@edpsy.u-szeged.hu

Czarnocha, Broni 1-95

Dann, Emily 1-324

Day, Chris 1-271
29 Frizingmall Road
Bradford BD9 4LA
UNITED KINGDOM
chris@krys.demon.co.uk

De Bock, Dirk 2-241, 1-346
University of Leuven
Center for Instructional Psychology and Technology
Vesaliusstraat 2
Leuven 3000, Belgium
dirk.debock@avl.kuleuven.ac.be

de Beer, Therine 3-305
MALATI
16 Stradford Village
Old Stellenbosch Road
7130 Sommerset West
SOUTH AFRICA
therine@malati.wcape.school.za

DeBellis, Valerie A. 2-249
East Carolina University
Dept. of Mathematics
129 Austin
Greenville, NC 27858-4353
USA
debellis@mail.ecu.edu

DeMarois, Phil 2-257
William Rainer Harper College
Mathematics Department
Palatine, IL 60067
USA
pdemaroi@harper.cc.il.us

Dickman, Nomy 1-267
7/A Laskov St.
Haifa 34950
ISRAEL
dickman@b.t.l.co.il

Doerr, Helen M. 2-265
Syracuse University
Math. Dept
215 Carnegie Hall
Syracuse, NY 13244
USA
hmdoerr@syr.edu

Donovan II, John E. 1-316
299 Berryman Drive
Snyder, N.Y. 14226
USA
jed3@acsu.buffalo.edu

Douek, Nadia 2-273
Rue Gaston Lauriau, 14 bis
Paris 93100
FRANCE
ndouek@worldnet.fr

Downs, M. 1-356

Dreyfus, Tommy 1-119
Center for Technological Education
Exact Sciences Department
P.O.Box 305
Holon 58102
ISRAEL
nttommy@weizmann.weizmann.ac.il

Drouhard, Jean-Philippe 1-272
IREM
UNSA
Parc Valrose
Nice Cedex 2, 06108
FRANCE
drouhard@unice.fr

Dubinsky, Ed 1-95
Holon Center for Technological Education
POBox 305
58102 Holon
ISRAEL
nttommy@weizmann.weizmann.ac.il

Edwards, Julie-Ann S. 2-281
8, Pentire Avenue,
Southampton SO15 7RS
UNITED KINGDOM
jsel@southampton.ac.uk

Edwards, Laurie D. 1-273
707 Gerard Ct.
Santa Cruz, CA 95062
USA
edwards@cats.ucsc.edu

Ell, Fiona R. 2-289
5 Wynyard Rd.
Mt. Eden
Auckland 1003
NEW ZEALAND
fiona.ell@clear.net.nz

Elliott, Sally J. 1-347
19 Westwood Drive
Inkersall,
Chesterfield S43 3DF
UNITED KINGDOM
selliott@globalnet.co.uk

English, Lyn D. 2-297
Queensland University of Technology
Victoria Park Road
Kelvin Grove, QLD 4059
AUSTRALIA
l.english@qut.edu.au

Escudero, Isabel 2-305
University of Seville
Dept. Didactica Matematicas
Avenida Ciudad Jardin 22
Sevilla 41005
SPAIN

Estepa, Antonio Castro 2-313
University of Jaen
Dto. Didactica de las Ciencias
Lagunillas S/N
Jaen 23071
SPAIN
aestepa@ujaen.es

Even, Ruhama 1-348
Weizmann Institute of Science
Dept. of Science Teaching
Rehovot 76100
ISRAEL
nteven@wiccmail.weizmann.ac.il

Ezer, Hanna 1-274

Fakir Mohammad, Razia 1-275
The University of Oxford
Green College
Woodstock Road
Oxford OX2 6HG
UNITED KINGDOM
razia.fakir-mohammed@edstud.ox.ac.uk

Feilchenfeld, David 2-321
Koreh Hadorot 32
Jerusalem 93393
ISRAEL
davidf@mofet.macam98.il

Ferreira da Silva, José Eduardo 2-329
Rua Sao Joao No. 83, Ap. 102 - Centro
Juiz de Fora - MG CEP: 36010-080
BRAZIL
jefsilva@net.em.com.br

Forgasz, Helen J. 3-217
Latrobe University
Graduate School of Education
Bundoora, VIC 3083
AUSTRALIA
h.forgasz@latrobe.edu.au

Forman, Ellice 1-75
University of Pittsburgh
5C01 Forbes Quad
Pittsburgh, PA 15260
USA
ellice+@pitt.edu

Francis, Nasif 1-287

Frant, Janete Bolite 1-276, 1-308
Rua Almirante Tamandaré 50/502
Flamengo
Rio de Janeiro 22210-060
BRAZIL
janete@unikey.com.br

Friedlander, Alex 2-337
Weizmann Institute of Science
Dept of Teaching
Rehovot 76100
ISRAEL
ntfried@wiccmail.weizmann.ac.il

Fuglestad, Anne Berit 1-277
Agder College, Faculty of Math.
Tordenskjoldsgate 65
Kristiansand 4604
NORWAY
anne.b.fuglestad@hia.no

Furinghetti, Fulvia 2-345
University of Genova
Dept. of Mathematics
Via Dodecaneso, 35
Genova 16146
ITALY
furinghe@dima.unige.it

Gafny, Rina 1-284
Saleit 45885
ISRAEL
rinag@beitberl.ac.il

Gal, Iddo 2-353
University of Haifa
Dept. of Human Services
Haifa 31905
ISRAEL
iddo@research.haifa.ac.il

Gardiner, John 3-1
5, St. Chaos Crescent
Uppermill, Oldham OL3 6MJ
UNITED KINGDOM
jandsgardiner@compuserve.com

Garuti, Rossella 2-137, 3-9
Via del Melograno, 7
Fossoli di Carpi 4101
ITALY
scucarnb@comune.carpi.mo.it

George, Elizabeth Ann 3-17
3709 Chadam Lane # 2 B
Muncie, IN 47304
USA
eageorge@bsuvc.bsu.edu

Gialamas, Vasilis 3-25
Alamanas 10
Agia Paraskevh 153 42
GREECE
egial@atlas.uoa.gr

Gilead, Shoshana 1-197
Kadima 17
Haifa 34382
ISRAEL
gilead@isdn.net.il

Goldin, Gerald A. 2-249
Rutgers UnivERSITY
Serc Bldg. Rm. 239
118 Ergling Huysen Rd.
Piscataway, NJ 08854
USA
gagoldin@dimacs.rutgers.edu

Goodchild, Simon 3-33
College of St. Mark & St. John
Faculty of Teacher Education
Derriford Road
Plymouth PL6 8BH
UNITED KINGDOM
staffsg@lib.marjon.ac.uk

Goroff, Daniel L. 3-41
Havard University, Mathematics
1 Oxford Street, # 325
Cambridge, MA 01238
USA
goroff@math.harvard.edu

Gottlib, Orly 1-338, 1-348

Graven, Mellony 1-278
Radmaste Centre, WITS
University, P.O.Wits 2050
SOUTH AFRICA
163mell@cosmos.wits.ac.z

Gray, Eddie 1-235, 2-113, 3-49, 3-153
University of Warwick
Math. Education Research Centre
Coventry CV4 7AL
UNITED KINGDOM
e.m.gray@warwick.ac.uk

Gwambe, Ribas 1-315
Rua de Kassvende 353-3÷
Maputo
MOZAMBIQUE
ribas@nambu.uem.mz

Hadas, Nurit 3-57
Weizmann Institute of Science
Dept. of Science Teaching
Rehovot 76100
ISRAEL
nthadas@wiccmail.weizmann.ac.il

Halai, Anjum 3-65
Green College
Woodstock Road
Oxford OX2 6HG
UNITED KINGDOM
anjum_halai@hotmail.com

Hall, Carrol A. 1-327
36 Tewson Road
Westfield Perth 6111
Western Australia
AUSTRALIA
chall@faroc.com.au

Hanna, Gila 3-73
University of Toronto
Institute for Studies in Education
252 Bloor Dt. West
Toronto, ON M5S 1V6
CANADA
ghanna@oise.utoronto.ca

Hardy, Michael D. 1-279
Univ.of Southern Mississippi Gulf Coast
730 E. Beach Blvd.
Long Beach, MS 39560
USA
michael.hardy@usm.edu

Harries, Tony 1-349
Bath Spa University College
Menton Park
Newton St. Loe
Bath BA2 9BN
UNITED KINGDOM
t.harries@bathspa.ac.uk

Har-Zvi, Shirley H. 3-81
6, Chativat Golani
Kfar-Saba 44408
ISRAEL

Hazzan, Orit 1-280
Technion
Dept. of Science Education
Haifa 32000
ISRAEL
oritha@tx.technion.ac.il

Healy, Lulu 3-105
University of London
Inst. of Education
20 Bedford Way
London WCIH OAL
UNITED KINGDOM
lhealy@ioe.ac.uk

Hegedus, Stephen J. 3-121
University of Oxford
Dept. of Educational Studies
15 Norham Gardens
Oxford OX2 6Py
UNITED KINGDOM
heg@ermine.ox.ac.uk

Heinz, Karen 4-201
Penn State University
270 Chambers Building
University Park, PA 16802
USA
krh10@psu.edu

Heirdsfield, Ann M. 3-89
Queensland University of Technology
Centre for Math. & Science Education
Victoria Park Rd.
Kelvin Grove
Brisbane 4059
AUSTRALIA
a.heirdsfield@qut.edu.au

Hejny, Milan 1-320
Charles University
Faculty of Education
M.D. Rettigove 4
Praha 1, 11639
CZECH REPUBLIC
milan.hejny@pedf.cuni.cz

Heliophotou, Maria 2-201

Heller, Rachel 1-267
Hashiloah 29
Moreshet 20186
P.O. Misgav
ISRAEL

Hernández Domíngues, Josefa 3-345
Univ. of La Laguna
Dept. of Mathematical Analysis
C/Astrofisico Francisco Sanchez
La Laguna 38257
SPAIN
jhdezd@ull.es

Hershkowitz, Rina 1-9, 3-57
Weizmann Institute
Dept. of Science Teaching
76100 Rehovot
ISRAEL
nthershk@weizmann.weizmann.ac.il

Hillel, Joel 1-119
Concordia University
Mathematics and Statistics
7141 Sherbrooke Street West
Montréal, Québec H4B 1R6
CANADA
jhillel@vax2.concordia.ca

Hirshfeld, Nily 1-348
Weizmann Inst. of Science
Rehovot 76100
ISRAEL
ninily@wis.weizmann.ac.il

Hoffman, Ronit 1-350
7 Raziel St.
Ramat-Gan 52244
ISRAEL
ronithof@mofet.macam98.ac.il

Horin, Nehama 1-284

Hoskonen, Kirsti 3-97
Kumputie 7
Heinaevesi 79700
FINLAND
kirsti.hoskonen@helsinki.fi

Hoyles, Celia 3-105, 3-353
University of London
Math Sciences
20 Bedford Way
London WC1H OAL
UNITED KINGDOM
choyles@ioe.ac.uk

Hudson, Brian 3-1, 1-347
Sheffield Hallan University
School of Education
Collegiate Crescent Campus
Sheffield S1O 2RP
UNITED KINGDOM
b.g.hudson@shu.ac.uk

Huhtala, Sinikka 1-281
Mustikkasuonkatu 6.E
Aanekoski 44120
FINLAND
huhsi@jypoly.fi

Huillet, Danielle J. G. 1-282
C.P. 2065
Maputo
MOZAMBIQUE
dany@zebra.uem.mz

Hungwe, Godwin 1-283
Univ. College of Distance Education
University of ZIMBABWE
POBox MP 167
Mt.Pleasant, Harare
ZIMBABWE
seitt@samara.co.zw

Ilany, Bat-Sheva 1-284, 3-113
Beit Berl, Center for Math Educ.
28 B, Agnon St.
Ra'anana 43380
ISRAEL
b7ilany@beitberl.ac.il

Irons, Calvin J. 3-89
Qld University of Technology
Centre for Maths & Science Education
Victoria Park Road
Kelvin Grove - Brisbane 4059
AUSTRALIA
c.irons@qut.edu.au

Irwin, Kathryn C. 2-161, 2-289
University of Auckland
School of Education
Private Bag 92019
Auckland
NEW ZEALAND
k.irwin@auckland.ac.nz

Iwasaki Hideki 2-49, 4-337
Hiroshima University, Graduate School for International Development & Cooperation
1-5-1 Kagamiyama
Higashi-Hiroshima 739-8529
JAPAN
hiwasak@ipc.hiroshima-u.ac.jp

Jahnke, Hans Niels 3-73
University of Bielefeld
IDM
Postfach 10 0131
Bielefeld 33501
GERMANY
njahnke@post.uni-bielefeld.de

Janssens, Dirk 2-241
University of Leuven
Department of Mathematics
Celestijnenlaan 200B
3000 Leuven
BELGIUM
dirk.janssens@wis.kuleuven.ac.be

Jaworski, Barbara 1-185, 3-121
University of Oxford
Dept. of Educational Studies
15, Norham Gardens
Oxford OX2 6PY
UNITED KINGDOM
barbara.jaworski@educational-studies.ox.ac.uk

Johnson, Peter 2-105

Jones, Keith D. 2-281
Univ. of Southampton,
Research & Graduate School of Education
Southampton SO17 18J
UNITED KINGDOM
dkj@southampton.ac.uk

Jones, Sonia 4-257
University of Wales Swansea
Dept. of Education
Hendrefoelan
Swansea SA2 7NB, Wales
UNITED KINGDOM
sonia.jones@swan.ac.uk

Kahn, Peter E. 1-285
Liverpool Hope University College
Hope Park
Liverpool L16 9JD
UNITED KINGDOM
kahnp@livhope.ac.uk

Kakihana, Kyoko 1-351
Tokyo Kasei Takuin Tsukuba Women's University, Computer Science
3-1 Azuma
Tsukubashi, Ibarakiken 305-0031
JAPAN
kakihana@cs.kasei.ac.jp

Kaldrimidou, Maria 1-286
Univ. of Ioannina
Dept. of Early Childhood Education
Ioannina 45110
GREECE
mkaldrim@cc.uoi.gr

Karaliopoulou, Margaret 3-25
Tsitoyri 5B
Xalndri 15231
GREECE
chronis@ath.forthnet.gr

Katalifou, Athina 1-352
Agias Varbaras 25A, Palaio Faliro
Athens 17563
GREECE
akatal@cti.gr

Kaufman Fainguelernt, Estela 1-353
Rua Frei Leandro 22, Apt. 401
Rio de Janeiro 22470-210
BRAZIL
estelakf@openlink.com.br

Kendal, Margaret Anne 3-129
P.O.Box 303
3340 Bacchus Marsh
AUSTRALIA
kendal@bacchusmarsh.net.au

Kent, Phillip 3-137
Imperial College (Univ. of London)ac.
Mathematics
180 Queen's Gate
London SW7 2BZ
UNITED KINGDOM
p.kent@ic.ac.uk

Keret, Yaffa 3-145
36 Levy Eskol St.
Tel Aviv 69361
ISRAEL
keretcpa@netvision.net.il

Khoury, Yousef 1-287
Alram 110/8
Nazareth 16000
ISRAEL
mathinfo@rannet.com

Kinzel, Margaret 4-201
Penn State University
Curriculum & Instruction
University Park, PA 16802
USA
mtk134@psu.edu

Kiro, Sara 1-289
Weizmann Inst. of Science
Dept. of Science teaching
Rehovot 76100
ISRAEL

Klaoudatos, Nikos 3-25
17 Noembrioy 16
Xolargos 15562
Greece
chronis@ath.forthnet.gr

Klapsinou, Alkistis E. 3-153
73A Craven Street
Coventry CV5 8DT
UNITED KINGDOM
alkistis@fcis1.wie.warwick.ac.uk

Klein, Ronith 1-288
3 Hameiri St.
Tel Aviv 69413
ISRAEL
ronitk@mofet.macam98.ac.il

Koirala, Hari P. 3-161
Eastern Connecticut State Univ.,
83 Windham St.
Willimantic CT 06226
USA
koiralah@ecsu.ctstateu.edu

Kopelman, Evgeny 1-354
Neve Yakov 405/9
Jerusalem 97350
ISRAEL
kopelman@cc.huji.ac.il

Kot, Lilya 1-289
Weizmann Inst. of Science
Dept.of Science Teaching
Rehovot 76100
ISRAEL

Koyama, Masataka 1-290
Hiroshima University
Faculty of Education
1-1-2, Kagamiyama
Higashi-Hiroshima 739-8523
JAPAN
mkoyama@ipc.hiroshima-u.ac.jp

Krainer, Konrad 1-159
University of Klagenfurt
IFF, Sterneckstr. 15
Klagenfurt A-9020
AUSTRIA
konard.krainer@uni-kla.ac.at

Kramarski, Bracha 1-291, 1-333
Bar-Ilan University
School of Education
Ramat-Gan 52900
ISRAEL
kramab@mail.biu.ac.il

Kratzin, Christian 1-292
University of Dortmund
FB Mathematik
Vogelpothsweg 87
Dortmund 44221
GERMANY
christian.kratzin@mathematik.uni-dortmund.de

Krupanandan, Daniel 1-293
1 El Burn
7 Wolseley Road
4001 Durban
SOUTH AFRICA

Kutscher, Bilha 3-169
29 Hashayarot St.
Jerusalem 92544
ISRAEL
bilhak@mofet.macam98.ac.il

Kynigos, Chronis 2-209, 3-177
19 Kleomenous St.
Athens 10675
GREECE
kynigos@cti.gr

Kyriakides, Leonidas 3-185
Institute of Education
Univrsity of Warwick
Coventry CV4 7AL
UNITED KINGDOM
l.kyriakides@warwick.ac.uk

Lagrange, Jean-Baptiste 3-193
5, rue Yves Montant
Saint Gilles 35590
FRANCE
lagrange@univ-rennes1.fr

Lane, David M. 1-355
Rice University
Psychology and Statistics
6100 Main St. MS - 25
Houston TX 77005
USA
lane@rice.edu

Larriva, Cesar 2-193
Michigan State University
Dept. of Teacher Education
116 T Erickson Hall
East Lansing, MI 48824
larriva@msu.edu

Latner, Lea 3-201
Mendel Zinger St. 4
Haifa
ISRAEL
yossil@rafael.co.il

Lawrie, Christine J. 3-209
University of New England
Curriculum Studies
Armidale 2351
AUSTRALIA
clawri2@metz.une.edu.au

Layman, John 1-318
University of Maryland
Physics Dept.
College Park, MD 20742
USA
jl15@umail.umd.edu

Leder, Gilah C. 3-217
La Trobe University
Graduate School of Education
Bundoora, Victoria 3083
AUSTRALIA
g.leder@latrobe.edu.au

Leikin, Roza 1-143, 4-241
Technion
Dept. of Education in Technology & Science
Haifa 32000
ISRAEL
rozal@tx.technion.ac.il

Lemut, Enrica 2-137
Istituto Matematica Applicate del CNR
Via De Marini 6
16149 Genova
ITALY
lemut@image.ge.cnr.it

Lerman, Stephen 1-75
South Bank University
Div. of Education
103 Borough Rd
London SE1 OAA
UNITED KINGDOM
lermans@sbu.ac.uk

Leron, Uri 1-135, 1-317
Technion Israel Inst. of Technology
Dept. of Educ. in Techn. & Science
Technion City
Haifa 32000
ISRAEL
uri@tx.technion.ac.il

Leu, Yuh-Chyn 3-225, 3-233
National Taipei Teachers College
Dept. of Math. Educ.ation
134, Secz, Ho-Ping E. Road
Taipei 106
TAIWAN ROC
leu@tea.ntptc.edu.tw

Levenberg, Leonid 4-49
Center for Educational Technology
Klausner St. 16
P.O.Box 39513
Tel Aviv 61394
ISRAEL

Lin, Fou-Lai 1-343
88 Sec.5, Roosevelt Road
Taipei
TAIWAN ROC
linfl@math.ntnu.edu.tw

Lin, Pi-Jen 3-241
521, Nan-Dah Road
30033 Hsin-Chu City,
TAIWAN ROC
linpj@nhctc.edu.tw

Linchevski, Liora 2-25, 4-161
43 Shmaryahu Levin St.
Jerusalem 96664
ISRAEL
liora@vms.huji.ac.il

Lozinskaia, Raissa 1-294
Mokrushin St., No. 22, apt.82
Tomsk 634034
RUSSIA
lozinsk@mpi.tomsk.su

Magajna, Zlatan 3-249
University of Ljubljana
Faculty of Education
Kardeljeva Pl. 16
Ljubljana 1000
SLOVENIA
zlatan.magajna@uni-lj.si

Malara, Nicolina Antonia 3-257
Univ. of Modena
Dept. of Math.
Via Campi 213/B
Modena 41100
ITALY
malara@unimo.it

Mamona-Downs, Joanna 1-356
University of Macedonia
Dept. of Educational and Social Policy
Egnatia St. 156
Tessaloniki 54006
GREECE
mamona@macedonia.uom.gr

Mann, Giora 4-345
Beit Chanan 76868
ISRAEL
giorama@mofet.macam98.ac.il

Maracci, M. 3-265
University of Pisa
Dept. of Mathematics
Via Buonarroti 2
Pisa 56127
ITALY
maracci@student.dm.unipi.it

Mariotti, Maria A. 3-265
University of Pisa
Dept of Math.
Via Buonarroti, 2
Pisa 56127
ITALY
mariotti@dm.unipi.it

Markopoulos, Christos 3-273
Univ. of Patras
Dept. of Educ.
Patras 26110
GREECE
cmarkopl@upatras.gr

Markovitz, Zvia 1-295, 1-339
Oranim School of Education
Centrer for Math. Education
Tivon 36006
ISRAEL
zviam@mofet.macam98.ac.il

Marku, Larisa 1-267
Hator 5 c/6
Natania
ISRAEL

Matrozos, Dimitrios 3-25
Mhtsakh 35-37
Athens 11141
GREECE
spapast@atlas.uoa.gr

Matumoto, Yosifumi 1-357
Nishi Nippon Institute of Technology
1633 Aratsu Kanda Miyako-gun
Fukuoka T 800-0394
JAPAN
ymatumot@mec.nishitech.ac.jp

Maull, Wendy 2-105

McGowen, Mercedes 3-281
601 Pleasant Place
Streamwood, IL 60107
USA
mmcgowen@harper.cc.il.us

McRobbie, Campbell J. 2-73
Queensland University of Technology
Victoria Park Road
Kelvin Grove Q 4059
AUSTRALIA
c.mc.robbie@qut.edu.au

Mekhmandarov, Ibby 4-49
Center for Educational Technology
Klausner St. 16
P.O.Box 39513
Tel Aviv 61394
ISRAEL
ibby_m@cet.ac.il

Mendonça Domite, Maria do Carmo 3-289
Av. P. Assanguaba, 237
Sao Paulo 04060000
BRAZIL
mcdomite@usp.br

Merenluoto, Kaarina 1-296
University of Turku
Dept. of Education
Lemminkaisenkatu 1
Turku 20520
FINLAND
kaamer@utu.fi

Meron, Ruth 4-49
Center for Educational Technology
Klausner St. 16
P.O.Box 39513
Tel Aviv 61394
ISRAEL
ruti_me@cet.ac.il

Merri, Maryvonne 1-297
B.P. 87
Castanet-Tolosan 31326
FRANCE
maryvonne.merri@educagri.fr

Mesquita, Ana Lobo de 1-298
6 rue d'Angleterre
Lille 59800
FRANCE
ana.mesquita@lille.iufm.fr

Mevarech, Zemira R. 3-81, 1-333
Ministry of Education
Rehov Shivti Israel 34
Jerusalem 91921
ISRAEL
mevarz@ashur.cc.biu.ac.il

Millet, Shosh 1-274, 1-358
Achva Academic College
Math Education
Beit-Elazari 76803
ISRAEL
lpmillet@weizmann.weizmann.ac.il

Mkhize, Duduzile 1-299
University of Witswatersrand
Radmaste Centre
Private Bag 3, Wits 2050
Braamfontein 2050
SOUTH AFRICA
163mkd@cosmos.wits.ac.za

Mochon, Simon 1-300
Centre for Research & Adv. Studies
Dept.of Math. Educ.
Avenida Instituto Politecnico Nacional 2508
Mexico City 07360
MEXICO
smochona@mailer.main.conacyt.mx

Möller, Regina D. 3-297
Univ. of Koblenz/Landau
Abt. Landau, Institute of Math.
Adolf-Kessler-Str. 53A
Landau 76829
GERMANY
rmoeller@uni-landau.de

Monaghan, John D. 2-105
University of Leeds
Centre for Studies in Science & Math. Education
Leeds LS2 9JT
UNITED KINGDOM
j.d.monaghan@education.leeds.ac.uk

Mousley, Judith A. 1-301
Deakin University
Faculty of Education
Geelong Victoria 3217
AUSTRALIA
judym@deakin.edu.au

Movshovitz-Hadar, Nitsa 3-201
Technion, Education Dept.
Haifa 32000
ISRAEL
nitsa@tx.technion.ac.il

Mulligan, Joanne 3-89
MacQuarie University
Scool of Education
North Ryde, NSW 2109
AUSTRALIA
jmull@ted.edu.mq.edu.au

Murray, Hanlie 3-305
University of Stellenbosch
Faculty of Education
Private Bag XI
Matieland 7602
SOUTH AFRICA
georglm@iafrica.com

Musicant, Bracha 3-313
37 P.O. 1045
Kochav-Yair 44864
ISRAEL
orenmusi@inter.net.il

Mutemba, Balbina 1-282

Nakahara, Tadao 1-302
Hiroshima University
Faculty of Education
1-2-2 Kagamiyama
Higashi-Hiroshima 739-8523
JAPAN
nakahar@ipc.hiroshima-u.ac.jp

Nardi, Elena 3-121, 3-321
University of East Anglia
School of Education
Norwich NR4 7TJ
UNITED KINGDOM
e.nardi@uea.ac.uk

Nasser, Lilian 1-303
Rua Leopold Miguez, 99, Apt. 601
Rio de Janeiro 22.060-020
BRAZIL
lnasser@dmm.im.uprj.br

Navarro de Mendicuti, Teresa 1-340
Benemérita Escuale Nac. de Meastros
Bosque De Montezuma 98
La Herradura
Naucalpan 53920
MEXICO

Nemirovsky, Ricardo 4-217
TERC
2067 Massachusetts
Cambridge, MA 02140
USA
ricardo_nemirovsky@pop.terc.edu

Nesher, Perla 1-75, 2-33
University of Haifa
School of Educ.
Mount Carmel
Haifa 31905
ISRAEL
nesherp@construct.haifa.ac.il

Newstead, Karen 3-329
Mathematics Learning and Teaching Initiative
PO Box 647
Beuville 7535
SOUTH AFRICA
newstead@malati.wcape.school.za

Ninomiya, Hiroyuko 1-304
4-5-7 Hachihonmatsu-higashi
Higashi-Hiroshima 739-0142
JAPAN
ninomiy@ipc.hiroshima-u.ac.jp

Nirenburg, Renata 2-33
Center for Educational Technology
Klausner St. 16
P.O.Box 39513
Tel Aviv 61394
ISRAEL
renata_n@cet.ac.il

Nisbet, Steven 3-337
Griffith University
Faculty of Education (CLS)
Mount Gravatt, Q 4111
AUSTRALIA
s.nisbet@mailbox.gu.edu.au

Noda, Aurelia M. 3-345
Univ. of La Laguna
Dept. of Mathematical Analysis
C/Astrofisico Francisco Sanchez
La Laguna 38257
SPAIN
mnoda@ull.es

Nohda, Nobohiko 1-351
Tsukuba University
Institute of Education
Tennoudai 1-1-1
Tsukuba-shi Ibaraki-ken 305
JAPAN
nobnohda@ningen.human.tsukuba.ac.jp

Nortvedt, Guri A. 1-305
Gaustadveien 203
Oslo 0372
NORWAY
g.a.nortvedt@ils.uio.no

Norwood, Karen S. 4-121
North Carolina State University
3108 Rutledge Court
Raleigh, NC 27613
USA
karen@poe.coe.ncsu.edu

Noss, Richard 3-353
University of London,
Mathematical Sciences
20 Bedford Way
London WC1H OAL
UNITED KINGDOM
rnoss@ioe.ac.uk

Nyabanyaba, Thabiso 1-306
University of the Witwatersrand
P.O.Box 257
Wits 2050
SOUTH AFRICA
036thabi@cosmos.wits.ac.za

Ojeda S., Ana-María 4-1
8 Dale Lane
Chilwell, Nottingham NG9 4EA
UNITED KINGDOM
gvzamos@pmnl.maths.nottingham.ac.uk

Olivier, Alwyn 3-305, 3-329, 4-161
University of Stellenbosch
Fac. of Education
Stellenbosch 7600
SOUTH AFRICA
aio@akad.sun.ac.za

Openheim, Esther 1-359
Weizmann Inst. of Science
Dept. of Science Teaching
Rehovot 76100
ISRAEL
ntesther@wis.weizmann.ac.il

O'Reilly, Declan 1-347

Outhred, Lynne 4-185
Macquarie University
School of Education
Sydney NSW 2109
AUSTRALIA
lynne.outhred@mq.edu.au

Owens, Kay 1-220
University of Western Sydney Mcarthur
POBox 555
Campbelltown 2560
AUSTRALIA
k.owens@uws.edu.au

Panizza, Mabel 1-262
Montaneses 1910 No. 15
Buenos Aires 1428
ARGENTINA
mpanizza@mail.retina.ar

Paola, Domingo 2-345
Via Canata 2
Alassio 17021
ITALY

Papastavridis, Stavros 3-25
Mhtsakh 35-37
Athens 11141
GREECE
spapast@atlas.uoa.gr

Parzysz, Bernard 1-212
22, Av. Du General Leclerc
Fontenay-aux-Roses 92260
FRANCE
parzysz@poncelet.univ-metz.fr

Patkin, Dorit 1-274, 1-358
12 Barazani St
Tel-Aviv, Israel
69121
patkin@netvision.net.il

Patronis, Tasos 4-9
University of Patras
Dept. of Math, Patras 26500
GREECE
valdemar@math.upatras.gr

Pawley, Duncan 4-17
11 Soudan St.
Merrylands NSW 2160
AUSTRALIA
dm.pawley@student.unsw.edu.au

Pegg, John 4-25
University of New England
School of Curriculum Studies
Armidale 2351
AUSTRALIA
jpegg@metz.une.edu.au

Pehkonen, Erkki 4-33
University of Helsinki
Dept. Teacher Education
P.O.Box 38 (Ratakatu 6A)
Helsinki 00014
FINLAND
erkki.pehkonen@helsinki.fi

Pehkonen, Leila 4-41
University of Helsinki
Dept. of Education
PL 39 (Bulevardi 18)
University of Helsinki 00014
FINLAND
leila.pehkonen@helsinki.fi

Peled, Irit 4-49
University of Haifa
Faculty of Education
Mount Carmel
Haifa 31905
ISRAEL
ipeled@construct.haifa.ac.il

Pence, Barbara J. 2-89
San Jose State University
Dept. of Math. & Computer Science
San Jose, CA 95192-0103
USA

Peter-Koop, Andrea 1-360
University of Muenster
Institute for Didactics of Mathematics
Einstein Str. 62
Muenster 48149
GERMANY
apeter@math.uni-muenster.de

Philippou, George 2-201, 4-57
University of CYPRUS
Dept. of Education
Box 537
Nicosia 1678
CYPRUS
edphilip@ucy.ac.cy

Pinto, Márcia M. Fusaro 4-65
Rua Trento, 160
Belo Horizonte 31340-460
Minas Gerais
BRAZIL
marcia@mat.ufmg.br

Piteira, Gisélia Correia 1-307
R. Brancanes, No. 11-4 B
Setubal 2900
PORTUGAL
giseliacpiteira@mail.telepac.pt

Pitta, Demetra 3-49
Iouliou Tipaldou 6
Rita 28 Court, Apart. 301
Nicosia 1077
CYPRUS
d.pitta@warwick.ac.uk

Post, Thomas R. 4-297
University of Minnesota
Curr.& Instr. 24O Peik Hall
159 Pillbury Drive S.E.
Minneapolis, MN 55455-0208
USA
postx001@maroon.tc.umn.edu

Potari, Despina 3-273
Univ. of Patras
Dept. of Educ.
Patras 26110
GREECE
potari@upatras.gr

Povey, Hilary 3-1
Sheffield Hallam University
20 Lawson Road
Sheffiels S10 5BW
UNITED KINGDOM
h.povey@shu.ac.uk

Pozzi, Stefano 3-353
University of London, Inst. of Education
Mathematical Sciences
20 Bedford Way
London, WC1H 0AL
UNITED KINGDOM
s.pozzi@ioe.ac.uk

Prabhu, Vrunda 1-95

Praslon, Frédéric 4-73
4, Allee des Camelias
Noisy Le Grand 93160
FRANCE
praslon@univ_mlo.fr

Price, Alison J. 1-361
The Vicarage
Elsfield Road
Marston
Oxford OX3 0PR
UNITED KINGDOM
ajprice@brookes.ac.uk

Pritchard, Lisa 4-81
13, Top Road
Barnacle, CV7 9LE
UNITED KINGDOM

Rabello de Castro, Monica 1-276, 1-308
Rua Barata Ribeiro 253/1101
Copacabana
Rio de Janeiro 22040-000
BRAZIL
rabello@unikey.com.br

Radford, Luis G. 4-89
Universite Laurentienne
School of Education
Ramsey Lake Road
Sudbury, Ontario P3E 2C6
CANADA
lradford@nickel.laurentian.ca

Rahmani, Levi 3-81

Rasslan, Skakre 1-309
Oranim School of Education
Tivon 36006
ISRAEL
shaker@mofet.macam98.ac.il

Reading, Chris 4-97
University of New England
School of Curriculum Studies
Armidale NSW 2351
AUSTRALIA
creading@metz.une.edu.au

Reid, David A. 4-105
Acadia University
School of Education
Wolfville NS B0P 1X0
CANADA
david.reid@acadiau.ca

Reisch, Christopher P. 1-316

Reiss, Kristina 1-310
University of Oldenburg
Dept. of Mathematics
Oldenburg 26111
GERMANY
reiss@mathematik.uni-oldenburg.de

Resnick, Tzippora 1-362
Weizmann Inst. of Science
Science Teaching
Rehovot 76100
ISRAEL
nttzippi@wis.weizmann.ac.il

Riives, Kaarin 1-311
University of Tartu
Fac. of Mathematics
Vanemuise
46-203, Tartu 51015
ESTONIA
riives@math.ut.ee

Robinson, Naomi 1-312, 1-348
Weizmann Institute of Science
Dept. of Science Teaching
Rehovot 76100
ISRAEL

Rojano, Teresa 1-300
Centre for Research & Adv. Studies
Nicolas San Juan 1421
Col.del Valle
Mexico DF 03100
MEXICO
mrojanoa@mailer.main.conacyt.mx

Rommelaere, Rebecca 2-241
University of Leuven
Institute of Psychology
Vesaliusstraat 2
3000 Leuven
BELGIUM

Ron, Gila 1-313
Givat Yoav
Ramat Hagolan 12946
ISRAEL
gilaron@kinneret.co.il

Rososhek, Sam 4-113
Tomsk State University
36 Lenin Avenue
Tomsk 634050
RUSSIA
rososhek@mpi.tomsk.su

Rossouw, Lynn 1-314
Univ. of the Western Cape
School of Science & Math. Education
Private Bag X 17
Bellville 7535 - Cape Town
SOUTH AFRICA
lrossouw@uwc.ac.za

Rowell, Denise W. 4-121
North Carolina State University
Math, Science & Tech Ed
326 Poe, Box 7801
Raleigh, NC 27695
USA
denise@poe.coe.ncsu.edu

Rowland, Tim 4-129
20 Bedford Way
London WCIH OAL
UNITED KINGDOM
t.rowland@ioe..ac.uk

Ruthven, Kenneth 1-56
University of Cambridge
Dept. of Education
17 Trumpington Street
Cambridge CB2 1QA
UNITED KINGDOM
kr18@bermes.cam.ac.uk

Ruwisch, Silke 4-137
University of Giessen
Karl-Gloeckner-Str. 21 C,
35394 Giessen
GERMANY
silke.ruwisch@math.uni-giessen.de

Sackur, Catherine 1-272
1 bis Rue C. de Foucauld
Nice 06100
FRANCE
sackur@unice.fr

Sadovsky, Patricia 4-145
Univ. de Buenos Aires
Facultad de Ciencias Exactas y Naturales
Ciudad Universitaria, Pabellon II Aula 14
Buenos Aires 1428
ARGENTINA
patsadov@mail.retina.ar

Safuanov, Ildar 4-153
Komarova,1, kv.24
Naberezhnye, Chelny-6, 423806
RUSSIA
safuanov@yahoo.com

Sakonidis, Haralambos 1-286
Democritus University of Thrace
Dept. of Primary Education
N. Chili
Alexandroupolis 68100
GREECE
sakonid@edu.duth.gr

San, Luis Weng 1-315
C.P. 72
Maputo
MOZAMBIQUE
lweng@nambu.uem.mz

Sánchez, Victoria 2-305
Universidad de Sevilla
Didactica de la Matematicas
Avenida Ciudad Jardin 22
Sevilla 41005
SPAIN
mvsanche@cica.es

Sánchez-Cobo, Franciso T. 2-313
University of Jaen
Dpto. de Math./Escuale Politecnia Sup.
Avenica de Madrid
Jaen 23071
SPAIN

Sandow, Dara 2-193
308 Michigan Avenue #112
East Lansing, MI 48823
USA
sandowda@msu.edu

Santos, Elvira Lazaro 1-363
Praceta Julio Dinis, No. 11 2. Esq.,
Baixa da Banheira 2835
PORTUGAL
elvirasantos@mail.telepac.pt

Sasman, Marlene C. 4-161
MALATI
2 Third Avenue
7800 Fairways
SOUTH AFRICA
marlene@malati.wcape.school.za

Saxe, Geoffrey 1-25
University of Califormia
Graduate School of Educ.
Berkeley, CA 94720-1670
USA
saxe@socrates.berkeley.edu

Schaeffer, Corinne M. 1-316

Schorr, Roberta Y. 4-169
14 Sweney Court
Neshanic Station, N.J. 08853
USA
schorr@rci.rutgers.edu

Schroeder, Thomas L. 1-316
State University of New York at Buffalo
Dept. of Learning and Instruction
505 Baldy Hall
Buffalo, New York 14260-1000
USA
tls7@buffalo.edu

Sela, Hagit 1-317
Amnon Ve Tamar 3
Haifa, 34791
ISRAEL
shagit@tx.technion.ac.il

Selden, Annie 1-364
1015 Melrose Drive,
Cookeville TN 38501-1554
USA
selden@tntech.edu

Selden, John 1-364
1015 Melrose Drive.
Cookeville TN 38501-1554
USA
selden@tntech.campus.mci.net

Setati, Mamokgethi 4-177
P.O.Box 4977
Halfway House 1685
SOUTH AFRICA
kgethi@icon.co.za

Sfard, Anna 1-75
Eyal St. 45/3
Jerusalem 96955
ISRAEL
sfard@vms.huji.ac.il

Shama, Gilli 1-318
The Israeli Open University, Mathematics,
16 Klauzner St.
POBox 39328
Tel Aviv 61392
ISRAEL
gillish@oumail.openu.ac.il

Shamash, J. 1-348

Shane, Ruth 1-319
Kaye College of Education
POBox 65
Beersheva 84100
ISRAEL
shane@bgumail.bgu.ac.il

Shaw, Pamela F. 4-185
Macquarie University
Statistics
Sydney 2109
AUSTRALIA
pamela.shaw@mq.edu.au

Shefet, Malka 2-25

Shimizu, Katsuhiko 1-351
Nat. Inst. of Educational Research
6-5-22 Shimomeguro
Tokyo 153
JAPAN
shimizu@nier.go.jp

Shmueli, Nurit 3-113
Nahal Shorek 14
Kfar Jona 40300
ISRAEL
nuritshm@beitberel.ac.il

Shmukler, Alla 1-267
Ofakim 8
Nesher 36770
ISRAEL
shmukler@tx.technion.ac.il

Shternberg, Beba 1-197
Center for Educational Technology
Klausner St. 16
P.O.Box 39513
Tel Aviv 61394
ISRAEL
beba_s@cet.ac.il

Siepinska, Anna 1-119
7141 Sherbrooke St. West
Montreal, Quebec H4B1R6
sierp@vax2.concordia.ca

Silveira, Corina 4-193
16 Twyford House
15 Hulse Rd.
Southampton SO15 2PY
UNITED KINGDOM
cfs@soto.ac.uk

Silver, Edward A. 4-241
University of Pittsburgh
729 LRDC
3939 O'Hara Street
Pittsburgh, PA 15260
USA
eas@vms.cis.pitt.edu

Simon, Martin A. 4-201
The Pennsylvania State University
Curriculum & Instruction Faculty
266 Chambers Building
University Park, PA 16802
USA
msimon@psu.edu

Simpson, Adrian 2-17, 4-81
Warwick University
MERC
Institute of Education
Coventry CV4 7AL
UNITED KINGDOM
a.p.simpson@warwick.ac.uk

Skott, Jeppe 4-209
The Royal Danish School of Educational Studies, Dept. of Mathematics
Emdrupvej 115 B
Copenhagen 2400
DENMARK
skott@dlh1.dlh.dk

Smith, Eddie 1-314
University of The Western Cape
Department of Education
Private Bag X17
7535 Belvill
SOUTH AFRICA

Smith, Margaret Schwan 4-201
Penn State University
Curriculum & Instruction
272 Chambers Building
Universiti Park, PA 16082
USA
mss160@psu.edu

Socas, Martin M. 2-1, 3-345
Universidad de la Laguna
Fac. de Matem□ticas
Depto. de Analisis Matem□tico
38271 La Laguna - Tenerife
SPAIN
msocas@ull.es

Solomon, Jesse 4-217
City On A Hill Public Charter High School
City On A Hill
320 Huntington Ave.
Boston, MA 02115
USA
jessesolomon@yahoo.com

Sproule, Stephen 4-225
POBox 8505
Edleen 1625
SOUTH AFRICA
163sls@cosmos.wits.ac.za

Stacey, Kaye 3-129, 4-233
University of Melbourne
Dept. of Science & Math Education
Parkville 3052
AUSTRALIA
k.stacey@edfac.unimelb.edu.au

Stehlíková, Nada 1-320
Charles University
Faculty of Educ.
M.D. Rettigove, 4
Praha 1, 11639
CZECH REPUBLIC
nada.stehlikova@pedf.cuni.cz

Steinberg, Ruti 1-321
Rabina St. 15/1
Tel Aviv 69395
ISRAEL
dms@math.tau.ac.il

Steinbring, Heinz 1-40
Universität Dortmund
Vogelpothsweg 87
44221 Dortmund
GERMANY
heinz.steinbring@math.uni-dortmund.de

Steinle, Vicki 4-233
University of Melbourne
Dept. of Science & Math. Education
Parkvill 3052
AUSTRALIA
v.steinle@edfac.unimelb.edu.au

Stevenson, Ian 3-137
University of London
Dpt. of M.S.C., Inst. of Educ.
20 Bedford Way
London WC1H OAL
UNITED KINGDOM

Stylianou, Despina A. 4-241

Sullivan, Peter 4-249
Australian Catholic University
Faculty of Education
P.O.Box 213
Oakleigh, Vic. 3166
AUSTRALIA
sullivan@christ.acu.edu.au

Sutherland, Rosamund 1-349
University of Bristol
School of Education
Bristol BS8 1JA
UNITED KINGDOM
ros.sutherland@bristol.ac.uk

Tabach, Michal 1-322, 1-362
Weizmann Inst. of Science
Science Teaching
Rehovot 76100
ISRAEL
nttabach@wis.weizmann.ac.il

Taizi, Naomi 1-323
Weizmann Inst. of Science
Dept. of Science teaching
Rehovot 76100
ISRAEL

Tall David 1-3, 1-111, 2-225, 2-257, 3-281, 4-65
21 Laburnum Avenue
Kenilworth CV8 2DR
UNITED KINGDOM
david.tall@btinternet.com

Tanner, Howard 4-257
University of Wales Swansea
Dept. of Education
Hendrefoelan,
Swansea SA2 7NB, Wales
UNITED KINGDOM
h.f.tanner@swan.ac.uk

Tarlow, Lynn 1-324
1860 East 13th Street
Brooklyn, New York 11229-2808
USA
ltarlow@bigfoot.com

Thomas, Noel 1-325
Charles Sturt University
School of Teacher Education
Bathurst 2795
AUSTRALIA
nthomas@csu.edu.au

Tinoco, Lucia 1-303

Tirosh, Chaim 1-326
13 Barth Street
Tel Aviv 69104
ISRAEL
chaim@mofet.macam98.ac.il

Tirosh, Dina 1-288, 4-265
Tel Aviv University
School of Education
Tel Aviv 69978
ISRAEL
dina@post.tau.ac.il

Tomazos, Dianne 1-327
58 Aldridge Road
Booragoon 6154
Western Australia
AUSTRALIA
tomazos@central.murdoch.edu.au

Trgalová, Jana 1-119
Concordia University
Maths & Stats Dept.
7141 Sherbrooke St. West
Montreal H4B1R6
CANADA
Trgal@vax2.concordia.ca

Trigueros, María 4-273
Eugenia 26
Mexico, D.F. 03810
MEXICO
trigue@gaoss.rhon.itam.mx

Truran, John M. 4-281, 4-289
PO Box 157
Goodwood, S. A. 5034
AUSTRALIA
jtruran@arts.adelaide.edu.au

Truran, Kathleen M. 4-281, 4-289
POBox 157
Goodwood, S.A. 5034
AUSTRALIA
kath.truran@unisa.edu.au

Tsai, Wen-Huan 3-241, 4-297
National Hsin-Chu Teachers College
Dept. of Math & Science Education
521, Nan-Dah Road
Hsin-Chu City 30033
TAIWAN ROC
tsai@nhctc.edu.tw

Tsamir, Pessia 1-328, 4-305
12, Tyrza St.
Ramat-Gan 52364
ISRAEL
pessia@post.tau.ac.il

Tzekaki, M. 1-286
Aristotle Univ. of Thessaloniki
Dept. of Early Childhood Educ.
Thessaloniki 54006
GREECE
tzekaki@nured.auth.gr

Tzur, Ron 1-169, 4-201
Penn State University
265 Chambers Building
Univ. Park PA 16802
USA
rxt9@psu.edu

Ubuz, Behiye 1-365
Middle East Technical University
Science Education / Education
Ankara 06531
TURKEY
behiye@tutor.fedu.metu.edu.tr

Ulitsin, Alexander 4-49
I.C.C. Ministry of Education
Theme of Mathematics
Jerusalem, 91911
ISRAEL
alex_u@netvision.net.il

Ursini, Sonia 4-273
Av. Universidad 2042 Ed. 9-907
Mexico D.F. 04360
MEXICO
sursinil@mailer.main.conacyt.mx

Vannier, Marie-Paule 1-297

Vaulamo, Jaana 4-33

Verschaffel, Lieven 2-241, 1-346
University of Leuven
Institute of Psychology
Vesaliusstraat 2
3000 Leuven
BELGIUM
lieven.verschaffel@ped.kuleuven.ac.be

Vidakovic, Draga 1-95
Georgia State University
Math & Comp. Sci
University Plaza
Atlanta 30303-3083
USA
draga@cs.gsu.edu

Warren, Elizabeth 3-337, 4-249, 4-313
Australian Catholic Univ.
Mcauley Campus
POBox 247
Everton Park, Qld 4053
AUSTRALIA
e.warren@mcauley.acu.edu.au

Watanabe, Tad 1-366
Towson University
Math. Dept.
8000 York Rd.
Towson MD 21252
USA
tad@towson.edu

Way, Jenni 2-41
University of Cambridge
17 Trumpington Street
Cambridge CB4 1QA
UNITED KINGDOM
jaw36@com.ac.uk

White, Paul 4-249
Australian Catholic University
Faculty of Education
179, Albert Rd.
Strathfield 2135
AUSTRALIA
p.white@msm.acu.edu.au

Wiliam, Dylan 4-321
King's College of London
School of Education
Cornwall House, Waterloo Road
London SE1 8WA
UNITED KINGDOM
dylan.wiliam@kcl.ac.uk

Wilson, Melvin (Skip) 1-269
Virginia Polytechnic Inst & State University
Dept. of Teaching and Learning
303 War Memorial Hall
Blacksburg, VA 24061-9313
USA
skipw@vt.edu

Winbourne, Peter 1-329
South Bank University
103 Borough Road
London SE1 0AA
UNITED KINGDOM
winboupc@sbu.ac.uk

Winsløw, Carl 4-329
Royal Danish School of Educational Studies
Math. Dept.
Emdrupuej 115 B
Copenhagen NV 2400
DENMARK
cawi@dlh.dk

Wu, Chao-Jung 3-233
National Tapei Teacher College
Dept. of Educ. Psychology & Counseling
134, sec. 2 Ho-Ping E. Road
Taipei 106
TAIWAN ROC
cog@tea.ntptc.edu.tw

Wu, Yuh-Yin 3-233
National Taipei Teachers College
Dept. of Education
134, Sec. 2, Ho-Ping E. Road
Taipei 106
TAIWAN R.O.C.
yuhyin@tea.ntptc.edu.tw

Yamaguchi, Takeshi 4-337
Fukuoka Univ. of Education
Dept. of Math. Education
729-1, Akama
Munakata 811-4192
JAPAN
ytakeshi@fukuoka-edu.ac.jp

Yamaguti, Kiyosi 1-330
Faculty of Engineering
Kyushu Sangyo University
Fukuoka 813-8503
JAPAN
ykiyoshi@ip.kyusan-u.ac.jp

Yerushalmy, Michal 1-197
Haifa University
Faculty of Education
Mount Carmel
Haifa 31905
ISRAEL
michalyr@construct.haifa.ac.il

Zamir, Smadar 1-331
Maimon St. 28
Haifa 32585
ISRAEL
zmau@netvision.net.il

Zangor, Roxana 2-265
Syracuse University
Dept. of Mathematics
215 Carnegie Hall
Syracuse, NY 13244
USA
rzangor@syr.edu

Zaslavsky, Orit 1-143, 1-265, 1-267, 1-313, 1-317, 1-331
Technion
Dept. of Science
Education
Haifa 32000
ISRAEL
orit@techunix.technion.ac.il

Zazkis, Rina 1-273, 1-332
Simon Fraser University
Faculty of Education
Burnaby, B.C. V5A 1S6
CANADA
zazkis@sfu.ca

Zehavi, Nurit 4-345, 1-359
Weizmann Institute of Science
Dept. of Science Teaching
Rehovot 76100
ISRAEL
ntzehavi@weizmann.weizmann.ac.il

Zeichner Orit 1-333
Bar Ilan University
School of Education
Ramat-Gan 52900
ISRAEL
orit_z@usa.net

Zevenbergen, Robyn 4-353
Griffith University
Fac. of Education
PMB 50, Gold Coast Mail Centre
Bundall, Qld. 9726
AUSTRALIA
r.zevenbergen@mailbox.gu.edu.au

Ziskin, Clara 1-267
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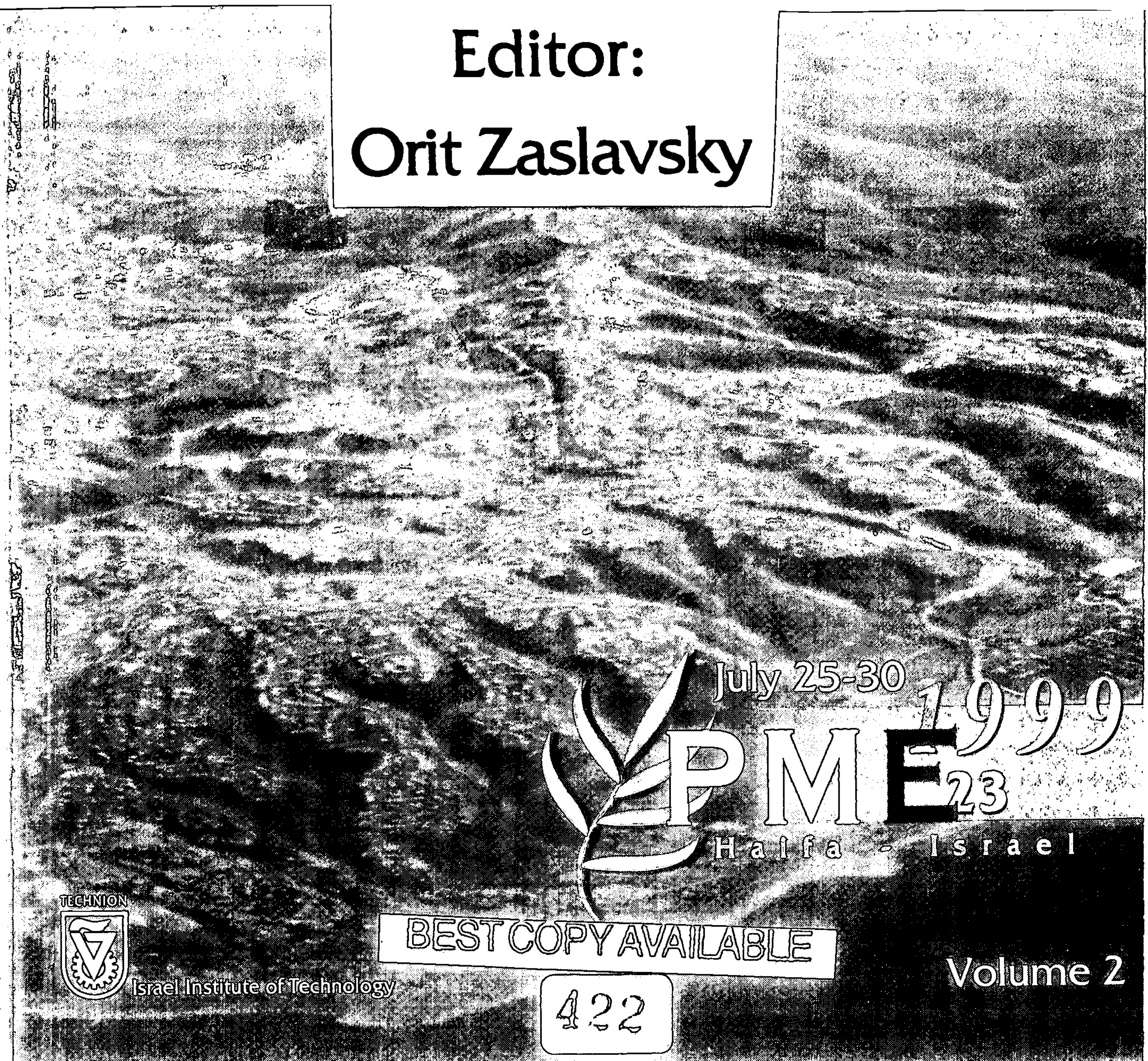

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Editor:
Orit Zaslavsky
Department of Education in Technology and Science
Technion - Israel Institute of Technology
Haifa 32000
Israel

Fax: +972 4 832 5445
E-mail: orit@tx.technion.ac.il

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Research Reports

RESEARCH REPORTS

TEACHER PROFILE IN THE GEOMETRY CURRICULUM BASED ON THE VAN HIELE THEORY

Afonso,C; Camacho, M. and Socas, M.M.
University of La Laguna. Tenerife. Spain

Abstract

In this paper we present a study of six in-service teachers. Using various instruments such as diagnostic tests to assess the level of the teachers' geometrical reasoning, structured interviews, learning unit notes and class session video recordings, we study the teachers' experiences and behaviour regarding the teaching/learning of geometry and we analyze whether the profile of the teachers in question is or is not in keeping with the profile of the ideal type of teacher who, according to current Educational Reforms (MEC 1989), is presumed to be prepared for successfully teaching the present innovative Matehematics curriculum which is based on an interpretation of the Geometry curriculum set forth in the Van Hiele model of geometrical reasoning. We conclude that, in order to implement these curricular innovations with some measure of security, it is necessary to set up in advance comprehensive teacher training programmes. These programmes should not be an isolated part of the curriculum nor a series of recipes about how to put Van Hiele programmes into effect, but rather an interpretation, justification and orientation arising from the teachers' practice itself.

Introduction

The role played by Geometry in the compulsory education curriculum has been under discussion throughout the school Mathematics community for the past few decades. Its importance within the curriculum has been clearly reflected in the various documents which at an international level outline the path to be followed in Secondary School Mathematics teaching (NCTM 1991); NRC, 1989). Among these discussions (Freudenthal, 1973; Clements and Battista, 1992) the van Hiele model of Geometric reasoning constitudes a theoretical framework that enables us to design and restructure the Geometry curriculum in compulsory education (Geddes, 1992; Geddes and Fortunato, 1993; Burger and Culpepper, 1993).

This frame of reference means our accepting a major curricular change in Geometry that has multiple effects, especially with regard to the subject (Geometry), the students and the teachers.

To date, research on Geometry from the van Hiele perspective has focused more on the structure and organization of contents and on a better understanding of the students' knowledge and behaviour (Clements and Battista, 1992) than on the those problems the teacher faces when putting this curriculum into practice.

According to followers of the Van Hiele theory, mathematical thought follows a concrete model made up of two parts. Firstly, a descriptive part whereby a sequence of types of reasoning ("levels of reasoning") can be identified; an individual's mathematical reasoning progresses through these various levels from the moment s/he begins learning until s/he reaches the maximum degree of intellectual development in

this field. Secondly, an instructive part whereby teachers are given guidance about how they can help students most easily attain a higher level of reasoning as they pass through "learning phases".

As is well known, the Van Hiele model (Fuys, Geddes and Tischler, 1984) is made up of five levels of reasoning –recognition (visualization), analysis, classification (informal deduction), formal deduction and rigour; and five learning phases – information, directed orientation, explication, free orientation and integration.

The aims of this empirical, descriptive study are as follows.

a) To compare the situation of a group of in-service teachers regarding their experiences and behaviour in Geometry teaching/learning situations.

b) To analyze whether the profile of the teachers in question is or is not in keeping with the profile of the type of teacher who, according to current Educational Reforms (MEC 1989), is presumed to be ready to teach successfully the present innovative Mathematics curriculum based on an interpretation of the Geometry curriculum set forth in the Van Hiele geometrical reasoning model.

Research Context and Methodology

The present study was carried out in six public schools in Tenerife during the academic year 1996-97. Six in-service teachers took part, all of them with more than 10 years of experience. These teachers taught Mathematics at Year 7 level (11-12 years old) and their places of work were distributed as follows: Teachers B and D (urban area), Teachers A and C (suburban area), and Teachers E and F (rural area).

An essentially qualitative methodology was used. Test instruments enabling us to determine the teacher's level of reasoning (Usiskin, 1982; Jaime, 1993) and structured interviews with closed protocols were employed together with instruments that allow us to undertake studies through a purely interpretive analysis of video-taped classroom sessions and an examination of the learning unit notes used by the teachers. We have considered that it is the suitable methodology to use because the research problem has to do with aspects related to teacher´s thinking. That presupposes that we have to obtain data, not only from the teacher´s geometrical knowledge but also from his oppinions and decissions in a giventeaching situation

Instruments

The data collection instruments used were:

1.- Structured interviews with closed protocols and open-answer questions. These enable us to obtain information about individual differences (D.I), institutional limitations (L.I), the nature of the task the teachers set their Geometry students (N.T.), their opinions of their students (J.P.E), the geometrical contents (J.P.C.) and the kind of decisions the teachers must make regarding teaching and learning (D.D.), as adapted from the model proposed by Shavelson and Stern (1981).

In the teaching decisions category (D.D), Geometry teachers' didactic practices are analyzed and are then related to their classroom styles. In order to identify the various styles of the teachers under study, we adapt the terminology used by Porlán (1993): traditional style (excessive concern with contents in their formal logical

aspect); technological style (excessive concern with operative and behavioural objectives); spontaneous style (excessive concern with the activities carried out by the estudents), and investigative style (concern to integrate desirable scholarly knowledge and the students' knowledge and interests). However, in this study we fuse together the "spontaneous" and "investigative" trends and call this amalgam simply "investigative", this being the term used in current Educational Reform programmes in the Mathematics field, as we do not possess enough elements to differentiate the amalgamated terms with the instruments we use.

2.- Tests to assess teachers' geometrical thought (Usiskin, 1982 –TU and Jaime, 1993-TJ). Although there are other diagnostic instruments to assess reasoning levels (Mayberry, 1981; Crowley, 1987); which have been used with more or less success in various research projects, we opted to choose both these tests for two reasons:

a) To determine the degree of acquisition attained by the teachers, in accordance with the continuity hypothesis set forward in the Van Hiele Theory.

b) To compare the information provided by two different diagnostic instruments deriving from contradictory theories (the discrete and the continuous) and thereby contrast results.

3.- Class notes required from the teacher before s/he performs in class. Taking into account these notes, we are able to study the type of decisions the teacher has to take during the class. The structure of the notes enables us to examine the type of organization the teachers make beforehand. We can identify two different tendencies towards either conceptual or curricular organization. By conceptual organization we mean the type of organization that treats contents as a basically instructive element which are then organized from the point of view of the internal logic of geometry. By curricular organization we mean the type of organization that treats contents as a basically educational element and which are then organized from a curricular point of view. In other words, contents are considered epistemologically and phenomenologically as an educational tool in order to attain skills that also require a pedagogic and didactic organization (Methodology) and an organization of the assessment process designed to measure the skills acquired.

4.- Video recordings of two one-hour sessions taken by the teachers and the observations made by an external observer in the classroom. The classroom study made by the external observer and the analysis of the transcriptions of the video recordings are carried out using the observation guidelines adopted by Walker (1984). In the present work we consider the following categories: students (groupings, motivation and participation in tasks), teachers (suitable mathematical vocabulary, teachers' answers to students's questions and class work distribution), resources (textbooks, written materials, graphic materials, manual materials and other resources) and performance of the learning unit (what is taught –concepts, procedures or attitudes; how the task is organized and what the role of the teacher is when the task is carried out).

Results analysis.

We should point out that it has been difficult to find teachers prepared to carry

out educational experiments which imply the implementation of innovations as proposed by Van Hiele. So we cannot claim that selection of the 6 teachers taking part in our research was random.

Based on the data shown in Table 1 we can see that the teachers use different teaching styles in the classroom: Teachers A, C and F usually employ an investigative style rather than the traditional, while Teachers B, D and E use the traditional style rather than the investigative. Except for Teachers B and E, the technological style is hardly ever used.

Teacher	A	B	C	D	E	F
D.I.: Individual differences: -Coordination wih other teachers -Importance of Geometry -Why?	.No .Very important .Formative; understanding of the real world	.No .Very important .Of use as a focus of interest	.No .Important .Develops the reasoning ability	.No .Important . Gives specific ideas about space	.Yes .Important .Usefulness	. Yes .Very important .Applications in life
N.T.: Task nature. They use : a) Textbooks b) Graphic materials c) Manipulatives	. Yes . Yes . Yes	. Yes . Yes . Yes	. Yes . Yes . Yes	. Yes . Yes . Yes	. Yes . Yes . Yes	. Yes . No . Yes
J.P.E.: Teachers' opinion of students: a) Students like Geometry? b) Why?	. Yes . Intuition and construction	. Yes .Construction	. Yes . As long as it's dynamic	Indifferent . Depends on how it's taught	. Yes .Sometimes	. Yes .No opinion
J.P.C.: Teachers' opinion of contents of Geometry: 1.-Practice. a) Much informal work b) Promoting spatial development c) Much deductive work d) Many activities and open questions e) Many manual activities 2.- Role in Mathematics f) Most important part of Mathematics g)Shows level of mathematical understanding.	. Yes . Yes . Yes . Yes . Yes .No opinion . No	. Yes . Yes . Yes . Yes . Yes . Yes . Yes	. No . Yes . No . Yes . Yes .No opinion . Yes	.No opinion .Yes .Yes .Yes .Yes .Yes .No opinion	. No .Yes .No opinion .Yes .Yes . No .No opinion	. No . Yes . Yes . Yes . Yes . No . Yes
D.D.: Didactic decisions about teaching: a) Investigative style b) Traditional style c) Technological style d) Groupwork.	.Yes . No . No . Sometimes	. No .Yes .Yes . Sometimes	.Yes .No .No .Almost always	. No .Yes . No .Almost always	. No . Yes . Yes .Almost always	. Yes . No. . Yes . Hardly ever

Table 1 *Structured Interview*

As was to be expected, real difficulties arise when subjects attempt to solve a test

such as that proposed by Jaime (1993), as this is quite hard and implies a great deal of work on the part of the respondent. In our study, one of the 6 teachers did not answer the last five questions in the Jaime test; so, with only the results from the Usiskin test to go on, this teacher (Teacher C) is placed, not very reliably, at Level 1, which is not very reliable. With regard to the remaining teachers, two of them (Teachers A and D) are placed at Levels 3 and 4, their results from both tests coinciding for the most part, while the other three teachers are at Levels 2 and 3 (Teachers B, E and F). The results of the tests are shown in Table 2. In the first column (TU) we give the number of correct answers/total number of questions for each teacher, and in the second column the degree of acquisition for each level according to the corresponding percentage (Complete acquisition = C, High acquisition = H, I = Intermediate acquisition, Low acquisition = L, No acquisition = N). To interpret the degrees of acquisition, see Gutiérrez and Jaime (1995).

Teachers	A		B		C		D		E		F	
	TU	TJ	TU	TJ	TU	TJ	TU	TJ	TU	TJ	TU	TJ
LEVEL 1	4/5	(C) 100	3/5	(C) 100	3/5	(C) 100	4/5	(C) 100	4/5	(C) 100	3/5	(H) 75
LEVEL 2	3/5	(C) 90'6	3/5	(H) 78	2/5	-	3/5	(C) 91'9	3/5	(C) 88	4/5	(A) 78
LEVEL 3	5/5	(C) 96'6	1/5	(I) 46'7	1/5	-	3/5	(C) 95'8	2/5	(H) 75'8	2/5	(I) 47'5
LEVEL 4	3/5	(H) 81'3	0/5	(N) 0	0/5	-	2/5	(H) 75	2/5	(L) 66'5	1/5	(N) 0
LEVEL 5	4/5	-	0/5	-	0/5	-	3/5	-	1/5	-	1/5	-

Table 2. *Geometric Reasoning Tests*

The Geometry taught in the Mathematics curriculum for students aged between 9 and 13 years involves levels of attainment of between 1 to 3 on the Van Hiele geometrical reasoning scale. Accordingly, as the subjects of our research teach the final two years of Primary and the first two years of Secondary education, we can see that only Subjects A and D attain the geometric reasoning levels deemed appropriate for carrying out the tasks at these stages of education.

The classroom notes handed in by the teachers (see Table 3) show a greater tendency towards conceptual organization (Teachers C, D, E and F) than curricular organization.

Based on analysis of the video- recorded sessions, groupings for undertaking tasks confirm the tendency towards group work in the case of Teachers B and C, and the tendency towards individual work in the case of Teachers A, D, E and F, in other words, most of the teachers involved in our research. However, the tendency expressed in the interview was group work (Table 1). Also, there is a greater tendency towards individual work (A, D, E and F) than group work (B and C).

Generally speaking, the teachers are quite rigid and exercise excessive control when it comes to class dynamics; nonetheless, they adopt more flexible postures when communicating course contents, suitable vocabulary and correct answers to students' questions and doubts. Although there is a tendency in teachers' answers towards an homogenized treatment rather than individualized treatment (Teacher F), the other teachers tend to answer the whole group (A, B and C) or establish some balance

between individual and group answers (D and E).

Teachers A and B organize classroom tasks from an investigative focus when carrying out the learning unit, and these teachers also play the role of guides in this area. The other teachers (C, D, E and F) set tasks in a routine fashion, playing the role of transmitters of knowledge. In these cases, although to a lesser degree in the case of Teachers A and B, organization is fundamentally curricular, as we have noted before.

Worthy of note is the way in which Teachers A, D and E confirm the expectations they expressed in their respective interviews, while the other teachers (B, C and F) fail to live up to these expectations.

Teacher	A	B	C	D	E	F
CLASS NOTES	Curricular organization	Curricular organization	Conceptual organization	Conceptual organization	Conceptual organization	Conceptual organization
VIDEO-RECORDING: Students a) Groupings b) Motivation c) Participation in the task	 Individual High High	 Group Low Low	 Group Medium Low	 Individual High Medium	 Individual High Medium	 Individual High Medium
Teachers a) Suitable vocabulary b) Teacher's answers (*) c) Student distribution	 Yes Group, E,G,M Individual	 Yes Group, E,G,M,W Group	 Yes Group, E,G,M Group	 Yes Group/Ind. E,G Individual	 Yes Group/Ind. E,G,M Individual	 Yes Individual, E, G,M Individual
Resources: a)Textbooks b) Materials: graphic, manual, written c) Others resources	 No Yes,Yes,No -	 No Yes,Yes, Yes -	 No Yes, Yes, No -	 No Yes, No, No -	 No Yes, Yes, No -	 No Yes, Yes, No -
Learning unit a) What is taught (**) b) Task organization c) Task role	 .C, P. Invest. .Guide	 .C, P Invest. .Guide	 .C, P .Routine .Transmitter	 .C, P, A .Routine .Transmitter.	 .C,P. .Routine Transmitter	 .C, P .Routine .Transmitter

Table 3. *Class notes and Video Recordings*

(*) Video recordings: Teacher answers: By means of examples (E); Using graphics (G);Using materials (M); Using written materials (W)

(**) Learning unit: Concepts (C); Procedures (P); Attitudes (A)

Conclusions

The reforms in the Spanish education system (MEC, 1989) involve major changes in teacher training and imply significant direct effects on teachers' classroom work. Such is the scope of these reforms that curricular proposals made on the basis of Van Hiele theories require a teaching community possessing certain skills and attitudes (teacher profile) that might lead to major changes in teachers'epistemological outlook, which can be summarized as follows:

1. Scientific training in Geometry to at least one level higher in geometric reasoning than the level teachers will work on with their students.
2. The concept of learning in terms of guided investigation.
3. Ability to work with sets of student with very different basic skills, interests and necessities as far as Geometry is concerned.

4. The idea of the Geometry curriculum as an educational tool that enables students to attain the various levels of geometric reasoning.
5. Positive valorization and use of group work.

The present research work forms part of a broader project we are undertaking in conjunction with in-service teachers (Afonso, Camacho and Socas, 1995 and 1997), which is designed to find out whether there is any relationship between the styles of teachers working in our education system and the profile of the ideal teacher capable of carrying through the curricular proposals based on Van Hiele's theories. Our aim is to establish the means for teachers to change their attitudes and help understand better the dynamics of the processes involved in curricular changes of this nature.

On the basis of the various instruments used in our research work, we have interpreted teacher profiles in the terms set out in Table 4. The data are taken from geometric reasoning tests (Table 2) and classroom notes and video recordings (Table 3). In Table 4 we do not refer to the data obtained in the structuredinterviews because of the contradictions apparent in the data obtained from the various instruments.

We can see that Teacher A would have the ideal profile for carrying out a curriculum designed from the Van Hiele persepective if we could aid him to foster group work and repect class dirrerences, though his teaching is effective and highly individualized.

Teacher profile	A	B	C	D	E	F
1. Scientific training (Table 2)	3-4	2-3	(TU) 1	3-4	2-3	2-3
2.Guided research (Table 3, items 4 and 5)	Yes	Yes	no	no	no	no
3.Respect the heterogeneity of the class (Table 3, item 3)	no	no	no	Yes-no	Yes-no	Yes
4. Organization of Geometry from a curricula (Table 3, item 1)	Yes	Yes	no	no	no	no
5. Team group (Table 3, item 3)	no	Yes	Yes	no	no	no

Table 4. *Teacher profile*

In spite of having a suitable level of geometric reasoning (3-4), Teacher D sets about classroom activities in a routine way, thus becoming no more than a transmitter of knowledge. Also, the teacher puts little value on group work, over-standardizes his classes and manifests an excessively conceptual idea of the curriculum. Such a teacher would appear to have difficulties in carrying out curricular proposals in the terms set out by Van Hiele.

The results we have obtained for the rest of the teachers lead us to believe that these teachers' epistemology might be a major obstacle when it comes to implementing a Geometry curriculum based on Van Hiele's theories. There are various imbalances between the five categories that go to make up the ideal teacher profile.

In order to implement these curricular innovations with some measure of security, it is necessary to set up in advance comprehensive teacher training programmes. These programmes should not be an isolated part of the curriculum nor a series of recipes about how to carry out Van Hiele's idea, but rather an interpretation, justification and orientation arising from practice itself (immersion). The term immersion should be understood as the performance and discussion on the part of the teachers of those very activities that they will later propose for their students in class,

as well as their knowledge of the most relevant aspects of research into the present area.

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DOING ALGEBRA TYPE STUFF: EMERGENT ALGEBRA IN THE PRIMARY SCHOOL

Janet Ainley
Mathematics Education Research Centre
Institute of Education
University of Warwick, Coventry, U. K.

There is a large body of research about the early stages of algebra which attempts to identify and explain pupils' difficulties. In this paper I take a different approach, exploring the potential of spreadsheets to provide new ways for children to be introduced to, and to appreciate the need for, an algebra-like notation, and illustrating 'emergent algebra' in primary school children familiar with spreadsheets.

Introduction

There is a large body of research about the teaching and learning of the early stages of algebra (sometimes referred to rather confusingly as 'pre-algebra') much of which seems to be about trying to explain why algebra is hard, identifying which bits of it are hard, and exploring pedagogic approaches which offer ways of overcoming the difficulties. A theme within much of this research has been to identify when 'real' algebra begins; to define a boundary between algebra and arithmetic. Filloy and Rojano (1989) define a 'didactic cut' between linear equations which contain only one use of a letter to represent the unknown and can thus be solved by essentially arithmetic approaches, and those equations in which the unknown appears more than once, so that arithmetic approaches have to be replaced by algebraic ones, involving the manipulation of symbols.

Other researchers have focused on the cognitive obstacles pupils encounter in the early stages of algebra. Herscovics and Linchevski (1994) argue that the divide between arithmetic and algebra lies not in the mathematical structures, but in the cognitive structures of pupils. They define the 'gap' as occurring when pupils are required to operate on or with the unknown. Many other researchers (too numerous to list without the risk of serious omissions) have explored in detail particular cognitive obstacles, sources of misinterpretation of algebraic symbols, pedagogic approaches which may introduce, or alleviate, potential difficulties, and theoretical models of learning processes which may illuminate our understanding of why algebra is hard, and when the hard bits start.

Brown et al (1998) draw on the work of Ricoeur to offer a critique of this research in early algebra as an example of research based on the notion of transition from one state (of knowledge) to another. They propose instead the need for multiple narratives in order to capture the complexity of the learning process. In the spirit of Brown et al., but with not drawing directly on Ricoeur's work, I intend in this paper to offer a

narrative of what I shall call *emergent algebra* in children in the last year of primary school, within the context of the use of spreadsheets.

Re-viewing the problem

In offering this narrative of children's emergent algebra, I want to take a view of the whole area which differs from that of many previous researchers in a number of ways. This may prove to side-step some difficulties, and to introduce others. In contrast to attempts to define the boundaries of algebra, I want to work within a fuzzy description of 'algebra type stuff': an expression given to me by a ten year old pupil. I take this view partly because I am unconvinced that a clear definition of what is and isn't algebra is particularly useful in looking at what pupils do, say and write, and partly because the use of technology, and particularly spreadsheets, may affect the kinds of algebra type stuff which are important and interesting. As I discuss in more detail in the next section, spreadsheet environments produce some interesting ambiguities in the ways algebraic ideas can be seen and used.

A different view of what makes the early stages of algebra hard is that it is very difficult for pupils to have any sense of the purposes of algebra, of what it is that algebra is useful for. Lave and Wenger (1991) have drawn attention to the important characteristic of out-of-school learning contexts, in which, '*learners, as peripheral participants, can develop a view of what the whole enterprise is about*'. Sutherland (1991), in a discussion of what she saw as some outstanding research questions in the teaching and learning of algebra called for '*a school algebra culture in which pupils find **a need** for algebraic symbolism*' (my emphasis). However the current school curriculum offers few genuine opportunities for pupils to develop a view of the whole enterprise, or a sense of the need for algebraic symbolism.

Investigations of number patterns in practical contexts are often used as a starting point for the introduction of algebraic notation. But, although it may seem clear to the teacher that a general algebraic expression arises naturally from such investigations, for pupils this may seem pointless. If you already have a rule for finding any term in the sequence, what is the point of expressing it again algebraically? An alternative approach often proposed as a way of providing meaningful contexts for algebra is the use of word problems (for example, Stacey & MacGregor (1997), Sutherland & Rojano (1993)). In practice, such problems can often be solved by arithmetic approaches, or the contexts themselves are largely ignored by pupils as a distraction (Ainley (1997)), so the usefulness of algebraic notation remains unclear for pupils.

One of the features of technological environments which use algebra-like notations (e.g. spreadsheets, graphic calculators, Logo or Basic programming) is that when the notation is used, it does something: there is an immediate point and pay-off for using it. This feature can be incorporated into the design of meaningful activities in ways which give opportunities for pupils to gain at least a glimpse of the purpose of the

whole enterprise (Ainley (1996), Ainley, Nardi and Pratt (in press)). Furthermore, computers and calculators will only accept inputs in particular forms without and capacity for interpretation, and so there is an additional need to adopt the conventions of notation. This opportunity to use an algebra-like notation in an active way, to produce results of various kinds (a Logo drawing, a numerical output from a Basic program, a spreadsheet of data which can be graphed), is in stark contrast to the more passive use of algebraic symbolism in more traditional pedagogic approaches, where the only feedback accessible to the pupil may be the teacher's approval.

The spreadsheet as an algebraic environment

Spreadsheets, which are becoming increasingly available both in and out of schools, offer an environment with some interesting algebraic opportunities. However, as spreadsheets are designed as commercial, rather than pedagogic, tools there are also some significant differences between what can be done on a spreadsheet, and what might be done in formal, pencil-and-paper algebra. In this section I will explore both the opportunities and some of the limitations of spreadsheets as an environment for learning about and using algebraic ideas[1].

Spreadsheets use an algebra-like notation, in which the cell reference is used ambiguously to name both the physical location of a cell in a column and row, and the number that the cell may contain. The spreadsheet thus offers a strong visual image of the cell as a *container* for a number, which may or may not be present in the cell when the cell reference is used to create a formula elsewhere on the sheet. The image of a variable name as a container whose contents can be changed is one which has been used successfully as a pedagogic device in a number of settings (see, for example, Tall and Thomas (1991)). However, the image offered by the spreadsheet is ambiguous in another powerful way: when a formula is entered in a column, it can be 'filled down' to operate not just on a single cell, but on a range of cells in a column. The cell reference can then be seen as both specific (the particular number I am going to enter in this cell) and general (all the values I may enter in this column).

When a cell reference is used within a formula, the cell in question may contain a number, or another formula, as shown in cell B2 in Figure 1. This means that operating on an existing operation or function, a well known area of difficulty for many pupils, is easy and intuitive. The spreadsheet notation allows the encapsulation of the operation as a single cell reference, but at the same time disguises what has happened (as in D2 in Figure 1).

	A	B	C	D
1				
2		=(A2-7)*5		=B2*C2
3				

Figure 1

1 Although there are some small technical differences between particular pieces of software, I shall ignore these in favour of concentrating on common features.

Spreadsheets lend themselves to activities in which the focus is on creating expressions to represent relationships. It is less obvious how they may be used to create equations, and since there is no facility, or need, for symbol manipulation on a spreadsheet, solving equations can only be done indirectly through trial and improvement approaches. Whilst I am aware that some researchers might claim that this is sufficient reason to disqualify activities on a spreadsheet from being 'algebra', I wish to propose a different view. Spreadsheets provide a powerful environment for what Kieran (1996) refers to as *Generational activities*: expressing general relationships arising from a variety of sources. Because of the ease with which large quantities of data can be generated and explored, they are also powerful for some aspects of *Global, meta-level activities*, such as problem solving and awareness of constraints in problem situations (Kieran (1996)).

Starting algebra type stuff with spreadsheets

Much of the research and curriculum development with spreadsheets in secondary school algebra which has been reported, even the inventive work of Sutherland and Rojano (e.g. 1993), has focused on using spreadsheets to tackle traditional problems. However, the cognitive accessibility of spreadsheets also offers exciting opportunities to introduce pupils to different kinds of activities which are essentially algebra-like, and thus to meet algebraic ideas in new ways. Within the Primary Laptop Project, children regularly use spreadsheets as tools within mathematical and scientific activities. Pupils currently in Year 6 (aged 10-11; the final year of primary school), have had access to spreadsheets during the last four years. Their work has included entering formulae for 'function machine' or 'guess-my-rule' activities, collecting and graphing data in problem solving activities, and entering and copying formulae to generate data. Within the primary school, the children have not been introduced to any 'formal' algebra, such as the use of letters in equations to represent specific unknowns, but they have had opportunities to use spreadsheet notation for representing unknown numbers, generalised relationships and ranges of numbers, in the context of meaningful activities. In the last part of this paper I report on the preliminary stages of research on how these experiences are translated into work with formal algebraic notation as children move from primary to secondary school.

Talking about Algebra Type Stuff

To provide a starting point for conversations with these Year 6 children about their algebraic ideas, I have used some of the questions selected by MacGregor and Stacey (1997) in their study of secondary school pupils' understanding of algebraic notation. Using these questions offered the opportunity for comparison with results from a large scale survey. Because of the experience the children, I decided to present the questions in both a standard algebraic form (as in MacGregor and Stacey's study) and in a spreadsheet version. While designing the question sheets, it also occurred to me that some of the questions could be expressed in 'everyday' language, without any use of a formal notation, and I designed a third sheet of questions phrased in this way. All the Year 6 children were asked to complete the three question sheets; first the

everyday version, then the spreadsheet questions, and finally the algebraic form. In the following discussion I shall focus on just one of these questions, HEIGHTS[2].

everyday version	algebraic version
David is 10 cm taller than Ellie. If you knew Ellie's height how could you work our David's height?	David is 10 cm taller than Ellie. Ellie is h cm tall. What can you write for David's height?

In the interviews, I introduced an extension to this: the *Christmas tree question.*

Let's imagine that David is a little boy, and he is going shopping with his mother (father) to buy a Christmas tree. He says, "I want a Christmas tree that is twice as tall as I am". What could you write down for the height of the tree?

From the written responses, it was clear that most children found the everyday version of HEIGHTS unproblematic, and the results for the algebraic version were comparable to those MacGregor and Stacey obtained for children who had had some introduction to algebra in the first year of secondary school. My main interest , however, was to use the written responses as the basis for interviews exploring the children's ideas. I made the selection for interviews on two criteria: I was interested to talk to some of the children who had apparently completed both the spreadsheet and algebra sheets correctly, and also to some of those who had given idiosyncratic responses. There is space here to give examples from only four conversations.

Livy had completed the everyday and spreadsheet questions correctly, but had left the algebra sheet blank except for one question. However, her explanation of this answer indicates that she is comfortable with using a letter to stand for an unknown number.

Res: ... you did this one with the triangle. Can you explain what you meant there? **[n x 3]**

Livy: Well, is it like a sort of algebra sort of thing, so [...] so, using n would be like having a number, would be some equivalent number, so that, what n is it's times by 3. [...]

Res: Let's just have a look at these others that you didn't try before. [We look at HEIGHTS.]

Livy: It would just be h add 10 or something.

Res: OK, you write it down, write it down how you think it might go. [Livy writes: **h+10**].

After looking at the rest of the page, I asked Livy the Christmas tree question.

Livy: Just do... Well you times David's height by 2, but would you put um a letter, 'cause you don't know what it is?

Res: Well we've got something written down for David's height.

Livy: Oh, he's 10 cm taller.

Res: Mmmm. ..... Can you think of a way we could write down David's height times 2?

Livy: times 2 on the end.

Res: OK have a go, write it over here in this space, how you think you might write it.

Livy immediately wrote **h + 10 x 2**, and without prompting correctly added brackets. Notice that Livy is familiar with the term 'algebra', and seems to have a reasonable notion of what this means.

2 In trying to devise a version of this question involving sensible use of a spreadsheet, I produced something unnecessarily complex, which I will not include here.

Wen was more tentative in his responses, completing the everyday sheet, but leaving the spreadsheet page blank. For the algebra version of HEIGHTS he had written **h 10cm**. He too had clearly heard of algebra.

Res: Tell me what you were thinking about if you can remember when you wrote that.
Wen: I was thinking of algebra, whatever, something ...
Res: OK good. [...] So what did you mean when you wrote that?
Wen: Ellie is h cm tall and if David is 10cm taller than Ellie, then you must do h 10cm. [...] It's 10 cm taller than h.

Wen did not see that he needed to add an operation to link h and 10; it was already clear to *him* what his answer meant. However, when I wrote **h + 10 cm**, he was happy to accept this notation, and said, '*It means a number add 10.*' At the end of the interview Wen confidently used this notation to write **(h+10) x 2** in response to the Christmas tree question. I asked him about his use of 'algebra'.

Res: You knew that name 'algebra'. Where did you learn that from?
Wen: Murderous maths. [A comic-style book]
Res: What do you think algebra means?
Wen: um it's a bit, it's where you use a letter for an unknown number.

Emm gave detailed and correct answers to the everyday and spreadsheet questions, but seemed to have fallen into a well-documented error in trying the algebraic versions. Her response to HEIGHTS was **i**, followed in small writing by the comment **including each letter adds 10cm**. This seemed to indicate that she was not simply seeing the letters as a code. As the following conversation shows, this was actually her way of trying to write 'h plus 10'.

Res: Can you explain what you meant here? I wasn't quite sure what you meant.
Emm: I'm not quite sure. I said if every like a, b, c, d if in between that letter was 10 cm, if Ellie was h it would be i next.
Res: Oh I see,[...] Can you think of any other way we might be able to write down something for David's height?
Emm: I couldn't quite think of anything, because that seemed kind of strange, because I didn't understand the h.

After discussing some other questions, I decided to simply show Emm the standard notation for David's height. Her face immediately lit up.

Emm: Oh! of course! (laugh and huge grin)
Res: Do you like that?
Emm: Yes, it's much easier, 'cause my mum says I work things out really hard, the hard way.
Res: Right, so you think that's an easier way to write it? Now you've seen that, could you think of a way to write some of the other questions?

Emm went on quickly and confidently to complete the rest if the questions on the page, and the Christmas tree question. She seemed to accept the need for this new notation to express her ideas, and her pleasure in using it was clear.

Kit responded to the everyday version of HEIGHTS by writing **= x - 10 cm.** Although he had got the wrong operation, he was clearly trying to give an algebraic response.

Res: Tell me about what you wrote here and why you wrote it. [...]
Kit: Well x ... we say x means any number so we, so if David is 10 cm taller than Ellie, Ellie would be the x, no, David would be the x and Ellie would be the answer, would be what came out, so we were doing like er function machines and stuff
Res: I see. [...] Where you put 'equals x'?
Kit: I did it as a formula.
Res: Right, so what you really meant was Ellie's height equals x minus 10?
Kit: No I meant if David's 10 cm taller ... David's, well I think I went wrong there, I think I should have put plus 10.

During a lengthy discussion of Kit's responses to the other everyday questions (all answered on the same style), and the spreadsheet questions, it was clear that he had some understanding that letters could be used in similar ways to cell references, to stand for any unknown number. His use of an equals sign at the beginning of his answers also indicated that he was thinking in terms of spreadsheet formulae in his algebra-style responses. However, when we got to the algebraic version of HEIGHTS he appeared to have changed his approach to use an 'alphabetic code'.

Res: 'What can you write for David's height?' You put a G
Kit: Yeah, I was doing it in like in algebra type stuff.
Res: Well, what we were doing here is certainly algebra type stuff! Why did you choose G?
Kit: Well I just thought um ...
Res: ... you'd just choose another letter?
Kit: No, I actually did something and like went through the stages but using letters instead of numbers.
Res: Oh ... Can you remember why you wrote G?
Kit: Erm, probably because ... actually it's wrong because it should be higher than H.
Res: OK, [...] Can you think of another way that we could write down David's height?
Kit: Well maybe h could have been like the same as n [n was used in a previous question]
Res: We could use h instead of n. What this is telling us really is to use h to stand for ...
Kit: ... yeah, a number
Res: ... a number, but we don't know what the number is, [...] but we're allowed to call that number h, to use h to stand for that number, just like you were using x before.
Kit: Yeah and then we used n, and now were going on to h.
Res: [...] So could you write like the formula for David's height, using h and saying what you have to do to h?
Kit: Yeah, you could put like just h plus 10.

Despite his apparent confidence when he had the freedom to choose a letter to represent an unknown number, Kit seemed to be confused when a *particular* letter was assigned in the question, perhaps feeling that this must have some significance. At one point in the interview he said in two consecutive statements that '*n means just the same as x*', and that '*n is nothing*'. He still seemed uncomfortable about the use of particular letters when he tackled the Christmas tree question.

Res: How would we write down the height of the Christmas tree?
Kit: n times 2, or 2 times n.
Res: Well, where's this n come from?
Kit: n is just his height.
Res: OK, we've got something that means his height.

Kit: h, so it will be ...

Res: Well we've got, all of that [pointing to **h+10**] means his height. h plus 10 means David's height.

Kit: Yeah, so it will be h + 10 brackets, bracket it times 2. [...] That's how I would write it [...] if I didn't know. I'd probably put x actually.

The picture that emerges from these interviews is not of children who are confused by, or failing to use, algebraic notation and ideas. These children are reasonably comfortable with talking about and representing unknown quantities, and with the idea of operating mathematically with them. They also accept the need for a way of expressing these ideas in writing. Many of them were clearly aware that this was a legitimate part of mathematics, and knew the name 'algebra' from sources outside school. Their written representations do not always match those expected in formal algebra, but this seems to be because they do not yet know the conventions, rather than because they cannot grasp the ideas. Once they were shown the notation, or reassured about their attempts to use it, all four children went on to use it confidently, even to express the more complex answer to the Christmas tree question.

Their position seems to me to be similar to that of young children who are learning to write their native language: they can already communicate in a limited spoken form of that language, and understand the purposes for which written language is used by adults. Young children's spontaneous attempts at writing are often described as *emergent writing*. By analogy, rather than using the term 'pre-algebra', I would like to describe much of what the children produced as *emergent algebra*, arising from their attempts to imitate and invent a written notation whose function they already, at least partially, understand. This has the enormous advantage of enabling me to see their attempts not as errors or misunderstandings, but as attempts to get it right.

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THE RIGOUR PREFIX

Lara Alcock and Adrian Simpson

Mathematics Education Research Centre, Institute of Education
University of Warwick, Coventry, CV4 7AL, UK.

This paper defines the rigour prefix, a way of dealing with mathematical categories in contrast to general cognitive categories. Illustrative episodes from interviews with beginning university students learning about convergent sequences highlight the contrast between students who are beginning to develop this prefix and those who continue to work with mathematical objects using general cognitive strategies. The importance of the parallel development of the object and rigorous ways of dealing with the object are discussed.

Introduction

Human beings categorise the world in order to deal with it. In this paper we shall be concerned with two different types of human categories; object categories and situational categories. Here "object" covers both concrete and abstract objects, although our principal example will be the mathematical object "convergent sequence". Situational categories are akin to frames (Minsky, 1975; Davis, 1984) in that they determine our expectations of, and our normal behaviour in a given situation. In order to deal with mathematical objects in an educational setting, learners develop situational categories which we call personal "maths frames".

Situational categories frequently act as prefixes in the sense of Lakoff (1987). Important aspects of a concept may be overridden by a prefix to the word denoting that concept, for example the prefix "white" in "white lie" overrides the idea that a lie is usually intended to be harmful. In a similar way finding ourselves in a situation belonging to a particular category may cause us to deviate from our usual behaviour. We will contrast expert maths frames, which act as a significant prefix, with non-expert maths frames which deviate less from general cognitive behaviour.

While our situational categories in general enable us to deal with the world, they do not necessarily always work to our advantage. Schoenfeld (1987) describes some common beliefs students have about doing mathematics problems, including that there is one and only one way to solve any given problem, and that none should require more than a few

minutes' work to complete. These are examples of expectations in the maths frame which may well inhibit a student's progress.

The Rigour Prefix

General cognition involves dealing with categories which are not "classical"; their membership is not dependent solely on some given common property (Lakoff, 1987). If categories were classical then there should be no gradience of membership, and categories should have well-defined boundaries. Instead categories can have "better" and "worse" members, for example the robin is generally considered a better example of a bird than the ostrich, and some have fuzzy boundaries, for example the category "tall man". This means that we are rarely consider any given category in its entirety. In order to reason about it we are compelled to operate on representative members of the category and then assume a generalisation to a large part of the category. Since efficiency is valued over accuracy (Balacheff, 1986), this strategy, which we call the "general cognitive strategy", causes few problems.

However, contrasting this with the formal work of expert mathematicians we find that they operate quite differently. Mathematical objects (at least at university level) belong to defined categories. By the nature of definitions the membership of such categories depends solely on common properties and therefore they *are* classical. In a formal mathematical sense there is no such thing as a "good" or "bad" example of a convergent sequence, and the category has well-defined boundaries. This means that in mathematics whole categories can be dealt with at once by working exclusively with the definitional properties. This is necessary if unambiguous communication between mathematicians is to be possible.

We claim that expert mathematicians have, as part of their maths frames, an encapsulation of this idea, which some call "rigour". This acts as a prefix, in the sense that it overrides the individual's general cognitive strategies in formal mathematical situations. The "rigour prefix" is essentially a directive in the maths frame allowing those who have it to exploit the classical nature of defined categories by using only the definitional properties in formal work.

Mathematicians who have developed the rigour prefix are thus able to access two ways of working: they can work with representative examples to develop intuitive ideas about object structure, but they are also aware of the need to work with the whole category. Those who have yet to develop the rigour prefix only have access to the former way of working.

At the beginning of a university mathematics degree, students are required to develop mathematically in two distinct ways: they learn new object categories (such as "convergent sequence"), but also need to enhance their maths frame as a situational category in order to cope with formal deduction and proof.

We can infer the different levels of development of the rigour prefix in beginning university students from the contrast in the kind of objects they use. In examining students' initial reasoning with convergent sequences, we can see a qualitative difference between those whose immediate reaction is to deal with the whole category (through the definition) and those who deal with an exemplar.

The Rigour Prefix in Action

The following excerpts are taken from interviews with students taking a first course in Analysis. Some were attending a standard lecture course (with three hours of lectures per week and a weekly assignment to be completed for credit), and others were attending a new course based on Burn (1992). Those on the new course had only one lecture per week and in addition attended four hours of classes; a class of 30-35 students worked in groups of about four to complete a weekly "workbook", with the assistance of a teacher (a member of staff or a graduate student) and second year students who had been successful on a similar course the previous year. The workbooks aimed to lead students through a carefully structured sequence of questions covering inequalities, sequences and series. These students proved most of the main results of the course for themselves.

Students from both courses were interviewed fortnightly, in pairs, throughout the term when the course took place. The interviews included a mathematical task-based section related to their recent work.

In both cases the students were asked to work on the question:

> For a sequence (a_n), which of the following is true?
>
> a) (a_n) is bounded $\Rightarrow (a_n)$ is convergent,
> b) (a_n) is convergent $\Rightarrow (a_n)$ is bounded,
> c) (a_n) is bounded $\Leftrightarrow (a_n)$ is convergent,
> d) None of the above.
>
> Justify your answer.

We present excerpts of interviews with two pairs of students whose behaviour is illustrative of the contrast between having and lacking the rigour prefix.

The first two students were taking the question-based course. Having decided that b) is true and produced a counterexample to reject a) and c), Adam and Ben immediately had the following exchange:

B: Right, b) is convergent. Right, our definition of convergence is that...well, there exists an N such that when n is greater than big N, there, modulus of a_n is less than

epsilon. So that leads to epsilon being a bound...plus or minus epsilon being a bound, about a. a_n. Is it?

A: Yes go on do you want to write that down? Right the, you have it bounded by, the plus or minus epsilon, thing, and that shows it's eventually bounded and then it is, bounded. By sort of, an earlier result, sort of thing.

[Ben then writes something down and is asked to read it out]

B: It's only sort of, vague. But, modulus of a_n will be less than epsilon if it's converging, when n is greater than big N. Hence minus epsilon is less than a_n, is less than epsilon, therefore epsilon is a bound. Since (a_n) is eventually bounded, therefore (a_n) is bounded by, whatever various proofs that we've done in the book!

I: Mm. What were you going to say, Adam?

A: If it's convergent, rather than converging, or tending to zero...that, the modulus of a_n you've written, should be modulus of, a_n minus a. And you have to say, if, the sequence (a_n) tends to a, then...and that needs to be the modulus of a_n minus a.

While their speech was informal, Ben's immediate recourse to the definition of convergence allowed them to make rapid progress. They went on to prove that a sequence which is eventually bounded must be bounded and wrote down and explained their whole proof without any significant difficulty. We claim that this is evidence of the development of a rigour prefix - the use of the definition means that the conversation is about *all* convergent sequences. It is not claimed that either had a fully functional rigour prefix at this stage; they may not have an explicit idea of why formal definitions are used in Analysis, but they certainly seem to know *that* they are used.

This contrasts with the behaviour of Wendy, a student on the lecture course. In the first excerpt she and Xavier had just established that they think b) is true, and Wendy had drawn a picture of a monotonic increasing convergent sequence:

W: It's convergent...yes so if it's convergent it's always, or, say it could be the other way round it could be, going down this way. It converges, so it's always above that limit.

X: Could you do, a_n minus l, mod is less than epsilon...thing? That goes to...

W: I wouldn't have done that, but you can have a go if you want!

[There follows a discussion in which they generate counterexample to disprove a) and c), and then Xavier comes back to his idea, and Wendy agrees.]

X: Erm, well the term in the sequence a_n, er minus the limit, or the bound, the modulus of that is less than epsilon, where epsilon is, any, er, real number, positive real number...

W: So that shows it's convergent. You've got *a*...(writing).... And therefore it's between those, and therefore it's bounded.

[They are asked to explain this in more detail which they do, and then to consider what happens "before big N". (There is some confusion here as they speak of "bounded" meaning only bounded above or below.)]

W: Yes if it's an increasing sequence, the limit'll be, up above. It'll be increasing up...so, what happens before big N doesn't really matter because it'll always be below that. And then if it's decreasing sequence it'll er...(*draws*)

X: Mm. So it doesn't matter what happens before, big N. It's what, after, what happens after,

W: As long as you know whether it's an increasing or decreasing,

X: Sequence.

Short pause.

I: What if it's neither of those things?

W: You're in trouble.

I: Oh dear...

Pause.

X: So if it's oscillating...(drawing)

[The interviewer then clarifies the meaning of "bounded" and the fact that a sequence cannot "shoot off to infinity" at any particular term, and suggests that the terms before a_N are our current problem and asks if we can fix that.]

W: Find out what the points before that point are...

I: Can we do that?

W: When you've found big N, yes.

I: Yes. And what would you do then?

W: You'd have to check that erm...well you could find out whether it was an increasing or decreasing sequence,

I: What if it wasn't either?

Pause.

Throughout this discussion (which lasted approximately 15 minutes) Wendy appears to be employing what we earlier termed "general cognitive strategies" in her reasoning about convergent sequences, based around the idea that a monotonic sequence is a good representative example of that category. It may be suggested that she does have a

version of the rigour prefix and is using this as intuitive work prior to introducing the definition. However we argue against this. Not only does she return to the monotonic sequence idea several times, despite repeated suggestions from the interviewer that other kinds of sequences also be considered, she also sidesteps Xavier's attempt to bring part of the definition into the discussion.

In Xavier's case we can see a contrast between his developing maths frame (which is beginning to show signs of a rigour prefix in his attempts to introduce part of the definition) and his development of the object category for "convergent sequence" which appears to be no more developed that Wendy's: both seem to have access to a very limited number of representative examples.

This highlights the relationship between mathematical object categories and the rigour prefix; the two may influence each other but they are not entirely dependent on each other. A student may have well developed object categories, for instance be able to cite many varied examples from a category without apparently assuming any extra properties or weighting them differently. However, such a student may not work with these categories in the way determined by the rigour prefix. Equally it is possible to have the rigour prefix in the maths frame but have a poor understanding of what any particular mathematical object category contains. We claim that changes to object categories are easier to make than changes to situational categories.

Discussion

Category change has many similarities to Skemp's ideas of how concepts come to be learned (Skemp, 1979). Skemp claims that concepts are altered as the learner acknowledges the importance of different examples of the object. In Wendy's case the examples of convergent sequences that she has been exposed to appear to have led her to believe that a good representative example of that category is monotonic. She is by no means atypical of those on the lecture course in this respect. We suggest that while a teacher may choose to exhibit a particular example on the basis of its simplicity of expression, intending it to serve as illustrative of the general principles under discussion, a student may attach inappropriate weight to it as a member of a category. Exposure to more varied examples may discourage this tendency to consistently work with one particular representative example, and increase the student's number of prototypical members. Using more representative examples enables the student to cover more of the category. However, this is not the same as developing the rigour prefix, without which the general cognitive strategy of working with representative examples is the only available option.

The rigour prefix is defined as a directive in the maths frame allowing those who have it to exploit the classical nature of defined categories by using only the definitional properties in formal work (although their informal reasoning may still rest on

representative examples). We claim that situational categories are modified in the same way as object categories, that is by the learner acknowledging the importance of different instances of the situation. This may be why the question-based course is more successful in encouraging students to develop the rigour prefix. These students are regularly put into a situation where they are asked to prove general results for themselves: the general cognitive strategy is not likely to lead to consistent success in producing such proofs. They are likely to find their current maths frame is no longer efficient and to be ready to explore alternatives to the general cognitive strategy.

We do not suggest that the development of the rigour prefix will be smooth. An interim stage for many in Analysis seems to be a period of manipulating definitional statements in order to please the authorities rather than as a reasoned alternative to the general cognitive strategy. The next stage in development may be a realisation that the definition corresponds to their own category for that object. As was put by one girl in the pilot study for this research:

> "I didn't see how it [the definition] related to what I thought, and I thought what I thought was a lot better than how they'd written it on the page, and then the more I saw how it, sort of works, the more I can see that that is actually just what I thought anyway, and it's just said better than I could say it."

This idea has obvious connections with those of concept image/concept definition (Tall and Vinner, 1981). A student without the rigour prefix works solely with some piece(s) of their concept image, although they may be able to state the definition quite accurately. In addition such a student may well display a propensity towards "proving by example", having what Harel and Sowder (1998) call inductive or perceptual proof schemes.

Conclusion

The way mathematics has evolved has led to experts in the field working with defined (and therefore classical) categories. These offer huge scope for generality and accuracy, but their use requires a maths frame which can override general cognitive strategies.

The development of the maths frame occurs in all learners throughout their mathematical education. However, the development of the rigour prefix is essential to the transition to advanced mathematical thinking. Prior to this, the learner may be able to cope with school mathematics by using a combination of a specialised version of the general cognitive strategy as their maths frame and by being able to create and modify object categories. At university, the student not only needs to be able to continue to create and modify object categories at increased levels of abstraction, they also need to develop a new and fundamental aspect to their maths frame: the rigour prefix.

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THE PROBABILISTIC THINKING OF 11-12 YEAR OLD CHILDREN

Gilead Amir, Liora Linchevski, Hebrew University, Jerusalem
Malka Shefet, Lewinski College, Tel-aviv

Abstract

This research explored the probabilistic thinking of 11-12 year old children in Israel. A questionnaire was developed, including scales that explore children's estimations of probabilities. 294 children completed the questionnaire, and 32 of them were also interviewed. Some of the main findings include: 'representative' sequences got higher estimations of chance than other sequences; most of the children did not discriminate between single sequences and classes of sequences; several new examples of children's use of the 'representativeness' and the 'availability' heuristics were identified.

1. Literature and theoretical framework

Piaget and Inhelder (1975, original in French, 1951) analyzed children's thinking about probability into the usual stages (pre-operational, concrete operational, etc.), culminating in a formal understanding of probability through combinatorics. Green (1982), in a survey of 3000 children aged 11-16 showed how their development followed a hierarchy which was consistent with Piaget's stages.

A focus of criticism of much of Piaget's research has been the gap in communication which may develop between researcher and child (see for example Donaldson, 1978). Borovcnik and Bentz (1991) similarly cast doubt on common interpretations of answers to commonly used probability questions.

Another focus of criticism on Piaget is in not taking enough into consideration non-formal, intuitive lines of thought. Fischbein (1975) showed that some intuitions in young children's thinking are important in helping (and hindering) their pre-formal probabilistic thinking. Kahnemann, Slovic and Tversky (1982) showed how adults reason in situations of uncertainty using intuitive 'heuristics', rules of thumb which seem to be developed to guide our behavior in daily living. Shaughnessy (1981) asked college students to compare the likelihood of two sequences of births of six children. Only 27% of them answered that there is about the same chance for each. 70% thought 'BGGBGB' is more likely than 'BBBBGB', probably applying 'representativeness' (expecting a sample to be similar to its parent population). Two heuristics we have found (Amir, 1994, Amir & Williams, 1994, Amir & Williams, 1995) relevant also to children: 'representativeness' and 'availability' (estimating odds according to memories of similar past

experiences). Beliefs also influence children's interpretations of probabilistic situations (Amir, 1994).

This knowledge of chance and probability that the child acquires informally, mainly outside school, is relevant to the learning of probability (see Fischbein, Nello & Marino, 1991). Unfortunately, our knowledge of children's preconceptions of probability is not systematic and not precise (Shaughnessy, 1992). This research intended to contribute to this knowledge both in methodology and in substance.

Thus, the aims of this research were:

1) To develop tools for research of probabilistic thinking.

2) To develop insight into probabilistic thinking of 11-12 year old children.

2. Method

Two questionnaires were used: the first is a translation of part of Green's questionnaire (1979) into Hebrew. The second questionnaire is new, aimed at mapping children's probability language, and quantitative estimations and comparisons of probabilities. The instruments' validity and reliability were checked by piloting them both in writing and in interviews.

Use was made of scales, which are frequently used in research of attitudes, beliefs, etc, but are not commonly used in estimations of chances. An example of a question:

3. On each of the scales mark the chances of the event.
For example:
What are the chances of 'Heads' when tossing of a usual coin?
If we think the chances are even we mark the number 50:

← →
Low chances High chances
0 100

A fair coin is tossed 6 times. Tick the chances to get the following results:	
a. 6 'Heads'	
b. 6 'Tails'	
c. 5 'Heads' and 1 'Tails'	
d. 1 'Heads' and 5 'Tails'	
e. 4 'Heads' and 2 'Tails'	
f. 2 'Heads' and 4 'Tails'	
g. 3 'Heads' and 3 'Tails'	

A similarly structured question with scales asked about chances for specific sequences:

A coin is tossed 6 times. Tick the chances to get the following results (in that specific order):

	0 — 100
a. 'Heads', 'Heads', 'Heads', 'Heads' 'Heads', 'Heads'	
b. 'Tails', 'Tails', 'Tails', 'Tails', 'Tails', 'Tails'	
c. 'Heads', 'Heads', 'Heads', 'Tails', 'Heads', 'Heads'	
d. 'Tails', 'Tails', 'Tails', 'Heads', 'Tails', 'Tails'	
e. 'Heads', 'Heads', 'Tails', 'Heads', 'Tails', 'Heads'	
f. 'Tails', 'Tails', 'Heads', 'Tails', 'Heads', 'Tails'	
e. 'Heads', 'Heads', 'Tails', 'Tails', 'Heads', 'Tails'	
f. 'Tails', 'Heads', 'Tails', 'Heads', 'Tails', 'Heads'	

A question with equivalent items dealt with series of births:

In a family of 6 children mark the chances of the following sequences (in that specific order):

	0 — 100
a. boy, boy, boy, boy, boy, boy	
b. girl, girl, girl, girl, girl, girl	
c. boy, boy, boy, girl, boy, boy	
d. girl, girl, girl, boy, girl, girl	
e. boy, boy, girl, boy, girl, boy	
f. girl, girl, boy, girl, boy, girl	
e. boy, boy, girl, girl, boy, girl	
f. girl, girl, boy, boy, girl, boy	

Other questions dealt with classes of sequences in families (such as '3 boys and 3 girls'), and with dice tossing.

When piloting these questions, in writing and in interviews, we found that the pupils used the scales without difficulties, and understood their meaning. Obviously the pupils, with no formal knowledge of probability, were not expected to offer precise values, or even rough values, of the probabilities. What the scales did enable is comparison of their various estimations: Does the pupil see different sequences in a specific question as having equal chances? Is there a trend in his estimations? Does the pupil give similar chances to a specific sequence (such as 'H,T,H,H,T,H') and to the unordered event (in this case '4 'Heads' and 2 'Tails'')? How similar are the chances the pupil gives to equivalent events in different contexts? Do the pupil's responses suggest an application of a certain heuristic? Data that emerged from the questionnaires was validated through interviews.

After finalizing the instruments the questionnaire was administered to 294 11-12 year old pupils from central Israel, chosen randomly. 32 of these children were also interviewed. Results were analyzed both quantitatively and qualitatively.

3. Results

3.1 Probability of sequences

When giving an estimation of the probability of the result '2,2,2,2,2,2' when tossing a die 6 times on a scale in the questionnaire the average was 0.24 (n=294), the lowest probability for the sequences in the question. The average estimation for the sequence '6,6,6,6,6,6' was close ($\bar{x}$=0.25). The highest probability in the question was for the sequence '6,5,3,6,2,4' ($\bar{x}$=0.59). As previously explained, the significance of the results is not in the absolute values of the probabilities, because the pupils have no tools for estimation of the correct probabilities. The results' significance is in the comparison between them: although mathematically each sequence has the same probability, the pupils assign to the second sequence a much higher probability. The same results were received also when dealing with coin tossing: 'H,H,H,H,H,H' and 'T,T,T,T,T,T' were the sequences estimated with the lowest probability ($\bar{x}$=0.26 and $\bar{x}$=0.25) when tossing 6 times a coin, and 'H,T,H,T,H,T' was the sequence estimated with the highest probability ($\bar{x}$=0.57). In the domain of birth series, the sequence 'B,B,B,B,B,B' received the lowest probability ($\bar{x}$=0.36), only slightly lower than the sequence of 6 girls ($\bar{x}$=0.36). The sequence 'B,B,G,G,B,G' received the highest probability ($\bar{x}$=0.63).

One possible explanation is that pupils expect different sequences to have different chances because they apply the 'representativeness' heuristic (a detailed discussion of this heuristic is in section 3.4). Another possible explanation deals with communication: maybe the pupils understand wrongly the question. Maybe they treat each sequence as a **type** of result, a representative of a class of sequences, rather than a specific sequence. If so, then although the answer is formally wrong, it is based on the correct line of reasoning: when tossing dice, receiving 5 different numbers with one repetition, and with no specific order, has a higher probability than receiving 6 times the same number!

This explanation can be checked by comparing estimations of probabilities of specific sequences with estimations of probabilities of unordered events including several possible sequences. For example – comparing the estimation for the probability of the sequence 'B,B,G,G,B,G' with the estimation for the probability of the event ' 4 boys and 2 girls in a family of 6 children'. This comparison is possible with the following table, containing results of sequences

and the matching classes of sequences of 6 coin tosses, and equivalent questions dealing with the order of births in families of 6 children.

Table no 1: Average estimations of probabilities for specific sequences versus classes of sequences of 6 coin tosses and of birth order in families with 6 children (n=294)

Tossing a coin 6 times				A family with 6 children			
CLASS OF SEQUENCES	AVERAGE	SPECIFIC SEQUENCE	AVERAGE	CLASS OF SEQUENCES	AVERAGE	SPECIFIC SEQUENCE	AVERAGE
6 HEADS	0.341	H,H,H,H,H,H	0.256	6 BOYS	0.359	B,B,B,B,B,B	0.357
6 TAILS	0.351	T,T,T,T,T,T	0.253	6 GIRLS	0.365	G,G,G,G,G,G	0.358
5 HEADS, 1 TAIL	0.420	H,H,H,T,H,H	0.357	5 BOYS, 1 GIRL	0.450	B,B,B,G,B,B	0.427
5 TAILS, 1 HEADS	0.419	T,T,T,H,T,T	0.346	5 GIRLS, 1 BOY	0.454	G,G,G,B,G,G	0.450
4 HEADS, 2 TAILS	0.579	H,H,T,H,T,H	0.488	4 BOYS, 2 GIRLS	0.549	B,B,B,G,G,B	0.503
4 TAILS, 2 HEADS	0.550	T,T,H,T,H,T	0.489	4 GIRLS, 2 BOYS	0.554	G,G,G,B,B,G	0.520
3 HEADS, 3 TAILS	0.604	H,H,T,T,H,T	0.556	3 BOYS, 3 GIRLS	0.649	B,B,G,G,B,G	0.632
		H,T,H,T,H,T	0.568			B,G,B,G,B,G	0.582

3.2 Probability of sequences versus complex events

The pupils' responses from the table indicate a high similarity when comparing the estimations for sequences with the estimations for classes of sequences.

This low discrimination was also supported by part of the interviews. For example, in a discussion of coin tossing, D. was asked if there is a question similar to 'H,H,H,T,H,H'. She answered that "1 'Tails' and 5 'Heads'" is similar, and that, yes, she would give each the same estimation. Similarly, A. claims that there is no difference between the question about the probability to get the sequence '6,4,5,3,1,2' and the probability of "to get 6 different numbers" when tossing dice. "It is the same, this is also 6 different numbers".

These results suggest that many children do not discriminate between ordered sets and unordered sets in probability. For these children the comparison between the probabilities of the sequences 'H,H,H,H,H,H' and 'H,T,H,T,H,T' is the same as the comparison between the probabilities of the events "getting 6 'Heads' when tossing a coin 6 times" and "getting 3 'Heads' when tossing a coin 6 times". These differences are especially small when dealing with birth orders. A possible explanation is that in families the relative frequencies of boys and girls are seen as important, and not so much the order.

Although, as previously stated, results of sequences and matching classes of sequences were similar, most children did give the unordered event a slightly higher probability than the sequence, i.e., there is some discrimination between the two, but it is low, and has no relationship with the number of sequences in the class of sequences.

3.3 Probability in different domains

Table 1 enables comparison also between answers about coin tossing and about order of births. Similarity in answers to equivalent questions is high, although in general results of birth orders are higher, especially when dealing with specific sequences. Perhaps the context of children in the family is more familiar, and so estimations for these questions are higher. In this case, the pupils are applying the 'availability' heuristic (offering higher estimations for familiar situations).

Interviews confirmed that while part of the pupils thought there is no difference between equivalent questions in different contexts, others thought there is some difference. For example, part of an interview between the child A. and the interviewer I.:

A.: "The questions are different. Because, everything depends on God, like, what he gives man."
I.: "And with coins?"
A.: "With coins it is luck."

A. attributes order of birth of children to God, and results of coin tosses to luck. Questions that are equivalent mathematically do not seem so to children, due to their beliefs. For similar results see Amir (1994).

Or in another interview, with D.:

D.: "No, this is really not the same question."
I.: "Why?"
D.: " Because it is a fact that last week a girl was born to a woman with 11 boys."
I.: " And with coins, to get 11 times 'Heads'?"
D.: "It is impossible."

The explanation of this pupil is based on memory: the pupil does recall memories of families with many boys, but does not have similar memories of long sequences of 'Heads' or 'Tails' when tossing a coin – perhaps because he

does not often toss coins! This type of generalization based on memories is, as mentioned before, an application of the 'availability' heuristic.

And with K.:

K.: "The two questions are different ... A coin you toss, here [i. e. with births, G. A.] you can get what you want ... With a coin you can control only if you cheat ... With children you do not compete."

3.4 Representativeness

Previous literature (for example Konold et al, 1993), and our pilot interviewing, suggested that the heuristic 'representativeness' includes two distinct and independent dimensions when applied within probabilistic situations: the tendency to expect a sample to reflect the numerical proportion of the parent population; the tendency to expect a sample not to be too orderly, i.e., to look 'random'. Order seems to be more special, thus with less chance to happen. These dimensions lead us to identify four types of responses that seem to reflect application of the 'representativeness' heuristic. A problem used in this research that can exemplify these different types is: tossing 6 coins.

Most of the pupils thought (correctly) that getting 3 'Heads' and 3 'Tails' has the highest chances of all given possibilities. A group of these children applied this view also when analyzing sequences: they gave all sequences with 3 'Heads' and 3 'Tails' a high chance, without taking into consideration the order.

Another group of children thinking 3 'Heads' and 3 'Tails' has the highest chance – when analyzing sequences expected the ordered sequence 'H,T,H,T,H,T' to have the highest chances.

A third group of the children thinking 3 'Heads' and 3 'Tails' has the highest chance – when analyzing sequences expected the 'randomly' ordered sequence 'H,H,T,T,H,T' to have the highest chances.

Another group of the children thought the highest chances are for a case which is **near** the expected numerical value, but not the exact value. In the case of tossing 6 coins they expect "4 'Tails', 2 'Heads'", or "2 'Tails', 4 'Heads'" to have the highest chance. When asked about the result with the highest chances when tossing 30 times a coin, these children answered that "14 'Tails', 16 'Heads'" or "14 'Heads', 16 'Tails'" have the highest chances.

4. Conclusions

1. New instruments based mainly on scales provided useful information about children's concepts of probability.

2. Children gave sequences different probabilities. 'Representative' sequences got higher estimations of probability.

3. Discrimination between sequences and classes of sequences was low.

4. Equivalent probability questions in different domains gave close results, although differences existed, especially when comparing order of birth of children to tosses of coins.

5. The 'representativeness' heuristic includes two distinct and independent dimensions: the tendency to expect a sample to reflect the numerical proportion of the parent population; the tendency to expect a sample not to be too orderly, to look 'random'. These dimensions led to the identification of 4 variations of 'representativeness'.

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WHAT CAN BE LEARNT ABOUT FRACTIONS ONLY WITH COMPUTERS

Ilana Arnon, Pearla Nesher, Renata Nirenburg, CET, Tel Aviv

In this report we will present the software "Shemesh", designed for the learning of mathematical concepts through concrete representations that can not be constructed by the students without the computer. The concept of ***Equivalence Class*** *plays a significant role in the structure of Rational Numbers. In a discrete Cartesian system an equivalence class of fractions is represented as a line through the origin. Other fraction concepts also have concrete representations in such a system. Fifth-graders who used "Shemesh" in their learning process were clinically interviewed several months later. It was found that they remembered these representations and could use them for solving conventional arithmetic fraction-problems.*

To compare two fractions means to find the order relation between them, such as the following problem: "Of $\frac{3}{4}$ and $\frac{7}{10}$, which fraction is larger, or are they equal?" How do we usually approach such problems in elementary schools? One way is to teach some algorithm, to be learned by heart. If we aspire to some meaningful learning, we might choose more lengthy methods. For example, we might teach the following method:

Find two new fractions: One equal to $\frac{3}{4}$, the second equal to $\frac{7}{10}$, yet both having the same ("a common") denominator. Compare the new fractions.

Some will argue that the easiest common denominator to find would be 40 (the product of the two given denominators). In this method, we expand each given fraction by the denominator of the other, to deduce that $\frac{3}{4}=\frac{30}{40}$, $\frac{7}{10}=\frac{28}{40}$, and hence, according to a previously learned rule of comparing fractions with common denominators, $\frac{28}{40}<\frac{30}{40}$, hence $\frac{7}{10}<\frac{3}{4}$.

Others will argue that there exists a simpler common denominator, to work with - namely 20. The followers of the first method will argue that 20, being a smaller integer, might be perhaps easier to use for the calculation, but not at all easy to find. To obtain it involves either using complex ideas of whole-number-theory, such as decomposition into prime factors and smallest-common-multiple, or else constructing lists of fractions equal to the original ones until hitting fractions with a common denominator: $\frac{3}{4}=\frac{6}{8}=\frac{9}{12}=\frac{12}{16}=\frac{15}{20}=\ldots$, $\frac{7}{10}=\frac{14}{20}=\ldots$ Hooray! All these methods lead the student to believe that he had found **the** appropriate common denominator (and respective representative solution), and not **one of several.**

Working on fraction concepts with 4-, 5-, and 6- graders we found that it was extremely difficult for students simultaneously to conceive of the components of this complex situation: The two given fractions, the equality of each of them to the members of its own list, and the three significant characteristics of the fractions with common denominator: that their denominators are equal, that each of them equals

one of the original fractions and that they usually do not equal each other. (On the contrary, we have to determine which one is larger and deduce from that the order relation between the original fractions).

The same holds for other arithmetic operations on fractions, such as addition and subtraction. If we do not wish to teach an operation as a technical algorithm, the solution always begins by searching for a replacement for each given fraction; We search for the replacement in the set of the fraction's equals. We choose the replacement according to convenience; the same arithmetic problem can often be solved by means of more then one choice. The idea that the solution of the problem does not depend on the choice we make (on the individual replacement we used in the solution-process) is very crucial for the understanding of fraction-operations, and too much neglected, in our opinion, in school mathematics.

There is a fundamental mathematical idea behind all these methods, namely, that a fraction is not a single pair of integers (numerator and denominator), but a class of equivalent such pairs. The arithmetic operations executed between pairs of fractions are in fact defined in terms of their equivalence classes. Here is an example of a definition of this idea:

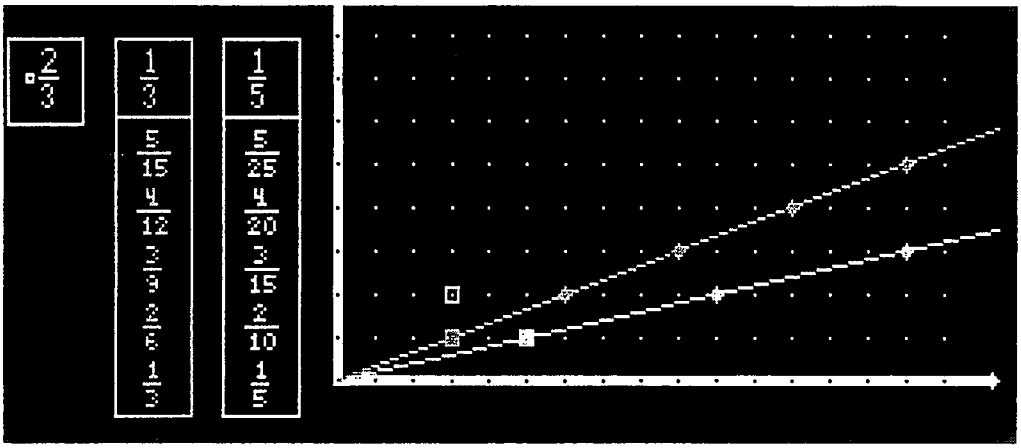
> *Let R be a commutative ring without zero-divizors.*
>
> *(a) We define a relation on* $R \times (R \setminus \{0\})$ *by* $(a,b) \sim (c,d) :\Leftrightarrow ad = bc$. *This is an equivalence relation. The equivalence class of (a,b) is denoted by* $\frac{a}{b}$.
>
> *b) The set* $Q(R) := \{ \frac{a}{b} \,/\, a \in R,\ b \in R \setminus \{0\} \}$ *of equivalence classes, endowed with the operations* $\frac{a}{b} + \frac{c}{d} := \frac{ad+bc}{bd}$ *and* $\frac{a}{b} \bullet \frac{c}{d} := \frac{ac}{bd}$, *is a field, called the* **quotient field** *of R.*
>
> (Spindler, 1994, V. II, p. 40).

Teaching fractions as equivalence classes means more than teaching the equivalence of two elements of a given class. It means getting the students to operate on the classes as **objects**. So, while the concept of fractions as equivalence classes remains difficult and very abstract it is essential that it be taught. Piaget taught us that students of this age develop abstract mathematical concepts by reflecting on their own **concrete** activities (Piaget, 1976). What if we find concrete representations for equivalent classes of fractions? Will that enable the teaching of this concept?

Resnick (1987) emphasized the necessity of strict and explicit **mapping** between a concrete representation and the mathematical ideas it represents. Nesher (1989) emphasized the need for well defined and (mathematically) closed **knowledge domains**, to be taught by the use of isomorphic concrete exemplification domains. Dubinsky (1991) emphasized that the development of a new mathematical concept

begins by an **action**, and Arnon (1997) investigated the improvement of the development of fraction concepts by the use of **concrete actions**. Arnon (1997) also showed the necessity of another condition: the ability of children to operate the concrete actions, after due interiorization, **in their imagination**.

We chose to represent fractions as equivalence classes of ordered pairs of integers in a discrete Cartesian coordinate system. An ordered pair is represented by a point whose vertical coordinate is the numerator and its horizontal coordinate - the denominator. The ordered pairs of each equivalence class are situated on a straight line passing through the origin (Kalman, 1985; Kieren, 1976):



Drawing 1

The use of computers is indispensable for this exemplification, because of the need for accuracy in drawing the different lines. The software "Shemesh" ([1]) was developed with a screen consisting of two separate parts: A number-domain and a drawing-domain. An equivalence class is represented in the number-domain as a list of fractions $\frac{a}{b}$, and is referred to as a "class". In the drawing-domain it is represented as a line through the origin together with all the points of the "class", a point for each numeral $\frac{a}{b}$ (see drawing 1). Users can construct points representing individual pairs of numerator and denominator and lines representing equivalence classes. Other options of the software are: The construction of additional types of fraction sets, the performance of arithmetic operations such as addition, subtraction, comparison, and more. The software and its optional activities were constructed according to the necessary conditions established by the researchers that were quoted above. Even so, the use of this representation for elementary school students poses the following questions:

- To what degree is this representation really **concrete** for young children?

[1]) "Shemesh", Fractions as Equivalence Classes, CET - Center for Educational Technology, Tel-Aviv, Israel

- Does the use of this software enhance the development of the concept of fractions as equivalence classes?

We set out to investigate these questions. In a preliminary experiment, thirty fifth-graders attended between one to four lessons a week. The sessions took place in a computer laboratory, in addition to the normal school schedule, and participation was voluntary. Some of the students dropped off along the way. Others were persistent and attended up to 30 lessons. The lessons consisted of group-work, individual work and large group discussions.

Three months after the teaching experiment was over we interviewed 21 students (those available at the time). One student dropped out because she was quite unable to work with the specific representation. We will present the data collected from the remaining 20 students in two parts, concerning the construction of the concept, and its further use.

A. The construction of the concept of equivalence class

1. The representation of a numerator and denominator pair as a point in the Cartesian system

All 20 students handled this representation correctly (although some needed a few "recollection" exchanges): Using pencil and paper drawings they drew points representing given fractions, and vice versa, and successfully accomplished even more sophisticated assignments.

2. The relation between the members of an equivalence class

All 20 students knew that the fractions whose points were on a line passing through the origin were equal to each other. They also knew that the fractions that appeared on a list in the number domain equaled each other.

> S8: "*...actually they are in the same class... because they are equal.*"
> S19: "*Because they are in the same class, and they equal each other.*"
> S20: "*First of all, they are all on this line. Second - they are all equal.*"

3. The link between a list on the number-domain (a "class") and a line on the drawing-domain

19 students expressed their certitude about the link between such a list and its line:

> S8: "*Because usually in the computer, when we have a line, and we have a class, then usually they belong to each other.*"
> S17: "*All the fractions of a class are on the line.*"

S10 expressed the integration of all three ideas:

> "*They* [refers to the list] *are all fractions of* $\frac{2}{3}$, *it is a line of* $\frac{2}{3}$ *and they all are expansions of* $\frac{2}{3}$"

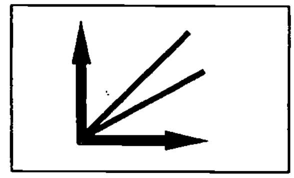
B. The use of lines in solving arithmetic fraction-concepts.

1. The use of lines for the comparison of fractions

In this representation the order relation between fractions is determined by their **lines**, and not by their individual **points**: The fraction whose line is higher is the larger fraction (see drawings 2 and 3). This rule is valid in the first quarter of the Cartesian system, and needs refinement with negative fractions (which was tried out with older populations, but will not be reported here). In the interviews we dealt with three situations of fraction-comparison, which denote a hierarchy in the development of students' ability to use the lines as problem-solving tools:

1a. The fractions were presented by their lines

The students were presented with a drawing of two lines in a Cartesian system devoid of points:



Drawing 2

The problem was:

"Here are the lines of $\frac{1}{1}$ and of $\frac{1}{2}$. Write each fraction next to its line."

$\frac{1}{1}$ and $\frac{1}{2}$ were chosen because we believed that most students knew which of them is larger. We expected students who had interiorized the representation of the relation ">" in the Cartesian system, and knew that $\frac{1}{1} > \frac{1}{2}$ will ascribe $\frac{1}{1}$ to the higher line, and $\frac{1}{2}$ to the lower. We found that 15 of the 20 students did so, and relied on the order relation between the fractions in their explanations:

S1: "*Because this is a whole, and it is larger than a half, therefor its line is higher.*"

S10: "*Half is smaller. It is a fraction smaller than one over one. And the lines that are higher, which rise upward, are the lines that are larger.*"

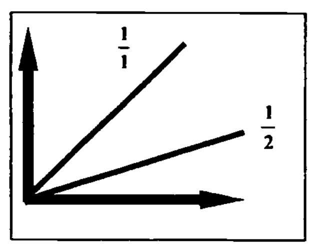
1b. The given fractions were presented as lines while the answers were requested as points:

The students were presented with this Cartesian system and were requested to draw points of fractions:

a. Smaller than $\frac{1}{2}$;

b. Larger than $\frac{1}{1}$;

c. Smaller than $\frac{1}{1}$ **and** larger than $\frac{1}{2}$.

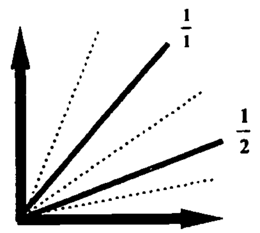


Drawing 3

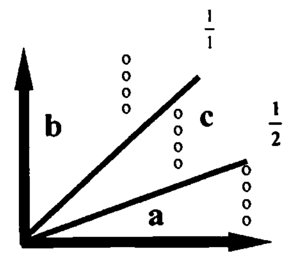
The same 15 students who succeeded to solve the previous problem, succeeded here too. S10 (for question c):

"*It can't be above the line of one, but it will be above the line of half.*"

2 students used lines for their answers:



14 students spoke of **zones of the plane:**

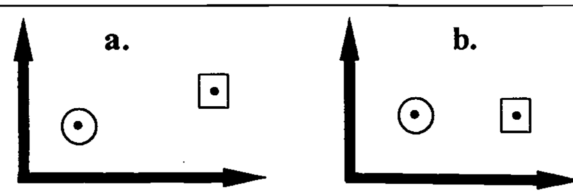


Drawing 5

1c. Both fractions represented by points

The students were presented with two Cartesian systems devoid of all but two points:

In each system the students were asked to mark the point of the larger fraction.



Drawing 6

In order to solve the problem they had to know that the order relation was determined by the relative position of the **lines**, and not of the points. 14 students used lines to solve the problem correctly. 4 of them actually drew the lines. 10 answered correctly with no visible action. When asked to explain they refered to lines in their explanations:

S4: "*If we take a line and we drew it from the origin-point up to the fraction, and we draw it, then we will find that the square will be beneath the line, so it will be smaller than it* [no line drawings, only hand gestures of imaginary line drawings in the air, accompanying the explanation].

S5: "*If I took a ruler and drew a line from here to there, the line would be higher than if I drew a line from here to there* [draws while explaining his answer] *and the higher the line, the larger it is.*"

S8: "*Because here if one draws of the circle, an imaginary drawing of the line, then the line is higher than the... if one draws an imaginary line of the square* [with each reference to a line "draws" with his finger in the air a line via the origin]."

S17: "*It has to do with the lines, not the fraction itself.*"

One might say that for these students the concrete action of comparing fractions by means of their representative lines was interiorized to a degree where they performed it in their imagination.

One student estimated the fractions corresponding to the given points numerically, and then compared them correctly. One student thought that the further the point was from the vertical axis, the larger the fraction. The others either could not solve the problem, or were not asked to.

2. The density of the rational numbers

The question was: "Find fractions (as many as possible) that are larger than $\frac{1}{5}$ **and** smaller than $\frac{1}{3}$: $\frac{1}{5} < \square < \frac{1}{3}$ "

16 students started with $\frac{1}{4}$. That was the only solution for 3 of them. Others added expansions of $\frac{1}{4}$. 2 students added (*"from my head"*) a fraction that was not an expansion of $\frac{1}{4}$ ($\frac{3}{13}$ and $\frac{3}{10}$) and used the software for checking. The last 11 used the Cartesion system in their search for more solutions: They drew the lines of $\frac{1}{5}$ and of $\frac{1}{3}$ (see drawing 1), and wrote many fractions whose points were located between these two lines. As for how many such fractions exist, they all answered "*many*" or "a *lot*" or "*infinite*":

S9: "[Draws the lines] *"All that is between this line and that line*" [writes $\frac{4}{13}$, $\frac{4}{14}$, $\frac{5}{16}$, $\frac{5}{17}$. Then she expands the drawing-domain to reach more fractions].

In.: "*How many are there?*"

S9: "***Up to infinity***".

S21: [draws the lines and writes $\frac{2}{7}$. Strolls with the cursor among the points within the lines, and continues to write: $\frac{2}{9}$, $\frac{3}{10}$, $\frac{3}{11}$, $\frac{3}{12}$, $\frac{3}{13}$.] "*there are a lot*".

Summary of findings:

The findings we described suggested answers to our two research questions. As to the question about the concreteness of this representation, we have shown that fifth-graders worked with the different components of the discrete Cartesian system - points, lines, axes, origin and zones. These were concrete to them to the extend that they were able to draw sketches by hand when needed.

As for the second question, about the development of the concept of fractions as equivalent classes, we have shown that:

- Students understood the mapping between the mathematical language and this representation (Resnick, 1987; Nesher, 1989): They could identify a fraction given by its point or line, draw sketches of a line or point of a given fraction, and even find the correspondence between sets of fractions of a given characteristic, and zones of the Cartesian system.
- The students knew the term "equivalence class of fractions" and its concrete representation - a line through the origin. They were also aware of the equality of any two elements of such class.

- The students used this concept and its representation to solve arithmetic problems. They did so when the **problem was presented by drawings**, such as when they found fractions smaller than $\frac{1}{2}$ beneath the line of $\frac{1}{2}$. They also did so when the problem was presented in formal **mathematical language**, such as when asked to find fractions larger than $\frac{1}{5}$ and smaller than $\frac{1}{3}$, they drew the lines of these fractions and found the required fractions between these lines. They could even make deductions about the abundance of such fractions.

Further findings, relating to the connections students make between this representation and other common concrete representations of fractions, will be reported in the presentation.

Conclusions

The findings of this research indicate that fifth-graders can develop the concept of a fraction as an equivalent class, provided that appropriate concrete representations and activities are used. Such development was made possible by the software "Shemesh".

The findings also indicate that this concept develops gradually. Further research is needed to obtain more insight into the nature of this development.

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DECISION-MAKING STRATEGIES IN PROBABILITY EXPERIMENTS: THE INFLUENCE OF PREDICTION CONFIRMATION

Paul Ayres
University of Western Sydney Nepean
Australia

Jenni Way
University of Cambridge
UK

Two groups of grade six students observed a video-recording of coloured balls being drawn from a box (sample space unknown). After every fifth selection, students were required to predict the colour of the next ball drawn. One group observed a sequence where the most frequently occurring colour (white) was drawn 80% of the time following prediction, whereas for the second group, a white only appeared 20% following prediction. Even though the accumulated experimental probabilities prior to prediction for both sequences had been manipulated to be identical, the former group chose white, more consistently than the latter group. Consequently it was argued that children may be influenced in their probability judgements by confirmation or refutation of their 'predictions'.

One focus area of probability research, particularly in the context of education, has been inappropriate decision-making strategies, or *misconceptions* in situations involving random events (for example: Peard, 1995; Shaughnessy, 1981; Tversky & Kahneman, 1982). Several of these strategies are attributed to the effect of sequences of randomly generated outcomes from probability experiments on the subject's expectations regarding the 'next' outcome. One such strategy that has received considerable attention is *representativeness* (Fischbein & Schnarch, 1997; Kahneman & Tversky, 1972; Shaughnessy, 1981), which is the expectation that a random set of outcomes should be representative of the composition of the known sample space. Related to representativeness is the type of thinking known as *negative recency* or *gambler's fallacy*, where there exists the expectation that as the frequency of a particular outcome increases the probability of that outcome occurring decreases. For example; when repeatedly flipping a coin, a run of heads would lead to the expectation of the next flip being a tail. The opposite and less common strategy (Fischbein & Schnarch, 1997) is referred to as *positive recency* (in this case, predicting a head because that's the trend).

Much of this research has been conducted with adults, using written tasks or 'tests', in which preconceived sets of outcomes have been presented to the subjects. However, it is quite common for researchers working with children to use real random generators to accommodate children's need for concrete experiences (for example: Truran, 1992; Way, 1996). Often in this type of study the children, following a number of experimental outcomes, are asked to state what they consider to be the most likely outcome of the next random event. This context gives rise to a little studied, possible influence on decision-making; that of the confirmation or refutation of the 'prediction' by the actual next outcome.

Truran (1996), working with a *known sample space*, analysed the changes in prediction of primary and secondary students in regards to the next outcome. One

finding was that when the more-likely outcome was predicted, it didn't really matter whether the next outcome confirmed or refuted that prediction. However, if the less-likely outcome was predicted, the subject was highly likely to change the prediction, particularly if the following outcome refuted the less-likely prediction. Similarly, Ayres & Way (1998b) working with *unknown sample spaces*, found evidence that upper primary-aged students would change their prediction patterns according to how successful they were in their predictions. Although, students would choose the most frequently occurring colour under specific conditions, they would change strategy if their predictions were not rewarded.

The findings (not directly tested) by Ayres & Way (1998b) suggest that children may be influenced in their probability judgements by confirmation or refutation of their 'predictions' rather than the overall picture. Consequently, this study was designed to explore this theory directly. Because random generators naturally produce sequences which vary, a video-recorder (see Ayres & Way, 1998a) was used to control the outcome sequences and provide a realistic medium for children.

METHOD

Participants. Fifty nine grade six students from a primary school in the state of New South Wales, Australia, participated in this study. Students had not been formally taught any chance and data topics in their mathematics classes.

Apparatus. A video recording of coloured balls being chosen from a box was made according to the following procedure. Ten coloured table-tennis balls (6 white, 3 blue and 1 yellow) were placed in an opaque brightly-coloured box (18cm x 18cm x 14cm) with no lid. To ensure that particular outcomes occurred, the box was fitted with three cardboard compartments which were not visible from the camera angle. Within each compartment was placed a ball so that the three colours were represented once only. In addition, the box was fitted with a false bottom in which the remaining seven balls were placed. Hence when the box was shaken (before each selection), the noise was consistent with a box containing a number of balls. Furthermore, the compartment design made it possible for a researcher to select a particular coloured ball at will, but give the appearance that the ball was selected at random. In this fashion, thirty selections were made, with replacement, and consequently a particular colour sequence of outcomes was made to occur. The researcher (positioned in front of a white board) was filmed making the selections. As each selection was made, a second researcher recorded the "colour" on the whiteboard in a 6 x 5 array format. At all times, the researcher making the selections and the box was visible, as was the record of the colours previously selected. In order to make the video as authentic-looking as possible (see Ayres & Way, 1998a), a clock was positioned close to the whiteboard to indicate a continuous time passage and avoid possible suspicions of video-splicing. After five selections, the researcher paused and asked for a prediction to be made. This process was continued a further five times and a prediction was made after every fifth selection.

Ayres & Way (1998b) found that students of this age are more likely to select the most frequently occurring colour if a high percentage of the sample space is represented by that colour. Consequently, the particular sequence selected included 19 whites (63%), 7 blues (23%) and 4 yellows (13%). To create a situation where the most frequently occurring colour (white), appeared consistently on prediction, four of the first five prediction were manipulated to be white. This sequence was called the *Typical Outcome* sequence (see Table 1). The positions where predictions were made are underlined in Table 1.

Table 1: Colour sequences developed for the Typical and Non-typical groups.

Outcomes	Typical Outcome Sequence	Non-typical Outcome Sequence	Experimental Probabilities (%) W : B : Y
First five	WYWBW	WYWBW	60 : 20 : 20
Second five	WWBWY	BWWWY	60 : 20 : 20
Third five	BWWWB	WBWWB	60 : 27 : 13
Fourth five	WYBWW	YWBWW	60 : 25 : 15
Fifth five	WBWWW	BWWWW	64 : 24 : 12
Sixth five	WYWBW	YWWBW	63 : 23 : 13
	W	Y	

Note: Underlined positions indicate where predictions were made.

In the same fashion, a second video-recording was made. In contrast to the first, the less likely outcomes (blue and yellow) appeared consistently at the prediction locations. This sequence was called the *Non-typical Outcome* sequence (see Table 1). To achieve this effect, the underlined colours in the first sequence were rotated with a different colour within the same subset of five colours. As a result, only one white appeared in the first five prediction positions. Students who predicted a number of whites would therefore not be very successful. Furthermore, the accumulated experimental probabilities after each set of five outcomes (see Table 1) for both sequences were identical, and approximately matched the theoretical probabilities (60: 30:10). Consequently, if students viewed either sequence and were guided by experimental probabilities alone, they would choose white as the most likely outcome in both situations. However, depending upon which sequence was viewed, students' success rates would vary considerably. It was therefore anticipated that this design would cause the two groups to adopt different selection strategies.

Procedure. Students were randomly assigned to two groups. One group was shown the typical outcome sequence (Typical Group), whereas the second group viewed the non-typical outcome sequence (Non-typical Group). The experiment was conducted with small groups of students each time. As an introductory instructional

phase, a student was asked to select a ball from the box (used in the video) with the partitions removed, show it to the rest of the class, before returning it to the box. The class was then asked to predict what colour would occur the next time if another ball was selected. This procedure was repeated twice, so that students became familiar with the idea of making predictions following a random selection. The experimenter used the following statement: "What do you think the next colour will most likely be?" The wording "most likely" was used to encourage students to make decisions based on their concepts of chance; however, it should be noted that children may interpret words, such as "likely", differently to what is expected (Konold, 1991). They were also informed that there were some white, blue and yellow balls in the box, but no clue was given to the proportions. Additionally, the students were told that it was a game and students should try to predict as many correct colours as possible. When the experimenter was satisfied that the students understood the nature of the task, they were shown one of the videos according to which group they had been assigned. The video recording was shown on a large TV monitor positioned at the front of the classroom. After each selection a record of the colours was also recorded on the classroom chalkboard. After the first five colours were observed being drawn, the video was stopped and students were asked to make their predictions, then given time to record them on answer sheets. This procedure was then repeated for five more subsets. The language used by the researcher was identical for both groups, as was the task that the students were required to complete.

RESULTS

For each student, a sequence consisting of six colour predictions, was recorded. Given the nature of the prediction tasks in this study, of particular interest was the number of whites chosen. If students were guided by experimental probability then it was expected that a high percentage of whites would be chosen. To investigate this, the mean number of whites chosen for each group was calculated (see Table 2). In addition, the number of whites chosen in the first and last three predictions was also recorded (see Table 2). Ayres and Way (1998b) found that students may not necessarily choose the most frequent colour after a small number of observations and may need more information before committing to a strategy. By comparing predictions over the two halves it was possible to analyse the extent to which students refined their strategies as more selections were observed.

Over the six trials, the mean number of whites predicted by the Typical Group (2.73) was not found to be significantly greater than the Non-typical Group mean (2.65) under a two-tailed t-test ($t(57) = 0.26$, $p > 0.05$). Although, there was no difference in overall means between these groups, substantial differences can be found in the prediction patterns. By comparing the change from the first three predictions to the final three predictions there was a significant difference between the Typical Group (+0.61) and the Non-typical Group (-0.24) under a two-tailed t-test of differences ($t\ (57) = 2.44$, $p < 0.05$). For the Typical Group, a significantly

greater number of whites was chosen in the last three predictions (1.60), compared with the first three (1.13), under a paired t-test: $t(29) = 2.31$, $p < 0.05$. As the mean number of predictions for white was greater than 50 % (1.6 out of 3) for the final three, there was a clear indication that this group had started to favour the most likely outcome. Nineteen of the thirty students chose at least two whites. In contrast, the Non-typical Group chose less whites in the latter half (1.21) compared with the first (1.45), although the difference was not significant under a paired t-test: $t(28) = 1.16$, $p > 0.05$. A value of 1.21 out of 3 indicates that the group was less likely to favour the most likely outcome in their predictions. Only six of the 29 students chose at least two whites. The above analysis indicated that by the end of the prediction sequence there were significant differences between the groups. The group, which was being rewarded with correct answers by choosing the most likely outcome, was increasing the use of this strategy; whereas, the group which was not being rewarded was moving away from the strategy.

Table 2: Mean number of whites predicted by the two groups

	Typical Group (n=30)	Non-typical Group (n=29)
Mean number of whites chosen on first 3 predictions	Mean = 1.13 SD = 0.73	Mean = 1.45 SD = 0.79
Mean number of whites chosen on last 3 predictions	Mean = 1.60 SD = 0.81	Mean = 1.21 SD = 0.86
Mean number of whites chosen over 6 predictions	Mean = 2.73 SD = 1.08	Mean = 2.65 SD =1.20

Prediction Patterns

To gain insights into the type of strategies employed, prediction profiles for each group were found by calculating the frequency of each colour selected at each prediction point (see Table 3). The profiles indicate that many students changed their choice of colour. For both groups the most frequent colour chosen for the first prediction was yellow. This may be example of the *negative recency* effect as the other two colours had more recently occurred (see Table 1). Following the first prediction, the group profiles changed considerably. For the Typical Group, white was the most frequent (61%) colour chosen on the second prediction. As a white occurred at the first prediction point and because of the high proportion of whites occurring, this group may have been influenced by experimental probability in this instance. However, these students were not rewarded with a correct prediction, as a white did not occur when the colour was drawn. This failure may have also influenced the third prediction, as yellow was the most frequently (55%) chosen colour in this position. Again, this may be an example of negative recency as a

yellow had not occurred for five selections. For the remaining three predictions, white was the most frequently chosen colour, with 84% of the students choosing white for the last prediction. Overall, white was chosen by more students than any other colour except for the first prediction (when a sequence had barely been established) and on the third prediction (following the only non-white occurrence at a prediction point).

In contrast, the Non-typical Group, only had a clear preference (41%) for white on the third prediction. As this followed a white occurring at the second prediction point, it may have been the group's only majority attempt to use experimental probability. Although there are some other notable preferences for this group, namely 66% blues on the fourth prediction and a high number of yellows on the second and fifth predictions, it is unclear why these preferences occurred. It could be argued that both negative and positive recency effects occurred at particular points. Equally, it may be argued that these profiles are just random. However, what is clear is that this group was reluctant to predict white.

Table 3: Colour selections (%) at each prediction point

	Typical Group			Non-typical Group		
Predictions	W	B	Y	W	B	Y
First	29	16	52*	38	21	41
Second	61	32	3*	41	3	55
Third	23	19	55*	41	28	31
Fourth	42	26	29*	31	66	3
Fifth	42	23	35	31	21	48
Sixth	84	6	10	34	34	31
Overall	47	20	31	36	29	35

* Note: a red selected by one student at these positions

The above analysis, based on the data reported in Table 3, indicated that students in both groups changed the colour of their predictions considerably. The extent of these changes was investigated further by calculating the changes of colour made by each student in their prediction sequences. For example, a student in the Typical Group predicted the following six colours- W B Y W W B. This sequence involves four changes of colour from one prediction to the next The maximum of number changes possible is five. The mean number of changes made by the Typical Group and Non-typical group was 3.52 (SD = 1.06) and 3.38 (SD = 1.32) respectively. Both means indicate considerable colour changing. However, there was no significant difference found between groups on this measure under a 2-tailed t-test: $t(57) = 0.13$, $p > 0.05$.

To investigate the extent to which changes are influenced by previous successful predictions, changes were categorised according to whether they followed confirmation or refutation of predictions. For each student the number of changes made following either outcome was calculated as a proportion. For example, if a student made four correct predictions which were followed by a total of one change in colour following these successes, then the student had a ratio of 0.25 changes per successful predictions. For the Typical Group, the mean ratio of changes made (0.92, SD=0.47) following successful predictions was significantly greater than the mean ratio (0.46, SD = 0.46) following unsuccessful predictions under a paired t-test: $t(28) = 3.15$, $p < 0.01$. For this group, the students changed colour twice as many times following an unsuccessful prediction than a successful one. For the Non-typical Group, no significant difference was found between the mean ratio (0.62, SD = 0.42) following successful predictions and the mean ratio (0.71, SD = 0.46) following unsuccessful predictions under a paired t-test: $t(27) = 0.77$, $p > 0.05$.

CONCLUSIONS

The main objective of this study was to test the hypothesis that a subset (consisting of student predictions) of a sequence of "random" outcomes, can affect selection strategies. Even though the accumulated experimental probabilities prior to prediction for both sequences were identical, the Typical Group chose white, more consistently than the Non-typical Group, especially in the latter half of the trials. The experimental design was therefore instrumental in forcing measurable differences between the groups. Consequently it was argued that children may be influenced in their probability judgements by confirmation or refutation of their predictions. Certainly students in the Typical Group were less likely to change colour following a successful prediction, than following an unsuccessful colour. In contrast, students in the Non-typical group consistently changed colour. These changes for the latter group may well have been caused by an underlying conflict between the overall observed experimental outcomes and the lack of white occurring in the prediction locations.

To a certain degree, this study has shown that children this age, without any formal probability schooling, can make decision based on likelihood. However, probabilistic reasoning may not be well developed as decision making seems to be highly influenced by unexpected outcomes. Consequently, any teaching activity in this domain needs to make a focus on this issue. The study has also shown the effectiveness of using a video-recorder as a research tool. No student doubted the validity of the videos. In fact, all students enjoyed the activity. From the perspective of future research, the study will be expanded to investigate other age groups and to include experiments when the sample space is known. In addition, interviews will be conducted to find out if their given reasons support the data trends.

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THE DEVELOPMENT OF MATHEMATICS EDUCATION BASED ON ETHNOMATHEMATICS: THE INTERSECTION OF CRITICAL MATHEMATICS EDUCATION AND ETHNOMATHEMATICS

Takuya BABA
IDEC, Hiroshima University, Japan

Hideki IWASAKI
IDEC, Hiroshima University, Japan

Abstract

Ethnomathematics so far has gained a certain degree of momentum in the developing countries, and some developed countries with multicultural backgrounds as well. On the other hand, critical mathematics education which originated from a developed country, has highlighted critical points of ethnomathematical research. In order to connect ethnomathematics to school mathematics as a part of educational endeavor, the complementary relation between critical mathematics education and ethnomathematics has been deliberated in both ways. Subsequently the foundational framework has been proposed to strengthen the critical nature by means of reciprocity and to develop mathematics education based on ethnomathematics.

1. Introduction

'Ethnomathematics', which was named by D'Ambrosio at ICME 5, in 1984, has resonated among researchers who tried to engage in cultural aspects of mathematics education, and has formed a productive field of research to enlarge the interpretation of mathematics education. Some examples shown in such as Ascher (1991) and Gerdes (1988, 1990) have gradually added concrete images to its general characteristics and materialized its abstract existence. However, the clearer its characteristics become, the more criticisms are made against its potential and applicability to mathematics education.

> 《Ethnomathematics does not amount to a set of general thinking tools since the mathematical activity is 'locked' into this practice , of which it is part, and it cannot function as a tool or basis to criticize that practice itself. Being critical towards the use of mathematics in the context of practice requires viewing that practice from an external perspective in a way that allows the mathematics to be distinguished in some way from the remaining aspects. It is this complete integration of mathematical activity with practice which marks it as distinct from the mathematics of the classroom.》 (Keitel, 1997, p.19)

In other words, her criticism is equivalent to the observation that ethnomathematical activities are closely linked with practices and there is no

necessity for the practitioners to explore the implication and method of their activities. On the other hand, the 'general thinking', which forms a core of school mathematics, demands the completely opposite direction, that is consideration of the reason and method of activity. Thus, the critical points in her words can be summarized as follows.

"Ethnomathematics cannot express itself in its own words."

"The practitioner of ethnomathematics is not necessarily conscious of its implication."

"The objectives of ethnomathematics and school mathematics are different in nature."

Since these points are taken from the perspective of critical mathematics education proposed by Keitel, the first objective of this research is,

① to consider the relationship between both ethnomathematics and critical mathematics education.

Applying this result, the second objective of this research is,

② to consider the preconditions such that ethnomathematics may provide some perspectives for the foundational reconsideration of mathematics education.

2. Critical Mathematics Education (CME)

Keitel has referred to Skovsmose from various perspectives in CME and Skovsmose has developed this theory more overtly and radically. In this research, therefore, the focus of consideration is upon the theory of Skovsmose.

CME is an educational theory which reflects critical theory of Frankfurt School and develops educational practice for the formation of critical citizenship. Its systematization is being made in northern Europe and Germany. The subject, mathematics, had a tendency of being excluded from the practice because of its formal and objective nature, but Skovsmose pointed out the importance of critical aspect which should be incorporated into mathematics education, as follows.

《It is necessary to increase the interaction between ME and CE, if ME is not degenerated into one of the most important ways of socializing students (to be understood as students or pupils) into the technological society and at the same time destroying the possibilities of developing a critical attitude towards precisely this technological society.》 (1985,p.338)

Here, we would like to clarify the objective of CME through the verification of its three key terms, instead of trying to define directly the meaning of 'critical'. These three key terms, which are listed below, play a central role for the fulfillment of the objective of CME.

(1) critical competence (2) critical distance (3) critical engagement

More concretely, (1) critical competence means that students need capability to think and judge by themselves what is important and they are presumed to have it for the active participation in the educational process. This is a human element of the educational process. (2) critical distance is to keep a distance from the given subject or curriculum, and the teacher and students are not supposed to take it for granted. This regards an element of curriculum in the process. (3) critical engagement means to direct the educational interest of classroom towards the outside of school and to relate it to the existing social problems. This concerns how to set the objective in the educational process.

《I conceive a majority of the developed examples of concrete materials to be used in mathematical education as being abstract from a social point of view, even if they are concrete in a physical sense.》 (1995,p.63)

Skovsmose pays much attention to the social implication of teaching material and what it realizes in mathematics education. Any practical examples in CME aim at the formation of critical citizenship through thematization and projectization strategies beyond the subject framework at present. Thus when the children are asked if it is reasonable to do something in order to get pocket money or what is a reasonable salary, they are naturally stimulated to deliberate on the issue of money more socially.

This perspective is very indicative when we consider the fact that many children just learn to do calculation without knowing the reason and yet this technological society is supported to be productive by high level mathematics. However, there seems to be little consideration from the viewpoint of mathematics in particular and this aspect will be discussed after the two-way consideration of ethnomathematics and CME.

3. Ethnomathematics

The term, 'ethnomathematics', sometimes causes confusion in terms of whether the definition refers to a single mathematical activity, a set of mathematical activities in each culture, or the research to analyze these activities. We think it is not very productive to adapt a narrow definition, and that awareness of multi-aspects of the definition should underpin the discussion.

《'ethnomathematics' can refer to a certain practice as well as to the study of this practice. In what follows we use 'ethnomathematics' in both senses, although we primarily think of ethnomathematics as including certain educational ideas and a research perspective.》 (Vithal, Skovsmose, 1997, p.133)

In this study, we will not differentiate these two 'ethnomathematics', as

practice and its analysis. The more explanatory terms, 'mathematical activities' and 'ethnomathematical research', will be employed to make a clear distinction only when necessity arises.

While many mathematical activities have been identified and unified into ethnomathematics, Bishop (1994, p.15) and Vithal, Skovsmose (1997, p.134-135) have recently tried to clarify the structural relationship among some strands of the ethnomathematical theme. Their classifications have similarities from the historical and socio-cultural point of view except that the latter has the additional category as a research to focus on the relationship between ethnomathematics and mathematics education. Naturally Bishop as a mathematics educator knows its importance but Vithal, Skovsmose state clearly that this has potential to integrate other strands.

The intention of this paper relates to this category which is to lay a foundation for the relationship between mathematics education and ethnomathematics. The ethnomathematics emerged as 'objection' from the developing countries against the predominance of Western mathematics in schools, but the fulfillment of this goal requires ethnomathematics to be self-referential, which will be discussed in depth in the next section.

Here we would like to take one example for further discussion. Gerdes (1990) deliberated on the application of sand drawing, Sona, to the mathematics curriculum.

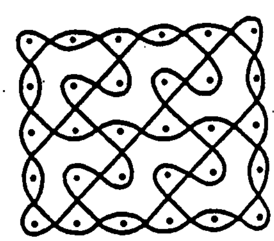

Fig.1 Sona

The Sona drawer has to draw efficiently and beautifully by any means. The method was invented to mark a set of equidistant points with a fingertip on the wiped-up ground and make a drawing by use of these points as reference. Many Sona drawings are done under the restriction of symmetry and in one stroke. And the laws such as symmetry and repetition incarnated in the drawing can represent arithmetic relation, sequence, symmetry and similarity in mathematics education.

Gerdes attempts to resurrect a cultural value with introduction of this Sona. Accordingly this creates a new research field by means of applying mathematical activities, immanent within African culture, to the curriculum development.

4. Bilateral Consideration of Ethnomathematics and CME

According to Vithal, Skovsmose (1997, p.132), while both ethnomathematics and CME have a common term as a reaction to modernization theory, the former has a cultural background and the latter a political one. In short they share the critical stance against the implicit belief which modern society holds but from different angles. However a further deliberation on this difference exposes the difficulty and complexity of this problem. One person, one group of people, or one society can criticize another party from a certain angle but this fact only exposes half of 'the truth'. Neither ethnomathematics nor CME are immune from criticism. That is why we would like to consider as a next step the relationship between ethnomathematics and CME in both directions and to find a complementary role to strengthen each other structurally.

Consideration of CME by use of ethnomathematics

As discussed earlier, three key terms of CME are critical capability, critical distance and critical engagement, which provide important views to the educational process. As for the first term, it is necessary to retain an inner standard for the criticism, and this standard and critical capability are fortified by the reflection of his/her own mathematical activities.

The second term implies 'to keep a distance from curriculum' and it requires a mathematics different from the one under discussion. Ethnomathematics substantiates this other mathematics in a practical way.

> 《whenever we increase our understanding of other cultures, we increase understanding of our own by seeing what is or is not distinctive about us and by shedding more light on assumptions that we make which could , in fact, be otherwise. Our concepts of space and time are, after all, only our ideas and not objective truth. And, there is no single correct way to depict objects in space, nor one correct way to orient a picture in order to comprehend its contents.》
> (Ascher, 1991,pp.186-187)

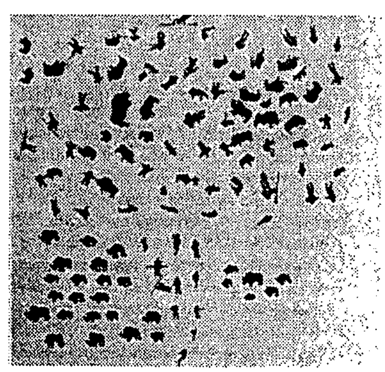

Fig. 2 Musk-ox hunting on North Somerset Island

Also Ascher has given this picture as an example. It enables us to imagine and investigate a mathematics beyond the horizon of our culture.

The critical engagement as the third term means 'to direct an interest toward the outside world of classroom'. It naturally brings about such awareness to stay within the situation of ethnomathematical practice, because the ethnomathematics is unavoidably intertwined with other activities in the society.

It has been shown from the above consideration that CME provides just the theoretical framework and ethnomathematics can concretize this framework of CME by providing practical examples.

Consideration of Ethnomathematics from the Perspectives of CME

Naturally ethnomathematics takes a critical stance against Western mathematics from its original background, but the first two of Keitel's points require ethnomathematics to verify itself critically. That concerns the expression of ethnomathematics in its own words and the awareness of ethnomathematics practitioners. At this junction, Gerdes offered a kind of solution in his educational consideration.

> 《The artisan, who imitates a known production technique, is, generally, not doing mathematics. But the artisan(s) who discovered the technique, *did* mathematics, was/were thinking mathematically. When pupils are stimulated to *reinvent* such a production technique, they are doing and learning mathematics.》 (1988, pp.140-141)

This means that the artisan of mathematical activities, with little mathematical consciousness, practices his own activity as a part of culture. However, it is this mathematical consciousness, and in other words the objectization of practice in mathematical perspective, that enables the incorporation of ethnomathematics into the educational practice. And this is not the viewpoint of practitioner but the one of creator.

In the example of the Sona, there exists a specific method 'to plot a matrix of points on the sand beforehand' in order to make a perfect Sona drawing without hesitation. This method transforms 'to draw Sona' to 'to plot a matrix of points' and 'to follow an algorithm to travel through these points'. However the repetition of drawing the Sona a thousand times will not automatically give birth to this method, but for this invention it is necessary to consider the process analytically and synthetically. This transformation provides justification for its applicability to education. Therefore, the critical verification of ethnomathematics has a close relationship with the fact that 'ethnomathematics is engaged in mathematics education' and we suppose that the last point by Keitel can find a solution in this connection.

As for this critical verification we would like to examine the Sona further by use of key terms of CME. In this case students are assumed to have critical competence basically and they are encouraged to do mathematics from the viewpoint of being creator. The critical distance from the ethnomathematics is required implicitly here. In other words, this example already includes some key terms of CME. The Sona may be able to cast the question to the applicability and eligibility of school geometrical materials, but in mathematics education its self-critical nature should be scrutinized. To view critically the Sona will induce the students to look around themselves, to analyze the mathematical activities in their environment and to develop these activities into an organized but still somewhat personal tool.

5. Conclusion

The first objective of this paper has been to consider the relationship between ethnomathematics and CME in both directions. As a result, it has been shown that mathematical activities substantiate three key terms in CME, and in return these terms can provide a structure and thus a rationale for the application of ethnomathematics to mathematics education. The second objective of this paper, that is the consideration of prerequisite for ethnomathematics to contribute positively to mathematics education, has also been addressed in this two-way consideration because of their complementary relationship.

So far, ethnomathematics has developed discussion from the viewpoint of critics to criticize the school mathematics, but the perspective of educational application necessitates it to be viewed critically as well. This means the incorporation of 'ethnomathematics as method' with 'ethnomathematics as object' can consolidate a foundation for the research field of ethnomathematics. The following framework will be proposed for the integrated approach in ethnomathematical research.

(1) to reflect critically mathematics education through mathematical activities and ethnomathematical research

a) mathematical implication b) social implication

(2) to reflect critically ethnomathematics from the perspectives of critical mathematics education

c) mathematical implication d) social implication

The distinction between mathematical implication and social implication is important to consider the characteristics of the subject, mathematics. The

mathematical implication here concerns the formality peculiar for the subject and, on the other hand, the social implication means the usage of the mathematical concept in the social context.

In this framework, the first component as (1) has potential to reveal the uncritical nature of the present mathematics education. The students are frequently required to develop mathematical thinking as an objective tool with what they find little meaning, but they are not encouraged to consider the relation between this objective tool and their mathematical activities. Here the integration of CME and ethnomathematics plays an important role to uncover what has been taken for granted. The second component as (2) will invite the students to reflect their activities and develop their thinking from there. Their own activities are the target of reflexive thinking and at the same time the source for higher level of reflection. For this development three key terms of CME will provide a guide in how to develop the critical thinking upon their mathematical activities.

Thus we think this integrated approach by use of the above framework plays a pivotal role in the practical and theoretical development of the ethnomathematical program so that ethnomathematics research will eventually produce a fruitful alternative to the present school mathematics.

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LACAN'S FOUR DISCOURSES AND MATHEMATICS EDUCATION

Roberto Ribeiro Baldino
UNESP, Rio Claro, SP, Brazil

Tânia Cristina Baptista Cabral
UNESP and EBP-SP, SP, Brazil

Abstract

In this paper, we use Lacan's four discourses in order to characterise current traditional mathematics teaching and suggest an alternative methodology from the point of view of psychoanalysis theory. The paper is a continuation of "Lacan and the school credit system", Proceedings of PME22, v. 2 p. 56-63.

Introduction

In Baldino & Cabral [1998A] we pointed out that the plenary conference of Shlomo Vinner in PME-21 [Vinner, 1997] highlighted a mismatch between students' discourse and action with respect to the *credit system*. We referred to it as a *mismatch phenomenon*. We remarked that it is in this gap opened by such a mismatch that Lacan places *desire*, and we approached this phenomenon from the perspective of psychoanalytical theory of *Lacan's four discourses* [Lacan, 1973]. This theory has helped us to understand what we have been doing for quite some time, namely learning mathematics as an experience of modification of the *desiring subject* [Cabral, 1998].

In that paper, we referred to discourse as a complex process entailing the participation of the talking subject in three registers: *imaginary* (pre-suppositions of the talking action), *symbolic* (language), and *real* (jouissance). Therefore, the discourse was to be understood as a *joint effort* of the students and the teacher in order to *sustain* a certain relation or statute of actions and utterances. According to Lacan, there are four possibilities for this statute: the *master's*, the *university's*, the *object's* and the *hysteric's* discourses. In the paper mentioned, we introduced four signifiers: the *master* S_1, the *knowledge* S_2, the *lost object* a (petit-a) and the *hysterics* $\$$ and four positions: the *agent*, the *work*, the *production* and the *truth*. The master's discourse is characterised by the distribution of these signifiers through the four positions according to the following diagram (figure 1). We invested the master's discourse in the analysis of a hypothetical traditional classroom. The positions are occupied respectively by the *teacher*, the *student*, the *credit system* and the *castrated* or *ignorant teacher* (figure 2). We also hinted that the three other discourses should be obtained by counter-clockwise shifts of the signifiers through the positions.

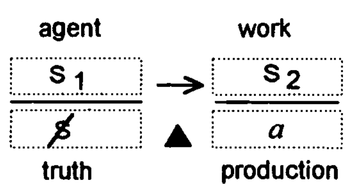

Fig. 1: the master's discourse

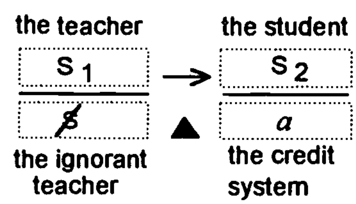

Figure 2: the master's discourse in the classroom

In the present paper, we shall invest the other three discourses comparing a hypothetical traditional classroom with a classroom organised according to an alternative teaching proposition called Solidarity Assimilation Groups (SAG) [Baldino, 1997]. We shall show that, in traditional teaching, these discourses alternate from the university to the object and then to the hysteric, whereas in SAG, this order is reversed. For further references about psychoanalysis, language and mathematics education, see Brown [1997], Atkinson & Moore [1998] and Baldino & Cabral [1997, 1998B].

The University's discourse

If we follow the student's way from elementary to high school and into the university, we shall find that it is not the master's discourse that is present in the classroom any more. The challenge of the teacher's authority has disappeared. In the university, the master's discourse happens only during a very brief initial moment when the teacher states the course's pre-requisites and the goals to be pursued, according to the syllabus. Honours students do not ask questions at this moment. The teacher minimises the discussion of the credit system and refers to it as a minor bureaucracy. He assumes that the student's grades are a consequence of the acquired knowledge. The students seem to readily agree. The teacher makes an effort to hover over the magisterial authority (S_1). He tries to build an image of himself, identifying himself with a scientist or researcher. The lessons are dominated by a continuous word-flux emanating from the blackboard and received by affirmatively-nodding frenetically writing students sitting on the second row. The first row is most of the time empty, since it is reserved for students considered geniuses.

In the mathematics classroom, the students frequently expect clear explanations in a steady voice and watch carefully for the teacher's vacillations. Questions are allowed within narrow limits. Questioning is the risky enterprise by which students establish their position in their ranking. The risk is pays off if the question embarrasses the teacher or reveals that he has made a mistake. Each class starts at the point where the last one stopped, following the thread of organisation of mathematical knowledge established by mathematicians who operate the scientific practice. "*Le savoir est, à un certain niveau, dominé, articulé de nécessités purement formelles, des nécessités de l'écriture, ce qui abouti de nous jours à un certain type de logique*" [Lacan, 1991, p. 53]. We say that an effort of linearisation of the signifiers' chain is evident mainly in mathematics. In order to perfectly understand and dominate this logic the teacher has to talk, to show and practice his ability as a juggler of knowledge. Such is the *scéance magistrale*. The teacher's alibi for putting on this sort of game is to pretend that he is teaching. He actually believes he is and that the students learn by listening to his voice. He believes that his talking can fill the gap in the students' knowledge. Actually, he is the one who is learning the most.

The foundation of this kind of discourse is "*une prétention insensé d'avoir pour production un être pensant, un sujet*" [Lacan, 1991, p. 203]. Indeed, signifiers have rotated one fourth of a lap counter-clockwise (figure 3). The signifier of the castrated subject **$\not{S}$** is now in the place of *production* and exerts the function of *loss*. It points to the students that were lost, either because they gave up the course's credit or because they will never use the acquired knowledge in their future lives, or because they did not actually learn what the institution claims they did.

agent		work
S_2	$\rightarrow$	a
S_1	▲	$\not{S}$
truth		production

Figure 3: the university's discourse

The *petit-**a***, the cause of desire that was the lost production of the student in the master's discourse, is now on the numerator and exerts a demand that puts the agent into action. Now the *petit-**a*** indicates the students' plenary, the big-Other to whom the teacher addresses. "*L'objet a, c'est ce que vous êtes tous, en tant que rangés là*" [Lacan, 1991, p. 207]. The **S_2** must exert its function with respect to desire from the position of the agent.

However, desire is generally the Other's desire. What the teacher desires is what is socially desirable. Hence, it is necessary to go after the teacher's desire on the right of the arrow, in the place of work, from where the Other, in this case the students' assembly, exerts the demand. In this position we now have the object a. As the cause of desire, the a is hidden in a gap of the Other, where the agent will have to look for it. In order to do that, the teacher tries to guess the student's doubts (or questions and motivations) for the knowledge that he wants to introduce. The arrow indicates that it will be impossible to fulfil this guess, but a renewed attempt has always to be made, in order to sustain the university's discourse. The students, on the other hand, look through the blackboard at their future career or at the admittance door to the math department.

Since the a is a lost object, its function of demand can only be exerted if a representative of it reaches the front stage and offers itself as a desirable *façade* of the a. This façade acts as a bung of the Other's gap: "*C'est là le creux, la béance que sans doute viennent d'abord remplir un certain nombre d'objets qui sont, en quelque sorte, adaptés par avant* (...) *ils sont priés de constituer avec leur peau le sujet de la science* (...)" [Lacan, 1991, p. 121]. The bung of the gap, the *façade* of the *petit-a*, is the smart faces of affirmatively nodding students.

In order to sustain the S_2 in the position of the agent, it is necessary to pretend that what the teacher says is true, not because *magister dixit*, but because of a logical reasoning stemming from solid epistemological foundations. The magisterial authority (S_1) must be repressed under the stage where he stands and must never be evoked as an argument. It is necessary to make believe that what is hidden under the stage is the deep secret of knowledge. It is the a-students who actually work, and as workers they have to produce something. The barrier in the denominator indicates that there is no possible contact between the secret of knowledge and the castrated subject. Such a contact would short-circuit the whole scheme. "*Comme sujet, dans sa production, il ($\$$) n'est pas question qu'il puisse s'apercevoir un seul instant comme maître du savoir*" [ibid. p. 203]. The university discourse is the main expedient used by the State, the Law and the Ideology in order to assure that the labour work-contract is made between equal parts that meet freely in the market.

The object's discourse

In general, at the end of the course there is a final exam. Students are expected to review the course material and organise their ideas in order to show a certain performance, the so-called "mathematical ability". However the exam invariably leads to credit and certificate. The students work in order to prepare themselves for the exam. However, the work is now done by the ignorant subject ($\$$) who is at home, at the student's house or in the library as the subject who tries to understand the course material. Consideration of classrooms in third-world countries shows that, for many students, the acquisition of the necessary knowledge is an impossible strategy to get credit [Baldino, 1997]. They lack the adequate background and study habits. They resort to several *ad hoc* strategies among which rote learning is the most widespread [Cabral, 1992, 1998]. Vinner refers to such strategies as "pseudo-conceptual" and "pseudo-analytical modes of thinking" [Vinner, 1997, p. 1-70]. The object of desire (a) is the credit system, now in the position of the agent. True knowledge (S_2) is repressed and provides an alibi for the credit system to operate. The result is a credit certificate, a meaningless sheet of paper stamped "*passed*" (S_1) without which the student cannot move into the next course. The certificate is just as

void as a king's signet, a mark without which the profession cannot be practised. The arrow indicates that no pseudo-learning strategy can assure that the student will pass. The black triangle indicates that a direct connection between the certificate (**S_1**) and the true knowledge (**S_2**) would invalidate the whole pseudo-learning strategy and substitute true knowledge for rote learning (figure 4).

agent		work
a	$\rightarrow$	$\cancel{S}$
S_2	$\blacktriangle$	S_1
truth		production

Figure 4: The object's discourse

The hysteric's discourse

It is evident that the school apparatus cannot function only on the basis of either the university discourse or this kind of object's discourse, since the production of such discourses are respectively the cultural eunuch and a piece of paper. Somewhere else in the university some other kind of discourse must be happening. "*The moment one knows the difference between analytical and pseudo-analytical he or she can reflect about their thought processes, abandon the <u>pseudo-analytical</u> and follow the <u>true-analytical</u>. I say abandon the <u>pseudo-analytical</u> because usually the pseudo-analytical comes first*" [Vinner, 1997, p. 1-74].

Indeed, the *scéance magistrale* is followed by the class of *travaux dirigés*. Later in the exam the students will be asked to reproduce this sort of work, without the help of the teacher. The students are now expected to ask *What is this? Why does it work? What is the ultimate reason for it?* They are expected to address these questions not to the teacher but directly to the organisation of mathematical knowledge expressed in the books. They are expected to reduce complex developments to symbols of their own. In the *scéance magistrale* the teacher has showed them that he can do it, hence this is possible. They are told: *Don't you see that now it is your desire that moves the process? What do you want from school? Your production should now be your own comprehension*. This means that now each student has to work ($\cancel{S}$) in order to supply his/her ignorance with respect to a small piece of mathematical knowledge (a) generally in the form of a problem to be solved or a proof to be understood. Placing oneself in the position of the one who does not know ($\cancel{S}$) is the fundamental condition for learning. The condition of possibility for this to happen is that this piece of knowledge becomes a representative of his/her private object of desire. In their effort, the students are comforted by the idea that there is an ultimate reason for each mathematical result to be true, since it fits into the organisation of mathematical knowledge (**S_2**).

The student knows that the exam questions will not be mere application of ready formulas and that some kind of extra difficulty will be involved. S/he will have to resort to the history of his own knowledge acquisition to be found in the under-pressed **S_2**. The effort is paid off by the *Aha! I got it!* This "aha" (**S_1**) means nothing, except that something has died in the subject. Indeed, an old (ignorant) subject has died and another one who dominates this piece of knowledge, has been born. Comprehension has no weight. The arrow means the impossibility of fully supplying the subjects demand for understanding. Some residue is always left. The triangle means the isolation: if there would be a passage between **S_1** and **S_2** in these positions, there would be no possibility of a *aha*, since **S_2** is already complete (figure 5).

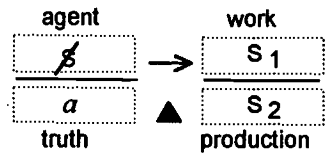

Figure 5: the hysteric's discourse

However, things do not generally happen as expected. The day before the exam the student is at home reviewing the course's subject-matter. Since s/he lost a long time as a good *a*-student, studying the *a*-theory for the exam of another course, s/he finds out that there will not be enough time, neither to do the exercises and understand the proofs nor to accomplish any fast rote learning. Many obscure points remain. S/he dangles completely lost between the notes in the binder and the textbooks. *Somebody please tell me what to do*; s/he asks herself. At mid-night s/he is tired. At this moment the exam loses its meaning, either as a warranty of knowledge for the future courses or as a means to pass this course. The Other's demand becomes concentrated on a single point. The student simply hopes to be lucky the next day. The exam becomes a pure S_1 that goes to the position of demand of the other in the form of a school's "*honour's code*", reminding the student that there are less honourable strategies.

The cause of desire (***a***) sustains the agent as the castrated subject whose truth, not to be revealed at any price, is that s/he is a distressed student ($\$$) who has an exam next day and to whom the institution did not provide enough time and conditions to learn. S/he should not be blamed for such a failure. The final production of this discourse is the last sprout of the traditional school credit system, the inclusion of *cheating know-how* ($\mathbf{S_2}$) among the strategies to get credit. The cheating document is going to be thrown away after the exam, eliminated as if it had never existed. The arrow indicates the impossibility of the above-depicted student to satisfy the divergent school's demand, no matter how hard s/he works. The triangle means that the truth of the hysteric cannot face the know-how that constitutes it as a truth. Knowing how to pass without learning, cannot be admitted as legitimate.

The four discourses in the SAG classroom

It is possible that what we are calling traditional teaching cannot be fully found anywhere. Each classroom has some traces of it. For the sake of the exposition we have assembled all these traces in a single exaggerated cartoon and have labelled it *traditional teaching*. It consists in a sequence of four moments. Briefly, an inaugural moment, based on plain authority (the course's introduction), a second moment dominated by a verbal flux (the *scéance magistrale*), a third moment centred on the credit system, and a final moment where the student only hopes for luck in the exam.

If we want the students to learn anything beyond learning how to get credit, we must do something different from the traditional teaching described above. We propose SAG as one among possibly many alternatives to face the difficulties pointed out by the preceding analysis of the school apparatus. It can certainly be said that such an analysis was biased. Which one is not? We explicitly admit that differently biased analysis would lead to other propositions.

In the alternative didactical and pedagogical SAG proposition, the four moments alternate in the reversed order: we go from the master's discourse to the hysteric's discourse, then to the object's discourse and end with the university's discourse. In addition, some roles that support such discourses will be reversed.

At the first moment of a typical SAG classroom, the master's discourse is present. However, the teacher is the one who knows ($\mathbf{S_2}$). She is informed by everything that we have just written. She knows that the students want to get credit and she resists co-operation. However, she does not display this knowledge. She works to organise the classroom in such a way as to frustrate the (re)production of the credit system distortions. A work contract explicitly including all rules to get credit is introduced and tried for a couple of weeks, before it is put to a vote. The contract is based on some non-negotiable principles. As a characteristic of SAG it includes specific rules to get credit for classroom group work, for collective classroom organisation and for tutorial sessions [see Baldino, 1997]. Insofar as the students are not used to similar work contracts, they show some degree of astonishment. However, they generally end up accepting the contract and abiding by the teacher's proposed classroom organisation. In this way they preserve their position as students (S_1). As the ones who are there to learn, students are ignorant by definition. This ignorance includes ignorance of classroom organisation. Such ignorance supports their consent (**S**). The production of this discourse is a certain classroom organisation (***a***). It may take a few weeks to get it going (Figure 6).

agent		work
$\dfrac{S_1\text{student}}{\cancel{S}}$	$\xrightarrow{\quad}$ ▲	$\dfrac{S_2\text{teacher}}{a}$
truth		production

Figure 6: The master's discourse in the SAG classroom

Next, the teacher distributes worksheets to the groups of students or assigns them a specific task from the textbook. The students naturally ask what they are supposed to do. Here is a typical dialogue:

Student: *What do you want me to do?*
Teacher: *Look at the task on the blackboard.*
Student (later): *May I do it this way?*
Teacher: *If it is right, you may; if it is wrong you may not.*
Student: *Is it right?*
Teacher: *Check with your peers.*

The teacher deliberately, but only to a certain extent, refuses to assume the function of the supporter of mathematical truth. She stands and observes the class, trying to decide which group she is going to visit next or which general instruction or advice she should write on the blackboard. Since the voted contract is prevailing, the student identifies the watching teacher's eye as a kind of demand ($\mathbf{S_1}$) to him/her from some point of a picture impossible to fully apprehend (**S**). The students know that credit is at stake in this gaze. However, they also know that the condition to get credit is not displaying mathematical ability. Instead, they are expected to *work-to-learn*, according to the established group work rules. These rules include listening to their peers and being able to explain how they have solved each problem, until a consensus is reached or a clear divergence is established. Rules to get credit for group work explicitly exclude right/wrong mathematical criteria. In order to sustain such a situation, mathematical knowledge has to appear as the hidden object of desire (***a***). The production of this discourse is the collective elaboration of the solution of the assigned exercises or questions. No possible action can completely free the student from the staring of the teacher ($\rightarrow$). No contact between the production and

mathematical truth is possible (▲); *credit is for working to learn and understand*, not for getting the right solution (Figure 7).

$$\begin{array}{ccc} \text{agent} & & \text{work} \\ \$\ \text{student} & \longrightarrow & S_1 \text{ the eye} \\ \hline a & \blacktriangle & S_2 \\ \text{truth} & & \text{production} \end{array}$$

Figure 7: the hysteric's discourse in the SAG classroom

At the end of the class, each group hands in a common worksheet with the result of their work. In the next class, the teacher gives these worksheets back to the groups, with added remarks to be further worked out. Mistakes are pointed out, but they do not weigh negatively on credit. The first task of the group is to correct the mistakes and answer to the teacher's specific remarks. The ignorant student (**$\$$**) who inquires into the mathematical objects (***a***) settles the demand.

Mathematical knowledge (**S_2**) is present and assures an underlying function of coherence indicating that this demand makes sense and may in principle be satisfied. The production is the student's exclamation: *Aha, I got it* (**S_1**). Of course, this situation is a mere hope. It never happens completely, as we would like it to happen (→). The student's "*Aha*" only marks the beginning of a necessary explanation and is not worth it (▲) (Figure 8).

$$\begin{array}{ccc} \text{agent} & & \text{work} \\ a \text{ mathematics} & \longrightarrow & \$\ \text{student} \\ \hline S_2 & \blacktriangle & S_1 \\ \text{truth} & & \text{production} \end{array}$$

Figure 8: the object's discourse in the SAG classroom

Finally the class is organised in such a way that students go to the blackboard to present the solutions of their groups to the students' assembly or to the peers who attend the tutorial meeting. This is certainly a typical university's discourse, but contrary to the situation described in the traditional teaching, now the speaker is the student (**S_2**). The demand is put forth by the audience (***a***), as before, but it is does not consist of affirmatively nodding students. The objective of the meeting is to give back to the student at the blackboard a reading of what s/he is saying. Students are instructed to watch for points where the comprehension of the colleague at the blackboard may look frail and to convey their remarks, not through statements of sapience addressed to the teacher, but through questions addressed to the student at the blackboard until s/he realises his/her weak point was. The teacher's authority assures the co-ordination of the process (**S_1**). The production is the castrated subject (**$\$$**) insofar as the students in the audience recognise their difficulties in the mistakes of the student who is at the blackboard (figure 9)

$$\begin{array}{ccc} \text{agent} & & \text{work} \\ S_2 \text{ student} & \longrightarrow & a \text{ plenary} \\ \hline S_1 & \blacktriangle & \$ \\ \text{truth} & & \text{production} \end{array}$$

Figure 9: the university's discourse in the SAG classroom

SAG's daydream is the following. If classroom group work is well done, if the students have the necessary pre-requisites for the course and if the assigned tasks are adequate, there should be no need for exams. Students may get credit only on the basis of group work assessment. This is not because SAG assures that the students will learn as much as the institution requires them to learn. It is because we will have learned as much as possible, since we will have engaged them in the tasks best suited for their learning and given them the best possible assistance. In reality, we may also say there is not such a classroom fully organised according to SAG. SAG is a principle to be followed: *credit for group-work, not for mathematical ability*; in one word, for the ethics of work in the classroom [Baldino, 1998]. In practice the work contracts have established a percentage (up to 30%) for group work assessment. This principle is necessary to bring and sustain the object's discourse in the classroom. This is not as easy as is supposed in Vinner [1997]: "*The moment one knows the difference between analytical and pseudo-analytical, he or she can reflect about their thought processes, abandon the pseudo-analytical and follow the true-analytical. I say abandon the pseudo-analytical because usually the pseudo-analytical comes first*" [Vinner, 1997, p. 1-74].

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ACTION RESEARCH: COMMITMENT TO CHANGE, PERSONAL IDENTITY AND MEMORY

Roberto Ribeiro Baldino

Antônio Carlos Carrera de Souza

Action Research Group in Mathematics Education, UNESP, Rio Claro, SP, Brazil

ABSTRACT

This paper reports on the activities of a five-year action research group whose goal is to develop a speech community committed to change the failure of mathematics teaching and the classroom routines that support it. Our premise is that accepting or challenging the stories told by newcomers to the community about their classroom experiences will interfere with the formation of their professional identities by preserving some of their recollections in the community's memory and dismissing others. Ms. Daniels, a personage of Borko et al [1992] is brought to the fore in order to provide a guiding thread for the discussion.

Introduction: change, commitment and identity

Taking into account the public concern and amount of investments made in mathematics instruction around the world, one conclusion is inevitable: mathematics teaching is a human activity haunted by failure. If we hope to banish the ghost of failure, *change* becomes necessary. In PME22, a Plenary Panel Discussion was dedicated to it. There are many meanings of this word. In some cases, *change* means to look for a "linguistically and culturally sensitive learning environment" [Khisty, 1998:101]. In other cases *change* means to concentrate on curriculum issues [Jaworski, 1998, Pence, 1995], or to introduce new technologies [Crawford, 1997], or to *change* the classroom norms and management [Cobb&Yackel, 1995, Tomazos, 1997]. It may also mean to *change* teachers' knowledge and practices [Simon & Tzur, 1997], or to *change* teachers' beliefs [Becker & Pence, 1996]. Still in other cases, *change* means to focus on ethnical issues or gender inequity [Breen, 1998] or, simply to improve learning and teaching [Konrad et al, 1998].

However, if the majority of the projects for *change* succeeded in effectively improving the learning of mathematics, there is no guarantee that the minimum standards would not rise automatically, so that the ghost of failure would still be around. If we look at the hundreds of existing teacher formation programs, it seems that all efforts to produce a *commitment to change* as an output also fail. What happens to the student teachers the "day after"? If the program closely follows them as they enter the school system, success can be reported [Shane, 1997]. However, as soon as the effects of the teacher formation program cease, or even before that, there has been dramatic evidence that teachers conceptions and practices are rapidly absorbed by the dominant traditional school ideology, and *change* is invalidated. [Borko et al, 1992, Ensor 1998, Schmidt & Duncan, 1998].

We shall concentrate on the case of Ms. Daniels, who "approaching the end of her student teaching, was still unable to provide a clear explanation of division of fractions" [Borko et al, 1992:209]. When a student, Elise, asked her why to invert the second fraction, she erroneously picked up the diagram for multiplication and got stuck. The authors are perplexed because Ms. Daniels "did not attempt to correct the representation the following day" [ibid. 198] and because she "did not learn the conceptual information and representations that she needed to produce an adequate explanation of division of fractions during the mathematics methods course" [ibid. 218]. She had to compensate her failure in answering Elise's question by drilling the inverse algorithm. "However, despite her realization that 'the explanation wasn't very good', she was basically pleased with the lesson" [ibid. 198].

How strongly can teacher formation programs impose commitments to change on the student teachers? If "teacher preparation programs respond more quickly to calls for reform than school classrooms, perspective teachers' field experiences are inconsistent with the expectations developed in their teacher education coursework" [Van Zoest, 1998:354]. Brown [1998] reports the case of a student, L, who revolted against the teaching methods. This case had a happy ending, since L reevaluated her written stories and reached "the calm, almost detached from the former state, knowingly living in the new realm where the new brand of stories are seen as fitting better" [ibid.]. However, can't we say the same of Ms. Daniels? As a result of her strategy she *got credit* [Vinner, 1997], became a teacher and went on living in school, where her brand of stories are seen as fitting better. Can the constraints of a credit-based system impose emancipatory commitments on the human subjects who constitute its output? Was this the contradiction that L revolted against?

However, suppose that all goes well and we finally obtain a teacher culturally committed to change. Here is an excerpt of a dramatic report.

"Keiko's philosophy of education was developed in Japanese culture and her (...) beliefs about mathematics mesh with the goals of NCTM (1989, 1990 & 1995) but not with the traditional classroom and ways of teaching mathematics in American Schools. (...) By the end of her two week field placement, Keiko concluded that she would not teach in public schools because the cultural and societal differences were too great; they demanded that she give up her *identity*" [Schmidt & Duncan, 1998:310-311, our emphasis].

The question is, should we help Keiko to abandon her *identity* and become a happy, accepted citizen of her new cultural environment? Or are we politically committed to change the school she taught in? Commitment is a consequence of the subject's personal *identity* as a social human being. Arguing from the analysis of a film (Bladerunner) Zizek [1993] has produced evidence that our identity depends on our memories.

"Stories are precious, indispensable. Everyone must have his history, her narrative. You do not know who you are until you possess the imaginative version of yourself. You almost do not exist without it" [Time magazine, quoted by Zizek, 1993 in Brown, 1998].

Brown [1998] asks, how do we build a sense of our own identity through the memories we hold on to? His answer is that our memories "are constructed through

our own particular understanding of the path we follow through the passage of time" [ibid. 2] and then he connects this "with the explicit task of pinning down bits of experience faced by teachers carrying out reflective practitioner research" [ibid. 2]. He finally suggests that the student teachers must find a space where their personal stories about their teaching practices can be told and will be listened to, so that these practices can be reevaluated and finally cast as part of their new teacher's personal identity.

Borko et all's [1992] report on the case of Ms. Daniels is an example where the control of the credit system did not work as expected. The authors are puzzled: why did she not investigate the topic later? For Ensor [1998] there is no puzzle at all; just "a human subject inserted into a range of different contexts, each of which defines competence differently" [286].

From this point of view, Ms. Daniels behavior is coherently human: she told the researcher interviewers what they would like to hear; she reproduced the instructor's demonstration in the exam, and she kept the class under control by drilling the inversion of the second fraction algorithm. One discourse for each occasion. This has nothing to do with faking and gulling but with typically honest human behavior. According to Lacan, we can say that a being is properly human if she fails to abide by what she honestly promises to herself. Such a mismatch is what he calls *desire*. I can't avoid recalling a tale by a famous Brazilian writer from the last Century[1]: a girl fell in love with a sailor and promised to wait for him. When he came back from his six-month trip, she was married to another guy. – *But you swore that you would wait for me*, protested the desolate sailor. The girl tried to comfort him. – *Of course, I did, but please, understand, when I swore it was true...*

This tale refers to a commitment to permanence that meant change. It elicits that the reported cases, with the exception of L's, refer to commitments to change that turned into permanence. We seem to know very well that school does not do what it promises. What we do *not* know is how it reacts to change, especially the change that compels it to do what it promises. Is the discourse for change covering up a desire for permanence? In one word, we have touched the dialectics of *change/permanence*.

Methodology: action research with self-regulated differential intervention

In 1993, guided by reflections like these, we endeavored to simultaneously challenge and produce conditions favorable to student teachers' commitment to change[2]. Somehow we had realized what we can now state clearly: commitment is a consequence of human identity; such an identity develops from recollections of memories; and such recollections are anchored in discourses involving the subject and a context. We also realized that the discourse starts with a demand from the context that precedes utterances and writing: "The discourse is the norm of what fits and what does not fit into the Other's ears, and consequently what can and what cannot be said

[1] Machado de Assis.
[2] At UNESP, Rio Claro, SP, Brazil.

by the speaker" [Baldino & Cabral, 1998:58]. Hence, the context is productive of discourses:

"(...) in each case the produced text is evoked within a particular context, by a specific invitation to speak. In this sense the *contexts are productive*. At the same time they are constraining insofar as each context, with its audience, both canalizes and silences expression" [Ensor, 1998:283 emphasis added].

So we succeeded in creating a *speech community* whose identity is now supported by a five-year history. The stories told by the newcomers about their classroom experiences are listened to and discussed from the point of view of the understanding of the community members. Our premise is that accepting or challenging such discourses interferes with the process of formation of the student teachers' professional identities by preserving some of their recollections in the community's memory and dismissing others. What is at stake is the feeling of belonging to this community or not. The objective is not to graft commitment onto this or that student, but to provide an identity for the community of students and in-service teachers committed to change.

Of course, this community has to be based in the university but should not be subjected to bureaucracy so as to keep its critical stand with respect to both school and academy. In particular it has to be free from any obligations to the credit system. Participation should be only on a volunteer basis. The task of constructing this community clearly required an action research method:

"Action research is small-scale intervention in the functioning of the real world and a close examination of the effects of such intervention (...) the ultimate objective being improvement of practice in some way or other" [Cohen & Manion, 1994:186].

Our project is both decision and conclusion oriented. Our two research questions are: 1) how to face the general failure of mathematics teaching at all levels? 2) What are the school and classroom routines that sustain this failure? There are no formalities for admittance; everybody who is touched by these questions are welcome to our Saturday-morning meetings. At the beginning of each semester we split into several subgroups according to the interests of participants. Each sub-group defines a project connected to the research questions involving some kind of classroom action. Projects may last for one semester or for several years. Every Saturday all projects report their weekly progress and get advice from the other community members. Themes have varied from children's songs and stories to the teaching of analysis, from games for integers to ecology and garbage collection. Evaluation of these projects consists of submitting their results to specialized reviewers. There are almost one hundred publications, including dissertations and papers in specialized journals and conference proceedings.

During these years we have counted on an average of 25 people in these meetings: professional teachers, graduate students in mathematics education, and undergraduate student teachers who are simultaneously taking courses in mathematics methods and pedagogy. Most of the classroom interventions employ the technique of *differential intervention*: once in the classroom, either as a regular, a

temporary or experimental placement, the teacher or teaching team does not do exactly what is expected from him/her/them but introduces some *change* that makes the school system uneasy and against which tradition does not find any ready available argument: introduction of group work, new topics, new instructional material, new classroom norms, new forms of evaluation, etc. We do not go to observe the student teachers' classrooms. Data is collected from the reactions of the school system to such changes as reported by the student teachers who decide what *change* should be made. We say that the *differential intervention* is *self-regulated.*

Research report and discussion

One way to report on the five-year production of this action research group would be to select one group and report fully on it. Another way would be to report on a hypothetical "average" group. We shall follow the second way, using an expedient: we shall suppose that Ms. Daniels has joined us in the beginning of this semester. In this way we will be able to accurately describe what has actually happened in most of our sub-groups. In what follows, many of Ms. Daniels speeches are copied or adapted from Borko [1992]. We shall omit the references.

In the first Saturday plenary meeting, Ms. Daniels declared her interest in fractions and was suggested to join the existing sub-group of rational numbers coordinated by a Mathematics Education graduate student [Izzi, 1998]. However, she managed to convince some of the participants about the importance of dedicating one semester specifically to operations with fractions. The newly-formed subgroup found a tutor among the university teachers and graduate students. They established the time-schedule for their weekly meetings and had no trouble finding a school where they could carry out some experiments. In fact the cooperating teacher was very pleased: – "I can relax for a few weeks", she said, meaning that this was her only motivation. The following Saturday the group reported their plans to the plenary. In the first sub-group meeting Ms. Daniels conveyed confidence on her mathematical knowledge:

– I already know all about rational numbers; I have successfully completed over two years of course work as a mathematics major. I only need some techniques that will hold the students' attention, some ideas that will work.

The tutor wrote on the blackboard one over ½ and asked Ms. Daniels to complete it. She wrote "2" and asked surprised:

– What do you mean? Do you want me to describe how I would teach the topic to a sixth-grade class?

– No. I want you to tell me why *you* have done that. Why did *you* invert the bottom fraction? It is you, not your student who is on the spot here.

Ms. Daniels tried several explanations, but none stood up to the tutor's inquiry. The other elements of the subgroup were called to help her, but nobody succeeded. The group became united around this mathematical difficulty. Ms. Daniels finally confessed in a low voice: – "I don't know why you invert and multiply; I just know that's the rule" [Borko et al, 207]. Her faith on her knowledge was finally broken. This is

the moment when Lacan says that the hatch is opened. It is the only moment when the student is receptive to information that the tutor can pass through the opening before it closes again. The tutor conducted the dialogue through the following questions, eventually supported by diagrams and cubes:

– If you give one sausage to each half-dog, how many sausages does one dog eat? And if you give one sausage to each 5/8 of a dog, how many sausages will *five* dogs eat? And how many sausages per dog? As soon as Ms. Daniels produced the answer "eight sausages to five dogs make 8/5 of sausage per dog", the tutor remarked: – How come you said you did not know why you invert. You have produced the answer yourself...

The group was amazed. Other examples were tried. Some students started explaining to those who had not yet understood. The tutor intervened:

– No explanations, please! You may only ask questions until they get what you want them to understand.

Finally everybody got the idea. The session was over. The next Saturday they reported their intention to take the subject into the sixth-grade class as soon as possible. In the next sub-group meetings they planned how they would organize the classroom for group work, how to establish a *work contract* precisely indicating how grades would be assigned to participation in group-work[3]. They drew cartoons with figures of dogs whose bodies could be stretched by inserting cards[4] and discussed how they would move from this material to the conventional written algorithm. When all was ready, one of the plenary sessions was dedicated to testing the instructional material with all the community members. The group was urged to write a report about their classroom experience. They reported that the most disruptive students were the ones who got the idea first and acted as tutors to the other groups. They decided also to report on the detached and sometimes deleterious attitude of the cooperating teacher. Their diaries looked very much like the ones reported in Brown [1997, Ch. 7]. Their individual written reports were discussed in the sub-group meetings. The tutor's remarks about these reports *were also written* and subjected to the sub-group members for further discussion. Results of these discussions were regularly reported and debated in the Saturday plenary sessions. The paper finally produced was considered to be of joint authorship, like the papers stemming from other sub-groups: Souza et al [1995], Leal et al [1996], Baldino et al, [1997]. Here is a typical discussion of a plenary session.

Ms. Daniels. – I wonder why the mathematics methods instructor never mentioned this to us.

Participant. – He could not. He insisted on the measurement interpretation of division of fractions and "there is no direct or concrete way to demonstrate using manipulative, the derivation of this algorithm" [Borko et al, 1992:214].

[3] This is important, given the degree of deterioration of public teaching in the State of São Paulo.

[4] This material was actually developed in Centro de Ciências, FAPERJ, Rio de Janeiro in 1983 and was named "Sispixa" ("Stretchy").

Ms. Daniels. – Why didn't he shift to the distribution representation like we did? Why did he insist on his representation knowing that it could not lead to an idea to use in the classroom?

Participant: He had no need for this shift. He was looking for "ways to represent mathematics concepts and procedures" [NCTM, 1991:151 in Borko et al, 1992]. He was not trying to build knowledge from a dialogue situation with you. He was merely trying to "explain that the derivation demonstrated that the algorithm produced a correct answer" [Borko et al, 1992:214]. Since he believes that knowledge has to be "represented", he naturally assumed the function of introducing this representation to a large audience. He talked, he explained, he demonstrated; students should have "followed" him. In the last PME Ron Tzur showed nicely how a teacher's epistemological conceptions determined the way he conducted his class [Tzur & Kinzel, 1998].

Ms. Daniels: Doesn't he realize that the theory he gives us is cut and dry and that most of us do not follow it?

Participant. For this he counts on the audience to which he is the *introducer of the representation* of knowledge. "(...) the bright ones who are good in math, will have a pretty good understanding of what's happening here. The rest of them I just have to take it on faith" [instructor's lesson in Borko et al, 1992:214]. What he was saying to you, Ms. Daniels, is that you are not included among the bright ones and that you will have to take it on faith. This is a very subtle way of blaming the victim. Apparently your instructor himself would not have been able to answer Elise's question better than you did. Your interviewers do not even believe that a concrete derivation of the inversion algorithm is possible. Yet you are charged because "you did not seem to feel that it was (your) responsibility to actively seek to improve (your) understanding of the mathematics (you) were teaching either by consulting resources or by engaging in hard thinking of (your) own" [Borko et al, 219]. What they have actually taught you is to set a good display of representations of knowledge and to charge those who do not follow you for not being bright enough. This is very different from the dialogue situation that we have here.

Ms. Daniels (exasperated): – This is a situation that I am certainly committed to change.

Participant (casually): Why? Why do you want to change it?

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KAREN AND BENNY: DÉJÀ VU IN RESEARCH

Annette R Baturo, Tom J. Cooper, Campbell J McRobbie
Centre for Mathematics and Science Education
QUT, Brisbane, Australia

An integrated learning system (ILS) is a computer-based tutoring program that provides students with learning experiences in many disciplines across many years of school. This paper reports on the fraction knowledge of Karen, a Year 6 student using an ILS to remediate her fraction knowledge, and compares her resulting knowledge constructions with those of Benny (Erlwanger, 1973), a student using an Individually Prescribed Instructional (IPI) program in the 1970s. Karen was interviewed on tasks involving the common and decimal fractions and the results showed that she was able to progress on the ILS with an impoverished understanding of fractions, a phenomenon that echoes the earlier findings with respect to Benny. This paper discusses reasons for this phenomenon as well as the propensity of learning systems (computer or paper) to focus on syntactical and instrumental understanding (Skemp, 1978).

One of the by-products of the growth of information technology in education has been the computer-based integrated learning system (ILS) which includes extensive courseware plus management software. An ILS has three essential components, namely, *substantial course content*, *aggregated learner record system*, and a *management system* which tracks learners' task responses and progress, and provides performance feedback to the learner and teacher (Underwood, Cavendish, Dowling, Fogelman, & Lawson, 1996, p. 33). An ILS marginalises the teacher's role and virtually removes students' initiative and autonomy in the system's learning process (Bottino & Furinghetti, 1996).

This paper reports on diagnostic interviews undertaken with Karen, one of several Year 6 students who were using an ILS for remediation purposes. Karen was singled out because the ILS system revealed that she had made the most "gains" (about 18 months) in mathematics in a 5-month period. The interview probed her structural knowledge of fractions to determine the thinking strategies she employed when processing fraction concepts. The paper provides the results of Karen's interview and relates her responses to her performance on the ILS and to her beliefs about the ILS. These results are then compared with those of Benny (Erlwanger, 1973), a student working on an IPI program in the USA a quarter of a century ago.

Individual Learning Programs

The ILS used by the students was a comprehensive instructional system powerful enough to deliver complex courses. According to the manufacturer, its courses were designed to foster the development of foundation skills and concepts and to promote the use of higher-order thinking skills. It should be noted that the manufacturers endorse the system only as a tool for teachers to use to consolidate already introduced material and to diagnose student difficulties. They argue that it is the teachers' role

to introduce the material to be practised on the ILS, and to remediate the difficulties identified by the ILS. Thus, they contend that the effectiveness of the ILS depends on the quality of teacher input and that any evaluation of the ILS should take into account the role of the teacher in relation to the program.

The ILS in this study was a closed system, that is, the curriculum content and the learning sequences were not designed to be changed or added to by either the tutor or the learner (Underwood et al., 1996). Its major feature was its management system which, according to the manufacturer, has three main functions: (a) to deliver courses to each student according to the teacher's instruction; (b) to manage all student enrolment and performance data; and (c) to provide the means for teachers, laboratory managers, and administrators to organise the use of the courses, and to monitor student progress.

The ILS core numeracy course is divided into a range of *topics* (e.g., numeration, addition, multiplication, fractions, space) which are then sub-divided into collections of tasks that are sequenced in terms of performance at different *levels*. The difference between levels was constructed so that high mastery at one level (approximately 85%) is the same as mastery (above 60%) at the next level. The core numeracy course is based on USA syllabi but correlates reasonably well with Australian syllabus requirements; individual tasks were developed and placed in levels as a result of large-scale trials in the US. For their initial placement on the ILS, students are given a large number of tasks at different levels until the system finds the level at which they have reasonable mastery (about 65-75%). When students achieve high mastery at one level, the system automatically raises them to the next level. To maximise the chance that task performance correctly represents level, the tasks within a level are presented randomly. Any reduction of randomness affects the accuracy of placement and, therefore, the potential for students to achieve mastery. Without mastery, students may not experience the continual success, and therefore the motivation for achievement, that lies at the theoretical heart of the ILS.

The ILS tasks are in the form of electronic worksheets which are generally attractive in their presentation and sometimes creative in the way they probe understanding. They attempt to encourage the construction of knowledge by providing 2-D representations of appropriate teaching materials in mathematics (e.g., Multi-base Arithmetic Blocks, Place Value Charts, fraction and decimal diagrams). Built into the core numeracy course are online student resources that enable students to get special help during a session should the need arise. However, use of the Help and Tutorial icons automatically grades performance as incorrect. The Toolbox icon makes calculators, rulers, tape measures and protractors available for student use and also provides complex tools (e.g., graphing and drawing) for advanced levels.

For some topics and levels, there appears to be insufficient task variety to prevent repetition. Furthermore, some tasks have novel presentation formats which students find difficult to interpret (e.g., spring scales used to determine number size, not object mass). Other tasks require inflexible and/or novel solution formats which result in

students' correct answers being marked incorrect (e.g., failing to type the units digit first in operations) as are responses which differ syntactically from the expected responses even when they represent semantic understanding (e.g., the omission of zero in decimal numbers such as 0.63). There is a tendency for questions to be closed (i.e., "find the right number") and a tendency to base performance on speed (although the teacher can vary the time limits on answers). Time delays (e.g., while an algorithm is completed with pen and paper) can lead to the ILS's defaulting to incorrect. For each level and topic area, there are worksheets that can be printed thus providing students with extra practice and teachers with a guide to the types of activities that need consolidating.

The ILS is designed to work with individual students, a feature which appears to be contrary to modern teaching/learning principles which encourage group work that promote verbal and kinaesthetic interaction. However, students normally work on the ILS in mathematics for a maximum of 45 minutes per week over three sessions. Therefore, the amount of time working alone is small when compared against the total weekly time spent on mathematics (usually 200-225 minutes).

A computer search found only two evaluations of the ILS used by the students in this study. In a review of 32 studies, Becker (1992) concluded that only manufacturer-based studies reported significant gains for the ILS but that these studies, had "used the unusual procedure of eliminating cases from data analysis that showed sharp declines" (p. 30) while retaining cases that showed large gains. In a review of 9 studies, Underwood et al. (1996) found a substantial positive gain for mathematics performance in computation but not for fractions (which is of interest to this study). However, to compare mathematics knowledge gains, both reviewers used standardised tests which do not provide explicit information on students' mathematical knowledge structure and thinking strategies.

Integrated learning systems are reminiscent of the Individually Prescribed Instructional (IPI) packages that proliferated in the US in the 70s with the ILS activities presented in electronic, rather than paper, form. Both systems have a management system which marks students' responses, directs unsuccessful students to other similar activities until "learning" (familiarity?) takes place, and directs successful students to another higher level, and the process is repeated continuously. The only real difference between the two systems is that, in any ILS session, activities cover a variety of mathematical topics whereas in any IPI session, activities are presented in finely detailed sequences within the one topic. The pedagogical flaws in IPI systems were exposed by Erlwanger (1973) when he undertook a series of interviews with a variety of students in an attempt to understand what mathematical knowledge students acquired from individualised instruction and that knowledge was acquired.

One student, Benny, had been perceived by his teachers to be "very good" at mathematics, a perception that had been gained entirely from his rapid rise through the levels of instruction. The interview with Benny revealed that he had constructed several misconceptions that enabled him to accommodate the variety of answers that were demanded by the discourse of the package. The following protocol provides an

instance of Benny's misconception with respect to decimal fractions and his "incorrect generalizations about answers" (Erlwanger, 1973, p. 15). He had previously solved $2 + .8 = 1.0$ (a revealed error pattern) and $2 + {}^{8}/_{10} = 2{}^{8}/_{10}$ and was explaining (unsolicited) how IPI's answer key (the ILS's marking system) would mark him if he interchanged the answers to the particular problems.

Wait. I'll show you something. If I ever had this one (2 + .8) . . . *actually, if I put* $2{}^{8}/_{10}$, *I get it wrong. Now down here, if I had this example* ($2 + {}^{8}/_{10}$), *and I put 1.0, I get it wrong. But really they're the same, no matter what the key says*. *(p. 15)*

Method

Subject: Karen was one of 60 Year 6 students who had been using the ILS for approximately 3 months. She was selected for further interviewing because, according to the system, she had made the largest mathematical knowledge gains (about 18 months) but an initial interview had revealed that her understanding of elementary fraction concepts was impoverished. Furthermore, she liked using the ILS and believed that it was helping her learn.

Instruments: Two diagnostic interviews were undertaken at the end of the study to probe Karen's structural knowledge of fractions. Interview 1 comprised tasks which involved: (a) probing understanding of the basic part/whole notion of fraction, that is, the relationship between fraction name, equal parts, and number of equal parts; (b) translating prototypic area representations (i.e., 10 × 10 grids) of tenths and hundredths to symbols (decimal fractions); and (c) translating tenths and hundredths, written in words to symbols, and vice versa. Interview 2 comprised tasks which involved comparing common and decimal fractions (see Figure 1). These tasks were based on a worksheet related to Level 4 (a level by which most concepts and processes related to fractions had been "taught" by the computer program), and which Karen had already completed successfully (according to the ILS).

In each set, circle the number that has the larger value.

A. $\frac{2}{5}$ $\frac{4}{5}$	D. $\frac{2}{3}$ $\frac{3}{4}$	G. 3.14 3.6	J. $\frac{9}{10}$ $\frac{99}{100}$
B. $\frac{1}{6}$ $\frac{1}{2}$	E. $2\frac{5}{6}$ $3\frac{1}{3}$	H. 3.84 3.7	K. $\frac{4}{10}$ $\frac{12}{100}$
C. $\frac{3}{5}$ $\frac{3}{8}$	F. 4.7 6.2	I. 2.08 2.8	

Figure 1. Comparison tasks used in Interview 2.

Procedure: For each interview, Karen was withdrawn from the classroom and the interviews audiotaped. In Interview 2, she was asked to read the given fractions before solving in order to address one of Erlwanger's (1973) major findings regarding students who were engaged in individually prescribed learning situations, namely, the lack of appropriate vocabulary required for discourse. Karen was asked

to provide an explanation for each selection she made to determine whether her comparisons were based on syntactic or semantic knowledge.

Analysis: The interview results were scored and Karen's responses and explanations were examined for error patterns. Inferences were thus drawn regarding Karen's structural knowledge of fractions.

Results

The pertinent results of Karen's first interview are provided to illustrate the impoverished fraction concepts and language that she brought to the second interview (and on which her progress through the ILS was based). Karen had a sound understanding of the need for equal parts when determining fractions but not of the relationship between the fraction name and the number of equal parts. For example, when asked why she had not selected the shape showing 3 equal parts (when identifying halves), Karen said she wasn't sure.

For the pictorial representations of tenths and hundredths, she wrote 2.10 for 2 parts out of 10 equal parts, 74.100, 40.50, and 4.100 (see Figure 2). With the exception of 40 hundredths, she was able to unitise the representations but was confused with decimal and common fraction recording forms. The representation of 40 hundredths seemed to invoke a part/part ratio notion instead of a part/whole fraction notion, indicating that she may have developed syntactic knowledge developed by overuse of prototypic representations.

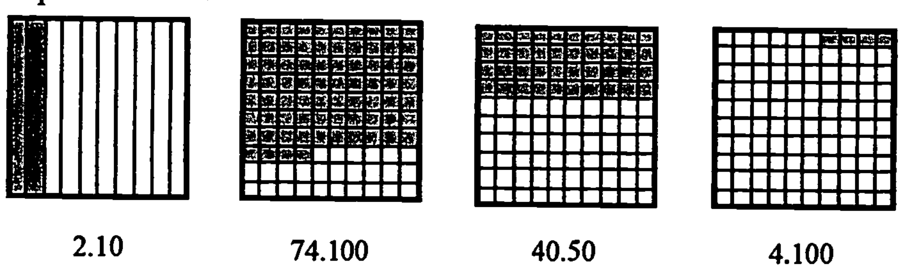

Figure 2. Karen's responses in Interview 1.

Karen was able to read the given decimal numbers syntactically only (i.e., as "6 point three nine", for example, instead of "6 and 39 hundredths"). For the remaining task (writing numbers given in word form in digit form), Karen gave the following responses: *eight tenths* (08.10); *four and six hundredths* (4.600); *eight and nine thousandths* (8.9000); *five and thirty thousandths* (5.30 000); *47 thousandths* (47.000 000). Thus underlying Karen's 2nd interview (and her progress on the ILS) was incomplete or unconnected fraction knowledge of the notion of equal parts and number of parts, confusion between common and decimal fraction recording forms, and an inability to read or write decimal numbers.

Interview 2. Karen read the common fractions semantically (i.e., so the fraction name could be heard, for example, 2 fifths. 1 sixth) but read the decimal numbers syntactically as the following protocol shows for 4.7: *4 point 7* [4 and 7 what?] *4 and 7 um . . . I've forgotten.*

Karen misinterpreted Task A, reading it as *5, take away 4 minus 5, take away 2.* When focused on the task, she selected $^4/_5$ *because 4 is bigger than 2.* In Task B, she selected $^1/_6$ *because 6 is bigger than 2.* When asked to show how she could prove this, Karen drew the following shown in Figure 3 and said that *there's more of them* (sixths). Karen also drew similar representations for $^2/_3$ and ¾.

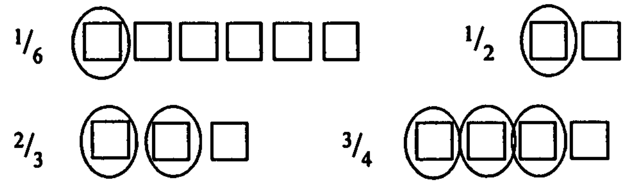

Figure 3. Karen's representations of fractions.

In Task C, Karen selected $^3/_5$ and then said she thought she'd marked the wrong one *because there's 8 on the bottom and 3 of them* ($^3/_8$) *and there's only 3 on the bottom here* ($^3/_5$) *and the same 3.*

Changing her mind turned out to be a common occurrence as shown by the following protocols. In Task D, she had correctly selected ¾, explaining that: *You've got 4 wholes and you've got a remainder 3* ($^3/_4$); *over here* ($^2/_3$), *you've only got 3 and a remainder of 2.* [So, how do you know one is bigger than the other in that case? Which part do you look at – that part (numerator) or that part (denominator) when you compare?] *That* (numerator) *but I also look down here* (denominator). *Um, I think it might be that one* ($^2/_3$) *now.* [Tell me why you think that.] *Well, there's 3* (drawing 3 small squares and circling 2 of them – see Figure 2) *and there's 4 and 3 of them* (drawing 4 small squares and circling 3 of them). [Well, there's the same amount left over (1 square). That makes it tricky doesn't it?] *Mm.* [So how do you decide if they've got the same amount left over?] (No response) [Are they the same value?] *They could be.* [You're not too sure about this one, are you?] *No.*

In Task E, Karen selected $2^5/_6$ as having a larger value than $3^1/_3$ because she said she just looked at the fraction parts. Similarly, in Task F, she initially selected 4.7 *because 7 is bigger than 2* (in 6.2). Unsolicited, she then went on to say *but then again, I was wondering about these two* (the whole-number parts). [So you can't make up your mind, then, whether you should look at these two (whole number parts) or these two (decimal-fraction parts). Is that what you're saying?] *Yeah, so I just chose one.*

For Task G (3.14; 3.6), she selected 3.14 but when asked why, she said, *Um, I think that one* (3.6) *should be the answer.* [Why?] *Because 3 point 6, um . . . I can't remember.* [You've done them before but you can't remember – is that it?] *Yes.*

Task H (3.85, 3.7) revealed the impact of prototypic tasks. Karen selected 3.7 *because that's* (3.7) *only got 1 number after it* (the 3), *and that's* (3.84) *got 2 numbers after it. I've had these on* – (ILS) *and there's usually only 1 number after.* This inappropriate strategy enabled her to make the correct selection in Task I (2.08, 2.8).

For Task J ($^{9}/_{10}$; $^{99}/_{100}$), Karen eventually selected $^{99}/_{100}$ *because there's only 1 more* (difference between numerator and denominator in each fraction)*so they're the same number. But this* (the denominator, 100) *is bigger than that* (the denominator, 10) *and this* (the numerator, 99) *is bigger than that* (the numerator, 9) so . . . [So you had a bit of a problem deciding then?] *Yep.* Karen used similar comparisons of numerators and denominators to make the correct selection in Task K ($^{4}/_{10}$, $^{12}/_{100}$). For future interviews involving comparison of fractions, a task such as comparing $^{8}/_{10}$ and $^{6}/_{100}$ will be included to challenge students who use a strategy similar to Karen's.

Conclusions and discussion

Karen, who had made the highest knowledge gains (according to the ILS) of all the Year 6 students using the system at that school, revealed that she had an impoverished understanding of comparison of fractions, tasks which were based on those undertaken on the ILS. Karen had constructed "rules" and the rule she applied depended on the pair of fractions being compared. Amongst her repertoire of rules for comparing common fractions was a "whole number rule" that was invoked when the numerators were the same but the denominators were different (e.g., Task B– $^{1}/_{6}$, $^{1}/_{2}$; Task C – $^{3}/_{5}$, $^{3}/_{8}$). According to Karen's rule, $^{1}/_{6}$ is larger than $^{1}/_{2}$ because "6 is large that 2" and, similarly, $^{3}/_{8}$ is larger than $^{3}/_{5}$. However, she had a different rule when the common fractions being compared had different denominators and a difference of 1 between the numerator and denominator in each pair (e.g., Task D– $^{2}/_{3}$, $^{3}/_{4}$; Task J – $^{9}/_{10}$, $^{99}/_{100}$). For Task D, Karen couldn't make up her mind whether the fraction with the larger value was $^{3}/_{4}$ (because 4 is larger than 3) or whether the fraction were equal (because of the difference of 1). However, there was no such indecision with Task J where she applied the whole-number rue to both the numerator and the denominator (and thus selected the correct fraction but for an inappropriate reason). Inconsistencies such as these are indicative of superficial syntactic knowledge (Hiebert & Wearne, 1985; Resnicket al., 1989). The problem of applying rules without reason is that it can led to as many correct solutions as incorrect solutions, depending on the tasks provided, thus adding to the confusion and leading to the development of thinking that Benny exhibited (Erlwanger, 1973), namely,*but really they're the same, no matter what the key says (p. 15).*

Prototypic tasks also tend to promote incomplete structural knowledge as evidenced by Karen's performance in Tasks G (3.14, 3.6) and H (3.84, 3.7). Karen consistently applied the "fraction rule" (Resnick et al., 1989) where the number with the fewer number of decimal places is the larger in value because "tenths are larger than hundredths". The ILS tasks (*there's usually only 1 number after)* had promoted Karen's construction of the fraction rule although it is doubtful whether Karen considered the place values of the digits, focusing instead on the "length" of the decimal-fraction component of the number. It is also hypothesised that the ILS comparison of fraction tasks (common and decimal) focused on same whole-number parts so that she "learnt" to consider the fractional parts only (see her performance in Tasks E and F – $2^{5}/_{6}$, $3^{1}/_{3}$ and 4.7, 6.2 respectively).

An interview with Karen's teacher revealed that he was unable to provide specific instances where Karen's ILS gains had been transferred to improved mathematical performance in the classroom. However, Karen believed that it was helping her learn which raises two issues that need to be addressed in future research, namely, associating success with "getting things right" irrespective of the means to the correct response, and associating effort (i.e., time spent) with learning.

The anomaly between Karen's progress on the ILS and misconceptions revealed by the interviews raises the spectre of Erlwanger's (1973) report regarding IPI in which he claimed that IPI programs develop an attitude to learning that is answer-oriented, syntactically based and delivery-process driven. In this learning context, students tend to develop the skills required for a correct answer without developing mathematical knowledge that is transferable outside the closed system (Erlwanger, 1973; Fuglestad, 1996; Jones, 1998).

In the time ($15^1/_2$ hours) that Karen had spent on the ILS, a teacher trained in remediation should have been able to overcome the deficits in Karen's elementary fraction knowledge. However, for Karen, neither traditional teaching (including the years leading up to Year 6) nor the ILS had facilitated construction of appropriate fraction knowledge.

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FRACTIONS, REUNITISATION AND THE NUMBER-LINE REPRESENTATION

Annette R Baturo and Tom J Cooper
Centre for Mathematics and Science Education, QUT, Brisbane, Australia

This paper reports on a study in which Years 6 and 10 students were individually interviewed to determine their ability to unitise and reunitise number lines used to represent mixed numbers and improper fractions. Only 16.7% of the students (all Year 6) were successful on all three tasks and, in general, Year 6 students outperformed Year 8 students. The interviews revealed that the remaining students had incomplete, fragmented or non-existent structural knowledge of mixed numbers and improper fractions, and were unable to unitise or reunitise number lines. The implication for teaching is that instruction should focus on providing students with a variety of fraction representations in order to develop rich and flexible schema for all fraction types (mixed numbers, and proper and improper fractions).

In summarising the literature, Behr, Harel, Post, & Lesh (1992) claimed that there are five subconstructs of rational number, namely, *part/whole, quotient, measure, ratio number*, and *operator*, and that comprehending rational numbers means having an understanding of each subconstruct as well as their interrelatedness.

Australian mathematics syllabi focus primarily on the *part/wole* subconstruct. Under this subconstruct, a *fraction* is a generic term used to denote a numerical amount that is a *part of a whole* (Kieren, 1983; Nik Pa, 1989; Payne, Towsley, & Huinker, 1990), where the whole is any continuous quantity (e.g., a region/area, a line or a volume) or discrete quantity (e.g., a set of objects). Thus, children's ability to interpret fractions is highly dependent on their notion of a unit (whole), their ability to partition the whole (Lamon, 1996; Pothier & Sawada, 1983), and to reconstruct units (Behr et al., 1992; Lamon, 1996; Nik Pa, 1989; Steffe, 1986). According to Steffe (1986), there are four different ways of thinking about a unit, namely, *counting* (or *singleton) units, composite units, unit-of-units* and *measure units,* with each type apparently representing an increasing level of abstraction There is a consensus in the literature (Behr, Harel, Post, & Lesh, 1992; Harel & Confrey, 1994; Hiebert & Behr, 1988; Lamon, 1996) that the cognitive complexity involved in connecting representations, symbols and operations can be attributed mainly to the changes in the nature of the unit. In particular, the complexity required to process unit-of-units and measure units has major implications for acquiring an understanding of rational numbers, particularly in relation to concrete and pictorial representations.

Whatever the representation of the whole, fundamental to the *part/whole* subconstruct is the notion of *partitioning* a whole into a number of *equal* parts and composing and recomposing (i.e., *unitising* and *reunitising*) the equal parts to the initial whole. According to Kieren (1983), partitioning experiences may be as important to the development of rational number concepts as counting experiences are to the

development of whole number concepts. Students, therefore, should be provided with several opportunities to partition a variety of fraction models in a variety of ways so that they come to understand that ½ (for example) always represents one of two equal pieces. *Partitioning, unitising* and *reunitising* are often the source of students' conceptual and perceptual difficulties in interpreting rational-number representations (Baturo, 1997; Behr et al, 1992; Kieren, 1983; Lamon, 1996; Pothier & Sawada, 1983). In particular, reunitising, the ability to change one's perception of the unit, requires a flexibility of thinking that may be beyond young children.

Australian syllabi advocate the use of the area model in developing the initial understanding of a fraction because of the conceptual and perceptual difficulties students have in interpreting the other models (Payne, 1976). For example, with the set model, students find it difficult to unitise a group of discrete objects (Behr et al., 1992; Nik Pa, 1989); with the linear model, children tend to see the marks as discrete points on a line instead of as parts of a whole unit and, again, the problem is related to unitising; with the volume model, the equal partitions are often not shown. Although the set, linear and volume models are not used in the initial development of the part/whole notion of fractions, they should not be avoided as full understanding of any notion requires an ability to abstract the salient features from a variety of materials (Dienes, 1969). In his study involving 220 college students, Silver (1983) reported on what he called *representational rigidity*, a limitation in the variety of mental models that was available to the students. This limitation appeared to be a major inhibitor of the students' ability to operate on fractions.

There has been a recent resurgence of interest in the number line representation, for place value (Bove, 1995), mental computation (Beishuizen, 1997), word problems (Okamoto, 1996), fractions (Maher, Martino, & Davis, 1994), percent problems (Dole, 1998; Parker & Leinhardt, 1995), and functions (Olsen, 1995). This would appear to be in conflict with the earlier literature (e.g., Payne, 1976; Payne et al., 1990) that stressed the conceptual difficulties students had in unitising and reunitising fractions represented by number lines. However, the number line appears to be an ideal representation to help students connect whole-number and fraction processes such as counting (e.g., 3 fifths, 4 fifths, 5 fifths, 6 fifths ... is isomorphic to counting whole numbers). A well, the partitions on a number line showing fractional parts can be recorded as improper fractions or as mixed numbers, thus strengthening the understanding that these two forms can be used interchangeably.

This paper reports on Years 6 and 8 students' responses to tasks involving placing a mixed number and improper fractions of a number line. The study's impetus was an interest in students' ability to use this representation, particularly with respect to unitising and reunitising, and the conflicting reports concerning number line success.

The study

Subjects. The subjects were 24 Year 6 students (12 girls, 12 boys) from 3 suburban and 2 regional primary schools and 10 Year 8 students (5 girls, 5 boys) from 2

regional secondary schools. The schools were generally in lower middle-class areas. The students were chosen by their teachers to represent a cross-section of abilities in their classes (not including extremes).

Instrument. There were three interview tasks (see Figure 1) which were designed to represent a sequence of cognitive difficulty (simplest to hardest).

Task 1: Show $2^1/_4$ on the number line below.

Tasks 2 and 3: Show $^6/_3$ and $^{11}/_6$ on the number line below.

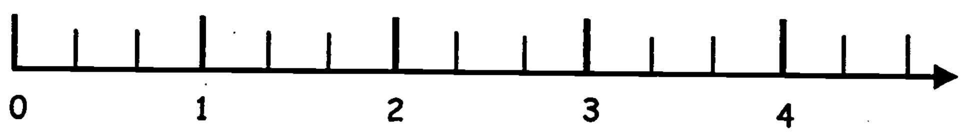

Figure 1. *Number-line tasks.*

Unitisation was required for all three tasks whereas reunitisation (in conjunction with physical or mental repartitioning) was required for Task 3. Task 1 was included to give all students the most chance of being correct in that it required unitisation only and the whole number provided a visual clue to the particular whole to be considered on the number line. Task 2 was considered to be more difficult that Task 1 because of the improper fraction recording of a whole number. It was thought that this would be a nonprototypic representation for most students. Task 3 was considered to be more difficult than Task 2 because, apart from the nonprototypic recording, it required reunitisation.

Procedure. The students were withdrawn from class and interviewed individually. They were asked to place the numbers and then to explain their responses. Before doing this, the tasks were read to the students to alleviate any difficulties that might arise from: (a) the use of the slash instead of the vinculum in the fraction recording, and (b) an inability to read fractions. The interviews were audiotaped.

Analysis. The students' responses and reasons were recorded and categorised into commonalities. Inferences were drawn with respect to the knowledge and strategies held by students giving certain responses.

Results

The results for each task were categorised in terms of correctness and closeness to correctness (see Table 1).

Task 1. When asked to explain their responses, students with correct placement tended to respond in terms of the existing parts on the number line and to

acknowledge that the 2 and the $^1/_4$ were separate [e.g., *because they're all quarters, and that's two* (the whole number) *and that's a quarter*].

Table 1
Number (%) of Task Response Categories for All Tasks by Year Level.

	Students	
Task response category	**Year 6 (n=24)**	**Year 8 (n=10)**
Task 1: $2^1/_4$		
Correct ($2^1/_4$)	54.2	60.0
Other quarters ($2^2/_4$, $2^3/_4$)	16.7	20.0
Eighths ($2^1/_8$, $2^5/_8$, $2^7/_8$)	20.8	20.0
Could not do	08.3	00.0
Task 2: $^6/_3$		
Correct ($^6/_3$)	25.0	10.0
Near ($1^5/_6$, $1^2/_3$, $2^1/_3$, $2^2/_3$)	20.8	20.0
High ($3^5/_6$, 4, $6^1/_3$, $6^2/_3$, $6^3/_4$)	29.2	10.0
Low ($^1/_6$, $^1/_2$, 1)	00.0	30.0
Could not do	25.0	30.0
Task 3: $^{11}/_6$		
Correct ($^{11}/_6$)	25.0	00.0
Near (2, $1^1/_3$, $1^2/_3$)	08.3	20.0
Mid ($3^1/_3$, $3^2/_3$, $3^5/_6$, 4)	20.8	30.0
High ($11^1/_3$, $11^1/_2$)	08.3	00.0
Low ($^1/_2$)	00.0	10.0
Could not do	37.5	40.0

The responses from the 6 students who marked incorrectly but focused on quarters (i.e., $2^2/_4$ or $2^3/_4$) were idiosyncratic. For example, of the 2 students who marked $2^2/_4$, one indicated prototypic thinking in that she seemed to associate the unit numerator with a half (*it has 1 here so it's in the middle*); the other student seemed to combine $2^1/_4$ with another quarter (*there's two up there and it goes up in quarters ... goes one quarter, two quarters, three quarters ... two quarters, so there).* Of the 4 students who marked $2^3/_4$, three seemed to associate the unit numerator with meaning 1 "space" less than the next whole number whilst the remaining student seemed to interpret $2^1/_4$ as having 2 wholes, 2 quarters, and then add on the ¼ [*cos there's two ... zero to two* (pointing at whole numbers) ... *and you go up four* (pointing to quarters) ... *go to two* (quarters) *and about there* ($^3/_4$)]. The responses from students who marked eighths generally appeared to ignore the relationship between the denominator and the number of partitions. They seemed to have developed a holistic view of mixed numbers so that $2^1/_4$ was *just a little past 2* (marking $2^1/_8$) or *just a bit before 3* (marking $2^7/_8$). The student who marked $2^5/_8$ indicated that he also associated the unit numerator with a half (*cos that's 2 and a quarter*) and then marked just past the ½.

The students who could not place the number were generally confused by the fourths. They counted the marks and, therefore, saw the intervals as thirds [e.g., *two* (pointing to the whole number) *and then it goes one, two, three instead of four* (pointing to the partitions)].

Task 2. Unlike Task 1, the Year 6 students' responses were more accurate than those of the Year 8 students. Students with correct placement tended to invoke the quotient (*2 threes are 6*) or operator (*6 thirds is 2*) subconstructs as well as the part/whole subconstruct (i.e., counting in thirds). Although correct, 2 of the latter students (part/whole) were unsure of their answers, for example, *I'm pretty confused about this but with thirds, there's 3 in each one of them* (the units) *but, if you count them like that, there's 6 there* (at 2). It seems as though this student had had little or no experience with renaming whole numbers (with the possible exception of 1) as fractions and that this was a nonprototypic task for her. The other student was confused between the 4 intervals (and therefore quarters) and the three partitions (and therefore thirds).

The explanations from students who marked "near" the correct response (e.g., $1^5/_6$, $1^2/_3$, $2^1/_3$, $2^2/_3$) showed that they tended to count thirds but either counted the partition lines starting at 0 ($1^2/_3$ or $1^5/_6 - 1^2/_3$ *with a little bit on*), counted an extra third (*I went 1, 2, 3, 4 5, 6 and then a third more* – $2^1/_3$), or thought that the partitions at the whole numbers could not be counted because they were not the same size as the internal partitions ($2^2/_3$). The responses from students who marked "high" (i.e., beyond 3) indicated that they thought the 6 in $^6/_3$ was a whole number. This belief was so strong that the students extended the number line to past the whole number 6 so that they could position the improper fraction [e.g., *six and a bit more* – marking $6^1/_3$; *you've got to put thirds* (between 6 and 7) *so you'd mark the 3rd one* – counting the partition at 6 as the first third and thus marking $6^2/_3$; *six, and three partitions for the third* – marking $6^3/_4$]. The responses from students who marked "low" (i.e., $^1/_6$, $^1/_2$, 1) showed very poor understanding of improper fractions and confusion between representations (e.g., *it's six threes ... six threes are eighteen ... so 18% of the whole* – marking $^1/_6$; *it's half* – interpreting the fraction as $^3/_6$; *I counted up six in a row and came back three* – marked 1).

Students who could not place the number tended to have the same misconception as those marking high (but without the initiative to extend the line), namely, equating the numerator with the whole number, 6 (e.g., *there's no six*; *not enough numbers*). One student couldn't interpret the improper fraction because it didn't have a whole number before it and therefore, there was no "range marker" [e.g., *there's 3 here* (counting the whole numbers) *but it didn't say a whole number to put it between, or, after or before*].

Task 3. As for Task 2, the Year 6 students were more accurate than the Year 8 (none of whom gave the correct response). Except in one instance, correct placement involved either overt physical repartitioning or mental repartitioning of thirds into

sixths (e.g., *well there's only thirds ... but if you count two of them, there'd kind of like be sixes - 1 sixth, 2 sixths; cut each third in halves; you'd have to draw them all ... put one in between*). The exception was the boy who invoked the quotient subconstruct (as he had for Task 2).

The "near" placements at $1^1/_3$ and $1^2/_3$ were based on idiosyncratic strategies, neither of which involved reunitising thirds as sixths [e.g., $^{11}/_6$ was interpreted as $1^1/_6$ and the partition after 1 was marked – $1^1/_3$; *2 sixes are 12 so take off 1* (third) – $1^2/_3$]. The "near" placement at 2 revealed erroneous thinking (e.g., $^1/_6$ – $^1/_3$ + $^1/_3$; $^1/_6 > ^1/_3$). Generally, the "mid" errors were based on either counting the thirds as sixths and consequently marking $3^2/_3$ (if parts were counted correctly) or $3^1/_3$ or 4 (if partition lines, not intervals, were counted). "High" placement at $11^1/_3$ and $11^1/_2$ resulted from associating the numerator with whole numbers on the number line and then partitioning the unit from 11 to 12 into thirds. Again, this belief was so strong that the students extended the number line to past the whole number 11. The "low" placement at $^1/_2$ was a consequence of the same confusion between percent and fractions as for Task 2

The students who could not place the number appeared not to comprehend the fraction, the number line or how the two could go together (e.g., *you can't get it on there* and *but that's not sixths*). They exhibited no confidence and little interest in attempting the problem.

Discussion

The poor results for the simplest task (Task 1) suggests that there is an inherent cognitive difficulty involved in conceptualising the number line representation of fractions. The difficulty appears to be compounded when the given fraction is recorded in improper fraction form. These arguments are supported by the increased number of students who could not attempt to answer Tasks 2 and 3 (see Table 2).

Incorrect responses were idiosyncratic, that is, few error patterns were discerned, and same responses nearly always were the result of different (and inappropriate) thinking strategies. Those error patterns that were discerned were: (a) counting partition lines rather than intervals and therefore counting included 0; (b) associating the numerator (in improper fractions) with a whole-number marker and therefore counting whole numbers rather than parts; (c) failure to reunitise when required because of lack of awareness (and therefore a metacognitive shortcoming) or lack of knowledge (e.g., 1 sixth is composed by doubling a third, rather than halving a third).

Furthermore, maturation appears to have no effect on performance. In fact, performance in this study dropped with age (particularly for the improper fractions), a phenomenon we found difficult to understand, particularly as we had expected the Year 8 students to have been much more exposed to this form of fraction recording in view of the fact that they would have encountered addition and subtraction of unlike common fractions requiring decomposition, as well as being exposed to percent conversions to decimal and common fractions, and to prealgebra tasks. We

tentatively suggest that, if appropriate structural knowledge has not been constructed (i.e., semantic knowledge), then students are forced to create "rules", the number of which increase as more and more knowledge needs to be accommodated. The result of this would be to have no means of solving nonprototypic tasks or to invoke as many "rules" as one can think of. One student (Year 8) exemplifies this latter situation as his protocol shows. In Task 2, he had placed $^6/_3$ (which he read as 6 threes) at $^1/_6$ and, in Task 3, placed $^{11}/_6$ at $^1/_2$.

Task 2: *Six threes are 18 so 18% of the whole* (0 to 1) – *about there* ($^1/_6$).

Task 3: *I timesed 6 by 11 – 66% in a hundred so I took it to the nearest part, point five* ($^1/_2$).

Conclusions

This study found that students have conceptual difficulties in placing proper (e.g., $^1/_4$ in $2^1/_4$) and improper fractions on number lines may have been partitioned into the appropriate number of parts. Thus, the results tend to support Payne's (1976) findings that students have difficulties with unitising units on a number line. In particular, there appears to be confusion between whole and part [e.g., not counting the wholes ($^6/_3$ is placed at $2^2/_3$), counting markings not spaces ($^6/_3$ is placed at $1^2/_3$), and counting wholes as parts ($^6/_3$ is placed somewhere after 6)].

The study also found that the placement problems were exacerbated when the number of partitions did not match the given denominator, indicating a continuing difficulty with partitioning and unitising/reunitising on a number line (e.g., counting thirds instead of sixths so that $^{11}/_6$ is placed at $3^2/_3$; not knowing how to reunitise thirds as sixths). This finding reinforces the consensus in the literature on the fundamental importance of children's notion of the unit with respect to representations (e.g., Behr et al., 1992; Harel & Confrey, 1994; Kieren, 1983; Lamon, 1996; Pothier & Sawada, 1983).

The major implications of the findings are that: (a) unless teachers are aware of the inherent conceptual problems students have in processing number lines, their effective use as a teaching and problem solving aid will be limited; and (b) further research which focuses on analyses of students' comprehension of number line fraction representations is required in view of the current resurgence of interest in the area (Beishuizen, 1997; Bove, 1995; Okamoto, 1996; Dole, 1998; Olsen, 1995). An exhaustive review of the PME proceedings dating back to 1994 produced very few articles in this area, supporting the need for further research.

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CLASSROOM COACHING: CREATING A COMMUNITY OF REFLECTIVE PRACTITIONERS*

Joanne Rossi Becker
Barbara J. Pence
San José State University, USA

Abstract

This paper reports on an observational study of 14 high school mathematics teachers who had been involved in one to five years of professional development, including intensive summer institutes, follow-up workshops, and purchase of resource materials and technology. The study was undertaken to determine the effects of the in-service programs on the actual classroom practices of the participants. Using a coaching model, the two authors observed over 200 classes of 17 sections. Seven categories of results were formed from observational data, interviews with teachers, informal interviews with students, and perusal of student work and other ancillary material. This paper reports on three of those categories and discusses critical dimensions of the model of coaching.

Perspective

The professional development programs upon which this research was based endeavored to adhere to promising practices identified by the California Post-secondary Education Commission in a study of projects it had supported from 1992 to 1996 (CPEC, 1996). These included the following aspects:

- *Successful projects found ways to create systemic change across entire school districts.* In the CPEC project we provided professional development to all high school mathematics teachers in two districts (about 150 teachers) over a three-year period. This project, coordinated with the national Equity 2000 project, aimed at helping teachers change their curriculum and instruction in algebra 1/course 1 as the districts implemented an "algebra for all" policy in ninth grade. In the NSF leadership project, we included teachers from several districts, but endeavored to incorporate at least two teachers from each school so that the teachers would have support as they went back to their sites and led curriculum reform.
- *While successful projects need a coherent and consistent set of goals and a reasonable theory of change, they also require strategies that allow participants the flexibility to meet their own personal needs.* The NSF project, in which some teachers participated for three summers four academic years, had the opportunity to incorporate this strategy by forming a small cadre of teachers to help plan subsequent workshops after the first year and by allowing teachers to work on projects of their choice.

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- *Successful grantees adopt their own system of internal assessment.* Such evaluation has been an ongoing aspect of both projects (Becker & Pence, 1998; Becker & Pence, 1996; Kitchen, Becker & Pence, 1997; Peluso, Becker & Pence, 1996).
- *Successful projects develop strategies that enable their teachers to achieve self-actualization.* Fundamental change in teacher behavior occurs when a teacher begins to think of her/himself as a professional and feels that authority is internal rather than conferred from an external authority. Without this orientation, Cooney has claimed (1994), teachers will be unable to exert control over their curriculum and even their pedagogy. Evidence of this level of professionalism was found in previous research (Becker & Pence, 1998).
- *Successful projects do much more than explain constructivism; they model it by involving their teacher-participants in well designed constructivist learning experiences.* As discussed in the next section, we modeled in the professional development a teaching approach that we advocated the participants use.
- *Successful projects know that time-on-task is an important determinant in teacher learning as well as student learning.* The projects utilized extensive summer institutes (three weeks or more) and at least 10 days of followup workshops during the academic year, repeated over several years.
- *Learning about new content or pedagogy is only a necessary condition for improving one's teaching; actually employing that knowledge in the classroom requires more.* The coaching model described in this paper was our approach to encourage teachers to use their new knowledge in the classroom.

The classroom coaching model we used had two main objectives as an extension of the professional development in which teachers had participated: to provide analytic and objective feedback to the teacher with regard to teacher-student behavior; and, to develop within the teacher the desire and ability to reflect upon her/his own behavior and evaluate the results of that behavior as a means of self-improvement. The coaching followed an adaptation of a clinical supervision model, providing teachers feedback that was descriptive rather than evaluative and always requested. The feedback was usually given immediately following the lesson observed, or as close to that as practical given the teacher's schedule.

Background

The professional development projects being studied in this research were based on the assumption that what a teacher believes and what a teacher knows both influence the teaching of mathematics (Fennema & Franke, 1992; Thompson, 1992). What a teacher knows is understood to include both content knowledge and pedagogical content knowledge (Cooney, 1994). Cooney (1994) has interpreted Shulman's (1986) original notion of pedagogical content knowledge in the discipline of mathematics. For Cooney, pedagogical content knowledge in mathematics involves integrating content and pedagogy, borrowing ideas from mathematics and from our knowledge about teaching and learning mathematics. he presents the example of the rational numbers, for which we have various interpreteations and a deep knowledge base about how children construct their

understanding of the rational numbers through these different interpretations. This integration of the mathematical and psychological domains defines pedagogical content knowledge.

As we structured the inservice education to enhance both content and pedagogical content knowledge, we were mindful that teachers themselves are constantly constructing knowledge, albeit knowledge about students' learning of mathematics, effective teaching of mathematics, as well as mathematical content. Therefore activities were structured to ensure that knowledge was actively developed by participant teachers, not passively received. Teachers were frequently involved in presentations, facilitation of small group activities, and even development of workshop foci as the inservice progressed. The professional development became a collaboration among university faculty, district curriculum coordinators, and participant teachers.

Methodology

A year-long observational study was undertaken to ascertain the impact of the two professional development projects on the classroom practice of participant teachers. A sample of teachers were invited to participate in this study. We selected the sample of 24 based upon the number of years of involvement in the inservice, which varied from one to five, and demographic factors such as gender, ethnicity, school district, and teaching experience. Fourteen teachers agreed to participate, and 17 different classes of the 14 teachers were observed by the authors for a combined total of 210 classroom observations over a six-month period. The authors alternated weekly visits of each teacher.

Courses observed varied from an algebra restart [for students who were unable to succeed in algebra the first semester] to algebra 2/integrated course 3. We observed one algebra restart, six algebra 1 classes, five geometry classes, one algebra 2, one integrated course 1, two integrated course 2, and one integrated course 3. The textbooks used varied from the traditional (3 classes) to "transitional" (10 classes), to integrated (four classes). The sample of teachers included five males and nine females from nine schools and four different school districts. Three teachers were Asian-American and the rest European-American. The participants varied in teaching experience from under five to thirty years, with inservice participation from one to five years.

In addition to the classroom visits, both informal and formal interviews were held periodically with each teacher. These interviews were based upon what had been observed in the teacher's class, probing such things as: goals for the lesson and the unit; planned assessment; use of technology; student understanding; plans for followup to the lesson observed; and curricular issues.

Description of Coaching

The classroom observations primarily used a clinical supervision model, in which the teacher and observer discussed before the lesson specific items on which

the observer would focus. The observer then endeavored to collect data to help inform the teacher about the teaching behavior(s) of interest to him/her. After the observation, the teacher and observer discussed the lesson; the observer shared information about the focus of the lesson with the teacher in a non-evaluative way. That is, the observer reported data on the behaviors of interest without assessing the merit of the teacher's classroom practices.

During each visit, the researcher acted as a participant-observer (McCall & Simmons, 1969), taking an active part in the class activities, helping students as they worked individually or in groups and asking questions of students to determine student understanding of new concepts. The observer acted as a true collaborator in the classroom. The debriefing sessions freqently included questioning and reflection on the part of the teacher, as we exchanged ideas for:

- further developing new mathematical concepts;
- reforming the curriculum;
- infusing technology into their teaching;
- ways to teach upcoming mathematical concepts;
- assessment techniques;
- equity issues in teacher-student interactions; and,
- ways to help recalcitrant or struggling students.

Results

In analyzing the data from observations and both informal and formal interviews, we formed seven categories on which we partitioned the classroom practices of the fourteen teachers. These categories included equity, multiple representations, and the use of technology, the foci of the observers. The additional four categories - student understandings, use of cooperative groups, alternative assessment, and reform curriculum - arose either from the observations or from questions raised by the teachers in debriefing sessions. Table 1 below delineates the partitioning formed, with "yes" indicating strong evidence of practices consistent with the emphasis of the professional development.

Due to space limitations, we will discuss three categories in this paper: equity, student understandings, and assessment. Each category is exemplified by instances from at least one teacher, instances which are representative of patterns discerned in that category.

Equity

Although more than half of the teacher sample showed evidence of inequitable treatment in the classroom, especially by gender, we want to highlight two teachers in particular. One male teacher, Damon, consistently called upon a small number of male students during questioning. In his geometry class, for example, in one observation Damon interacted with males on 16 occasions (seven of which were callouts), but only 3 with females (two of which were callouts). This class was only 37% female; however, the young women were only involved in 16% of the interactions with the teacher. This disproportionate pattern of

interactions was typical in this class. In his algebra 1 class, evenly split between young men and women, Damon usually interacted about twice as much with the young men as with the young women in the class. Damon learned about the observers' areas of interest because he asked and we answered honestly. Following this discussion, Damon asked one observer to share data relative to equity. He was most chagrined to learn about the inequities in his response opportunities, and was determined to monitor and change his own behavior.

A female teacher, Shana, was much more equitable in her teacher-student interactions, with most of these occurring during small group activities or as pairs worked on computers. However, she initiated discussions about this issue on her own. She had a student assistant tally her interaction patterns as a way of monitoring her behavior. And Shana, on her own initiative, devised a plan to ensure equitable interactions in her class. She planned to use the computer to randomly generate a new sequence of names for her to use each day as she called on students.

Student Understandings

Some concepts were difficult for teachers to teach for understanding. In an integrated course 1, Polly was introducing the concept of standard deviation, which she did rather traditionally, putting an extensive table on the board and having students calculate all the intermediate steps on a set of data. Then she showed the students how to find the standard deviation on a graphing calculator, and she allowed them to use this tool consistently thereafter. In this introduction, Polly did not attempt to provide an intuitive understanding of what the standard deviation measures. After the class, she remarked to the observer that although she thought students understood the steps in the algorithm, she did not feel they understood what the standard deviation meant from her instruction. This was confirmed by the observer in a later observation by questioning students as they worked in groups. Polly told the observer that she would think about how to provide meaning for this concept. In a later lesson, Polly presented a nice activity in which students collected data on the heights of students' navels from the floor and again while standing on identical chairs. Students then compared various measures of central tendency; they were not surprised that the mean changed by the height of the chair, but were very surprised that the standard deviation did not change at all. This activity seemed designed to help students begin to develop understanding of the standard deviation, and it seemed to meet its goal. Polly's concern about student understanding was, we felt, indicative of new knowledge Polly had gleaned from the inservice programs in which she participated. Although the text did not provide any guidance, Polly had the confidence and persistence to develop an instructional strategy that she felt would develop the understanding she valued.

Assessment

Shana had established a detailed scheme of assessment in her geometry class in which she included performance standards for homework, performance standards for projects, a rubric for computer work with a sample format, a culminating computer investigation for the semester's work, and a unit portfolio which was peer

reviewed. We should note that Shana had been working on this scheme over three years of her inservice involvement, revising it as experiences suggested needed changes. The coaching helped support Shana in this unusual assessment plan and assisted her in reflecting on its effects on student understandings and reporting to parents and her peer teachers.

CATEGORY	YES	NO
Equity	6	8
Multiple representations	5	9
Technology	9	5
Student understandings	8	6
Use of cooperative groups	8	6
Alternative assessment	5	9
Curriculum	10	4

Table 1: Observation Categories

Discussion

The intensive inservice program over three to five years impacted the teachers in a number of ways. The programs established a common base of content and pedagogical content knowledge (Cooney, 1994) and a common language between teachers and researchers which allowed for a challenge of traditional beliefs about the teaching and learning of mathematics. In addition, the teachers received a considerable amount of resource materials and technology, such as graphing calculators and computer software, essential for implementing new knowledge in the classroom setting. Perhaps most important to the teachers was the strong network of peers formed over the years. This network supported teachers as they attempted to implement change in their curriculum and instructional practices (Becker & Pence, 1996).

Past evaluations of the inservice documented these positive effects of the programs (Becker & Pence, 1996; Peluso, Becker & Pence, 1996; Peluso, Pence & Becker, 1994). In this work we used a variety of self-report measures to determine the effect of our inservice programs. However, we are well aware of the limitations of such data. Therefore, we initiated this observational study to ascertain the impact of the two professional development projects on actual classroom practices of participant teachers.

But, we were surprised that many of the 14 teachers reported that the **coaching itself** was a critical extension of the professional development program. The two-to-one contact over six months with each teacher served several purposes. First, it provided the opportunity for both observers to experience the development of full units of mathematical content. As the content developed, we were able to see how student work progressed and how student understandings grew. Weekly visits enabled the teachers to identify a conflict or concern, ask questions about student experiences and glean insight into their teaching from our feedback. Second, although we had a good working relationship with these teachers before

they assented to participate in the study, the coaching helped establish a stronger rapport and true collegial collaboration. We were not the authorities, but rather sounding boards with whom teachers could formulate instructional questions, extend those questions, and work out solutions for themselves.

Finally, our presence supported the teachers who were trying to make real change in their curriculum, instruction, and assessment. At some school sites, parents, other teachers, or administrators hindered reform efforts. Observer feedback and encouragement ameliorated such challenges. We acted as another voice, counteracting negativity these risk-takers sometimes faced (Peluso, Pence & Becker, 1994).

Summary

We should point out that the classroom coaching described in this paper requires a great deal of time and thus financial resources to effectuate. However, the results of this work indicate that such coaching may well be a critical component to consolidate changes in classroom practice as a result of professional development.

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CONSTRUCTING AN UNDERSTANDING OF DATA GRAPHS

Dani Ben-Zvi
The Weizmann Institute of Science, Rehovot, Israel

I describe episodes of two 13-year-old students working on Exploratory Data Analysis (statistics) developed within an innovative curriculum. I analyze the microevolution of their incipient understandings of some features of graphs as data representations. The description includes the role of the instructional materials, the students' discussions and collaborative attempts to solve the tasks, and the teacher's intervention. Although her intervention seemed to be a miscommunication, it appears to have helped the students to make sense of their tasks.

BACKGROUND

The teaching of *Exploratory Data Analysis* (statistics) is mostly based on: (a) organization, description, representation and analysis of data, with a considerable use of visual displays (Shaughnessy et al., 1996); (b) a constructivist view of learning (Garfield, 1995); and (c) incorporation of technological tools for making sense of data and facilitating the use of various data representations (Biehler, 1993).

With these perspectives in mind, we developed a middle school statistics curriculum[1] (Ben-Zvi & Friedlander, 1997a), implemented it in schools and in teacher courses, and undertook research on learning (Ben-Zvi & Arcavi, 1997; Ben-Zvi & Friedlander, 1997b). The curriculum is characterized by: (a) a use of extended real (or realistic) problem situations; (b) collaboration and communication in the classroom; and (c) a view of the teacher as "a guide on the side" (Hawkins et al., 1992). The students pose, collect, analyze, interpret data, and communicate (Graham, 1987) using a spreadsheet. The classroom activities are semi-structured investigations, in which students, working in pairs, are encouraged to hypothesize about possible outcomes, choose tools and methods of inquiry, design or change representations, interpret results, and draw conclusions.

THE STUDY

A pair of 13-year-old students (*A* and *D*) was videotaped at different stages of their learning statistics (20 hours of tapes). I focus here on 15 minutes of their work with brief teacher interventions. The students were considered by their teacher to be both very able and very verbal. They were asked to talk aloud and explain their actions.

The purpose of the following analysis is to study how students construct their understanding of graphs as displays of real life data, and learn to design them to support certain claims. I used interpretive microanalysis (see, for example, Meira, 1991, pp. 62-3) to try to understand students' discussions, considerations, difficulties and solutions. In this analysis I consider socio-cognitive aspects, taking into account verbal, gestural, and symbolic actions, in the context in which they emerged --

[1] The project is part of *CompuMath*, an innovative and comprehensive curriculum (Hershkowitz & Schwarz, 1997).

comparing and contrasting the data with other pieces of data, written records, and conversations with the teacher.

The Problem Situation

The extended (four lesson) activity - *The same song, with a different tune* - occurs early in the curriculum. The context is the Olympic 100 meters race. The students were given, in a spreadsheet, the men's 100 meters record times, and the years in which they occurred (from 1896, the first modern Olympiad, to 1996). In the first part of the activity, the students were introduced to the context of the investigation and were asked to describe the data graphically and verbally. In the second part, the students were asked to manipulate data graphs, i.e., change scales, delete an outliar, and connect points by lines. In the third part, they were asked to design graph to support the following claim: *"Over the years, the times recorded in the Olympic 100 meters improved considerably"*.

In the following, I present and analyze the students' work through the activity.

DEVELOPING UNDERSTANDING OF DATA GRAPHS: THE 'STORY' OF *A* AND *D*

In this section, I present three parts of the activity chronologically: (a) getting acquainted with the context, (b) acquiring tools, and (c) designing graphs.

(A) Getting Acquainted with the Context

In the first part of the activity, *A* and *D* analyzed the table of results, compared the records of consecutive Olympiads, considered the issue of extreme data, sorted the data, and created a graph with a spreadsheet (Figure 1). In their written summaries, they wrote that (a) the best record is 9.48 sec. and the worst is 12 sec., (b) the greatest improvement is from 10.25 to 9.48 sec., and (c) the differences between records are not constant. The first two conclusions are wrong: the best record is 9.84 sec., and the greatest improvement is from 12 to 10.8 sec. When requested to describe the data *patterns*, they did not seem to understand the meaning of the question. With the teacher's help, they concluded correctly that "the record times seem to improve, yet there was occasionally a lower (slower) result, than the one achieved in previous Olympiads".

Although *A* and *D* seemed to notice the general trend of improvement in the records, their view was mostly local and focused on discrete data points, or, at most, on two consecutive records. I claim (based on data not detailed here) that their difficulty to discern general data *patterns* was caused by: (a) the students' lack of experience with the notion of *pattern*; (b) the discrete nature of the graph; (c) the non-deterministic and disorganized nature of statistical data, which is very different from the deterministic formulae, they had met in algebra.

(B) Acquiring Tools

In the second part of the activity, the students became acquainted with three strategies for manipulating graphs (changing scales, deleting an outliar, and connecting points), and considered the effect of these changes on the shape of the graphs. The objective was to prepare for the *design* task (Part C below).

Changing Scales

The following transcript describes the students' comments on the effect of changing the vertical scales of the original graph from 0-12 to 0-40 (Figures 1& 2):

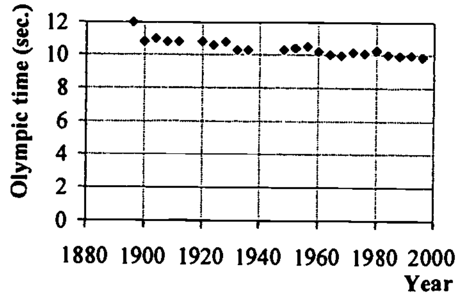

Figure 1: Given graph

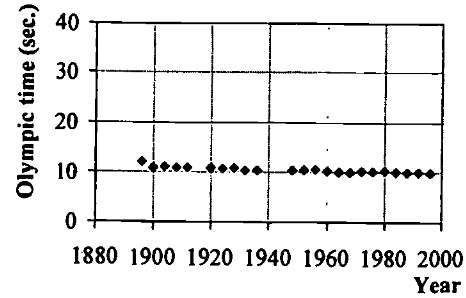

Figure 2: Manipulated graph

A. *Now, the change is that... that the whole graph stayed the same in shape, but it went down...*

D. *The same in shape, but much, much lower, because the column* [the y-axis] *went up higher. Did you understand that?* [*D* uses both hands to signal the down and up movements of the graph and the y-axis respectively.]

A. *Because now the 12, which is the worst record, is lower. It used to be once the highest. Therefore the graph started from very high. But now, it* [the graph] *is already very low.*

The students' perception of the change is restricted to the overall relative position of the graph; they considered the shape itself as remaining "the same". Their description includes: global features of the graph ("The whole graph ... went down"), an interchange of background and foreground (the graph went down and/or the y-axis went up), and local features (12 as a "starting point" of the graph). These descriptions are linked and complement each other. *A* wrote the following synthesis in his notebook: "The graph remained the same in its shape, but moved downward, because before, 12 - the worst record - was the highest number on the y-axis, but now it is lower".

Deleting an Outliar

In this task, the students were asked to delete an outlying point (the record of 12 sec. in the first Olympiad, 1896) from the graph (Figure 2), and describe the effect on its shape. First, *D* justified why 12 can be considered an outliar:

D. *It* [the record of 12 sec.] *is pretty exceptional, because we have here* [in the rest of the data] *a set of differences of a few hundredths, and here* [the difference is] *a whole full second.*

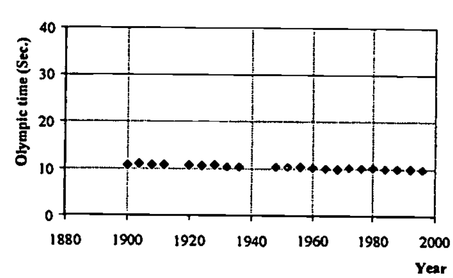

Figure 3: Outliar deletion

Then, they struggled to interpret the effect of the deletion on the graph (Figure 3):

D. *The change is not really drastic ... Now,*

however, the graph looks much more tidy and organized.

A. *One point simply disappeared. The graph... its general shape didn't change.*

They wrote in their notebooks different descriptions of the change: "The graph became straighter" (*D*); "One point in the graph disappeared" (*A*). Thus, the students struggled between different views of the effect: global and significant change (the graph is tidy and organized), no change at all (the general shape didn't change); or just a mere description (one point disappeared).

Although the dispute about the outliar was not resolved, it served another purpose: it drew *A*'s attention to a mistake in their conclusions in the first part of the activity, and corrected it: "the greatest improvement is from 12 to 10.8 seconds".

<u>Connecting Points</u>

In the third task, they were asked to connect the points to obtain a continuous graph. The new graph (Figure 4) elicited many comments from the students, who tried to make sense of what they saw. They were particularly intrigued by the fact that the connected graph included both the original points, and the connecting line.

D. *OK. You see that the points are connected by lines. Now, what's the idea? The graph did not transform to one line. It transformed to a line, in which the points are still there. It means that the line itself is not regarded as important.*

A. *This line is OK. We previously thought that if we connect the points with a line, they might disappear. But now, there is a graph, and there are also the points, which are the important part.*

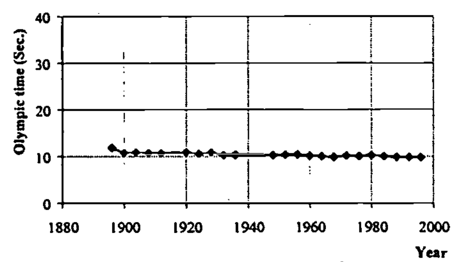

Figure 4: Connecting line

In their view, the connecting line (as provided by the spreadsheet) not only did not add any new meaning, but also contradicted the context, as *D* observed: "Olympiads occur only once in every four years" (namely, there is no data between the points). The students did not see the line as an aid to detect or highlight patterns in the data, and this is consistent with their previous difficulties in recognizing data patterns.

So far *A* and *D* were practicing manipulations (changing scales, deleting an outliar, and connecting points), and discussing their effect on the graph's shape. The intention was to provide students with the means to design a graph, in order to support a particular claim. In the following section, I discuss in what sense this preparation helped them achieve this purpose.

(C) Designing Graphs

I present here a fragment of the students' work on the third part of the activity. The students were asked to design a graph to support the statement: "*Over the years, the times recorded in the Olympic 100 meters improved considerably*". I bring first a teacher intervention, which eventually helped the students understand the task. Then, I focus on five attempts (*Stages 1-5* below) to obtain a satisfactory form of the graph.

The Teacher Intervention

A and *D* did not understand the task and requested the teacher's (*T*) help:

T. [Referring to the 0-40 graph displayed on the computer screen -- see Figure 4.] *How did you flatten the graph?*

A. [Surprised] *How did we flatten it?*

T. *Yes, you certainly notice that you have flattened it, don't you?*

D. *No. The graph was like that before. It was only higher up* [on the screen].

The teacher and the students are at "loggerheads". The teacher assumes that the students (a) had made sense of the task, but just did not know how to perform it, (b) had acquired the necessary tools, and understood their global effect on the graph's shape to be used to support the claim. Thus, her hint consisted of reminding them of what they had already done (scale change). However, the students did not regard what they had done, as changing the graph's shape. Although this intervention seemed to be a case of miscommunication, it apparently had a catalytic effect, as reflected in the dialogue, which took place immediately afterwards:

T. *How would you show that there were very very big improvements?*

A. [Referring to the 0-40 graph -- see Figure 4.] *We need to decrease it* [the maximum value of the y-axis]. *The opposite...*[of what we have previously done].

D. *No. To increase it* [to raise the highest graph point, i.e. 12 sec.].

A. *The graph will go further down.*

D. *No. It will go further up.*

A. *No. It will go further down.*

D. *What you mean by increasing it, I mean - decreasing.*

A. *Ahhh... Well, to decrease it... OK, That's what I meant. Good, I understand.*

Even though their use of language is not completely clear, their previous perception that the graph shape remains the same was not mentioned at this stage. Moreover, *D* expressed what appears to be a new understanding:

D. *As a matter of fact, we make the graph shape look different, although it is actually the same graph. It will look as if it supports a specific claim.*

At this point, *D* seems to discern that a change of scales may change the perceptual impressions one may get from the graph. Thus, they seemed to understand the purpose of the activity, and started to focus on its goal. In the following, the students' five attempts to design corresponding graphs are presented.

Stage 1 (The scales are changed to x: 1880-2000; y: 0-5)

D suggested (Figure 4) changing the scale on the y-axis to 0-11. It seems that he chose 11, since he had previously deleted the outliar, making 11 the maximum data point. They didn't implement this change, because he immediately proposed another scale change: 0-5. This suggestion seems to be based on his assumption that

the smaller the range the larger the decline in the record time would look *(Idea I)*. However, when they implemented this change, the graph disappeared.

A. *We don't see the graph at all, since there is no graph in 5.*

Stage 2 (x: 1880-2000, y: 0-12)

Having failed to present a new graph, they returned to the 0-12 range (see Figure 1):

A. *The graph looks more curved, because the difference between records is much bigger, since we increased the... now the "Olympic time in seconds"* [y-axis] *is from 0 to 12, and every record– as much as it descends – it is bigger than the record... the line is more...*[*D.* interrupts] *Wait a second, the line is bigger than it used to be from 0 to 40.*

The effect of changing scales on the graph's global features (straight, curved), which were not noticed initially, and started to be considered after the teacher's intervention, were now being fully considered. Still, the students struggle to verbalize and explain what they do, or want to do.

Stage 3 (x: 1896-1996, y: 0-12)

At this stage, it seemed that *A* and *D* had exhausted the changes on the y-axis. So they turned to the x-axis. *D* suggested changing the upper limit of x from 2000 to 1000 (*Idea I* above). They realized, however, that this would cause the graph to disappear again (the year's range is 1896-1996). Thus, *D* proposed using 1996 (instead of 2000) as the upper limit of x. Although the effect[2] was marginal, *D* commented:

D. *One can really see, as if there are bigger differences in the graph... Very interesting!*

Although they had presumably understood how changing scales effects the graph's shape, *D*'s wrong impression of this horizontal change, seems to originate from his ambiguous distinction between vertical or horizontal "differences" and/or distances. However, having focused their attention on the x-axis, they realized that it does not start at zero, which triggered the following idea *(Idea II)*.

Stage 4 (x: 1896-1996, y: 8-12)

A transferred attention from the x-axis to the y-axis, and suggested changing the lower limit for y from 0 to 8 (to get a scale of 8-12). Observing the resulting sharp visual effect, he reacted immediately:

A. *It looks much bigger.*

Stage 5 (x: 1896-1996, y: 9.48-12)

D suggested applying *Idea II* to the x-axis, but withdrew, when *A* indicated that it already started at a non-zero value. Instead, *A* suggested using the minimum

[2] The lower limit for y changed automatically to 1896, resulting in a final range of 1896-1996, instead of 1880-2000, which were the default values provided by the software.

record time (9.48 sec.) as the lower limit of y *(Idea III)*. The resulting graph (Figure 5) satisfied them, and they made the following final comments:

> D. *This way we actually achieved a result* [graph] *that appears as if there are enormous differences.*
>
> A. *To tell you the truth, this booklet is lovely.*
>
> D. *Right, it is nice!*

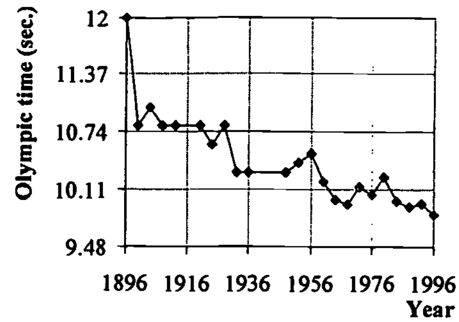

Figure 5: Final design

DISCUSSION

This 'story' of *A* and *D* traces the microevolution of incipient understanding of some features of graphs as displays of real life data (see also Bright & Friel, 1997). It describes the students' perceptual development from a stage in which they did not understand the requirements of the task and the notion of data *pattern*, to the final successful completion of the *design* task. The following elements seem to have contributed to the construction of students' understanding of certain characteristics of data graphs.

Careful instructional engineering. The students worked with semi-structured guidance to solve open-ended questions. First, they acquired tools to modify graphs and then, they employed these tools in the design of graphs, to support a certain claim.

Close collaboration between the pair of students. The students:

a) verbalized almost every idea that crossed their minds. At times this spontaneous verbalization produced mere descriptions, but later served as stepping stones towards a new understanding, and at times, it served as self-explanation (Chi et al., 1989) to reinforce ideas;

b) complemented and extended each other's comments and ideas, which seems to have "replaced" some of the teacher's role in guiding their evolution;

c) decided to request the teacher's help when faced with a difficulty, which could not be resolved among themselves; and

d) transferred and elaborated, in iterative steps, ideas of changing scales, from one axis to the other.

The teacher's main intervention. At a first glance, the teacher's intervention to help the students make sense of the task, can be considered unfortunate. She did not grasp the nature of their question, misjudged their position, and tried to help by reminding them of their previous actions. The students, however, did remember the acquired tools, but perceived them differently.

Nevertheless, this miscommunication itself contributed to their progress. At first, *A* and *D* were surprised by her use of the notion of *flattening the graph* as a description of what they had done. Then, they started to direct their attention to the

shape of the graph, rather than to its relative position on the screen. Although puzzled by the teacher's language, the students appropriated (Moschkovich, 1989) her point of view on what to look at. Their previous work and their "struggle" with language seems to have prepared them for the reinterpretation of what they had done, triggered by their teacher's comments.

In sum, the microevolution of the students' understanding of data graphs was influenced by the instructional engineering, the students' ways of making sense (descriptions, self-explanations, questions to a colleague and the teacher, transfer of ideas, etc.), and the teacher's intervention and the use students made of it.

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ROUTINE QUESTIONS AND EXAMINATION PERFORMANCE

John Berry & Wendy Maull — University of Plymouth, UK
Peter Johnson & John Monaghan — University of Leeds, UK

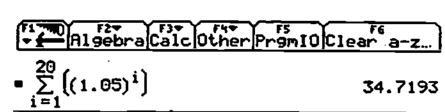
Abstract *This study concerns student performance in pre-university examination questions. In particular whether lower attaining students in mathematics examinations generally gain their marks on routine parts of questions? This is an important issue because routine questions could be awarded fewer marks if algebraic calculators are allowed in examinations Students' scripts in a recent mathematics examination were examined in an attempt to evaluate this question. The results are not conclusive but indicate that a problem of this type does exist, though the nature and location of the problem is not as straightforward as expected.*

Introduction

Our starting point is the question: do A-level Mathematics[1] students who attain lower pass grades (D and E) generally obtain these grades by answering 'routine' parts of A-level Mathematics questions? Routine questions may be viewed as those for which students may be expected to execute a rehearsed procedure consisting of a limited number of steps. Problems in characterizing routine questions are considered later. The next three paragraphs explain the rationale for and import of the study.

During the period 1994-1996 the then Schools Curriculum and Assessment Authority (SCAA) set up a number of working groups investigating possible consequences of student use of a new generation of algebraic calculators on A-level Mathematics questions and papers. One important debate was whether such use would accentuate the difference between higher and lower attaining students, e.g. between those attaining A-level grades A & B and those attaining grades D & E. An example should clarify matters.

A typical question on geometric series may start by a request to evaluate $\sum_{i=i}^{20} 1.05^i$ and then proceed to a question on compound interest, e.g. "If I invest £550 at a rate of 5% per annum, how many years must I wait until I have more than £1000 in this account?" It should be noted that the new generation of algebraic calculators can perform the first part of this question, e.g. the TI-92 screendump below. In the ensuing discussion everyone initially assumed, as a generality with exceptions, that students



[1] Advanced level (A-level) Mathematics is the most common senior public examination for students in the UK. It covers considerable algebra and calculus of a single variable. There are five pass grades, A to E. Examinations are set and marked by independent institutions called Examination Boards. Examination sheets are called papers.

attaining lower A-level grades learnt how to do the first routine part but would have difficulty with the second non-routine part. At the next meeting, however, the discussion continued with several people saying they were not sure that this really was the case. There appears to be no literature of direct relevance in this area.

Now if lower attaining students generally obtain the majority of their marks on routine questions and if, as seems likely, such questions are allocated a relatively smaller share of the total mark scheme when algebraic calculators are permitted in examinations (see Monaghan (to appear) for a discussion of this issue), then these students will find it more difficult to pass these examinations. Mathematics is already considered a difficult subject at senior school level (see Fitz-Gibbon & Vincent (1994, p.23) for UK data) and we would be extremely concerned if mathematics examinations became more difficult to pass.

Methodology

To address the question we analysed the performance of students with different A-level grades (A, B, C, D & E) in questions which have routine and non-routine parts. An A-level Examination Board provided us with the scripts from a recently marked Pure Mathematics paper. A Pure Mathematics paper was chosen, rather than a Mechanics, Statistics or Discrete Mathematics paper, since Pure Mathematics is the core for all mathematics options and because it is the area most likely to be affected by algebraic calculators (see Monaghan (to appear) for a discussion of this). Over 300 scripts from an almost equal number of male and female students who obtained scores at the boundaries of the A, B, C, D and E grades were provided.

Each question part was coded as routine or non-routine (further details below) and the marks for the question parts were adjusted in accordance with our expectations of what future marks, where routine questions were allocated a relatively smaller share of the total mark scheme, would be like. Students' scripts were then remarked.

The A-level paper and mark scheme(s)

The paper used was the first of six papers. It was one of three papers that all students following a popular modular A-level scheme had to take. There were four questions each worth 15 marks. Questions 1, 2 and 4 had four parts. Question 3 had five parts. We reproduce question 2 below as an example of the kind of question asked. The original marks are given in square brackets.

The gradient of a curve is given by $\frac{dy}{dx} = 3x^2 - 8x + 5$. The curve passes through the point (0, 3).

(i)	Find the equation of the curve.	[4]
(ii)	Find the coordinates of the two stationary points on the curve. State, with a reason, the nature of each stationary point.	[6]
(iii)	State the range of values of k for which the curve has three distinct intersections with the line $y=k$.	[2]
(iv)	State the range of values of x for which the curve has a negative gradient. Find the x-coordinate of the point within this range where the curve is steepest.	[3]

The other questions concerned: Q1, trigonometry in context; Q3, coordinate geometry (lines, circles and ellipses); Q4, integration in context (comparison of exact and numeric methods). We classified each question part as routine (R) or non-routine (N) and obtained: Q1 (R, N, N, N); Q2 (R, R, N, N); Q3 (R, R, N, N, N); Q4 (R, R, R, N). The division of marks for routine and non-routine parts was 30 marks each. Various alternative mark schemes were developed, all adjusting the mark ratio so that routine parts of questions scored fewer marks. The agreed version left each question with 15 marks, left the mark allocation of Q1 unchanged but adjusted the others so that routine parts totalled 23 marks and non-routine parts totalled 37 marks. The parts of Q2, for example, were allocated 3, 4, 4 and 4 marks respectively.

The grade boundaries for this paper were: A, 40; B, 33; C, 27; D, 21; and E, 15. In keeping with A-level conventions A, B and E grade boundaries were determined by examiners' judgement while C and D boundaries were fixed at equal intervals between B and E grade boundaries.

Results

311 scripts (63, 63, 62, 62 and 61 at grades A, B, C, D and E respectively) were remarked to the new mark scheme. The use of statistics in this study must be carefully examined for much of the data is far from independent (consider, for example, the relationship between the total on the original mark scheme and the total on the revised mark scheme). The statistics which follow are intended to give the reader a feel for the general patterns in the data. Three aspects are examined here: the overall scores; the proportion of marks to routine and non-routine parts of questions; and factor analytic results suggesting that students follow through whole questions.

The scatter diagram, figure 1, of old and new totals illustrates the general pattern. The ranges overlap but the ranges from the original grades retain their hierarchical structure. The new totals are generally lower than the original totals. In fact of the 311 student scripts examined 297 obtained lower scores from the new mark scheme, 11 scores remained the same (3, 3, 2 and 3 from A, C, D and E grade students respectively) and three obtained higher marks (a B and a D grade student obtaining one more mark and a C grade student gaining three more marks). This indicates that increased emphasis on non-routine questions leads to a general lowering of the overall marks obtained.

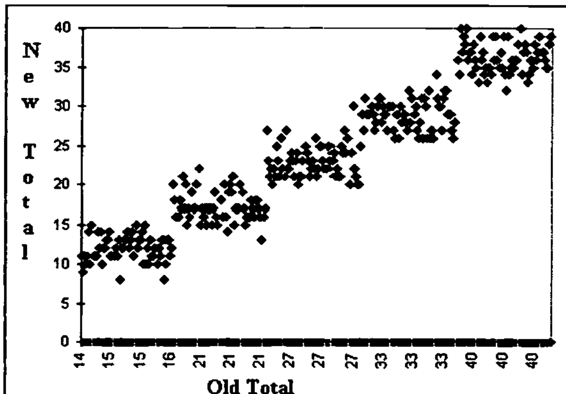


Figure 1, scatter diagram showing total scores under both mark schemes

Figure 2 displays the mean marks obtained by the groups of students at each grade level for:

- Old routine (O-R) parts (out of 30)
- Old non-routine (O-N) parts(out of 30)
- New (revised) routine (N-R) parts (out of 23)
- New non-routine (N-N) parts (out of 37)

	O-R	O-N	N-R	N-N
A	26.3	13.7	19.9	16.4
B	23.2	9.3	17.4	11.2
C	19.9	7.1	14.8	8.1
D	15.8	5.2	11.1	6.2
E	11.8	3.2	8.3	3.6

Figure 2, mean marks over grades

Note that each column decreases with decreasing grades (hardly surprising) and that mean marks obtained for routine parts of questions are consistently greater than mean marks obtained for non-routine parts of questions even though there were more marks for non-routine parts of questions in the revised mark scheme. This may be interpreted as evidence that students attaining at all pass grade gain more marks on routine parts of questions An examination of the ratios O-R:O-N and N-R:N-N is interesting. For grades A-E we get, respectively: 1.9, 2.5, 2.8, 3.0, 3.7 and 1.2, 1.6, 1.8, 1.8, 2.3. The decrease in the second list, relative to the first list, clearly mirrors the higher weighting given to non-routine parts of questions in the revised mark scheme. The increase in both lists, however, may be interpreted as evidence that higher (respectively lower) attaining students gain proportionally (to their overall mark) more (respectively less) marks on non-routine parts of questions.

Principal component analysis of both the old and the new scores yielded 6 factors with eigenvalues greater than one. In both old and new scores the question parts which loaded significantly on the factors were as follows: all parts of Q2; all parts of Q4; Q1 parts i, ii, and iii; Q3 parts i, ii, iii, and iv; Q3 parts ii, iv and v; Q1iv, Question 1i (negatively) and Q2i. The first five factors suggest an interpretation that the correlations of the scores of the parts within a question dominate the analysis, i.e. if you do well on one part of Q2, you tend to do well on all of it.

Grades and routine questions

So, do lower attaining students gain a substantial proportion of their marks on routine parts of questions, compared to higher attaining students? The results are not conclusive but they are not without interest. Before considering them we address an aspect of their surface validity. The results arise from at least two semi-arbitrary decisions: the categorization of parts of questions as routine or not (and the dichotomy implicit in this categorization); the weightings given in the revised mark scheme. These are important factors to bear in mind but it should be noted that both decisions were made after considerable debate by a group of people with considerable experience of the type of examination paper in question.

The overall lower scores obtained from the revised mark scheme and the distribution of mean marks over routine and non-routine question in both mark schemes clearly indicate that all students score substantially more marks on what we have designated routine parts of questions. The increasing ratios of routine to non-routine mean

marks over grades in both mark schemes, however, does provide evidence that lower attaining students do obtain proportionally more of their marks on routine parts of questions. Looking at the scatter diagram in figure 1, however, it would not appear that this would make a substantial difference to the overall grades (if they were still determined by the same judgement/equal interval rubric) given the small 'new total' overlap over old grades. Indeed, several experiments at reallocating grades under the revised mark scheme were conducted and none of these resulted in more than a 10% reallocation of grades. The outcomes are not recorded in this paper for fear of introducing another semi-arbitrary element.

The principal component analysis results alerted us to a possibility we had not, but should have, anticipated: that many students, at all levels of attainment, exhibit a propensity to follow a question through. This may have many bi-causal connections with other influences, including a familiarity with a specific content area – indeed a dialectic may exist between attainment and familiarity with a range of content areas. Again we must view these results with caution, due to regularity conditions implicit in the analysis and a lack of prior hypotheses, but five out of six located factors indicating such a propensity in students certainly deserves consideration.

What this and other results lead to is the need for further investigations. Two such investigations are a refinement of what is meant by the term 'routine question' and a more realistic interpretation of curriculum development vis a vis assessment development. We turn to a considerations of these now.

Problems in characterizing routine questions

A problem with this study is that it implies that routineness is located in a question rather than being a psychological construct of the relation between an individual, or group, and a question. The latter appears more valid. Indeed, the psychological relation is likely to be a socio-psychological relation. Routineness as located in a question is, however, one way to approach this study. This study's origins grew from a SCAA working party examining examination questions and we had the opportunity to explore this issue by examining completed A-level scripts without exploring classroom situations. Rather than viewing our approach as flawed we choose to view it as exploring one avenue of routineness.

There are very few readily available references to routine questions in the mathematics education literature. An important paper, which addresses a similar level/type of mathematics to this study, is Seldon, Mason & Seldon (1989). They make a distinction between problems and exercises and view, in a similar way to the above socio-psychological relation, a problem having two components: task and solver(s). They view 'cognitively non-trivial' problems as those where "the solver does not begin knowing a method of solution". and note that traditional calculus courses contain few cognitively nontrivial problems. They note that 'tasks' require skills and are divided into parts, algorithms, sample solutions and examples and that many problems are made routine in this way. This accords with our own experiences

of UK A-level mathematics classes. Seldon et al, however, do not define what they mean by 'routine'.

Boaler (1997) explored a range of issues from two schools with strikingly contrasting ethoses and teaching methods. We focus here on her analyses of pre-A-level students' performance on conceptual/procedural questions, which may be viewed as a form of the non-routine/routine division. She defines procedural questions as "those questions that could be answered by a simplistic rehearsal of a rule, method or formula." Conceptual questions were viewed, in contraposition, as questions which require "the use of some thought and rules or methods committed to memory in lessons would not be of great help". Boaler claims that conceptual questions are more difficult, for students, than procedural questions, and descriptive statistics support this view. Notwithstanding the fact that cursory summaries do not do justice to Boaler's work it can be said that performance on conceptual and procedural questions shows that similar overall results in examination performance may be obtained in different ways (ratios of success in the two types of questions) under different school ethoses and subject teaching methods.

Boaler's work raises the obvious question of the relationship between routineness and teaching methods. Nagy et al (1991) examine the relationship between test content and instructional content at the level of High School calculus. It would be in injustice to characterize their study as purely quantitative as their foci are intent of instruction, nature of materials and operations. However, their analysis of assessment activities on a six-category system, ranging from skills to situational problems, showed wide variation in teachers' emphasis, especially at the skill level. This calls attention to the importance of further studies on instruction which our study cannot contribute to.

A parallel question to the 'routine/non-routine' distinction is "What makes one exam question more difficult than another?" (Fisher-Hock et al. (1997)). This is the starting point of the Question Difficulty Project which examined UK mathematics and other subject examinations taken by 16 year old students. The project examined a model of question answering based on reading, application and communication through protocol analysis. The study notes the difficulty of both social and mathematical language, to the presentation of answers and the concomitant recording of steps (the latter two being particularly important for future work with algebraic calculators). Trials of variations in mathematics questions revealed 22 sources of difficulty, from command words to irrelevant information. The import of this for our study is the sheer number of factors impinging on what might make a question difficult (routine) or not.

This study and the work of Boaler alert us to the issue of context. Context is a term in mathematics education that is particularly difficult to define. Indeed, we believe that no definitive definition can be produced. It was the original intention of our study to examine contextual/non-contextual questions as well as routine/non-routine questions but the problems of finding real context questions and the problems of

characterizing context questions forced us to focus on the more manageable routine/non-routine distinction. We leave the question of context to further research.

Three further thoughts on routineness:

The relation between experience and the examination question is metonymic, at the level of syntax rather than meaning[2], i.e. the form of words used in a question, evokes a particular response. Arguably the most routine question at this level in the UK is, *find the equation of the tangent to __ at the point __.*

Students opinions of what are routine and non-routine questions are important, though it must not be assumed that students share a collective meaning of the term. We asked 100 students who had studied the paper we examined, in the course of their revision, to categorize the question parts, see figure 3. These responses generally show: only partial agreement with our categoization; several parts are neither generally perceived of as either routine or non-routine; greater apparent accord on routineness than on non-routineness.

	1i	1ii	1iii	1iv	2i	2ii	2iii	2iv	3i	3ii	3iii	3iv	3v	4i	4ii	4iii	4iv
total 'routine'	98	81	47	4	99	97	32	40	95	87	71	47	49	93	69	87	70
total 'non-routine'	2	19	52	94	1	3	68	59	5	13	29	53	51	7	31	13	30
Our categorization	R	N	N	N	R	R	N	N	R	R	N	N	N	R	R	R	N

Figure 3, student classification of routine and non-routine parts of questions

Finally we discussed routineness with the relevant mathematics officer of the Examination Board whose paper we used. They had a policy for question design: "In a standard question of four parts, the first two parts require an application of a standard algorithm, the next two parts require insight." This may be construed as a form of the routine/non-routine division and accords quite closely with our categorization in the paper considered.

Curriculum development and assessment development

This study is located in what may be termed 'traditional' examinations. Common themes of contemporary research in assessment are the purpose and validity of assessment and alternatives to traditional forms of assessment (see Niss, 1993). We applaud this but traditional forms will continue to persist in many countries and investigations into how students perform in them are important. This will be the case with the UK A-level examination (see http://www.qca.org.uk/aframe.htm and go to A/AS subject core consultation, then to Mathematics).

In the medium and long term scenarios, when algebraic calculators become commonplace in developed countries' classrooms, it is possible that the curriculum will develop to incorporate the potential offered by these calculators (see Browne & Ellis, 1997). Ideally examination questions, if not the form of examinations, will change in line with curriculum developments. In this projected setting the current

[2] Suggested by T Rowland in personal correspondence on the nature of routine questions.

study is partially misplaced as an attempt to second guess the future without taking account of parallel curriculum and assessment development. It is difficult, however, to envisage how a realistic 'future' experiment might be conducted for traditional examinations.

In regions that permit less rigid senior school examinations the potential problems exhibited by UK examinations are not necessarily present. Where teacher-generated examinations are allowed there is scope for simultaneous curriculum and assessment development. In some schools in Austria, for example, where regular use of algebraic calculators is made teacher-generated 'parallel' calculator and non-calculator examinations take place (Browne & Ellis, 1997).

Conclusion

There are many issues in this study which require further investigation: the effect of teaching and learning styles and of instructional content on students' performance in questions and question parts; an analysis of what makes an A-level question difficult regardless of apriori categorizations such as routineness; an exploration of the factors involved when students follow a question, of several parts, through.

Regardless of further research there is a need, in countries with traditional senior school examinations, to take the indicators in this study seriously lest, when algebraic calculators are allowed in examinations, that the examinations become more difficult to pass for a class of students. Although this study in not conclusive the indicators are such that we cannot ignore the equity issue present.

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PUPILS' IMAGES OF TEACHERS' REPRESENTATIONS

Chris Bills and **Eddie Gray**
Mathematics Education Research Centre, University of Warwick
COVENTRY CV4 7AL, UK.
email: chris.bills@warwick.ac.uk and serah@warwick.ac.uk

This paper suggests that learner's procedural and conceptual knowledge of mathematics is constructed, in part, from their mental representations of teachers' external representations. These mental representations reproduce the learner's perceptual experiences and are thus referred to as images. In the first instance individual images may be quasi-sensory or language-like in format. Some pupils form a general image of the structure inherent in the external representations whilst others have specific images which embody the surface characteristics of those representations teachers use to communicate the mathematics. For such pupils the medium becomes the message.

INTRODUCTION

Teachers use a variety of representations, involving verbal, written, pictorial and concrete material presentations, to communicate their mathematics to pupils. Pupils are asked to replicate the mathematical activity through the use of these 'external' representations. Their talk, writing, drawing or actions indicate the nature of their knowledge construction. This paper considers the mental representations pupils form from these varied stimuli and seeks to answer the question "What kinds of mental representations do pupils form from the representations given and how do they make use of them?" A preliminary study in an English school for 5 to 11 year olds suggests that pupil's internal representations are initially 'image-like' in that they reproduce, in part, their previous perceptual experiences. These images may not be visual or tactile, yet, when they are evoked for mental calculation, pupils use language associated with spatial or motor aspects of the external representation. The study also suggests that the pupils' construction of mathematical meaning appears to be based on their internal representations of the teacher's external representations. It is conjectured that the way in which these internal representations 'stand-in' for the original experiences varies between pupils of different abilities. It is also conjectured that individuals' mathematical knowledge construction may be transformed from a quasi-sensory mental representation (visual, auditory, tactile) to a more general, language-like, representation over different time periods.

The paper provides a brief review of some of the literature on representations and images which has influenced and informed the study and gives some detail of the Phenomenographic methodology. The focus for the initial classroom observations and semi-structured pupil interviews has been whole number place value and addition and subtraction of two digit numbers. The implications of these findings and recent research at the University of Warwick are considered and the conclusion is drawn that a longitudinal study of teachers' and pupils' representations is needed to investigate the influence of the one upon the other.

REPRESENTATIONS AND IMAGES

Lesh, Post, & Behr (1987) suggest that five types of representation are available to mathematics teachers: world contexts, manipulatable models, pictures and diagrams, spoken language and written symbols. Such representations are not *the mathematics* but transformations of the mathematics into communicable form, a process that Kang & Kilpatrick (1992) have termed 'didactic transposition'.

The way in which pupils form mental representations from their mathematical experiences is open to debate. The Piagetian view is that mental representations are constructed by the individual. This is, in a sense, a compromise between theorists such as Fodor, who assumes an *innate* representational language of thought, and others, such as Vygotsky, who insist on the priority of public external representations that are *copied* to become the internal representations (For a fuller discussion see Olson & Campbell, 1993).

The study reported in this paper starts with the same premise as Lesh & Kelly (1997), that the learner's internal conceptual structures cannot simply be received from others but are developed from, and expressed using, external systems of representation. The assumption is also made that aspects of the internal representation can be inferred from the children's use of external representations, in this case through their language and the physical movements that they use (see also Thomas, Mulligan, & Goldin, 1996). However, no precise claims can be made about the nature of these internal representations (Kaput, 1992).

The pupil's encounter with a teacher's representations is a multi-sensory experience and the first interest of this study is the image formed from the child's perception of this experience. An image "reproduces in part some previous perceptual experience in the absence of the original sensations" (Russell, 1956) or, more generally, it is a quasi-sensory experience that is a "concrete re-presentation of sensory, perceptual, affective or other experiential states" (Richardson, 1969). In this sense an 'image' of Dienes Blocks might be the almost tangible sense of feeling them or moving them, a recalled vision of them on a desk, the recall of a teacher talking about them, a memory of thinking how to add tens and ones, or remembered pleasure in piling them higher than anyone else.

In the field of visual imagery Kosslyn (1980, 1996) has demonstrated that images can have depictive, picture-like, qualities that could not be explained by purely propositional, language-like, mental representations. His model suggests that images are short-term memory representations generated from long-term memory representations that may have a depictive or propositional form. He also proposes a 'Representational-Development hypothesis' which has strong parallels with the development from procedural to proceptual thinking in some individuals (see for example Gray & Tall, 1994). Kosslyn suggests that the type of internal representation that is predominantly used changes with age and that later types of internal representation, being more powerful, supplement and eventually overshadow older ones.

Kosslyn further suggests that visual imagery will necessarily be used in response to a question about a concrete object where the information has not been stored as part of a propositional representation or can not be deduced from propositional representations. As propositional knowledge increases and deduction becomes easier then visual imagery may be used less. Indeed, he suggests that with the increasing use of facts the image is more likely to take on a propositional format. If the child has few representations in other formats he has little choice but to use his visual image. Imagery generation may be preferred because it is quicker than propositional reasoning which requires more or different processing resources. A similar point is made by Intons-Peterson & McDaniel (1991) that the less familiar we are with a task the closer is our imagery to the original perception.

Paivio (1986) refers to 'proto-typical' mental representations of conceptual categories that are either the best example or a composite of typical features of that category. Kosslyn (1996) suggests that proto-typical images tend to be stronger because they have been accessed more often but that a particular exemplar may be stronger if it has been refreshed frequently. In the context of this paper children may form proto-typical images from the variety of teacher's representations, for example a sense that 62 is 6 tens and 2 ones without reference to columns or blocks. Others may have an image based on the most frequently used exemplar (for example 'counting-on' for addition) or most recently used exemplar (for example Dienes Blocks for addition).

METHOD

The assumption made in this exploratory study is that the images formed by pupils can not be studied in isolation from the context of the classroom or the interaction between the pupils and teachers. It is therefore regarded as essential to observe the common experiences of the learners as a basis for the analysis of their different conceptualisations. The focus is on what the teachers and pupils say and do in the lessons and how the pupils make reference to their experiences when questioned in interviews.

The research approach adopted is a naturalistic qualitative one which can be termed 'Phenomenographic' in that it is an investigation of people's understanding of phenomena and it seeks to categorise and explain the qualitatively different ways in which people think about the phenomena. The initial discovery of previously unspecified categories of thinking may be peculiar to the researcher and context but the test of their validity is in their applicability for other researchers and as a source of explanation of differences in learning outcomes (Marton,1988).

The study was conducted in a school for 5 to 11 year olds in a large middle-income village near Birmingham, UK in the period October 1997 to July 1998. One lesson per week was observed with follow-up interviews with individual children. Audio transcriptions of children's interviews were supported by detailed field notes. The teacher observed is an experienced male and his 33 pupils were those judged by their previous teachers to be the most able of the 80 Y2 pupils (6 to 7 years old) in the school. Samples of these pupils were interviewed in October, March and July.

RESULTS

The discussion of the results will draw upon the several representations used by the class teacher (Mr. K.) to demonstrate 2-digit addition. These included a Number Track, ranging from 1 to 105, a Hundred Square, Dienes Base Ten Blocks, Numeral Cards, printed with single digits, and the written algorithm. The pupils practised a representation-specific procedure with each of the materials. Though each representation is structure-oriented, in the sense used by Resnick & Ford (1981), the 'transparency' of the correspondence between the material and the structure is variable (Meira, 1998). The validity of using such a variety of representations of the same mathematics has been questioned (Dufour-Janvier, Bednarz, & Belanger, 1987) but the intention of this study was to observe the effect of the variety not to criticise it. To trace the relationship between teacher's representation and pupils' internal representations over the 9-month period two themes are examined:

- The common experience of the pupils and their individual conceptualisations from the experience.
- The medium term proto-typical representations that are formed by the pupils.

Conceptualising from Experience

A typical lesson prior to the October interviews shows one of the teacher's representations involving Dienes Blocks used in demonstration mode:

> The teacher gave Mandy two tens (In response to the question "Another way of putting it?" Mandy said "twenty") and 4 ones. When asked "How many altogether?" Mandy said "Twenty-four". A similar demonstration was used with Nina. She was given 1 ten and 2 ones and responded correctly to similar questions. The teacher requested Mandy and Nina to "Now put them together in my hands". In response the two children put the *tens in one hand* and the *ones in the other*. The teacher then requested "How many altogether?" adding "Look how easy it is to add them instead of all individual cubes".
>
> Pupils were then directed to work on two-digit addition questions presented in the textbook as pictures of Dienes Blocks with written numerals

One week later, after some further practise with the representations, 11 pupils were asked to work out one question, 24 + 53 (presented on paper), in their heads . They were then asked how they had worked it out. Though none of the pupils spontaneously mentioned visualising Dienes Blocks, Elspeth's response has a clear trace of the teacher's representation:

> Elspeth: Well you add the tens together then you add the units because its like *in one hand* you have the tens and *in the other* you have the units

Elspeth was one of only 2 pupils who gave the correct answer to this problem. The other pupil counted on from 53 using fingers to help.

Though the above example lesson involves the teacher's use of Dienes Blocks a dominant representation, previously presented to the children, had been the Hundred Square used for two-digit addition. When the children were prompted to think about either a Hundred Square or Dienes Blocks (their choice), none could mentally

manipulate them without considerable assistance. Max's responses, however, suggest that he had formed an image related to the Hundred Square:

Max If it was 24 add 11 it would be 35 ... Because it was diagonal like that. (moves head to the right and down)

Asked what 24 add 10 was he replied "34" and explained

Max because you just go down and it would just go back under there .. 'cos that would go in tens and then that would stay in the same place and the tens would just go back under there.

In each instance it sounds as if he is describing a visual image yet when requested to move down 5 squares from 24 he could not do so. A possible explanation is that he has a 'global' image (Kosslyn, 1996) of a Hundred Square that lacks the visual resolution for him to scan very far over it.

Another instance where the internal representation is a consequence of what has been attended to and extracted (see Kosslyn, 1980) is given by Neal

Neal You get 24 in your head then add on the 5 and the 3. .. 32

Here he has attended to the separate adding of tens and ones.

A second indication of a relationship between the teacher's representation and the child's conceptualisation is provided from an example obtained in March. During a lesson that focused upon adding one to a three digit number a temporary teacher summarised previous teaching in the following way:

"You have been throwing dice and all sorts of things. You also looked at rolls of raffle tickets like this... $\boxed{}\boxed{186}\boxed{}$ (Drawn on board) What comes next? Why wasn't the 1 or the 8 changed?" A pupil replies "Because you are not adding tens or hundreds."

The teacher next wrote 199 in the middle position and again asked "What comes before? What comes after?" Going on to comment: "But that means I'm altering the tens and hundreds. That's because I can't have more than 9 in any column."

The teacher continued to talk to the group of pupils who had experienced difficulty. To illustrate "*going to*" the next number she held up 3 Numeral Cards and then *changed* the units digit card for a different one. She indicated that only the units digit changed except when the 9 is changed for a 0 and then the tens are changed as well. The children were invited to use similar cards to help them with additional exercises concerning raffle tickets. In the exercise which followed half of them made mistakes by altering the wrong digit.

The usual class teacher also makes use of these Numeral Cards to illustrate changing digits to increase a number by one, ten or a hundred. Interviews conducted during March included a question focusing upon children's conceptualisation of adding one. Fifteen children were asked: "What comes next after three hundred and seventy nine?" and then "How did you decide that?"

The results of these interviews suggested that three quarters of the children were influenced by these experiences. The separate digit reasoning of the pupils has traces

of the single-digit representation involving changing individual digits of a number. Two thirds of this 'influenced' group (9 children) obtained the correct solution:

Hazel	Because um three hundred and seventy nine we have to *change* the um ten, we have to change the 7 to an 8 and we take 9 to an oh.
Elspeth	Well when you add, you said that it was seventy nine, and add one on and it equals to seven, so, eighty because when you add one on to 9 it has a nought at the end.
Brian	Well you just um well um you don't think about the first number, well, you don't *change* the first number but you um you have to change the second two numbers because it's *gone on to* a nine and it's like *going on* to a ten

However, one third of those who made use of separate-digit reasoning attempted to add one to the wrong digits:

John	… 389 … I *changed* the ten to the next
Ann	4 hundred . 480 'cause if you just keep it on 3 it wouldn't be right
Christine	... 4, … 5 hundred, and, … fifty. Well I was thinking about, I added one more on to, the units makes ten and then I, added it on so it was 4 made 5, the hundreds, and 7 would … uh… becomes 5 little dots

The use of words such as 'change', 'changed' and 'gone on to' suggests that these particular pupils' representations are image-like in that there are spatial allusions to the teacher's representation and an echo of the teacher's words. It is conjectured that these representation-specific words are not those that the children would use if their mental representation were simply based on counting.

A final group of children, approximately one quarter of the number interviewed, reflected a more powerful form of knowledge construction based upon a deeper conceptualisation of numeracy.

Jack(1/4)	'cause you know 79, eighty , just add a hundred to it, 380.
Clara(20/4)	'cause um I know, I know the hundred well what comes after 79 is 80
Steph(20/4)	... 380 I added one on .... I added another one on.

Representational Consistency

The pupils were interviewed again in early July. In the period between the two interviews the children had many lessons in which adding tens by increasing the tens digit was demonstrated and practised with Dienes Blocks and Numeral Cards. Nine pupils were given verbally-presented questions requiring addition of ten in both interviews. There are striking similarities in the language they used in response. For example

	March interview: 81 add on 10	July Interview: 38 add 10
Cora	*ninety* one. You just add one, I know nine comes after 8.	I just add one onto the tens
Jack	*ninety one*. If you just go 8, 9 then you just make it into a ten and you put the 1 on, *ninety one*	… forty-eight. I know my ten times table and then I just put the eight on
Christine	well I, … sort of ignore the units for a minute and just added like a ten on.	I would leave the un… units and just add on ten

Mandy	well cause in Mr K's we did adding on ten and I remembered cause 81 add on 10 is 91	Well we did it in Maths and he just said you add on ten so it would be the same
Brian	well I just change the tens and I just leave the units.	Well 'cause all you have to do is add the ten. You leave the units

In July questions on 2-digit addition and subtraction showed that half of the pupils used the same form of words for each, indicating reference to similar images or perhaps a single image of 2-digit manipulation. Some children's representations were very close to the teacher's written algorithm which he based on Dienes Blocks. There is evidence here of a proto-typical image being recalled and described to explain their calculation

DISCUSSION

Gray and Pitta (1996, 1997) indicate that there are qualitative differences in the images formed and used by pupils of different abilities. They suggest that some children continue to mentally reconstruct the numerical procedure rather than encapsulate it. Lower achievers tend to use pictorial and iconic representations and attempt to mentally manipulate images of these. High achievers more often have symbolic images that appear to act as thought generators and memory aids. The divide between high and low achievers is also apparent in the images they project of concrete nouns (ball, car) and abstract nouns (five, fraction). Pitta goes on to conjecture that children may have a disposition toward different kinds of mental representations which transcends arithmetical and non-arithmetical boundaries (Pitta, 1998).

In this paper it is assumed that initially the image a learner uses when thinking about a piece of mathematics is an amalgam of verbal and non-verbal information derived directly from perceptual experience of the teacher's representation. Paivio (1986) notes that such a representation of past episodic experience has been refered to as a 'memory trace'. The 'image' is subsequently augmented, however, by the learner's experiences of using both external and internal representations. Furthermore an image may no longer be evoked when automatic recall of a known fact replaces it. It would appear that some pupils seem more capable than others of constructing an efficient proto-typical image that embodies the structure inherent in the representations whilst disregarding surface characteristics. Some pupils mentally manipulate digits to decide 380 follows 379, for others it has become a known fact. Some continue to count-on for 2-digit addition, others use a mental analogue of the written algorithm and yet others count on in tens and ones.

The preliminary study in this paper provides evidence of the relationship between the teacher's representations and the pupils' images. It also suggests that the images can remain unchanged over a short time scale. Knowledge construction in arithmetic and algebra requires a cognitive shift to encapsulate active aspects of arithmetic into numerical concepts. The abstraction, which is the essence of this shift, requires that the surface characteristics of, and actions on, representations that are didactic transformations of the mathematics be eventually overshadowed. It remains to be

seen whether or not Kosslyn's representational-development hypothesis is applicable to images used for mental calculation and place value and whether mental representations formed in mathematical and non-mathematical contexts follow similar development in individuals. A longitudinal study may resolve these issues.

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Problem solving processes in geometry.
Teacher students' co-operation in small groups: A dialogical approach

Raymond Bjuland, University of Bergen, Norway

This project investigates how students' mathematical understanding develops in a social context and when using mathematical language in small group dialogues. The empirical material has been collected by means of an ethnographical research method, while the analysis of group discussions and the reconstruction of the students' problem solving processes are based on a dialogical approach. The theory is presented in two categories: problem solving and cooperative learning. The participants were first term teacher trainees at a Norwegian college of education in the autumn of 1996. In this paper two episodes which illustrate some aspects of the dialogues are presented. The students have created an open atmosphere in the groups. Their reflections during the solution process and their social skills play an important role in order to develop their mathematical understanding.

1. Introduction

In this paper I will give a brief overview of my ongoing research. I am working on a doctoral thesis and the preliminary title is: '*Problem solving processes in geometry. Teacher students' co-operation in small groups: A dialogical approach*'. I focus on two perspectives.

The first perspective is to identify the students' mathematical thinking and understanding, their social skills and different affective characteristics. My research questions concerning the first perspective are: What mathematical thinking and which social skills can be identified? Which affective characteristics, like joy, frustration, willingness and endurance to work on the problems are prominent in the small group dialogue?

The second perspective is to analyse how students in three small groups develop their mathematical concepts through social interaction and to analyse how the students reflect on the solution process. Here I investigate similarities and differences in the three groups of students as far as the following research questions are concerned: How do the students reflect on the solution process? How do the students develop their understanding of mathematical concepts through the social interaction?

The empirical material is based on a group project on problem solving in geometry. The project was carried out on teacher trainees in their first term at a college of education in the autumn of 1996. One hundred and five students attended the course, and they were divided into groups of five.

The material consists of fieldnotes from observation in three randomly chosen groups of students with 8 lessons in each group, tape recordings from the same lessons and the reports of all the groups. So far, I have started a qualitative study of three groups of students where the students' dialogue in the groups and the reconstruction of the solution process for the mathematical problems are analysed.

In this paper two episodes which illustrate some aspects of the dialogues based on my first perspective are presented.

2. Theoretical Background

The theoretical material consists of five categories: problem solving, cooperative learning, affective aspects, classroom research, and social and cultural aspects. In this paper I focus on two of the categories: problem solving and cooperative learning. These areas play an important role when it comes to pupils' work on problems in a social context. The research literature is quite comprehensive for each of these areas and different traditions of research have different approaches. I therefore limit my presentation to some important studies related to my work.

2.1 Problem solving

Much attention has been paid to the concept of problem solving in the American literature from 1970 and onwards (Silver 1987; Schoenfeld 1985, 1992). In the literature the concept has been used with multiple meanings that range from 'working rote exercises' to 'doing mathematics as a professional' (Schoenfeld, 1992, p. 334). Schoenfeld defines a mathematical problem in the following way:

> 'For any student, a mathematical problem is a task (a) in which the student is interested and engaged and for which he wishes to obtain the resolution, and (b) for which the student does not have a readily accessible mathematical means by which to achieve that resolution.' (Schoenfeld, 1993 p. 71)

Schoenfeld (1993) also emphasises that it depends on the pupils' prior knowledge if a task is a real problem. He claims that most exercises in textbooks are not real problems since they often can be solved by means of a well-known algorithm. He points out that a problem must confront a student as a difficulty. It is this understanding of a mathematical problem that forms the basis of my work.

Metacognition is another concept which has played an important role in the American literature during the 80s and the 90s (Silver 1987; Schoenfeld, 1985, 1992). This concept, which is related to problem solving, also has different definitions in the literature. According to Barkatsas & Hunting (1996) there is however one definition which is generally accepted, having incorporated two important aspects of metacognition: to monitor and regulate ones own cognitive processes. The students' monitoring and reflections on the solution process are an important starting point of my analysis.

Two models are of vital importance as far as the mathematical solution process is concerned (Polya, 1945, Borgersen, 1994). The models show different stages in the problem solving process. The stages are not linear, but they must be seen as dynamic and cyclic.

Schoenfeld (1992) is concerned with developing a theoretical framework in order to analyse complex problem solving behaviour. He raises two important questions: What does it mean to think mathematically? How can we help pupils to think mathematically? His framework consists of five categories (the knowledge base, problem solving strategies, monitoring and control, system of beliefs and socialization). These categories form the basis of my analysis of the students' mathematical thinking.

Lester (1994) gives an overview of the problem solving research during the 70s, 80s and into the 90s. He is worried about the decline of research within this field in the USA. He emphasises that there is still a need for more research on problem solving instruction. Lester suggests that research should focus on the role of the teacher, the interaction between teacher-student and student-student and on groups and whole classes rather than individuals.

2.2 Cooperative Learning

The social psychologist, Kurt Lewin was concerned with cooperation in groups in the 30s and 40s. Morton Deutsch (1949) continued to work within this field. The tradition from Lewin and Deutsch went on in the 60s, 70s and 80s in the direction of more practical, methodological implications of teaching on all school levels. A method called 'Learning Together' which is developed by David and Roger Johnson at the University of Minnesota, has inspired Norwegian researchers to develop cooperative learning in Norwegian schools (Haugaløkken, 1987; Digre & Solerød, 1993).

Johnson & Johnson (1990) give some advice how cooperative learning can be used in mathematics. They suggest some basic elements in cooperative learning. These elements (positive interdependence, promotive interaction, individual accountability, interpersonal and small group skills, and group processing) have formed the basis of my analyses when it comes to the identification of the students' social skills.

In recent years, researchers of different theoretical traditions have focused on the activity of the classroom where the social interaction between teacher-student and student-student plays an important role. Cestari (1998) has used a dialogical approach in order to explore communication in the classroom. The aim of her research was to identify how mathematical concepts develop through language, in a dialogue between teacher and student (see Cestari, 1997a, 1997b). Such a dialogical approach has been the basis of my analysis of the communication in the small groups.

3. Method

Three groups of students were observed, and 8 lessons were tape-recorded in each group. I was a non-participating observer. The empirical material consists of these observations, fieldnotes and group reports. A dialogical approach has been used to interpret and analyse the conversation of the students. A dialogue is characterised by

'an interaction, in temporal and spatial immediacy, between two or more participants who face each other and who are intentionally conscious of, and orientated towards each other in an act of communication' (Markova & Foppa, 1990, p. 6).

This definition is the basis of my analysis of the dialogue between the students in their learning processes. I analyse each utterance by examining the turns both before and after the utterance. In this way, every single utterance is analysed and put into a wider context. An utterance is maybe caused by an earlier idea or a statement in the dialogue, and we get an idea of how the utterance is related in the context. By using this dialogical approach, I try to identify what kind of social interaction is established by the members of a group and how they verbalise their mathematical reasoning in this social context.

The transcription of the dialogue of each group session has been divided into episodes which are related to the research questions on the two perspectives. A new episode starts when there is a natural change in the conversation, maybe a pause or a new idea, a statement or a question that generates new thinking processes between the students.

Every single episode will be analysed on three levels. The first level analysis describes what the students say. The second level analysis is my interpretation of each utterance. On the third level I discuss my description and interpretation and link it to related literature. The levels are not separated, but together they will form a unit for each episode.

4. The context of the study

A new, private college of education was founded in the autumn of 1996 in Norway. The first year 105 students attended the school, and they all participated in the problem solving project on geometry in small groups. The students attended the course in their first term at the college. The students were divided into project groups of five by the administration (21 groups), and three groups were randomly selected for observation. In the project period all groups got their own room.

5. Presentation of data

In Bjuland (1998) I introduced four episodes from the dialogue of group A, which show some important aspects of my study. In this paper I present two episodes. The first episode from group A identifies some of the student reflections during the solution process, while the second episode from group B is concerned with the interaction of mathematical understanding and social skills. Both episodes are selected from the first group meeting of the groups, and the students are in the process of solving the following problem:

Choose a point P in the plane. Construct an equilateral triangle in such a way that P is an inner point and the distances from P to the sides of the triangle are 3, 5 and 7 cm respectively.

5.1 Student reflections as part of the solution process

The students of group A have only been working on the problem for some minutes. They have drawn a model, and they have constructed three circles with radius of 3, 5 and 7 cm respectively, with P as centre. This episode focuses on the students' first reflections on the solution process.

134. Unn: So, we have to construct the tangents...
135. Roy: Eeh...yes
136. Unn: Haven't we done this in an earlier problem?...
137. Liv: What?
138. Unn: Constructed the tangents?...(6 sec)...

.

. Roy and Liv study the figure of the triangle that Roy has drawn

.

143. Unn: We are able to construct those tangents...aren't we?...(9 sec)...
144. Roy: An equilateral triangle where P is an inner point and the distances from P to the sides
145. are three, five and seven centimetre respectively...
146. Liv: So, it is an equilateral...
147. Roy: Yes, that's true...you have a point there...
148. Unn: We have to read the problem...(Liv is laughing)...
149. Roy: Wait a moment...is it when all sides are equal?...
150. Liv: Yes...
151. Roy: Yes isosceles is two sides...this isn't correct then...(Liv is laughing)...then we have to
152. give up what we have done so far...
153. Liv: I think I write it on the blackboard (READ THE PROBLEM)
154. Roy: But...eeh...since it is an equilateral...then all angles must be sixty degrees...

Unn starts the episode by focusing on the construction of the tangents (134). She defines in a way the next step in the solution process. Roy has just drawn a model, and perhaps his brief response (135) suggests that he is busy studying it. The next question from Unn (136) shows that she tries to invite the other students to take part in the discussion on how to construct the tangents. It seems like the students have solved a similar problem before, and Unn is now reflecting upon whether the other problem could be helpful in the solution process. By looking back on an analogue problem, Unn tries to put the problem into a familiar context. According to the research literature (Polya, 1945; Schoenfeld, 1985; Borgersen, 1994), such a reflection could be an important problem solving strategy in order to succeed in solving a problem. The brief question from Liv (137) invites Unn to repeat the idea of constructing the tangents (138). The pause (138) could suggest that Unn has to wait for a response to her last utterance (138). Roy and Liv are still studying the figure of the triangle.

The dialogue shows that Unn repeats the idea of constructing the tangents (143). She has now repeated this idea three times (134), (138), (143). When her utterance is ended by a question (143), she is inviting a response. On the other hand, it seems as if Unn is patient and gives the other students some time (143) to reflect on her idea. Leder (1987) refers to some studies where teachers after introducing a question, only allowed students to think for one second before giving an answer.

Instead of responding to Unn's question (143), Roy breaks the silence by reading the problem once more (144), (145). We see that the students' solution process is not linear and straighforward, but it is dynamic and cyclic. When they look back and study the problem, Liv recognises that it has to be an equilateral triangle (146). The section (144) - (148) shows that the students have discovered that they missed important information the first time they read the problem. Roy reads the problem (144), Liv discovers the missed information (145), while Unn emphasises how important it is to read the problem carefully (148). We see how all the students contribute in the mathematical discussion.

Roy's question (149) may show that he is not certain of the differences between an equilateral and an isosceles triangle, but on the other hand, he does perhaps invite the group members to be aware of the characteristics of these triangles (149) - (151). Roy recognises that they have to give up what they have done so far (152), and Liv's writing on the blackboard suggests that she really wants to stress how useful it is to read a problem carefully before starting the implementation (153). It seems that the students have made a useful experience. Roy's statement (154) suggests that the group members have got a better understanding of these special triangles.

We see how the students' reflections could help them to put the problem into a familiar context. When they look back and read the problem once more, they also experience how important it is to study a problem carefully before starting work on it. The reflections may help the students to develop their understanding of an equilateral and an isosceles triangle. We also see that the students' solution process is not linear and straightforward, but dynamic and cyclic.

5.2 The interaction of mathematical understanding and social skills

In group B the students have read the problem, and they have constructed three circles with radius of 3, 5 and 7 cm respectively, with P as centre. The mathematical discussion shows that the students intend to construct tangents to the circles by constructing angles of 60 degrees and displace parallels to each of the circles. The students are not sure how to do it. Jon and Maj discuss how to continue the construction, while Eli helps Siv to construct an angle of 60 degrees.

263. Jon:	Are we ready to work together again?..(inaudible talk)...
264. Eli:	If you just try without the circle once...you just take a line...
265. Siv:	Yes...
266. Eli:	Then you put a dot on that line...

267. Siv:	Mmm...(5 sec.)...
268. Eli:	Then...
269. Siv:	Yes
270. Jon:	Yes, that's nice
271. Eli:	Then you put the compasses...
272. Siv:	We have some basic learning...
273. Jon:	Yes, that's nice

Jon's question (263) suggests that he and Maj may have got an idea of how to go on with the construction, and now he is concerned with introducing the idea to the other students. On the other hand, Jon's question could also be an invitation to the rest of the group to participate in the same discussion. It seems as if Jon is aware of how important it is to work together on the problem.

Jon does not get any response to his question. Eli goes on helping Siv with the 60-degree construction (264). It is important to notice that Eli does not want Siv to focus on the circle, but just to focus on the angle. Eli has perhaps seen that doing this construction could be a difficult exercise for Siv. It seems as if Eli tries to simplify the situation for Siv by suggesting that Siv should not think of the circles but 'just take a line' (264). The brief response of Siv (265) shows that she has understood what to do, and Eli then continues the explanation (266). The dialogue shows how Eli gives help to Siv step by step, by 'just take a line' (264), then 'put a dot' (266), and Siv makes brief responses (265), (267). The pause (267) could suggest that Siv gets time to think and reflect on each step.

It is interesting to emphasise that when Eli breaks the silence (268), she does not continue the step by step instruction (271) until Siv has confirmed that she is ready (269). The social interaction between the two girls shows an open atmosphere where Eli, who operates in the role of a teacher, is really concerned with helping Siv with the angle construction. Jon (270) indicates that he has been listening to the instruction and is aware of what the two girls are doing. Instead of being impatient to continue the construction, Jon praises the work of Siv and Eli. This also suggests how the group members try to establish an open learning environment. Siv informs the other students of the 'basic learning' (272) Eli gives her. It is possible that Siv feels her lack of knowledge is an obstacle for the progress of the solution process, but the response of Jon (273) shows that she has nothing to fear. He still appreciates what the two girls do.

The dialogue shows how one student is concerned with helping another student to understand a basic construction. We see that the group members are determined to have a common starting point in the solution process so they are able to participate in the mathematical discussion. We see the students helping, listening to, respecting, and encouraging each other. This suggests that the students have developed their interpersonal and small group skills, which according to Johnson & Johnson (1990), are basic elements in cooperative learning.

6. Summary

In this paper I have given a brief overview of my research on problem solving processes in geometry as teacher students cooperate in small groups. Two episodes are presented to illustrate some aspects of the group dialogues. The episodes suggest that the students really want to participate in the mathematical discussion, and it seems as if they already at the first group meeting have created an open learning environment in the groups. The students' solution process is not linear and straightforward, but dynamic and cyclic. The dialogues show that the students' reflections on the solution process and their social skills play an important role in order to develop mathematical understanding.

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CHALLENGING THE ESOTERIC: LEARNING TRANSFER AND THE CLASSROOM COMMUNITY.

Jo Boaler, Stanford University, California.

In this paper I use the perspective of situated cognition to analyse the results of paired interviews with 76 students from six schools. The students' perceptions about the individual, abstract or 'esoteric' nature of school mathematics environments are used to challenge traditional models of teaching, on the grounds that they problematise movement from the communities of practice of the classroom to those of the socially constituted World.

The Oxford Dictionary defines the esoteric as that which is accessible only to 'the initiated, not generally intelligible, private (and/or) confidential'. I intend in this paper to show that a range of features of the traditional mathematics classroom (Boaler, 1997a; 1998), contribute to the esotericism of the classroom environment and, in so doing, limit the usefulness of the mathematics that students learn. I shall draw upon data from an ongoing research project that is studying the mathematical learning of students in six schools in England (Boaler, Wiliam & Brown, 1998). The aim of the project is to monitor the impact of teaching method and ability grouping and the interaction between the two, upon students' understanding of mathematics. In a series of paired interviews with 76 students from the six schools, a range of factors emerged that the students cited as limiting their understanding. The relationship between these factors and the peculiar or esoteric nature of the mathematics classroom will be the focus of this paper.

Situated perspectives on learning (Lave, 1988; Wenger, 1998) move the focus of research away from individuals and their construction of knowledge (Lerman, 1996) towards the broader communities of practice in which people operate and the relations formed between people and systems of their environments. Lave has challenged traditional notions of 'learning transfer' as they suggest that knowledge exists, independently of the World, and may be taken from one place to another, impervious to context, situation or process of travel. But whilst the 'transfer' term may be inadequate because it suggests a view of cognition that is distinct and separate from the social world in which it is constituted, it is clear that people function in the World through a process of using, applying and adapting learned knowledge. One of the main purposes of school is to prepare students to use the knowledge they learn in the classroom, in the rest of their lives. Mathematics education appears to be particularly problematic in this regard, as a range of research projects have shown that students are unlikely to use the mathematics they learn in school in any other places (Masingala, 1993; Nunes *et al*, 1993), resorting instead to their own invented methods. I will suggest in this paper that this problem arises, in part, from the fact that mathematics teaching is often based upon narrow models of the mind and learning transfer.

Research Methods

The six schools in the study are located in five local education authorities in England. Some of the school populations are mainly White, others mainly Asian, while others include students from a wide range of ethnic and cultural backgrounds. Approximately 1000 students are being monitored as they move from year 8 to year 11. Research methods have included approximately 120 hours of lesson observations during years 8 and 9 (ages 12-13) and 38, in-depth, interviews with single-sex pairs of year 9 students. This has included 4 students each from a high, middle and low 'ability' group in the 4 schools that use ability grouping and students from a comparable range of attainment in the 2 schools that teach mathematics to mixed ability groups.

In interviews with the 76 students, students were asked, amongst other things, to describe what they liked and disliked about mathematics lessons, they were asked to describe particularly good and bad lessons, and they were asked to contrast their current experiences of mathematics lessons with experiences in previous years. In most cases the same questions were asked of students, but as interviews were open, allowing the interviewer to respond to issues that the students raised as important, some questions were not asked of all students. The conversations with students were coded, using a process of open coding (Glaser & Strauss, 1967). A number of issues emerged from the interviews that the students cited as significant to their learning of mathematics; four of the themes related to the unusual, particular or esoteric nature of mathematics classrooms in a particularly significant way and these themes will be the subject of this paper.

Research Results

Monotony

In the UK, mathematics teaching is characterised by a strict adherence to a particular scheme, with a scheme usually comprising a series of mathematics textbooks or workcards. In the six schools that are being studied, two use the SMP 11-16 scheme; one uses Oxford mathematics textbooks; one uses the SMILE workcard scheme, one uses 'Task Mathematics' textbooks, another uses NMP textbooks. All six of the schools rely upon their particular scheme to a large extent, with 90% or more of lessons requiring students to work through books or cards. In approximately 120 hours of observations, researchers observed students working through books or cards, with no practical, investigational or group work; although students did report that they were given occasional investigations or open-ended tasks each term.

At the beginning of the interviews all of the pairs of students were asked to describe their mathematics lessons. Fifty-two of the 76 students immediately communicated the lack of variety they experienced, with words like 'just' and 'every' being used in almost all of the student descriptions, for example "we just work through books every lesson". Sixty of the students were also asked if they could describe a lesson that was

particularly good, a lesson that stood out for them as being enjoyable. Twenty-two of the students simply answered that they could not. Two students laughed at the suggestion that a mathematics lesson could be *particularly* good, one said that she would have to "have a really long, hard think" and most explained that they could not think of such a lesson because mathematics lessons were "all the same". Twenty of the 22 students who said that there were no particularly good lessons were girls. Nine of the 19 students who could think of a good lesson, chose one in which they had abandoned their normal work and completed a project or investigation. Three students chose lessons when they 'didn't do any work', four students chose lessons by the same teacher who was popular, mainly because he used a variety of approaches. Only three students chose lessons involving the books or workcards that they used in the vast majority of their lessons. For example:

P: Every day we was copying off the board and just doing the next page or the next page or the next page and it gets really boring. (Paula, School A).

I: The lessons can be a bit tedious, the same thing every lesson.
J: Just the same thing for weeks on end. (Isaak & Jake, School F).

When students were asked to describe subjects that they particularly liked or that contrasted with mathematics, many of their descriptions centred upon variety:

I: For instance in English you're doing different topics, like once we did Shakespeare, now we are doing a magazine and stuff like that. (Ishak, School F)

The monotonous nature of school mathematics lessons was an important, distinguishing feature of mathematics for the students.

The Individual Learner

Grouping decisions are commonly made in UK schools, with individuals being moved into teaching groups according to perceptions of 'ability' or some other factor, with the assumption that groups are made up of separate individuals and that relationships between students have minimal impact upon their learning. Yet many of the students interviewed, cited their relations with other members of the group as the most important factor influencing their predilection towards mathematics. Four of the six schools in our study had recently changed the grouping of students from mixed ability, to 'setted' ability grouping, with students regarded to be of similar ability placed into the same groups for mathematics and taught work at particular levels. This meant a change in teacher and teaching method, as well as level of work, for the majority of students. Seventy of the students were asked whether they preferred working in mixed ability or setted groups. Fifty-one (73%) chose mixed ability groups, 19 (27%) chose setted (11 of these students came from intermediate groups, neither high nor low groups were popular with students – see also Boaler, 1997b). What was particularly significant for this analysis was that 31 of the students cited the relationships they had formed within groups as the main reason for their preference:

N: Some people, they don't like see what set they are in, they see what people are in their class. (Nigel, School R).

The impact of students' relationships with classmates, upon their attitudes towards and learning of mathematics, was a totally unexpected outcome of the research:

A: I prefer being with my class because you know everyone and you get on with more people. In this class you don't know everyone and it's difficult. (Aisha, School W).

P: I think it makes us better when we are as a form, because when we are as a form, that is you learn. Like, if you know that's like your group of people you don't feel shy to do anything in front of them or anything. (Paula, School A)

The importance of the relationships formed between students also emerged when students were asked about the way they moved forward in mathematics when they encountered a difficult problem. Forty-five out of 50 of the students asked, said that they found it more helpful to ask other students for help than the teacher. This suggests that the relations formed between students were formative at an important point in their learning, when they needed to learn something new and possibly experience cognitive conflict. Another indication of the importance of student relationships was revealed when students were asked to describe their favourite lessons. Many of the students' descriptions centred upon the rare opportunities they received to work with others:

R: I like the ones when we do experiments, when we are in a group, again. So you can work in a group, so if anybody is stuck on anything you can help people and if you are stuck you can ask people for help." (Ruby, School C)

C: Frogs (investigation) was good because everyone was involved.
A: It was fun because everyone like joined in with it and everything. We all had a go with it didn't we? (Carla & Ann, School R)

The significance students placed upon their relationships with other students is perhaps unsurprising, given that most adults would probably cite relationships with colleagues as important factors impacting upon job success and satisfaction. The formation of student relationships is however, a factor that is rarely considered by schools and absent from many analyses of learning. This seems to be particularly significant for mathematics education as the majority of mathematics classrooms in the UK place a premium upon individual work. In the six schools in our study students were allowed to talk to each other as they worked, but none of the teachers encouraged discussion as a form of mathematical thinking or learning, except for during occasional lessons. It seems significant that the social relations formed between students and the discussions they held with each other, were cited by many students as the most important feature of their learning, yet this social dimension was largely downplayed or ignored in the schools, by virtue of the mathematics approaches

employed. The students located their learning of mathematics within a broad, social domain, which is entirely consistent with situated perspectives on learning, whilst the schools regarded the students as individual learners.

Lack of Meaning

Many students regard the purpose of mathematics lessons to be the memorisation of procedures (Boaler, 1997; Schoenfeld, 1985). Teachers of mathematics introduce methods and procedures to students in the hope that students will learn and understand the procedures, as well as link the different procedures to the broader mathematical domain. But, as Mason points out, this does not always happen: 'To the teacher they are examples of some good idea, technique, principle or theorem. To students they simply are. They are not examples until they reach examplehood.' (Mason, 1989 p29). The distinction between the teachers' intention to demonstrate examples of a broader phenomenon, and the students' inclination to view the examples as facts to be learned, is revealed by the fact that teachers rarely regard mathematics as a subject which involves a lot of memorisation, whereas students often do:

> *F: It's because maths is different from other subjects. You have to know the facts and remember them, (...) remember the rules and stuff, remember which way goes that way and there's just a lot to remember. (Fiona, School W)*

In the six schools in our study, many of the students appeared to regard mathematics as a vast collection of rules and equations that held little meaning for them:

> *A: It's because there are so many equations and stuff.*
> *L: It's hard and it's boring.*
> *JB: It's different to other subjects then?*
> *A&L: Yeah.*
> *L: Some of the questions are so hard and so weird. (Aisha & Lena, School W)*

> *C: I look at it right, and it looks like Greek on the page, sometimes and it's like what? (Cheryl, School C)*

Conversely, when students talked about subjects they liked, they often related their preferences to the meaning the subjects held, and their relationship with the World:

> *H: (In science) you learn about normal things in life, that you don't really know, like energy. Energy and stuff and acids and all that, stuff in your own homes." (Harnack, School W)*

> *P: (In geography) you learn about people and places and you get to research and stuff – researching places, statistics from countries and things. (Peter, School H)*

> *C: History is like learning what happened in the past and how its affected us now. (Charlie, School W)*

The students talked about working hard in other subjects because they were genuinely interested in the content of the subject. Those students who were motivated in mathematics seemed only to be inspired by the prospect of gaining correct answers. None of the students' descriptions of mathematics gave any indication that the students were encouraged to appreciate the beauty of the subject, the creativity possible in the exploration of problems, or the links between mathematics and life:

DW: Do you ever work hard on something just because you are interested in it?
C: Yeah, but not in maths. (Colin, School R)

The distance between the World of school mathematics and anything that was meaningful or real for the students carries obvious implications for students' enjoyment of mathematics, but the students' perceptions also convey the esotericism of the school mathematics community within which they were required to operate. This may carry a significance that extends beyond enjoyment. Being good at mathematics in such a community, appeared to some students to involve being less than human:

M: Like we are robots. All we want to do is work like. But in the other classes it's different. (Mitch, School A)

The students' perceptions of the unrealistic nature of school mathematics may account for the fact that 30 of the 42 students asked, said that they could see no links between the mathematics of the classroom and the rest of their lives, for example:

S: It's got no connection. It's just something to make you think. (Suthida, School C)

M: I just think I am never going to see this again – you look at some things and you think – I am never going to see this again, so what is the point? (Moynur, School H)

A: You learn stuff that you think, oh God, what am I going to do with this? Why am I learning this? (Amy, School C)

Discussion and Conclusion

Situated theories posit learning as an 'aspect of changing participation in changing "communities of practice" everywhere' (Lave, 1996, p150). Students do not just learn cognitive structures and forms in mathematics classrooms, they learn to '*be*' school mathematics learners, becoming inducted into specialized and institutionalised forms of knowledge (Dowling, 1996). My concern for the students in the 6 schools in this study, as well as students in other specialised, esoteric mathematics environments, is that the students regarded the mathematics classroom as sufficiently strange and other-worldly that learning to '*be*' a mathematics learner, involved adopting the identity (Wenger, 1998) of an 'alien' (Mitch, school A) or, at the very least, someone who could abandon natural human desires to attain meaning and interact, socially, with others. In a previous in-depth study of students learning mathematics in two

schools (Boaler, 1997a; 1998; *forthcoming*), I found that students who learned mathematics in a traditional environment found it difficult using their school mathematics in non-classroom settings. They related this difficulty to the fundamental differences they perceived between the environments of school and the 'real world'. Such environmental differences impacted upon the students' use of mathematics. The results of these two studies both suggest that even when students learn a mathematical procedure in the classroom, if the community in which they learned mathematics is abstract, individualistic and generally esoteric, they will find it difficult adapting their participation in that community to any other.

In the late seventies and early eighties many schools and textbook publishers responded to the awareness that students face difficulties using school learned mathematics in their jobs and everyday lives, by placing mathematical examples 'in context'. But the use of pseudo-realistic contexts (Boaler, 1993) appears to have done little to enhance students' mathematical competencies in the 'real world'. More recently researchers have advocated the use of meaningful problems that provide the kinds of realistic constraints and affordances (Masingila, Davidenko and Prus-Wisniowska, 1996) that students are likely to meet in their lives. This is an important development, but I would like to suggest that our focus as mathematics educators should extend beyond the mathematics problems given to students, to the communities of the mathematics classrooms and the identities students develop in relation to these. In addition to providing students with the opportunity to use mathematics, and to choose, adapt and apply methods, we must recognise that students' learning is socially constituted and that students need to interact with the people and systems of their environment (Greeno & MMAP, 1998), in the mathematics classroom as they do elsewhere.

The suggestion that mathematics teaching approaches should offer varied, realistic constraints and engage students in discussion and negotiation is far from new. But the situated perspective adds another dimension to such proposals. For if learning mathematics entails more than the construction of cognitive forms, but of changing participation in a range of communities, then a classroom community that lacks the human and worldy qualities of social interaction and meaningful engagement, may 'bound' (Siskin, 1994) students' knowledge. Thus it is not the form of knowledge that is in question, but its accessibility. Classrooms that appear 'alien', esoteric or other-worldy to students may simply condemn their mathematical knowledge to nether reaches of their minds, producing learning identities that lack compatability with any other places.

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ABOUT THE GENERATION OF CONDITIONALITY OF STATEMENTS AND ITS LINKS WITH PROVING

Paolo Boero
Dip. Mat. Univ. Genova

Rossella Garuti
Scuola Media "Focherini" Carpi
and IMA CNR Genova

Enrica Lemut
IMA CNR Genova

Abstract *: Conditionality of statements (i.e. the fact that statements of theorems are implicitly or explicitly shaped according to the* "if A then B" *clause) has been a peculiarity of theorems throughout the history of mathematics and of various related fields. The aim of the research partially reported in this paper is to detect and describe a set of processes of generation of conditionality in statements (PGC) that is wide enough to cover the majority of PGCs that occur in different fields of mathematics. In this paper we will describe four kinds of PGCs, along with some productive links between these PGCs and the processes of construction of proof.*

1. Introduction

In our previous investigations we considered the conditionality of statements (i.e. the fact that statements of theorems are implicitly or explicitly shaped according to the *"if A then B"* clause) as a peculiarity of theorems throughout the history of mathematics and various related fields. (see Boero and Garuti, 1994). We also considered two possible ways of generating conditionality in the geometry field, and posed the problem of finding other ways (see Boero et al, 1996). In one case (that of generation of conditionality by a "time section" in the exploration of the problem situation), a strong link was detected in students' protocols between the process of generation of conditionality (PGC) and the process of construction of proof (see Garuti et al, 1996).

The aim of the research reported in this paper is mainly to detect a sufficiently wide set of PGCs and describe them in order to cover the majority of PGCs that occur in different fields of mathematics. We will describe four kinds of PGCs; we may add that no other PGC was detected in the examined protocols (see 4.1). In addition, as part of our continuing research on the cognitive unity of theorems (see Garuti et al, 1998), we will here describe some productive links between the PGCs and the processes of construction of proof (see 4.3.).

This research may have important implications for mathematics education: it seems to be possible (through suitable tasks) to let students experience different kinds of PGCs that are important in mathematical activities concerning theorems (see 5.).

2. Background Research

Psychology has always devoted much attention, in a developmental perspective, to reasonings concerning conditionality (a landmark contribution in this direction is the early scientific production of Piaget: (see Piaget, 1924, Chapter 2). More recently, psycholinguistic research has explored in depth the acquisition of the *"if... then..."* clause, analysing its context-dependence and its links with other aspects of mental development, in particular those related to mastery of causality (see French, 1985 for a survey).

On the mathematicians' side, processes related to producing conjectures and proving theorems have for decades been a fundamental point of attention: we may quote Hadamard (1949), Polya (1962) and, recently, Thurston (1994).

Progressively, this attention has shifted from descriptions of personal experiences or very general statements to more precise hypotheses.

Recently, research in the fields of logic, foundations of mathematics and artificial intelligence have converged on the need for understanding of how humans actually produce conjectures, prove theorems and exploit the knowledge thus acquired: *"We do not yet see how humans are able to discover proofs, we cannot yet explain how they affect the human mind"* (Robinson, 1998).

Educational research too has focused on the topic of analyzing processes of production of conjecture and construction of proof, in order to create suitable learning environments and tasks to enhance them. Recent contributions in this direction are the theoretical constructs of *"transformational reasoning"* by Simon (1996) and *"transformational proof scheme"* by Harel (1998).According to Simon,

"Transformational reasoning is the physical or mental enactement of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated" [...] "[...] transformational reasoning is a natural inclination of the human learner who seeks to understand and to validate mathematical ideas. The inclination, like many other inclinations [...] must be nurtured and developed [...]."

"It seems that transformational reasoning can serve several cognitive functions including, theorem generation, making of connections among mathematical ideas and validation of mathematical ideas".

In our research about historical-epistemological, cognitive and educational aspects of conjecturing and proving (see Boero & Garuti, 1994; Boero et al., 1995; Boero et al, 1996; Garuti et al, 1996; Garuti et al, 1998) conditionality of statements was a point of major concern.

Conditionality has been a crucial peculiarity of theorems throughout the history of mathematics and of all related fields. Heath (1956) points out how conditionality is always present in Euclid's "Elements" theorems, whether in explicit terms or in implicit terms. In the latter case, the statement can be reformulated in order to make the *"if A, then B"* clause explicit (for instance in the case of Pythagoras' well known theorem, the usual statement *"in a rectangular triangle, the square built up on the hypotenuse... etc"* can be reformulated as follows: *"If a triangle is rectangular, then..."*). We may remark that, today, statements of theorems do not differ from Euclid's as concerns conditionality. After Hilbert's revolution the epistemological perspective has changed considerably as concerns the nature of truth expressed by the statement of a theorem, the nature of postulates, the requirements of proof. However, the formulation of a statement in (explicitly or implicitly) conditional terms remains a peculiarity of theorems. Moreover, when we consider the conditionality of statements we do not limit ourselves to the textual property of statements. Its substantial importance in mathematical activities concerning theorems lies in the fact that the proving process keeps the *"if A then B"* clause as a crucial orienting reference for validating the statement. The difficulty is to match this evidence about the importance of the conditionality of statements from the epistemological point of view with a cognitive analysis of how it is generated during mathematical activity of conjecturing and how it is linked to the proving process.

Our research work on some PGCs detected in students' protocols (see Boero et al, 1996) pointed out some peculiarities of those processes, related to management

of virtual time and space variables in students' "inner visual space" (Vygotskij, 1978, Chap. I). In particular, we described in the following way a particular kind of PGC detected in students' protocols:

"the conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process that will be subsequently detached from it and then "crystallize" from a logic point of view ("if..., then...)".

We also found some links with the proving process(see Garuti et al, 1996).

Our next research work was aimed at finding other PGCs and determining more precise links between PGCs and proving processes. In this paper we will describe some kinds of *"transformational reasonings"* (Simon, 1996) that intervene in producing and proving conjectures (see 4.1.).

3. Method

Following occasional hints, a systematic investigation was performed on students' protocols concerning conjecturing and proving. We considered:

- beginners' written protocols (grades from V to VIII), in order to explore some basic PGCs not yet influenced by known patterns and complex analytic techniques; these protocols were used as sources of ideas about possible generative processes;
- undergraduate mathematics students' protocols, in order to validate the definitions deriving from preceding analyses and make them more precise and content-independent. Most were written protocols but some recorded dialogues with the teacher were also considered. Students were attending mathematics education courses on problem solving in the last four years.

A common production condition for all protocols was that in all cases the educational setting should stimulate students to write or orally express their thinking processes. In most cases this was done as real-time wording of their intuitions and endeavors, in others as on the spot reports about their reasoning. In the case of undergraduate students, this was done by systematically exploiting their written (or possibly recorded oral) solutions as anonymous texts to be discussed by their fellows, without any evaluation about correctness. In this way students recognized exhaustive wording of processes as a necessity in order to get interesting material for discussion. In the case of beginners, writing down reasoning was a part of the didactical contract in the classes engaged in Genoa Group Projects for primary and junior high school.

The fields of mathematics involved were:

- elementary plane and space geometry, arithmetic (properties of natural numbers) and elementary algebra for VII-VIII graders: five tasks with more than 20 protocols for each task;
- mathematical analysis, euclidean geometry, algebra and theory of numbers for undergraduate mathematics: nine tasks, more than ten protocols for each task.

The analysis of students' protocols was performed following these steps:

- first, detecting and trying to describe PGCs and their links with proving processes as they arose in single, clear protocols (see later for some examples);
- then, challenging the description through the comparison with other protocols (possibly by different students and in different fields of mathematics) that presented similarities as concerns PGCs and their links with proving; and

subsequently improving the description in order to make it content-independent;

- finally, trying to establish a common style of description among the different PGCs that had been detected, trying to get an overall vision of them and show any possible relationships, symmetries, etc. among them.

The reported results represent a final summary of what emerged during these analyses. They are quite complete as concerns the four detected PGCs, they are far from being exhaustive as concerns links with the proving processes.

4. Some Results

4.1. Processes of generation of conditionality

In order to make the presentation easier to understand, examples will precede definitions. The examples will be given related to different fields of mathematics so that "invariant" elements are highlighted. The following kinds of PGCs were detected in the students' protocols, covering the different fields of mathematics.

PGC1.

For some examples concerning VIII graders, see Boero et al, 1996. Here are some others.

EX.1.1.: geometry field, undergraduate students. Task: *"In the euclidean environment, formulate and demonstrate, a conjecture concerning the possible existence of a minimum area among the areas of all triangles obtained by closing an angle with straight lines passing through a point on the bisecting line of the angle itself".*

One student draws the configuration angle/bisecting line/point on the bisecting line and then draws several straight lines passing through that point. Initially, these are sharply very inclined with respect to the bisecting line, and on the same side; then come other lines close to the perpendicular line and finally lines on the other side that strongly diverge from the perpendicular line. Afterwards, the student states: *"It seems to me that the areas of the triangles decrease as they approach the position ... the perpendicular line ... I see triangles growing and growing on one side without any balance on the other side." (She shades in one large triangle emerging from the isosceles triangle and the corresponding smaller incoming triangle. Perhaps the conjecture is: if the passing through line is perpendicular to the bisecting line, the area gets its minimum".*

EX.1.2.: algebra, undergraduate students. Task: *"Let ax+by be an expression where a and b are positive integers, x and y integers; find out under what conditions on a and b the expression ax+by can assume its minimum positive integer value.".*

A student writes: *"Let me try: a=4 and b=6: 4*1+6*1=10; 4*2+6*1=14; 4*2+6*2=20. The results are increasing; but...I can also use negative values for x and y: for instance, 4*2+6(-1)=2; 4*2+6(-2)=-4; 4*3 + 6(-2)=0; 4(-4)+6*3=2. It seems to me that the results can not go lower than 2. I try with 3 and 5: 3*1+5(-1)=-2; 3*2+5(-1)=1. I reached 1, which is the minimum positive integer value. It is easy now, perhaps because 3 and 5 do not have any common divisor (but 1).*
The conjecture: if a and b do not have any common divisor (but 1), the minimum value is 1."

Generally speaking, a PGC1 can be described as a time section in a *dynamic exploration* of the problem situation: during the exploration one identifies a configuration inside which B happens, then the analysis of that configuration suggests the condition A, hence *"if A, then B"*.

PGC2.

EX. 2.1.: the study reported in Boero & Garuti (1994) concerned VII-graders who had to express in general geometric terms "Thales' discovery" (i.e. the anecdote concerning the determination of the height of a pyramid by exploiting the proportionality between the heights of objects and the lengths of the shadows they cast).

The following kind of reasoning was identified in 3 out of the 34 students: *"The length of the shadows is proportional to the height of the sticks;* sunrays *are parallel. But straight lines might not be parallel. If the straight lines are parallel, the lengths of the segments cut by another two lines will be proportional"*

EX. 2.2.: undergraduate students. Task: *"Can we always represent f(x)=sin(Ax+B) by finite linear combinations of products of integer powers of sinx and cosx?"*

"It seems so, by applying the trigonometric formulas. For instance, if A=2 and B=3, I can write: $\sin(2x+3)=\sin 2x.\cos 3+\cos 2x.\sin 3=2\sin x\cos x.\cos 3+(\cos^2 x-\sin^2 x).\sin 3$.

But A might be π also: we cannot write sinπx in that way. And also A=1/2 does not work: we would need roots. On he contrary, if A is an integer, it works".

Generally speaking, a PGC2 can be described as: noticing a regularity B in a given situation, then identifying, by exploration performed through a *transformation* of the situation, a condition A, present in the original situation, such that B may fail to happen if A is not satisfied.

PGC3.

EX. 3.1.: algebra, undergraduate students. Task: *"Generalize the following property: the sum of two consecutive odd numbers is divisible by 4". Demonstrate the property found".*

"Let me consider 3 consecutive odd numbers, e.g. 3+5+7=15 or 5+7+9=21. It seems to me that only divisibility by 3, which is the number of addenda, emerges. I shall try with 4 consecutive odd numbers: 3+5+7+9=24; 5+7+9+11=32; 1+3+5+7=16. What do 24, 32,16 have in common? They are divisible by 8. I shall try with 6 consecutive odd numbers: 1+3+5+7+9+11=36; 3+5+7+9+11+13=48. Both sums 36 and 48 are divisible by 12.

If there are 4, the sum is divisible by 8. If there are 6, sum divisible by 12. If there are 2, we have seen that the sum is divisible by 4. It seems to me that what is emerging is that the sum of an even number of consecutive (odd) numbers is divisible by its double, by the double of the numbers of addends I am adding.

EX. 3.2.: V-graders (cf. Bartolini Bussi et al., to appear). Task: *"Ascertain what happens when the number of cog-wheels, engaged and arranged in a ring configuration, increases, having already found that three can not turn all together, while four do".*

"(One pupil draws 5 wheels and indicates the rotation direction with arrows) "*5 wheels can not turn*; (draws 6 wheels) *6 can turn* (draws 7 wheels*). It could turn with 4 but with 3 it could not. So, if the numbers of cog-wheels is even, they can turn. If it is odd, they can not".*

Generally speaking, a PGC3 can be described as a 'synthesis and generalisation' process starting with an *exploration* of a meaningful sample of conveniently *generated* examples.

PGC4.

EX. 4. 1.: (undergraduate students) In the task presented in EX. 3.1., a student begins considering 8 consecutive odd numbers and finds out that the sum is divisible by 16. He writes: *"It may be that the double of how many numbers I am adding is influential in someway, but it might depend on the fact that 8 is a power of 2"*. He considers ten consecutive odd numbers, and finds out that their sum is divisible by 20. *He concludes conjecturing that: "If n is even, the sum of n consecutive odd numbers is divisible by 2n".*

EX.4.2.: (V graders) In the task presented in EX.3.2. a student acts as follows: He draws 6 engaged cog-wheels, and marks each of them with a clockwise or counter-clockwise arrow alternatively. *"With 6 wheels, it all turns well, but if I put one more* (he draws a small wheel between one pair of wheels and draws two arrows beside it (one clockwise and the other counter-clockwise) very close to the two wheels it is contacting*; this wheel prevents the others from turning. It is an odd number. It is like with 5 wheels with respect to 4. If they are odd, they can not turn."*

Generally speaking, a PGC4 consists in a reasoning which can be described as follows: the regularity found in a particular *generated* case can put into action "expansive" research of a "general rule" whose particular starting case was an

example; during research, new cases can be generated (cf. Pierce's “abduction” ; see Arzarello et al. 1998)

4.2. Some Comments

We may observe that PGC1 and PGC2 are, to some extent, dual processes. Indeed, in the first case mental exploration (centered on B) leads to detection of A as an arrival point, while in the second case the starting point is the regularity, and then dynamic exploration starts (by transforming the situation where the regularity occurs). We may wonder whether there is a common underlying cognitive background. N. Douek (personal communication) suggests that in PGC1, exploration leads to the "cause" that originates B, while in PGC2, exploration reveals the "cause", whose lack may make B fail to occur. In this way links emerge with the hypothesis of "causality" as one of the possible backgrounds of conditionality (cf French, 1985).

PGC3 and PGC4 too are, to some extent, dual processes: in PGC3 extensive exploration leads to intensive insight; in PGC4 intensive exploration leads to a local insight, which in turn gives rise to extensive exploration that may confirm it and make it more exhaustive. N. Douek (personal communication) suggests that PGC3 implies the passage from the analytical description of several cases to an expression able to synthetize (some of) them while PGC4 involves the passage from a more or less synthetising expression of a particular case to a more general one suitable for wider application. In both cases, the passage from one representation to another seems to play a major role.

Bearing in mind preceding descriptions of PGCs and comments, we may expect that apriori analysis of the task (formulation and content) could to some extent predict the PGCs that will be produced by students. In particular, in a task aimed at discovering a singularity, we may expect that most PGCs will be of the PGC1 and PGC2 type, while in a "generalization" task most PGCs should be of the PGC3 and PGC4 types. The examined protocols confirm this prediction. For instance, in the case of the "generalizing and proving" task of EX. 3.1 and EX.4.1. only PGC3 and PGC4 were detected in the 43 protocols examined (with the exception of one student who produced his conjecture through a PGC2-type exploration).

In the examples considered, students produce only one PGC; in general, we observed that in many cases the same student tries and abandons different PGCs before getting a conjecture he/she finds satisfactory. But in the case of the undergraduate students we also noticed quite frequently that a generation of conditionality can be reached through a sequence of coordinated steps, each of which bears a peculiar PGC (possibly different from those found in the other steps).

4.3. Some links between PGCs and construction of proof

We have detected an important link between the PGCs described in the preceding subsection and students' proving processes under the same task: *frequently the same mental exploration which leads to the conjecture is re-started by the student with entirely different functions during their proving process.*

For example, as concerns PGC1, exploration can move from a support to the selection and the specification of the conjecture (in the conjecturing phase), to a support for the implementation of a logical connection (in the proving phase): some

examples are reported in Garuti et al (1996). Here is reported the beginning of the proof produced by the student of EX. 1.1.:

"I draw again the situation in a careful way. (she draws angle, bisecting line and the perpedicular passing by the chosen point, then another straight line passing through this point. She shadows the outcoming and incoming triangles) *I see that the outcoming triangle is much bigger that the incoming triangle. But why is it bigger? Let us see. The area is base multiplied by height* (she draws the heigths coming out from the vertex lying on the bisecting line). *Yes, the heigths are equal and the bases are very different (etc.)* " (underlying puts into evidence the point where the resumed exploration of the situation becomes explicitly functional to proving).

As concerns PGC4: during the proving process, some students re-start from the particular case in which the regularity was detected, and then extend to other cases in order to find appropriate links between the hypotheses and the thesis; the function of the exploration changes from *"what regularity this is a case of "* to *"what link this is a case of"*. An example is the proof by the student of EX. 4.1.:

"We start again from 8 consecutive odd numbers; we try to write in that case the formula;: $2K+1+2K+3+2K+5+...+2K+15=(2K).8 + 1+3+5+...+15$. I see that I obtain the sum of 8 consecutive odd numbers; also this sum must be divisible by 16; but this fact remind me that in the sum of 15 numbers there is the factor 16... No, here we do not have 15 numbers; but we have all the odd numbers from 1 to 15. I have to find the formula that fits this case. I am remembering perhaps that it is n^2 . Let us check. With 2 odds it is 4. With 4 odds it is $1+3+5+7=16$. It is OK. The square of 4. But how can the square of 8 be divisible by 16? Yes, it is. $64=16x4$. Let us check if it is true in general. n is even. Hence $n^2=(2m)^2$ is divisible by 2n".

(underlining puts into evidence "abduction" phases during the proving process).

Some students who had produced conditionality through a PGC3 also revealed this kind of behaviour in the same task; and (on the contrary) some students who had produced conditionality through a PGC4 realized, during the proving process, an exploration similar to that exemplified in EX. 3.2. This seems to confirm the existence of deep links between PGC3 and PGC4 (as dual processes).

5. Possible research developments and educational implications

The content of the preceding subsection raises interesting research problems about the links between the conjecturing process and the proving process in the perspective of the cognitive unity of theorems (see Garuti et al, 1998). In general, the exploration underlying a PGC and the exploration performed during the proving process are very similar in "nature" but differ in "function". What is the precise meaning of these two words? Another interesting research development concerns modeling of the possible links between PGCs and proof construction processes, especially when the task *"Demonstrate that.."* requires the appropriation of a conjecture produced by others and then the production of new lemmas through PGCs, with related demonstrations (a typical situation in advanced mathematics).

And, naturally, the problem of identifying possible PGCs that differ from the four described in this paper still remains open.

Some connections with results produced by other researchers emerge from our analyses. We would particularly point out the need for in depth comparison of PGC2 and PGC4, and Balacheff's "crucial experiment" and "generic example", although the two criteria of analysis, ("cognitive" in our case, and "epistemological" in the case of Balacheff) are different. Emerging connections bear deep, potential points of contact between epistemological and cognitive analyses.

As to the educational implications of this study, preceding analyses can be exploited to find appropriate tasks for students in different grades, so as to allow

them to experience processes which seem to be relevant in mathematical activities concerning theorems. Indeed, we have remarked that the formulation and content of the task may influence students' PGCs (see 4.2.). Naturally, the interest lying in these considerations is related to an hypothesis of "educability" of the capacity to produce PGCs by experiencing them (cf. Simon, quotation in Section 2.).

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Working with pupils' meanings: changing practices among teachers enrolled on an in-service course in South Africa

Karin Brodie
Department of Education, Wits University

This paper reports on the second-phase of a research project which investigates the changing practices of teachers on an in-service course at Wits University in South Africa. The paper looks at the teachers' mediational strategies, in particular how teachers elicit and work with pupils' mathematical meanings. It focuses in detail on the teachers' use of groupwork, their responses to pupils' ideas, and their use of questioning, highlighting some of the difficulties they experience. It reflects on the role of the courses in facilitating teachers changing practices.

Introduction

In this paper, I will present some interim results from an ongoing research study which focuses on the changing practices of mathematics teachers who are studying on an in-service course, the Further Diploma in Education Programme (FDE), at Wits University. The main aim of the study is to investigate the influence of the FDE programme on teachers' practices. This paper will focus on some of the teachers' mediational strategies, in particular those that might be seen to be aspects of a learner-centred approach, and will reflect on the ways in which the FDE programme influences teachers' development of more learner-centred practices.

The FDE Programme

The FDE Programme is a two-year, mixed mode (distance and residential), in-service programme for mathematics, science and English teachers, and school managers. Students enter the programme with a 3-year, post-matric, teaching qualification. Successful completion of this course gives equivalence to a 4-year qualification. An aim of the course is to enable teachers to improve their qualifications through undertaking studies which are of direct relevance to their teaching. The programme's goals are to contribute towards improving teaching and learning in South Africa through: extending teachers' educational, subject and subject teaching knowledge; developing teachers as competent, reflective professionals; and enabling teachers to work with curriculum innovations (among others). The programme started in 1996, and about 130-150 teachers register each year.

Mathematics teachers on the FDE programme take five courses over a period of two years. Three of these are mathematics courses, two of which focus on mathematical content and one on teaching approaches. The other two courses are general Education courses, which all FDE students take as core courses.

In the next section, I will give an overview of two of the courses. *"Theory and Practice of Mathematics Teaching"* (Dikgomo et al, 1996) was developed and is taught by a team of mathematics educators. Its focus is on the teaching and learning of mathematics from theoretical and practical perspectives. *"Curriculum and Classrooms"* (Brodie and Purdon, 1996) is a core education course which was developed by two members of the Wits Education Department and is taught by a team of education tutors, led by myself. It focuses on generic issues (across subjects) relating to curriculum, curriculum innovation, and teaching and learning.

The FDE courses

A broad overview of the course materials for these two courses shows that both take a broadly social-constructivist view of learning. They attempt to counter behaviourist approaches to learning and present some of the ideas of Piaget and Vygotsky (to different extents) as alternate ways of understanding learning.

"Theory and Practice of Mathematics Teaching" deals explicitly with classroom management, and with groupwork under the title of "co-operative learning". The section on classroom management attempts to develop a sense of a classroom culture where the learners are disciplined and take responsibility for their own learning, and where mathematical thinking is developed. It includes a section on classroom questioning, pointing out the benefits of open questions and giving examples of them. Much practical advice is given to teachers on how to develop a culture of mathematical thinking and learning in the classroom. The section on co-operative learning provides a short justification for co-operative learning based on social-constructivist theories, and then gives different ways in which teachers might use and develop groupwork and co-operative learning in the classroom. Other issues dealt with in the course are the use of resources (blackboard, OHP, textbooks, "manipulatives", calculators, computer software); problem-solving and investigations; and professionalism.

"Curriculum and Classrooms" deals with the notion of learner-centred teaching as teachers eliciting, listening to and attempting to understand pupils' meanings and building on these to develop pupils' knowledge. This notion is developed out of Piagetian and Vygotskian theories, and examples of teachers engaging and not engaging with pupils' meanings are discussed. Bruner's notion of scaffolding is developed in detail, whereby teachers listen to and work with pupils' ideas and provide guidance and support to help pupils develop their ideas. Questions are seen as part of scaffolding and the kinds of questions and responses that might be useful are discussed. Many examples are given and teachers are asked to identify examples of good and poor scaffolding in their own lessons. Both courses emphasise the role of the teacher in learner-centred classrooms. The teacher is seen as mediator of knowledge and as crucial to the development of mathematical meaning on the part of pupils. Both courses also work substantially from transcripts and case studies, and students have reported that these are very helpful as models for their own teaching (Adler, Lelliott and Reed et al, 1998, pg 156-157).

The FDE teaching staff attempt, very consciously, to model teaching approaches that we think are useful. During the residential session, students work in groups for much of the time, particularly where there are classes with large numbers of students. However, lectures are also given, even in classes of up to 200 students. We structure tasks carefully, ask open-ended questions and attempt to give time for students' interests, difficulties and concerns to be taken up in tutorial sessions. Mathematics content is taught in investigative and problem-solving ways, in disciplined and relaxed environments.

The research

The research project is a three-year study of a sample of mathematics, science and English teachers who enrolled on the FDE programme in 1996, the first year in which we ran the programme. Data was collected by a team of researchers. We visited the teachers in their classrooms in August 1996, 1997 and 1998. The 1996 visit provided for baseline data (Adler, Lelliott and Slonimsky et al 1997) and the 1997 visit focussed on changes and continuities in

teachers' practices, in the context of their schools (Adler, Lelliott and Reed et al, 1998). This paper focuses on data from the 1997 analysis.

The school contexts in which the teachers work vary considerably, both materially and in relation to the culture and atmosphere established for teaching and learning. Some schools, predominantly those in urban areas, are relatively well-off, with electricity, laboratories, and functional classrooms, where there is usually enough space for the number of pupils. Most rural schools do not have electricity nor running water, and many classrooms are without windows, floors or ceilings. In many cases there is not enough space to accommodate the number of pupils. Some schools function well, with clear time-tables and procedures. At others, pupils and teachers are not always in class, with much late-coming and absenteeism. During the data-collection week at least two days of teaching were lost in some schools, due to cultural events, teacher- and civil-service strikes.

There are ten mathematics teachers in the sample[1], five primary and five secondary teachers. Observations of 3 lessons of each teacher were made according to a structured observation schedule, and narratives of each lesson were written. A video was taken of each teacher teaching one lesson. Each teacher was interviewed at length about her/his teaching and about conditions in the school. Principals in each school were also interviewed. Pupils' books were examined in order to get a sense of coverage throughout the year, and to see whether written work differed from oral work in any way. Tests and/or exams were also looked at in order to investigate formal assessment strategies.

The data for each teacher was analysed qualitatively by different members of the research team according to a set of categories developed by the team. An overview was then written looking across the teachers[2]. This paper will report on part of the overview, looking across the teachers through the category of mediation, with subcategories: groupwork, working with pupil responses and questioning.

A shift to groupwork

Perhaps most significant among the mathematics teachers in 1997 was a shift to groupwork. In 1996 only one teacher included organised group activities in her lessons, and three others had pupils seated in rows but working in pairs[3]. In 1997, seven teachers explicitly organised groupwork and nine teachers, in at least one of the observed lessons, had learners either seated in groups and working together, or seated in rows but working in pairs. Only one teacher was coded as having pupils work individually in all three lessons. All teachers expressed positive attitudes towards groupwork, though they varied in whether and how groupwork is implementable in their classrooms.

Ways in which groupwork is structured and mediated varied across levels and teachers. The

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[1] See Adler, Lelliott and Slonimsky (1997) et al for details about how the sample was chosen.

[2] The mathematics data was analysed by myself, Mamokgethi Setati, Phillip Dikgomo and Jill Adler. An overview analysis was then written by myself and Jill Adler.

[3] Interestingly, the one teacher who used groupwork last year, did not do so this year. She had lost a lot of school time, and was working to try to catch up with her Grade 11 and 12 classes.

secondary teachers who used groupwork did so with standard textbook tasks. In one case these were written up on the board, and in the second a worksheet was distributed, one to each group. The groups were relatively small, with four to six pupils in a group. In most cases, the pupils interacted substantially with each other in the groups, there was a "buzz" in the classroom, and most pupils seemed to be on task. The two teachers circulated and interacted with the groups, clarifying instructions and helping with difficulties. One of these teachers had 68 Grade 10 pupils in a classroom built for 40. She was able to organise the class very quickly into groups of four, indicating that pupils were used to groupwork. Her pupils involved themselves in the task set, she circulated among them (not easy with so little space), and in so-doing demonstrated that successful organisation of group work is not necessarily a function of class size. In the smaller class, pupils from each group wrote their solutions on the board, these were compared and discussed, and mistakes corrected.

Groupwork in the primary school was very different from the secondary school. Groups were more explicitly organised and structured, pupils generally sat in their groups most of the time, and there were usually at least 6 pupils in a group, sometimes more, depending on the size of the class. The primary teachers' ways of working with groups also differed from the secondary teachers, and were worrying in some ways.

In one lesson, a Grade 3 teacher called one group up at a time to her desk to demonstrate measuring the table. While she worked with each group, the rest of the class sat unoccupied. She only managed to work with two groups during the lesson. So, although she explicitly organised the class into groups, her method of working with the groups ensured that there was very little pupil participation or pupil-pupil interaction in the lesson, and that very few pupils actually spent time on the task. In this teacher's second lesson, each group worked on a different pen and paper task (standard algorithmic tasks such as long subtraction, fractions etc). There was absolute silence while only one pupil in each group worked on the task. The children who were not working watched the child who was. This means that the benefits of interaction were not achieved, while the benefits of individual work, ie each child getting a chance to work on a task were diminished. Report backs from these groups involved a pupil from each group writing up the solutions on the board. Some were repeated by the teacher and mistakes were corrected. No comparisons were possible because the task done by each group was different. Moreover, each group's work was rubbed off before the next one began. So most pupils had no record of the work in their own group nor the work of the other groups.

A second primary teacher (Grade 7) also had pupils sitting in groups with very little working together and interaction. At times pupils interacted with each other covertly, by whispering, which suggests that pupil-pupil interaction is frowned on in this class. The pupils had cuisenaire rods, and there were enough for each pupil, but many did not do anything with them. Thus again, many pupils were not on task.

The other three primary teachers who used groupwork enabled more interaction between pupils in the groups. Resources were shared between pupils, and they worked together. However, even in these classes, it is not clear that all pupils benefitted. In one grade 7 class, where about 40 pupils were divided into 6 groups, it was clear to the observer that at most half of each group was on task and contributing at any time. The teacher knew this, but her only way of mediating in the classroom was to tell everyone to contribute and listen to each other (advice which comes from the FDE courses). This advice was not very helpful for pupils who did not know how to contribute or how to allow others to contribute.

In looking across primary and secondary teachers, the following features of groupwork are evident. Primary teachers have clearly organised groups who sit together. They tend to give less standard tasks, using resources brought in from outside the classroom (measuring tape, cuisenaire rods etc...). None of the primary teachers intervened to deal with the mathematics while the pupils were working. They preferred to wait for report-back sessions to do this, ie they are concerned with maintaining a particular form of organisation of groupwork, rather than with the substance of what happens in the groups. In the following section, I will show that they struggled to deal with unexpected responses from pupils which makes it difficult to mediate effectively in groups. The secondary teachers on the other hand, mediate the work in the groups more actively. They seem more able to deal with pupils' productions, probably because they have not attempted to vary the tasks substantially, and because they are more confident mathematically. However, even in secondary classrooms, there are questions about actual pupil-participation in groups. In the case of one of the secondary teachers (Grade 12), the observer noted that the teacher did not know how to enable girls to participate, even though she was aware of less participation from girls.

The above analysis suggests that although most of the teachers are taking up the use of groupwork from the courses, they may be doing so without attention to the details of how to work with groups, particularly how to intervene mathematically. Moreover, none of the teachers spoke explicitly to pupils about how to work in groups during the observations, and it is not evident that they had done so previous to our visits. None of the techniques for co-operative learning discussed in *"Theory and Practice of Mathematics Teaching"* (eg jigsaw, think-pair-share or pairs-check) were evident in any of these lessons. This suggests that teacher education programmes need to think carefully about how to work with teachers so that they don't only "do groupwork", but use it well. We need to identify difficulties and concerns that teachers have, such as those described above, and deal with them explicitly in our courses.

Working with pupils' meanings

In the above section we have seen teachers who did not attempt to work with the pupils' constructions, for example the Grade 3 teacher in the report-back session. Other teachers however, did attempt at times to mediate and work with pupils' ideas during report-backs or in whole class discussions. Here too, they experienced difficulties as the following example will show.

The teacher is working in Grade 7 with cuisenaire rods. She has structured the activity by getting the pupils to make up a bigger rod with a number of smaller rods. After one has been made the pupils chant: "two yellow rods make one orange rod". The teacher then asks what fraction the yellow rod is of the orange rod and the answer "half" is given. Then, working in their groups, the pupils make up other combinations of rods and the teacher asks some pupils to show the class. Here is an example which was discussed in the whole class:

<table>
<tr><td colspan="3">green</td><td></td><td></td></tr>
<tr><td>white</td><td>white</td><td>white</td><td>white</td><td>white</td></tr>
<tr><td colspan="5">yellow</td></tr>
</table>

The pupil who constructed these rods said: "one green and two whites makes one yellow", and "the green is a "third" because "it takes three whites". This pupil demonstrated a particular understanding of the task, ie put any combinations of smaller rods together to make a bigger rod, they do not necessarily have to be equal. He also displayed some confusion relating to naming

fractions, ie he called the green rod a "third", rather than the white. The teacher reinforced his contribution with "very good" and asked the other pupils what he had said. Here she got: "The green rod is a third because three white rods are equal to the green rod", which emphasises the conceptual confusion. At this point the teacher tried to work with the pupils' meanings, by saying that the "green rod is now the whole" and the "three whites stands for the parts". In fact she reversed their meaning in order to bring them in line with her understanding. Then she concluded with "one whole and three parts, therefore this is a third" (and it is not publicly clear what 'this' is - but given the prior articulations, pupils could think that "this" refers to the green).

In this lesson, the use of concrete apparatus in the groups did not enable the development of mathematical concepts because the teacher's way of talking about the part-whole relationships obscures the fraction concept. Moreover, she did not explain the prior conceptual link: between the number of small rods that make a big rod, and the name of one of the small parts. Given some of the pupils' mistakes, this link was not understood by the pupils. Even more worrying is that the mathematics that is spoken about is incorrect. The teacher and some pupils might well understand that it is the whites that are the thirds of the green and not the other way round, but this is not explicitly dealt with. An incorrect way of naming mathematical relationships in English remains in the public domain, and was moreover confirmed by the teacher.

Although this teacher manages to achieve some pupil participation in the class, she does not know how to deal with offerings that she does not expect, for example the fact that one green and two whites make a yellow. She does not take this opportunity to work with the notion of fractions as equal parts of a whole. This teacher cannot utilise pupils' offerings to deal with aspects of the concept that they are struggling with. The teacher confirmed in her interview that she was aware of a problem with this example, and had tried to turn or shift the expression around, but nowhere was there any clear rearticulation of appropriate and relative wholes and parts.

Similar examples were observed in at least three other classrooms, two primary and one secondary (Adler, Lelliott and Reed et al, 1998). In all cases, the teacher struggled to know what to do with pupils' ideas and meanings. The *"Curriculum and Classrooms"* course explicitly deals with engaging pupils meanings in the section on learner-centred teaching. The students often use the examples presented there to illustrate points about learner-centred teaching in assignments and exams. However, when they are confronted with similar situations in their classrooms they struggle to manage them. An important question for the study is why this is the case. Is what teachers do and don't recognise and admit into classroom discussion related to their own mathematical knowledge? Or are they still working with conceptions of teaching which suggest that they must give the pupils the knowledge? If the latter is the case, then they will be working with two contradictory notions simultaneously, that of the courses which says they should elicit and hear pupils meanings, and another, more transmission-oriented view which does not facilitate their making use of pupils meanings. Perhaps the courses do not focus enough on how to work with pupils' ideas, focussing rather on the fact that this is important.

"Opening up" and "closing down"

In the Grade 7 example above, we have seen a teacher who attempts to allow meaningful participation in her class, but struggles to deal with the responses that pupils give. "Opening up" the classroom to pupils' meanings does not ensure that the discussion will remain open. Teachers can easily "close down" what started as open-ended discussion.

Our findings overall regarding questioning were that most teachers' questions remained narrow and required one word answers, factual recall or procedural explanations. Two of the primary teachers and three of the secondary teachers did ask questions that required elaboration or explanation, but these were generally answered procedurally by pupils. Four teachers (three primary and one secondary) did ask more open-ended questions or set open-ended tasks. However, even in these cases, only the secondary teacher was really able to probe the pupils' understandings through these kinds of questions. Even in this case, the pupils' explanations were often about definitions or were procedural, so deep conceptual thinking was not probed.

The following example of "closing down" occurred in one of the primary lessons. The teacher was trying to make links between tessellations and tiles in pupils' houses, and asked pupils where they see shapes outside of school. One pupil answered "tins", meaning that he saw circular shapes in coldrink tins. This was so different from what the teacher wanted to hear, that she thought he was struggling to express himself and asked him to explain in his main language, TshiVenda. Eventually, after much prompting by the other pupils, she acknowledged his answer and gave her answer, "tiles". So her open question had a closed answer, a perfectly acceptable response from a pupil could not be heard by the teacher because she was expecting something different.

In one of the secondary classrooms, the pupils were working on a problem on the board and obtained the solution -0. This happened twice and the pupils were interested in finding out about it. However, the teacher ignored their questions and continued to explain how to cancel algebraic fractions. In this case a possibility for interesting discussion was closed.

In contrast to this are two other secondary teachers. One teachers' questions are not open-ended, however she uses them effectively to scaffold pupils' knowledge. She does work with pupils' meanings, although these are more likely to be what she expects, because her questions are less open. Another teacher encourages pupil questions and therefore enables pupils to set the agenda for discussion. This teacher listens to pupil questions and answers them, which may qualify as "closing down", but since it is in response to pupils' own questions, he is working with pupils' meanings.

The teachers' use of open and closed questions suggests that the situation requires more complex analysis and practice than a mere shift from closed to open questions suggests. Two teachers use closed questions effectively and teachers who try to use open questions encounter difficulties. Wood (1992) argues that research (in the Northern hemisphere) has shown that teachers struggle to raise the level of cognitive demand of questions. This research confirms this finding in a different context, and suggests that a more nuanced view of what questions actually do and don't achieve and the range of purposes for asking them, may help teachers to use them more effectively.

Discussion and Implications

For the FDE Programme there is much to reflect on. The most commonly used 'new' approach is groupwork, which is used with varying success. Secondary teachers seem to use it more effectively, perhaps because they keep the tasks standard. Primary teachers, and some secondary teachers, struggle to deal with pupils' meanings. Important questions for the programme are: why does this happen and are the courses able to deal with it. Can we anticipate teachers' difficulties in our courses or can we only provide a basis from which teachers can embrace the difficulties they experience along the way as a natural part of changing? On the basis of my analysis here it

seems clear that some "fine-tuning" of ideas, introducing nuances and texture are important.

This paper provides evidence that teachers who have been involved in almost two years of an INSET programme which attempts to help them to develop their practice, experience difficulties in doing so. They certainly do manage to try out some new ideas. In relation to learner-centred practices, some elements are easily taken up, for example organising pupils into groups, while others are more difficult to deal with, such as asking open questions and dealing with pupils' meanings without closing down the discussion. The difficulties that these teachers experience suggest that teachers with less time to think about change or less input as to what it might mean would struggle even more. This adds to my argument (Brodie, 1998a) that it is not helpful to exhort teachers to change to learner-centred practices without clarifying what these might be. More work has to be done, with teachers, based on research into their teaching, on what new concepts and practices might mean in a South African context, and what happens as teachers begin to try them out in their classrooms.

All of the teachers in our research study showed strengths and weaknesses in their uptake of the ideas in the various courses and in their practice. Some of the differences across teachers can be explained by the levels at which they teach, what other aspects of change they are trying to manage simultaneously, their own personal preferences for which courses are the most useful (for example some teachers found mathematics content courses empowering while others found the education courses more useful), and by their contexts, some contexts are clearly more enabling of change and development than others (see Adler, Lelliott and Reed et al, 1998). How they have managed to work with certain ideas in their classrooms will come from an interaction between the individual teacher, her context, what she has learned from the courses, and what she has learned from other sources, including the prominent new Curriculum 2005 discourse. It is not possible to untangle the effects of all the disparate influences on a teacher, to be able to attribute particular changes to a particular programme or course. Nor is it desirable to do so, because the teacher-in-context is always part of and contributing to a range of influences on her practice. However, research into teacher practices can be formative for the development of courses, identifying what can be assumed and what needs to be further developed in more sophisticated ways.

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NEEDING TO USE ALGEBRA - A CASE STUDY

Laurinda Brown, University of Bristol, Graduate School of Education

Alf Coles, Kingsfield School, South Gloucestershire, UK

In the UK there has been a move away from teaching algebra to pupils aged 11. Kieran (quoted in Sutherland, 1997) has suggested three components of algebraic activity: generational, transformational and global meta-level. A comparison of the work of two high achieving 15 year olds with two high achieving 18 year olds gave evidence for Kieran's model as a useful way of describing algebraic activity and prompted the question: would it be possible to work meaningfully with 11 year olds on all three components of algebra? We link this question with Sutherland's (1991) call 'Can we develop a school algebra culture in which pupils find a need for algebraic symbolism to express and explore their mathematical ideas?' This paper analyses, as part of a project funded by the UK Teacher Training Agency (TTA), the work of one 11 year old pupil who has needed to use algebra.

Background

Issues relating to algebra have formed conclusions to two recent reports (Winter et al. 1997, Sutherland, 1997) into mathematics teaching and learning at secondary schools in the UK. Winter et al (1997) is a national report in which algebra is identified as a key component in facilitating a smooth transition for pupils between school and higher education in mathematics. Pupils' algebraic skills were often found to be in need of attention by teachers at the start of higher education courses. The Sutherland (1997) report was the outcome of a Royal Society (RS) and Joint Mathematical Council (JMC) working group, set up in 1995 *to make recommendations about the teaching of algebra* (p. ii), partly in response to *the apparent lack of articulation between mathematics taught at school and that required by higher education* (p.ii).

In the RS/JMC report a key conclusion is that: *the National Curriculum is currently too unspecific and lacks substance in relation to algebra. The algebra component needs to be expanded and elucidated - indeed rethought* (p.iii). A further conclusion is: *that more research is needed to understand the relationship between what algebra is taught and what is learned* (p.iii).

Both these conclusions sparked our interest in looking at algebra in secondary schools. In particular, we wanted to explore Sutherland's (1991) challenge:

Can we develop a school algebra culture in which pupils find a need for algebraic symbolism to express and explore their mathematical ideas? (p.46).

What is algebra?

In asking ourselves the question: *what is algebra?* we wanted to find a definition with which we could work to try to understand what pupils actually did which could be described as algebraic activity when they were engaged in doing mathematics. In reviewing current research on algebra strands emerged to do with context, meaning-making, complexity and control which we found useful in our thinking. Introducing and using algebra in a context is talked about from a view which we support that:

Traditionally, algebra in schools has been dealt with at a syntactical level; the students have no 'meta-control'; they know that they are allowed to do some things and not others, and obviously they sometimes make mistakes ... to improve the situation one can call to mind an algebra which is always linked to a context; not necessarily to the (often unreal) 'real world problems', but to the properties of numbers, or to the manipulation of functions, in all cases where it is necessary to interpret the result (Menghini 1994, p.13).

We see the important task as making symbol representation meaningful rather than as a submission to the Cockcroft report (1982) expressed that: *Mathematics lessons are very often not about anything. You collect like terms, or learn the laws of indices, with no perception of why anyone needs to do such things* (para 462).

... it was the lack of this (linking symbols to the situations they represent) that led to failures in the past teaching of algebra: the children who failed thought of x and y as meaningless marks that had to be played with by peculiar rules (Sawyer, quoted in Anderson 1978, p.20).

One argument is that this meaning might be achieved through working with pupils on thinking mathematically, where algebra is one component:

One major part of the effort to reform secondary school mathematics is the project of changing the goal of studying school algebra from mastery of symbolic manipulations to the ability to reason mathematically (Yerushalmy 1997, p.431).

Pupils need some fluency in symbolic manipulation, however:

The manipulation of symbols is only a small part of what algebra is really about, the traces that are left behind after mathematical thinking has taken place (Mason 1992, p.5).

One implication of this is that:

symbolic manipulation should be taught in rich contexts which provide opportunities to learn when and how to use those manipulations (Arcavi 1994, p.32).

In other words algebra should arise from complex situations:

Algebraic symbolism should be introduced from the very beginning in situations in which students can appreciate how empowering symbols can be in expressing generalities and justifications of arithmetical phenomena ... in tasks of this nature, manipulations are at the service of structure and meanings (Arcavi 1994, p.33).

There is never an end-point in this conception of learning mathematics. If I am learning to reason mathematically to structure my thinking about problems, then what I learn is in an ongoing state of complexification and enrichment.

Here we had our link to the challenge (Sutherland, 1991) of creating a school algebra culture in which pupils find a need for algebraic symbolism. The need we envisage here is for expression of awarenesses within complex situations. This clearly places onus on us as teachers to create a classroom culture in which there is the possibility for

pupils to work at and attempt to express what they are aware of. What we are prepared to notice and able to perceive is to a large extent dependent on the culture around us, and the language available to us.

We view the developing culture and ethos of our classroom as a community of practice (Lave and Wenger, 1991), where the practice is mathematical inquiry (Schoenfeld, 1996). The learning of algebraic thinking is part of learning mathematics and is situated in the classroom interactions.

A unifying strand through all these quotes is the sense of algebra as an evolving language that can emerge from situations and contexts that are already laden with meaning. Algebra can be used to express and offer insights into those situations. It is in this emergent expression and consequent empowerment that pupils can discover a need for algebra.

We have taken the following defintion of algebraic activity from the Sutherland (1997) report:

(i) Generational activities which involve: generalizing from arithmetic, generalizing from patterns and sequences, generating symbolic expressions and equations which represent quantitative situations, generating expressions of the rules governing numerical relationships.

(ii) Transformational activities whch involve: manipulating and simplifying algebraic expressions to include collecting like terms, factorizing, working with inverse operations, solving equations and inequalities with an emphasis on the notion of equations as independent 'objects' which could themselves be manipulated, working with the unknown, shifting between different representations of, function, including tabular, graphical and symbolic.

(iii) Global, meta-level activities which involve: awareness of mathematical structure, awareness of constraints of the problem situation, anticipation and working backwards, problem-solving, explaining and justifying (Kieran, quoted in Sutherland, 1997, p.12).

Within the discussions of the working group set up to write the report (Sutherland, 1997) this definition was the one which covered every member's interpretation of what algebra is. Such a broad definition allows us, as teachers, to work on our recognition of what algebra is in what the pupils do.

Methodology and methods

The definition of algebraic activity that we have chosen supports our need for a way of looking at what pupils do which will in turn transform our perspectives of how to work with the pupils in the culture of the classroom. We work within what Bruner (1990) called a 'culturally sensitive psychology':

(which) is and must be based not only upon what people actually do but what they say they do and what they say caused them to do what they did. It is also concerned

with what people say others did and why ... how curious that there are so few studies that (ask): how does what one does reveal what one thinks and believes (p.16-17).

We are interested in focusing on what we and the pupils in our classroom do and, consequently, we use an enactivist methodology (Reid, 1996, Brown and Coles, 1997, Hannula, 1998) the two key features of which are firstly: *the importance of working from and with multiple perspectives, and the creation of models and theories which are good-enough* for, *not definitively* of (Reid, 1996, p207).

theories and models ... are not models of. *That is to say they do not purport to be representations* of *an exisiting reality. Rather they are theories* for; *they have a purpose, clarifying our understanding of the learning of mathematics for example, and it is their usefulness in terms of that purpose which determines their value* (Reid, 1996, p208).

The second key feature of enactivist methodology is that we take multiple views of a wide range of data:

The aim here is not to come to some sort of "average" interpretation that somehow captures the common essence of disparate situations, but rather to see the sense in a range of occurrences, and the sphere of possibilities involved (Reid, 1996, p207).

We see our *research about learning as a form of learning* (Reid, 1996, p208) where our learning is gaining a more and more complex set of awarenesses about our teaching of mathematics.

In a pilot project two pairs of high achieving pupils, one pair of 15 year olds and one pair of 18 year olds were interviewed as they worked on a problem set by Alf. This problem could be tackled algebraically. The major difference between the two pairs of pupils was the control with which the older ones first explored the problem numerically, until they had some sense of what was going on, and then moved effectively to an algebraic representation and solution showing evidence of all three of the components of algebraic activity. The 15 year olds, on the other hand, reached for the symbolism quickly but became bogged down in the transformational work.

This experience led to our asking the question: would it be possible to create a classroom culture of 'being a mathematician' with 11 year old pupils so that when they themselves were aged 15 they would be operating as the 18 year olds did? We started work in September, 1998 on a project, funded by the TTA, to investigate this question with one mixed ability class taught by Alf. In order to gain evidence of the pupils' developing algebraic awarenesses we have stressed their need to write about their ideas and conjectures when doing mathematics and periodically we have asked them to write about 'what I have learnt?' both in terms of mathematical content and 'being a mathematician'. We have been surprised at the sophistication with which this class has developed and extended the culture of 'being mathematicians' but that is part of a larger project and not our concern here. In this paper we focus on the evidence of developing algebraic competence within one child as mathematician.

The case of Alex: needing to use algebra

This case study, told in three stages, illustrates one eleven year old pupil's developing use of the three components of algebraic thinking (Kieran, quoted in Sutherland, 1997) over the first term in secondary school, specifically leading to an example of the pupil (Alex) finding their own need for algebra. The interviews with Alex at the beginning and end of term, his exercise book, his writing on *what I have learnt?*, a half-term review, base-line entry test data and notes from observations of class lessons form the data set from which the following three incidents have been selected.

The first activity the class tackled was a rich numerical problem that lasted for seven lessons. In that time the teacher was using strategies to allow pupils to raise many questions within the group although everyone also had a lot of practice with the processes of basic addition and subtraction.

Stage 1: Algebra introduced by teacher but not used by pupil

Some of the questions pupils raised involved wanting to know why 9s fell in particular places in the calculations. The teacher recognised that one way of answering these questions was to use an algebraic demonstration, since no pupil was using algebra. Alex had not used algebra before. On being interviewed after the seven lessons he remarked that *basically all of it in my primary school was sums* and further that ideas of proof were not used at primary school. After the demonstration 11 out of the 27 pupils could recreate the manipulations and 8 were able to extend the techniques to show other similar results within the problem. We did not, however, expect pupils to be able to reach for algebraic technique in a different context (the situatedness of learning, Lave and Wenger, 1991) nor were we concerned that the majority of pupils might not be able to reproduce the original demonstration at this time. The possibility of using algebra to know why things work as they do was now within the community of practice (Lave and Wenger, 1991) and from another viewpoint the zone of proximal development (Vygotsky, 1978) of the pupils and this was our main purpose in introducing the algebra. At this stage Alex thought of thinking like a mathematician as *you've just got to ask yourself why is it doing this?*

In the first interview Alex was invited (by Alf the interviewer), to try numbers in a problem which he had not seen before. He quickly spotted a difference of 3 (Fig 1):

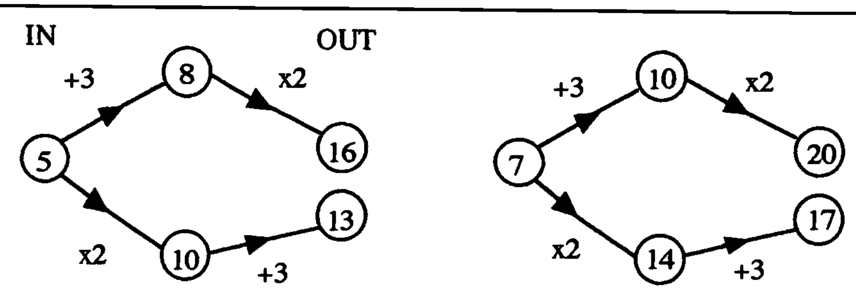

Fig 1: Alex's first two 'both-ways'

When Alf asked: *You said that being a mathematician is about asking questions so what's your immediate question?* Alex replied: *Does that happen with every number you put in?* In working on this question he tried out 'minus numbers' and decimals. It was evidently not natural to let a letter stand for any number and explore the consequences.

There is evidence of generational activity here since the pattern of 'there's always a difference of three' was spotted. But despite recognising 'why' as being a mathematical question Alex does not ask himself *why* in this context and consequently does not display global meta-level awareness within this problem. Algebraic symbolism was not used so there was no evidence of transformational activity either.

Stage 2: Algebra used by pupil in response to teacher's question

Fig 2 below is taken from a half term review given to the class which involved some questions to explore how they were getting on with algebra and an end of half-term 'what have I learnt?'. Here, in response to the prompting in the text of the question, Alex is able to work through the problem using a general letter N (even though there is no explicit invitation to use N in the statement of the task) demonstrating some transformational skills. We believe he is able to share 2N + 4 by 2 to get N + 2 because of awarenesses formed through the numerical process of trying a few examples first. Alex is effectively using the skill of multiplying out brackets, but no algorithm for this has yet been taught. Alex recognises that the sequence of instructions always results in 2 and so uses generational activity.

3) Try out this trick with different numbers ... write down anything you notice ... can you prove anything about this trick?

Think of a number	1	5	100	N
↓ Double it	2	10	200	2N
↓ Add 4	6	14	204	2N+4
↓ Halve your answer (share by 2)	3	7	102	N+2
↓ Take away the number you first thought of	2	2	2	+2
↓ ANSWER	2	2	2	2

The Answer always comes out as two. If you look at the sequence most are alimaneting its selve eg Think of a number, Take away the numer you first thought of!!! same with Double it and Halve your answer, if there was no "add 4" it would come to 0 but there is a "add 4" when that is halve it leaves you with "2" the answer.

Fig 2: Question 3 from Alex's half term review

In commenting: *most are alimaneting its selve* (we think this means 'eliminating themselves') we would intepret a global meta-level appreciation of the structure of the trick and in reaching for the N also a global meta-level awareness of the power of using symbols, although this happens in response to another's questioning.

Stage 3: Algebra needed to answer a question posed by pupil

In the second interview with Alex, at the end of the first term, Alf posed the same problem as in the first interview, but with different numbers. Alex tried one more example and commented: *The one I've just done was 6 difference and the same for that one there.* As in the first two incidents, Alex displays generational activity in noticing a numerical pattern. In response to: *What questions are around for you as you notice a pattern like that?* He replies in a similar way to before: *Does it work for all of them?* Previously this statement led him to try out decimals and 'minus numbers' but after one more numerical example, without speaking, this time he produced the following algebraic solution (Fig 3):

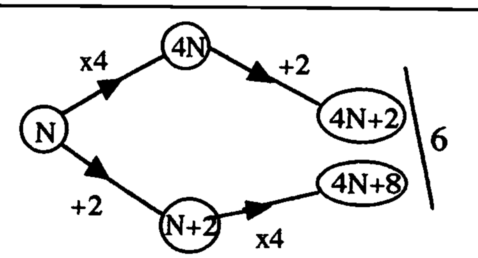

Fig 3: Alex's algebraic 'both-ways'

There is certainly evidence here of transformational activity and this feels like the 18 year olds' interview because Alex gains control of the process before using algebraic skills. Even more surprisingly Alex returned to a numercial problem and said: *I know what's making it 6 difference now, with the N. Because the bottom way - I can't say it. But that 7 it's going to be more than just timesing it by 4 straight away and adding two on the end. Really you're timesing the 2 plus the 5 by the 4 that way. It's hard to explain. So, that one would be 4n plus 8. So, these two cancel out each other leaving 6 behind. So now you know every one's going to go to 6.*

Alex clearly shows evidence here of insight into the structure of the problem, a global meta-level awareness, which, unlike at the half-term review, is also articulated. The difference that strikes us here, compared to the first two incidents, is that the algebra has arisen from a question of Alex's. In recognising a pattern and asking himself 'why?' in this new context he creates a need. His experiences over the term allow him to answer this need with the use of a letter N to stand for a general number. As he worked through the general case the structure of the problem was illuminated: *I know what's making it 6 difference now, with the N.*

In commenting on the process of his solution, Alex recognised the power of N standing for any number: *I should have done that first off.* Alf, in reply during the interview, tells him that it is good to start with the process and we would argue that Alex's need for algebra came through the posing of his own question: *why?* and that this came out of a pattern spotted (generational activity) after the process of doing a few examples.

His transformational skills, in contrast to the second stage, appear less dependent on numerical awarenesses since it is in the transforming that he gains structural insight. It is beginning to feel as though Alex will not need to be taught algorithmically many of the transformational skills needed in secondary school eg how to multiply out brackets.

Conclusion

The question we are working on for the TTA is whether 11 year old pupils would be able to operate algebraically like the18 year olds in our pilot study by the time they were 15 years old. Evidence so far would suggest that some of the pupils will be able to achieve this facility much earlier than age 15. Alex has developed over 15 weeks from no experience of algebraic thinking to using algebra to illuminate his thinking in relation to a problem. The cognition here is in the developing practices and language of 'being a mathematician' and the theories *for* our practice as teachers are in the stressing of the importance of writing to encourage the pupils' awareness of awareness and working together on why the patterns within problems and structures exist.

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TEACHERS' DOUBTS ABOUT INVENTED ALGORITHMS

Anne Buzeika and Kathryn C. Irwin

Auckland College of Education and University of Auckland

Two teachers experimented with allowing their students to invent ways of doing multi-digit calculation. The students showed considerable success and number sense in doing this, impressing their teachers. The methods that the students chose were closely related to teachers' statements or example. Despite seeing the value of this to the children, the teachers were uneasy about continuing to allow them this opportunity in the face of existing socio-historical practice and beliefs. We argue that the strength of socially and institutionally sanctioned mathematical practices prevented them from seeing this activity as central to children's developing number sense.

In many parts of the world, the concern that children understand calculation has led to an emphasis on developing number sense rather than mere fluency in the use of algorithms. It has been well documented that children develop a meaningful understanding of numbers if given the opportunity to use their own procedures (e.g. Carpenter, Frenke, Jacobs, Fennema & Epsom, 1998; Kamii, 1989). However, teachers vary in their willingness and ability to teach in a manner that encourages children to understand numbers rather than use an algorithm.

This study followed two teachers as they encouraged their children to develop their own ways of doing multi-digit computation. It views not only the classroom as a site of social construction of concepts, but also the teachers' wider community as a site in which beliefs and pedagogical practice are constructed. We argue that the beliefs of this wider community had a strong influence on their ability to allow children to develop their own calculation procedures. Using a Vygotskian analysis, the group knowledge of teachers in this country is of a pedagogy that leads only to use of a traditional algorithm. The underlying number sense behind this pedagogy has not become sufficiently individualised for individual change. This point is

related to that made by Ensor (1998) who discusses teachers' belief systems as social rather than individual.

New Zealand has had a mathematics curriculum since 1992 that suggests that students develop their own ways of calculating and develop flexibility and creativity in applying mathematical ideas (Ministry of Education, 1992). No where does it recommend teaching the traditional algorithm. However, the vast majority of teachers continue to teach algorithmic methods for multi-digit addition, subtraction, multiplication, and division. The debate in teaching circles in this country continues to centre on the use of regrouping as opposed to the equal additions method that the teachers themselves used as students. This persists despite the fact that the work of authors recommending emphasis on number sense and invented algorithms is well known by mathematics educators in this country.

For many years in New Zealand it has been the tradition to teach children how to do multi-digit calculation through the use of place value blocks. The method of instruction required students first to add double-digit numbers where no regrouping of the blocks was required. The next step was to solve a problem in which there were too many unit blocks, which then needed to be regrouped. Although educators were initially exited about this teaching procedure, it appears to have become an algorithm in itself rather than being seen as one way that place value could be demonstrated.

Discussion of Procedure

The first author volunteered to provide professional development for a school, demonstrating invented algorithms as a way of encouraging number sense. The school had a policy of encouraging independent thought, the principal welcomed the offer, and two teachers volunteered to introduce multi-digit computation in this manner. This professional development took place over six months, with three initial meetings with interested teachers followed by a period in which teachers were

observed and aided in their classrooms, with interviews and less formal contacts thereafter. The two classes that are the focus of this report were a Year 3 class (aged 7 – 8) and a Year 3 & 4 class (aged 7 – 9).

Both of the teachers were experienced. Ms N, who taught the Year 3 class (Class 1), had 14 years of experience, but relatively low confidence in her ability to teach mathematics. Mr C, who taught the Year 3 & 4 class (Class 2), had 20 years of experience and an above average level of confidence in mathematics. He was a popular teacher who could be described as having a certain charisma with the children.

The methodology of the study was a case study of two classrooms, with a participant observer. Classes were videotaped and audiotaped for a five-week period when the children were working on multi-digit computation. Samples of children's work were collected every 3 or 4 school days. These samples form the basis of the data in Tables 1 and 2.

Ms N chose to introduce this unit of work through revising basic facts and recording various ways a number could be written, for example 9+10=19. She then discussed two-digit place value with the class (52 is 5 tens and 2 ones). She presented problems for the children orally, primarily as word problems. Initially she asked children to add two-digit numbers in which the units always summed to less than 10. Children were encouraged to use any method that worked to get the right answers, and to share their methods with their peers. She then focused the children's attention on how basic facts could be used to solve problems with larger numbers, for example if 3+4=7 then 30+40=70 and 300+400=700. Only after she saw that the students were successful with their own methods of addition did she give the children problems for which the units equaled more than ten. Similarly, she only introduced subtraction after children had been working with addition for four weeks, and even then, because it was suggested to her. When introducing subtraction she

said, *"Part of me wants to get out the equipment and show you how to work it out, but I'm going to let you have a think about it first"*.

It is interesting to note that in her planning she was restricted by the same factors that would have governed her introduction of multi-digit calculation with the conventional algorithm using place value blocks. She had always used the place value blocks to model this exercise, and her thinking continued to follow this pattern. She doubted that the children would be able to do more difficult calculations, and did not want them to fail. She did not support flexible thinking in which addition and subtraction were seen as inverse operations. However, she expressed considerable interest in children's methods of working, encouraging them to demonstrate different methods to their peers.

Mr C started his program at a more advanced level. He initially assessed multi-digit addition and subtraction through word problemsthat required combining quantities such as 53 and 39 or taking a group of 27 from 62. These weregiven as homework, and Mr C emphasised that the children could get the answer in any way they chose. Next he had them work on different ways in which numbers could be decomposed through an exercise in which children thought of four numbers that could be added to make a given number, such as 42 = 14+12+10+6. After this exploration with numbers he went on to practice with basic facts in all four operations, emphasising the inverse relationship of multiplication and division. This was followed by a discussion of the strategies that many peopleused, based on knowledge of doubles, derived facts and decomposing numbers to the nearest tens. He gave verbal problems or provided a Lucky Dip from which children picked numbers for addition, as well as some open-ended problems from a common text. He encouraged students to use mental methods as well as record the way in which they worked, and the students took delight in being able to do problems mentally. Although he said that he was committed to students developing their own ways of calculating, he lost his confidence and commitment at the time of the fourth data

collection point. He was concerned that he was wasting time and that parents and the next teacher would find his children ill prepared in their algorithmic skill. At this point he demonstrated the way that he would do a problem, on the board. This demonstration had a marked effect on students' methods.

From his plan it appears that this teacher was much more flexible and connectionist in his use of numbers than was Ms N. He was not tied to the procedures used with place value blocks. However, his students became less flexible in their operations than were the children in the other class.

Discussion of Results

Children in both classes demonstrated a range of strategies in their calculation: counting on, addition by place value, using compensation, various types of decomposition, etc. Some children in Mr C's class used conventional algorithms in their first sample, done as homework. Tables 1 shows the way in which children placed the numbers to be calculated, and Table 2 shows whether the students started with the tens (or larger values) or the units when calculating.

Data Collection Point	1	2	3	4	5	6	7
Class 1 Horizontal	91%	100%	100%	95%	95%	77%	95%
Class 1 Vertical	9%	0	0	0	0	0	4%
Class 2 Horizontal	39%	70%	78%	43%	43%	56%	56%
Class 2 Vertical	57%	30%	17%	57%	57%	43%	43%

Table 1. Percent of children using horizontal and vertical formats for calculating at each data collection point, over 5 weeks. Class 1: n=22, Class 2: n=23.

Most of the children in Class 1 used a horizontal format throughout, with a slight dip at point 6 when subtraction was introduced and several children did not show their working. The children in Class 2 started with work done as homework, and about half of them turned in standard vertical calculations, presumably taught by parents or siblings. The percentage using this method decreased when developing

individual ways was praised in class, but as soon as their teacher demonstrated the way in which he worked, the percentage of children using his vertical format increased.

Data Collection Point	1	2	3	4	5	6	7
Class 1 Tens before units	59%	82%	82%	73%	77%	27%	41%
Class 1 Units before tens	14%	4%	4%	0	0	0	4%
Class 2 Tens before units	39%	22%	35%	4%	0	0	0
Class 2 Units before tens	26%	22%	0	4%	13%	17%	26%

Table 2. Percentage of children in each class who calculated the tens before the units versus the units before the tens in each class, at each data collection point, over 5 weeks. Class 1: N = 22, Class 2: n=23.

Table 2 demonstrates dramatically the effect of the teachers' statements or demonstrations on children's working and accuracy. The majority of children in Class 1 operated with tens before units until they started subtraction, when the majority stopped showing their method of working. An example of their method of working was, for 47 + 28: *"I put the 40 and the 20 = 60 and 7 + 8 = 15 and the 10 from the 15 = 70 and 70 + 5 = 75"*. At Point 6 when Ms N expressed her doubts about the students' ability to do subtraction, they attempted to use the same strategy that they had used for addition. This direct transfer led to a drop in accuracy from a range between 86% - 100%, to an accuracy of 59%. Despite the encouragement of the author and students' subsequent improvement, she felt that her doubts about children's ability were confirmed.

In Class 2, 22% - 39% of the children worked with the tens before the ones on the first three samples. The marked change came at the fourth data collection point, when he demonstrated how he would do vertical calculation, starting with the units. After one demonstration of the conventional algorithm no one showed working with tens before units. Students were eager to calculate in his way, which probably was that demonstrated by their parents. While the accuracy of these students had been 91%-100% at the first three data collection points it dropped to

35%-82% for the next three points. Few children showed their working, making it very difficult for the teacher to gauge their number sense.

Discussion

Despite the fact that the teachers had seen the benefit of children basing their multi-digit calculation on their own number sense, neither teacher was sufficiently convinced of its usefulness to change their traditional way of teaching. This appears to have been because the method was in opposition to that expected by their wider community. Both teachers feared that future teachers and especially parents would be displeased if students did not know how to use conventional algorithms. As Yackel, Cobb and Wood (1992) point out, teachers work in a context in which procedures must be institutionally sanctioned. Teachers construct their own beliefs about teaching practice in their wider community, just as children construct their own mathematical concepts. These constructions must be useful to them, in this instance, more useful than the need to please parents and other teachers. Despite the support of the principal and some of the other teachers in this school, these teachers did not see this pedagogy as useful to them, despite acknowledging its usefulness to the children.

They were also concerned by new political demands for national assessment of numeracy at age 9, which translated in their minds and that of the parents to the use of algorithms. One year later, Ms N had reverted to teaching the traditional algorithm for multi-digit calculation, using her previous methods. Although she was very impressed with the number sense that her children showed, she felt more confident in using the institutionally sanctioned methods based on place value blocks rather than encouraging wider number sense. Mr C had the confidence to understand the children's number sense, but he saw more disadvantages than advantages to altering his teaching style.

It may be important to note that this was not an extended teaching experiment similar to those of Carpenter et al (e.g. 1998) or Cobb, Wood & Yackel (e.g. Yackel, 1998) and did not have the same degree of external support. The fact that the teachers' commitment to the beliefs of the wider community was greater than their commitment to the development of number sense in their children emphasises the need for any change to be the focus of a much wider community than that of the classroom.

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THE AMBIGUITY OF MATHEMATICS

BILL BYERS
DEPARTMENT OF MATHEMATICS AND STATISTICS
CONCORDIA UNIVERSITY
MONTREAL, CANADA H4B 1R6

This paper deals with mathematical rigor and the notion of ambiguity in mathematics. It takes the counter-intuitive position that ambiguity is of central importance to the mathematical endeavor—that it is essential and cannot be avoided. In our view, rigor and ambiguity form two complementary dimensions of mathematics—what we characterize as the surface versus the depth dimensions of the subject. This position has major implications for the teaching of mathematics since we hold that the philosophical position of the working mathematician is the single most important impediment to the improvement of the teaching of mathematics at the university level.

Introduction

One of the most intractable teaching problems faced by university mathematics departments revolve around the appropriate role of rigor in the undergraduate curriculum. Hovering around the teaching of university mathematics is the unquestioned axiom which holds the that there is a definitive version of any area of mathematics and that this definitive version of the subject is more or less identical to its rigorous presentation. In her interviews with mathematicians Sfard (1994) concludes that "there seems to be another mode of thinking about mathematical concepts, a mode which has little to do with systematic deduction." Nevertheless, in their role as classroom teachers, mathematicians often revert to a formal, rigorous presentation. Even if a teacher of an elementary course, such as calculus, teaches little theory, nevertheless, in the background there is the rigorous theory which is, for the teacher, definitive. This notion that rigorous mathematics is definitive infiltrates the classroom in many ways and contributes to making it the sterile learning environment that it unfortunately often is.

In some other writing (Byers 1983, 1984) I have tried to address the discrepancy between the subject as it is taught and the subject as it is understood. In particular, the rigorous, static, formal version of the subject does not match the dynamic and human dimensions of learning, understanding and creating mathematics. One eminent mathematician said that what we are doing as mathematicians is "constructing better ways of thinking," (Thurston 1994), thus thrusting to the fore the

idea that mathematics is something that cannot be understood independently of the human beings who create and use mathematics. Attempts to demystify mathematics (and Platonism is surely a myth) and bring it back into the human realm is an important tendency in mathematics education (Triadafillidis 1998). It is consistent with the movement to revive the philosophy of mathematics by setting aside questions of foundations and focusing instead on actual mathematical practice (Hersh 1997, 1998).

To the working mathematician it is evident that mathematics is *not* merely tautological. Thurston's statement that, "...what we are doing is finding ways for *people* to understand and think about mathematics" (1994), raises the question of where this understanding comes from? Surely it cannot be derived from the formal structure of the subject. As a consequence when the mathematician, as teacher, identifies the subject with its formal presentation he often does not consciously make a goal of developing understanding or of providing a fertile ground in which insight might flourish.

In order to dramatize the point that mathematics is not logic *and* to indicate an approach to the subject which would include understanding and to what I call below the depth dimension of mathematics, I have, in all seriousness, put the word ambiguity into the title of the paper. Mathematics is deep and powerful because it is multi-faceted. Thus mathematics transcends logic yet logic is an essential ingredient in it. Similarly one could say that mathematics is neither completely objective (as a formalist or a Platonist would claim) nor is it completely subjective or constructed by the individual (c.f. Lakoff and Johnson 1980).

Ambiguity and Depth

Anyone who has done some creative work in mathematics will agree that some pieces of mathematics are "deeper" or more profound than others. Often in a piece of mathematics or in a proof one asks questions like, "What is really going on here?" or "What is the basic idea?" These questions go in the direction of depth. The most complimentary thing that one can say about a mathematical idea is that is "deep." So mathematics has more than one dimension. On the one hand there is the dimension of the logical structure, what we will call the "surface structure", (which we will take to include instrumental or algorithmic aspects) but on the other there is the dimension of depth. Of course the division between the two is not so simple but for the purposes of this discussion the distinction is clear enough to talk about. When one says that mathematics is basically tautological or that logic is the essence of mathematics one is referring to the surface structure (which mathematicians usually

take for granted). The power of mathematics clearly comes from the other dimension, that of depth. When we ask what mathematics is we must specify which dimension we are talking about. When we teach mathematics we must also specify in which domain our teaching objectives lie. When we talk about the ambiguity of mathematics we are trying to get a handle on the phenomenon of depth.

What is ambiguity?

People often take ambiguity to be synonymous with incomprehensibility. However we shall primarily focus on the following part of the dictionary definition of ambiguity: "admitting more than one interpretation or explanation: having a double meaning or reference."(Oxford 1993). The definition of ambiguity also has a secondary meaning, that of being "indistinct, obscure, not clearly defined". The kind of obscurity we are thinking of is, for example, when one is doing research and one feels that there is a theorem lurking somewhere but one doesn't yet see exactly what it is. The sense in which we use the term "ambiguity" will be further clarified by the examples which follow. To return to the notion of depth, the relationship between ambiguity and depth can be understood by considering the metaphor of binocular vision. Seeing through one eye produces only a flat, two-dimensional image; it requires two images to produce depth. The existence of a double perspective creates a situation which may lead to understanding or even creativity.

Ambiguity in mathematics: strength or weakness?

We often understand ambiguity as mere confusion or lack of clarity which we consider to be undesirable. Sometimes this confusion is unnecessary and should be clarified (Hillel 1989). But is it always possible to avoid ambiguity? We tend to react to every presence of ambiguity by attempting to remove it rather than by working with it. We maintain that ambiguity, viewed as the existence of a multiple perspective, can be an opportunity and not just a problem. Ambiguity functions in mathematics in a way which is analogous to the poetic function in language (James, Kent and Noss 1997). Consider the following situations:

<u>Square roots</u>

$\sqrt{2}$ is ambiguous. Is it an arithmetical, counting number or a geometric, measuring number? Which world does it belong to? The irrationality of $\sqrt{2}$ is one instance of "a continuous feature of the history of mathematics. . .the prevailing tension between the arithmetic and the geometric" [Dunham 1990] We claim that the tension arising from the fundamental arithmetic/geometric ambiguity was a spur to the development

of much of mathematics. Though the irrationality of $\sqrt{2}$ destroyed once and for all the hope of the Greek mathematician/philosophers for a "rational" universe, it was an opportunity as well. The problem which the ambiguity of $\sqrt{2}$ presented to the mathematical world was ultimately resolved by the creation of the real number system.

Matrices

A matrix leads a double life. On the one hand it is a collection of numbers arranged in a rectangular array, on the other, it is a function, a linear transformation. (It also has many other interpretations but these two are sufficient for us to make our point.) Whereas we add matrices as though they were collections of numbers, we multiply matrices in the way that we do because we are composing them as functions. Often, in linear algebra, we jump back and forth between these two points of view. Rank, for example, can be looked at from both points of view. Or, think of the representation of a linear transformation, **T**, as a matrix relative to a certain pair of bases. Since **T** is usually given by a matrix the whole situation is fraught with ambiguity. It is the existence of this multiple perspective which gives the student so much trouble. They often ask: "When do you think of a matrix in one way, when in the other? How do you know which way to think of a matrix in a given problem?" However, it is precisely this ambiguous point of view which gives the concept of a matrix its depth. The successful student has learned to alternate easily between these two ways of looking at a matrix. In fact when we think of a matrix it has become a mathematical concept with an independent existence which can be looked in a multiplicity of ways. No one of these ways is the exclusive or the correct way of understanding what a matrix is.

Infinite Series and Real Numbers

Even the notation for an infinite series is ambiguous. The summation notation, $\sum_{n=1}^{\infty} a_n$ conventionally stands both for the formal series **and** for the sum of the series (if it converges). Even the word "sum" is used ambiguously since it describes both an operation, the verb 'to sum', and a thing, the noun, 'the sum'. This ambiguity is compounded when we write real numbers as infinite decimals, as in .999... . As above the real number is a thing, either a quantity or a point on the real line. But it is also a process, the sum of a series or a series of successive approximations. (Sfard 1994) would refer to this as the problem of reification, of "treating a process as its own product" but we would add that the problem is not only that the "process" of the infinite sum is replaced by the "product" of the real number but that process and

product are equated (as in .999... = 1) in a statement which is ambiguous. One could say that .999... is only a representation for 1 but then how can one write that this representation is "equal" to 1. If you say that the numeral "1" is also a representation for 1 then this gives rise to yet another ambiguity: that between the numeral and the number represented by that numeral.

The difficulty here is similar to the difficulty that has been pointed about by various authors (e.g. [Kieren 1981]) concerning children's propensity to understand the equality sign in simple sums like '2 + 3 = 1 + 4' in operational terms. Gray and Tall [1994] have discussed these "process-product" ambiguities in mathematical notation. They stress that the learner's grasp of these ambiguities is central to their success or failure in mathematics.

Functions

The notion of a function is ambiguous. There are many equivalent definitions but let us focus on two . There is the ordered pair, graphical definition of a function. This is a static definition: the function is a set (of ordered pairs) or a picture (the graph) or a table.. However there is also the mapping definition, which is related to the black box, input-output definition. This latter is a dynamic definition. Here the 'x' is transformed into the 'y'. This definition is the one which is used in thinking of a function as an iterative process or a dynamical system or a machine.

Again mathematicians go back and forth from one of these representations to the other. New developments in mathematics may entail looking at a concept in a new way. The input-output model was crucial to looking at functions as the generators of iterative processes. It came into its own with the development of computers. The graphical representation of a function is of little value when one wishes to study the orbit structure which the function generates.

At a higher level one puts sets of functions together to form function spaces. In fact one of the conceptual breakthroughs in analysis is the idea that a function may be considered a point in such a function space. Here again the initial barrier to understanding, namely that a function could also be thought of as a point, turns into an insight. That is, it is precisely the ambiguous way in which a function is viewed which is the insight. Once a function is seen as a point in a metric space, we can talk about the distance between functions, the convergence of functions, about functions of functions, etc. This sort of dual representation is often present in situations of mathematical abstraction.

Fundamental Theorem of Calculus

The fundamental theorem is a non-trivial application of the above discussion on ambiguity. Differential Calculus and Integral Calculus can (and historically were) developed independently of one another. The Fundamental Theorem says, of course, that these processes are inverses of one another. This means that differentiation is not more fundamental than integration nor is the opposite true (at least for functions of one variable). Actually the theorem says that there is, in fact, one calculus process which is integration when we look at it in one way and differentiation when we look at it in another. That is, there is a multiple perspective which is essential to an understanding of calculus.

How is this multiple perspective used? Consider, for example, the proof of the existence theorem for the differential equation

$$\frac{dy}{dx} = f(x,y); \quad y = y_0 \text{ when } x = x_0$$

One proof proceeds by rewriting the equation as an integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$$

and then seeing that the solution is a fixed point of the contraction mapping

$$T(y)(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \quad .$$

This proof is possible because of the dual representation of the calculus as derivative/integral. Mathematics is full of such dualities . Each of them adds depth and power to mathematics.

In summary these examples bring out the following points:

1. Many familiar mathematical concepts have an ambiguous or multi-dimensional nature. For example Thurston in the above quoted paper lists eight different ways of "thinking about or conceiving of the derivative." He insists that these are not different logical definitions. They are, however, different *insights* into the concept of derivative. Importantly Thurston warns us that "unless great efforts are made to maintain the tone and flavor of the original human insights, the differences start to evaporate as soon as the mental concepts are translated into precise, formal and explicit definitions." I interpret his comments to mean that the concept of the derivative is ambiguous. We may have many insights into the notion of derivative each of which teaches us something new about what a derivative is. These different insights may reduce down to the same formal definition but in doing so something of value is lost. That is, while the precision of formal, logically precise mathematics is

valuable, it is obtained at the expense of the loss of some insight or intuition which is also mathematically valuable.

2. This ambiguity is neither accidental nor deliberate but an essential characteristic of the conceptual development of the subject as well as of the person attempting to master the subject.

3. The ambiguity is not resolved by designating one meaning or one point of view as correct and then suppressing the others (although this is usually the student's preferred course of action). The ambiguity is "resolved" by the creation of a larger meaning which contains the original meanings and reduces to them in special cases. This process requires a creative act of understanding or insight.

4. Thus ambiguity can be the doorway to understanding, the doorway to creativity.

Conclusions

In conclusion let me point out that the whole of the above discussion is self-referential: not only is ambiguity part of mathematics but mathematics *itself* is ambiguous. Its nature is also multi-dimensional. There is the logical surface structure and the deeper dimensions of understanding, insight and creativity. It is not possible to imagine mathematics without its computational and formal aspects but to focus exclusively on them destroys the subject. Ambiguity, even paradox, push us out of our air-tight logical mental compartments and open the door to new ideas, new insights, deeper understanding.

Even the implicit "model" which might seem to be lurking in these pages is not correct. The different aspects of mathematics which we have described are in continual interaction, continual evolution. An idea like derivative is formalized. Thus in a sense the multiple possibilities contained in the informal idea are reduced to one. Then the formal idea can be understood in various ways, some of these retrieving some of the viewpoints that were inherent in the original preformal situation, others arise out of the interpretation of the formal definition of derivative. These new ideas can themselves be formalized and so the whole chain is set in motion again. So a more complete way of looking at the situation is to say that though logic does tend to rigidify a situation, it also contains the seeds of further development.

Logic moves in one direction, the direction of clarity, coherence, structure. Ambiguity moves in the other, that of fluidity, openness, release. Mathematics

moves back and forth between these two poles. Mathematics is not a fixed, static entity which can be structured definitively. It is dynamic, alive: its dynamism a function of the relationship between the two poles which we have described above. It is the interactions between these different aspects which gives mathematics its power, its "unreasonable effectiveness." (Wigner 1960).

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DISCOVERING THE STORY BEHIND THE SNAPSHOT: USING LIFE HISTORIES TO GIVE A HUMAN FACE TO STATISTICAL INTERPRETATIONS

Jean Carroll RMIT University, Melbourne, Australia

This paper reports on a study of primary school teachers' views of their own knowledge of and feelings about mathematics teaching and learning. A survey and statistical methods of analysis were used to gain a broad view of the different dispositions of 100 teachers in suburban schools in Melbourne, Australia. Eight different teacher types were identified and represented in a teacher type table. Life histories were then collected from five teachers, representing five of the teacher types, in an attempt to further understand the professional development of the teachers and the origins of their current cognitive and affective views. The interaction between the qualitative and quantitative methods of data collection and analysis are discussed in the process.

Not all questions that an educational researcher might be interested in answering can be answered by statistical study designs. Even where statistical methods are applicable, the educational researcher will often want to use qualitative data to help formulate interpretations of any particular statistical analysis. Interpretations of statistical studies in education are generally interpretations about the experience of actual people. This is a major reason for some form of qualitative data analysis to be a part of any educational study. The researcher's understanding of the lived experiences of the people she is researching is a significant contributor to the relevance of the findings to the practice of teaching.

The study presented here was designed to investigate the question: What are primary school teachers' views of mathematics and mathematics teaching and how do these views change? The importance of primary school teachers' understanding of and feelings about mathematics and mathematics teaching have been widely discussed (Carroll, 1997; Fennema and Franke, 1992; Kanes & Nisbet, 1994). One of the constructs at issue in this study was, what is the relationship between cognitive and affective factors in the teaching and learning of mathematics for primary teachers? Relevant literature (Leder, 1993; McLeod, 1992) and the research reported in this paper suggest that cognitive and affective studies remain incomplete and are theoretically reductive if the interaction of the two isn't acknowledged.

The initial phase of this research was quantitative and the results were presented in the form of a teacher type table (Table 1) that shows tendencies in primary teachers' perceptions of their ongoing professional ongoing learning and feelings about teaching mathematics. To understand the actual experiences referenced by the teacher type table, I collected a number of personal life stories which, when read in conjunction with the teacher types, gave a very human face to the teacher type table.

It is the process of seeing the human face behind research results that I wish to illustrate in this paper. To do this I will: briefly outline a methodological rationale for

taking life histories; describe the collection of the data; give the thematic analysis of the life histories; present the teacher type table; and then give a reading of the table in terms of the themes uncovered in the life histories.

Taking Life Histories

The stories of our experiences that we tell ourselves or others are a large part of what we take as our identity. There are a great many problems for a researcher in finding useful and controllable ways of accessing these stories. No person's full story is going to be exhausted in one interview session or indeed in the fullest written autobiography. In approaching the problem of collecting this data I made use of the methodology offered by Van Manen (1990). The method of data collection I chose was what Van Manen called "protocol writing". He defined protocol writing as, "the generating of original texts on which the researcher can work" (1990, p. 63). These "original texts" are ideally descriptions of experiences without causal explanations, generalisations, or abstract interpretations. They are not meant to be works of literature. How adequate the data is will depend on how well the researcher has conveyed her intentions for the piece to the participant and how able the participant is to respond.

Five teachers who worked at suburban primary schools in Melbourne expressed their willingness to participate in this study. They were asked to write a mathematical life history (Chapman, 1993) and given the following instructions:

> I would like you to write about your mathematical life history. Could you describe your experiences and feelings as you were learning maths at school and college/uni (etc) and your feelings about teaching mathematics to children over the years. I am interested particularly in the times when your feelings or understanding changed (either for the better or worse) and what or who you attribute the changes to. If you can remember any events that seem significant to you, please describe them in as much detail as you can remember. The mathematical life history is like a story of your recollections about maths and maths teaching. You should make it as long or as short as you feel is suitable.

Thematic Analysis of Life Histories

I analysed the data using a thematic approach. The notion of what a theme is and how one actually identifies it is not at all straightforward. Indeed themes, as used here, are as murky as lived experience itself. Uncovering a theme in a piece of protocol writing requires the empathetic understanding of the researcher. To understand why two humans can understand each other is to enter to the very heart of the present debate about the validity of human science research. An interpretation of text may, in fact, say more about the interpreter than the text or its author (for example, see Pimm, 1994). Thankfully, there is a practical solution to these concerns: if the meaning I see in a text, you also see in the text then we will take it to be there. What I'll mean by a theme is a phrase or word that seems to capture the point of a sentence or group of sentences as they are found in a number of stories.

Van Manen gave three practical approaches to discovering themes: the holistic or sententious approach; the selective or highlighting approach and the detailed or line by line approach. (Van Manen, 1990, pp. 92-93). The approach I adopted was what he called the highlighting approach. For each of the stories I would read them quietly and highlight key phrases, that is, the phrases I found to be particularly apt in expressing the experience being described.

Before presenting the analysis of this data I will give a summary of the results of the statistical analysis of a questionnaire, the Mathematics Attitude and Knowledge Scale (MAKS), designed for and administered to 100 Melbourne primary school teachers (88 females and 12 males). The MAKS was constructed to probe the interrelationships between cognitive and affective factors in the mathematics teaching of primary teachers (for a detailed account and analysis of this questionnaire see Carroll, 1998).

The Development of a Teacher Type Table

Factor analysis of the questionnaire data led to a four factor solution (using oblimin rotation on the pattern matrix of the factor analysis) which provided information about how the teachers: felt about mathematics teaching (Factor F); viewed their knowledge and feelings about mathematics (Factor M); perceived their knowledge of mathematics teaching (Factor K) and conceived of mathematics and mathematics teaching (Factor C). The teachers' scores on each of these factors were used to develop a teacher type table which identified different types of teachers and described tendencies related to mathematics teaching and learning. Only the first three factors (Factors F, M and K) were considered in developing the teacher types because these three contributed to the underlying construct; knowledge and feelings about mathematics and mathematics teaching, as identified in a principal components analysis. Factor C relating to the teachers' conceptions of mathematics and mathematics teaching was not included in the analysis of teacher types, since the items included in it showed little correlation with the principal construct.

The teachers' scores for each factor were said to be positive if they were above the mean factor score and negative if they were below the mean. These statistics were used to allocate teachers to one of eight teacher types which are shown in Table 1. The percentages indicate the proportion of teachers in each type.

Themes in the Life Histories

Life histories were collected from Ann, Betty, Cathy, Dot and Ellen who represented the teacher types; F-M-K-, F-M+K-, F-M-K+, F+M+K- and F+M+K+ respectively. These were the largest categories. The small size of the other categories made selection of teachers willing to participate difficult. The five teachers were female as none of the male teachers volunteered to take part in this aspect of the study. To report on the data contained in the life histories is a verbose procedure. To accommodate this to the present space constraints, I will briefly illustrate the themes

Table 1 Teacher Type Table

F-M-K- (23%) **F-** Negative **feelings about teaching mathematics** including lack of confidence and enjoyment and finding it threatening **M-** Knowledge and feelings about **doing or studying mathematics** are negative; have not done well at maths, maths is not the best subject and find doing maths problems frustrating. **K-** Lacking in **knowledge about the approaches for teaching mathematics to primary school children**	**F+M-K-** (3%) **F+** Positive **feelings about teaching mathematics** including confidence, enjoyment, excitement and finding it non threatening. **M-** Knowledge and feelings about **doing or studying mathematics** are negative; have not done well at maths, maths is not the best subject and find doing maths problems frustrating. **K-** Lacking in **knowledge about the approaches for teaching mathematics to primary school children.**
F-M+K- (13%) **F-** Negative **feelings about teaching mathematics** including lack of confidence and enjoyment and finding it threatening **M+** Knowledge and feelings about **doing or studying mathematics** are positive; have done well at maths, better in maths than other subjects and find maths problems interesting and challenging **K-** Lacking in **knowledge about the approaches for teaching mathematics to primary school children**	**F+M+K-** (8%) **F+** Positive **feelings about teaching mathematics** including confidence, enjoyment, excitement and finding it non threatening. **M+** Knowledge and feelings about **doing or studying mathematics** are positive; have done well at maths, better in maths than other subjects and find maths problems interesting and challenging **K-** Lacking in **knowledge about the approaches for teaching mathematics to primary school children**
F-M-K+ (11%) **F-** Negative **feelings about teaching mathematics** including lack of confidence, lack of enjoyment and finding it threatening **M-** Knowledge and feelings about **doing or studying mathematics** are negative; have not done well at maths, maths is not the best subject and find doing maths problems frustrating. **K+** Knowledgeable about the approaches for teaching mathematics to primary school children	**F+M-K+** (6%) **F+** Positive **feelings about teaching mathematics** including confidence, enjoyment, excitement, challenging and finding it non threatening. **M-** Knowledge and feelings about **doing or studying mathematics** are negative; have not done well at maths, maths is not the best subject and find doing maths problems frustrating. **K+** Knowledgeable about the approaches for teaching mathematics to primary school children.
F-M+K+ (5%) **F-** Negative **feelings about teaching mathematics** including lack of confidence and enjoyment and finding it threatening **M+** Knowledge and feelings about **doing or studying mathematics** are positive; have done well at maths, better in maths than other subjects and find maths problems interesting and challenging **K+** Knowledgeable about the approaches for teaching mathematics to primary school children.	**F+M+K+** (31%) **F+** Positive **feelings about teaching mathematics** including confidence, enjoyment, excitement, challenging and finding it non threatening. **M+** Knowledge and feelings about **doing or studying mathematics** are positive; have done well at maths, better in maths than other subjects and find maths problems interesting and challenging **K+** Knowledgeable about the approaches for teaching mathematics to primary school children.

found in the data. A summary of the key dispositions expressed in each of the life histories in respect of each of these three themes is presented in Table 2.

Experiences as Students Each of the teachers wrote of significant experiences which occurred during their own school years in which their remembered perceptions of an event has seemingly influenced their self concept. For example, Ann wrote, "I remember reciting tables, however, to my father, and I know I knew them well...My father said I knew them well, but only because I had a good memory - he was right! - and I really didn't have a mathematical mind, as he did." Betty spoke about her school experiences, "My early years at the local technical school give me memories of challenging and enjoyable maths sessions and I seemed to 'breeze through'. I think in about year 9, I undertook Maths A and B and did very well in both."

For Ann, who has been teaching for over 30 years, the experience of learning tables with her father seems to have left her feeling that even though she could do her tables she couldn't do maths. On the other hand Betty's report gives the impression of thinking herself capable mathematically.

Personal Philosophy "Personal philosophy" in this context is considered here to be the expression of positions that convey a sense of coherence in self-understanding regarding teaching practice and personal history. For example, Betty says, "In conclusion, I think that it wasn't until I was teaching maths myself that I realised that there were better ways to teach/learn maths. As a student myself I don't think I knew any different... Maths skills are essential to our everyday lives so we have to ensure that students want to participate and learn the concepts involved." Betty's belief that mathematics should be personally relevant appears to stem from the lack of relevance of her experiences as a learner.

Ellen, in discussing the student teachers who work with her, says, "Comments from student teachers are interesting. 'I never knew why you did that until being here', is a common one and applies to basic concepts such as subtraction (decomposition). People still don't know why they do things! Rote still goes on!" It seems that Ellen does not believe in rote learning and is continually surprised that student teachers unquestioningly accept their own rote learning until being shown the reasons for procedures they have learned. Each of these quotations expresses a position and a reason for giving it. Insofar as position and reason are connected I take these quotations to be expressions of personal philosophies.

Significant Influences Significant influences are taken here to mean the descriptions of experiences that the writer takes as having informed an on-going change in her teaching practice and consequent self-understanding. For example, from Dot, "Lecturers during my teacher training were quite influential in helping me to develop in my teaching of maths, the qualities and approaches which had lacked in my own maths teachers." Ann commented on notable events that had improved her teaching, "Really getting into the team teaching area, guided by a very gifted and tactful coordinator. We did this for terms at a time, and although he took the maths measurement component,... he explained his operation in detail, and gave us such useful notes, that we were able to follow a similar model in future years, when he had gone into admin."

Interaction of themes In any particular piece of writing the themes are often woven very closely together and it would be a mistake to see them tied to particular sentences or paragraphs. The nature of a life history as a present expression of a complex past is illustrated well in this longer quote from Cathy:

> I also remember very clearly being very frightened in my maths in grade 4 because the teacher would come around with a ruler and the ruler was on its side and if you got things incorrect he would take the ruler and smash it against your knuckles so what I tended to do was be very very quiet, and try not to participate too much so that he would forget I was there, because I was very very scared of making a mistake. So of course, there was no push to have a go at it, like making mistakes are a part of life and I think that is something that I have learnt through my own experiences and its something I'm very conscious of in my own class - actually encourage the kids to be risk takers and they get rewarded for the tries that they have even if they're nowhere near correct, because otherwise they're going to do what I did and just go into the background and that's the end of that.

In this quote we see woven together the themes of "Experiences as Students" (grade 4...) "Personal Philosophy" (making mistakes is part of life...) and "Significant Influences" (in my own class...).

Table 2 Summary of Life Histories

Type	Ann F-M-K-	Betty F-M+K-	Cathy F-M-K+	Dot F+M+K-	Ellen F+M+K+
Training	2 years	4 years	2 years	3 years	2 years
Years teaching	31 years	7 years	17 years	10 years	30 years
Highest maths	Year 10	Year 12	Year 11	Year 12	Year 12
School experience	Negative episode recalled vividly. believes she is not mathematicall y minded	Positive, did well	Felt that she did not understand maths, all rote learning, vivid recollection of upsetting experience.	Positive experiences at school left feeling confident of her knowledge	Loved maths and did well although she didn't always understand
Personal philosophy	Not evident	Has ideas about how maths should be taught but finds them difficult to implement	Clearly developed based on own negative experiences a learner.	Reflection of lecturer's philosophy	Evident, well developed
Significant influences	Colleagues, team teaching, inspiring EMIC tutor, gifted tactful coordinator, the experience of teaching	Inservices, team teaching, publications, own experiences as a learner	Teachers' college, experience as a teacher, experiences as a learner, curriculum documents	Lecturer at teachers' college, own experiences as a learner EMIC program	Loved maths at school, very involved in professional development, her role of maths coordinator

A Reading of the Teacher Type Table in Terms of Selected Life Histories

The data collected in the life histories and thematised in Table 2 allows us to see the complex procedure of identity formation which is reflected in the snapshot of dispositions caught in the teacher type table. Connecting the themes with the types is necessary if the teacher type table is to be of use as an instrument of change for practising teachers. The life histories give a human underpinning to the scales expressed in the teacher types.

The life history data suggests that school experiences were important determinants of the teachers' present attitudes towards mathematics and views of their knowledge of mathematics. Ann and Cathy, whose scores were negative on Factor M (knowledge and feelings about doing or studying mathematics) in the teacher type table, described negative experiences of learning mathematics at school and described themselves as lacking in confidence in their knowledge of mathematics. They recalled mathematics learning as a predominantly negative experience involving memorising procedures of which they had little understanding. The frustration and lack of enjoyment that stemmed from these experiences was apparent in their histories which were written years later. Betty, Dot and Ellen, who were positive on Factor M, recalled school mathematics learning as involving more enjoyable experiences and rated their mathematical knowledge more highly. They also inferred that their knowledge of mathematics today is adequate or better.

Schuck (1997) identified different voices when teachers speak, which are apparent in the words of these teachers. She discussed "self as student" and "self as teacher" as two of the voices that teachers use. It is evident from the life histories and the teacher type data that the "self as student" continues to speak many years after the teachers have ceased to be students. The feelings about the "self as student" of mathematics appear to remain relatively unaffected by subsequent experiences of " self as teacher". Professional development for teachers like Ann and Cathy may need to allow them to acknowledge and understand this voice which continues to influence their views so strongly.

These views of mathematical knowledge and the associated feelings described by the teachers reinforce the notion that affective and cognitive factors are interwoven in the learning of mathematics. The interaction of cognitive and affective factors was also evident in the factor analysis when items loading on factor M consisted of those concerning feelings about learning and doing mathematics and as well as items related to knowledge of mathematics.

In discussing the influences on their professional development, the five teachers described situations in which significant personal relationships were established between themselves and a more knowledgeable person. The histories suggest that effective professional development occurs in a context of personal interaction. The interactions described were with lecturers, presenters, tutors, coordinators, peers, principals and other colleagues and were mostly the result of ongoing relationships, which ranged in duration from several months (in the case of lecturers and tutors) to several years (for relationships with peers and colleagues). These relationships were

important when they valued the teachers' experiences, included a climate of mutual respect and enabled learning to be collaborative.

The development of a personal philosophy appears to be linked to the development of the teachers' confidence in their knowledge of the approaches for teaching and learning mathematics. The life histories of Cathy and Ellen contain clearly enunciated personal philosophies for teaching. These two teachers were positive on Factor K (Knowledge of the approach for teaching mathematics). Ann, Betty and Dot's philosophies were less well developed and their teacher type indicated they lacked confidence in their knowledge of mathematics pedagogy.

Conclusions

The teacher type table was built from a statistical analysis. As pure research it could have been left as a completed entity. What I hope I have shown in this paper is that reading the table in terms of the actual lived experience of some of the teachers who participated in the study, gives the teacher types a meaning beyond the abstract and has the power to make the types personal for other teachers and provide information for those interested in professional development. Personal stories bring an understanding beyond the cognitive and their affective dimension often gives access to changed practice where a purely cognitive understanding does not.

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RESEARCHING MATHEMATICS TEACHER THINKING

Olive Chapman
University of Calgary

This paper is a theoretical essay on researching mathematics teacher thinking. The focus is on a humanistic perspective of defining and researching teacher thinking and research tools that can be used to gain new insights about mathematics teachers= thinking and actions from the teacher=s perspective. Data collection using story and role play and the analysis process are highlighted and illustrated. This research approach has potential to contribute to understanding teachers and the teaching of mathematics in ways that could lead to more meaningful and effective teacher development experiences in order to facilitate meaningful reforms in mathematics education.

Introduction

If we accept that teacher thinking determines how the curriculum gets interpreted and delivered to students, then the nature of mathematics teachers= thinking becomes a key factor in any movement to reform the teaching of mathematics. A shift to a problem-solving approach to the teaching of mathematics, for example, will unlikely occur if teachers= thinking about the nature of mathematics and the teaching and learning of mathematics, consciously or unconsciously, is different from the intended theoretical perspective of such an approach. In recent years, there has been significant recognition of the teacher as the ultimate key to educational change. Teachers are not inert conduits through which the curriculum is delivered. Instead, it is what teachers think and do in the classroom that ultimately determines the kind of learning that students acquire. At a time when major reforms are being advocated in mathematics education it is, thus, of significant importance to focus on the mathematics teacher in order to facilitate the successful implementation of these reform recommendations. This paper is a theoretical essay on researching mathematics teacher thinking. The focus is on a humanistic perspective of defining and researching teacher thinking and research tools that can be used to gain new insights about mathematics teachers= thinking and actions from the teacher=s perspective.

Research on the Mathematics Teacher

Studies on the mathematics teacher have traditionally focused on deficiencies in teachers= behaviors and knowledge, i.e., what teachers do not do or do not know. One limitation of many of these studies is that they employed universal measures of teachers= knowledge that were not directly related to instruction in the mathematics classroom. Thus, it was often easy to find deficiencies since teachers= actions and knowledge were not necessarily considered in the context of the explicit goal of the curriculum or standardized assessment of students. For example, teachers would be tested on their conceptual understanding of mathematics when they came from a system that taught and tested computational skills. In recent years, most of these studies have focused on preservice teachers (e.g., Ball, 1990; Even, 1993; Graeber, Tirosh & Glover, 1989; Simon, 1993). But there has also been a shift to focusing on the beliefs of inservice teachers (Ernest, 1989; Thompson, 1992).

There now seems to be growing interest in understanding the mathematics teacher in terms of her or his beliefs/conceptions and the relationship between these beliefs and her or his teaching. But the tendency is still to judge the teacher by highlighting what are considered by researchers to be inappropriate and inconsistent beliefs. Although these studies produce useful insights about the mathematics teacher, they tend to provide a fragmented and decontextualized view of mathematics teachers= thinking. One reason for this is the narrow scope in which teacher thinking is defined and the nature of the research process being used. The intent of this paper is to suggest alternative ways for considering teacher thinking and its investigation that could make a meaningful contribution to our understanding of the teaching of mathematics as a lived experience in classrooms.

Defining Teacher Thinking

The description of what constitutes teacher thinking is not simple because of the philosophical and psychological considerations involved, and the growing number of viewpoints that are being adapted. In mathematics education, teacher thinking seems to be be equated with teacher beliefs/conceptions and teacher knowledge. But more generally, from a humanistic perspective, teacher thinking is defined to reflect the lenses teachers construct and use to make sense of their teaching. It refers to ideas in the mind of the teacher and ideas in practice. These ideas are the meanings the teacher uses to organize his or her knowledge of teaching and his or her behavior in the classroom. Thus, in addition to beliefs/conceptions, teacher thinking has been described in a variety of related ways, for e.g., frames (Barnes, 1992), images (Clandinin, 1985),personal practical knowledge (Connelly & Clandinin, 1988),practical knowledge (Elbaz, 1983), perspective (Janesick, 1982), and personal knowledge (Lampert, 1985).

Barnes defined frame as the clustered set of standard expectations through which all adults organize, not only their knowledge of the world but their behavior in it. Clandinin defined image as something within our experiences, embodied in us as persons and expressed and enacted in our practices and actions. Connelly and Clandinin defined personal practical knowledge as a moral, affective, and aesthetic way of knowing life=s educational situations. Elbaz defined practical knowledge as theoretical and practical components of teacher=s knowledge; knowledge as experiential, embodied and based in the narrative of experience. Janesick defined perspective as a reflective, socially derived interpretation of experience that combines beliefs, intentions, interpretations, and behavior and serves as a basis for subsequent actions. Finally, Lampert defined personal knowledge as knowledge used by a teacher in accomplishing what she/ he cares about, what students want, and what the curriculum requires. The underlying assumption to these viewpoints, then, is that what teachers do and think within their professional lives depends on the meanings they hold and interpret within their personal, social, and professional realities.

Humanistic Perspective of Research on Teacher Thinking

Research on teacher thinking can be considered in terms of an analytic/positivistic perspective or a humanistic perspective (Brown, Cooney, & Jones, 1990)where the former focuses on discovering reality in the form of value-free theory and the latter on understanding the contexts that shape a person=s perception of his or her reality. The

focus in this paper is on the humanistic perspective (e.g., Ellis & Flaherty, 1992).

Genuine humanistic studies on teacher thinking tend to focus on a teacher=s perspective of his or her classroom behavior, instead of a theoretical researcher=s perspective, in order to make sense of teaching. The theoretical researcher=s perspective considers teacher thinking in terms of predetermined categories imposed by the researcher.It focuses on a researcher=s view of how teachers think about their classroom behaviors based on models of teaching and teachers generated by researchers independent of teachers.These categories or models are generally embodied in surveys, questionnaires, observation instruments, and coding schemes. Teachers are generally seen to fit well within these models that are often set prior to data collection.

Studies based on the teacher=s perspective, in contrast to the theoretical researcher=s perspective, seek to understand teachers from their own perspective. The intent is to understand how particular individual teachers understand their work, for e.g., how does a teacher makes sense of the teaching of mathematics or of implementing mathematics reform? In this context, Chapman (1997) investigated how three teachers made sense of the teaching of mathematical problem solving. In general, studies based on the teacher=s perspective view teachers in a humanistic way, i.e., as persons who have something of value to contribute and not as objects of study. Teachers= actions are seen to have meaning in their situations or contexts. Thus, the focus of these studies is on conceptualizing the experiential knowledge of teachers and providing plausible explanations of teaching processes as they are for the teacher. In particular, teaching behaviors have to be understood in relation to the intentions of the teachers and to the situational complexity. For e.g., it is not the frequency of the questions in the classroom that is important, but, rather, what questions about what content is asked at what moment to what student.

Humanistic Research Process

In this section, the humanistic research process used in the studies on mathematics teacher thinking on the teaching of problem solving (Chapman, 1997; 1998) and an ongoing project investigating teacher thinking in teaching mathematical word problems will be used as a basis to consider some specific aspects of this process in the context of mathematics education. Given the constraint on space, the focus will be on general descriptions of two humanistic tools for collecting data (i.e., stories and role play) and of the analysis process.

Data collection

Whereas teachers= actions are observable, their thinking is not and must be inferred from what they say they do, what they say about what they do, and what they actually do in their classrooms. Thus, studies tend to depend on in-depth, open-ended interviews and classroom observations. Interviews are used, for e.g., to probe the constructs or meanings which teachers bring to their teaching and the relationships among these constructs. The teachers give their account on their own terms and not on terms imposed by the researcher. However, since taken-for-granted, underlying meanings of teachers= thinking are not readily accessible by the teacher and have to be accessed indirectly, it is important that these accounts include situations that embody such implicit meanings. Two ways found to be useful in the Chapman studies in this endeavor are

story and role play.

Story as Data: Story has been described as a Asymbolized account of actions of human beings that has a temporal dimension≅ (Sarbin, 1986, p. 3). A[It] is concerned with the explication of human intentions in the context of action≅(Bruner, 1986, p.100). Based on this view, in recent years, story has been promoted as a relevant form for expressing teachers' practical understandings because teachers' knowledge is event structured and stories would provide special access to that knowledge (Carter, 1993). Story/narrative has also been conceptualized as, Aa [cognitive] scheme by means of which human beings give meaning to their experience of temporality and personal actions ... a framework for understanding the past events of one's life and for planning future actions≅(Polkinghorne 1988, p.11). In general, then, the stories we tell reflect who we are and what we may become. They provide a basis for meaning recovery and meaning construction of our act-ions. Thus, they can facilitate interpretation and understanding of teachers= experiences.

Participants of a study could be asked to write or tell stories (and anecdotes) about past, present, and future experiences with mathematics and the teaching of mathematics. The stories should focus on the teachers= learning/doing of mathematics and teaching/ facilitating the learning of mathematics in the classroom. The participants could tell stories of their choice, for e.g., from different periods of their teaching career; of personal experiences with Agood≅ and Abad≅ teaching of mathematics; of mathematics lessons they enjoyed/liked/did not like teaching. They could also tell stories to support generalized claims they make during the research interview.The stories should describe a specific situation or event as they lived through it. So they should avoid causal explanations, generalizations, or abstract interpretations and describe the experience from the inside, i.e.,including feelings, emotions, and thoughts in action. These stories should include accounts of complete mathematics lessons that involve the teaching of a mathematics concept for the first time to a particular set of students. Such accounts should describe the lessons from beginning to end and provide as much detail as possible on what the teacher and students did and said in dealing with the mathematics and how the mathematics content was dealt with or presented.

Role Play as Data: Role play has similar characteristics to story telling in that it involves acting out the story and not just telling it. Thus instead of living out a situation only mentally by telling the story, one lives it out physically and mentally. The role play allows the teacher and researcher to experience and capture the teacher=s thinking and instructional strategies from different angles. It also provides opportunities to magnify specific aspects of the teacher=s classroom behaviors that could reveal underlying meanings of the behaviors. The role play, then, should follow observations of the participant=s behavior in the classroom. Such observations should be used to get a sense of the tone of the participant=s classroom and teaching and to identify specific situations that set this tone. Examples of such situations are:

(i) teacher modeling a mathematics concept or procedure;
(ii) teacher-student discourse used to develop a mathematics concept/procedure;
(iii) teacher intervention: (a) when students are not experiencing difficulties with the concept or procedure, (b) when students are experiencing difficulties with the

concept or procedure,(c) during individual student work, (d) during group work. These situations become themes for the role play. Thus the role play is guided by the participant=s lived experiences. For example, one participant=s teaching may involve situations (i) and (iii) a, b, c, while another may involve (ii) and (iii) a, b, d.

Using situation (i) (i.e., modeling a mathematics concept) as an example, the role play can unfold as follows: First, the participant teaches a mathematics concept from her or his curriculum in the way she or he would to a whole class. The researcher observes the teaching process focusing on what he or she would do in order to reproduce it. Second, the researcher plays the role of teacher and teaches the same concept under the direction of the participant in order to teach the lesson like him or her. Third, the researcher re-teaches the lesson without any help from the participant, using the participant=s approach. The participant observes the teaching process focusing on points of conflicts with his or her expectations. Finally, the researcher teaches the lesson using an alternative approach while the participant focuses on identifying points of conflicts with his or her teaching. Each of these stages is accompanied by a discussion with the participant on his or her intentions or reason behind his or her thinking.

Analysis

Making sense of the data collected is a crucial and difficult aspect of qualitative inquiries. One large area of choice is that of level of interpretation to be employed. However, to be prescriptive regarding the analysis process is problematic because, for humanistic studies, the context of a particular case significantly influences how the analysis actually unfolds and such contexts cannot be generalized (i.e., to generalize will be to revert to a positivistic framework). Thus, what is presented here is one way in which the analysis can unfold based on the approach that evolved from Chapman=s work on the mathematics teacher. This approach consists of four related phases.

The first phase makes explicit the researcher=s original meanings. These meanings are the researcher=s initial, spontaneous interpretation of how the teacher thinks and acts in the classroom in teaching mathematics. This interpretation tends to evolve during data collection while the researcher is listening to or observing the participant, and before reflecting on the transcripts. It is usually judgmental, based on a view unintentionally imposed by the researcher in response to an apparent, familiar pattern of behavior that seems to be obvious. Thus this interpretation attends to surface meanings and does not take into account the teacher=s assumptions and intentions. It forms a baseline beyond which the researcher aims to reach in order to understand the teacher=s meaning and not simply to justify that of the researcher. The second phase of the analysis focuses on the participant=s explicit or espoused meanings. These meanings are determined by reviewing the data in order to identify explicit statements about the teacher=s beliefs, intentions, and expectations. These statements are then clustered to form themes that are characteristic of the teacher=s thinking and behavior. These themes become the basis for understanding the teacher=s perspective but are not in themselves necessarily the underlying meanings.

The third phase of the analysis focuses on the participant=s implicit meanings. This involve reflecting on the data in order to identify plausible explanations and descriptions of the meanings underlying the themes from the second phase. Since these

meanings are implicit, the stories and role play become very important for providing indirect access to them and to make them explicit. Thus, the stories, anecdotes, and role-play transcripts are reviewed in order to identify, for e.g., similarities in plots for different stories, points of strong emotions, and statements and actions that convey personal meanings (e.g., those that reflect personal judgement, intention, expectations and values of the participants) that reoccur in the various events described. During this review, new themes can emerge from the data in addition to those in the second phase. This review leads to particular meanings behind the teacher=s instructional behaviors and understanding of the themes in terms of the teacher=s lived experiences in the classroom. Finally, the fourth phase of the analysis focuses on the configuration of meanings and themes. This involves identifying coherent patterns from separate meanings and themes, as understood by the researcher, in order to provide a holistic perspective of the teacher=s thinking.

It is important that the participants play a role in the data analysis if the goal is to gain understanding from their perspectives. This role can be very involved, as in Chapman (1997), or minimally involved, as in Chapman (1998). In the former, the researcher collaborates with the participants in the third and fourth phases. Thus, for example, the participants can independently identify meanings/themes implicit in their actions, which the researcher use to compare with hers or his and negotiate differences with the participants. In the minimum involvement, the participants comment on the findings by the researcher in the third and fourth phases. In general, the outcome of the analysis should make sense to the participants in that if it provides a plausible way of understanding their thinking, they should be able to resonate with it explicitly and implicitly/intuitively.

An Example

This section presents a very small part of Tad=s case as an example to illustrate the methodological considerations previously discussed, focusing on the four phases of the analysis process. Tad is one of the participants in an ongoing project on teacher thinking in the teaching of mathematical word problems. Tad is an experienced high school teacher who is considered to be a very good teacher.

Phase 1 of analysis: The focus of the researcher=s original meaning was that Tad=s teaching seemed to be very traditional. He would model two examples of the word problems then have students work individually on practice questions while he circulated and provided help. During the modeling he seldom asked questions but students could ask questions for which he provided the answer.

Phase 2 of analysis: One of Tad=s explicit meanings was that his teaching was interactive, i.e., he liked to get the students involved in the lesson. Tad=s explanation of what he meant by interactive was that it involved getting students to ask questions. In general, Tad saw himself as a student-centered teacher. What interactive meant from Tad=s perspective was considered a theme for further investigation.

Phase 3 of analysis: Tad=s implicit meaning for this theme was determined by examining teacher-student discourse in the data, e.g., the nature of questions he asked/encouraged, the nature of his responses to questions, the nature of his intention of

his responses, and his emotions associated with questioning. The outcome of this examination included the following:

Tad viewed interactive teaching in terms of the quantity and nature of questions asked by the students and not by the teacher. He felt good when the students asked a lot of questions about the mathematics concept being presented and disappointed when they did not. He liked to be asked questions that involved higher level thinking, e.g., Awhy≅ questions. The role play created conflicts for him when no Awhy≅ questions were asked. Although Tad did not ask questions to reveal students= thinking during the presentation of the mathematics concept, his teaching was dependent on the students= questions that he encouraged. Tad used the nature of the students= questions as an indication of their thinking. But he also wanted to be asked questions in order to display his thought process and not just factual knowledge. He wanted students to learn from his thought process and not just his telling. He expected students to resonate with his thought process in order to compare their thinking, but not to just mimic a specific set of procedures. Students could use their own procedures that made sense to them. Thus it was during individual work, when students needed help, that Tad would ask them to reveal their thinking in order to allow him to intervene effectively. The Awhy≅ questions asked by students allowed Tad to elaborate on his thinking in terms of what the students wanted to know. In this sense, his teaching was student-centered in that the students determined the scope of what he presented. From his perspective, students will learn mathematics with understanding by seeing the thought process of the teacher in solving a problem in terms of what both the teacher and students considered to be important. Thus his teaching was a collaboration between him and the students.

Phase 4 of analysis: The analysis of Tad=s case is still in progress, therefore a configuration of themes has not been determined. However, in Chapman (1997), the configuration of themes was presented in the form of metaphors. For e.g., Aadventure≅ was used to portray one of the participants= teaching of problem solving in a holistic way in terms of the underlying meanings framing her classroom behavior. The configuration of themes could also be presented in the form of case narratives.

This sample of the analysis of Tad=s teaching, although only a small part, suggests that while his teaching seemed to be traditional on the surface, the underlying meanings provided a different understanding that begins to reflect how and why his approach was considered successful in helping students to learn senior high school mathematics. Tad=s case also begins to provide insights about teacher-student discourse that could contribute to understanding teaching approaches for high school mathematics and form a basis of future research or teacher development activities.

Conclusion

A genuine humanistic perspective to researching mathematics teachers aims at understanding the teaching of mathematics from a teacher=s perspective. Thus it requires the researcher to not be judgmental, but to seek understanding of the teacher in a holistic way. This research approach has potential to contribute to understanding teachers and the teaching of mathematics in more realistic ways than research that studies the teacher as objects. It can also provide insights to enhance professional development experiences of mathematics teachers in order to facilitate meaningful reforms in mathematics education.

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WHAT KIND OF MATHEMATICAL KNOWLEDGE SUPPORTS TEACHING FOR "CONCEPTUAL UNDERSTANDING"? PRESERVICE TEACHERS AND THE SOLVING OF EQUATIONS

Daniel Chazan, Cesar Larriva, & Dara Sandow

Michigan State University

In this paper, we explore the appropriation of categories for describing classroom activity and student understanding which exist in the literature for use in the task of describing qualities of teachers' substantive knowledge of the mathematics they teach. In characterizing the resources which teachers have for use in their teaching, such a description of teachers' mathematical knowledge seems potentially more useful than cataloguing the amount of coursework taken. In particular, we use tasks which have been explored in the existing literature on the solving of equations and questions about teaching students to solve linear equations in order to surface preservice teachers' substantive knowledge of this sort of task. In the paper, we raise the question of whether finer distinctions are necessary for describing teachers' substantive knowledge of the mathematics they teach.

Objectives/purpose

When discussing the mathematical preparation of teachers, the amount of coursework taken is often used as a crude measure of prospective teachers' content knowledge (Ball, 1992) and of the mathematical resources which they bring to teaching tasks. This exploratory study builds on work describing students' understandings of mathematical topics with the goal of learning to describe qualities of teachers' substantive knowledge of mathematics (not their pedagogical content knowledge or their knowledge about the nature of mathematics). As part of a larger study of the impact of different clinical settings on the nature of preservice mathematics teachers' substantive mathematical knowledge, we seek a vocabulary for describing differences in the qualities of teachers' knowledge of a particular mathematical topic, rather than relying on the quantity of coursework they have completed. In this part of the study, we raise questions about categories currently available in the literature for this task.

We have chosen to examine the issue of the qualities of teachers' knowledge of mathematical content by exploring preservice teachers' knowledge related to teaching the solving of equations. We would like to explore connections between the nature of a teacher's own understandings of solving equations and systems of equations and their notions of how to help students understand why some linear equations have no solution or infinitely many such solutions (for example, why solving such equations symbolically results in equations like $0 = 0$ or $2 = 0$). With the current North American secondary school focus on graphing calculators and

functions-based approaches to algebra, preservice teachers may choose to view the literal symbols in such equations as either variable quantities or unknown numbers; they may have seen approaches to solving linear equations by graphing each side of the equation, as well as methods which focus on the writing of equivalent expressions. As a result, this particular topic challenges us to develop descriptions of teachers' content knowledge which are applicable across different approaches to conceptualizing the same content, rather than simply assuming that a particular approach necessarily is an indication of a particular type of mathematical understanding (Masingila, 1998). For this reason, this topic is especially valuable and challenging as a locus for the study.

Perspective or theoretical framework

As part of attempts to change the nature of mathematics classrooms, there have been efforts to indicate ways in which mathematical activity in classrooms can differ. Skemp (1976) proposes that there are two sorts of understandings of mathematics that are the goals of classroom activity (Incidentally, this reading was used with the preservice teachers (interns) studied in this project and influences some of the interview comments presented below.). For Skemp, *relational understanding* includes an *instrumental understanding* of what to do in order to solve mathematical problems plus an understanding of why such procedures work. In passing, he suggests that "nothing else but relational understanding can ever be adequate for a teacher" (p. 13). Hiebert and Lefevre (1986) use different terms. They suggest that *conceptual knowledge* "is characterized most clearly as knowledge that is rich in relationships" (p. 3), while *procedural knowledge* consists of knowledge of the representations systems used in mathematics and algorithms for completing mathematical tasks (p. 6). They view these two types of knowledge as intimately connected. Similarly, in order to describe differences in classroom discourse and in teachers' instructional goals, Thompson, Philipp, Thompson, and Boyd (1994) distinguish between *calculational* and *conceptual* (as well as, in passing, *computational*) *orientations in the teaching of mathematics*. Building on this work, Cobb (1998) uses the terms *calculational* and *conceptual discourse* to describe differences observed in classrooms.

All of these sets of terms are aimed at creating categories to describe differences in classroom activity which are experientially vivid to the authors. And, in general, they have been used to encourage greater attention to issues of relational understanding or conceptual knowledge. Perhaps because examples of attention in classrooms to such understanding is relatively rare, it is easier to describe the instrumental, procedural, or calculational part of the divide. These adjectives describe orientations towards teaching, individual knowledge, understanding, or classroom discourse which are prevalent in many classrooms and focus on procedures for calculating the solutions to given mathematical tasks. It is the relational, or conceptual, side of the divide whose definition is more complex. But, it is precisely this side of the divide that we require in order to understand whether

teachers have subject matter knowledge which will support teaching for conceptual understanding. For example, Skemp (1976) and Hiebert and Lefevre (1986) use a criterion of interrelationship. Yet, this criterion is clearly not sufficient for distinguishing conceptual from procedural knowledge. Procedures can become quite complex and related to a host of other procedures and symbols systems without necessarily involving a conceptual understanding or orientation.

When faced with this difficulty, Thompson et al. (1994) use vignettes involving a traditional word problem to distinguish between calculational and conceptual orientations to mathematics teaching. In the context of this sort of problem, they point to ways in which one teacher continually asked students to refer their calculations back to the situation described in the word problem. The other teacher let the classroom discourse focus on the calculations and not their meaning in the situation (p. 86). However, these criteria are limited to discussion of classroom activity that results from a problem that is situated in some sort of extra-mathematical context.

Chazan (in press) takes a different approach. In discussing standard exercises not situated in an extra-mathematical context, he focuses on whether or not an approach to teaching a particular content area identifies the mathematical objects of that content area and provides students with task instructions that identify the goal of the task in terms of these objects.

Methods of inquiry

In this study, we explore whether the sets of terms proposed in this literature seem useful in describing preservice teachers' understandings of tasks involving the solving of equations and their notions of how to help their students with such tasks. To explore these issues, we have created an interview focused on the solving of equations and systems of equations. The interview consists of two parts. The first part focuses on a phenomenon of teaching. Early in their internship year, the preservice teachers were asked how they might help a student who does not understand why one sometimes gets 0=0 or 2=0 as the solution of a linear equation when solving with algebraic symbols.

The second part of the interview asks the preservice teachers to tackle mathematical problems which have been used in research on student understanding of the solving of equations (From Sfard & Linchevski (1994, p. 218): Will the following system of equations always have a solution? $k-y=2$ and $x+y=k$; from Schoenfeld (1985): What is the solution set for $x^2y+y^2x = 1$?). This part of the interview attempts to elicit whether the prospective teachers view the x's and y's in equations as variable quantities or unknowns and to explore the nature of the representations which they feel can be used to solve such problems.

A final question asks the interviewees to indicate what equations are and what it means to solve equations.

This paper will describe the data from one interviewee whose understandings are difficult to categorize. We will use this data to raise questions about the utility of distinctions made in the literature for the task of identifying qualities of teachers' content knowledge.

Presentation and analysis of the data

At the end of the interview, when the intern was asked what she would tell students an equation is or what solving equations means, this intern's perspective was based in the interpretation of literal symbols as unknown numbers. She said: "...that's how I define an equation, is that it has an equal sign in it." and similarly "To solve an equation means you find the unknown value that makes both sides of your equation equal, that's what solving an equation is.... You're looking for the unknown value ... that makes both sides of the equation true." Yet, in other parts of the interview, she used techniques that might be viewed as in tension with this perspective.

When faced with the question of helping a student understand the meaning of the solution to $3x+7 = 2(x+5)+x-1$, this intern indicated that such questions were about to come up in one of the classes she is teaching. Her responses involved the use of graphical representations using a two dimensional Cartesian plane. Such an approach was suggested by her textbook, but she had her own preferences which differed slightly from the book:

> ... what the textbook does is teaches them to subtract everything over so that one side is zero and then graph it so that the solutions will be the x-intercepts.... Actually, just the x coordinates.... I'm going to be a rebel, I think and I'm going to make my students do this instead, graph y equal to this side as one equation and y equal to this side as one equation and see where the lines intersect.... That's where the x and the y are both the same. If the y's are the same, then these two have outputs that are the same.... So looking at these two [the expressions on either side of the equation as she's rewritten it, $3x+7 = 3x+9$], I have parallel lines. I have both slope 3 and different y intercepts. So that's how I'm getting, how I'm not getting the solution for x.

Her focus on graphical representations seemed to flow from a desire to help students develop a deeper understanding of the solving of equations: "If they were solving this equation, I think they're just solving for an unknown, and if they're doing this algebraically, it's really procedural and I don't know if they have a great understanding of what's going on." She seemed aware that the use of this sort of graphical technique involved a different interpretation of literal symbols. On the other hand, the notion of functions was not explicitly part of her discussion; $y=3x+7$ and $y=2(x+5)+x-1$ are both viewed as equations and not as functions. Thus, in her view, when graphing these two "equations," one now has a system of equations. However, by considering the solution to the "system" of equations $3x+7=2(x+5)+x-1$ as the x coordinate of the intersection of the two "equations," she

is treating the "system" like a single equation whose solution is a set of values, rather than a system of equations whose solution would be the coordinates of a set of points.

> I guess it's thinking about a variable in a really different way, so. But when you're solving this kind of a system, the x is an unknown, but when you're graphing, you're thinking about y and x, all possible solutions, kind of different concepts. I wonder if that makes it confusing at all for students.... Now [indicating $y=3x+7$ and $y=3x+9$], I have a system of equations.

But, it wasn't that these sorts of graphical techniques were a "trick" that was somehow new to her repertoire. Graphing seemed a strong and integrated part of her approach to the solving of equations that were more complex mathematically. Thus, when approaching a system of equations in two variables with one literal coefficient, $k-y=2$ and $x+y=k$, she solved the problem graphically. Her solution was quick and confident, though she only answered the question as posed and did not go on to characterize the nature of the solution set (as do some of the interviewees in Sfard (1994)).

> ... what you're asking is for every k, is there a solution x and y? Okay, well I was looking at this and thinking this is a system of two lines and then I just noticed that I have $k - y=2$. That line is a specific y equals line... $y= k-2$. ... if this one has a slope zero, so that's going to make it a lot easier to think about this one as $y = -x + k$. So if I'm thinking about graphing this ... that hits, my intercept is k and the slope is negative, so I'm going to get a line like this, so there is a specific point of intersection. So there is a solution, now is there a solution for every k? Well, if I vary k along the y axis, it seems like there would always be a solution to me.

Similarly, when approaching the single equation in two variables in which the variables cannot be untangled, $x^2y + y^2x = 1$, she immediately thought of graphing each side on a three dimensional coordinate system. Though she did not produce a solution, she was not overwhelmed by the problem and made productive strides towards describing the nature of the solution set.

> ...either one of them can't be zero, so I'm never going to have that type of a solution. Okay, this looks three-dimensional.... It does because, well then I can graph this [the left side of the equation] and it's some kind of plane intersecting with a plane $z=1$. So maybe if I fooled around, I could figure out, you know, what x and y would have to be to equal one, but maybe there' s not any solutions, or maybe there's a lot of different ones that occur different places. Cause this type of thing being $z = x^2y + y^2x$, might be really wavy. I'm not really sure what it looks like. Because I have, my variables are multiplied together, so that's kind of something that I'd have to sit and think about (laughs)

> for a really long time before I could get a picture. I think that I would go right to my computer and graph that... I have a 3-d graphing calculator on my computer at home.... It would probably be easier if I chose certain planes for y. Say, y equals one, and then look at what I'm left with, I'm left with $x^2 + x$ and $x^2 + x$ is just a parabola, so I have for the plane $y = 1$. That's this plane, I get some kind of a parabola. I think I did that right. And if I chose $x =1$, then I would just be getting in the $x =1$ plane, I would be getting $y + y^2$, which is also an upward parabola. So it looks like there is a couple different places where it's going to cross.

Frustrated by her inability to present the interviewer with a solution, the intern then said:

> I am thinking maybe I did some stuff like this in multivariable calculus. And that I am just not clicking with remembering the procedures for figuring out. I think that I have a relational understanding of what's going on with the situation.

To what extent does this intern's substantive mathematical knowledge support teaching for a conceptual or relational understanding of the solving of equations?

At the level of espoused beliefs, this intern has a conceptual orientation towards teaching. In discussing the solving of equations, she explicitly refers to a relational understanding and would like to offer her students more than what seems to her to be procedural skill. Though we may argue about exactly where to draw the line for conceptual understanding, arguably this intern has a conceptual understanding of this topic. She is able to solve a variety of problems involving the solving of equations. In doing so, she makes use of a similar strategy throughout and seems to have a well-connected understanding of relationships between problems of different types. She strives to answer the question of why a particular equation in one variable would not have a solution and why a particular system of equations will have a solution for any value of k. She can explain the goal of solving linear equations and how graphing both "sides" of the equation helps one find the unknown number for which one was searching. Though there are some tensions between the method that she chooses to solve equations and the way in which she defines the nature of the task, perhaps this is only natural. After all, Sfard and Linchevski (1994) suggest that competence in algebra involves versatility and adaptability in the interpretation of symbols. They suggest that in solving algebra problems a person oscillates between operational approaches and a variety of structural ones.

However, we are concerned that this intern's substantive mathematical knowledge does not provide sufficient resources for the development of her students' conceptual understanding. As she pointed out, and we concur, one might be concerned that her students will become confused. If any string of symbols with an equal sign is an equation, and thus some functions can be represented by equations,

how does this fit with the notion that only some equations are functions? If one solves an equation in one unknown by adding an implicit unknown and then solving a "system" of two equations in two unknowns by looking at the intersection point of two lines, how will students understand why the solution to such "systems" are different than the solutions to problems which begin with a system of equations in two unknowns?

In addition to the potential for confusion, we wonder whether, by teaching this graphical strategy, this intern would simply be teaching a new procedure for the solving of equations, rather than helping her students develop a conceptual understanding. Would she be able to articulate why this procedure works and why it is a legitimate way to seek the solution to an equation? We are not sure. We might argue that though this intern may have a conceptual orientation towards teaching, there is a fundamental confusion within her conceptual understanding. Contra Sfard and Linchevski, we might argue that there is a contradiction between her definition of an equation and its solution and her methods for solving equations.

Is it that a functions-based approach reduces simply to a different procedure for solving equations and doesn't offer a qualitatively different understanding of the domain (as we felt in reading Massingila, 1998)? We would disagree (see Chazan, in press). Perhaps, in this case, if the intern had made more explicit connection of this graphing technique to functions rather than equations, we might be more sanguine. Alternatively, perhaps our interview's focus on a particular problem type was too limited. Maybe if we had asked the intern how instruction over the course of the year would help students become prepared for this sort of task, then she might have been able to articulate how they would come to learn this technique in integrated way and not as an isolated procedure.

These concerns make us wonder about using descriptions like conceptual or procedural understanding for an examination of teachers' substantive knowledge of mathematics. Perhaps the difficulty is that conceptual understanding is not an "achievement," that is, something that one either has or does not have. Instead maybe one can have conceptual understandings of different kinds, including partial, or confused, conceptual understandings. Perhaps, discussions of the sorts of understandings useful for supporting teaching for conceptual understanding might be more usefully organized around a set of dimensions, for example: To what degree is the teacher able to articulate the goals of a problem in terms of relevant mathematical objects? To what degree is the teacher able to relate situations and the mathematics used to model situations? To what degree is the teacher able to provide justification for why procedures work? To what degree are there matches between the teacher's definition of the goal of a task and the procedures used to reach those goals? To what degree is the teacher aware of inconsistencies in his/her own understandings? In this sense, having a conceptual orientation to teaching and a conceptual understanding of a topic might not mean that one has sufficient subject matter resources for teaching that topic.

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A RECIPROCAL MODEL RELATING SELF ESTEEM AND MATHEMATICS ACHIEVEMENT

Constantinos Christou*, George Philippou*, Maria Heliophotou**
*University of Cyprus, P.O Box 537, Nicosia Cyprus, **Cyprus College

In this study we developed a model relating general self-esteem and mathematics achievement through basic intervening variables supported by social cognitive theory. A structural equation model was developed to examine the paths from mathematics achievement to general self-esteem and vice versa trough mediating variables. The data, collected from 308 preservice primary school teachers, provided a good fit to the developed model, indicating an indirect reciprocal causal relationship between self-esteem and mathematics achievement.

Background and aims

Knowledge, skill, and prior attainments are often poor predictors of subsequent success because self-perceptions powerfully influence behavior. The beliefs that people have about themselves are key elements in personal self-evaluation and behavior prediction. Bandura (1986) considered self-reflection and self-evaluation to be the most unique human abilities and believed them to include perceptions of self-esteem. The concept of self-esteem is not uniquely defined in the literature. Kahne (1996) considers self-concept (SC), self-esteem (SE) and values to comprise the three dimensions of self-perception. SC refers to the descriptions we hold for ourselves and SE refers to the level of satisfaction we attach to those descriptions. Similarly, Kohn (1994) defines SE in terms of the "personal judgement of worthiness that is expressed in attitudes the individual hold for himself" (p.273). On a rather different line, Byrne seems to equate SC to "perceptions of ourselves" and in specific terms as "our attitudes, feelings and knowledge about our abilities, skills, appearance and social acceptability" (1984, p.429).

Self-esteem judgements reflect evaluations that can be either task oriented or ego oriented (Kahne, 1996). Task oriented evaluations are based on self-efficacy beliefs while ego oriented beliefs derive from perceived social differential characteristics. The issue in the former case is to enhance skills, while in the latter case, the issue is to establish superiority over one's mates. In either case SE is postulated to affect behavior, as people tend to engage in tasks in which they feel competent and avoid tasks in which they do not. High SE helps to create an environment in which individuals choose to undertake difficult tasks and activities, while feelings of low SE develop a sense that the obstacles are insurmountable, foster stress, and narrow the vision. Consequently, SE can be considered both as a determinant and a predictor of the level of accomplishment that individuals finally attain.

Studies of the causal relationships between academic achievement (AA) and general self-esteem (GSE) have tended to focus on connections between AA and

GSE via academic self-concept (ASC). Early research reports have mentioned causal connections leading from AA to ASC and then to GSE (Byrne, 1984). On the other hand, causal paths have also been found flowing from GSE or ASC to AA (Marsh, 1990). An increasing number of reports have come to agree that the causal relationships are reciprocal rather than unidirectional (Marsh, 1993).

For some time attempts to examine reciprocity paid little attention to subject specificity (Marsh & Yeung, 1997). Though a generalized academic affect may be appropriate in some situations, there is an implicit underlying assumption that measures of affect vary substantially over different school subjects. Marsh and Yeung (1997) rejected the idea of a model that posited a single higher order dimension of academic self-concept, and described a model in which SC is subject specific. Moreover, Marsh (1993) referred to a growing body of research showing that verbal and mathematics SCs are nearly uncorrelated. He concluded by rejecting the usefulness of measures of general academic self-concept as a summary of SCs in specific school subjects, when there is no correlation among them.

In this study, we examine the reciprocal relationships between MA and GSE of preservice teachers, considering that the relationship between AA and SE is subject specific. What is new in this study is the inclusion of intervening variables that refer to preservice teachers' confidence in doing and teaching mathematics. Thus, the main purpose of this study was to examine how specific variables of teaching mathematics influence the GSE of teachers by including intervening variables that are more suitable for the pre-service teachers. To this effect, we developed another construct, the teacher's self-esteem of mathematics, which is hypothesized to influence their GSE. Specifically, the study addressed the following questions: Are the relationships between MA and GSE reciprocal? If yes, what variables mediate this reciprocal relationship? Are there differences in the structure of pre-service teachers' GSE in terms of their gender? The latter question is related to earlier findings that general and academic self-concepts are more highly correlated with the mathematics SC for boys and more highly correlated with the verbal SC for girls (Marsh, 1993).

Method

To answer the research questions we estimated a theoretically informed multivariate causal model in which the hypothesized reciprocal relationships are decomposed through the introduction of mediating psychological constructs. The proposed model is based on the theoretical assumption that views SE as both a "social force" and a "social product". In the context of mathematics learning and teaching, SE is seen both as influencing and being influenced by the mathematics achievement of pre-service teachers. The model assumed that MA (F4) affects students' GSE (F1) through four intervening latent factors:

- Students' perceptions of teachers' appraisals about their own capabilities in mathematics (F5),

- Students' perceptions of their comparative performance in mathematics (F6),
- Students' perceptions of their ability to teach mathematics as compared to their classmates (F7),
- Students' confidence in teaching mathematics (F8).

Similarly, we proposed that GSE enhance MA through two mediating factors: motivation (F2), and anxiety (F3).

The model was tested against the data collected from all freshmen and sophomore preservice teachers admitted at the University of Cyprus in 1977 on the basis of competitive examination scores. Using listwise deletion of missing values, the final sample included 308 students. Of these 28% were males and 72% females. Mostly we used three observed variables to identify each latent variable (F1 to F8), since a larger number is technically unnecessary. Table 1 shows the latent variables, and the loadings on each item for males and females.

Results

All indicators load strongly and distinctly on each of the latent constructs for both groups. Table 1 shows that the standardized loadings are all above .35 with the exception of the first variable on the Motivation factor, which are quite low for both groups (.142 for females and .114 for males). These findings indicate that the hypothesized structures can be adequately represented through the first-order factors. More precisely, we found that the reciprocal relationship between GSE and MA can be represented by the hypothesized 8 first order factors. The negative sign of the third indicator of the GSE factor, which holds true for both the male and the students, reaffirms previous results that students' GSE is lower when they feel that they should respect themselves more than they do.

The analyses were conducted with covariance matrices, since the focus of the study was on the testing of the invariance of factor loadings and factor regressions across the male and female pre-service teachers. We began with the least restrictive model in which only the form of the model, i.e., the pattern of fixed and non-fixed parameters is invariant across groups. The initial baseline model is "totally non-invariant", as no between-groups invariance constraints were imposed on estimated parameters (Table 2). This model provided the basis for all subsequent models in the invariance hierarchy.

In the first stage we tested the ability of the model to fit the data separately for each of the two groups with no invariance constraints. The parameter estimates were reasonable for both groups in that all factor loadings were large and statistically significant and the patterns of correlations were logical and consistent with previous research. Moreover, the goodness of fit index was good in relation to typical standards. Table 2 shows that the Comparative Fit Index (CFI) for the total sample was .914, which indicates a "good fit".

We pursued two more specific tests imposing invariance constraints for sets of parameters (factor loadings, and regression correlations and factor variances) across the two groups, to test for gender differences. We began with tests of the

Table 1: The Factors, and the Factor Loadings of the Observed Variables

Factors	Items	Factor Loadings	
		Females	Males
F1: General Self-esteem	On the whole, I am satisfied with myself.	.666	.736
	I take a positive attitude toward myself.	.689	.756
	I wish I could have more respect for myself.	-.427	-.436
F2: Motivation	I have had the feeling of wanting to quit school.	.142	.114
	I have had the feeling of wanting to quit university.	.445	.450
	I was not interested in school maths at high school.	.771	.791
F3: Anxiety	I feel nervous when doing mathematics.	.890	.886
	I feel my hands sweating when working with problem solving.	.864	.860
	During the past year, I felt that I was going to have a nervous breakdown due to mathematics.	.764	.787
F4: Mathematics Achievement	Grades in mathematics at high school.	.747	.709
	Grades in mathematics in university math courses.	.365	.228
F5: Perceived Teachers' Appraisals	My math teachers were not interested in what I did in maths.	.355	.325
	By my teachers' standards I was not good in maths.	.751	.924
	My teachers did not appreciate my abilities in maths.	.606	.614
F6: Perceived Comparative Performance in Mathematics	I believe that in maths I am better than many of my classmates.	.797	.857
	I believe that I can understand mathematics better than my classmates.	.940	.911
	I believe that I am one of the best students in maths	.741	.673
F7: Confidence in Teaching Mathematics	I feel confidence in teaching maths.	.677	.658
	I feel confidence in explaining maths.	.884	.876
	I believe that I will become a good math teacher	.758	.777
F8: Perceived Comparative Ability of Teaching Mathematics	I believe that I can teach mathematics better than my colleagues.	.883	.911
	I believe that I will become a better math teacher than my colleagues.	.909	.890

equality of factor loadings across the two groups, followed by tests of regression correlations and factor variances. The "totally non-invariant" model indicated a good fit for the whole sample as well as for each of the two groups separately: $\underline{x}^2$ =

648.9, with 426 df (degrees of freedom) for the total, x^2 = 335.7 with 213 df for females, and x^2 = 313.3 with 213 df for the male model ($p < 0.001$ in all cases). The introduction of factor loadings invariance resulted in a change of x^2 by 2.796 and a change in df of 15 (p>.001) with a good fit (CFI = .919). Finally, the introduction of the invariance of regression correlations resulted in a change of x^2 by 11.694 and a change in df of 14 (p>.01) with a better fit (CFI = .920). The conclusion is that the measures between the two samples are equivalent, meaning that the model has an equally good fit for females and males.

Table 2. Goodness of Fit for Separate Solutions for Each Group with no Invariance Constraints and for Invariance Constraints Imposed across the Two Groups (males and females).

Model	x^2	df	CFI	p	x^2_d	df_d	p
No invariance							
Males	313.259	213	.909	.001			
Females	335.689	213	.918	.001			
Total	648.948	426	.914	.001			
FL invariance	651.744	441	.919	.001	2.796	15**	p > .001
FL, FC, invariance	663.438	455	.920	.001	11.694	14**	p > .001

Overall, the fitting indices are almost the same for each step in the hierarchy, none of the changes in the chi-square values were found to be statistically significant. This means that the constrained model can adequately explain the structure of interrelationships among the factors and the directions of the paths in both groups. In other words, the restrictive model seems to be identical in the two groups.

Figure 1 shows that the teachers' GSE is influenced by their MA through the four intervening variables. The first path indicates that GSE is influenced indirectly by MA through the perception of teachers' feedback so that high MA elicits positive teacher appraisals: regression correlations (RC: .760 and .780 for females and males, respectively). These responses contribute to positive feelings with respect to the mathematics SC and perceived comparative performance in mathematics (RC: .850 and .930), and also through the teaching confidence (RC: .486, .369). The former leads to positive math teaching SC (RC: .800 and .901), though the perceived relative MA leads to GSE through the confidence in teaching mathematics (RC: .870 and .780) and to the perceived comparative ability of teaching (RC: .985 and .990). The latter factor (relative teaching ability) leads directly to confidence in teaching mathematics (RC: .416 and .369) and indirectly to GSE. What is interesting to note is that all the three intervening variables: teachers' appraisals, perceived relative performance in mathematics and perceived relative ability of teaching mathematics, lead directly to teaching mathematics

confidence, which in turn leads directly to GSE. The direction of each indirect causal path is consistent with the hypothesized positive effect of MA on GSE and confidence in teaching mathematics.

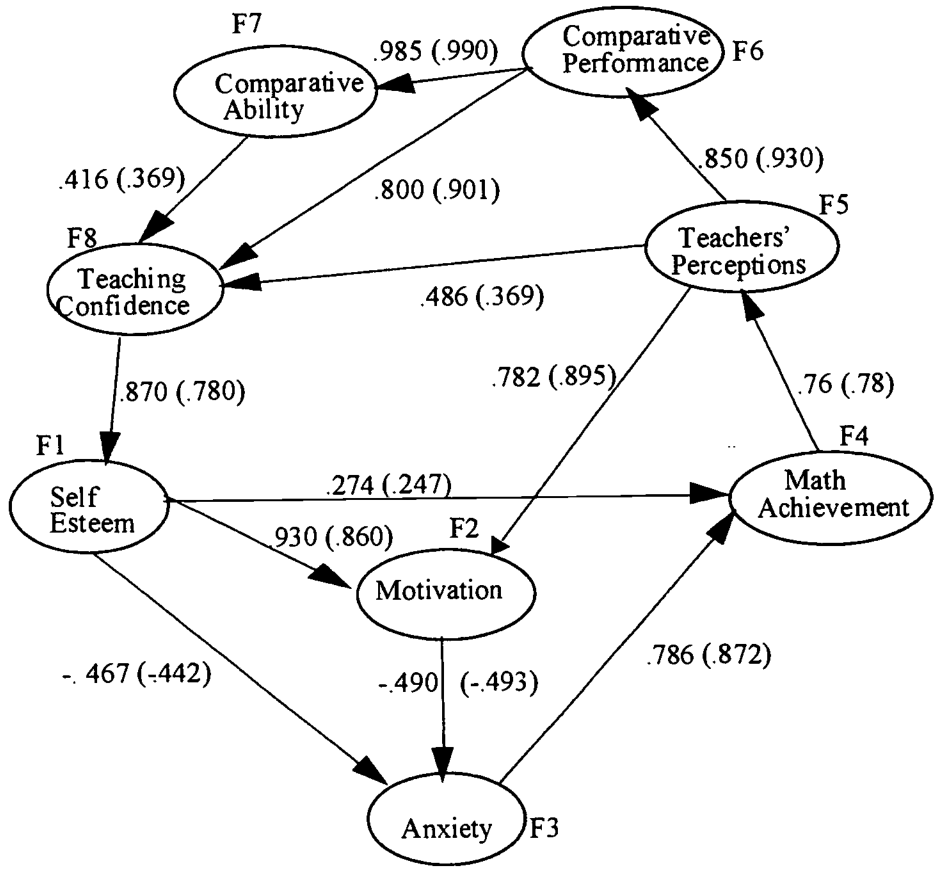

Figure 1: The Model with Regression Correlations.(Numbers in parentheses indicate males' parameters).

The hypothesized paths involving GSE, mediating variables, and MA are all significant. GSE has a significant negative effect on anxiety (RC: - .467 and - .442 respectively) so that low general self-esteem leads to high anxiety. Anxiety in turn has a direct effect on MA, indicating that high anxiety contributes to positive mathematics achievement, reaffirming previous studies. GSE has a significant positive effect on motivation (RC: .930 and .860), which in turn affects MA indirectly through anxiety. Specifically, motivation has a significant negative effect on anxiety (RC: - .490 and - .493 respectively) meaning that low motivation leads to high anxiety. Furthermore, GSE affects MA in a direct way, indicating that high GSE leads to high MA (RC: .274 and .247). The directions of all these paths are again consistent with the hypothesized effect of GSE on MA. Examination of the modification indices suggests an indirect effect of teachers' appraisals on mathematics anxiety through motivation (RC: .782 and .895) reflecting the idea that negative feedback from teachers results in an increase in the mathematics anxiety of students.

Conclusions

In summarizing the results of the present investigation, the most important findings are the critical paths relating mathematics achievement and general self-esteem of pre-service teachers through intervening variables that are closely related to the professional development of teachers. The present study is in line with the results of previous studies indicating that self-esteem is formed and influenced by evaluations of significant others for one's own behavior (Marsh & Yeung, 1997). In this context, the results provide a strong case for the role of educators in the formation of pre-service teachers' perceptions of themselves both as students and future teachers of mathematics. These perceptions are the outcome of mathematics achievement and consequently influence the way pre-service teachers think about themselves as mathematics learners and mathematics teachers.

In agreement with this perspective, the nature of the model developed in the present study indicates that general self-esteem is important for pre-service teachers both as an outcome and as a mediating variable that helps to explain other outcomes such as mathematics achievement. The results of this study are consistent with most longitudinal studies which support the model in which prior academic achievement influences subsequent self-esteem and prior self-esteem influences subsequent achievement (Byrne, 1996). However, the present study is more specific than previous ones in that it examines the nature of a particular school subject and general self-esteem, and explores the specific intervening variables through which general self-esteem influences mathematics achievement.

A major finding of the present study is that pre-service teachers' general self-esteem is formed in relation to an external social comparison reference point on the basis of which teachers compare their self-perceived performance in mathematics with the perceived performance of other students in the same subject. The relationships found are consistent with principles of social comparison and reflected appraisals, which postulate that one's self concepts and self-attitudes are outcomes of social relationships and processes. The principle of reflected appraisals argues that people are to a great extent, influenced by the evaluations or judgments of significant others (Bishop, Brew, Leder, & Pearn, 1996). In social interaction, not only do people tend to perceive themselves on the basis of feedback received by others, but they are also likely to internalize these responses and evaluate themselves, in part influenced by the responses of others that are communicated to them. In the same way, their feelings and attitudes towards themselves will be gradually formed on the basis of perceptions and observations made by them. In mathematics achievement, students tend to perceive their own successes or failures from the various approvals or disapproval of teachers as these are reflected in grades and comments. These in turn become the bases on which the students judge themselves and form their self-concept as students. The specific judgment that the students form of themselves, based on their academic achievement, will also contribute to their overall self-feelings. For many pre-service teachers the mere possession of high ability signifies self-worth, which in

turn leads to a high mathematics teaching self-concept and mathematics confidence.

The second important finding of the present study is that the reciprocal nature of pre-service teachers' relationships between self-esteem and mathematics achievement is the same for both males and females. Self-esteem and self-concept researchers evaluate gender effects in mean levels of self-esteem and self-concept (Hattie, 1992), but insufficient attention has been given to gender differences in the factor structure of self-esteem and mathematics achievement among pre-service teachers. Thus, one of the main concerns of the present study was a more fundamental issue relating to the factor structure underlying self-esteem and mathematics achievement, i.e. as to whether responses to the same instrument have the same meaning for male and female pre-service teachers. It was found that the same reciprocal model represents the responses of both male and female pre-service teachers as far as mathematics achievement and self-esteem are concerned.

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TEACHERS' VIEWS ON PUPIL COLLABORATION IN COMPUTER BASED GROUPWORK SETTINGS IN THE CLASSROOM

A. Chronaki* and C. Kynigos**
Open University, Milton Keynes, UK*
University of Athens, School of Philosophy, Dept of Education and Computer Technology Institute, Athens, Greece**

Abstract: *Pupils' collaborative work has recently regained attention due to an increase in classroom use of modes of group and project work often required by computer based environments. The significance of this type of social structuring in lessons has been appreciated as an important factor for teaching and learning mathematics. The present paper discusses the issue of 'collaborative work' from the teachers' point of view and focuses on identifying and interpreting the types of teacher's views concerning the collaborative work of pupils. In particular, it examines how teachers themselves describe pupils' collaboration and how they see their role in shaping interactions and managing collaboration amongst pupils within a group situation.*

1: Theoretical Orientation

Learning in collaborative settings has attracted a lot of interest as researchers are moving on from constructivist to interactionist and socio-cultural viewpoints to interpret mathematical teaching and learning processes and are appreciating the role of the social setting in which learning takes place. Recent studies, however, attempt to combine Piagetian and Vygotskian perspectives, rather than to generate theory focused on learning in specific collaborative situations (Kynigos and Theodosopoulou, in press, Yackel and Cobb, 1996). In this study we espouse a socio-cultural view of teaching and learning in the sense that we conceptualise this double process as a set of social interactions taking place within the cultural environment of a classroom. Learning is seen as culturally embedded and it occurs in dialogic exchanges between peers and more knowledgeable members of culture (Bauersfeld, 1990, Lerman, 1993 and Confrey, 1995). We address learning situations where small groups of pupils collaborate in computer-based mathematical projects with their normal teachers within a school based innovation programme. Collaboration in computer based learning environments can be used to augment ways of acting which generate common meanings regarding the activity. For instance, in a study carried out by Hoyles et al. (1992), pupil activity generated by groupwork based on the use of exploratory software for mathematics included the negotiation of goals and processes, the need for justification of ideas and actions to partners, the development of a shared language for communicating actions to be taken and the brainstorming of solution strategies. In this study we focus on the meanings for the nature of pupil collaboration created by their teachers during their involvement in this school innovation programme, as they express them in the course of a set of semi-structured interviews combined with extensive lesson observation[1].

[1] The study was carried out in the framework of project YDEES: "Development of Popular Computational

In the last decade there has been a major shift in research paradigms involving the teacher and the teaching profession (Hoyles, 1992). Early studies took the stance that teaching is the technical implementation or conveying of a curriculum or pedagogy designed outside the classroom and that the study of teaching is directly informed by the extent to which the process and the results are close to the prescribed ones. The studies were also restricted to studying the teacher-pupil dyad rather than settings involving a variety of social interactions in the classroom. These studies were followed by ones addressing the teacher as a reflective practitioner directly shaping teaching process and content, and took teacher beliefs, views and epistemologies seriously into consideration (Olson, 1989, Lerman, 1992). Teachers´ personal images (thinking, views, beliefs, intentions, ideologies), are influencing the shaping of the context of teaching and learning. The reason for studying the views of teachers is grounded in the assumption that these have a significant influence on their planning and actions during lessons. Views and attitudes act as a sort of filter and they can be indispensable in forming and organising the meaning of things, but on the other hand they can block the perception of new realities and the identification of new problems and solutions. As a result, researching into how teachers themselves conceptualise pupils´ collaborative learning may offer insights concerning the motives they carry along to their interactions with the pupils. One could then be in a position of making some sense about the potential type of collaboration that teachers can ultimately structure for their pupils. Adopting this theoretical orientation in the present study, we look at the views on the nature of pupil collaboration, formed by teachers involved in implementing pedagogical innovation in a weekly course on computer based mathematical projects.

2: Research setting

The research took place in a Greek primary school project which has been going on for over a decade and involves the use of computer technology for a weekly "investigations" hour from year 3 to 6. From the start, the project was seen at the school level as the infusion of a pedagogical innovation together with the use of computer technology. The researcher played the role of teacher educator and consistently held seminars at the beginning and the end of the school year and meetings of a varying frequency during the year. The project set off explicitly focused on the idea of providing pupils with the opportunity to collaborate in small groups, gain some autonomy from the teacher and become more active in their thinking, constructing and problem solving (Kynigos, 1992). This was socially mediated and agreed upon by the school's direction and staff and all teachers took part with their own class. They used Logo as a means of expressing ideas, constructing and experimenting and a word processor and a drawing application for composing reports on their projects. A study

Tools for General Education: The Computer as Medium for Investigation, Expression and Communication for All in the School", General Secretariat for Research and Technology, #726, E.P.E.T. II, 1995-1998.

of teacher strategies six years into the project showed that it was informative to describe these in terms of a) the aspects of the learning situation addressed by the teacher and b) the kind of pupil activity their comments intended to encourage (Kynigos, 1996). For instance, more than half of the teachers´ comments (54%) did not focus on specific mathematical content but dealt with process related issues (such as social interactions, use of tasks and resources).

We thus felt that it was important to ask how the teachers themselves interpreted and experienced the meaning of their pupils´ collaborative work. We did this by means of semi structured interviews with five of the more experienced teachers, at a time when - in the framework of a larger project - we were participating in teaching experiments in the classroom. This shared experience of planning and delivering lessons helped us increase our understandings and insights concerning the meaning of their narratives and minimised the effect of contextual factors (i.e. the nature of different tasks) on interpretations. The interviews aimed to address questions of the type: What is the 'mental context´ of collaborative learning that teachers construct through their narratives about pupils´ work and interactions? What different styles of descriptions about collaboration exist amongst teachers? How do they view their interaction with pupils in collaborative activity? Coding of teachers´ answers was oriented about the two main interview questions: a) what collaborative work meant to them and b) how would they explain their interventions in group settings so that to encourage collaboration. More details concerning the methodology for data collection and analysis can be found in Chronaki (1998).

3: Three ways of viewing pupils´ collaboration

All teachers were relatively confident in using the computer as a tool for teaching, but a few still expressed reservations about using this medium at all times. Difficulties with various technicalities and lack of time in planning and organising lessons were mentioned. When asked about their views regarding the value of collaboration, they all agreed that it is a vital component for pupils´ learning (this can also be attributed to an influence of their training). However, looking in more detail at the answers provided, one can discern variations in their mental images about what true collaborative work may mean for them. In terms of differences, three modes of teachers´ talking and thinking over their pupils´ collaborative work were identified through the data of the interviews (collaboration as aiming towards the work-outcome, as based on human relations and as a common-sense activity). There is always a difficulty with putting textual data (transcripts of interviews) into neat categories. Hence, there is no claim here that the three distinctive clusters of views, discussed below, are in any way exhaustive in describing teachers´ perceptions about collaborative learning in a general sense. However, these three modes characterise the main themes of teachers´ thinking met in the particular school community studied.

4.1: Collaboration is geared towards the work-outcome

Kostas and Michalis are both senior teachers, well respected by the pupils and both can be described as firm and formal in their relations with pupils. These two teachers perceive collaboration as a means to accomplish a particular mathematical activity. For them, pupils´ goals within collaboration should be the completion of mathematical tasks in the lessons. Kostas: '*Collaboration is based on effective work relations*´, and Michalis: '*They have an aim, and they use the views and help of the others so as to achieve this aim. The aim and the task need to be the focus and not the relations*´. Both stressed, as can be seen in the extracts below, that pupils need to learn to distinguish between friendship and collaboration and they felt strongly responsible for assisting them in realising this. For example, Kostas said: '*Pupils need to realise the difference between friendship and cooperation. Cooperation means that I can work with somebody on a particular subject even though s/he is not a friend*´. Their belief is that the pupils in the group need to put their feelings and sentiments aside and to get on with organising their roles and activities. For them collaboration is solely devoted towards the completion of the tasks in hand. The goal needs to be oriented towards work. Moreover, they saw this as a main pedagogical gain, implying that people need to collaborate almost at any cost. The view of collaboration as a gathering of people working towards the end product was also reflected in the ways they saw their intervening role in fostering pupils´ collaboration, which can be described as a managerial one. Kostas explained: '*My role is to show them that friendship is different from collaboration. This is very stressful for children. For example, it´s very difficult for them to realise that they haven´t dealt well with a situation, they are not mature enough for controlling their emotions*´. And Michalis: '*I see my role as supporting the pupils to carry out their activities. In terms of the team´s structure, I believe that the teacher needs to control its function. The teacher needs to make sure that things work smoothly and the pupils work towards achieving something*´. In short, they saw themselves as providing explicit explanations, as redirecting and focusing pupils´ energy towards the goals and objectives of the activity. However, their focus is on managing pupils´ work as end-product, not managing the relations of the pupils who work (i.e. the process of working).

4.2: Collaboration is about human relations, too.

Petros and Natasa took a different stance. They described 'collaboration´ as deeply rooted in pupils´ relations and feelings. Although, they have different experiences in teaching, Petros is a relatively new teacher (less than 5 years of teaching experience) whilst Natasa is more senior, they both are creative teachers who like to get involved with new ideas in their teaching. They have relaxed and friendly relations with their pupils and at times they are not hesitant to express affection. These two teachers talking about collaboration was lengthy and they used a rich and sophisticated

vocabulary to describe their thoughts. They saw collaboration as being based on pupils' human interactions, but they also talked about the work as a core of attention in the collaborative activity. For example, Petros said: '*It's very difficult to define collaborative learning, to describe it. It is never the same. It changes and develops all the time. There exist interactions and relations between pupils and the participants need to get new roles. Many times these roles are not stable during the lesson. They change (and they need to change) depending on what a particular pupil has to offer and also on what the specific task demands. One basic drawback is that pupils hesitate to make decisions and to organise their roles. Effective collaboration for me is to encourage the listening and discussion of all different views. To give equal opportunities to different voices. It is important that all pupils have the opportunity and also the responsibility to make explicit and communicative their thoughts with the other members of the team. These thoughts and views then need to be respected by the others, they need to find processes, ways and routines for making decisions. It is true that it is not easy to reach some consensus. It is difficult for all three pupils to agree. There are always diverging views. But, the issue is that with the realisation of these differences, they can find specific mechanisms and methods so as to synthesise and to construct a commonly agreed line. Then, they all need to follow this line without feeling rejected. And this is the most difficult part*'. And Natasa:'*Collaborative learning means that there is no pupil who tries to control the team at all times. It is important that all participants discuss the problems of the task and also the problems, the difficulties of their collaboration. In this way they are called on to provide solutions through their collaboration. These solutions need to be the outcome of discussion and not the imposition of the pupils or even the teacher*'.

With regard to their role in structuring pupils' collaborative work, these two teachers seemed to possess a repertoire of skills and tools for intervening with groups and showed a flexibility in their approaches. During observations, they talked using examples from their teaching in varied lessons and narrated a variety of ways they had tried in the past to encourage collaboration in groups. Amongst the tools that they often used they mentioned: restructuring the groups when necessary, focusing their pupils attention on the process of collaboration, its change and development, discussing with pupils, listening to their concerns, encouraging the opening up of personal aims, thoughts and motives to others, and respecting each others feelings. Overall they emphasised a strong and genuine concern about pupils relating to each other.

4.3: Collaboration is collaboration

Erato, is also a senior teacher who describes herself as a traditional teacher with modern ideas. By this she means that she prefers to mix and match new and old methods in her teaching according to what fits better into her lessons. Talking with her about collaboration, it was difficult to get her to articulate and unpack a description of the term. Her response was along the line of, 'collaboration is really collaboration',

implying that collaboration is a common sense word (everybody should really know its meaning) that does not need further explanation. Erato says: '*It is very simple. If you are not able to collaborate, you cannot produce. And this is something that all kids need to realise very well, especially when they cannot achieve their goals during the lesson. The lesson depends on their cooperation, and there are times that they were not able to finish their project because of their inability to collaborate´...I had to rearrange the teams many times. They couldn´t fit. They couldn´t. It was impossible. They were too competitive with a negative sense. This resulted in them being very mean to each other. For example, one would say to the others. I am clever! You, stop talking now! You are stupid! Stay out!. Further more they are often unable to sustain their roles or they mix up their roles. They cannot organise themselves within the team´.*

Talking more with her about her ways for intervening with pupils in assisting them to collaborate one could notice that although Erato was aware about the difficulties that pupils have in collaboration, she is not confident with her ways for dealing with these difficulties. She spoke very emotionally about her attempts (or even struggles) to put pupils into order when they worked in a group. She confessed that helping them cultivate good relations with each other was almost an impossible task.

4: Discussion

Three modes of teachers´ viewing of collaboration were presented in the previous section; collaboration as work-outcome, as relations and as common-sense activity. A common theme all teachers talked about was how difficult, stressful and painful the experience of collaboration was for pupils. For instance, they noted that pupils can be harsh to each other, immature and sometimes irresponsible for work and feelings. They held the view that constructive collaboration amongst pupils in group settings is rare, and they emphasised the difficulty, the complexity and also the importance for encouraging collaboration as a tool for fostering mathematical exploration and discussion. This commonly expressed concern about the problematic status of collaboration (and communication) in maths lessons in this educational setting suggests the need for further study into the nature of teachers´ views on such social interactions in classroom situations. Apart from this common concern about pupils´ difficulty to collaborate, teachers differed in their views about the nature of collaboration and about their role in coping with this situation, which they portrayed as one where the unexpected reactions of pupils within the group, 'disturbed´ the smooth process of the mathematical activity during the lesson. Some would portray a work-oriented view of collaboration and see their role as 'managers´ who need to employ clear-cut methods, preferring to distance themselves from pupils´ personal lives and to focus on doing the mathematical tasks instead (e.g. as in the case of Kostas and Michalis). Others, like Petros and Natasa, would adopt a 'human relations´ orientation manifested by getting

involved with their pupils´ lives and feeling enthusiastic about exploring possibilities for dealing with challenging situations. Finally, others like Erato, held a non-explicit and non-articulated view of collaboration conveying feelings of frustration about the dead-ends presented by difficult pupil interactions.

The first and third mode are 'work-outcome´ oriented and teachers seemed to have a certain 'agenda´ in mind about how the pupils should do tasks in the group (i.e. not being disruptive, not wasting time, producing what is required). Teachers in the first mode (the 'managers') felt they knew the tricks for dealing with pupils and through exercising their authority could eventually manage to get the groups sitting quietly and producing the expected work. The 'ground rule´ that they make explicit for their pupils is of the type: *'I want you to finish the task´.* Their structuring of pupils´ collaboration is mainly focused on allocating tasks and making sure that the group completes the work. And even though the work will finally be produced by the group, they placed no emphasis on guaranteeing that all pupils have contributed. Other studies have shown that it may well be the case that: a) pupils are seated as a group but work individually (Bennett, 1991), or b) pupils rely on a few competent ones in the group to do the work for all (see Hoyles and Sutherland, 1989). These teachers seemed to adopt a role of directing their pupils rather than collaborating with them, focusing on the end product of their activity and thus de-emphasising the importance of collaboration itself. The teacher in the third mode, not knowing what to do with pupils´ disruptive behaviour, got very disheartened. Apart from her statements that she did not want to be directive (or authoritative) with pupils, she did not have any explicit strategies to foster their collaboration. As a result, she felt frustrated because she realised that she could not 'control´ the situation (i.e. get the group working smoothly together). This could be due to a) her interpersonal relations with the pupils in the group and b) her own perception about collaborative work itself. In the first and third mode 'ground rules´ about collaboration itself are not expressed explicitly to the group and attempts to communicate them with pupils were not made. The reasons for this can be either a) lack of awareness about its features and therefore not being in a position for making them explicit (see Erato) or b) non-appreciation of the importance of talking and exploring collaboration due to over-focus on the end product (see Kostas and Michalis).

The views of teachers in the second mode could be described in terms of the notion of 'communities of practice´ coined by Lave and Wagner (1991). In this case, an important part of the object of teacher-pupil discussion was collaboration itself. Both teachers (Petros and Natasa) realised the importance of exploring collaboration with their pupils and revealed their enthusiasm and willingness to learn more about their pupils´ lives, relations and problems. Teachers in this mode have an explicit, wide-angled, image in mind about pupils´ collaboration and expressed an ability to talk about

its features and explore its complexities. Placing sensitivity on the process of collaboration itself suggests that these teachers are in a better position to encourage pupils´ exploratory activity.

This study is part of an ongoing research into teachers views on aspects of learning situations and is thus used to generate further enquiry into their view of collaboration and the respective nature of their teaching. Knowing more about how teachers themselves view such situations (through their experience) and how they perceive their role in dealing with the entailed complexities 1) may be used as instruments for reflection and re-orientation of perceptions and conceptualisations about collaboration and 2) may enable them to identify and suggest instructional intervention.

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Abductive Inference: Connections between Problem Posing and Solving

Victor Cifarelli

Department of Mathematics
The University of North Carolina at Charlotte

Using the Peircean construct of abductive reasoning, this paper examines the novel problem solving actions of a college student. The analysis documents and explains how the solver's solution activity is constituted by an intermingling of problem solving and problem posing, with the solver's abductions providing the "cognitive fuel" needed to sustain their co-evolution.

Introduction

Accounts of mathematics learning have long acknowledged the importance of autonomous cognitive activity, with particular emphasis on learners' ability to initiate and sustain productive patterns of reasoning in problem solving situations (Cobb, 1988; Mason, 1995; Schoenfeld, 1985). Nevertheless, most accounts of problem solving performance have been explained in terms of inductive and deductive reasoning, containing little explanation of the novel actions solvers often perform prior to introducing formal algorithmic procedures into their actions. For example, cognitive models of problem solving seldom address the solver's idiosyncratic activity such as the generation of novel hypotheses, intuitions, and conjectures, even though these seen processes are seen as crucial tools through which mathematicians ply their craft (Anderson, 1995; Burton, 1984; Mason, 1995).

In contrast to inductive and deductive reasoning, Charles Saunders Peirce (1839-1914) asserted the existence of another kind of reasoning, *abduction*, which furnishes the reasoner with a novel hypothesis to account for surprising facts. It is the initial proposal of a plausible hypothesis on probation to account for the facts, whereas deduction explicates hypotheses, deducing from them the necessary consequences, which may be tested inductively. According to Peirce, abduction is the only logical operation which introduces any new ideas, "for induction does nothing but determine a value, and deduction merely evolves the necessary consequences of a pure hypothesis" (Peirce, 1891, p. 303).

The Generation of Hypotheses to Facilitate Problem Posing and Solving

While few studies of mathematical problem solving have specified precisely the role of abductive actions in the novel solution activity of solvers, the research on problem posing (Silver, 1994; Brown and Walter, 1990) suggests ways that hypotheses play a prominent role in solvers' novel solution activity. According to Brown and Walter (1990), problem posing and problem solving are naturally related in the sense that new questions emerge as one is problem solving, that "we need not wait until *after* we have solved a problem to generate new questions; rather, we are logically obligated to generate a new question or pose a new problem in order to solve a problem *in the first place*" (Brown and Walter, 1990, p. 114). Furthermore, Silver (1994) asserted that this kind of problem posing, "problem formulation or

re-formulation, occurs within the process of problem solving" (Silver, 1994, p. 19). Finally, the cognitive activity of "within-solution posing, in which one reformulates a problem as it is being solved" (Silver and Cai, 1996, p. 523) may aid the solver to consider "hypothesis-based" questions and situations (Silver and Cai, 1996, p. 529). This illustrates both the dynamic, yet tentative nature of solvers' solution activity as well as the propensity of solvers to abduce novel ideas about problems while in the process of solving them.

Objectives

The purpose of the study was to analyze the problem posing and solving processes of learners in mathematical problem solving situations, with particular focus on ways that the learner's emerging abductions or hypotheses help to facilitate their novel solution activity. The perspective taken here is that problem solving situations are self-generated by solvers, arising from their interpretations of the tasks given to them. Their interpretations of a particular task may suggest to them additional questions and uncertainties, the consideration of which helps them construct goals for purposeful action. Successful completion of the task may involve many such problem posings, all generated by the solver in the course of their on-going activity and each having the potential to alter the solver's current goals and purposes. In the course of generating problems, the solver may monitor potential solution activity for its usefulness. In this way, problem solving can be viewed as a form of abductive reasoning through which solvers mentally reflect upon and contemplate viable strategies to relieve cognitive tension, involving no less than their ability to form conceptions of, transform, and elaborate the problematic situations they face.

In an earlier study (Cifarelli and Sáenz-Ludlow, 1996), examples of abductive reasoning activity were discussed, highlighting its mediating role in the mathematical activity of learners. This work has been elaborated and extended in a series of research studies which have analyzed the qualitatively different kinds of abductions that mathematical learners demonstrate (Sáenz-Ludlow, 1997), the evolving structure of solution activity that results from abductive reasoning in problem solving situations (Cifarelli, 1997a), and the transformational influence of abduction in problem solving situations (Cifarelli, 1997b). The current study sought to extend these results by explaining the ways that learners' abductions foster an intermingling of problem posing and problem solving activities.

Methodology

Five graduate students in Mathematics Education participated in the study. The students were enrolled in a class, taught by the researcher, the Use of Technology to Teach Middle and Secondary Mathematics. The students were interviewed on 3 occasions throughout the course. These interviews took the form of problem solving sessions, where students solved a variety of algebraic and non-algebraic word problems while "thinking aloud". All interviews were videotaped for subsequent analysis. In addition to the video protocols, written

transcripts of the subjects' verbal responses as well as their paper-and-pencil activity were used in the analysis.

Based on the analysis of the verbal and written protocols, a case study was prepared for each solver. The solvers' protocols were examined to identify episodes where they faced genuinely problematic situations. Previous studies conducted by the researcher characterized abduction as a structuring resource utilized by problem solvers (Cifarelli, 1997a, Cifarelli, 1997b). Specifically, while resolving problematic situations, the solvers were inferred to have generated abductive inferences which served to organize, re-organize, and transform their mathematical actions. These structuring actions, which often introduced the formation of new problems or re-organization of previous problems, were interpreted as acts of problem posing that had profound influence on the solvers' overall solution activity, thus establishing a connection between solvers' problem posing and problem solving (Brown and Walter, 1990).

The current study examined more thoroughly the novel actions of solvers, with particular focus on identifying additional interconnections between problem posing and problem solving processes.

Analysis

The following paragraphs contain episodes from interviews conducted with Jessica. Jessica was a secondary mathematics teacher in her second year of teaching and proved to be among the strongest mathematics students in the class, achieving high scores on all class exams and assignments. She demonstrated strong problem solving activity throughout the interviews, as indicated by the novelty of her actions in completing the tasks.

Jessica's Abductive Activity. Jessica was required to solve a variety of non-algebraic problems during the initial interview. One of the tasks involved a person paddling a canoe on a river:

> **Sally, an avid canoeist, decided one day to paddle upstream 6 miles. In 1 hour, she could travel 2 miles upstream, using her strongest stroke. After such strenuous activity, she needed to rest for 1 hour, during which time the canoe floated downstream 1 mile. In this manner of paddling for 1 hour and resting for 1 hour, she traveled 6 miles upstream. How long did it take her to make this trip ?**

Upon reading the problem, Jessica commented that she had seen a similar problem before but had not solved it.

Jessica: *I have had one like this ... and I'm not sure. I had a similar one in Dr. L's class. Upstream-downstream, airplane flying with the wind behind them. Professor L gave us a list of 100 problems. I looked them over and did not choose this one. I didn't do it, but I did watch other students do it. So I have not technically done this problem.* **(appears confident she can do it[1]).** **(re-reads the problem; several seconds of reflection)**

[1] Comments in boldface describe the non-verbal actions of the solver as inferred by the researcher.

Jessica: *Okay, distance is 6 miles. Let's see ...total time is 2 hours ... we have to modify this because upstream means you are getting help and downstream means you're not ... Oh, wait ...* ***(reflection)*** *1 hour she travels 2 miles up ... and she rests 1 hour ... so it is not total is 2 hours. I read the last sentence ... and I totally forgot what I was supposed to find .. the total time . Okay ... distance equals rate x time, so 1 hour, okay the distance is 2 miles, time is 1, and rate ...* ***(long reflection)*** *... resting distance is -1, equals rate ...1 um ... so* ***(reflection, appears frustrated)*** *...I know I have to set up an equation then ... I could ...* ***(reflection; facial expressions suggest she is puzzled)***

Jessica's comments indicate that even though she had seen others solve the problem before, she still had some difficulty solving the same problem. She continued to reflect upon the situation and then had an idea to do something different to solve the problem:

Jessica: (long reflection, makes motions with her hands) Okay! So she <u>paddles</u> first, then she rests. She goes +2, then -1, she goes +2, -1, she goes +2 and 1, 3, and she goes + 2 again. So that's 1,2,.. .9 hours she makes the trip. That's <u>not</u> how they did it in class.

The interviewer questioned Jessica about her reasoning:

Interviewer: *Ah, so you were thinking back to how they did it ?*

Jessica: *Well this reminded me of that problem. I was trying to do what they did. But when I tried to do it their way, and try to get some equation going, it didn't work. I had to try something else. So just apply logic to it, it's +2, -1, +2, -1, then set up an equation (sic)* <u>*to see if it works*</u>.

In summary, Jessica experienced cognitive tension when her initial strategy of generating an equation did not appear to work. Her explanation indicated both the provisional aspect of her reasoning as well as the belief on her part that her ideas still needed to be verified to "to see if it works". More precisely, she abduced an idea of what the problem might be about and then initiated appropriate solution activity to test her abductive hypothesis.

To further probe her understanding, the interviewer asked Jessica to solve an extension of the canoe problem:

Suppose after 4 hours on the river, Sally took a lunch break for 1 hour, during which time she floated downstream. How long did it take her to go the 6 miles up the river ?

Jessica solved the follow-up task routinely. However, her solution surprised her and she demonstrated abductive reasoning in "making sense" of her solution.

Jessica: *Okay, ... so paddling is +2, resting is -1, so she rests another hour for lunch that's another -1 so first hour is +2, -1, +2, -1, and she did lunch, so that's another -1. So, 1 there ....she rests an hour, and another hour, so those 2 cancel out going back to 1 hour. So we have +2, -1, +2, -1, ... 1, 2, ... 6 ... 11, 12 hours to make trip with lunch break.*

Jessica: *What!? **(She appears surprised by her result; long period of reflection)** Yeah, I guess that 1 hour sets you back. **(several seconds of reflection)***

Interviewer: *What are you thinking ?*

Jessica: *Well, I was going to say that it would have been 10 hours, but I guess ... maybe you have to add a whole 'nother cycle? **(reflection)** Let's see. **(she annotates her diagram)** Yeah, you add +2, -1 to make up for that resting time, that one -1, to put an extra 2 plus -1 in there, cause that just cancels that whole one out there, and gives 3 more than 9 total. So I guess it is 12, yeah! ... Sally's crazy! 12 hours.*

While Jessica's interpretation of the problem posed no difficulty for her in generating a solution, her results clashed with what she had initially expected (i.e., that the one hour of rest would add only one additional hour to her previous solution, making the solution to the follow-up task 9 + 1 = 10 hours). Her expression of surprise was followed by her abduction that the discrepancy (between what she initially expected, a solution 10 hours, and what she actually computed, 12 hours) had to do with the cumulative effect of inserting the one hour rest period in the middle of the schematic she used as a diagram. She commenced to test her hypothesis and verify her hypothesis and confirm some certainty on her solution.

In solving both the Canoe and extension tasks, Jessica's abductions helped her make sense of surprising results. Specifically, in solving the initial task, her abduction helped her make sense of her realization that the way she had seen others solve a similar problem would not work. And upon solving the extension of the canoe problem, her abduction helped her make sense of the surprising fact that inserting a one-hour rest time into the previous task changed the solution by 3 hours (and not a mere 1 hour like she initially expected).

Jessica was asked to solve several algebraic and proportion problems during the second interview, including the following proportion problem:

At a Chinese dinner every 4 guests shared a dish of rice, every 3 guests shared a dish of vegetables, and every 2 guests shared a dish of meat. There were 65 dishes in all. How many guests were there ?

Jessica: *Okay, let's see. **(reflection)** Every dish of rice has 4 guests, every dish of vegetables has 3 guests, every dish of meat has 2 guests. **(draws diagram and reflects on it)** Okay, let's see ... Um ... That doesn't help. **(re-reads)** Okay, so rice plus vegetable plus meat is 65.*

After reflecting on the situation, she commented:

Jessica: *I have no idea what to do, how to start this.* **(reflection)** *I guess my problem is that I feel like I need another relation between rice, vegetables, and meat. I'm trying, in my own version to set-up a system of equations, then I could get something out of it. But with 3 of them* **(points to diagram of the 3 dishes)**, *I'm not sure where I'm going ... because I don't know ...*

Interviewer: *We can come back to this one later if you like ?*

Jessica: *Wait.* **(more reflection)** *! Let's see. If I have 12 guests that means ... I have 3 dishes of rice, 4 vegetables, and 6 meat dishes, ... Ah!* **(smile appears on her face)**

Interviewer: *What are you thinking ?*

Jessica: *Um ... I'm trying to find a nice even number of guests, so that I can figure out exactly how exactly how many dishes of rice, vegetables, and meat they need. I'm going backwards from the way I thought ... I thought I needed to start with the number of dishes. But if I have 12 guests, then I have* **(Points to her diagram and counts)** *3, 4, 5, 6, ..., 12, 13 dishes. So if there's 13 dishes, ... then that means 13 times 5 is 65 dishes. So I multiply guests by 5, that would be 60 guests. So I have 65 dishes ...* **(long reflection)** *... that doesn't seem right.*

Jessica: *Let's see. 13 went into 65 5 times, if I just multiply that by 5, I get 65. So, I should just be able to multiply that* **(points to number guests on her diagram, 12)** *by 5 to get 60 guests.*

Jessica: *Yeah! .. But it seems like you should have more guests than dishes ....* **(reflects, appears to "run through" her reasoning)** *I guess not!*

As was the case in solving the canoe problem, Jessica generated abductions to both solve the problem as originally stated and, upon having constructed a solution, to make sense of her solution.

Discussion

Table 1 summarizes the researcher's inferences about Jessica's problem posing and solving for the Canoe task.

Table 1: Jessica's problem posing and solving

Task	Goals and Purposes	Result of action
Canoe	· construct equations of form d = rt (no expectation of problem)	surprised when she cannot construct viable equation (she has a problem!)
	· H_1: the problem might involve linear displacement	uses linear iteration scheme to compute distance traveled (hypothesis testing -- problem solved)
Follow-up	· integrate new information into into linear scheme (no expectation of problem)	confident of her reasoning but surprised with result (she has a problem!)
	· H_2: the problem might involve ways incremental lengths were computed	re-constructs prior solution activity incorporating new information (hypothesis testing -- problem solved)

Table 1 characterizes Jessica's solution activity in terms of a series of episodes that involved problem posing, problem re-formulation, hypothesis generation, and hypothesis testing. Jessica experienced problems whenever her expectations of how she would proceed clashed with her actual computed results. For example, when her initial goal of generating a viable equation to solve the Canoe task was not achieved, she re-formulated, or *re-posed* the problem, whereby she transformed the original problem situation into one that involved linear displacements. With this re-formulation of the problem Jessica also expressed a hypothesis, $\mathbf{H_1}$ (of what the problem might be about), which was followed by her intention to explore the implications of adopting her hypothesis ("to see if it works"). The results of her 'testing' served as feedback to her hypothesis, thereby contributing to her subsequent solution.

Jessica's solution activity to complete the Canoe task was interpreted as involving *within-solution problem posing*, whereupon she re-formulated her goals and purposes in the course of her on-going activity, transforming the situation into a new problem for her solve. This result is compatible with previous research on problem posing (English, 1997; Silver, 1994; Silver and Cai, 1996). However, in solving the follow-up task, Jessica demonstrated a form of novel problem posing that has not been addressed in the problem solving research literature. Specifically, in completing the follow-up task, Jessica computed a result that was a surprise to her. In particular, when she incorporated the new information (that Sally rests for a one-hour period after 4 hours on the river) into her linear iteration scheme, she computed a result of 12 hours, which clashed with what she initially expected (a solution of 10 hours). She then hypothesized the possible reason for this apparent discrepancy ($\mathbf{H_2}$), in the process formulating a novel problem for her explore. She commenced to test her hypothesis by re-exploring her prior solution, focusing on the implications of incorporating the new information into her linear iteration scheme. This scrutinizing and evaluating of her prior solution in view of the new information was interpreted as a case of Jessica achieving a heightened awareness concerning the efficacy of her prior solution activity. This type of problem posing, which lead Jessica to re-examine her prior solution in the face of a new problem situation, suggests that problems are never <u>completely</u> solved; new situations present opportunities for solvers to connect directly with previous problem solving through the generation of and consideration of new questions and problems.[2]

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The Roles of Cognitive Units, Connections and Procedures in achieving Goals in College Algebra

Lillie Crowley

Lexington Community College
Kentucky, USA
e-mail: Lillie@pop.uky.edu

David Tall

Institute of Education
University of Warwick, UK
David.Tall@warwick.ac.uk

The purpose of this paper is to develop a means to illustrate and analyse the cognitive paths taken by students in solving problems. The approach is built upon the notion of cognitive unit (small enough to be consciously manipulated). Our interest is in the nature of the student's cognitive units and the connections between them. We find that a student may have an overall strategy and even formulate goals to achieve all or part of a solution. However, if conceptual structures are too diffuse, the student may concentrate on procedures that occupy most of the focus of attention. This may cause them to lose touch with the ultimate goal and be faced with sequences of activity that are longer, more detailed, and more likely to break down.

Introduction

Why is it that some students find algebra so essentially simple, yet others struggle so badly that they fail in school and need to take remedial algebra courses in college? The literature abounds in distinctions between the conceptual thinking of some students and the procedural thinking of others (e.g. Hiebert & Lefevre, 1986). But *why* does this occur? What is the nature of procedural thinking that makes it the default position for so many? Hiebert and Carpenter (1992) suggest two metaphors for cognitive structures, as *vertical hierarchies* or as *webs:*

> We believe it is useful to think about the networks in terms of two metaphors … structured like vertical hierarchies or … like webs. When networks are structured like hierarchies, some representations subsume other representations, representations fit as details underneath or within more general representations. Generalisations are examples of overarching or umbrella representations, whereas special cases are examples of details. In the second metaphor a network may be structured like a spider's web. The junctures, or nodes, can be thought of as the pieces or represented information, and the threads between them as the connections or relationships. Hiebert & Carpenter (1992, p. 67)

Such ideas have long been part of mathematics education. However, they are often used as general philosophical structures rather than explicit techniques to analyse empirical evidence. Our plan here is to extend these ideas and use the extended theory to analyse the specific solution processes for specific individuals in specific contexts. Here we focus on the activities of students working in college algebra.

Varifocal webs and cognitive units

Skemp (1979) proposed a "varifocal learning theory" in which the nodes of webs are themselves subtly connected schemas when viewed in detail. With this in mind, webs

and hierarchies may occur within the same model. As an example, consider the equation "$y=mx+b$". As a concept it can be viewed in more detail with a network of internal ideas: that m is the slope, b the intercept; that any linear equation can be represented by substituting numbers for the parameters m and b; that the graph can be drawn if one knows two points on it, or one point and the slope, etc. Some students therefore may see "$y=mx+b$" as a *single structure* with rich connections easily brought to the focus of attention.

Barnard & Tall (1997) introduced the notion of "cognitive unit" as "a piece of cognitive structure that can be held in the focus of attention all at one time". We see cognitive units as forming the nodes of a cognitive structure linked to other units using the web metaphor of Hiebert and Carpenter, incorporating the varifocal element of Skemp. There is a great deal of flexibility as to how the units and their connections may be laid out in a diagram. The notion of whether a link is "internal" within a unit, or "external" between units is largely a matter of personal choice. The actual connections within the brain are not topologically divided into an inside and an outside.

However, there are situations in which the idea of "inside" and "outside" can be helpful as a metaphor to represent the different strengths of connections, as we now consider. For instance, any of the following:

- the equation $y = 3x+5$,
- the equation $3x-y = -5$,
- the equation $y-8 = 3(x-1)$,
- the graph of $y = 3x+5$ as a line,
- the line through (0,5) with slope 3,
- the line through the points (1,8), (0,5),

may be considered as cognitive units which can be linked together as representing the same underlying concept—the single straight line or equivalent linear relation between x and y. This may be represented diagrammatically as six separate nodes with appropriate connections between each. In this sense the connections are *external* to the six cognitive units. However, an alternative, more powerful, view is to consider all six ideas to be various aspects of the same phenomenon, the linear relation/equation or straight line which all of them represent. This allows the separate ideas to be seen as different aspects of a *single entity* that is itself a single node in a larger network.

The move from conceiving of separate ideas to a single idea with different aspects is called "conceptual compression" (Thurston, 1990, Gray & Tall, 1994). For conceptual compression to occur, the individual's cognitive structure must have matured in such a way that the separate elements have an intimate connection enabling the individual to move flexibly from one to another. It is not just that there is a cognitive link between, say, the line through (0,5) with slope 3 and the line with equation $y = 3x+5$, but that both describe exactly the same thing—they are different aspects of the same entity.

In terms of Skemp's varifocal theory, this entity is itself a concept which has internal links as a schema in its own right. What is important to be able to compress a collection of related ideas into a cognitive unit is that the whole entity can be conceived as a unit that is "small enough" to be considered consciously, all at one time. The way that the human mind usually copes with this is to give it a *name* or *symbol*. The name or symbol (assuming it is "small enough") can be held in the focus of attention and manipulated. Such a concept has rich interiority through carrying "within" it many powerful links that enable it to be manipulated and invoked to solve problems.

If the diverse elements are not connected sufficiently fluently, then it may be impossible for the individual to regard the totality as a cognitive unit. It follows that it may be impossible for the individual to make links to it, simply because there is no "it". Any links that are made by such an individual are not made to a flexible conceptual entity but to one element in a loosely connected structure. We conjecture that it is this situation that underlies the often-heard cries of the remedial student saying "don't explain it to me, just tell me how to do it." An explanation—which may be perfectly clear to the teacher with a rich personal cognitive structure—is not perceived as an "explanation" to the student hearing words which do not link to adequate cognitive units in the student's mind.

Focus of attention and working memory

The way in which the human brain works enables certain ways of thinking and constrains others. Crick (1994) views the brain as a complex, multi-processing system which can be used coherently only if much of its activity is suppressed at any given time to focus consciously on a small number of important ideas (cognitive units). These in turn are linked to others that can be brought into focus as appropriate. This idea was expressed succinctly over a century ago:

> There seems to be a presence-chamber in my mind where full consciousness holds court, and where two or three ideas are at the same time in audience, and an ante-chamber full of more or less allied ideas, which is situated just beyond the full ken of consciousness. Out of this ante-chamber the ideas most nearly allied to those in the presence chamber appear to be summoned in a mechanically logical way, and to have their turn of audience.
>
> (Galton, *Inquiries into human faculty and its development*, 1883)

The "presence-chamber" of Galton is the current focus of attention and its "ante-chamber" extends it to the *working memory* consisting of closely linked cognitive units that can be evoked for problem-solving. However, it is important not to allow the physical metaphor of a "chamber" to suggest a single fixed area of activity in the brain. The "focus of attention" may be spread over many disparate areas currently resonating together in conscious thought. It therefore remains susceptible to other activities that can interrupt and override the current thought process. Such interruptions may result from unrelated external sensations, such as hearing a school bell ring to end the mathematics class, or more intimately linked strategic activities, such as a mental process monitoring whether a longer-term goal is being achieved.

Skemp (1979) theorizes that a specific problem-solving context provides a goal to be achieved, in which sub-goals may be formulated to achieve parts of the solution process. He hypothesizes that a *comparator* activity occurs at various times which considers whether the solution process is getting suitably close to the goal or to one of the intervening sub-goals. When following a routine sequence of actions we conjecture that the focus on successive remembered steps may be so great as to temporarily fill the focus of attention and suspend the activity of any comparator. This would suggest that the inflexibility of procedural thinking can become so dominant as to cause the individual to lose sight of the goal and so fail to solve the problem. Skemp also suggests the dual idea of an "anti-goal", something to be avoided—such as the anti-goal of avoiding failure—bringing with it a sense of anxiety that may negatively affect creative activity.

We therefore hypothesise that the difficulties encountered by remedial students relate to the nature of their ideas: that powerful concepts—which others can compress into manipulable cognitive units—remain, for them, as more cumbersome structures too diffuse to employ in a novel context. Our empirical evidence reveals that remedial students may have goals to achieve, indeed may articulate sub-goals, but the dominant procedures they use to attempt to achieve these goals seem to take up so much conscious thought as to prevent them from making necessary cognitive links to complete the exercise. While the successful mathematical thinker may have flexible cognitive units with powerful internal relationships which allow them to be used in diverse productive ways, the less successful may therefore be faced with longer procedural routes which actually make the mathematics *harder*. In other words, the *weaker* students are following longer more detailed cognitive paths that cause *greater* cognitive stress and further increase the chance of failure.

An example

As an example consider the following problem from a college algebra course:

> Find the x-intercept and y-intercept of the graph with equation $3x+4y=12$.

For students with a sense of the symmetry between the occurrences of x *and* y in this equation, it may be possible to "see" the answers in the equation itself. For instance, to obtain the y-intercept, imagine the "$3x$" part to be zero and focus on $4y=12$ to *see* the solution $12/4=3$ (Figure 1). A similar route for the other intercept gives a compressed solution of the problem as two immediate links without any need to write down intermediate steps. However, students who do not see this instant solution may resort to formulating sub-goals using lengthier procedures.

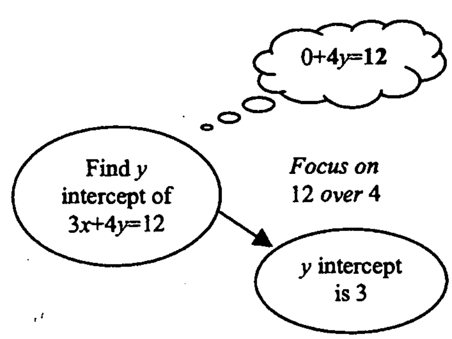

Figure 1: compressed solution

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Kristi

Kristi is a community college student taking a remedial Intermediate Algebra course using a graphing calculator to produce tables and graphs. She needs to pass it before she can attempt the college mathematics courses required for her degree in psychology. She had met the concept of a straight-line equation in its various forms before the course and when interviewed afterwards she was able to discuss problems dealing with lines, their equations, slopes, graphs, etc. However, she had a strong focus on the equation in the form "$y=mx+b$", not least because she had been taught to use it to type into her graphing calculator to draw a graph. She could also read off the slope as the number before the x, and the y-intercept as the number at the end. So when asked for the slope of $y = 3x+5$ she could see this as 3, and the y-intercept as 5. For her, this standard form was the starting point for many solutions to problems, and she was frequently successful using it. She therefore began to use the sub-goal of "putting the equation into the form $y=mx+b$" before attempting the question under consideration, whether or not this was appropriate.

Her second major strategy stemmed from the first. If the standard form is known, it can be typed and the graph drawn on a graphing calculator. Kristi frequently used a graph–either a mental one, a graph on a piece of paper, or one on a calculator screen.

> If I were to just look at it, to **visualize** it in my mind—it's a line ...

The interviewer said, "what's the y-intercept on the graph?" Kristi responded

> that's where the . . . it intercepts the y- ... I know it's just a line, so I know it's going to have to cross up here somewhere.

She had a piece of paper with axes drawn on it and pointed to a spot on the y-axis of the grid on the paper, above the origin. Kristi tried to visualize it—she had a mental graph—but seemed unable to use it to solve the problem at this point. The interviewer said "Can you graph it?" and she replied:

> Yes, if I have my graphing calculator ...

She has had success graphing with her graphing calculator, and was comfortable with it. Without it, however, she could still have some success . . .

> it's like . . . I need a point. ... zero? [she seems to seek support, but then proceeds on her own] . . . if x is zero, then . . . okay, x is zero. Zero, five. Okay.

She plotted the point (0,5). Implicitly she had found the y-intercept she was seeking, *but she failed to recognise it*. Either her comparator is failing to operate or she does not (at this moment) link the point she has found to her ultimate goal, the y-intercept. She continued in her strategy to produce a line by evaluating a second point. She let x be 1, and wrote the point (1, 4). She plotted the points, drew the line through them, and decided that the y-intercept was 5.

The interviewer then asked her to find the y-intercept for $2y + x = -6$. Using her "general strategy", Kristi began to put it into slope-intercept form, "move the x over", "divide by 2". When asked to do it without putting it into slope-intercept form, she said

> I don't know what to do . . . I can't visualize it in my mind . . . like, if I get back the value, I don't know what to do unless I divide everything by 2. So far, that's what I know to do . . . put it into slope intercept.

Asking her to do the problem without putting it into slope-intercept form severed her links with her coping strategies. She attempted once more to graph the equation by plotting points.

Later in the session the researcher asked her to find both the x- and y-intercepts of $3y + x - 12 = 0$. When asked, "What would you do here?" she replied:

> Divide everything by 3. In my mind I'm visually moving everything, and dividing x by 3, its ... one third x plus ... , so the y-intercept is 4.

Once again she put the equation into slope-intercept form to find the y-intercept. Had she had the conceptual link to do so, it would have been much simpler to set x to zero to find the y-intercept. She was then asked, "What are you trying to do? What do you graph?" and she immediately plotted the point (0,4). When she was then asked how to find the x-intercept, she replied:

> on the calculator screen, where x is . . . if y is what, then hit intersect and try to find where the x is.

Her general strategy of attack is represented diagrammatically in figure 2.

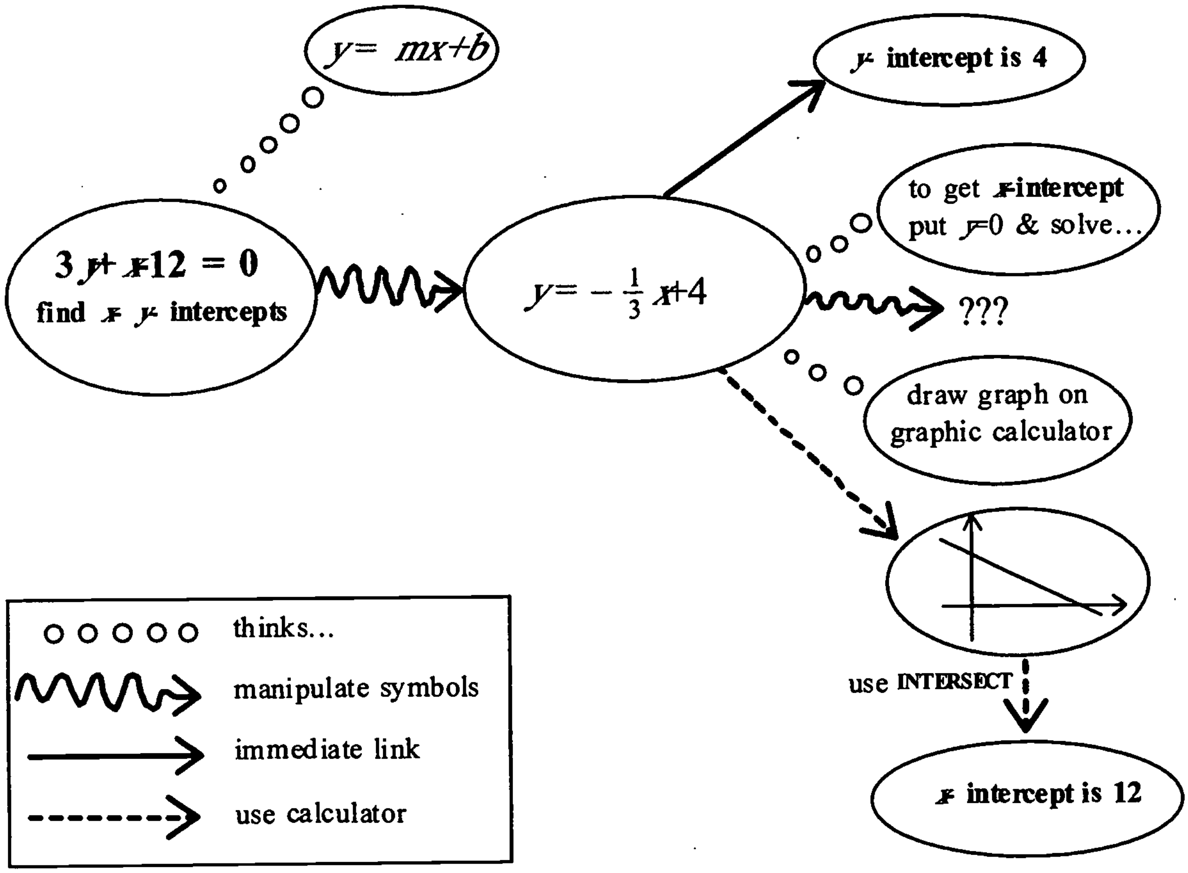

Figure 2: Kristi's strategies for finding x- and y- intercepts of $3y+x-12=0$.

This interview shows the complications that can appear when the student uses perfectly legitimate procedures to solve a problem. In this case, a compressed solution to find the x and y intercepts need involve only two very short computations in a symmetrical manner. However, the student's experience of the graph as a function provides an *asymmetric* relationship in which the roles of x and y (as input and output) are radically different and in which the methods of finding the corresponding intercepts are radically different. Kirsti thinks about the sub-goal of putting the equation into her favored slope-intercept form, itself a procedure requiring effort. From this the y-intercept is easily read off but the x-intercept requires a second lengthy procedure. The structures of the compressed solution and the more lengthy procedure are represented in figure 3.

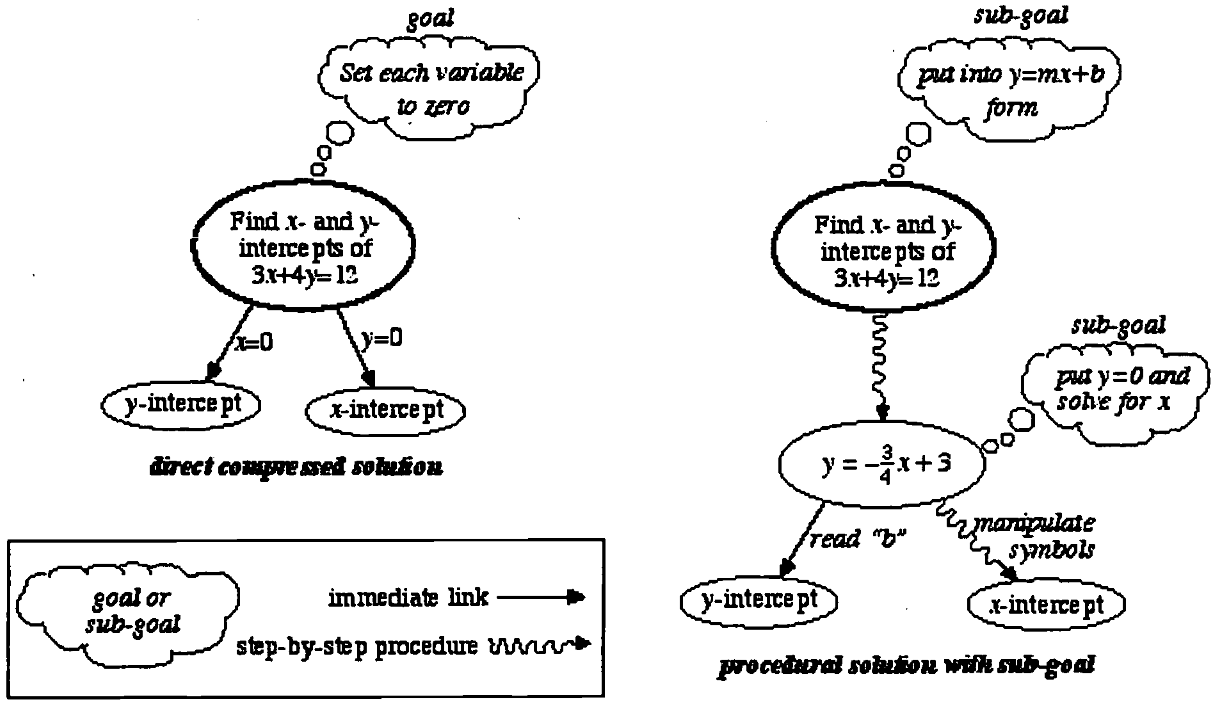

Figure 3 : compressed and procedural solutions for finding intercepts for a specific equation

This use of familiar uncompressed processes with sub-goals occurred repeatedly in Kristi's work. For example, she was asked to write the equation of the line through (1,4) and (4,–2), which she did successfully. She was then asked whether the three points (1,4), (4,–2), and (5,2) were on the same line. Rather than check (as the interviewer expected) whether the third point satisfied the equation of the line she had just found, she calculated the line through (1,4) and (5,2) and compared it with the one she had, saying:

> The way I know how to do it is to take the slope that I got, and get the line through these two points, and see if they are the same. That's the only way I know how to do it.

She used the idea of a line through two points again, repeating a familiar activity that had just been successful. However, she did not exhibit the flexibility that she needed to cope with different problems in new contexts.

The inflexible use of procedures occurred in many other students. Sometimes they were even more diffuse and error-prone than those attempted by Kristi. Kim, for instance, solved the equation $3y+x-12=0$ to obtain:

$$y=-\tfrac{1}{3}x+\tfrac{12}{3}.$$

For this student the equation was doubly difficult; it involved not only fractions, but also negative numbers. We can hypothesise that the notions of fractions and negatives have not become cognitive units that can be used fluently. Kim therefore compounds (at least) two levels of difficulty. First there are the uncompressed, inflexible procedures that are onerous to handle. Within these are uncompressed conceptual structures for negatives and fractions that render the difficulties even more burdensome.

Summary and reflections

In this paper we have highlighted the difference between the use of flexible cognitive units on the one hand and more diffuse uncompressed structures on the other. We give evidence that a student who has yet to compress external relationships between concepts into tight cognitive units with strong internal links will find it more difficult to cope with problems requiring their use. The case studied here showed that a simple problem of finding intercepts of a linear equation contains subtleties easily handled by a student with a compressed cognitive unit encompassing the properties of algebra and the graph of a linear equation. The student with a more diffuse cognitive structure is at a serious disadvantage; this places a strain on the focus of attention at this stage and may prevent powerful theory building for the future. In this way there develops a spectrum of performance in which those who are struggling use even more complicated solution processes that place them in greater danger of failure.

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MEASURING STUDENTS' PROVING ABILITY BY MEANS OF HAREL AND SOWDER'S PROOF-CATEGORIZATION

Csaba A. Csíkos

Department of Education

Attila József University, Szeged, Hungary

In this paper proving ability - according to the Lakatosian sense of proof - refers to the ability to make something evident. A test of proving ability was administered to 2572 students in Hungary. A dichotomous evaluation system allows for both hierarchical and non-hierarchical evaluation of proving ability. An hierarchical evaluation can be based on Harel and Sowder's proof-categorization. The results show that there is a positive correlation between proof types of different content domains (using the above-mentioned hierarchical model), suggesting that Harel and Sowder' taxonomy can be a powerful tool for measuring proving ability.

AIMS

Proofs are used in mathematics, in philosophy, in jurisprudence, and in everyday life. This study addresses the evaluation of proving ability. In this paper the term 'proof' will be used in the Lakatosian sense of 'making something evident' (Lakatos, 1976), and proving ability will refer to the ability to construct proofs.

There can be different approaches to characterizing proving ability. Proving ability processes can be defined as a combination of 'simplier' human abilities. On the other hand they can be a metacognitive-metadeductive ability. For the purpose of this study, proving ability will not be thought of as either a combination of abilities appearing in different well-known taxonomies (see Johnson-Laird and Byrne, 1991; Carroll, 1993), or as the results of 'meta-' processes (Johnson-Laird and Byrne (1991). Instead, it is the structure of proofs that is used to characterize proving ability.

The structure of proofs can be characterized by three variables: 1) the statement to be proven, 2) the axioms and other (formerly proven) statements used in the proving process, and 3) inference rules used in the proving process. With regards to the first variable, 'making something evident' may require similar reasoning processes regardless of the great variety in the content of the statement to be proven. Similarly, there is great diversity among proofs with respect to the number and type of axioms and other (formerly proven) statements which are used in the proving process.

Finally, the third source of variability in the structure of proofs is the use of deductive versus inductive inference rules. Models of human reasoning often use deductive inference rules to describe reasoning processes (Braine, 1978, 1990; Rips, 1983, 1994). However, even in mathematics it is not only permitted but suggested to let students „have the joy of discovery" (Saul, 1992, p. 11.; see also Pólya, 1954), that is, to encourage inductive reasoning. Thurstone (1995) went so far as to criticize the use of the DTP-model (definition→theorem→proofs) because it does not allow for inductive inferences. Also Mariotti, Bartolini Bussi, Boero, Ferri & Garuti (1997) emphasized the importance of 'dynamic exploration' in learning mathematical proofs. The three variables described above indicates a wide spectrum of proofs from axiomatic mathematical proofs to everyday verifying arguments. Even within mathematics there is a disagreement on what criteria a proof must meet. Hanna (1995, 1996), Martin and Harel (1989), Hersh (1993) and others emphasized the importance of distinguishing formal and informal proofs in mathematics.

The basic assumption of this study is that, regardless of the content of the statement to be proven, different proofs constructed by the same person will be similar with respect to the second and third variables of the structure. It is also hypothesized that students can use the reasoning skills necessary for constructing mathematical proofs in other contexts as well; the better mathematical proofs a student can construct, the better he/she can construct everyday proofs.

There are at least three approaches to the evaluation of proving ability. One means of evaluation is to start out from the so-called 'paradigm task' approach (Girotto & Light, 1993). This approach uses well-defined experimental tasks to study the nature and development of human reasoning, e. g., the Wason selection task, or logical puzzles (Johnson-Laird & Byrne, 1991; Rips, 1989). Another evaluation approach is the above-mentioned inference rule approach which concentrates on universal mental inference rules. In this study a third approach is used: Harel and Sowder (1998) proposed a model for classifying mathematics proofs that can be considered to be a combination of the approaches referred to above. There are other holistic evaluation methods for mathematics proofs (see, for example, Thompson & Senk, 1993), however, Harel and Sowder's proof categorization seems to be the most widely generalizable for non-mathematical contents.

The evaluation of proving ability calls forth special problems and difficulties. All of these approaches presume some kind of hierarchy of cognitive abilities: certain inference rules are more difficult to use, certain patterns of solutions are more advanced than others. However, in mathematics, as in other fields, there can be more than one proof of a certain theorem, and these proofs cannot be easily ranked in a hierarchy of difficulty. Hoyles (1997) pointed out that any hierarchy in evaluating proving ability can be a methodological artifact.

The aim of this study is to investigate the relationships between proof types of different contents, mathematics achievement, and school marks. It is hypothesized that there is a correlation between proving ability (using Harel and Sowder's

hierarchical classification), mathematics achievement, and school marks. The existence of a positive correlation between proof types of different contents may support the basic assumption that Harel and Sowder's proof categorization for mathematical proofs is a powerful means for measuring proving ability.

METHODOLOGY

Within a larger investigation called 'Development of mathematical abilities' six tests were administered to 2572 students, in 3 counties in Hungary, between April and May, 1998. The sample consisted of children of the 5th, 7th, 9th, and 11th grades (with ages ranging from 11 to 17 years). There were two additional questionnaries assessing personal data, school marks, and mathematics and physics academic self-concept. The tests were developed for this study, and two of them were previously piloted.

Table 1. *Arrangement of the „Development of mathematical abilities" investigation*

Test	Grade			
	5th	7th	9th	11th
Personal data (including school marks, sex etc.)	✓	✓	✓	✓
Mathematics and physics academic self-concept		✓	✓	✓
Creativity	✓	✓	✓	✓
Mathematical problem solving	✓	✓	✓	✓
Proving ability	✓	✓	✓	✓
Mathematics word problems			✓	✓
Physics achievement		✓	✓	
Mathematics achievement		✓	✓	

Among the tests of abilities, one measured proving ability. It consisted of six tasks of various types. One version of the test was administered to 5th and 7th grade students, and a second version to 9th and 11th graders, however, there were tasks common to both versions enabling better comparisons between groups.

Example I (task for 5th and 7th graders): „How can you prove that 6332 is not divisible by 3?"

(task for 9th and 11th graders): „Prove that 2 is the only prime number which is an even number!"

Example II (task for all grades): „For a long time people did not know that the Earth is round. How can you prove this for a person who does not believe this?"

Example III (task for all grades): „In an imaginery town there are people of three types: The truthful people always tell the truth, the liars always lie, and the mixed people alternately tell true and false sentences. One night somebody phones the fire-department:

- *Hello, fire department.*
- *The city hall is burning.*
- *Are you truthful?*
- *I am mixed.*

Is the city hall burning? Give your reason!

A dichotomous categorization system has been developed for each task, by which both hierarchical and non-hierarchical evaluations of students' proofs can be performed (see Figure 1).

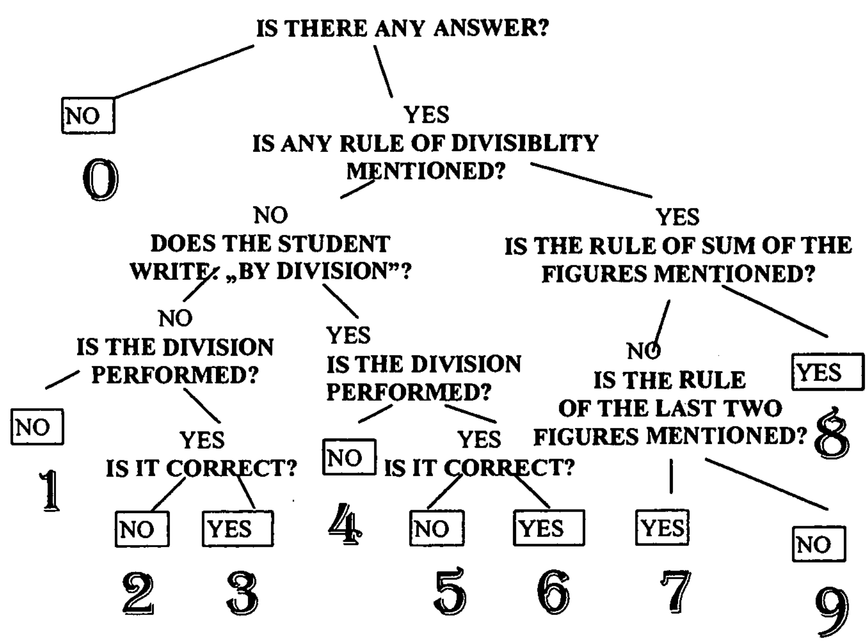

Figure 1. *Dichotomous evaluation of proof types for the task „How can you prove that 6332 is not divisible by 3?"*

The nominal categories of this dichotomous system can serve as a basis for an hierarchical evaluation of proof types: An ordinal scale measure can be developed

based on Harel and Sowder's proof-categorization system. Three hierarchically ordered stages can be identified in their model: 1) externally-based proofs, 2) empirical proofs, and 3) analytic proofs. The categories of our dichotomous system were transformed into ordinal scale categories on the basis of agreement among experts. Three expert raters independently recoded the nominal categories using an ordinal scale derived from Harel and Sowder's taxonomy. Kendall's coefficient of concordance (W=.909, p<.001) indicates a high level of agreement. In each case it was possible to recode the nominal categories into ordinal scale in 3:0 or 2:1 rate.

Table 2. *Result of the recoding process for the task „How can you prove that 6332 is not divisible by 3?*

nominal category	ordinal category (0=no response, 1=external, 2=empirical, 3=analytic)
0	**0**
1	**1**
2	**1**
3	**2**
4	**2**
5	**2**
6	**2**
7	**1**
8	**3**
9	**1**

Additional data collection is in progress investigating mathematics teachers' judgements of certain frequent proof-patterns. The ordinal scale measure constructed on the basis of teachers' judgements will provide another method for the hierarchical evaluation of proving ability.

RESULTS

Currently available data suggest that there are large within- and between-age-group differences in response-patterns: from external to analytical (using Harel and Sowder's taxonomy), and from social rule-based to logical rule-based (using a non-hierarchical taxonomy).

Kruskall-Wallis and Jonckheere-Terpstra analyses were computed on the ordinal scale measure of proof types, where the grouping variable was the grade level.

Table 3. *Results of Kruskall-Wallis and Jonckheere-Terpstra analyses of proof types measured by Harel and Sowder's taxonomy*

Task		Kruskall-Wallis analysis		Jonckheere-Terpstra analysis	
	N	χ^2	p	std. J-T	p
Round-shaped Earth	1178	7.39	.060	2.07	.038
Burning city-hall	2327	68.43	.000	8.01	.000

The results suggest that there is a tendency to construct higher-order proofs even in non-mathematical domains as a fuction of school grades.

Spearman correlation coefficients were computed to investigate relationships between proof types of different domains, and school marks in mathematics. Table 4 shows that results on the *round-shaped Earth* and *burning city-hall* tasks are significantly correlated with school marks in mathematics ($p<.001$ in each case)

Table 4. *Spearman correlation coefficients between results on two tasks, and school marks in mathematics*

	Round-shaped Earth	*Burning city-hall*
Burning city-hall	.176 (N=1178)	
Marks in mathematics	.149 (N=1120)	.242 (N=2225)

Note: p values for all correlations are less then 0.001.

Additional analyses (including latent trait analyses) will be conducted to investigate relationships between proving ability and background variables (e. g., mathematics achievement, mathematics self-concept, school marks, sex, etc.). Further non-parametric analyses will be conducted as data collection is completed.

THEORETICAL AND EDUCATIONAL IMPLICATIONS

The results presented here support the use of Harel and Sowder's proof-categorization as an effective measure of proving ability. By means of this taxonomy proofs from various content domains can be evaluated. The developmental curves of proving ability drawn from this cross-sectional study have different inflection points which appear to be determined by the content of the proof, e. g., whether the proof is a mathematical, a scientific, or an everyday proof. Further studies should address the hypothesis that mathematics proofs are the 'leaven' to foster the development of the *general* proving ability.

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WHAT CAUSES IMPROPER PROPORTIONAL REASONING: THE PROBLEM OR THE PROBLEM FORMULATION?

Dirk De Bock* **, Lieven Verschaffel*, Dirk Janssens* and Rebecca Rommelaere*
University of Leuven* and EHSAL, Brussels**; Belgium

Because of its wide applicability – both in everyday situations and in scientific contexts – proportional reasoning is a major topic in mathematics education. But according to several researchers and educators, the attention given to the proportional (or linear) model may have a serious drawback: it may develop in pupils a tendency to use the linear model also in situations in which it is not applicable. In three related studies by De Bock, Verschaffel & Janssens (1998a, 1998b) this "illusion of linearity" was empirically investigated among 12–16-year old pupils working on non-linear scaling problems. But what caused these pupils' improper use of linearity? Did they incorrectly believe that the linear model was appropriate or were they simply misled by a problem formulation which they associated with the proportional scheme? In this study we found a significant influence of this formulation factor: pupils confronted with non-linear scaling problems resisted more easily the trap of proportional reasoning when these problems were formulated as unfamiliar "comparison problems" than when formulated as traditional "missing-value problems", which they have learned to associate with proportional reasoning throughout their school career.

THEORETICAL AND EMPIRICAL BACKGROUND

Pupils' tendency to apply proportional reasoning in non-proportional problem situations has been frequently described and illustrated in the literature on mathematics education. Examples of this "illusion of proportionality" relate to different domains of mathematics, such as algebra (Berté, 1993), probability (Fischbein & Schnarch, 1996; Freudenthal, 1973) and geometry (De Blocq-Docq, 1992). Best-known is the misuse of proportionality in scaling problems (Feys, 1995; Freudenthal, 1983; Rouche, 1989). In order to determine the area or volume of an enlarged (or reduced) geometrical figure with the same shape, it appears that pupils frequently use the linear scale factor instead of its square or cube. In the American *Standards*, for instance, we read in this context that "... most students in grades 5–8 incorrectly believe that if the sides of a figure are doubled to produce a similar figure, the area and volume also will be doubled" (NCTM, 1989, pp. 114–115).

Recently, this tendency towards inappropriate proportional reasoning as well as its resistance to change were empirically investigated in three related studies by De

Bock, Verschaffel & Janssens (1998a, 1998b). Large groups of 12–13- and 15–16-year old pupils were administered the same set of proportional and non-proportional items about length and area of similar plane figures under different experimental conditions. The experimental items were constructed around different types of plane figures (regular figures such as squares and circles, and irregular figures) and formulated as traditional word problems. From a structural point of view, all items were "missing-value problems" in which three of the four data were given and the task was to find the missing one. Table 1 lists an example of a proportional item (item 1) and a non-proportional item (item 2) – both dealing with square figures – used in these studies.

1. Farmer Gus needs 4 days to dig a ditch around his square pasture with a side of 100 m. How many days would he approximately need to dig a ditch around a square pasture with a side of 300 m? *(Answer: 12 days)* 2. Farmer Carl needs 8 hours to manure a square piece of land with a side of 200 m. How many hours would he approximately need to manure a square piece of land with a side of 600 m? *(Answer: 72 hours)*

Table 1. *Two examples of missing-value problems (De Bock et al., 1998a)*

The major results of these studies can be summarized as follows. First, the tendency to apply the linear model in the solution of non-linear scaling problems proved to be extremely strong in the age-group of 12–13-year olds, and was still very influential among 15–16-year olds: overall percentages of correct responses on the non-proportional items varied between 2% and 7% in the group of 12–13-year olds and between 17% and 22% in the group of 15–16-year olds. Second, the type of figure involved played a significant role: pupils performed better on the non-proportional items when the figure involved was regular (a square or a circle), but as a drawback they performed worse on the proportional items about these regular figures because they sometimes started to apply non-proportional reasoning on the proportional items too. Third, the provision of adequate visual as well as metacognitive support, respectively in the form of given drawings made on squared paper and an introductory task that forced the pupils to read and solve one representative non-linear item in a mindful way, yielded a significant, but unexpectedly small effect on pupils' performance on

the non-proportional items and, once again, this positive effect was compensated by worse performances on the proportional items in the supported conditions.

While these studies revealed pupils' almost irresistible tendency to apply proportional reasoning in problem situations for which it was totally inappropriate, the question remains *why* so many pupils fell into this "proportionality trap", even in the case of visual and/or metacognitive support. Most likely, there is no unique nor uniform explanation for this phenomenon. As in many other subdomains of mathematical thinking and problem solving, it seems that pupils' errors were the result of the interaction between several task and subject variables. The present study focuses on the role of one particular task variable on pupils' tendency towards unbridled proportional reasoning – namely the problem formulation – in interaction with some of the subject and task variables from our previous studies. As already mentioned, in these previous studies all proportional and non-proportional items were presented as *missing-value problems* (see, e.g., Reiss, Behr, Lesh & Post, 1985; Tourniaire & Pulos, 1985). In this problem type, three numbers (a, b and c) are given and the problem solver is asked to determine an unknown number x. In a proportional missing-value problem (such as item 1 in Table 1), the unknown x is the solution of an equation of the form $a/b = c/x$. Arguably, non-proportional tasks of the missing-value problem type (such as item 2 in Table 1) are rather unusual for most pupils; the vast majority of the missing-value problems they encountered in the upper grades of the elementary school and the lower grades of secondary school, are problems for which the linear model suits perfectly. Therefore, it could be argued that pupils' extremely weak results on the non-proportional items may not be due to intrinsic difficulties with the mathematical concept involved in these problems – namely understanding the effect of a linear enlargement on area – but merely the result of a misleading problem formulation (namely the missing-value type which calls up the overlearned solution schemes and procedures of proportional reasoning). In order to find out to what extent pupils' weak performances on the non-proportional items can be due to this formulation issue, we set up a new study in which the formulation of the problems was experimentally manipulated while keeping their intrinsic conceptual difficulties constant.

METHOD

Hundred-and-sixty-four 12–13-year old pupils and hundred-and-fifty-one 15–16-year old pupils participated in the study. All pupils were administered the same paper-and-pencil test consisting of 12 experimental items (4 proportional items and 8 non-

proportional items) about the relationships among the lengths, areas and volumes of similar figures of different kinds of shapes, and 3 buffer items. In both age-groups we worked with two equivalent subgroups of pupils that were matched on an individual basis and that were given a different version of this test. In the first subgroup (the MV-group), all 12 experimental and all 3 buffer items were presented as missing-value problems, while in the second subgroup (the CP-group), the items were formulated as comparison problems. Table 2 contains the "comparison problem" version of the two missing-value problems from Table 1.

1. Farmer Gus dug a ditch around his square pasture. Next month, he has to dig a ditch around a square pasture with a side being three times as big. How much more time would he approximately need to dig this ditch? *(Answer: three times more)*
2. Farmer Carl manured a square piece of land. Tomorrow, he has to manure a square piece of land with a side being three times as big. How much more time would he approximately need to manure this piece of land? *(Answer: nine times more)*

Table 2. Two examples of comparison problems

HYPOTHESES

First, in line with the results of De Bock et al. (1998a, 1998b), we hypothesized that the vast majority of the pupils would suffer from the "illusion of proportionality", and that they would therefore apply proportional reasoning to solve not only the proportional items but also most of the non-proportional items. Consequently, we predicted that the pupils' performance on the proportional items would be very high, while their scores on the non-proportional items would be very low.

Second, we assumed that four extra years of mathematics education should have a positive effect on pupils' ability to resist and overcome the tendency towards improper proportional reasoning. Moreover, the Flemish mathematics program for the ninth and the tenth grade (15–16-year old pupils) offers several opportunities to bring pupils into contact with non-proportional reasonings and problem types. Therefore, we predicted that these 15–16-year olds would perform better on the test in general and on the non-proportional items in particular than the 12–13-year olds.

Third, we hypothesized that the pupils of the CP-groups would perform better on the non-proportional items than those from the MV-groups. When a non-proportional task is formulated as a familiar missing-value problem, it's more likely that pupils will associate it with the kind of proportional thinking which proved to be the adequate solution strategy for the vast majority of the missing-value problems they encountered in their school career so far. With the unusual formulation as a comparison problem there is a greater chance that pupils will no longer fall back on this kind of routine-based, superficial thinking and invest more mental effort in the analysis of the problems, leading to better performance. Besides, in this "comparison condition" each problem is given with only one number, so pupils cannot take refuge in the execution of all sorts of arithmetical operations; so to speak, they have no other choice than putting their minds on the mathematical content of the problem. In line with our previous studies in which better performances on the non-proportional items in a given experimental condition were always parallelled with weaker performances on the proportional items, we hypothesized as well that the pupils in the comparison condition would perform worse on the proportional items. Accordingly, we predicted for both age-groups higher scores on the non-proportional items and weaker scores on the proportional items for the CP- than for the MV-group.

ANALYSIS

The hypotheses were tested by means of a "$2 \times 2 \times 2$" analysis of variance with "Proportionality" (proportional vs. non-proportional items), "Age" (12–13- vs. 15–16-year olds) and "Group" (MV-groups vs. CP-groups) as independent variables, and the number of "Correct answers" as the dependent variable. In addition to this quantitative analysis, in which no distinction was made between different types of correct and incorrect responses, we also executed a fine-grained analysis of pupils' errors and solution strategies, but, because of space restrictions, this more qualitative and process-oriented part of the study will not be reported here.

RESULTS

Table 3 gives an overview of the percentages of correct responses for the two groups of 12–13- and 15–16-year olds on the proportional and the non-proportional items in the test.

	12–13-year olds		15–16-year olds	
	Proportional items	Non-proportional items	Proportional items	Non-proportional items
MV-group	87	12	86	34
CP-group	62	31	75	52

Table 3. *Percentages of correct responses for the two groups of 12–13- and 15–16-year olds on the proportional and the non-proportional items in the test*

The results provided a very strong confirmation of the first hypothesis. Indeed, the analysis revealed a strong main effect of the task variable "Proportionality" ($p < .01$): for the two age-groups and the two experimental groups together, the percentages of correct responses for all proportional and for all non-proportional items were 78% and 32%, respectively.

The second hypothesis was confirmed too: the factor "Age" had a significant main influence ($p < .01$): the 15–16 year olds performed better on all experimental items than the 12–13-year olds; percentages of correct answers were, respectively, 62% and 48%. Furthermore, the predicted interaction effect between the "Age" and "Proportionality" was found too ($p < .01$): while the 15–16-year olds answered nearly twice as much non-proportional items correctly than the 12–13-year olds (43% and 22% correct responses, respectively), they outperformed the 12–13-year olds only slightly on the proportional items (81% and 75% of correct responses, respectively).

Third, the analysis of variance did not reveal a main influence of the factor "Group" on the pupils' performance: in both experimental conditions the overall percentage of correct responses on the test as a whole was exactly the same (55%). However, a significant "Group × Proportionality" interaction effect ($p < .01$) was found. The MV-groups performed considerably worse than the CP-groups on the non-proportional items (23% and 41% correct answers, respectively), but this worse performance of the MV-groups on the non-proportional items was parallelled with much better scores on the proportional items (i.e. 87% correct answers versus 68% in the CP-groups). Apperently, the item-formulation used in the CP-groups prevented pupils for falling into the proportionality trap, but as a result these pupils sometimes began to question the correctness of the proportional model for problem situations in which that model *was* appropriate – a finding that is very similar to the one obtained in our previous studies and that has been observed in several other studies about strategic and conceptual change (Siegler & Jenkins, 1989; Vosniadou, 1994). Finally,

the analysis of variance did not reveal an interaction between the factors "Group" and "Age". Nor was there a "Group × Proportionality × Age" interaction effect, which means that the reported interaction between Group and Proportionality manifested itself equally in both age-groups.

CONCLUSION

Recently, the omnipresence, strength and resistance to change of the "illusion of proportionality" with respect to scaling problems presented in a school context was demonstrated in three related studies (De Bock et al., 1998a, 1998b). In these ascertaining studies, the majority of the pupils failed on the non-proportional items, because they routinely applied proportional reasoning in a situation wherein it was not at all appropriate. However, these studies did not allow to provide an explanation for pupils' alarmingly strong tendency towards improper proportional reasoning.

The present study involves a first step in our effort to unravel the factors fostering the occurrence of this illusion of proportionality. Typically, this phenomenon is qualified as a wrong belief, as exemplified in the quotation taken from the American *Standards*, mentioned in the Introduction of this research report: pupils apply the linear model because they incorrectly believe this model is appropriate for a given problem situation. In the present study, we demonstrated that pupils' tendency to apply proportional reasoning in problem situations for which it is not suited, is – at least partially – caused by particularities of the problem formulation that pupils learned to associate with proportional reasoning throughout their school career. The significantly better results of the CP-groups on the non-proportional items made it clear that for a lot of pupils, what lured them into the trap of proportional reasoning was not their belief in an overused mathematical model – in this case the linear function – but rather their illicit confidence in a link between that model and a certain type of problem formulation – in this case the missing-value type.

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ASPECTS OF AFFECT: MATHEMATICAL INTIMACY, MATHEMATICAL INTEGRITY

Valerie A. DeBellis
Department of Mathematics
East Carolina University, Greenville, NC 27858-4353 USA

Gerald A. Goldin
Center for Mathematics, Science and Computer Education
Rutgers University, Piscataway, NJ 08854-8019 USA

Affect may be regarded as an internal system of representation, interacting meaningfully with cognitive representation during learning and problem solving. This paper provides a concise theoretical overview, and then explores two aspects of affect that we propose as fundamental to the development of mathematically powerful processes: "mathematical intimacy" and "mathematical integrity." Illustrative examples are drawn from videotapes of elementary-school children solving problems during task-based interviews.

> "Thus, in a way, there is a value problem here. You can even consider it as a moral dilemma: to pretend and to get some credit or not to pretend and get zero credit? The vast majority of people that I know, including myself, will solve this dilemma without much hesitation: We will pretend. ... I would like to return now to the above situation in which an individual is not aware of the fact that he or she uses pseudo-knowledge in order to get credit. ... [This] is harder from the cognitive point of view, because the individual has no idea what a true knowledge is. He does not pretend. He assumes." (Vinner, 1997, p. 69)

Researchers of mathematical learning and problem solving increasingly recognize the importance, complexity, and depth of the affective domain. This domain goes beyond personal traits such as attitudes, beliefs, and self-concepts, and beyond the fleeting emotions accompanying cognition. We see affect as a highly structured system that encodes information, interacting fundamentally--and reciprocally--with cognition (Zajonc, 1980; Rogers, 1983; Goldin, 1988; McLeod & Adams, 1989; DeBellis & Goldin, 1993, 1997; Leder, 1993; DeBellis, 1996, 1998). Other cognitive psychologists, neuroscientists, and even information processing theorists are reaching similar perspectives: e.g., Picard (1997) suggests computers cannot achieve "genuine

intelligence" unless designed with capacity to build knowledge through "emotional" connections. But "mathematical affect" is complex and difficult to study--we have as yet only a preliminary theoretical framework. In this paper we discuss and refine the characterization of two sorts of affective structures we have inferred from observing elementary school children in task-based interviews: "mathematical intimacy," and "mathematical integrity." We offer definitions and illustrative examples, hypothesizing that these two aspects of affect are essential to the development of powerful mathematical ability.

Affect as Representation: Theoretical Framework and Interview Data

McLeod (1989, 1992) describes three components of the affective domain: emotions (unstable, intense), attitudes (reasonably stable, moderately intense), and beliefs (slowly developing and highly cognitive). Ortony, Clore, and Collins (1988) regard emotions essentially as reactions to cognitively construed events. The eliciting conditions of emotions include the cognitive representations resulting from such construals--a perhaps obvious but nevertheless essential cognitive basis for emotion. We agree this far, but we see the emotional system as much more than a reaction to cognitive inputs. Emotions occur in structures that themselves have a symbolic function--i.e., the eliciting conditions of cognitions in turn include affective representations. For instance the emotion of curiosity (crucial in our view, but essentially omitted by Ortony et al.) makes immediate sense as a condition eliciting cognition. Goldin (1987, 1988, 1998a) considers affect as one of five kinds of interacting internal systems of representation in the mature individual; the others are (1) verbal/syntactic, (2) imagistic (including visual/spatial, auditory/rhythmic, and tactile/kinesthetic), (3) formal notational (structured mathematical symbol systems), and (4) heuristic planning/executive control. Affective configurations have in our view representational capability--they can stand for, evoke, and generally interact with cognitive configurations as well as other affect, in highly context-dependent ways. The affective system includes changing states of feeling (local affect), as well as more stable, longer-term constructs (global affect). Individuals construct complex networks of affective pathways (sequences of states with accompanying meanings). These form complex networks including, but not limited to, what we call meta-affect (DeBellis, 1996; DeBellis & Goldin, 1997): emotions about and within emotional states, emotions about and within cognitive states, and the monitoring and regulation of emotion. The resulting structures influence mathematical problem-solving ability through affective interactions.

For example, feelings of frustration with a mathematical problem may evoke anxiety and fear in some students; in others, frustration may be associated with renewed or deepened determination. Frustration feelings may encode cognitively-essential information regarding the outcomes of strategies to that point--e.g., failure in a succession of trials to fulfill problem conditions, or the absence of effective record-keeping of information. The frustration feelings may in some students evoke heuristic strategies involving self-distancing and "pseudo-knowledge" (as in the quote above from Vinner). In others, they may trigger constructive heuristics for problem understanding, such as "solving a simpler related problem." Local frustration may sometimes reinforce global constructs regarding the student's sense of self or ability to do mathematics--e.g., "Nothing I ever do is going to work anyway." The immediate feelings of frustration during problem solving may occur in a meta-affective governing context of fear of failure; alternatively, the context may be the anticipation of success, so that the sense of frustration actually increases the problem solver's enjoyment as s/he realizes that the problem is a deep and interesting one. Our earlier research led us to extend McLeod's description to include a component of values/ethics/morals (Kohlberg et al., 1983), that provides the psychological sense of what is good and bad in doing mathematics, feeling in the right or justified, feeling wrong, or judging others. This assists the problem solver to evaluate internally if a mathematical argument is convincing, a proof valid, a solution correct, an understanding adequate, or an expression of approval deserved. The aspects of "mathematical intimacy" and "mathematical integrity" addressed in this paper relate to the values component of the affective domain.

We draw our illustrative examples from a series of 5 specially-designed individual task-based interviews with children aged 8 to 12, part of an exploratory longitudinal study (DeBellis, 1996; Goldin, 1998b). Inferences about affect are difficult, and no claim of reliability is made. We used ten different sources, each a kind of "window" on affect: (1) individual general background information, (2) affective verbal expression (tone of voice, timed pauses in speech, interjections, exclamations); (3) affective non-verbal expression (hand and body movements, posture, facial expressions; (4) instances of affect interacting with executive control, inferred from protocol analysis; (5) overall impressions about affect from mathematics educators who viewed the tapes; (6) cognitive analysis, with special emphasis on affective interactions, in a non-routine problem; (7) an independently-developed affective coding scheme for facial expressions; (8) evidence of instances of meta-affect; (9) evidence relating to the construct of mathematical integrity; and (10) inferring and describing affective pathways.

Mathematical Intimacy

Psychological literature on intimacy usually considers it to be interpersonal, but for some researchers the focus is on intrapsychic experience (Maslow, 1970). For us "mathematical intimacy" describes a possible intrapsychic psychological relation between an individual and (internally represented) mathematics, that connects with his or her sense of and value of self. It entails deeply-rooted engagement, not as an observer but as an emotional participant, that builds personal meaning and purpose. It involves intimate engagement through interaction, and eventually intimate relationship built on multiple intimate interactions (Prager, 1995).

Intimate mathematical interaction (DeBellis, 1998) is characterized by behaviors indicative of intimacy, and by intimate experiences inferred from behaviors. The former, depending on context, may include: the problem solver's self-placement especially close to the physical work, cradling the work with the arm or hand as if to say "this is mine," hesitation in sharing the work, closing the eyes as if to "feel" the mathematics, breathing deeply, tending with great care, or speaking in an especially halting, quiet, or excited way. Intimate experiences may include feelings of warmth, excitement, amusement, affection, sexuality, time suspension, deep satisfaction, "being special," or esthetic appreciation accompanying understanding. They may involve the person's internal representation of loved or respected ones, e.g., a sense that "My father would be proud of me for this." They are more than enjoyable; they build a bond between the learner and the mathematical content. To relate intimately to mathematics is ultimately to have access to and comprehend its "inmost" structures in a personal way. Mathematical intimacy may foster positive outcomes through powerful affect: a willingness to take risks (since intimacy may provide a sense of safety); perseverance (since intimate experiences may include feelings of loyalty, devotion, and passionate commitment); and confidence (since intimacy enhances a sense of well-being).

But intimate engagement does not guarantee a positive long-term relationship. The problem solver may feel disappointed, angry, or betrayed in the intimacy by unexpected mathematical outcomes, failures, negative reactions from loved ones, rebuke from a trusted teacher, or scorn from peers. Such "intimate betrayal" does not distinguish between the mathematically talented and the mathematically challenged individual. Even among professors, graduate students, and professional scientists--and certainly in mathematically gifted children--one finds a great deal of pain in relation to mathematics.

An illustrative example is Jerome (children's names are changed), male, age 10, in the

5th grade. In the third of five interviews, an extra videocamera recorded his facial expressions. [Small corrections to DeBellis & Goldin (1993) and additional important affective observations are included here.] Two glass jars, one with 100 green and one with 100 orange jelly beans were placed on the table. Clinician (C): "Suppose you take 10 green jelly beans from the green jar and put them into the orange jar and mix them up. Then suppose you take 10 jelly beans from the mixture and put them back into the green jar. Which jar would have more of the other color jelly beans in it?" Jerome initially indicated there would be more green jelly beans in the orange jar. Later on, he conjectured that if the number of jelly beans in the transfer is even the numbers would be the same, but if the number of jelly beans in the transfer is odd then one container will have more of the opposite-colored jelly beans than the other. Jerome then does experiments. Transferring 10 jelly beans, he finds each container has the same number of opposite-colored jelly beans. Later transferring 11 jelly beans, he counts: "Now two, four, six, eight, nine. Well they're both, they're both ... they keep on equaling the same amount even if they're odd. But ... two ... two" [inhales deeply, brushes back his hair] "green went over ... and ... two green with that and nine orange and then so nine green are left ... well, they, they still stayed the same. The same amount in each one." [4-second pause, raises his eyebrows, opens his eyes wide, shrugs his shoulders, smiles] C: "How can that be?" Jerome: "I don't know." [15-second pause, stares at jelly bean containers, raises his eyebrows, shakes his head from side to side, sits back in the chair] C: "What do you think is going on?" Jerome: "I dunno ..." C: "What are thinking about?" Jerome: "Uhhh ... I'm just trying to figure out how did this happen." [17-second pause, raises eyebrows, furrows his eyebrows, presses his lips, looks upward, shakes his head from side to side] "I dunno." Jerome's interactions, from which we infer intimate engagement, include his close proximity to the jelly beans when performing the experiments, his raised voice, his deep breaths, the gesture of brushing his hand through his hair, his shrugging of shoulders, his smiling, and the silent pauses. He sits back in his chair as if to push himself away from the experiment, to distance himself when the outcome contradicts his expectation.

Mathematical Integrity

Mathematical integrity describes an individual's affective psychological posture in relation to when mathematics is "right," when a problem is solved satisfactorily, when the learner's understanding is sufficient, or when mathematical achievement is deserving of respect or commendation. Integrity is associated with insistence on sufficient

understanding and resolution of uncertainties, and reliance on understanding in justifying the "rightness" of the work or the adequacy of a solution. It entails honesty and a degree of openness. In our view, a strong integrity structure of has the potential to enable powerful learning and problem solving--especially in interaction with mathematical intimacy.

The notion of "mathematical self-acknowledgment" (DeBellis, 1996; DeBellis and Goldin, 1997) refers to the person's ability (or willingness) to acknowledge insufficiency of mathematical understanding. Important components of this affective construct are: recognition of the insufficiency, a possible decision to take further action, and the nature of the action. The problem solver may admit that something does not make mathematical sense only to himself or herself, or to someone else (e.g., a teacher or another student). Either acknowledgment may pose specific value, moral, or ethical dilemmas. Mathematical performance may be hindered or helped by the choice of action in response to an acknowledged insufficiency: surface-level adjustments (e.g., "mathematical bluffing"), explicit efforts at deeper understanding, or a combination of both. Vinner (1997) describes "pretending" as a behavior of students trying to get credit when they "know [they] do not know," and the moral dilemma posed by educational systems that reward rapid obedience to mathematical rules over understanding. When "an individual is not aware of the fact that he or she uses pseudo-knowledge in order to get credit," the issue of integrity is more difficult. In our view, the student is not sufficiently intimate with the mathematics to recognize insufficiency in understanding.

An example is Jacqueline, female, age 9 in the 4th grade, in her first task-based interview. She was asked what the 50th card would look like, in a certain card sequence containing dots in a chevron pattern. Prior to this question she had correctly given the numbers of dots on the 4th, 5th, and 10th cards (7, 9, and 19 respectively, obtained by repeatedly "adding two" to previous cards). She appeared intimately engaged, but had not created the geometrical chevron pattern when asked to "show" what she meant. C: "How many dots do you think would be on the 50th card?" Jacqueline: [opens eyes wide, raises eyebrows] "Fifty?" C: "A huh, 50th card?" Jacqueline: [6-second pause, sits back in the chair, arches her back, opens eyes wide, raises eyebrows, presses lips together] "I think we're gonna have to multiply" ... "because we can't write 50 cards." [raises eyebrows] ... [smiles] "because that's too much. And you can't do this all the time. Sometimes you got to multiply to get finished easier." She decides to multiply 19 x 5 (19 dots on the 10th card) ... [11-second pause] "This is what I'm not good at." [smiles] ...

"Think that's right?" C: "What do you think?" Jacqueline: "I don't think so." ... "Maybe if we tried um ... maybe divide it" [looks up] "no ... this is 19." [points to the 10th row of her figure] C: "A huh." Jacqueline: "We got to 19 by 5 so it should be like" [furrows brow, sighs] "... this is ..." [points to the 5th row of her figure] "ooooohh, so like each, each 5 you add 10 to the 19." As the interview proceeds Jacqueline acknowledges an insufficiency of understanding (with integrity), makes a change, proposes a new strategy, and tries to use it, a total of 10 different times--showing great perseverance though ultimately she is persuaded by an incorrect solution: "Keep adding two and two and two and it came to a hundred, all that easy and I did all that for nothin'." [writes 100]

The Interplay between Intimacy and Integrity

Mathematical intimacy and integrity confer mathematical power, and interact to foster perseverance. With them the problem solver no longer needs cognitive and affective capacity for dissembling about insufficiencies, and no longer avoids but favors heuristics that lead to deeper understanding. Absence of integrity is a huge obstacle to intimacy. Bluffing (or pretending) blocks the individual not only from understanding but also from the experience of intimacy in relation to mathematics. Similarly the absence of intimacy reduces the individual's need for integrity, as non-intimate interaction is less likely to be experienced as posing a moral dilemma or value conflict. Lack of mathematical intimacy may also impede the problem solver's ability to distinguish knowledge from pseudo-knowledge.

To sum up, powerful affect is complex and consists of far more than positive feelings or high confidence levels. It entails structures of intimacy, integrity, and meta-affect that promote deep mathematical inquiry and understanding. Excellent teachers of mathematics appreciate, at least tacitly, the need for attention to their students' affective development. In our own research we have come to view affect as the most fundamental, and the most unrecognized in importance, of internal representational systems.

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Function: Organizing Principle or Cognitive Root?

Phil DeMarois

William Rainey Harper College
1200 W. Algonquin Rd.
Palatine, IL 60015, USA
e-mail: pdemaroi@harper.cc.il.us

David Tall

Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL. UK
e-mail: D.O.Tall.csv.warwick.ac.uk

The function concept is often used as an organizing principle for algebra and beyond. Here we consider its value as a cognitive root (a concept which serves as a basis for cognitive development). current theories of multiple representations and theories of encapsulation of process as object are used to build a view of function in terms of different facets (representations) and different layers (of development via process and object). Results of interviews with three students in developmental algebra will be used to highlight the model and to discuss the value of the function concept as a cognitive foundation to growth in mathematical understanding.

Introduction

The function concept is often suggested as an organising principle in mathematics:

> We believe that function is the fundamental object of algebra and that it ought to be present in a variety of representations in algebra teaching and learning from the outset.
>
> (Yerushalmy & Schwartz,1993, p.41)

It has become a central concept in school and university curricula around the world. We agree that the function concept can be a powerful foundation for logical organisation, but we question its suitability as the basis for a cognitive development.

Tall (1992, p. 497) defined a *cognitive root* as a starting concept with the "dual role of being familiar to students and providing the basis for later mathematical development". He considered the function concept as a possible cognitive root, counselling that there were serious obstacles such as the encapsulation of function as a manipulable object (eg Dubinsky & Harel, 1992; Sfard, 1992) and the complexity of coordinating alternative representations (Cuoco, 1994). Here we consider these two dimensions—the links between various representational *facets* of the function concept, and the *layers* or levels of compression in process-object encapsulation (Demarois & Tall, 1996). These are traced through a remedial college algebra course based on the function concept.

Framework

The facets studied will include the function *notation* (including the meaning of $f(x)$), the *colloquial* use of a function machine as input-output box, the standard *symbolic* (algebraic formulae), *numeric* (table) and *geometric* (graphic) facets, with the *written* and *verbal*. These will be represented as sectors of a disc (figure 1) in which movement towards the centre is seen as compression through the layers *pre-procedure*, *procedure*, *process*, *object*, and *procept*. Pre-procedure denotes that the student has not attained the procedural layer. Students at the procedure layer are dependent on carrying out a sequence of step-by-step actions. Students at the process layer can accept the existence

of a process between input and output without needing to know the specific steps, and see two procedures with the same input-output as the same process. The object layer denotes the capacity to treat the idea as a manipulable mental object to which a process can be applied. The procept layer indicates the ability to move between process and mental object in a flexible way.

To allow each facet to be linked directly to any other, the picture should be seen as having individual slices (facets) that can be moved and connected in any way.

An alternative representation (figure 2) is used to show the direct links between selected facets, some of which may be non-existent or in one direction only for individual students.

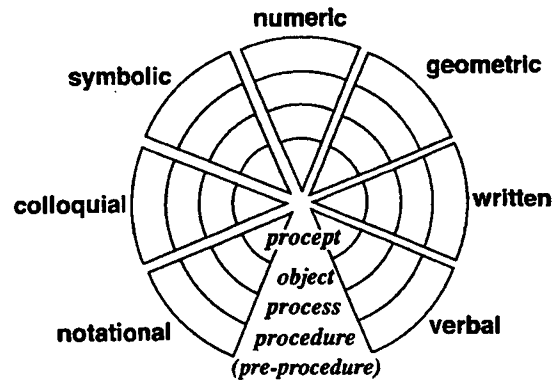

Figure 1: facets and layers of the function concept

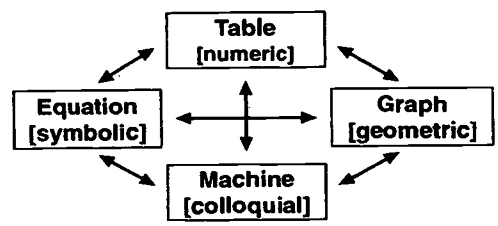

Figure 2: possible links between function facets

Student Conceptions of Function

DeMarois (1998) studied students taking a developmental algebra course at a community college. The students completed pre- and post-course function questionnaires and several participated in a post-course interview. Her, we focus on three students AF, BF and CM, where the first letter denotes the grade achieved (A, B, C) and the second denotes the gender (M or F). AF is a liberal arts student between 21 and 25 years of age. BF is a business student between 26 and 30, CM is a biology student over 30. AF had studied 1.5 years of algebra before college, BF and CM had taken 1 year. AF and BF were taking their first college mathematics course, CM had previously attended a basic mathematical skills course

. Function machines were used to analyse the colloquial facet. The majority of students displayed some understanding of function machines on the pre-test. In the individual post-course interviews, one question provided data on colloquial, verbal, numeric, and symbolic facets (figure 3).

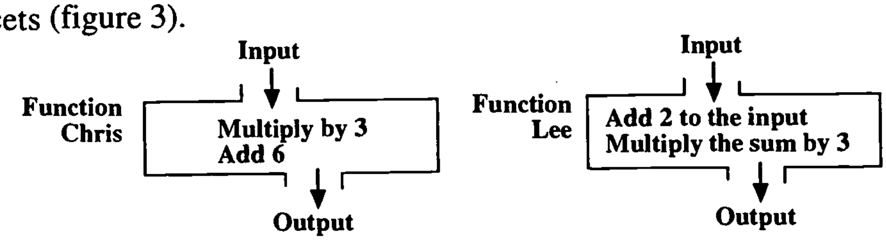

Figure 3: equivalent functions

Students were asked to write expressions for each function machine and asked whether the two function machines represented the same function (table 1).

	Chris	Lee	Are the functions Chris and Lee equal?
AF	$3x+6$	$3(x+2)$	Yes, if I distribute the 3 in Lee, I get the same function as Chris
BF	$x3+6$	$(x+2)3$	Yeah, but different processes
CM	$3x-6$	$x+2(3x)$	No, you come up with the same answer, but they are different processes

Table 1: Function machines as procedure, process and mental object

The three responses show AF speaking in terms of a mental *object*, BF in terms of *process* and CM in terms of *procedure*. AF easily links the colloquial and algebraic facets. BF gives a literal translation of both function descriptions showing less flexibility moving from colloquial to algebraic. CM sees Chris and Lee as different procedures (in our terminology). He also gives a literal translation of the second function as "x+2 three times", revealing that he is less comfortable relating the colloquial facet to the algebraic.

Further research into links between symbolic, arithmetic, geometric and colloquial facets was performed by asking the students to respond to the following questions:

- given a specific equation, create a table, a graph, and a function machine;
- given a specific table, create an equation, a graph, and a function machine;
- given a specific function machine, create a table, a graph, and an equation; and,
- given a specific graph, create a table, an equation, and a function machine.

They were encouraged to create the other forms in any order they wished. Tables 2-4 display the results where "√" indicates a successful attempt and the numbers indicate the order in which the representations were created.

AF From ↓ to →	Equation (symbolic)	Table (numeric)	Function machine (colloquial)	Graph (Geometric)
Equation		√ (1)	√ (2)	√ (3)
Table	√ (2)		√ (3)	√ (1)
Function machine	√ (1)	√ (2)		√ (3)
Graph	√ (2)	√ (1)	√ (3)	

Table 2: Creating representations: AF

Although AF was able to start with any representation and eventually get to any another the routes taken were not always direct (see figure 4). Given the equation, AF said:

> I am much more comfortable with the function machine and the table as opposed to creating a graph on my own. I'm not as comfortable doing a graph on my own.

Given the table, AF first created the graph, but went back to the table to create the equation. She used the graph to determine the type of equation but then used the table to determine the slope using finite differences:

> I'm trying to find the finite difference. I know from the graph it looks like it will be a line so I think it will be linear which I know is $y(x) = ax+b$. So for that I need the slope and the 0 input which I already have which is −3. It looks like the slope is 2 so I get $y(x) = 2x-3$.

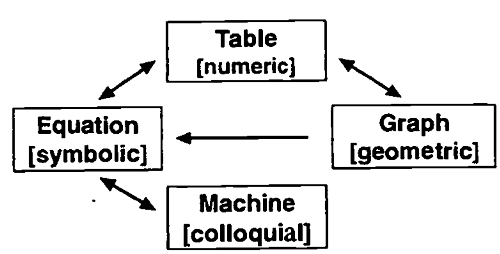

Figure 4: direct links between facets for AF

BF From ↓ to →	Equation (symbolic)	Table (numeric)	Function machine (colloquial)	Graph (Geometric)
Equation		√ (2)	√ (1)	√ (3)
Table			√ (3)	√ (1)
Function machine	√ (1)	√ (2)		√ (3)
Graph		√ (1)		

Table 3: Creating representations: BF

BF proceeded as in table 3. She could start from equation or function machine and generate all other facets, but was only able to move between table and graph when starting from one or the other. She kept trying to generate equations or function machines using only one point. She was thus unable to find the slope and could not make other links to equation or function machine (figure 5).

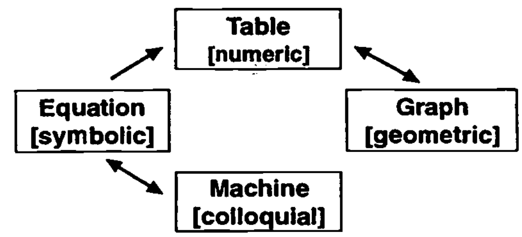

Figure 5: direct links between facets for BF

CM was also able to start from the equation or function machine and generate all other facets. Starting with a table he drew a graph, but could not cope the other way (table 4).

CM From ↓ to →	Equation (symbolic)	Table (numeric)	Function machine (colloquial)	Graph (Geometric)
Equation		√ (1)	√ (3)	√ (2)
Table				√ (1)
Function machine	√ (2)	√ (1)		√ (3)
Graph				

Table 4: Creating representations: CM

He had a limited ability to pass directly from one facet to another (figure 6). He said:

I'm not real sure on equation or function machine.

If you had to choose between the two, which would you prefer?

It doesn't matter. I don't like either. I really don't like anything that has to do with math.

[The pained look on his face and the nervous body language speak volumes.]

You like tables?

Yeah. Tables are a little bit easier for me. I trust those more than having to figure out stuff.

Given a graph he drew a table outline and said:

No. I can't do it.

You started to do a table.

Yeah ummm. If I were to sit down and think about it for a while I probably could. That's the way a lot of math is to me. I just keep trying different ways until I hit upon one that works. To save my life I probably could, but I'm not real sure.

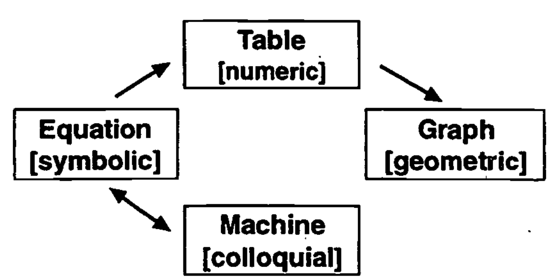

Figure 6: direct links between facets for CM

CM struggled throughout the course using inflexible procedures and limited connections between representations. He became frustrated and gave up easily, particularly where graphs were involved.

Overall, AF's performance on this series of questions was flawless. BF demonstrated good connections between symbolic and colloquial and between numeric and geometric, but only from the first of these pairs to the second. CM established a connection between symbolic and colloquial, but any connection to graphs was tenuous at best.

Student profiles

Visual profiles (Figure 6) of the concept images of function at the end of the course were created for each of the three students through analysing all the collected data. The shading indicates layers of each facet attained by the end of the course.

AF demonstrated knowledge during the interview that was at least equivalent to that displayed on the post-course survey. Her knowledge of the verbal facet matched her written facet since her verbal and written descriptions of function were identical. She was able to assimilate alternate definitions easily into her own concept image. AF did exhibit difficulty during the interview dealing with implicit equations as functions of one variable in terms of the other. She did not use the "uniqueness on the right" condition (Breidenbach et al., 1992, for example) in her selection of functions from a set of equations. She initially denied the constant function is a function, but later changed her mind. She displayed proceptual abilities working with both tables and function machines. She is easily able to think of them as functions (static objects) and as processes (dynamic objects). Her understanding of graphs was developing even as we conducted the interview. She did not need to know a specific procedure, recognizing each graph as representing a set of input-output pairs. She was not prototype-driven and although she did not initially seem to know how to apply the "uniqueness to the right" condition, after some instruction, she

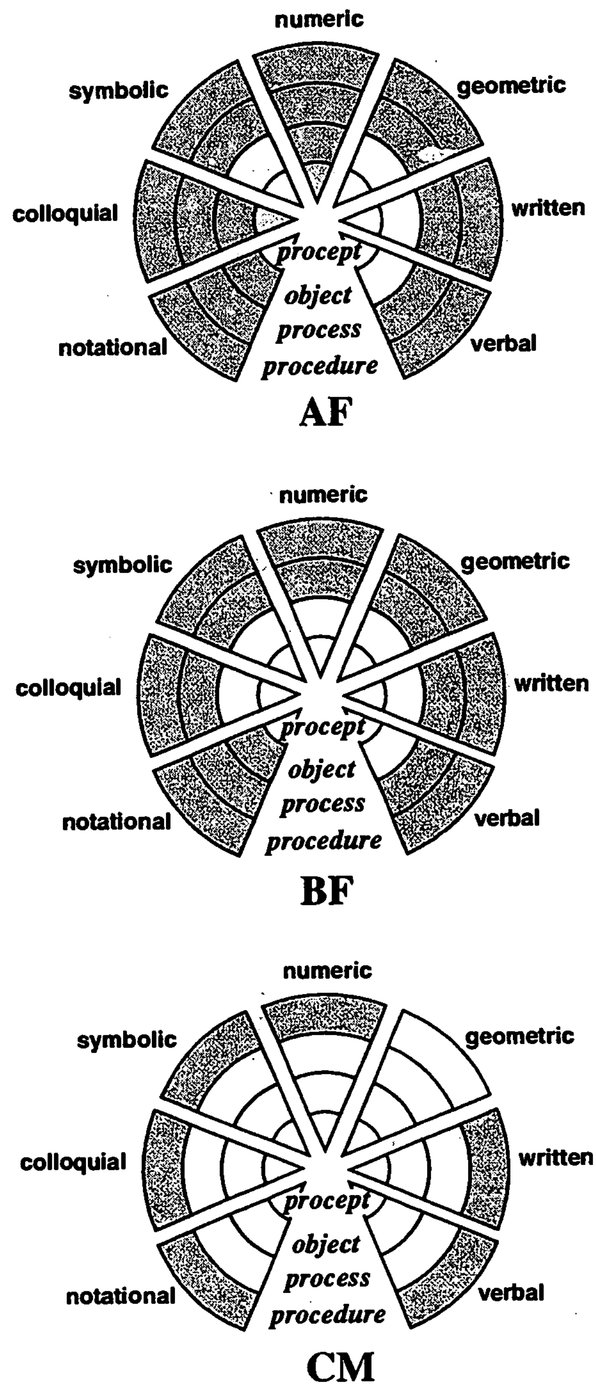

Figure 6: Student profiles

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was able to use it coherently. She was not placed in the object layer for the geometric facet because she demonstrated a process-orientation looking at graphs rather than to seeing it as a function object. Her knowledge of the notation facet (for instance the meaning of $y=f(x)$) appeared strong and consistent except for an occasion when she was asked to substitute 44 for y in an equation containing $y(x)$ and said "44 of x." She quickly withdrew this statement and described 44 as replacing $y(x)$. AF was the only student interviewed able to distinguish between $3f(2)$ and $2f(3)$.

Of the three students, BF exhibited the most growth during the course. At the beginning she was judged to be at the procedure layer only on the symbolic facet. By the end, she appeared to be at or near the process layer on all facets surveyed. The numeric and colloquial facets showed some difficulties with process. She was highly procedural in creating an equation from a function machine writing down the steps of the function machine literally. This result carried over to the interview. Her choice of tables that represent functions focused on those tables in which a clear procedure or pattern was present. Her strongest facet seems to be notation which she interpreted flexibly in both post-course survey and interview although she exhibited difficulty interpreting $3f(2)$ and $2f(3)$ and substituting 44 for y in an equation involving $y(x)$. In the interview she was placedin the object layer for notation because of her ability to discuss the notation as an object. On the symbolic facet, she accepted the constant function as a function, but had trouble with piecewise-defined functions. She was the only student of the three that was able to correctly apply the vertical line test to graphs both on the post-course survey and during the interview. While consistent in her verbal and written definitions, BF was not as comfortable as AF in adopting alternate definitions. She had more difficulty crossing boundaries between facets. She did not easily move from a function machine to an equation and was procedural in using equations. This caused difficulty when given a variable input. She was unsure what to do and was not sure the output made much sense.

CM was the least successful of the three. At the beginning of the course he demonstrated procedure layer knowledge in both numeric and colloquial facets placing him slightly ahead of BF. By the end of the course, he was procedural in every facet except for some movement into the process layer of the symbolic facet. On the post-course survey he showed some ability to reverse a table and some hints of process when selecting tables as functions. The interview suggested that CM was at the procedure layer on all facets except geometric where he remained pre-procedural. In addition, his interview answers in the symbolic, geometric, numeric, and verbal facets were highly inconsistent with those on the post-course survey. He looked for specific procedures when identifying equations or tables as functions and was unable to identify any usable rule when looking at graphs. His written and verbal definitions of functions varied and he could not assimilate any alternate definitions of function into his own. At best, he indicated some use of prototypes when looking at graphs and demonstrated some knowledge of function notation relating only to procedural aspects of equations and the function machine. Neither written nor geometric facets seemed connected to any other facet at all.

Quantitative Data

The class as a whole reflected this spectrum from procedure to mental object conceptions of function. On the pre-test in the colloquial, symbolic and numeric facets, around 70% were able to cope with input-output as procedure or process but only 3% were at this level handling graphs (table 5).

(N = 92)	Colloquial (Function Machine)		Symbolic (algebra)		Numeric (Table)		Geometric (Graph)	
	pre	post	pre	post	pre	post	pre	post
pre-procedure	32%	10%	26%	8%	30%	9%	97%	50%
procedure	20%	18%	54%	51%	14%	11%	2%	21%
process	49%	72%	20%	41%	55%	80%	1%	29%

Table 5: Changes in levels of responses for four facets between pre-survey and post-survey

The table reveals improvement in all four facets. Other data collected during the project implies a corresponding improvement in the verbal, written and notational facets. This suggests that the function concept is accessible as procedure or process for many of these remedial students. The function machine appears to be a sufficiently primitive structure to serve as a cognitive root on which to build the function concept. However, the manner in which these students link the function machine to other facets suggests real difficulties in building sophisticated ideas upon it. All three students AF, BF and CM moved to other facets via algebraic symbolism and only AF used standard algebraic expressions. Many others in the class exhibited similar difficulties moving from the function machine to other representations. Although the function machine is a good candidate as a cognitive root for the full function concept, for many of these students the total concept is too complex to allow a full development.

For instance, student competence with the geometric facet was almost non-existent at the beginning of the course and difficulties persisted throughout even though (or perhaps in part *because*) students had regular access to graphic calculators. While there was a significant increase in response handling graphical problems, by the end of the semester less than half the students were able to use a graph to find output given input and only 19 percent were able to reverse the process. Of our cross-section of students, AF showed good depth in understanding of this facet, but BF and CM had enormous difficulty.

Function notation was also interpreted inconsistently, with many students (including AF) using it correctly in some settings yet unable to translate it to a new, similar setting.

Students are often competent at "plug and chug" mathematics and use this ability to hide weaknesses in their understanding. CM, for example, used the more abstract symbolic facet when the more primitive table failed him. He indicated little understanding of the symbolism, but demonstrated several times that he could evaluate a function. This appears to be an example of "pseudo-conceptual" understanding where he attempted to respond in a manner he sensed was desired by the teacher, yet failed to make appropriate internal connections (Vinner, 1997).

Summary and Reflection

This study underlines the complexity of the function concept. Its inherent richness allows it to be considered as an *organising principle* in mathematical courses such as algebra. The use of function machines provides a new approach in remedial algebra which does not simply reproduce the procedural errors of earlier experience, There are gains in moving students to procedural and process levels of thinking in several facets, but the graphic facet and some of the links between different facets remain problematic. The function machine provides a primitive idea that the majority of the students recognised at the beginning of the course, at least at a procedural level. Theoretically it contains the basic idea of long-term growth – as an input/output procedure and potentially as a mental object that can be operated upon. However, for many students, the complexity of the function concept is such that the making of direct links between all the different representations is a difficult long-term task and, in the case of this course using graphing calculators, the development of graphical ideas had to start almost from nothing and only partial progress was made. An organising principle in theory: yes, but is it a cognitive root for general long-term development? In our judgement the jury is still out.

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CREATING A TOOL: AN ANALYSIS OF THE ROLE OF THE GRAPHING CALCULATOR IN A PRE-CALCULUS CLASSROOM

Helen M. Doerr and Roxana Zangor

Syracuse University

Abstract

In this paper, we report the results of a qualitative, classroom-based study on the relationship and interaction between the role and beliefs of the teacher and the patterns and modes of students' use of graphing calculators in support of their learning of mathematics. This interaction led to the creation and development of a set of ways the tool was used in the classroom and related mathematical norms. We found that the teacher's confidence, flexibility of use and her awareness of the limitations of the technology, led to the establishment of (a) a norm that required results to be justified on mathematical grounds, (b) multiple ways for visually checking hypothesized relationships between variables, (c) a shifting role for the calculator from graphing to checking, and (d) the use of non-calculator strategies for periodic transformations.

Introduction

Functions and graphs have been the focus of numerous research studies over the past decade. The study of students' understanding of the concept of function, and their abilities to create and interpret graphical representations, was given strong impetus by the advent of computers and their ready availability in some classrooms. This led to many computer-based studies that analyzed students' reasoning with and about linked, dynamic multiple representations of functions (e.g., Confrey & Doerr, 1996; Yerushalmy, 1991). Furthermore, while the graphing calculator has limitations when compared to a full-screen computer program, its low cost, portability and ease of use have resulted in its widespread use for teaching about functions and graphs in secondary schools in the United States.

Teachers, as well as the developers of standardized tests (such as college entrance examinations), have moved to adapt the graphing calculator into their practice. The National Council of Teachers of Mathematics's (NCTM) curriculum standards (NCTM, 1989) recommend using the graphing calculators to provide students with new approaches, including the use of multiple representations, to the investigation of mathematical ideas. While it might appear that practice has moved independently of and more quickly than research, there is in fact no shortage of research studies on the use of the graphing calculator (Dunham & Dick, 1994; Penglase & Arnold, 1996). However, many, if not most, of these studies are quasi-experimental in design and seek to answer the question of whether or not graphing calculators are effective in achieving certain instructional objectives, which are often left unchanged from traditional

paper-and-pencil approaches. Many of these studies compare the use of the graphing calculator to the use of paper and pencil on the same set of tasks. Such studies give little insight into how and why students use graphing calculators. Furthermore, few studies have attempted to understand the role of the teacher in a classroom where students have ubiquitous use of the graphing calculator and where the tasks have been changed to potentially take advantage of the graphing calculator's functionality (Penglase & Arnold, 1996). Teachers' attitudes and beliefs about the use of the graphing calculator in classroom practice are largely unexamined in the research literature (Tharp, Fitzsimmons & Ayers, 1997). In this paper, we report the results of a study on the relationship and interaction between the role and beliefs of the teacher and how students used graphing calculators in support of their learning of mathematics.

Theoretical Framework

Changes in curriculum are necessary to create an environment in which students can develop the new problem solving strategies that graphing calculator technology makes possible. Such curricular changes must be accompanied by changes in instruction so that teachers can develop new pedagogical strategies that are appropriate for the learners, the curriculum and the technology. In this research, we see teachers' pedagogical strategies as integrative of their views of how students learn, the mathematical content as embodied in the curriculum, and capacity of the technology to support learners in understanding mathematics.

The theoretical framework guiding this research follows the perspective described by Cobb and Yackel (1996) in which psychological and sociological aspects of learning are coordinated as an active process in which students reorganize their thinking through their interactions in the social context of the classroom. This social context includes the tools and representational systems which are shared among students and teachers. The meaning and the role of the graphing calculator as a tool for mathematical learning within the classroom are constructed by both teacher and students through their interactions, communications, and shared use of the tool. As Hiebert et al (1997) have observed, "Students must construct meaning for *all* tools. ... As you use a tool, you get to know the tool better and you use the tool more effectively to help you know about other things." (p. 54,emphasis added). In this study, we seek to describe how the teacher's beliefs about the graphing calculator were reflected in her pedagogical strategies. We then describe how these strategies led to the co-construction, with the learners, of a particular set of ways in which the graphing calculator became a tool for mathematical learning in the classroom.

Methodology and Data Analysis

A pre-calculus curriculum based on modeling problems in an enhanced technology environment (using graphing calculators, calculator-based measurement

probes for motion, temperature and pressure, and computer software) provided a rich setting for studying the role of the teacher and the patterns and modes of graphing calculator use by the students and the teacher. This classroom-based, qualitative case study took place in two classes of pre-calculus students in a suburban setting, taught by the same teacher. The classes were observed over two units of study on exponential functions (ten weeks) and trigonometric functions (seven weeks). The teacher had 20 years of teaching experience and was skilled in the use of the graphing calculator.

In this study, all of the students had either TI-82 or TI-83 graphing calculators. These devices are rich in graphing and statistical functionality, although lacking in symbolic algebra capability. In general, the students had used their calculators for well over a year before taking this course and were quite familiar with its features. The classroom was equipped with a computer, printer and a "graph link cable" that could be used to transfer pictures of the calculator graph to the computer for printing. The link cable could be used to transfer data and programs between the computer and calculator, but this feature was rarely used. On the other hand, students readily transferred data and programs between calculators using the calculator-to-calculator link cable. The classroom was equipped with a view screen that allowed the calculator screen (but not the keystrokes) to be projected using a standard overhead projection unit. The view screen did not allow for individual student's calculators to be projected, but only the particular calculator that was attached to the view screen. The students and the teacher both used that calculator during class discussion.

Classroom instructional activities regularly alternated between modeling problems investigated by the students within a small group and whole class discussion for sharing progress, discussing solution methods and extending results. All class sessions were observed by two or more members of the research team. Extensive field notes, transcriptions of audio-taped group work, transcriptions of video-taped whole class discussion, and interviews and planning sessions with the teacher constituted the data corpus for this study. These data were analyzed and coded for the patterns and modes of graphing calculator use by both the teacher and the students throughout the instructional units. In this paper, we present some of the results of this analysis, beginning with a description of the role and beliefs of the teacher as they were reflected in her pedagogical strategies. We then present some of patterns and modes of graphing calculator use that emerged as the students interacted with the teacher, with each other and with the problem situations.

Results

The teacher was particularly skilled in using the graphing calculator, as was demonstrated during both instructional units by her own use of the calculator and by her ease in answering the students' occasional questions on how to use the calculator for some particular task. The teacher's confidence in her own knowledge about the

calculator's capabilities and its potential uses was reflected in the willingness with which she encouraged the students to use the calculator in their individual and group work. She actively encouraged the students to take over the use of the calculator on the overhead projection unit during class discussions.

The teacher showed flexibility in her use of the calculator, as she would shift among the different representations of functions as tables, graphs, or equations, or shift from various lists to different kinds of regression equations. The teacher had a special preference for table - graph switching, particularly in conjunction with setting the viewing window for the graph. The settings for the window were generally determined via table values. An explicit decision to set the window for a better view of the "complete" graph was often made through an interpretation of the table for the given function. In this way, the table served as a scaling tool for the calculator's viewing window.

The teacher used the table for examining the numerical patterns that identified the structural features of a function. For example, she would ask the students to identify either the constancy of first order differences or the constancy of successive ratios as the defining characteristic of the linear or exponential structure for a given numerical relationship. The teacher also used the table to determining pointwise or local behavior of a function by looking for the intercepts with the axes or for the coordinates of extreme points. This table - graph switching by the teacher constituted opportunities for mathematically normative discussion with the students about the meaning of a complete graph of a function. This meaning came to include the local and global behavior of the function such as zeros, y-intercept, numerical patterns, symmetry, asymptotes and end behavior.

The teacher also believed that the calculator presented certain technological and mathematical limitations. In some instances, she talked about incompatibilities between the two different kinds of calculator, differences in how expressions were evaluated and limitations related to storage space such as the number of lists and the number of elements in one list. But the most important issue she raised concerned the validity of the calculator results. She asked questions such as "Does the calculator always tell the truth?" and "To what extent should we believe the calculator?" These kind of questions were most often posed by the teacher in specific situations when the calculator provided results in contradiction with the mathematically accepted truth. As an experienced teacher and a skilled user of the calculator, she was always aware of the calculator-generated errors and she pointed them out every time, especially because the students were tempted, as some of them said, "to go with what the calculator says."

Over the course of several months, the students became increasingly aware of these "mismatches" between the calculator results and mathematically accepted truth.

They started to develop a skepticism about calculator results. For example, in investigating a decay experiment where approximately half of the M&M candies were removed in each trial (Doerr, 1998), the students decided that an exponential function models this decay process. One student observed that even though in their experiment they ended up with zero candies, the exponential model did not attain a zero value since "you can divide by two infinitely without getting zero." But another student, who was manipulating the calculator with the overhead screen, scrolled down the table for their exponential function until, for very large values of x, the function appeared to take the value zero. This generated considerable discussion about the calculator having limitations and "not always telling the truth." From this situation as well as others, the students developed a reasonable skepticism and learned to interpret attentively and critically calculator-based results. The students began to see the calculator as a tool that should be checked based on their own understandings of mathematical results. Maintaining a reasonable skepticism about the mathematical truth of calculator results thus became established as part of the socio-mathematical norms of this classroom culture.

The teacher believed that the calculator would be a helpful tool for the students to use in finding meaningful responses to problem situations and for extending their mathematical thinking. The norms of the classroom culture came to define what kinds of calculator-based responses were and were not acceptable within this classroom. Two specific calculator-based methods (regression analysis and curve fitting by modifying parameters) were regularly used to solve problems where part of the task was to find an equation of a function to represent the data set of a given phenomena. These two methods were not very popular among the students as a whole, but rather were used extensively by three students. The teacher did *not* explicitly discourage these methods by telling the students not to use them, but rather she required a meaningful explanation of how the numerical results related to the problem situation. This meaningful interpretation of the result could not be given by the students who used either regression analysis or a curve fitting approach. These students did not see their findings as estimates of a mathematically determined model or of particular parameters directly related to the problem situation. Those students who used the calculator's regression functions were focused on the immediacy of obtaining some numbers (coefficients) to use in an equation and not on making sense of the result's meaning through a more mathematical analysis of the problem situation.

The calculator took on a role as a visual checking tool to evaluate how well an equation matched a data set from an experiment. The students developed three approaches for visual checking; these approaches depended on how they found their equation in the first place. In the first approach, the students had determined a function through a meaningful mathematical analysis. Graphical mismatches

between their hypothesized function and the data set of the phenomena often led the students to discover mistakes in their symbolic representation or in a computation. In some instances, the students evaluated the "goodness" of the match through the number of points from the original data set that actually lie on their graph. When they found that this graphical checking was not entirely convincing, the students usually switched to table checking. They then compared the numerical values of their relationship with the data set of the phenomena and evaluate their "closeness".

A second approach (taken by only a very few students) was used when an appropriate regression had minor graphical mismatches with the data set. The students dismissed these mismatches and tended to use "what the calculator says." The third approach for visual checking was that taken by those students who solved this kind of problem through a parameter-based curve fitting strategy. These students started with an equation of some general form, such as y=Asin(B(x+C))+D, and then systematically varied the parameters. In this case, graphical mismatches were crucial in their decision to reject the current graphical representation and search for a better one. This process was iterated until the students were satisfied with the goodness of their match. We found that students varied widely in their persistence in attempting to find a good fit. Criteria for the adequacy of a fit were not made explicit. Occasionally when the data set was small and discrete, some students were explicit about their criteria for the closeness of a fit. In that case, the adequacy was usually judged by the number of points actually on the visual representation of the curve.

The use of the calculator as a visual checking tool also supported the students' thinking about the idea of the non-uniqueness of an algebraic representation for an exponential or trigonometric graph. The activities on transformations of functions in both units were designed bi-directionally, going from the equation to its graph and from the graph to a non-unique equation. Initially, in exploring the relationship between graphs and their equations, the students used the calculator as a efficient graphing tool. The students quickly graphed the equations on their calculator and then sketched the graphs on paper. Later, as the teacher actively encouraged the students to use their knowledge of transformations to sketch a graph or find an equation, they relied more on their knowledge of transformations and of the shape of the parent function to sketch the graph on paper. The calculator's use then shifted from a graphing tool to a visual checking tool. The students merely graphed the equation on their calculator, traced the graph on the screen through relevant points and checked if those points matched the given or expected values.

This shift in the role of the calculator from a graphing tool to a visual checking tool was even more pronounced in the case of the trigonometric functions. The "look alike" feature of the graphs of these functions and the infinite number of

possible algebraic representations became particularly problematic with the periodic functions. (We note, of course, that the exponential functions studied earlier have these same features. However, these features were much more salient for the students with the class of trigonometric functions.) An artifact of the calculator screen is that the scaling on the axes of the graphs is not labeled. This made it particularly difficult for the students to visually identify what portion of the graph they were looking at. This in turn led to a shift to a limited use of the table and sometimes of the "Trace" command to numerically check crucial values on the graph. But this too became problematic for the trigonometric functions, since the values that are most often used for the independent variable are integer or common fraction multiples of π. But in the table these values appear as decimal numbers, which cannot be directly compared to the multiples of π, without a numerical conversion. Hence in exploring the transformations of the trigonometric functions, the teacher and students came to rely on their shared knowledge of transformations, with only the most limited use of the calculator.

Discussion and Conclusions

The teacher and the students in this study created a set of ways in which the graphing calculator supported their mathematical investigations and reasoning. The teacher was confident in her own knowledge of the calculator (especially table - graph switching) and believed that it could be a helpful tool for the students to use in finding meaningful responses to problem situations. She also believed that the tool presented certain technological and mathematical limitations. As a consequence of these beliefs, as they were enacted in her interactions with the students and the mathematical problem situations of the curriculum, the graphing calculator became a particular kind of tool in this classroom.

We found that the teacher and students developed a flexible use of the graphing calculator as a tool that could be used to investigate the complete view of the properties of a function's graph. Both teacher and students easily switched between the table and the graph to find local and global properties of a function. This in turn led to discussions about what constitutes a "complete" view of a graph. The teacher's emphasis on meaningful mathematical reasoning in problem situations led to a de-emphasis on the use of regression equations by all but a few students who did persist in their use. But the norms of the classroom came to require that meaningful coefficients had to be justified in terms of the problem's context, leaving little room for regression equations or trial and error curve fitting.

The teacher's recognition of the limitations of the graphing calculator led the students to develop a reasonable skepticism about calculator-generated results. This in turn led to the establishment of a norm that required results to be justified on

mathematical grounds, not simply taken as calculator results. The students' interactions with the tool led to the development of a set of ways of visually checking hypothesized relationships between variables. The role of the graphing calculator was an emerging role, as the students' use of the tool shifted from using it as a graphing tool to using its table and trace features as a checking tool. In the case of periodic functions, the teacher and the students came to rely on their shared strategies for transforming functions rather than use the calculator.

The graphing calculator was not a tool with some independent role and existence in this classroom. But rather, the teacher's attitudes and beliefs, as reflected in the role she took in the classroom, led to the calculator being used by the students and by the teacher in a particular set of ways that created and then reflected the graphing calculator as a tool that supported the mathematics learning in this classroom.

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ARGUMENTATIVE ASPECTS OF PROVING: ANALYSIS OF SOME UNDERGRADUATE MATHEMATICS STUDENTS' PERFORMANCES

Nadia DOUEK
I.U.F.M. Creteil

Abstract: *This paper examines conjecturing and proving in mathematics through analysis of texts written by undergraduate mathematics students. These students reported their reasonings while trying to generalize a property concerning natural numbers and prove the generalized property. Important reference knowledge remained implicit, and non-standardised, appropriate representation of explicit reference knowledge played an important role in students' performances. Referring to semantically rooted arguments was crucial for many students. Subordinating the proving process to the formal requirements of proof as a final product had negative consequences for some students.*

1.Introduction

During this century, the specificity of mathematical proof has frequently been an object of heated debate among mathematicians and philosophers. Particularly, keeping mathematical proof (and, more generally, mathematics) free from recours to "meaning" has been upheld as a possibility or even a necessity by someone (see Whitehead, 1925: *'Mathematics is thought moving in the sphere of complete abstraction from any particular instance of what it is talking about'*). By contrast, others opposed it as an illusion or even a danger (see Hardy, 1929:*"A formal proof is a kind of X-ray picture of an actual or possible piece of reasoning, revealing the bones [the form] but making the flesh [the content, the meaning] invisible.")*. Cognitive aspects of mathematical proof were not so extensively investigated. And in mathematics education it was only in the eigthies that a systematic effort was made to establish links between epistemological, cognitive and educational perspectives while tackling the specificity of mathematical proof in relationship with argumentation (see Balacheff, 1988; Hanna, 1989; Duval, 1991).

The study reported in this paper is part of a personal research project concerning the comparison between argumentation and mathematical proof and its implications for teaching. In Douek (1998) I sought to outline some possible guidelines for this kind of investigation, mainly considering a modern-day mathematician's reflection about his own work (Thurston, 1994) and Duval's analysis of the cognitive functioning of formal mathematical proof (Duval, 1991). In doing so, I considered the distinction between ordinary mathematical proof and formal mathematical proof (i.e. proof reduced to a logical calculation); and the distinction between the process of proof construction and its product (the final text of proof- see section 2. for more details).I sought to support the following position:

In spite of the undeniable epistemological and cognitive distance between ordinary argumentation and formal mathematical proof, argumentation and ordinary mathematical proof have many aspects in common, both as processes and as products.

In particular, I sought to show analogies between argumentation and ordinary mathematical proof, especially as concerns the use of both implicit and explicit reference knowledge, its dependence on social (and historically evolutive) constraints, and the need for semantically rooted arguments. Concerning the processes (arguing and proving), I sought to show how both are generally built up

through *'transformational reasonings'* (Simon, 1996) and heuristics. The analyses were mainly based on "evidence" from the history of mathematics, mathematicians' testimony or from what usually happens in school.

The aim of the research reported in this paper is to analyse in depth the mathematical activity of conjecturing and proving by exploiting a corpus of texts written by Italian undergraduate mathematics students; they wrote their reasonings while trying to generalize a property concerning the system of natural numbers and then prove the generalized property. In particular I will try to seek for the ways students exploited and represented their mathematical knowledge.

2. Theoretical Background

In this paper I will analyse students' protocols concerning production of conjectures and construction of proofs in an open-ended problem; in addition I will study how students exploit their mathematical and meta-mathematical knowledge in this activity. For these purposes the theoretical construct of **Theorem**, by M. A. Mariotti, seems to be appropriate. According to her (see Bartolini et al., 1997) a "theorem" is a statement, its proof and the reference theory - distinguishing between axioms, definitions and theorems of the specific theory in play, on the one hand, and general meta-knowledge about proving and theorems, on the other. In the same perspective I will consider **"Cognitive unity of theorems"**: this theoretical construct of Garuti's (Garuti & al, 1998) concerns the links that exist between the activity of conjecturing (especially as concerns the production of arguments for the plausibility of the conjecture) and the activity of proving.

I will consider argumentative aspects of proving. We cannot accept any discourse as an argumentation. Henceforth in this paper, the word **argumentation** will indicate two things: the process that produces a logically connected (but not necessarily deductive) discourse about a given subject (from the Webster Dictionary: *" 1. The act of forming reasons, making inductions, drawing conclusions, and applying them to the case under discussion");* and the text produced through that process (Webster: *" 3. Writing or speaking that argues"*). On each occasion, the linguistic context will allow the reader to select the appropriate meaning. The word "argument" will be used as *"A reason or reasons offered for or against a proposition, opinion or measure"* (Webster), and may include verbal arguments, numerical data, drawings, etc. So, an "argumentation" consists of one or more logically connected "arguments".

Argumentation is frequently opposed to **formal proof**, i.e. a proof reduced to a logical calculation. According to Duval (1991), in argumentatative reasoning, *"semantic content of propositions is crucial"*, while in deductive reasoning *"propositions do not intervene directly by their content, but by their operational status"* (defined as *"their role in the functioning of inference"*).

But what are the relationships between formal proof and what has been in the past and is today recognized as mathematical proof by people working in the mathematical field (for this reason, I will refer to it as **"ordinary mathematical proof"**)? My research work has been strongly influenced by the position of Thurston (1994):

"We should recognize that the humanly understandable and humanly checkable proofs that we actually do are what is most important to us, and that they are quite different from formal proof. For the present, formal proofs are out of reach and mostly irrelevant: we have good human processes for checking mathematical validity."

In the analysis of students' protocols I will distinguish between the **process of proof construction** (i.e. "proving") and its **product** (as a socially acceptable mathematical text): for a discussion, see Douek (1998, Section 4). Ordinary mathematical proof can be considered as a particular case of argumentation.

Argumentation and proof use references, and I will analyse how students do it. The expression **reference knowledge** will include not only reference statements but also visual evidence, etc. assumed to be unquestionable (i. e. "reference arguments", or, briefly, "references", in general). In Douek (1998, Section 4.1.) I have discussed the necessary existence (in ordinary argumentation as well as in ordinary mathematical proof) of references which are not made explicit.

3. Method

3.1. The educational context

I study written production of conjectures and their proofs in a task related to elementary number theory. The output in question was produced by 43 university students over four consecutive years (from 1995 to 1998) while completing their undergraduate studies in mathematics at the Genoa University. At this level, the students are capable of mastering the mathematical knowledge and the rules of algebraic calculation they must deal with. They are following a mathematics education course and work under a contract (explicitly established with their teacher) that requires them to write down every idea that come to them during their work, even if they change their mind about its validity or its usefulness. This contract is intented to obtain productions regularly for use by the whole group for didactical and cognitive analyses of problem solving activities.

3.2. The task

The students were to generalise a proposition (*"The sum of two consecutive odd numers is divisible by four")*, then prove the generalised proposition. The fact that they had to build up their own conjectures makes their work very different from ordinary school proving, where students have to gather arguments to support a proposition they might never have thought of before. In our case we may suppose that the act of forming a conjecture fixes the conjecture very firmly in their minds, and the proof can be strongly influenced by the steps that led to the insight of the conjecture (see Garuti et al., 1998: "cognitive unity of theorems").

3.3. Modes and criteria of analysis of students' performances

I considered 14 texts (by the 1997/98 students) in particular detail, and then checked analogies and possible differences with the whole set of 43 texts. Reference will only be made to the 14 texts analysed in detail, but the aspects described are recurrent in the other texts as well. Some excerpts from two texts (by Students [1] and [2]), chosen as representatives of opposite behaviours, are reported(see Annex).

Bearing in mind the aim of this study and the theoretical framework, each text has been analysed according to the following modes and criteria:

A) overall account of student's conjecturing and proving (global effectiveness of their performance, etc.);

B) implicit and explicit reference knowledge backing students' argumentation. I distinguished (see Bartolini et al, 1997: "theorems") between:

- content reference knowledge;

- meta-knowledge about the operations that the task called for (generalising, etc).

I also analysed the external representation of explicit reference knowledge. Concerning this issue, our attention focused particularly on personal (verbal, schematic, etc) expressions that would be unusual in a normally acceptable written mathematical production.This kind of analysis was needed in order to explore in depth how these undergraduate mathematics students used their knowledge;

C) occurrence of algebraic-syntactic or semantically based steps of reasoning and the relationships between them. This analysis was needed in order to understand better how the two kinds of reasoning are functionally interlinked and connected to the solution of the problem.

D) relationships between the proving process and the proof as a product (and the consequences of matching the former to the latter).

4. Students' behaviour

4.1. Overall account of students' work

Within the 14 texts, only four (Students [1], [2], [11], [13]) tried to prove something distinctly: two (Students [1] and [11]) prove their conjectures; and Student [13] a partial result of a confused conjecture. Student [2] (see Doc. 2) tries to prove a result that is stronger than the conjecture expressed in words; his proof lacks a fundamental step (justification of the formula used, which derived by generalisation from numerical examples). Let us call these four students the "proof group". But as we can hardly distinguish the processes of construction of conjectures from construction of proofs in the work of the students, we may as well study more texts from the perspective of proof construction. Another important argument to support this shift in the study from proof to conjecture construction is that five students do not achieve their proofs (even though they were on the right track) probably because of a lack of active mathematical practice combined with the unusual situation of having to build their own conjectures. So we can consider the constructive work of nine students (we may call "conjecture group", which includes the "proof group") and take, as comparative examples, elements of the work of the other five ("failure group").

4.2. Reference knowledge and its representation

The task called for elementary <u>content reference knowledge:</u> elementary arithmetics, algebraic language and its rules of calculation. Some students tried to use other reference knowledge such as functions and series. Concerning algebra, we may remark that the process of formalisation (i.e. the passage from content to formula) was not easy for many students, especially when they wanted to write the sum of K odds: for instance, some of them wrote $(2n+1) + (2n+3) + \ldots + (2n+?)$

and then stopped; few were able to express *?* as *2K-1*: see **(E)** in Doc.1. Writing the result of the sum was not easy either: it demanded a semantically rooted conversion of a known formula (the formula for the sum of the first n natural numbers - cf. Szeredi & Torok, 1998), or the re-construction of an ad-hoc formula: see **(F)** in Doc. 1

As concerns the external representation of content reference knowledge, I have found many organisations of data and schemas with visual effects that reveal regularities and help to express algebraically some arithmetic relations; we also found symmetries in the disposition of data and formulas, which provide hints for the calculus (see figures in Doc. [1] for two examples).

We may remark that these behaviours are related to knowledge which is not always recognised as an important tool for solving problems, though it is itself constructed knowledge (cf. Briand,1993, for similar remarks concerning counting strategies). We may also remark that in other fields of mathematics (such as numerical analysis or category theory) schemas and organisational schemes are crucial tools.

Meta-mathematical knowledge was made explicit especially when it was almost algorithmic (see Student 2) or referred to the task (*"What does it mean 'to generalize'"*), but appeared only implicitly when it was complex (actually richer) and nearer to the mathematicians' behaviour (see Thurston, 1994). Summing up the analyses performed, I may say that, concerning meta-mathematical knowledge, shared explicitable knowledge was much narrower than the actual knowledge used globally by the group. I found that eight students referred explicitly to methods for solving problems of this kind, but, to take an example, "organisation of data" was never mentioned even in partial explicitations of methods though it was a key strategy for four students and useful for three of them. Only one of the fourteen students (Student [12]) seemed to have no idea of possible strategies for solving problems of this kind: she seemed lost, mixed up different steps undertaken and produced several unfinished propositions. For Students [1] and [13] ("proof group") I detected very rich implicit meta-mathematical knowledge about how to solve the problem.

The implicit problem-solving methods I could detect globally were: change of representation; interpreting calculations in words and vice versa; visually organising data and calculations, up to a geometrical regularity. I could also detect changes of mathematical frames: arithmetic, algebra, series, etc: this is common in the process of proving for mathematicians.

4.3. Algebraic-syntactic or semantically based steps of reasoning

I have listed numerous breaks during calculations, which were needed to re-interpret the mathematical content of calculus in words. This can be seen as a sign of the primacy of semantical content over algebraic calculation during the process of conjecture and proof construction. As an example, we can consider the need of Student [1] to express algebraic propositions in words when seeking to recognise possible conjectures. This attitude displays the search for a semantically consistent

grasp of the algebraic signs. We can interpret it by saying that constructive work in mathematics cannot evolve only within formal expression.

On the other hand, if we observe the students who did not express the results of their calculation in words richly (five of the fourteen students), three (Students [3], [4], [12]) are in the "failure group" of five students and two ([2] and [13]) in the "conjecture group" of nine students. So the majority of the "conjecture group" (seven out of nine) needed semantic interpretations to pursue their work. I recall that Student [2] did not recognise the strong result obtained, and that [13] was confused in expressing his conjecture - it was not clear to this student what was proved by the calculation.

4.4. Proof as product and proof as process

Let us compare two examples that are representative of some others in the whole group of 43: in the first, proof as a product is close to proof as a process, while in the other the distance is very great.

Student [2] is considered skillful (good notes, etc), but sticks very closely to her explicit method and her presentation is very close to that of a formal presentation. This approximation to a formally correct mathematical text (cf Hanna, 1989) bears negative consequences on the productivity of the student's work: her research is linear and no change of strategy is found at any level. There are long repetitive arithmetic calculations, quite astonishing for the only student in the group who usually managed algebraic tools very well; more remarkably, the student arrives algebraically at a strong conjecture and interprets it in words as much weaker. And finally she does not produce a complete proof.

Analysing the text of Student 1, we can observe frequent changes of strategy, organisation of data and calculations, as well as a frequent effort to interpret in words. This variety, this need for change might help technically, but these were also "interpretation" efforts. They helped understanding and often stimulated the developement of new ideas. This could be called a "transformational reasoning attitude" (see Simon, 1996; Harel and Sowder, 1998). Some of these very useful forms disappeared in the final draft of the proof (**P**), where the logical link between the propositions became a priority. In addition, justification of the research method disappeared from the product (while examples of the interwoven presence of meta-mathematical arguments in mathematical reasoning were frequent in the construction stage). Her conjecture is strong and her proof is almost complete.

5. Conclusions

We have seen that important reference knowledge remained implicit in the students' proving processes and that some of the different references concerned the content, while others related to the meta-knowledge about the activity to be performed. We have also seen how non-standardised, appropriate representation of explicit reference knowledge had an important role in the students' performances. We have seen that when elaborating a productive process many students found syntactic arguments insufficient, and so semantically-rooted arguments became critical. Finally, we have collected some experimental evidence about the negative

consequences of subordinating the proving process to the formal requirements of proof as a final product.

As concerns the educational implications of the analysis performed in this paper, it can be argued that formal proof (which is sometimes imposed or proposed to students of any school level as a rule of construction of mathematical proof: see Hanna, 1989) is very distant from the effective activity of conjecturing and proving. This is true even for undergraduate mathematics students facing a new, challenging situation. Furthermore, the effectiveness of their activities seems to depend on intellectual qualities that are fully developed even during ordinary, demanding argumentative activities other than proving.

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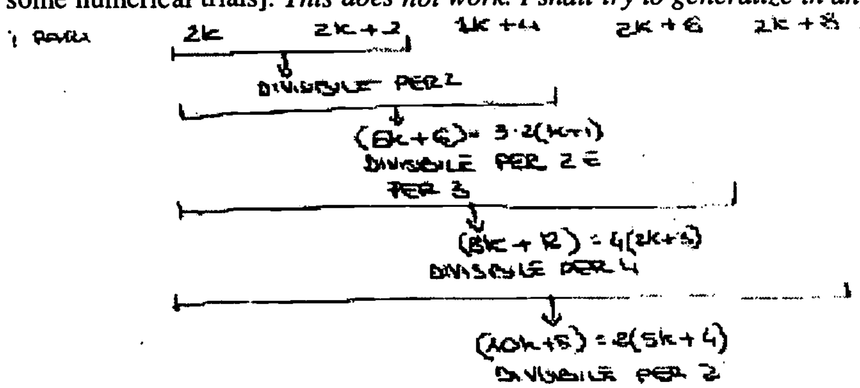
ANNEXE.

DOC.1: Excerpts from the text of Student [1]; it contains seven large, spatially organized pieces, like the two reported below, and many arrows, connecting lines and encirclings.

"I have some difficulties in understanding in what direction I must generalize. It might be: 'by adding two odd or even consecutive numbers I get a number divisible by 4' [she performs some numerical trials]. *This does not work. I shall try to generalize in another way:*



I was looking for something that could help me [...] but I got nothing.

[other trials, with a rich spatial organisation: two consecutive even numbers, two consecutive odd numbers - here she gets divisibility by 4; then three, four, five, six, seven consecutive odd numbers. By performing calculations, she gets the following formulas: *3(2K+3); 8(K+2) 10K+25=5(2K+5); 12K+36=12(K+3); 14K+49=7(2K+7)*]. *Is the result of the addition of n consecutive odd numbers (n odd) divisible by n? (2K+1)+(2K+3)+..(2K+ What must I put here?*

(E)

Vediamo negli esempi:

$$\underbrace{(2k+1)+(2k+3)+(2k+5)}_{3\ \text{numeri}} \quad (3\cdot 2)-1$$

$$\underbrace{(2k+1)+(2k+3)+(2k+5)+(2k+7)}_{4\ \text{numeri}} \quad (4\cdot 2)-1$$

$$\underbrace{\qquad\cdots\cdots\qquad +(2k+9)}_{5\ \text{numeri}} \quad (5\cdot 2)-1$$

[she performs an unsuccessful trial by induction; then she considers n numbers in general]

n numbers: (2K+1)+(2K+3)+...(2K+(2n-1))=2nK+1+3+5+...+(2n-1)=2nK+ (I am thinking of the anecdote of "young Gauss":

(F) *it makes* $2n.n/2=n^2$*)* $= 2nK+n^2= n(2K+n)$ *OK!!*

[Trials performed by applying the preceding formula $2nK+n^2$ in the cases n=2, n=4, n=6, n=8: she gets: *4K+4 divisible by 4; 8K+16 divisible by 8; 12K+36=12(K+3); 16K+64=16(K+4) divisible by 16*]. *Then if I add n consecutive odd numbers (n even), I get divisibility by 2n. Let us try a proof:* **(P)** *(2K+1)+(2K+3)+...(2K+2n-1)=2nK+(1+3+....2n-1)=2nK+(2n.n)/2=* $2nK+n^2$
[....]= 2n(K+n/2); n even implies that n/2 is an integer number: so I get divisibility by 2n. [...]

DOC.2: Excerpts from the text of Student [2]; spatial organization is almost linear, like that in the following trascript.

Student [2] starts her work by checking (on numerical examples: 3+5; 5+7; 101+103) the validity of the given property, then proves it. Then she writes: *"When I must tackle a problem, I try to see how it works in particular cases and then I generalize, as I have done in this case - although I knew the solution. I reason in this way because the particular case allows me to understand better how I can reach the solution of the problem in general (and this method works even when I do not know the solution). Thinking in arithmetic terms and then in algebraic terms helps me to solve the problem. For the original property the generalization comes fairly automatically, because* [she explains why in detail].

What does it mean 'to generalize' ? It means considering a property in which there are some closed variables (two odd numbers, or divisibility by 4) and getting a property in which variables are open. I change the number of odd consecutive numbers to add. For instance, I consider 3 [crossed out] *4 consecutive odd numbers 2n+1, 2n+3, 2n+5, 2n+7 and make the addition: 2n+1+2n+3+2n+5+2n+7=8n+16=8(n+2)=4(2n+4). Then I find a number that is divisible by 8, so it is divisible by 4. I perform the addition of 6 consecutive odd numbers* [similar calculations]*=24n+37=6(2n+6).*

Then I find a number which is divisible by 12, so it is divisible by 6. I try with 8: [similar calculations]*=8.2n+64=8(2n+8) Then I find a number that is divisible by 8, so it is divisible by 4. Following my reasoning, for an even number K of odd consecutive numbers I get: 2n+1+2n+3+....+2n+15+....=K(2n+K)=2K(n+K/2); but K is an even number, so it is divisible by 2 and (n+K/2) is an integer number. Then 2K is divisible by 4 (because K is odd). So I have found that the given property is still valid if I add up an even number of odd consecutive numbers.*

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Students' Views of Learning Mathematics in Collaborative Small Groups

Julie-Ann Edwards and Keith Jones

The University of Southampton, United Kingdom

Approaches to mathematics teaching which offer an alternative to stating facts and demonstrating procedures have been criticised for undermining the base for teachers' sense of their own effectiveness. Data from an ethnographic study of the classroom practice of an experienced teacher of mathematics who has developed an inclusive (or emancipatory) pedagogic approach indicate that while establishing collaborative groups in the classroom may take some time, students across the attainment range come to appreciate the effectiveness and efficiency of working in such a way. This is in some contrast to research findings about using cooperative groups, a quite different method of teaching. Such findings may support other teachers of mathematics developing an alternative pedagogic approach.

Introduction

The ways in which the actions of the teacher impact on the learning of the students in their class is reasonably well-documented, at least in general terms (see Brophy 1986 or Sylva 1994 for reviews). The development in mathematics education of a model of inclusive pedagogy (Murphy and Gipps 1996, Solar 1995) entails the teacher employing such actions as open-ended, problem-based learning within collaborative small groups. This pedagogical approach is designed with the intention of securing the success of all pupils in mathematics.

Such an approach is quite different from what Smith (1996) calls teaching by "telling", where the teacher's main role is stating facts and demonstrating procedures. Smith argues that teaching by "telling" provides a clear-cut basis on which teachers can build a sense of efficacy, the belief that they can affect student learning. Basing teaching on "telling", Smith suggests, builds a sense of efficacy for teachers by defining a manageable mathematics content and providing clear prescriptions for how to teach that content. In Smith's terms this means that adopting an inclusive pedagogy "undermines the base for teachers' sense of efficacy that teaching by telling provides" by de-emphasising "telling". This suggests that research is needed on how teachers who have adopted an inclusive pedagogic approach build new foundations for their sense of efficacy in teaching mathematics.

The research results presented in this paper may contribute towards what Smith has called a central question for empirical studies of mathematics teaching: how teachers who have moved away from teaching by telling are able to "reconceptualise their causal agency in teaching mathematics". The conclusions also point to what might be a fundamental difference between collaborative and cooperative group work in mathematics. The data come from a collaboratively designed and carried out

ethnographic study of the classroom practice of an experienced teacher of secondary mathematics (see Edwards and Jones, under consideration). The aim of this component of the study was to document the views and opinions of students who had experienced collaborative small group work as a means of learning mathematics through being taught by a teacher who had an inclusive pedagogical approach. In this paper we show how well the full range of students understood the effectiveness and efficiency of collaborative small group work as a means of learning mathematics. Yet, it seems, such understanding took some time to develop. These findings may prove useful for other teachers of mathematics seeking to adopt an inclusive pedagogy by suggesting a basis upon which they can judge their efficacy.

Theoretical Framework and Related Research

In attempting to understand the complexities of learning in schools, knowledge of the student perspective has come to be seen as crucially important. As a result, children's understanding of classroom processes and their own role in learning have become an area of increasing study (for examples, see Brown 1995, Christou and Philippou 1998).

An inclusive (or socially-just or emancipatory) pedagogy is being developed from work in feminist and other emancipatory endeavours. With such an approach, the teacher is intent on recognising and valuing a plurality of forms of knowledge and ways of knowing (Becker 1995, Povey 1996, Solar 1995). In the mathematics classroom, this might entail using open-ended, problem-based learning based on social and environmental curriculum contexts using collaborative team approaches within a diversity of teaching and assessment methods.

Some of the theoretical basis for this pedagogic approach comes from the socio-cultural, Vygotskian field. For example, collaborative group work, in which students work jointly on the same problem at all times, is linked with ideas such as situated cognition, scaffolding, and the zone of proximal development. As Damon and Phelps (1989) make clear, this is fundamentally different from cooperative learning which refers to distinct principles and practices such as specific role assignments in a group, and goal-related accountability of both individuals and the group.

A good deal is known about cooperative small group learning (for reviews, see Good, Mulryan and McCaslin 1992, or Cohen 1994). Much less is known about collaborative small group work (Lyle 1996). As a result, little has been reported about a range of issues such as how the composition and dynamics of groups affects their ability to function effectively (for a recent report, see Barnes 1998), or whether the students themselves find it an effective way of working. What is known is that the composition of collaborative groups needs careful consideration, and that there is a vital role for the teacher in establishing collaborative group practice, planning such work, and choosing and structuring appropriate tasks.

The study reported in this paper was designed to elicit the views on collaborative group work from secondary school students who had been taught for varying lengths of

time (from two to four years) by a teacher who had developed an inclusive pedagogical practice. A study of students' perception of *cooperative* small group work in mathematics by Mulryan (1994), which involved interviewing students in secondary mathematics classrooms, was designed to gauge the consistency of their understanding of the processes of cooperative work with that of their teacher. Mulryan found that with cooperative group work the perceptions of high achieving students were more in line with those of their teacher than those of low achieving students. Such a finding might suggest that *cooperative* group work could *increase* the separation between high and low achieving students, a possibility implied in other studies of cooperative learning (Good, Mulryan and McCaslin 1992 p172-173 and 176-177). One aim of the study we report in this paper was to examine the perspectives of both high and low achieving students who had experienced *collaborative* group work in secondary mathematics for a considerable period of time to see whether there was a difference in their perceptions of working in such a way.

Methodology

An ethnographic case study using semi-structured interviews was most suitable for this research for two reasons. First, it allowed the students to say what they wished about their experiences of collaborative group work within the framework of the interview schedule (Hammersley and Atkinson 1995 p25). Secondly, semi-structured interviews are known to be suitable for gathering information and opinions and exploring people's thinking and motivations (Drever 1995). Strict procedures were adopted for the interviews in order to minimise any potential bias introduced by the interviewer.

The sample

A random sample of seven students were chosen for the study, selected from the classes of a teacher who taught in a UK inner-city comprehensive secondary school whose mathematics results in national testing were approximately in line with the national average. The classes from which the students were chosen were a Year 11 low attaining class (students aged 15-16) who had experienced small group collaborative work in mathematics for the previous four years, a Year 10 high attaining class (students aged 14-15) who had experienced small group collaborative work for the previous three years, and a Year 8 middle attaining class (students aged 12-13) who had experienced two years of small group collaborative work. The seven students were selected in the following way: two from the low attaining Year 11 group, three from the high attaining Year 10 group, and two from the middle attaining class Year 8 class (attainment was defined by the school in terms of performance on standardised non-verbal reasoning tests). All the students had been taught by the same mathematics teacher throughout their experience of collaborative group work in mathematics.

The interview

An interview schedule based on the headings used by Mulryan (*ibid*) was utilised as a set of general prompts. Questions were based around the following pupil perceptions:

- perceptions of the purpose and benefits of collaborative small group work in mathematics

- perceptions of teacher expectations for appropriate student behaviour during small group work
- perceptions of the characteristics of small groups that are important for successful groups
- perceptions of the extent to which individual and group accountability exist in small groups
- perceptions in relation to the stability of membership of small groups

The opportunity was also offered to the students for more open comment on their experiences of collaborative small group work.

Analysis of data

Following transcription of the audio tapes, each response was systematically coded for a particular category or categories. These categories were developed in an on-going way as new student respondents contributed different categories until there was a stable set of categories. This process of grounded theorising was necessary as the sample size was too small to use the particular categories devised by Mulryan, who, even with a sample of 48 students, had no more than 5 responses in any one category. As part of this analytical process, some categories were grouped to reflect similar themes.

The following grouped categories were amongst those identified from the interview transcripts:

- Benefits of working together/ collaborating/ working as a team/ working as a group. This theme was evident in all seven respondents' descriptions of their experiences of collaborative group work. For example, R (low attaining Year 11) said "I think its really good, because we're able to work ... as a team ... you just understand more about maths than you do just by writing down on pen and paper". S (high attaining Year 10) said "you might only look at a problem one way, but ...if you give lots of different people a problem, and they look at it in .. different ways". V (middle attaining Year 8) said "it's lot easier to work in a group because you can help each other and you can find out the answers and make sure yours are right".
- Putting ideas together/ contributing/ using different skills (described as a process). As for the theme above, this was widespread throughout the transcripts. R (middle attaining Year 8) said "you put all your ideas together, and by putting everyone's ideas together, you come up with good ideas and just get good knowledge". R (low attaining Year 11) describes a similar experience, "and even if one person did say ... this is the right answer, we wouldn't just write it down, you'd, you know, make it more deeper and everybody'd put more to extend the answer". J (high attaining Year 10) related that "K came up with an idea once, and then we .. started working on that, and then other people ... put in other ideas on top of it, so we were always building up".
- Listening to/ respecting others in the group/ sharing knowledge.

This theme is distinct from merely recognising the skills offered by others. It is described by R (middle attaining Year 8 class) in the following way: "We can all listen to people's ideas, which I think is good and ... we all bring up our own ideas," and in the high attaining Year 10 class by S: "people come up with different ideas ... and you get to explore other people's ideas which helps". Z (low attaining Year 11) said "someone would say [something].. and then we all would .. put our different words in and talk about it".

- Confidence building/ feeling successful/ being motivated

  Some pupils, including the higher attaining students, described collaborative group work as a vehicle for increasing their mathematical confidence. For example, L (high attaining Year 10) said "I think in my case, ... if I know someone else thinks the same thing, I'm more confident about what I think". There were several instances of pupils describing the experience of group work as making them feel more successful. J (also high attaining Year 10) explained "I just think its better than working by yourself, really. I think you learn a lot more". Pupils also seemed to find the group dynamics a more motivating learning environment. Z (low attaining Year 11) affirmed "we just didn't want to leave it ... we used to stay behind lessons ... we wanted to get the work done ... I prefer doing maths ... with group work ".

- Friendship/ knowledge of collaborators/ stability of groups.

  Questions about group structure revealed that all the pupils believed that their performance in a group was positively affected by working with others who were well known to them. Friendship seemed to provide successful working relationships in the view of all those interviewed. V (middle attaining Year 8) explained "If you're not friends with somebody, ... you might not get along with them, and they might start getting into a bit of an argument about the answers". R (low attaining Year 11) said "no others could be as good as working with some friends". S (high attaining Year 10) said "well, obviously, you've all got to get on quite well, you've got to know ... I think it's easiest if you know each other first".

- Speed/ volume of learning.

  Students across the age and achievement range thought that collaborative learning in small groups allowed learning to happen more quickly and that they could learn more. J (high attaining Year 10) summed this up: "I think you learn a lot more, ... I think if people ... work together you can get a lot more done and you ... understand a lot more ... I think its probably quicker, because if you're working by yourself, it's you that does all the work,". R (middle attaining Year 8) said "it's easier if you do group work because you can get through it quicker and .. get to know a lot more". R (low attaining Year 11) offered a more reflective comment "I don't think it's quick or slow, it's in the middle, but because it's like that, you get a deeper meaning, you know what you're doing, you don't just skim it over the top".

Other categories of student response included: helping one another, thinking hard, enjoyment, autonomy and independence, and awareness of the possibilities of distraction. All the students were also aware of the expectations of the teacher in terms of what was appropriate for successful collaborative group work in mathematics.

Discussion

This analysis of the interview transcripts for the categories described above allows some comment to be made, both on emerging global patterns in the student responses and on local patterns within groups. Examples of such local patterns relate to the age of the students and the length of their experience of collaborative small group work.

Overall, the full range of students in this study seemed to recognise the benefits of collaboration. They realised the necessity of listening to one another, felt collaborative working made them confident and successful, and judged that they learnt more mathematics more rapidly by working in that way. There also appears to be clear indications that working with friends, that is working with those with whom you get on well, is important. It may be that this helps with the sharing and respecting of each others ideas and that, in the end, this helps with learning. These benefits of working with friends are noted by Zajac and Hartup (1997) in their review. Whicker and Nunnery (1997), in their study of cooperative groups in secondary school mathematics, found that their students "disliked having groups pre-assigned and permanent, and suggested alternating group membership".

Yet the responses of all the groups were not identical. In particular, the responses of the younger students from the Year 8 class, who had only experienced collaborative small group work in mathematics for two years, were different in several respects. These students found it more difficult to articulate their perceptions of collaborative group work. Overall, their responses during the semi-structured interviews were much shorter, less reflective, and demonstrated less understanding of the pedagogic process, than the older students. In addition, the younger students seemed more orientated towards outcome, rather than process or understanding. For example, student V, middle attaining Year 8, said that working in a group means "you can find out the answers and make sure yours are right", and, later in the interview, that it was more enjoyable to work in a group because "you can get more accurate answers from it".

Such responses from the younger students, and the contrasting answers from the older students, may indicate that, in addition to maturation, it takes quite some time for the teacher to establish fully collaborative groups. The research on cooperative groups has already established that simply placing the students in groups does not mean that group work will take place. Indeed a frequent complaint about common practice in UK primary schools is that the pupils are arranged in groups in the classroom yet they do essentially individual work. Training in cooperative working was found necessary for successful cooperative group work, and research on collaborative learning suggests that for collaborative group work some form of teaching of relevant skills is required (Gillies and Ashman 1996). A range of other factors is likely to influence the

successful development of collaborative group work, including, in secondary schools, the experience of the students in other curriculum subjects.

Finally, unlike Mulryan (1994), we found no difference between the perceptions of high attaining students and those of low attaining students. All the students in our sample felt that collaborative group work had a positive effect on their rate of learning and depth of understanding. The reason for this difference, however, may not lie solely with the grouping structure. In our study it is likely that the philosophical and epistemological stance of the teacher, in developing a strongly inclusive pedagogy, is the influencing factor.

Concluding comments

Smith (1996), in calling for research on how teachers, who have moved away from a pedagogic approach based on "telling", build new foundations for efficacy in teaching mathematics, suggests that studies should focus on "how teachers themselves *see and understand the effects of their practice on students*" (emphasis in original). In the case of the teacher in our study, one of the ways the teacher judges her efficacy is in terms of the success of the collaborative group work for all her pupils. Hence our focus in this paper on the student perspectives of working in collaborative small groups.

It is not the intention of this study to produce a typology of categories of student responses, nor to test a theoretical model. Our aim has been to describe the perspectives of secondary school students who have had considerable experience of collaborative small group work in mathematics. It is, in both the sense of the case size itself and in the sense of the time scale used, a "microethnography" (Hammersley and Atkinson, *ibid*, p 46). The lack of comparative cases "necessary for developing and testing an emerging set of analytic ideas" (Hammersley and Atkinson, *ibid* p205) is one difficulty of using a naturalistic situation to study. Furthermore, the data comes from one UK school and hence its generalisability is greatly limited.

Nevertheless, we hope we have provided a useful contribution to research both on collaborative group work in mathematics and on inclusive and emancipatory mathematics pedagogy. Such a pedagogic approach, given its coherent philosophical and epistemological basis, provides the teacher in this study with a strong anchor with which to judge her efficacy.

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PLAYING OR TEACHING?

THE INFLUENCE OF DYAD FRAMEWORK ON CHILDREN'S NUMBER EXPERIENCES IN MATHEMATICS GAME PLAYING AT HOME

Fiona R. Ell and Kathryn C. Irwin

University of Auckland

Families' use of a Maths Games Library was investigated using a socio-cultural framework. The Maths Games Library was a home-school communication project, initiated by parents and teachers in a primary school of 421 pupils. Analysing the interaction between parents and five-year-olds on audio tapes revealed two frameworks for game playing activity: a game-focused framework and a mathematics-focused framework. The characteristics of these frameworks resulted in differential experiences with number for the children playing the games. Teacher-pupil dialogue suggested an evaluative framework. The study highlights the significance of activity as the agent of development, indicating that the nature of interaction between players in a game can make a difference to children's learning in mathematics.

Introduction

A Maths Games Library is used as the focus of this study. It is a collection of games and activities which children take home weekly to share with their parents. This Maths Games Library was established in 1996 at North Primary, a predominantly white, middle-class school in Auckland, New Zealand. A committee of parents and teachers worked together to make the games, and a class set is used in every classroom from Year 1 to Year 3. This study was conducted in 1998 and the use of the Maths Games Library had become an established routine for parents and teachers. North Primary emphasised home-school communication and was eager to meet parents' needs for information.

Parents of children in New Zealand primary schools receive a reading book to share with their child each day. This provides an important link between home and school. The Maths Games Library was intended to provide a similar link for communication about children and their mathematics learning. What was unknown was what the parents and children did with the games in their homes. As a joint activity between an adult and a child, the game playing setting appears to have rich potential. This study aimed to look beyond whether or not the games were used at home and consider what happened between the parent and child as they used the games, using a socio-cultural framework.

Lerman (1998) describes the use of a cultural, discursive psychology to view mathematics education. He presents the 'zone of proximal development' as a research tool "...for analysis of the learning interactions in the classroom (and elsewhere)" (Lerman, 1998, p. 71). This study employs Lerman's proposed psychology in considering the interaction of parents and children in the zone of proximal development. Lerman (1998) hints at this in his addition of 'elsewhere' to places for analysis, but describes mathematics education as beginning in classrooms. Literacy research conducted within the socio-cultural framework outlined here suggests that the family is a powerful site for the development of emergent understandings and for the support of learning after entry to formal schooling. This study explores that notion, using a school-based initiative to explore children's experiences at home.

A theoretical viewpoint which highlights the importance of social interaction in development sees this game-playing setting as a potentially critical site for the formation of concepts and the growth of cognition. It involves an expert and a novice working together to construct meaning from materials. Language is used to establish intersubjectivity and to construct new ideas. Parent and child work together, co-constructing understanding through the progress of the game. The interaction of the three key elements - parent, child and game - result in a unique pattern of activity. This activity is situated within the home organisation and routine, and may constitute a family practice.

The social interaction between the players may be internalised as personal understanding through the game playing (Vygotsky, 1978; Wertsch & Stone, 1985). The role of participating in activity as the agent of learning is defined by Rogoff (1995), who suggests that settings such as the game playing described here can be seen as guided participation. This guided participation leads to participatory appropriation by individuals, who each form their own ideas about the activity based on their experience. Learning can thus be seen in the activity, rather than within the child. Activity therefore becomes the unit of analysis.

The game playing activity described in this study has several important features in terms of this framework. The presence or absence of an adult expert will affect the way the learning is mediated. There is potential to observe the transfer of responsibility from the expert-adult to the novice-child, or to see the use of scaffolding to support the child's learning. Establishing intersubjectivity through activity and language, as discussed by Lerman (1998) should be evident. Situating the game-playing within the home may have a key influence on how it proceeds; using the same materials at school might invoke a different set of rules.

Abreu (1998) explores this in considering Brazilian children and their home and school mathematics. While she focuses on the mathematics of 'real life' outside school, rather than on game playing, within the culture under investigation in this study game playing is a common form of interaction between parents and children (Wylie & Thompson,

1998). Abreu (1998) notes the diversity of the children in their engagement with home mathematics and their success in mathematics in school, despite apparent similarities in the community. The study presented here begins to look for the origin of such differences within activity; in culturally and situationally determined interaction patterns and systems, which may critically influence children's mathematics learning.

Analyses of adults and children working together suggest that contingent responsiveness may be a key element in exchange which results in development. Pratt, Green, MacVicar and Bountrogianni (1992) view parents solving division problems with their children. Although more directive than teachers, results show parents as responsive to the child's level of ability and to the difficulty level of the material. Lehrer and Shumow (1997) investigate the alignment between home and school after teachers have completed inservice to reform practice. While parents agreed with many of the new techniques, as viewed on video, they did not use them with their children. They provided more direction and instruction while solving word problems than the children's teachers did. Responsiveness to everyday mathematics has been found to be important in developing mathematical understandings. Young-Loveridge (1989) describes a literacy-oriented family with a child who is making slow progress in mathematics. This family had focused on literacy to the exclusion of numeracy, and did not utilise the opportunities presented by the environment or the child. Observing interaction rather than using an interview (Young-Loveridge, 1996) revealed that very little contingent responding about mathematics occurred in the context of cooking. This highlights that it is not involvement in cooking which builds mathematical understanding, but having mathematical ideas posed and responded to in this context. This may hold true for the Maths Games Library - it may not be using the materials that promotes understanding, but the interaction that takes place. Examining the nature of this interaction is therefore crucial.

Methodology

This study used activity as the unit of analysis, as proposed by the socio-cultural framework (Wertsch, 1995). The activity of game playing was captured using small tape recorders, which were unobtrusive and resulted in collection of the verbal interaction between players. This interaction formed the basis of the analysis, being the evidence of activity and the proposed mode of learning and development, as this social interaction becomes internalised. While studies of problem solving (Lehrer & Shumow, 1997; Pratt, Green, MacVicar & Bountrogianni, 1992) have used hierarchal analysis of support in order to show scaffolding, a game does not necessarily have this structure. Thus categories of utterance, with distinct purposes, were derived, rather than a measure of directiveness.

Method

Participants

Thirteen families recorded their game playing. Five of the target children were boys and eight were girls. The study child's place in the family varied from being an only child, to being the sixth of six children. Only two of the children had no pre-school experience, with seven children attending private pre-school in addition to public kindergarten. The children were all aged five at the time of the study, and had been at school no longer than two terms prior to the study.

Materials

Two games were selected from the Year One Maths Games Library. 'More Dots' was selected as an easy game, and 'Oops!' as a hard game. The games were brought home in a brightly coloured bag, complete with equipment and instructions. Isomorphs of the games were made for play with the teacher. These used dice instead of cards, but followed the same rules.

Families were provided with a small tape recorder and audio tape on which to record their game playing. They also completed a questionnaire after playing the game.

Procedure

The games were sent home as part of the usual routine for sending home games. Tape recording equipment was added to the bags of the target games, and the researcher specified which families should receive the games each week. The teacher played the games with two of the target children during a mathematics teaching session. This was recorded in the same way as the families'.

Data Analysis

Both games proceeded by the players turning over a card. This was used as the unit of interaction, and utterances after each card turn were coded and numbered. Contributions were numbered and attributed to the parent or the child. Thus the interaction could be reconstructed from the coding, in terms of the purpose and order of the utterances. The length of interaction after a card turn could be determined, as could the format of the interaction - be it turn-taking or several utterances by one person.

Results

Analyses of the families' tape recordings revealed two distinct patterns of interaction, or frameworks. These are summarised in Table 1. A case study example illustrates that the tone and content of the exchanges while the game progressed also reveal the characteristics of these two frameworks. These examples are from the same tape, where the child plays the game with her mother and her father. Her father is in a game-focused framework, her mother in a mathematics-focused framework. In game-focused play over half the time was spent playing in silence. In mathematics-focused play turns are punctuated by comment and discussion.

Game-focused play - Examples of interaction

Example 1.

P: What do you think this helps you with T.? Does it help you with counting?

C: Yup.

Example 2.

C: Dad, this is how you learn numbers isn't it?

P: (no response)

Example 3.

C: Two and two. Look Dad!

P: (no response)

Example 4.

C: Who wins that one?

P: It's two pairs.

These examples indicate the type of interaction occurring during game-focused play. These examples represent the additional mathematics talk undertaken by the father and daughter. The father attempts once to link the game as a whole to his daughter's mathematics learning (example one). This is done in an abstract way, rather than as part of the game play. T. later reflects this back to her father by asking if this is how you learn numbers (example two). This is not picked up on by her father. T. attempts to initiate dialogue in the two other examples. Both instances go no further than the examples given here, despite openings for further dialogue, such as what two and two might be, or what a 'pair' is. From the laughter and other talk present on the tape, it is clear that both father and daughter are enjoying the game, and enjoying it within a game-focused framework.

Mathematics-focused play - Examples of interaction

Example 1.

P: How come I win that?

C: 'Cause you got 3 and I got 1.

P: And how many more do you need to make 3 dots?

C: 3 more.

P: No.

C: 2 more.

P: Good girl. So if you had two more dot, ow many would you have?

C: 3

P: And how many would I have? Three as well. I've got three as well.

Example 2.

P: You've got to say how many there are

C: 4.

P: Four dots.

Example 3.

P: How many dots does that make altogether?

C: Five.

P: So 4 plus 1 equals

C: Five.

P: OK. So four plus one equals five. Right? So if I take one away, how many does that leave?

C: Four

P: If I take four away, how many does that leave?

C: One.

These three examples from the mathematics-focused play of the mother and daughter are typical of a pattern of exchanges which continue throughout their playing of 'More Dots'. The mother poses questions for T. to answer, using the cards turned over. These questions relate only superficially to the playing of the game, but use the cards to expand T.'s ideas about addition and subtraction. The mother then alters the rules of the game, to make it the lower card that wins on each turn. This leads to discussion about what 'the lower' is, and who will now win. T. and her mother work towards clarifying the language and the concept behind the idea of the 'lower number' over several turns. This drive towards making the game more difficult and drawing the mathematics out of it is not seen in game-focused play. The mathematics-focused play described here results in a different experience for the child.

Table 1: Characteristics of the game-focused and mathematics-focused frameworks.

Game-focused	Mathematics-focused
There are may one-contribution turns and fewer long turns	There are a greater proportion of long turns and fewer one-contribution turns.
Few questions are asked.	Many questions are asked.
The inherent mathematics content is covered.	Additional mathematics content is covered.
Silent turns are common.	Few, if any, silent turns.
There is a low percentage of turns with contingent contributions.	There is a high percentage of turns with contingent contributions.
Players take their own turns, doing their own 'work' on their turn.	Parents ask children to help them with their turns.

This overall pattern reveals a trend in parent support that is not obvious in closer analysis. The two target games were chosen to be of different difficulty levels, but in practice the children found them both easy. There is thus little failure by the children, and therefore no chartable pattern of adjustment in adult level of support. However, parents who rated 'Oops!' as less than very easy for their child on the questionnaire, all adopted a

mathematics-focused framework for their interactions in this game. While some parents adopted the mathematics-focused framework in games that their child found easy, no-one adopted a game-focused framework with anything other than a very easy game. Twelve playing sessions followed a mathematics-focused framework, and thirteen followed a game-focused framework.

In the game-focused framework, players simply play the game. Issues which are directly related to the game's progress are discussed. Many turns pass unremarked, or are punctuated by laughter or sighing. There is less discussion, and each turn of the card is more likely to yield a short exchange. Players who are working in a mathematics-focused framework use opportunities to introduce and discuss aspects of mathematics which are not essential to the game's progress. More questions are used, and the playing of the game is commented on frequently. A 'running commentary' develops between the players. Longer exchanges occur more frequently, and play partner's comments are more often followed by a contingent response.

The teacher data is limited, as she was only able to play with two of the target children, but her interaction pattern suggests a third framework - an evaluative framework. It follows the performance of the child closely and includes elements which allow the teacher to view independent performance. The teacher in these interactions does not attempt to teach anything. She follows the lead of the child's responses to explore their knowledge, but does not add to this. Her stance is evaluative, and responsive to the child.

These results suggest that children have different mathematical experiences with these games, depending on the framework selected. Children who play the games in a mathematics-focused framework are exposed to more mathematical concepts, explain their ideas and are questioned more often. Language is used by both parents and child to discuss ideas. Other analyses not presented here suggest that key influences on framework selection are the difficulty level of the game and the child's attitude. Further work with teachers and other groups of parents needs to be undertaken to consider issues of 'fit' between home and school. While games are used here as the 'bridge' between home and school, other issues of cultural, language and experience can be considered in a similar way. What is taken to and from school may be less visible than a maths game bag, but may have important consequences. The use of the socio-cultural framework to view this game playing reveals that important differences in interaction do occur, and that these may impact on children's learning.

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Profiles of Development in 12-Year-Olds' Participation in a Thought-Revealing Problem Program

Lyn D. English
Queensland University of Technology

Abstract

Five classes of 7th-grade students participated in an 11-week program of thought-revealing problem activities, comprising problem-posing and model-eliciting experiences. This paper presents profiles of development of three students who displayed different levels of achievement in number sense and novel problem solving. Among the issues addressed are developments in the students' facility with problem structures, including recognising related structures and posing new problems from given structural elements, students' development of divergent and flexible ways of thinking, and their processes in conceptualising and working a model-eliciting task.

Introduction

The ability to reason and converse mathematically in a range of problematic situations is becoming increasingly important in students' development. As highlighted in the recent *Principles and Standards for School Mathematics* (discussion draft; NCTM, 1998), students need to develop a mathematical disposition to analyse situations more carefully in mathematical terms, to formulate new problems, to explore mathematical structure, to investigate situations, and to make and test conjectures. Such development requires learning experiences which go beyond the usual scenario of students applying learned strategies to produce a "short-answer," predetermined solution to a well-defined word problem. While such problem experiences have an important place in the curriculum, they nevertheless are inadequate for students' problem solving and decision-making in the forthcoming era (Burrill, 1998; Greeno & Hall, 1997).

Thought-Revealing Activities

One approach to redressing this situation is to incorporate "thought-revealing" (Lesh et al., in press) problem activities within the mathematics curriculum. Such activities require students to externalise their thinking and reasoning processes by describing, explaining, constructing, modifying, and refining their mathematical understandings and viewpoints, *while* they are dealing with a problem situation. Thought-revealing activities thus enable us to learn more about the nature of students' developing mathematical knowledge and the thinking processes that produce this knowledge. At the same time, these activities *promote* development because they support the "productivity of *ongoing* learning or problem solving experiences" (Lesh et al., in press).

The thought-revealing activities of the present program encompassed problem posing and model generation. The importance of mathematical problem posing in the curriculum has been well documented, yet research in the domain is limited (English, 1998; Silver, 1994; Silver & Cai, 1996). Likewise, model-eliciting activities have not received the attention they warrant. These activities engage students in important mathematical processes (such as quantifying, organising data etc.) and require them to produce a model that describes certain relationships, patterns, and operations inherent in a real-life situation. A variety of concrete, graphic, symbolic, or language-based representation systems may be used to portray these relationships and patterns.

Model-eliciting activities provide rich learning experiences because they (i) require students to develop explicit mathematical interpretations of meaningful situations, (ii) develop important mathematical understandings (e.g., proportional ideas), (iii) emphasise the kinds of problem understandings and abilities that are needed for success in real-life situations, and (iv) cater for, and promote, a broader range of mathematical competencies (Lesh et al., in press).

Theoretical Framework

The program was designed within a framework that has been developed and refined over several years of research (e.g, English, submitted). This socio-cognitive framework has three main components, as follows:

(i) Understanding and Reasoning (e.g., understanding and utilising problem structures, and recognising related structures; understanding problem design; thinking and reasoning in mathematically constructive ways, including reasoning by analogy (English, in press) and thinking in flexible and divergent ways);

(ii) Mathematical Self-Awareness (e.g., students' perceptions of, and dispositions towards, problems, problem solving, and problem posing; expressing mathematical ideas, opinions, and beliefs; and applying metacognitive abilities such as planning and monitoring one's actions); and

(iii) Participation in Philosophical and Inquiry-Oriented Communities (e.g., where students engage readily in open questioning and mathematical inquiry, participate freely in constructive dialogue and debate, provide constructive feedback on one another's creations, and work collaboratively in group situations; Baroody, 1998; English, Cudmore, & Tilley, 1998; Stein, Silver, & Smith, 1998). Establishing this learning community was considered essential to the program. Such an environment has the potential to motivate students to explore mathematical situations that are intriguing, problematic, challenging, and inviting. Students have opportunities to build on, shape, and modify one another's ideas, to offer and analyze reasons for arguments put forward, and to help one another formulate questions and generate mathematical problems and models (cf. Splitter & Sharp, 1995).

Methodology

A qualitative research paradigm was followed in this study, with videotaping as the primary means of data collection. Small groups of students, individuals, and teacher-led classroom discussions were taped. Iterative refinement cycles for videotape analyses of conceptual change are being used in the data analyses (Lesh & Lehrer, in press). In addition to the videotape transcripts, data sources include students' journals documenting their responses (including their critical analyses of one another's work), other student artefacts, classroom field notes, notes from informal interviews with students and teachers, and feedback on the program from the students and teachers. The components of the above theoretical framework provide the windows through which the data are being analysed.

Participants and Selection of Case Studies

Five classes of seventh-grade students (12 years) from a non-state boys' school participated in the study, which was conducted during 1998. Nine students were selected for case studies. These students were selected on the basis of their responses to assessments of number sense and novel problem solving, which were

administered during the second term of the school year prior to program implementation. The tests were modelled on examples that had been used successfully in previous, related studies (e.g., English, 1997, 1998). The number sense test focused on facility with number and routine computational problem solving, while the novel problem-solving test included problems that required a range of reasoning processes (e.g., deductive, combinatorial, spatial reasoning), as well as general problem-solving processes.

The nine students chosen for case studies displayed one of three types of achievement on these tests: (i) high in number sense, but low in novel problem solving (i.e., at least one SD above/below the mean), (ii) low in number sense but high in novel problem solving, and (iii) high in both number sense and novel problem solving. Prior to commencing the program, each of the nine students was individually interviewed on a comprehensive set of problem activities. A parallel set was presented after the program. These sets addressed several of the components of the framework (e.g., students' attitudes and perceptions, their problem preferences, recognition of problem structure and related structures, problem posing).

Program Implementation

During the first half of the school year, meetings were conducted with the class teachers to discuss the philosophy and content of the program, and to seek their feedback. The program was then implemented in each class during the third and final terms of the school year (1.5 hours per week). Individual work, small-group activities, and whole-class discussions were incorporated in the program. The students maintained journals of their responses.

The program commenced with a strong focus on developing the students' mathematical self-awareness, which was maintained throughout. The first two weeks of the program were devoted to problem exploration and general discussions on problems, problem solving, and problem posing. Problem sorting (according to structural similarity) was also included. These beginning weeks were designed to improve students' understanding of, and attitudes towards problems and problem solving, to improve their confidence in dealing with mathematical problem situations, to help them become more creative and flexible in their approaches to problems, and to increase their confidence in, and willingness to talk openly about mathematics.

During weeks 3-6, the students explored the structures of a wide range of problems, identified similar structures, discussed how structures differed, why some structures were more difficult than others and so on. The students then posed new problems based on familiar structures (use of analogical reasoning), and progressed to posing problems from selected structural components (i.e., from open-ended statements, such as, "Mrs Mack has a blue pot, a red pot, and a green pot for her new plants. Her new plants are a rose bush, a gardenia, and a gerbera," and from other sources, such as travel brochures and newspaper items.) An important part of the students' problem posing was completing critical analyses of one another's problems. These analyses were followed by problem improvement and extension.

The remaining weeks of the program engaged the students in model-eliciting activities, where they worked on real-life situations (e.g., those reported in the print

media) to construct models that would deal with these situations mathematically. Three activities were included, with the first one, "The Good Old Daze," based on a timely newspaper report comparing the cost of living today with that of a previous era. Related tables of data on the costs of everyday items now, and 12 years ago, were also supplied for the students. Based on their discussions and research, the students were to write a short article for their school magazine where they debated whether or not 12-year-olds have a better standard of living today than their counterparts of a selected bygone era. In supporting their arguments, the students were encouraged to make use of effective representations (e.g., tables, graphs, student-generated mathematical procedures/rules). The main reasoning pattern required in working this activity is proportional reasoning, of the form, $a/b = c/d$.

After deciding on a bygone era, the students had to determine which issues to address in framing their argument. They then had to determine how to examine these issues. In doing so, the students were faced with decisions such as, "Which items will we consider?" "Why?" "What has happened to the cost of items? Have all items increased in price?" "How will we organise our data?" "What kinds of patterns or relationships appear to exist?"

Profiles of Development

Profiles of development of three case studies are addressed here. The students are (1) Nathan, who displayed high achievement in number sense, but low achievement in novel problem solving, (2) Tom, who was the reverse of this, and (3) Homer, who achieved highly in both domains (all names are pseudonyms). Data are drawn from the two sets of problem activities (presented to each student before and after the program) and from one of the model-eliciting activities completed during the program (samples of student artefacts to support the following will be included in the paper presentation).

Nathan (High in Number Sense, Low in Novel Problem Solving)

Nathan's views on his mathematical competence reflected his levels of achievement. For example, Nathan claimed that he enjoys mathematics and thinks "it is fun doing all the sums and working all the answers out," but did not enjoy solving mathematical problems very much, "because I usually get them wrong." Following the program, Nathan stated that he enjoyed the program because "it is very interesting to actually do maths differently." He also considered he was "definitely" better at problem solving after participating in the program, "because I find them easier to do; I have never been really strong at problem solving but it is starting to get easier now." Likewise, Nathan felt he was better at problem posing because he now knows "how to write a question."

Not surprisingly, Nathan preferred problems that involved computations and was adept at doing quite complex problems mentally. Prior to the program, he displayed competence in explaining the structural features of such problems, yet when asked to sort a set of problems, Nathan focused on contextual features. That is, he grouped problems that "asked the same question," irrespective of the operation required for solution. He was, however, able to correctly match the two combinatorial problems. On the post-program activities, Nathan still had trouble

sorting the computational problems into groups of similar structure, and displayed some difficulty in distinguishing addition from subtraction in comparison situations. Nevertheless, Nathan could match the two deductive and the two combinatorial cases.

After participating in the program, Nathan displayed a distinct improvement in his ability to turn open statements into problems, and to model a new problem on a given problem. Initially, Nathan lacked diversity of thought in his problem creations, was unable to pose a division problem, and could not create problems from a travel brochure. After the program, however, Nathan showed diversity of thought in his use of both mathematical structure and context.

Tom (Low in Number Sense, High in Novel Problem Solving)

It was interesting to note that Tom was very confident in his mathematical abilities, despite the fact that his number sense was rather poor. Tom claimed he liked mathematics because, "mostly, I am pretty good at it." In particular, he enjoyed solving mathematical problems the most, but "not ones with fractions." In reflecting on his participation in the program, Tom stated that he had not improved in problem solving because "I already knew all the problems," but felt he had improved in problem posing because he had not done any of this before.

Tom developed his ability to identify and work with problem structures during the program. On the pre-program interview, Tom sorted most of the problems by context (e.g., "They're about collecting."). After completing the program, however, Tom could readily match the corresponding deductive and combinatorial problems, and could clearly justify his actions. Interestingly, though, Tom placed all of the remaining problems in the one group because they all involved operations: "Well, we are given information and you have to work it out--like you have to minus $63 or times it or something like that to get another number that you need, like how much did class 7c raise." It was evident that Tom was sorting the problems according to whether or not computational procedures were needed, as he could clearly explain the structures of each problem and could show how they corresponded. Tom was no doubt influenced here by a previous class discussion on ways of classifying problems.

Developments were also noted in Tom's ability to pose problems from open statements. Prior to the program, Tom had some difficulty in posing such problems, and displayed quite convergent thinking in doing so. On the other hand, although Tom interpreted a multiplication situation in terms of repeated addition, he nevertheless was able to create two different division problems from a given division statement (e.g., *A ruler was 56 cm long. If someone chopped it into quarters, how long would each piece be?*). While Tom was also unable to generate a problem from a travel brochure, he could model a new problem on a given problem (albeit, retaining the same context). After completing the program, however, Tom was readily able to pose problems from open statements, and showed great diversity of thought in doing so. Likewise, Tom could easily pose problems from a travel brochure, and created diverse contexts when modelling new problems on given examples.

It is particularly interesting to note that the problems Tom posed were more diverse and more structurally complex than those generated by other members of his group (e.g., Tom was the only member to incorporate fractional ideas in some of his

problems). In fact, one of the problems that Tom generated from the travel brochure was chosen to be critically analysed by all class members. The nature of Tom's responses reflect previous findings, in which children classified as low in number sense and high in novel problem solving displayed divergent thinking and posed problems that were structurally complex (English, 1998; English, submitted).

Homer (High in both Number Sense and Novel Problem Solving)

Homer was a keen student who adopted leadership and monitoring roles in group activities. However, when his team members would not cooperate, he would withdraw and work independently. Homer was an enthusiastic participant in class discussions on problems and problem solving.

When questioned initially on whether he enjoys mathematics, Homer said that he does not enjoy it all the time, especially "when we are going over something, over, and over again; I don't like that." However, he did like working all types of mathematical problems, but stated that he does not like problems involving fractions, because "I am no good at vulgar fractions. I like decimal fractions though." Homer also considered he was "not very good at spatial problems."

It was pleasing to see Homer express a good deal of satisfaction with the program, stating that he liked the problems because "they are real life and were on paper." He felt more confident in both solving and posing problems after participating in the program, "because I have had a look at the different types of problems. . . and we have all done lots of better problems."

Homer's responses to the problem sorting activity were interesting, as was the case with Tom. Homer was easily able to sort the problems prior to the program, and could justify his method. Following the program, however, Homer sorted the problems into two groups, namely, the deductive examples and those involving operations (including the combinatorial problems where he stated, "You could use a tree to work these out, but it is a lot easier just to times it.") In justifying his grouping method, Homer explained, "I put all these together because you can use a number sentence to work them out." In other words, Homer sorted the problems into routine and nonroutine problems, as did Tom. Nevertheless, Homer was able to identify and explain corresponding structures, improving on his initial performance.

Homer also improved in his ability to pose problems from open-ended situations. Although prior to the program, Homer was able to pose deductive and combinatorial problems from the open statement, "Mrs Mack has a blue pot . . .," he had difficulty in making other open-ended situations into problems. For the problems he could create initially, Homer displayed little divergent thinking. After completing the program, though, Homer was readily able to construct different types of problems from a given open-ended statement, and could clearly explain the problem's structure and its solution. Furthermore, Homer was far more divergent and flexible in his thinking when creating several problems from other open-ended statements.

Model-Eliciting Problem Activity During the Program

Of the three case studies, Homer was the only student who expressed initial interest in working such problems. Like many of his peers, Nathan found it difficult to accept these as mathematical problems, and did not consider himself to be competent in

working them ("because I'm not a very good imaginer"). Tom considered these model-eliciting problems to "take too long, and you have to explain everything, and you have to put it down on paper, which is very boring."

Consideration is now given to the beginning responses of the students to the model-eliciting task, "The Good Old Daze." These responses reflect elements of the modelling cycles described by Lesh et al. (in press). It is only possible here to address the first two phases of problem conceptualisation, in which the students attempted to (1) determine which bygone era to address and the issues to examine, and (2) draw comparisons between selected items of data.

During the first phase, the teams in which Nathan, Homer, and Tom worked, began the task with a good deal of unconnected and, frequently, off-task discussion. The goal of the task was quickly ignored as the students became bogged down with irrelevant detail and procedural considerations. It was also during this phase that the students established roles for each member of the team, as can be seen in the following excerpt from Nathan's and Homer's team (Homer's role is clearly evident.)

Lachlan: *What year do we want to talk about?*

Nick: *Let's talk about last year because that's when I turned 12. I'm older than you.*

Homer: *Let's talk about a reasonable time.*

Alex: *Let's talk about when Lach was born!*

Homer: *<u>When</u> are we going to decide on a time?*

Following further irrelevant discussion, Homer stated: *Order in the court." "What are we going to do? Are we going to do the 1950s or the 1960s?*

Nick: *You make the decision, Homer.*

During this phase, the students also spent time trying to decide which issues they should address in developing their arguments:

Greg: *Let's look at university education, and health too.*

Tom: *OK, or life expectancy.*

Greg: *Do life expectancy.*

Chris: *Like expectancy for a male today is 75, or is it 70?*

Greg: *Tell me what to write down.*

Chris: *Life expectancy for a male is 70. . . . .*

Tom: *So what are we doing? Oh, male expectancy. I could do a bar graph that goes up, to show this* (his intention was to compare it with life expectancy in the 1960s).

Tom: *The male expectancy is one more year than a female life expectancy in 1960.*

When Homer and Nathan's class teacher intervened in the team discussions, the students identified issues such as crime, the cost of basic commodities, and wages as worthy of examination. Using the data sheets provided, the students began identifying various items for comparison, such as the cost of a loaf of bread in the 1985 with the cost today. Discussion then led to the students commenting that, to compare the prices, they needed to look at "the amount of pay." One student further commented that "you have to look at the ratio." The students then resumed their team discussions, which led them into the next phase of conceptualisation:

Alex: *Oh, let's look at the Coke and the lollies!* (off-task discussion followed)

Lachlan: *Men's dress shoes in 1985 were $49.95 and in 1997, $140.*

Homer: *Hey guys! Look at the miscellaneous items. Some of the things have gone down! This one's* (Fuji film) *gone down by a dollar!*
Chris: *CD players have gone down by $300.*
Alex: *A microwave was $359 and now it is $140. . . . well, it's gone down because most of the time when it first came out, everyone thought it was a success and they bought it, but now there is a better range . . .*
Lachlan: *Wow, the US Open tickets were $500 and now they are $2500! That's a really big jump!* ....(Students in other groups also noticed that some prices remained the same.) To this point, the goal of the task was still in the background for these students. They were more interested in comparing prices per se, rather than considering these in relation to the wages of the two eras, that is, the students were still engaged in primitive pre-proportional reasoning.

In sum, the program appeared to be a positive and productive learning experience for the students. Given that most of the activities were new to the students, it would seem that such a program can be effective in fostering some of the understandings and processes required for their success in the coming years. At least this appeared to be the case for the present sample of students.

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THE RELATIONSHIP BETWEEN PROFESSIONAL KNOWLEDGE AND TEACHING PRACTICE : THE CASE OF SIMILARITY.

Escudero, I., Sánchez, V. (Universidad de Sevilla. Spain)

Abstract: This is part of a research study into the relationship between professional knowledge and teaching practice for the secondary school Mathematics teacher, when he/she teaches specific mathematical topics, in our case, similarity. Here, we show what happens when a mathematical content appears in mathematics classes other than the one scheduled by the teacher, the reasons and justifications provided and how the teacher handles this content. The results highlight the strong interrelation between the structure of the lesson and the way of understanding the mathematical content, allowing us to observe the cognitive integration of the different domains of knowledge in decision-making.

The relationship between the professional knowledge of the Mathematics teachers and their practice is one current research topic that has been dealt with from different perspectives. From a cognitive perspective, Leinhardt and her collaborators consider teaching as a "complex cognitive skill", that is supported by two basic systems of teacher knowledge: knowledge about the structure of the lesson and knowledge about the subject matter that he/she teaches (Leinhardt and Greeno, 1986). For these authors, the first system of knowledge includes "the skills needed to plan and run a lesson smoothly and to pass easily from one segment to another, and to explain material clearly" (Leinhardt and Smith, 1985, p. 247). They use the term *schemata* to refer to knowledge about the set of organised actions relating to activities of teaching. The schemata for activities of teaching includes structures that these authors call *information schemata*, which allow the teacher to retain and subsequently use the available information in the course of the lesson. For these authors, the lessons in mathematics classes are not homogeneous with respect to teacher or student activity. They are segmented into discernible parts, called "segments" or "activity structures" (Leinhardt, 1989; Stodolsky, 1988).

The second system of knowledge considered by these authors is subject-

matter knowledge. This knowledge is understood by Leinhardt as: "the knowledge that a teacher needs to have or uses in the course of teaching a particular school-level curriculum in mathematics" (Leinhardt et al., 1991, p.88). The teacher's knowledge about the subject matter that he/she teaches influences his/her explanations. In this sense, they consider that the two systems of teacher knowledge proposed are interconnected and integrated. These authors propose that the conduct of a lesson is based on the teacher's agenda, understood as "the teacher's dynamic plan for a lesson. It is a mental plan that contains the goals and actions for the lesson" (Leinhart et al., 1991, p. 89). We may consider the agenda as a dynamic plan whose components may be modified during the course of the instruction.

We have also taken into account the studies that conceptualise practice as the work that the teacher faces when performing his/her professional tasks (Bromme, 1994; Bromme and Tillema, 1995), and the works by Llinares (1995) that consider teacher knowledge to be within the framework of the situated cognition (Brown et al., 1989). We admit teacher knowledge is generated and developed through the interaction with the situations, with a cognitive integration of different domains of knowledge being performed (García, 1997). All of the above-named is a reference framework in which to situate our work in relation to the teacher's professional knowledge.

This paper forms part of a research work which aims to obtain information about the relationship that exists between the professional knowledge and teaching practice for the secondary school mathematics teacher, when he/she teaches specific mathematical topics, in our case, similarity. Here, we are going to show: a) what happens in mathematics classes when mathematical content appears different to that scheduled by the teacher, b) the reasons and justifications provided and c) how the teacher handles this content.

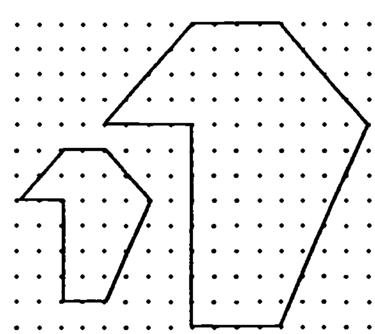
METHOD

The participants in the research were two secondary school mathematics teachers, deemed as "expert" teachers by their peers, who offered to collaborate voluntarily. The pupils belonged a two classes without any special characteristics (3^{rd} and 4^{th} years of Obligatory Secondary Education, 14-15 and 15-16 years of age, respectively). The mathematical content of the teaching was similarity.

In the design of the research the following were used: video recordings of all the lessons about similarity, observations of the classroom and several interviews:

interview for planning, interviews prior to and after each recording and a final interview. The interviews and the video recordings were transcribed in full. The teacher's agenda was described from the interviews. Using the transcriptions of the video recordings and of the screening of the videos, we identified different segments of teaching. We characterised these segments in relation to the notion introduced and the specific actions that he/she carries out. This allowed us to observe different aspects of the teacher's handling of the mathematical content, such as: use of an example for achieving a definition, a property, a theorem, with the constant intervention of the teacher and pupil, explanation through an example for reaching the definition, a property ..., with minimum or nil intervention from the pupils, amongst others. It also allowed us to observe the way in which the teacher incorporates the information that is generated during the course of the lesson for keep the class flowing, in order to achieve the objective of the scheduled mathematical content.

In relation to the subject matter, we identified the aspects of the concept that are relevant for the teacher (similarity as a teaching-learning object). Henceforth, we describe and analyse the content of the different components of the teacher's professional knowledge that take part in the situations studied. The form in which the teacher handles the mathematical content, during the teaching, may show us the role played by the different components of teacher knowledge. We consider the appearance in the class of a mathematical content other than that originally scheduled by the teacher to be a suitable space for obtaining information about the relationship between professional knowledge and the teaching of Mathematics.



Teaching episode: Emergence of radicals in the teaching of the ratio of perimeters in similar polygons.

The class starts out with the correction of some tasks that the pupils had to do at home. The tasks consisted of the construction of shapes similar to an irregular polygon drawn on a square grid. Once the tasks had been corrected, the two similar figures that are shown (see Figure) were drawn on the blackboard. The teacher asks the pupils to calculate the perimeter and the area of both figures. His objective is to introduce the ratio of perimeters and the ratio of areas for similar figures.

T. Calculate the area and the perimeter of the figure [...] in the original and for that one. We shall see the area, do you remember last year in the grid how we did it? How did we get the area?
S1. By dividing the biggest squares in other squares.
T. By counting the little squares that were left inside, that was the easiest way for the figure that was strange ... counting nothing more than ...

(The pupils start working on the problem, whilst the teacher carries on answering any questions from the pupils who had not used the grid in previous academic years).

S2. Is this the way we did it?
T. What? These are the square root of five, ... one, two, three, four square root of five. Don't be scared of the radicals that you haven't studied yet ... don't be frightened, those of you who studied them with me last year shouldn't have any problem ... the only thing is that you can measure ... if you do this ... these diagonals are not a single one, it is the diagonal of a square ... what is the square root of ...? What will this be?

From then on, the teacher applies Pythagora's theorem to explain how the diagonal of a square and a rectangle can be calculated. Carrying out calculations with radicals takes up the main part of the presentation. After this explanation, the pupil who had known how to carry out the calculations, jointly with the teacher, calculates the perimeters of the two polygons. To do so, he counts on the grid, using *the side of the small square as* a unit of measurement for the length, and using Pythagora's theorem for the measurements between points that are in a diagonal position. The teacher finishes off this process with the following comment:

T. You can all see it, can't you? Well, he got it right ... look carefully, What ratio is there between the two perimeters?. What are the perimeters like?. The figures were ... What were they like? (On the blackboard the pupil has written the values: $P1 = 10 + 4\sqrt{2} + 2\sqrt{5}$; $P2 = 20 + 8\sqrt{2} + 4\sqrt{5}$).

In this way, he establishes the ratio between the perimeters as a quotient.

The previous example shows us how the teacher handles some pupils' difficulty in dealing with the grid and the appearance of radicals.

As he states in the interview for planning, the teacher knows that the pupils may have difficulties with the grid, since only part of the pupils studied with him in the preceding academic year, in which they worked with grids. According to the teacher, one of the problems that the pupils may encounter is how to keep on the grid the equality of the corresponding angles. However, he considers the use of the grid to be useful. This may be the reason why in the correction of homework, the task selected uses precisely the grid as the "medium". Hence this episode shows how the teacher starts off a new segment of the presentation for the introduction of the ratio of the perimeters and the ratio of areas in similar figures. In the interview for planning, the teacher had already pointed out: "I'll start off by telling them to draw a two similar figures and that they should choose the size that they want ..." That is to say, the teacher has planned to use different figures that the pupils themselves may choose, without any type of restriction, though he foresees: "... I think that they will draw squares and rectangles".

However, either because the teacher feels pressurised by the time already used up or by the difficulties with the grids provided by the pupils in the correction of the tasks (information that the teacher receives through the schemes for information), he takes the decision to use the two similar figures that had come out in the corrected task (drawn on the grid), asking them to calculate the perimeter and the area for them both.

This decision changes the way of introducing these concepts, and it has several knock-on effects:

1) When trying to obtain the perimeter, the pupils continue to have difficulties when using the grid. Although the teacher tries to make these difficulties disappear, he ends up converting the use of the medium (the grid) into the objective. This gives rise to the introduction of contents that the teacher had not scheduled: use of Pythagora's theorem, the appearance of radicals and the operations with them. A new segment of presentation appears with a new mathematical content in order to account for an objective that was not originally scheduled (measuring in the grid and work with radicals). This new segment of teaching has some characteristics for the management of the mathematical content by the teacher (<u>explanation using an example</u>) different to that of the main segment (<u>use of an example for achieving it</u>).

The teacher justifies, in the interview following the recording, the decision to introduce the new content because "it seemed to me that since we were already dealing with grids that they could see, those who had not had classes with me last year, how you can calculate the area on a grid without having to use any formula...

that could be used to study at home and that the others would see something that they had not seen before". Moreover, he also justifies it to establish connections with the calculation with radicals, that "the fact of putting the radicals on the blackboard again is to remind them a little, because later we are going to study radicals ...". That is to say, "he unifies" the pupils' knowledge in relation to the grid and tries to connect it to a content that he is going to introduce later.

2) The teacher uses a great deal of time in dealing with this new content (measuring on the grid and work with radicals). This implies that the introduction of the ratio of perimeters and ratio of areas does not stand out very much (which is noted in the development of subsequent classes).

DISCUSSION:

All that we have showed till this point is an example of how the "action" developed in the classes is not a linear process. It does not involve assessing the quality of the planning by comparing what has been planned and what has been taught, but rather it means studying the reasons why new contents appear, how the teacher handles this and the arguments given by the teacher himself. The teacher's knowledge about the difficulties that the pupils may encounter with a specific teaching material, and the possibilities that he sees in that material, makes him select, for the correction of homework, a concrete task. In the pupils-content-teacher interaction, the difficulties that arise provide data that the teacher incorporates in his schemes of information and that, on the one hand, corroborate the difficulties foreseen and, on the other side, tell him that these difficulties have not been solved when correcting the tasks.

The teacher's action in the subsequent presentation highlights the fact that, either because he is pressed for time or through the information incorporated beforehand, he changes in the action the scheduled way to introduce the ratio of the perimeters, opting for a different introduction. The teacher for introducing the scheduled content and, at the same time, for going deeper into the difficulties perceived, he takes advantage of part of the same task that had been corrected. This indicates the teacher's capacity to adapt to the situation that is posed.

In the subsequent interaction, the teacher carries on adding information (from the schemes of information), which tell him that there are still difficulties with the calculations of the lengths of the segments on the grid. In order for the flow of the class continues in the proper manner, the teacher takes the decision in the action to go on to introduce the content that is causing the difficulties

(measuring in the grid used as a medium) as a new content (measuring on the grid and calculus with radicals), not scheduled originally. The way of handling it has the usual structure that he uses for contents that he deems that the pupils do not know. His curricular knowledge also intervenes in this decision, since the teacher knows that that content will be used later.

In relation to the main objective, the decision taken meant a change in the way for introducing the concept (for presenting the ratio of perimeters and ratio of areas in similar figures through different examples proposed by the pupils themselves to presenting it through a given example, selected by him), that differs from that stated in the original planning, and that shows the difference between what was "espoused" and what was "enacted".

To sum up, so far we have shown how the different systems of teacher knowledge are interconnected and integrated, and the strong interrelation between the domains of teacher knowledge and practice. It has been observed how decision-making in action, in relation to the mathematical content, is one of the basic skills for a teacher, with some characteristics of the integration of the different domains being noted in that decision-making.

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STUDENTS' UNDERSTANDING OF REGRESSION LINES [1]

Antonio Estepa, **Franciso T. Sánchez-Cobo** *University of Jaén (Spain)*

& Carmen Batanero *University of Granada (Spain)*

In this paper we present and discuss undergraduates' difficulties in finding the mean of a variable from the following data: mean of a related variable, slope and intercept of the regression line. Difficulties of the students in interpreting regression lines are described and implications for the teaching of regression are finally suggested.

1. INTRODUCTION

Correlation and regression are highly relevant in the Statistics curriculum for introductory level University students, not just in itself, but as a prerequisite in understanding other statistical concepts and procedures, such as multiple regression, analysis of variance and most multivariate methods. Despite this importance, mathematics education researchers have carried out very little research on this topic, though some research work on correlation can be found in Psychology.

Most psychological research has only concentrated on 2x2 contingency tables. Some psychologists have studied people's ability to estimate correlation from scatter plots or from a set of paired values of two variables (Erlick, & Mills, 1.967; Jennings, Amabile, & Ross, 1982; Lane et al., 1.985). The general conclusion is the adult's poor ability to estimate correlation, the better performance with positive and strong coefficients and the effect of previous beliefs on intuitive estimates. Within mathematics education, Estepa & Batanero (1994, 1996) studied the students' strategies in judging correlation in scatter plots, as well as their misconceptions concerning association. Truran (1997) described the understanding of association and regression by first year economics students. Morris (1997, 1998) studied the conceptions and understanding of correlation by undergraduates, as well as changes after a teaching experiment based on LINK (a computer assisted learning program for correlation). Evolution of students' understanding of association after teaching experiments based on computers have also been described in Batanero, Estepa, & Godino (1997), Batanero & Godino (1998), and Batanero, Godino, & Estepa (1998).

In this paper we analyse the students' performance in solving a problem about regression and their understanding of the regression lines. This is part of a wider study on the meaning of the correlation and regression in undergraduates (Sánchez, 1999), which include the assessment of conceptual and procedural knowledge on this topic.

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2. RESEARCH AIMS AND METHODOLOGY

In Estepa & Sánchez-Cobo (1998) we found scarce attention in textbooks as regards the centre of gravity in scatter plots, and the distinction between dependent and independent variables. This distinction was a main difficulty in our previous studies of understanding of association in contingency table (Batanero, Estepa, Godino, & Green, 1996). The current research was aimed to confirm our previous findings, and assess the learning of these concepts, as well as of the meaning of the parameters in regression lines after an introductory statistics course in the first year of University studies.

The content of course included the fundamentals of descriptive statistics, distribution tables and graphs; location, spread and order statistic; skewness and kurtosis; two-dimensional statistical variables, contingency tables, covariance and correlation, linear and polynomial regression, sampling, sampling distribution, interval confidence and hypotheses testing. Planning of the lessons by the lecturer and notes taken from two students along the course were analysed to get an insight of the type of teaching received by the students. This followed a traditional approach based mainly in lecturing and solving computation problems. Scarce time was devoted to interpretative and application problems.

Sample:

104 University students in their first year of Business studies (37 boys and 67 girls) and 89 University students in their first year of Nursing studies (20 boys and 69 girls) answered a written questionnaire. There were a total of 193 20-year-old students (average age). 109 of them had followed a scientifically oriented curriculum at secondary school, whereas the remaining 84 students had followed a humanities-oriented curriculum. Most students (117 or 60.6%) had not studied statistics in their secondary education. We asked the students about whether they consider statistics to be useful for their future professional work. 162 students (83.9%) found statistics to be sufficient, quite or highly interesting for their professional training; 155 students (80.3%) found correlation and regression to be sufficient, quite or highly interesting as a component of statistics.

Questionnaire

The complete questionnaire consisted of 12 multiple- choice items, 6 tasks where the student should estimate the value of the correlation coefficient and two application problems. In this paper we discuss the results concerning one of these problem:

Problem. The slope of a regression line is 16 and its intercept is the point y = 4. If the mean value of the independent variable is 8, which is the mean value of the dependent variable?

This problem was taken from Cruise & al. (1984, p. 288). As the students had never solved a similar problem along the course, this could be considered to be an mathematical problem, for them, as it was *"a situation that involves a goal to be*

achieved, has obstacles to reaching that goal, and requires deliberation, since no known algorithm is available to solve it. The situation is usually quantitative or requires mathematical techniques for its solution, and it must be accepted as a problem by someone before it can be called problem" (House, Wallace and Johnson, 1.983, p. 10).

In the Problem the students were asked to find the mean of the dependent variable, from the following data: i) the regression line slope, ii) the regression line intercept, and, iii) the mean of the independent variable. Since the students had not been explicitly been taught a standard procedure to solve this type of problem, they had to use their statistical, algebraic and geometric knowledge to find the solution. In particular students needed to remember the meaning of the parameters in the equation of the right line, and Cartesian representation, and to relate them to the notions acquired in the teaching received on linear regression. They had also to remember that the point with co-ordinates $(\bar{x}, \bar{y})$ belongs to both regression lines and discriminate the meaning of the independent and dependent variables, as well as interpret correctly the meaning of the slope and intercept (Truran, 1997).

Table 1. Frequencies and percentages of correct and incorrect solutions, according to solving procedures

Solving Procedures	Solutions		
	Correct	Incorrect	Total
1. $y = bx + a$	31 38.3	50 61.7	81
2. $y-\bar{y}=\frac{\sigma_{xy}}{\sigma_x^2}(x-\bar{x})$	8 34.8	15 65.2	23
3. Using both 1) and 2)	2 100.0		2
4. $x = b'y + a'$	2 11.1	7 88.9	9
5. $x-\bar{x}=\frac{\sigma_{xy}}{\sigma_y^2}(y-\bar{y})$		4 100.0	4
6. Using both regression lines Y/X and X/Y	2 66.7	1 33.7	3
7. Using a parameter		7 100.0	7
Total	45 34.9	84 65.1	129

Below, we analyse the students' answers, grouping them according to the procedures used. We then discuss the obtained results, where percentages are referred to the 129 students that provided a solution (67% of the total sample). The remaining

students gave no answer, which is indicative of the task difficulty for these students. Results are shown in Table 1.

4. RESULTS AND DISCUSSION

Results in Table 1 show that the majority of the 129 students answering the problem used one or both regression lines to solve the problem (94.6%). 106 students (82.17%) chose the correct regression line Y/X, while 13 students (10.1%) confused the regression line and used X/Y instead. Finally 3 students (2.3 %) used both lines. Below we analyse the student's errors and misconceptions depending on the procedure followed.

4.1. Use of the regression line Y/X.

From the 106 students using only the regression line Y/X (the first three rows in table 1), 81 students used, exclusively, the explicit equation $y = bx+a$, 23 students used only the point-slope expression $y-\overline{y}=\frac{\sigma_{xy}}{\sigma_x^2}(x-\overline{x})$ and 2 students employed both equations. Below we describe the thinking process involved in each of these procedures and the difficulties involved in the same.

Using the explicit equation of the regression line, $y = bx + a$

Giving the data in the statement of the problem, this procedure can lead to the immediate correct solution which was reached by 31 students (38.6% of those using this procedure), in the following way:

$$"a = 4 = intercept,\ \ b = slope = 16,\ \ \ y = a + bx,\ \ \ \ \overline{x} = 8; \overline{y} = 4 + 16.8 = 132;"$$

The main cause of failure with this procedure was confusing the dependent and independent variables, which led 27 students (33.3% of those using this procedure) to exchange the mean of X with that of Y, and, consequently, to an incorrect solution. This confusion was also described in the studies by Batanero, Estepa, & Godino (1.997) and Batanero, Godino, & Estepa (1998). An example of this confusion is

$$"a = 4,\ \ b = slope = 16,\ \ \ y = a + bx,\ \ \ \ \overline{y} = 8"$$

shown in the solution given by the student n. 46:

A second difficulty arises when the student did not realise that the gravity centre $(\overline{x}, \overline{y})$ is a point belonging to the regression line, and, consequently, this point must verify the equation $\overline{y} = b\overline{x} + a$. In the teaching carried out, this fact was not explicitly taught, though the gravity centre was defined as being the intercept of the two regression lines.

A third main difficulty was exchanging the meaning of the parameters b and a in the equation $y = bx + a$, as well as assigning them incorrect values (9 students). For example, the student 146 gave the following solution: *"y = a + bx, y = 4, a = 16, b=8, x = ?, 4 = 16 - 8x, 12 = 8x, x = 12 / 8 = 1.5"*.

<u>*Using the point-slope expression*</u> $y-\bar{y}=\frac{\sigma_{xy}}{\sigma_x^2}(x-\bar{x})$

Because in this case $\frac{\sigma_{xy}}{\sigma_x^2}$ = 16, we obtain $4-\bar{y}=16(0-8)=128; \bar{y}=132$. The main difficulty when using the point-slope equation of the regression line of Y/X was finding the point *(x, y)* needed. Only 8 students (34.8% of those using this equation) realised that this point was the intercept with the ordinate axis. Other difficulties arose from using an inadequate equation of the regression line Y/X, with a frequency of 7 students. For example student number 4 gave the following answer, where the slope and intercept of the equation are also confused:

$$"y-\bar{y}=\frac{\sigma_{xy}}{\sigma_x^2}(x-\bar{x}), \bar{x}=8, y=4, y-\bar{y}=16-8, \bar{y}=8-4, \bar{y}=4"$$

Here we found again the confusion between the dependent variable and independent variable in 7 students, as Student number 57, who gave the following solution: "$\frac{\sigma_{xy}}{\sigma_x^2}=16,(0,4),\bar{y}=8, y-8=16(0-\bar{x}),-4=-16\bar{x}, \bar{x}=\frac{1}{4}$"

The type of the equation of the regression line used seemed to influence the interpretation of the slope, since when the equation used was $y = bx + a$, we found more difficulties in interpreting the slope than when using $y-\bar{y}=\frac{\sigma_{xy}}{\sigma_x^2}(x-\bar{x})$, where most students seemed to understand that the slope is expressed by $\frac{\sigma_{xy}}{\sigma_x^2}$.

<u>*Other procedures*</u>

Two students use both the explicit and point-slope equations. These students did not realise that the gravity centre belongs to the regression line $y = mx+n$. However, they observed that the mean of the dependent variable can be found from the point-slope equation. Then the students compared both equations and determined the mean of the dependent variable, like in the following case (Student 33):

$$"y=bx+a, y=16x+4, y-\bar{y}=\frac{\sigma_{xy}}{\sigma_x^2}(x-\bar{x}), b=\frac{\sigma_{xy}}{\sigma_x^2}=16$$

$$a = \overline{y} - \frac{\sigma_{xy}}{\sigma_x^2}\overline{x}, a = \overline{y} - 16.8, \overline{y} = 132"$$

4.2. Use of the regression line X/Y

13 students (10.1% out of the 129 answering the problem) used the regression line of X/Y. From the 9 students who used the explicit equation of the regression line, $x = b'y + a'$, only 2 got the correct answer, with the following procedure: "$x=a' + b'y$; $y=8$; $0 = 16 \cdot 4 + a', = a' = -64, \overline{x} = -64 + 16.8,\ \overline{x} = 64$"

Four additional students used the point-slope equation of the regression line $x - \overline{x} = \frac{\sigma_{xy}}{\sigma_x^2}(y - \overline{y})$. The difficulties described in the previous section were repeated, such as the confusion between dependent and independent variable. Below we reproduce this difficulty in Student 105's answer, who, also confuse the meaning of the intercept:

"$x = a' + b'y$, $x = 8$, $y = 4$, $b' = 16$, , $8 = a' + 16 \cdot 4$, $8 = a' + 64$, $a' = 64 - 8 = 56$"

Other mistake was exchanging the meanings of the intercept and the slope, such as shown in Student 89's answers: "$a' = \overline{x} - b'\overline{y},\ 16 = 8 - 4\overline{y}, \overline{y} = \frac{8-16}{4} = 2$"

With respect to the point-slope equation of the regression line X/Y, the only confusion shown was that between the dependent and the independent variable, such as in student 122: "$x = 0, y = 4, \overline{x} = 8, x - \overline{x} = \frac{Cov(X,Y)}{\sigma_y^2}(y - \overline{y}), x - 8 = 16(4 - \overline{y})$, $0 - 8 = 16(4 - \overline{y}), 16\overline{y} = 16 \cdot 4 + 8 = 72,\ \overline{y} = 4.5$. *The equation of the line is defined as $x - x_1 = b'(y - y_1)$ being m the slope of the line. As we know the slope, the value of y, and we know that the line intercept the axis Y in the point 4, where x = 0, when substituting, the answer is 4.5"*

4.3. Use of both regression lines Y/X and X/Y

Three students used both regression lines Y/X and X/Y. We consider that this is the best procedure, since, both variables X and Y can play the role of dependent or independent variable if the context of the problem is not specified. Thus Student number 67 provided both solutions:

"*Lets suppose we take the line Y/X: then,* $x = 0, y = 4, y = bx + a, 4 = 0 + a, a = 4, a = \overline{y} - bx$, $4 = \overline{y} - 16 \cdot 8, 4 = \overline{y} - 128, \overline{y} = 132$. *Lets suppose we take the line X/Y : in this case,* $x = b'y + a', 0 = 16 \cdot 4 + a', a' = -64, a' = \overline{x} - b'\overline{y},\ -64 = \overline{x} - 16 \cdot 8,\ \overline{x} = 64$"

Finally, 7 students out of the 129 who attempted to solve this problem used other statistical concepts, such as those of mean or covariance getting an incorrect solution.

5. FINAL REFLECTIONS

From the above analysis we conclude that the most frequent difficulty for the students in our sample was distinguishing the role of dependent and independent variables. This lead to failure to 36 out of the 84 students who got incorrect solutions (42%), which confirms our previous findings that this is a main problem in understanding association (Batanero, Estepa, Godino, & Green, 1996). The fact that correlation ignores the distinction between independent (explanatory) and dependent (response) variables, whilst in regression this difference is essential might have received not sufficient attention in the teaching received by the students. This teaching should take into account that relationship in regression depends of the type of covariation presented in the problem context. If there is a causal dependence, the explanatory and response variables are univocally determined. However in other type of covariation - interdependence, indirect dependence, concordance and spurious covariation - the student must decide the best regression to employ.

Though both the explicit and point-slope equations of the regression line are, formally, equivalent, the later served better to our students to identify the slope that in the explicit equation; their more frequent use of the explicit equation was probably due to stated of the problem. Confusions about the meaning of the parameters and unability to relate the problem data to the regression line equation was also found.

We finally conclude that more research on the teaching and learning of association and regression is needed for achieving a teaching of quality. This research should be based on the didactical an epistemological analysis of the topic for searching their internal structure, emphasising the fundamental contents, and relating new findings to previous research work. Thus we can contribute to a better planning of the teaching that facilitate students' construction of a meaningful learning.

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The Motivation to Learn Mathematics

David Feilchenfeld

The Hebrew University of Jerusalem

This paper characterizes the motivation to learn mathematics among first year practical engineering students studying electronics. The motivation is directly measured by the students' self evaluation, relatively to their motivation to study other subjects. In addition, an analyze has been held of the manner in which students attribute the reasons of success in mathematics studies in comparison with the reasons of success in studying electricity. It was discovered that the motivation to study electricity is higher than the motivation to study mathematics. This result is ascribed to two reasons: I. The incentive value of achievement in electricity is considered to be higher than the incentive value of achievement in mathematics, because of its immediate relevance to electronics. II. According to the students, success in electricity is dependent on effort, more than success in mathematics. This is why it is more appropriate to devote to electricity more time and effort than to mathematics.

Introduction

Motivation, Attribution of Causes for Success, and Self Esteem

The motivation to learn mathematics is not different from the motivation to achieve in any other domain. The motivation to achieve is usually ascribed to three factors: the incentive value of the achievement, the subjective expectation for success, and achievement-related needs (Atkinson, 1957, 1964). The achievement-related needs are individual emotional drives and will not be discussed here. The incentive value of achievement is based upon the manner it is perceived: as a trifling matter or as something of importance. For example, the value of winning a chess game against a weak or inexperienced opponent is lower than the value of winning chess game against an excellent opponent. Weiner suggested that self-esteem following the incentive value of success are determined by the locus of the source of a cause (locus of causality) (Weiner, 1986 p. 128; 1992 p. 271). When the source of a cause is internal (within the student) the incentive value of success is higher, and when the source of a cause is external the incentive value of success is lower. As the incentive value of success is growing up, there is a tendency to attribute the cause of success to oneself, and thereof the pride and self-esteem improve. The success in a mathematics exam in which everybody succeeded (because it was easy, or because the grades were distributed generously) is considered to be of low value. On the other hand, success in a mathematics exam in which everybody failed is considered to be of high value, since the cause of the success is attributed to oneself, thus improving one's self esteem. In this principle, Weiner suggests that the effect of success or failure on self-esteem is due to the locus of causality.

Weiner had also treated the expectancy of success proposing what he called the "Fundamental Psychological Laws" (Weiner, 1985 p. 558). Weiner's Expectancy Principle says: "Changes in the expectancy of success following an outcome are influenced by the perceived stability of the cause of the event." As the expectancy of success is considered to be an important factor of motivation, Weiner's principle deserves some attention. This principle has three corollaries: "1. If the outcome of an event is ascribed to a stable cause then that outcome will be anticipated with increased certainty, or with increased expectancy, in the future. 2. If the outcome of an event is ascribed to an unstable cause, then the certainty or expectancy of that outcome may be unchanged or the future may be anticipated to be different from the past. 3. Outcomes ascribed to stable causes will be anticipated to be repeated in the future with a greater degree of certainty than outcomes ascribed to unstable causes." In this principle and its corollaries Weiner suggests that an individual subjective expectancy for success or failure is a consequence of his perceived stability to the cause for success or failure.

In order to analyze more finely the casual attribution, Weiner suggested an additional dimension to those two which were described (Weiner, 1979). This dimension of controllability was formed mainly to separate the causes perceived both as internal and stable into two sub-groups. Controllable causes were separated from arbitrary causes. The controllability level of a cause is linked to the level of responsibility it imposes upon the individual, and to the level of intentionality of the cause.

The manner in which a cause for success or for failure is perceived, is illustrated accurately and in all simplicity with the aid of the three dimensions of controllability, stability and locus of causality. The categorization of a cause has implications on the achievement's incentive value, the subjective expectation for achievement, the individual's motivation, and the arousal of emotions. The attribution of successes and failures to causes arouses various emotions depending on the category the cause is related to (Weiner, 1986), as described in table 1.

Attribution to	Emotions Aroused Following Success	Emotions Aroused Following Failure
External Controllable Cause	Gratitude	Anger
Internal Controllable Cause	Pride and High Self Esteem	Guilt and Low Self Esteem
Internal Uncontrollable Cause	Pride and High Self Esteem	Shame and Low Self Esteem
Stable Cause	Hope	Fear and Hopelessness

Table 1: The emotions which are aroused following a casual attribution (according to Weiner, 1986).

Method

A. The construct is a combination of three different and independent questionnaires. The motivation to learn mathematics may be changed for a concrete persona in different conditions. Thereof, it was only natural to try to examine the way the students themselves estimated their own motivation to learn mathematics. This was done by the students' estimations of their actual effort, in the variety of topics they learn, and grading them. Every student was requested to grade the first three subjects in which his attendance at class was highest. Similarly, he was requested to grade the first three subjects to which he dedicated the longest duration of studying, the first three subjects in which the largest amount of work was demanded, and the first three subjects which were most challenging.

B. The second questionnaire was based on a former one written by Fennema (Fennema, et al. 1979) which relied on what was then considered to be Weiner's Theory of Attributions (Weiner, 1974). I introduced few changes in respect of Fennema's questionnaire. Some changes are due to bringing it up to date according the changes which went over Weiner's theory (i.e. introducing the controllability dimension). Other changes are a consequence of emphasizes in this research. The questionnaire is related to eight events, half of which describe success in mathematics and half describe failure. Five causes have been attributed to each event. The five proposed causes for each event are considered to be part of five distinct categories as described in table 2.

	Category	Locus	Stability	Controllability
1	Aptitude of Deeper Understanding	Internal	Stable	Uncontrollable
2	Procedural Aptitude	Internal	Stable	Uncontrollable
3	Long Term Effort, Laziness, Industriousness	Internal	Stable	Controllable
4	Objective Task Characteristics, Friends, Lecturers	External		
5	Temporary Exertion, Mood, Chance and Luck	Internal External	Unstable	

Table 2: The five categories of causes for success and failure in mathematics, which were included in this research, and their classifications according to the locus of causality, stability and controllability.

The separation between the first two categories was not held in reliance on the attribution theory, but rather on a distinction, which exists among the mathematics teaching researchers community, between conceptual versus procedural understandings. (Different researchers relate to it in a variety of terminologies).

The students were asked to write down the degree of their consent to each of the forty causes (8 events x 5 causes) regarding the learning of mathematics, and the degree of their consent to the same causes regarding the learning of electricity. The answers were given on a four level scale. Attribution rates were extracted from the two levels of agreement (i.e. agree and absolutely agree). The differences in attribution rates of the causes regarding mathematics and electricity have been calculated.

C. The third questionnaire was destinated to map the characteristics the students attribute to each of the 40 causes. It is customary to assume that the causes fall within discrete and dichotomic categories such as internal or external, stable or unstable, and controllable or uncontrollable. This assumption may oversimplify the model because different people attribute to any concrete cause different values for the locus of causality, the stability and the controllability. In order to check how, in the students' view, each cause falls within a category, there were eight types of a second questionnaires distributed, each concerning one event (of success or failure in mathematics) and five possible causes for it. The questionnaire was based on Russel's Casual Dimension Scale (Russel, 1982). This scale, which relates to a unique cause, contains nine items, three of which are pertinent to each of the casual dimensions. Russel has demonstrated that this scale has the properties of an "acceptable psychometric instrument." For each of the nine items, a nine level scale (1 – 9) was presented to the students with opposite statements attached to the two ends.

Results

I. Motivation

Four parameters were used in order to evaluate the motivation to learn mathematics: attendance at class, duration of studying, amount of work demanded, and challenge. The picture received from all four parameters was similar. In all parameters the two dominant subjects were electricity and mathematics, and electricity over dominated mathematics. According to the students, the rate of attendance in electricity class was higher than in mathematics class. Likewise, the actual duration of studying electricity was higher than mathematics, the amount of work that was demanded in electricity class was higher than in mathematics class, and electricity was found more challenging than mathematics. This trend was observed throughout the whole population examined.

The group of students that perceived themselves as successful both in mathematics and electricity (75% of the population) deserve special attention. When looking at the grading of these students, a more powerful picture is obtained. These students have graded their attendance in electricity class in first or second places in a higher rate (85%) than the rest of the students (69%). At the same time these students have graded their attendance in mathematics class in first or second places in a lower rate (48%) than the rest of the students (64%). While there is no meaningful difference (5%) in attending electricity and mathematics classes among other students, there is one (16%)

among the students, which perceived themselves as successful both in mathematics and in electricity. Among these successful students there is a remarkable preference of attending electricity over mathematics classes, and utilizing more time in favor of electricity than in favor of mathematics studies, despite all lecturers of these subjects were very much appreciated by the students. Within the limits of their resources, they have actually spent less time on mathematics and more time on electricity, compared to the rest of the students. Within this group, the rate of grading first the time spent on studies of electricity was more than three times the rate of grading first the time spent on studies of mathematics. In spite of this powerful and impressive picture, it is impossible to evaluate, on the ground of these gradings, the extent of which electricity studies are more time consuming than mathematics. This is due to the questionnaire itself, which does not relate to the differences' values, but only to the gradings.

II. The Effect of the Hedonic Bias

The most striking result from the third questionnaire was the difference in the students' attitudes towards causes for success and causes for failure, particularly in the dimension of stability and to a lesser extent in the dimension of locus of causality. The average perceived stability of causes for success was 5.05, in contrast to a 2.93 average of causes for failure. In addition, the causes for success were treated by the students as more internal (average of 6.41) than the causes for failure (average of 5.43). This result may evolve from the well-established phenomenon known as The Hedonic Bias. This term relates to the tendency people have to credit themselves for successes, in contrast to their unwillingness to accept responsibility for failures. It is generally accepted, among the psychologist's community, that this behavioral pattern maximizes the pleasure of success as well as it minimizes the pain, which derives from failure. This is the source of the term. On the basis of this bias, it may be expected that students will perceive the causes for success as internal and stable, while the causes for failure will be perceived as external or as unstable. The students' desire for hope, and their reluctance from fear or helplessness, may have brought the students to perceive all the causes for failure as unstable, and all the causes for success as stable. The dominance of the Hedonic Bias in the stability and locus of causality dimensions overshadowed possible differences which may have been found out between attributing causes for success in mathematics and in electricity.

Average Perceived Stability of Causes:			
Regarding	Aptitude of Deeper Understanding	Procedural Aptitude	Effort
For Success	6.13	5.83	5.57
For Failure	3.56	3.21	2.68

Table 3: Averages of perceived stability for different categories of causes according to the students' views.

In spite of the above, it was found that the perceived stability of causes (for success or failure) regarding aptitude of deeper understanding was higher than the perceived stability of causes regarding procedural aptitude, which itself was higher than that of causes regarding effort, as is shown in table 3. Although the differences in perceived stability between the three categories are less meaningful then those between success and failure, they exist and seem to be consistent.

III. The controllability Dimension

The controllability value of each cause is an average of the students' perception of the extent the cause is controllable, intentional, and imposes responsibility. It has been clarified that the longer a cause for success concerning effort is lasting, the more it is perceived by the students as controllable and responsibility imposing. In addition to that, meaningful and consistent differences have been found in all three parameters between causes concerning aptitude and effort. The causes concerning effort were perceived as more controllable, more intentional, and more responsibility imposing than the causes concerning aptitude. This result is illustrated in chart 1, where the controllability values (averages of the three parameters) of causes concerning effort are higher (6.50-7.50) than of the causes concerning aptitude (4.94-6.51).

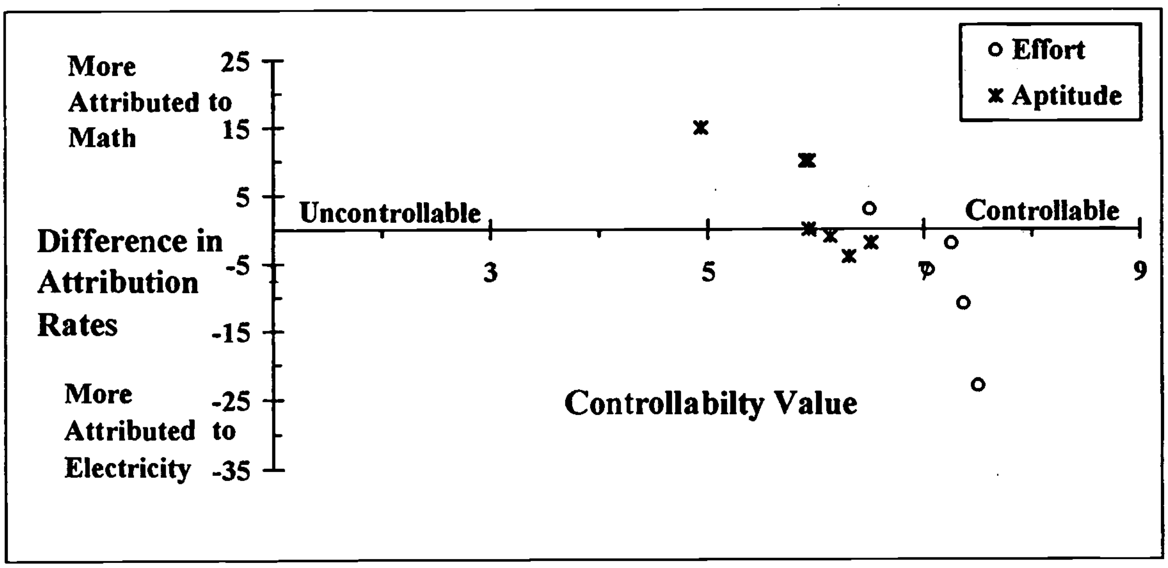

Chart 1: The relation between the controllability level (average between the students' perception of the extent the cause is controllable, intentional, and imposes responsibility) of internal causes for success and the rate differences (in percentage) in attributing the causes to mathematics or to electricity studies.

An interesting result (that is illustrated in chart 1 too) is the tendency to attribute causes concerning effort more to the studies of electricity, and causes concerning aptitude more to the studies of mathematics. Regarding the causes concerning effort and aptitude I found a relation between the controllability value and the differences in attributing them to electricity and to mathematics studies. The higher the perceived

controllability value of a cause, the less it is attributed to mathematics, compared to electricity studies. Because the results came from two different questionnaires, correlation between the two series of results could not be calculated. Nevertheless the correlation between the two series of averages (i.e. the average difference of attribution rates for every cause, versus the average perceived controllability value of the same causes) was calculated and found to be –0.86. These results show that students attribute causes concerning aptitude (which are perceived as less controllable) more to success or failure in mathematics, than to success or failure in electricity. On the other hand, students attribute causes concerning effort (which are perceived as more controllable) less to success or failure in mathematics, than to success or failure in electricity.

Discussion

As one of the research aims was investigating the motivation to learn mathematics, it is interesting to compare the results concerning mathematics with a different subject. The objective level of motivation to learn mathematics, the degree of affection towards mathematics, and the importance the students attach to it, beyond the rhetorical declarations, are not measurable. The reason to it is not the absence of an acceptable measure, but a more essential one. When a student has a high motivation for mathematics, it means he prefers studying mathematics to other alternatives available to him at that moment. The mathematics motivation level should therefore be relative to the motivation levels of the alternatives, and should not have absolute values. It is more than reasonable to compare it to other topics and their motivation levels. The two subjects, for which the highest motivation levels were found, among practical engineering students in electronics during their first year of studies, were electricity and mathematics.

It can be concluded that the motivation for studying electricity is higher than the motivation for studying mathematics. This difference is partly due to the higher value of importance attributed to electricity compared to mathematics, and to different strategies needed for success in the different areas. According to the students, success in electricity is more effort dependent than success in mathematics, while success in mathematics is more aptitude dependent than success in electricity. It is appropriate therefore to exploit in favor of electricity studies a larger slice of the limited resources of time and energy, and only a smaller slice in favor of mathematics studies.

Whereas the strategy taken by the students who perceived themselves as successful both in mathematics and electricity had proved itself, it should be closely examined. These students attended electricity classes more than mathematics classes, and spent more time studying electricity than mathematics. The students in this group have taken this strategy, in a more extreme route than the rest of the students, in spite of their estimation of the amounts of work that were demanded in the courses, which was similar to the other students' estimations.

It is only natural to perceive causes regarding aptitude and ability as more stable than causes regarding effort. Aptitude is usually considered to be a constant personal trait, and sometimes even as an inborn trait. On the other hand, effort is an outcome of personal traits like laziness or industriousness, as well as unstable conditions like motivation or social milieu, and is therefore considered to be less stable. Of special importance is the differentiation made by the students in the way they perceive the stability of causes concerning different kinds of aptitude. The perception of aptitude of deeper understanding as more stable than procedural aptitude (in mathematics) is an evidence for a distinction ability, as well as an evidence for a whole trend of perceiving different personal traits in different degrees of stability. Regarding Weiner's Expectancy Principle, it may be concluded that success as a result of causes concerning aptitude of deeper understanding, is expected by the students more than success as a result of causes concerning procedural aptitude. This corollary is very interesting considering the fact that it came out of students that most of their mathematics studies are procedural.

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AN ALGEBRAIC APPROACH TO ALGEBRA THROUGH A MANIPULATIVE-COMPUTERIZED PUZZLE FOR LINEAR SYSTEMS

José Eduardo Ferreira da Silva
Universidade Federal de Juiz de Fora, Colégio de aplicação João XXIII, MG, Brazil

Roberto Ribeiro Baldino
Grupo de Pesquisa-Ação em Educação Matemática – GPA – UNESP, Rio Claro, SP, Brazil

ABSTRACT

Relying on Marx's maxim "human anatomy contains a key to the anatomy of the ape" we argue that the relation between arithmetic and algebra is a difference of essence, *hence a dialectical discontinuity much more radical than is conveyed by words such as "gap" or "cut". We argue that the transition from arithmetic to algebra is impossible, and we characterize arithmetic as an* obstacle *to algebra. We present a manipulative and computerized puzzle to introduce linear systems as a possible alternative approach to the teaching of introductory algebra.*

Introduction

The literature about mathematics education seems to indicate a *discontinuity* between the arithmetic and algebraic domains. Filloy & Rojano [1989] consider "cut points" separating the two kinds of thinking; Linchevski & Herscovics [1996] indicate a "cognitive gap" that they endeavor to "cross" and overcome"; Schmidt&Bednarz [1995:84] refer to a "dichotomy" between arithmetic and algebra. Kieran [1981] distinguishes between arithmetical and algebraic uses of the equality sign.

However, the vocabulary used to express the relation arithmetic/algebra also indicates continuity. At least in one case, the word *continuity* appears explicitly: the Argentinean school programs "emphasize continuity with arithmetic" [Panizza, Sadovski & Sessa, 1996:107]. Continuity is also hinted at by signifiers such as "progress" (from informal to formal level of doing algebra) [Reeuwijk, 1995:1–143], "transition" (from arithmetic process-oriented thinking to proceptual algebraic thinking) [Graham & Thomas, 1997:10], "evolution" (from arithmetic to algebraic language) [Filloy & Rojano, 1989:19], and "transition from arithmetic to algebra" [Bouton-Lewis et al 1997:185]. In spite of arguments that "algebra cannot be considered as a arithmetical generalization" [Bodin & Capponi, 1996: 587], expressions denoting continuity like "generalized arithmetic" are still current [Wong, 1997:285; Graham & Thomas, 1997:9; Sfard & Linchevski, 1994:195,197; Kutscher & Linchevski, 1997:169].

The aim of this paper is 1) to argue that the discontinuity between arithmetic and algebra and, in general, between *operational* and *structural* ways of thinking [Sfard, 1991], is more radical than announced by words such as "cut", "gap", "dichotomy" or "duality"; 2) to argue that attempts to teach algebra starting from arithmetic [Linchevski & Herscovics, 1996] lead to difficulties, if not to impossibility; 3) to

argue in favor of a manipulative-computerized puzzle to solve linear systems of two equations in two unknowns to teach introductory algebra courses.

The dialectical discontinuity: a difference of essence

According to Sfard [1991] the literature on epistemology of mathematics teems with allusions to various dichotomies: abstract/algorithmic, declarative/procedural, process/product, dialectical/algorithmic, figurative/operative, conceptual/procedural, instrumental/relational. She proposes another opposition, operational/structural, that should be considered not as a dichotomy but as a *duality*. She argues:

> "The structural approach should be regarded as the more advanced stage of concept development. We have good reasons to expect that *in the process of concept formation, operational conceptions would precede the structural*" [10]. "The history of numbers is "a long chain of transitions from operational to structural conceptions" [14].

We have some difficulty in conceiving of a duality relation in terms of "precedence" and "transition". However, if we take the idea of precedence in the sense of ancestry, hence of *genesis*, we cannot avoid verging upon Marx's famous aphorism: "Human anatomy contains a key to the anatomy of the ape" [Marx, 1973:105]. Can we infer that structural thinking contains a key to operational thinking that "precedes" it? What does this mean?

> "Although it is true that the categories of structural thinking possess a truth for all forms of thinking, this is to be taken only with a grain of salt. They can contain them in a developed or stunted, or caricatured form etc., but always with an *essential difference*. The so-called presentation of cognitive development is founded, as a rule, on the fact that the latest form regards the previous ones as steps leading up to itself, and since it is only rarely and only under quite specific conditions able to criticize itself (...) it always conceives them one-sidedly" [The paragraph is a parody of Marx, 1973:106] ([1]).

Since "twentieth-century mathematics seems to be deeply permeated with the structural outlook" [Sfard, 1991:24], and since the structural way of thinking is "the more advanced stage of concept development" [ibid. 14], we infer that structural thinking (for example, developed algebraic thinking) will tend to regard all forms of computational thinking preceding it (for instance arithmetic) as steps leading up to itself. From this point of view, algebraic thinking contains arithmetic regarded as a

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[1]. Here is the original text. "Although it is true, therefore, that the categories of bourgeois economics possess a truth for all other forms of society, this is to be taken only with a grain of salt. They can contain them in a developed or stunted, or caricatured form etc., but always with an essential difference. The so-called historical presentation of development is founded, as a rule, on the fact that the latest form regards the previous ones as steps leading up to itself, and since it is only rarely and only under quite specific conditions able to criticize itself –leaving aside, of course, the historical periods which appear to themselves as times of decadence– it always conceives them one-sidedly" (...)" Although Marx's categories refer to political economy, the parody is valid because only its logical content is considered here.

step leading up to itself (hence continuity) but also as a caricatured form *essentially different* from itself (hence discontinuity). In order to overcome this one-sided view of arithmetic as a step leading to itself, the upper form will have to "criticize itself". What does this mean? We resort to another author who has thoroughly discussed the quote from Marx cited above from Hegel's perspective.

"This *essential difference* –and here is the decisive point– should be considered as crossed by destruction and generation. The movement from a continuity to a discontinuity perspective corresponds to the movement from a naive to a critical focus on the upper form" [Fausto, 1987:18].

Therefore, in its self-criticism, structural (algebraic) thinking should recognize itself as being generated insofar as the previous operational (arithmetical) thinking is being destroyed as such, and incorporated into it as something *essentially different*. The mathematician's movement back and forth from computational to structural approaches to a problem is not autonomous because the structural outlook is not reversible.[2] According to this argument, no "progress", "transition", "evolution" or "generalization" can lead from arithmetic to algebra. Simultaneous destruction-genesis (death/birth is just a non-simultaneous approximation) is the decisive dialectical concept to understand how arithmetic is embodied in algebra: the tree has to die so that the flower can be born from its rotting trunk. This conclusion will be important for teaching.

Looking at the continuity/discontinuity ambiguity pointed out at the beginning, from the point of view of the Piagetian theory of equilibration, what our argument amounts to is that instead of thinking about the relation arithmetic/algebra as a *completive generalization* [Piaget and Garcia, 1984, p. 10] we should think of it as an *abstractive reflection* [Piaget, 1975, p. 39] which implies a difference of essence.

Didactical difficulty of the continuity point of view

The prevalent teaching strategy in introductory algebra courses is to conduct the student step by step from procedural to structural thinking following a supposedly continuous path. Research is then organized to observe, evaluate and encourage progress along this path using teaching experiments, mathematical ability tests and interviews [Linchevski & Herscovics, 1996]. Instructional materials are supplied as needed: geometric models [Filloy & Rojano, 1989], balances [Linchevski & Herscovics, 1996, Aczel, 1998, da Rocha Falcão, 1995] or spreadsheets [Arzarello, Botazzini & Chiappini, 1995].

In their teaching experiment, Linchevski & Herscovics [1996] adopted a clear continuity strategy. They accepted the initial use of the inverse operations in reverse order naturally employed by the students for solving equations with a single occurrence of the unknown and let them proceed until their method became "lengthy and tedious". Then they assumed that the students were "ready to be exposed to new

[2] This is clear from Sfard's examples [p. 25, 26]; once we take a look into the structural solution, we cannot pretend that we have not seen it and just come back to the computational approach. The relation is not of duality but of irreversible spiral.

points of view" [40] and started teaching them a decomposition-and-cancellation method to solve equations with two occurrences of the unknown on the same member. Next, they introduced equations with the unknown on both sides. To the students they "pointed out that none of the methods they knew could effectively solve this type of equation" [53] and introduced the balance model to help them. However, in spite of all efforts, the students continued to solve the equations with only one occurrence of the unknown by performing reverse operations in reverse order. "It should be pointed out how stable this procedure remained in the seven months since our initial assessment" [59]. So, the step by step strategy did not lead to the expected change. Why?

How much do these students praise the arithmetical undoing strategy that they have learned for solving one-unknown equations? When they found it "lengthy and tedious" and claimed that "there must be another way" [62], were they actually ready to be "exposed" to another point of view? What is the effect of "pointing out" to a student that his/her method is not "effective"? What is the nature of the change expected by this strategy?

The authors do not seem to acknowledge that the students may have a special taste for wrong answers as long as they are *their* answers, obtained by *their* methods and make *them* big, strong and worthy of love for the gaze that they imagine to be cast upon them. Will they recognize in the instructors eyes the referential gaze for which they are willing to play the scene of their lives? What should be the *nature of the demand* provoked by these eyes if the expected change is to be produced?

Drawing on previous research, the authors attribute certain students' answers in ability tests about first degree equations to a "*limited view* of algebraic expressions", "*failure* to grasp the meaning of operations", and "*inability* to spontaneously operate with or on the unknown" [39-40, emphasis added]. Next, inability is identified with an *obstacle*: "This inability to spontaneously operate on or with the unknown constituted a cognitive obstacle" [41]. The teaching experiment was designed to cross this obstacle.

Thus, a too restrictive conception of *obstacle* [Brousseau, 1997:82] considered to be a mere "failure", seem to have presided over the whole teaching experiment. The students had to conform to new methods or... fail. Would they be willing to recognize in the instructors eyes the referential point of their imaginary identifications? Or did they recognize in these eyes the gaze of the school system, always ready to classify their answers as "failure" and "inability"? "The students experience we conjecture is not of a straightforward switch from arithmetic to algebra; their storying backdrop needs to be extended at the same time" [Brown & Wilson, 1998:171].

> "So far as *obstacles* appear as repetitions of failure, they provide a measure of the persistence of the subject's *jouissance organization*: the subject hesitates to abandon what worked well in previous situations and insists on justifying his statements in terms of the notions that he does not want to give up. This is why the nature of the demand makes a big difference in the didactic situation" [Baldino, 1997: 237].

So, not only did Linchevski & Herscovics' [1996] teaching attempt based on continuity assumptions reveal itself to be difficult but it also indicated that *the obstacle to pass from arithmetic to algebra is arithmetic itself.* Arithmetic is the knowledge that the students refuse to *destroy* in order for structural thinking to be *generated*. For the students, learning has the dimensions of death, this is why it is difficult. Insofar as the authors' teaching method started by recognizing the students' arithmetical procedures, they could only reinforce the obstacle and make students more confident of their arithmetical knowledge, more attached to their past life stories, instead of developing the courage of reformulating these stories from an algebraic point of view. Paradoxically, the more we focus on the supposed "gap" in order to consciously try to bridge it, the wider it becomes. If there is a "gap" between arithmetic and algebra, it is not to be crossed; it has to be *ignored, forgotten, dissipated*. If we want the students to think algebraically, we have to start by assigning them typical tasks of the algebraic domain. We have to seriously take into account that reflective abstraction implies that "every cognitive system relies on the following one for guiding, and the achieving its regulation" [Piaget, 1975:40]. There is no path to the top of the mountain; we have to parachute the students up there. This is the objective of the following puzzle in its settings: manipulative and computerized.

An algebraic puzzle: the *doublequal*

The material consists of a board with two pairs of squares connected by equality signs, and black and white pieces of three different shapes, say black and white stones (▲, Δ), black and white buttons (●, O), and black and white Montessori cubes (❑, ■). To start the activity, four handfuls of randomly chosen pieces are spread in each of the four squares. This will be called the *initial situation*. Examples of initial situations are in figures 1a and 1b.

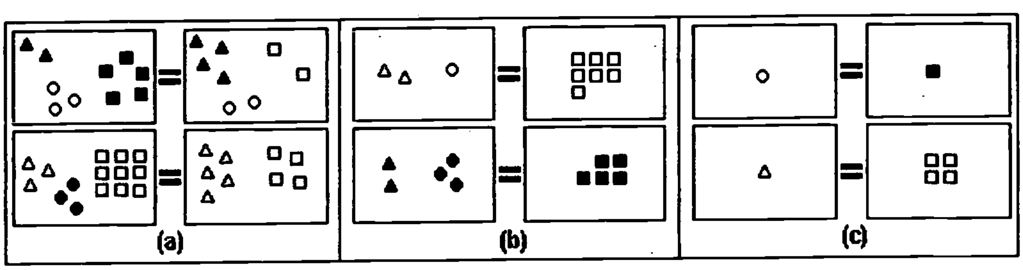

Fig. 1 - (a) Randomly chosen initial situation
(b) Simplified initial situation
(c) Final situation

The objective is to pass from the *initial situation* to a *final situation* through a series of *intermediate situations*. In the final situation, square A must contain only white stones (Δ), square C must contain C only white buttons (O) and squares B and D only Montessori cubes, either white or black (❑ or ■). See figure 1c.

The activity abides by this one single rule: one passes from a situation to the next by simultaneously adding (or removing) *equivalent handfuls* of pieces to (from) squares A and B or to (from) squares C and D.

The equivalence of handfuls is given by the following rules:

1. two handfuls of pieces are equivalent if they are made of the same number of pieces of each shape and color (general equivalence) or if:
2. they occupy squares A and B or squares C and D in any of the situations (local equivalence).
3. A handful of pieces consisting of equal number of pieces of different colors is considered equivalent to the empty handful (cancellation).
4. Dividing or multiplying the number of pieces of two equivalent handfuls by the same integral number leads to equivalent handfuls.

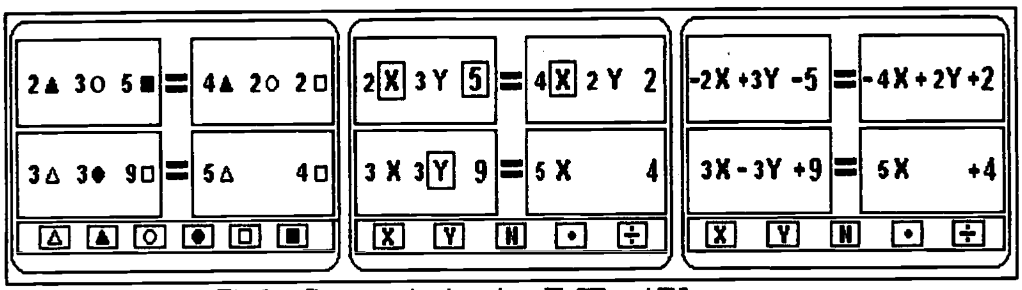

Fig.2 - Computerized settings II, III and IV.

Besides the manipulative, there are four computerized settings. The first one reproduces the manipulative setting (fig. 1). The following settings are shown in figure 2. Notice that in the last setting, *predicative signs* are turned into *operative signs*.

Discussion of pilot studies

Unlike Dienes and Gategno, we are not "trying to realize a perfect correspondence between the structure of the mathematical knowledge involved and the structure of the educational material" [Szendrei, 1996:420], nor are we assuming that the puzzle hides any kind of "hidden or frozen mathematics" [Gerdes, 1996:914]. We are assuming that buttons, stones and cubes are three-dimensional signifiers whose meanings are given by positions, movements and gestures. The material should be regarded as an amplifier of language resources, nothing else.

Pilot studies on the manipulative material were carried out with mathematics teachers, pedagogy and computer undergraduate students and high school 6th–graders. The aim of the preliminary studies was to: 1) adjust the written form of the rules; 2) verify the amount of extra help that should be given for the players to understand the rules; 3) verify whether the students became engaged in solving the puzzle.

The single rule of the *doublequal* is the pivotal point offered to the students around which they can start developing a new life story; a story of operational proficiency instead of one of failure. Further studies may determine what will become of the arithmetical domain for students who have passed through the doublequal: will

they use algebraic strategies to solve former arithmetic problems? Will they spontaneously group multiple occurrences of the unknown and operate simultaneously on both sides?

The pilot studies[3] revealed that the nature of the demand makes a big difference. In the first experiments, we made reference to a balance in order to introduce the operational rules. Whenever we tried this simplification, most of the groups started developing trial and error strategies. We tried to assign more difficult tasks to these groups, forbade them to use pencil and paper, and proposed situations where the solutions would not be integers. Instead of looking for another method, they stubbornly went on specializing their method of systematically investigating solutions with denominators 2, 3, etc. When we finally showed them the substitution methods, they revolted and complained that we were "cutting their creativity". However, even in groups with expert mathematics teachers, it took quite some time to identify the presence of a linear system behind the idea of the puzzle.

The computerized setting is programmed according to the mathematical rules of the linear systems, not according to the step-by-step manipulation to pass from one situation to the next, as stipulated in the rules. Therefore, the players may skip situations by condensing several transformations into one. Mathematically, this amounts to performing composition of operators *in action*. Operations on operators are necessary in order for them to become reified as objects.

"A person must be quite skillful at performing algorithms in order to attain a good idea of the 'objects" involved in these algorithms; on the other hand, to gain full technical mastery, one must already have these objects, since without them the process would seem meaningless and thus difficult to perform" [Sfard, 1991:32].

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COGNITIVE PROCESSES IN A SPREADSHEET ENVIRONMENT

Alex Friedlander
The Weizmann Institute of Science, Rehovot, Israel

This paper investigates the way two students used spreadsheets to solve an algebra task and analyzes some cognitive aspects of their work. Frequent attempts to generalize the problem situation and to justify these attempts and repeated revisions of their solution, led the students along a rather intricate solution path. Some cognitive skills seem to have been promoted, but also some difficulties were raised, by the use of spreadsheets.

Introduction

Various studies have reported on significant cognitive processes that occur when students work with spreadsheets in the early stages of learning algebra. Sutherland and Rojano (1993), from their own and colleagues' findings, found that processes such as naming a variable, representing and testing mathematical relationships, generalizing from arithmetic, and extending informal arithmetic strategies, are facilitated by work with spreadsheets. Rojano (1996) concluded that current studies on learning algebra in a spreadsheet environment "show that spreadsheets can be a substantial support in the development of essential aspects of algebraic thinking".

In this study, I examine cognitive and metacognitive processes supported by the use of spreadsheets, as students solve investigative algebraic tasks. The analysis is based on the observation of a pair of students in a beginning algebra class, working with spreadsheets (Excel) on an activity. The particular videotaped section was selected from fifteen videotapes of pairs of seventh graders working on algebra tasks, recorded during a whole year of weekly observations. The section is taken from three lessons considered by the four observers as representative of both the nature of the tasks, and of the learning processes observed throughout the year, and therefore worth analyzing in depth. I will focus on processes of generalization and justification, in general, and of algebraic modeling and explaining in a spreadsheet environment, in particular. In addition to the role of the computer, the social interaction between the two students was an important factor in the development of the solution path. However, due to the limits of this paper, the social aspect of the students' activity will not be analyzed.

Background

The spreadsheet environment described in this study is part of a beginning algebra course in a technology-based curriculum development project for grades 7 to 9. The course for grade seven includes the use of Excel, at the rate of one or two (of five) lessons per week. Most of the learning units are based on open-ended problem-situations. The use of computers in this project in general, and of spreadsheets during the first year of learning algebra, in particular, is aimed to support

- students' active construction of knowledge and cognitive abilities such as experimentation, prediction of results, modeling, generalization of patterns and justification of outcomes (Friedlander et al., 1989).
- students' ability to produce and analyze a variety of representations of a problem situation and of its solution.
- the development of students' metacognitive abilities, such as reflection upon observed phenomena, awareness to their own thought processes, self-regulation and control (Hershkowitz & Schwarz, in press).

The cognitive and metacognitive processes involved in working with Excel will be discussed by analyzing the videotaped work of two 13 year-old students (a boy and a girl) on a quite complex problem. The two students attended an urban, selective (although not aimed for the mathematically gifted) school, and were described by their teacher as having high mathematical ability.

The twenty-minute section discussed here occurred during the last period in a sequence of four lessons, devoted to the *Chocolate Cake* problem-situation. Figure 1 presents the problem and a generalized pattern of its solution.

A baker in the *Land of Oz* bakes cubical cakes. The edge length of the cube is called *the cake number*. He covers the cakes with chocolate icing (including the bottom...), cuts them up into unit cubes (called *slices*) and sorts them into four categories, according to the number of sides covered by icing. Then, for each cake, the baker packs *all* the slices of the same kind into a bag. The price of a bag is determined by the number of slices and by their kind. The chocolate is the only expensive ingredient, and it costs 1/2 zooz (an extinct, ancient currency) per A.U. (area unit). The completely uncoated slices are "given away to charity" and will not be considered.

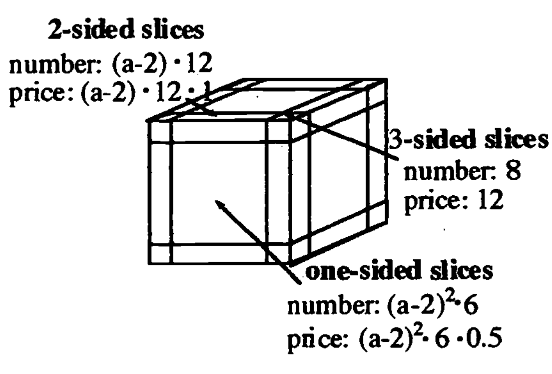

Figure 1. *The Chocolate Cake* Problem and its generalized solution for an a x a x a cube.

During the first three lessons, the students investigated the structure of a cube and the location and the *number* of slices in a whole cake. The goal of the last lesson in this sequence was to use previously obtained results about the *number* of slices, to model and investigate ways of pricing slices of different categories. Figure 2 presents possible expressions (written in spreadsheet syntax) that students are expected to use in order to complete the price table.

	A	B	C	D	E
1	Cake No.	Price of 3-sided slices	Price of 2-sided slices	Price of one-sided slices	Total price
2	1	0	0	0	3
3	2	12	=(A3-2)*12	=(A3-2)^2*6*0.5	=B3+C3+D3
4	3				

Figure 2. Price table for *The Chocolate Cake* Problem.

Cognitive Processes

The observed students worked relatively quickly and their familiarity with general expressions was apparent, as expected. However, they followed a rather "zigzag" solution path, in which the repeated attempts to generalize the price of the slices of a certain kind were not followed through systematically. Figure 3 presents an overview of their solution -- the formulae produced, their rationale and the reason for each revision. The task required the students to make four generalizations -- three for the price of different kinds of slices and one for the total price. For each generalization the students made two to four revisions, until they obtained a formula which they considered satisfactory. The revisions were caused by a variety of reasons, such as unreasonable computer output, peer discussion or intervention by a neighbor or by the teacher.

In the following, I analyze cognitive aspects of the students' work, in three categories: (a) generalizing, (b) analyzing results and (c) explaining and justifying. The numbers in square brackets refer to the versions of the formulae in Figure 3.

a) *Generalizing*. The activity took place after two months of weekly sessions with Excel. The students were familiar with the language of Excel and employed it naturally in their work and for communication. The observed pair produced their expressions in Excel format, from the start. Occasionally, the discussion of a formula was accompanied by verbal interpretations, as in [2.2]:

Boy: *Equals [parenthesis] A2 minus 2 -- this gives the... what's on each edge... and now multiplied by 12...*

Girl: *and divided by 2, because this is the price.*

Topic	Version #	Formula	Rationale	Cause for revision
Price of 3-sided slices	1.1	8	Considering the number of slices (and not their price).	Peer discussion (first attempt).
	1.2	4	Applying the principle "*all* slices cost 1/2".	Neighbor intervention.
	1.3	12	Adjusting the price of a slice to 1 1/2.	Neighbor intervention.
Price of 2-sided slices	2.1	(A2 - 2)*12	Considering the number of slices (and not their price).	Peer discussion (first attempt).
	2.2	(A2 - 2) * 12 / 2	Applying the principle "*all* slices cost 1/2".	Peer discussion.
	2.3	(A3 - 2) * 12 / 2	Adjusting the variable for the third line of the table.	Peer discussion (confronting unreasonable computer output).
	2.4	(A3 - 2) * 12	Adjusting the price of one slice to 1.	Peer discussion.
Price of one-sided slices	3.1	(A3 - 2) ^ 2 / 2	Considering the price of 1/2 per slice (ignoring the cube's 6 faces).	Peer discussion (first attempt).
	3.2	(A3 - 2)^2 * 6 / 2	Considering the fact that the cube has 6 faces.	Teacher intervention.
Total price	4.1	A3 + B3 + C3	Totaling the first three columns.	Peer discussion (first attempt).
	4.2	B3 + C3 + D3	Adjusting the sum to the second to fourth column.	Peer discussion (confronting unreasonable computer output).

Figure 3. Solution of the *Chocolate Cake* problem.

Even constant numbers were sometimes introduced as formulae (e.g., =8 in [1.1]). In most cases, however, the students employed the language of Excel as a means of communication, without further interpretation. From their discussions, it is clear that both students were aware of the general nature of their expressions. Sometimes, they stated this specifically (*G: For the three-sided slices, the price is always 8.*) The students' awareness of the general nature of their solution can also be deduced from their copying of the formulae for Cake Numbers 2 to 20, as opposed to their separate handling of Cake Number 1.

However, the students' familiarity with generalizing in a spreadsheet format did not prevent some technical errors. Thus, the use of cell names as variables caused some difficulties. During their first steps, the students used A2, rather than A3, as a variable in their expressions [2.1, 2.2]. This shift in the referred cell was probably due to previous experiences in which the students wrote their formulae in the second row (with the first row containing the column headings). Later on, the boy also attempted to find the cakes' total price by summing the numbers in columns A, B and C [4.1], rather than those in columns B, C and D.

Some of these errors could be prevented, by pointing and clicking with the mouse to the referred cell, rather than keying the referred cell's name into the formula. Sutherland and Rojano (1993) observe that pointing at cells can be advantageous and cognitively easier. However, our project staff frequently observed that a pair of students working together on a computer tend to divide between them the manipulation of the mouse and the keyboard. This division of work makes the method of reference by pointing and clicking almost impossible.

b) *Analyzing results*. The students scrutinized the numbers produced by each of their expressions. If the numbers seemed reasonable, they moved on to the next step of their solution, without any further remarks. There were two instances, however, when numerical results in the table created conflicts. In one case, the girl noted that the total price of Cake Number 2 should be 12 and not 14, as it appeared on their screen. This triggered a further inquiry that led to the necessary correction of the formula for the total price [4.1, 4.2].

In the second case, both students were puzzled by the negative numbers produced by their (wrongly referenced) formula for the price of the two-sided slices [2.2] First they wrote it in the cell corresponding to Cake Number 2. When they received a negative number (-6) they attributed this to the fact that, in this case, the number of two-sided slices is zero. They replaced the formula by zero and wrote the same formula again [=(A2 - 2)*12/2] in the next line for Cake Number 3, and copied it downwards. Again, they received a sequence of numbers increasing by 6 and starting with -6. The following conversation took place at this stage:

Both: *Why -6 !?!*
Girl: *Let's see, 3 less 2 is 1, multiplied by 12 it's 12, divided by 2 it's 6. Why -6?*
Boy: *I'm sure that the computer is wrong...*
Girl: *[Pointing at the lower part of the same column] Look... look - here it's positive.*
Boy: *Let's see till where it's negative.*
Girl: *What's this negative?*
Boy: *This doesn't make sense. Here it's zero? At [Cake No.] 4 it's zero?!*
Girl: *4 less 2 is 2, multiplied by 12 it's 24, divided by 2 it's 12.*
Boy: *[Looking at the edit section of the spreadsheet that shows the formula for Cake No. 4] A3... A3... Why A3?!*
Girl: *Wow!! because A2 is this [points at the top of the first column]*
Boy: *Then, we should have done here...*
Girl: *[Referring to Cake No. 2] Right! Write A3.*

As a result of this revision, they wrote a formula that had the correct reference and produced positive numbers, but still gave an incorrect price [2.3].

The episodes described above provide additional evidence that problem solving with a computer, naturally shifts the traditional emphasis from computational work to the design of a corresponding model for the problem situation, and to monitoring the out coming results (Heid, 1995).

However, as shown in three of the cases described above [2.3, 3.1, 4.1], monitoring and reflection on the nature of the numerical output is not always sufficient. For example, the pair did not detect their error regarding the price of the one-sided slices [3.1], since the formula produced positive and reasonably sized numbers. This error was corrected only as a result of teacher intervention. In this and many other cases, we have seen that the fact that computers provide large quantities of data in a short time may cause an over-reliance on "reasonable" data, as opposed to monitoring solution processes and reasoning.

c) *Explaining and justifying*. During a relatively short period of 20 minutes, the observed pair produced 26 explanations of various kinds and length. About two thirds of these explanations were given by one of the students as answers to the other's question, or in order to help the partner understand a certain aspect of the solution. The issues discussed in these explanations can be categorized as follows:

- General explanations -- relating to the (correct or incorrect) generalizations made about the price of different kinds of slices. These explanations employed general terms relating to the cube's structure, the pricing policy etc.

For example, when dealing for the third time with the price of the two-sided slices, the boy realized that there is no need to divide by 2, as they had done before, when they considered the price of 1/2 for one unit area, but disregarded the fact that these slices are iced on two area units. He provided the following explanation:

Boy: *Why divide by 2?*
Girl: *To find the price.*
Boy: *Why do we have to divide by 2? Are we stupid!!*
Girl: *Why not?*
Boy: *Because each of them is 1 in any case.*
Girl: *But thus we get the number of slices - right?*
Boy: *But each of them equals one A.U. [He means one zooz] and this is how it should be - without dividing by 2.*
Girl: *Right, right...*

- Local explanations -- relating to Cakes Number 1 and 2. The first case requires a separate solution. The general formulae can be applied for Cake No. 2. However, whenever a general formula did not produce a reasonable output, the students used numbers and tried to generalize for Cake No. 3. They explained their local solution by the fact that Cake No. 2 has "zero" one-sided, and two-sided slices -- thus implying that these are exceptions.
- Explanations of context -- were provided as reasons to correct errors that resulted from disregarding the characteristics of the problem (for example, when they considered the price of any slice to be 1/2 or disregarded the fact that the cube has six faces), or to remind the partner about a context-related conclusion, already reached in a previous stage of the solution (for example, the girl constantly reminded the boy to divide by 2, to get the formula for the price).
- Technical explanations -- relating to some computer technicalities (e.g., how to drag two cells in order to get the sequence of natural numbers), the change of the variable from A2 to A3 (see the section on generalizations), or the boy's passing remark that "the computer is wrong", since it produced negative numbers.

More than 40 percent of the explanations were of a general nature (i.e., they referred to all the cakes, they employed general terms and many of them were related to algebraic expressions) and the rest were equally divided among the three other categories. The large number of explanations can be attributed to the complex nature of the investigative task and to the pair's interactive work, but a main cause for their high level is the nature and the requirements of a spreadsheet environment.

Conclusion

The observation of the cognitive processes involved in solving an algebraic problem with Excel, showed some significant advantages for using spreadsheets in algebra.

Work with Excel allows a natural transition between the world of particular numerical cases and the general world of algebra. The need to use formulae as a means to produce many numerical examples, emphasizes the general nature of the algebraic expressions. The observed students considered spreadsheet formulae a natural tool, that creates numerical data needed to understand, analyze and solve a problem situation. As a result of using spreadsheets, they could develop cognitive and metacognitive skills (transitions between numerical and algebraic representations, generalization and justification of patterns, discussions of solution methods, analyzes and evaluation of outcomes) of a wide variety and at a high level. The students were also released from the burden of calculations and algebraic manipulations.

Besides these advantages, the observed students encountered some cognitive difficulties. Some of these can be attributed to spreadsheets. The spreadsheet syntax, although relatively simple, seemed to cause some difficulties -- such as the use of an incorrect reference as a variable. Spreadsheet capabilities can also turn out to be obstacles. Thus, the creation of numerical data in large quantities can cause an over-reliance on the "reasonability" of the output and diminish the need for an in-depth understanding of the problem at hand.

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EXPLORING STUDENTS' IMAGES AND DEFINITIONS OF AREA

Fulvia Furinghetti* & Domingo Paola**
*Dipartimento di Matematica dell'Università di Genova. Italy
**Liceo Scientifico 'A. Issel', Finale Ligure (Sv). Italy

ABSTRACT. *This study examines several aspects of the images and definitions that eight students of high school (16 years old) have regarding area. The analysis is performed through 10 open questions to which students answered through written statements, drawings, concept maps. The protocols shed light on the ways used by students to communicate their ideas and on the role they ascribe to definitions in their mathematical experience.*

INTRODUCTION

In studying the various aspects of proof we became aware of the crucial role of definitions. In particular, we have often observed that some cause of failures are due to the role students ascribe to them and how they deal with them. What happens can be explained considering the statement to prove as having a hypertext structure: it contains 'hot words' that the student has to single out and on which to 'click' when necessary. The operation of clicking establishes a link between hot words, concept images and concept definitions (henceforth called images and definitions) behind them to get the information useful to go on in proving. If one of these procedures (recognising hot words, clicking on them, getting information) is not activated the statement is obscure and to prove is a hard task. The research reported in (Furinghetti & Paola, 1997) may be an exemplification of this hypertext metaphor. We have found that the statement 'Prove that the product of any three consecutive natural numbers is divisible by 6' resulted difficult to prove for most students (aged from 14-17) because they did not recognised that 'divisible' was a 'hot word' in the statement to prove or, when they did, were not able to use the definition of 'divisible' that has been introduced to them previously. In that case we had the impression that students did not ascribe cognitive value to definitions, they seemed to perceive them only as labels which are not relevant to the mathematical work. Other authors have observed analogous students' behaviour. Rasslan and Vinner (1998, p.33) write that the student 'does not necessarily use the definition when deciding whether a given mathematical object is an example or non example of the concept. In most cases, he or she decides on the basis of the concept image, that is, the set of all the mental pictures associated in his/her mind with the name of the concept, together with all the properties characterising them'. Bills and Tall (1998, p.105) have introduced the expression 'formally operable' for a given individual to indicate a (mathematical) definition or theorem which an individual is able to use 'in creating or (meaningfully) reproducing a formal argument'. We can say that for our students the definition of 'divisible' was not operable.

We quote the following passage, taken from Wheeler (1991, p.1), to summarise some important points on the role of definitions in proving:

'When we talk about proof, it seems very often that we shift quite unconsciously between talking of proof as something technically proven and proof as a sign for formalism. I wish we would give more attention to the business of definition rather than that of proof *per se*, or at least that we would really take into account of the act of making definitions, because if one's talking about the normalization of mathematics it seems to me that it's in the definitions that we find the vectors of mathematics: these are things that we choose to define this way because they have a future, because they will go somewhere, because we can do something with them. Now, proof when it's finished, is finished. Of course, there are proof techniques that we can apply in other cases, so I mustn't be unfair to proof and say that we can't do anything with it other than just prove one particular theorem, but definitions *always* have to be shaped in order to point us onwards, so that we can go somewhere, otherwise they would be bad definitions that we would abandon, that somebody subsequently would change'.

OUR STUDY: METHODOLOGY AND AIMS

On the ground of the previous observations we consider very important to explore students' behaviours in defining and to promote classroom activities which lead students to reflect on definitions. In the present paper we report an activity of this kind centered on the concept of area. In carrying out this activity our aim was twofold: from one hand it was to investigate which images students have elaborated of the concept of area and how make them explicit, from the other hand our aim was to promote cognitive activities about area. The concept considered is recognised as very difficult, see (Douady & Perrin, 1989), but is not studied so much from the educational point of view, in particular at the level of high school.

The population is constituted by eight 16 years old students of a Scientific Lyceum, an Italian high school in which mathematics plays an important role. They were requested to answer the questions reported in Table 1 (next page), using written statements, drawings, concept maps. We explained them our purposes and asked for an active collaboration. They worked with good willingness and fulfilled our expectation. The allowed time was 50 minutes.

We tried to structure the questionnaire in such a way that students' thoughts and possible inconsistencies could emerge. Question Q.1 is aimed at verifying whether students' images of plane regions are only based on elementary patterns (polygons, circle) or include any kind of shapes. The distinction is significant since in the first case students may have difficulties in thinking to situations in which a formula to compute area based on elementary operations does not exist. Question Q.2 is aimed at outlining the nature of the concept in question through links established with other mathematical concepts. Questions Q.3, Q.4, Q.5 and Q.6 were conceived to stress the specificity of the languages used in different school situations. These questions were inspired by Austin and Howson (1979) who observe that in classroom there are different mathematical languages: the language used with mates, with the teacher and the language of mathematics. The remaining questions are mainly aimed at orienting students towards activities of metacognition, that is to say, using the Schoenfeld (1987) expression 'thinking about thinking'. In particular

Question Q.8 is focused on the possible origin of students' misunderstandings. Question Q.9 is addressed to see whether new forms of representation of knowledge may act as a stimulus to make explicit concept images.

Q. 1. When one talks about the *area* of a plane figure, which images are evoked to you? Represent some of them (three or four) in your protocol.
Q. 2. Which parts and which topics of mathematics do you link to the notion of *area of a plane figure*? Write in your protocol and try to give an explicatory example for each issues you refer to.
Q. 3. To explain to a fourth-fifth grade student what *area of a plane figure* means, what would you say?
Q. 4. To explain to an eighth grade student what area of a plane figure means, what would you say?
Q. 5. To explain to a mate of yours what *area of a plane figure* means, what would you say?
Q. 6. To say to your teacher what *area of a plane figure* means for you, what would you say?
Q. 7. Are you surprised about the questions 3, 4, 5, 6? In other words is it possible to use different characterisations of the same concepts according on the person you are addressing to?
Q. 8. Are there analogies and differences between the use and the meaning of the word *area* in the common and in the mathematical language.
Q. 9. Construct a concept map about the concept of *area*.
Q. 10. Reconstruct the didactic path followed in your school career until now as for the concept of *area*.

Table 1. The questionnaire on the students' images and definitions of area

ANALYSIS OF THE ANSWERS

In analysing the protocols we have singled out key elements and orientations. We have reported them in Table 2 (which takes the next two pages) to give a synoptic view of answers. In abbreviating and translating sentences some nuances of meanings are lost, but we feel that the basic issues of the students' reactions are kept. Table 2 has two entries: reading horizontally we have the answers to the same question by the eight students, reading vertically we have the picture of each student's images and definitions as emerged from protocols. In the following we give our interpretation of the findings reported in Table 2.

Question Q.2 offers insights on the *students' difficulties* about area. Only two students (Protocols 3 and 6) mention the 'measure'. Nobody refers to real numbers, which for us was the most obvious association. Another missing link for all students except one (Protocol 7) is that with the concept of function. In comparing the answers to Q.1 and Q.2 we find inconsistencies (see rows referring to Q.1 and Q.2 in Table 2) which are generated by the passage from the simple geometrical/visual

Table 2. Schema of the main points in students' answers

Q	Protocol 1	Protocol 2	Protocol 3	Protocol 4	Protocol 5	Protocol 6	Protocol 7	Protocol 8
1	Polygons, circle, other figures	Polygons, circle	Polygons, circle, other figures	Polygons, circle	Polygons	Polygons, circle	Polygons, circle	Polygons
2	Intersection of at least 3 half-planes	Algebraic inequations (first degree, absolute value)	Word definitions, formulas to compute areas, unit of measure	Concept of measure, number, Eucl./anal. geom., algebra, computer	Algebraic inequations and systems of inequat. (I and II degree)	Plane geometry, computing the measure of an area	Set intersection, set theory, inequations, functions, linear programming	Geometry
3	Portion of the plane contained in curved lines	The space occupied by the figure	Quantity of substance covering up the figure	I would draw a figure and would hatch the area	Space delimited by segments or closed curved lines	Space inside to the segments used to draw the figure	Space contained in the perimeter of the figure	I would draw and color with a pencil
4	Portion of the plane contained in segments or curved lines	Portion of the plane contained at the interior (of the figure)	Part of the plane defined by the sides of the figure	Part of the plane delimited by a closed broken line	Set of points interior to the perimeter	Measure of its surface	Space occupied by the figure in the plane	Definition: area is all is contained in the segments of the figure
5	The same as in 4	Portion of the plane contained at the interior	I don't know	Portion of the plane delimited by a closed broken line with only two dimensions	Intersection of half-planes defined by inequations	Measure of its surface which is delimited by its perimeter	Measure of the surface of the figure	It is called area of a plane figure the part of the plane contained in the perimeter of the figure (concave or convex)
6	Portion of the plane which is the intersection of many half-planes delimited by a system of equations	As in 5	I don't know	As in 5	As in 5	As in 5	As in 5	As in 5

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Q	Protocol 1	Protocol 2	Protocol 3	Protocol 4	Protocol 5	Protocol 6	Protocol 7	Protocol 8
7	Not surprised because it is necessary to use known concepts and to consider the capacity of understanding. I don't remember what I knew	Not surprised. It is correct to use different languages according to the age	Not surprised. It is correct to use different languages according to the age	Not surprised. It is correct to use different languages according to the age	Not surprised. It is correct to use different languages according to the age	Not surprised. It is correct to use different languages according to the age	Not surprised. It is correct to use different languages according to the age	Not surprised. It is correct to use different languages according to the age
8	Analogies: an interior surface. Differences: in the common language is only a measure	In the common life the area is only a surface and not a portion of plane	No: the concept is always one, the differences between the common life are in the fact that there is a practical use	Analogies: it is a zone delimited by a broken closed line. Differences: it is of 3 dimensions in common language, of 2 in math	No answer	I am not able to answer	In the common language the area is confused with the surface, while in math are different concepts	I don't know
9	Half-planes, systems of inequations, Cartesian plan. *Double arrows, single in-out*	Figures, formulas, the four arithmetic operations, transformation of figures in rectangles, pi. *Double arrows*	Figure, definitions rules, unit of measure. *Single arrows*	Equations, real numbers, measure, geometry, drawing, computer. *Double arrows, single in-out*	No answer	Plane geometry, definition of plane figure, concept of surface. *Single arrows*	Figure, plan, surface, measure. *Double arrows*	Def. of plane figures, def. of area of a figure, formulas to compute area. *Single arrows, single in-out*
10	I have not retentive memory	In primary: formulas without explanation. In High School: the teacher tells where formulas come from	I don't remember	Both in primary and secondary: measure. In High School there is more abstraction	No answer	Examples, definitions of figures, concept of area and surface. I don't remember other things	In primary: a precise concept, but easy to understand. Increasing the school level definitions are more complete, but more abstract	I don't remember
Q	Protocol 1	Protocol 2	Protocol 3	Protocol 4	Protocol 5	Protocol 6	Protocol 7	Protocol 8

The italic indicates the style of the signs in concept maps. *Single in-out* means at most one arrow going in and out from a node

context to richer contexts encompassing arithmetic, algebra, analytical geometry. In Protocols 1, 2, 5 the geometrical shapes mentioned in the row of Q.1 are not in relation with the answers to Q.2. In principle, we consider important to encourage students to go across different parts of mathematics: this could promote the flexibility advocated by Tall and Gray (1993). Our findings show that this process needs of careful control. Going across contexts does not always create meaning, in some cases may cloud the existing one so that what is generated is not fruitful communication between contexts, but rather *contamination of contexts*.

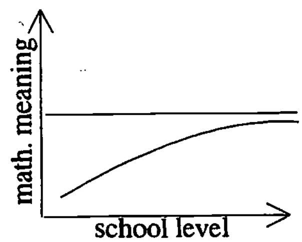
Questions Q.3, Q.4, Q.5, Q.6 shed light on *the evolution of the way used to communicate mathematical ideas according to the interlocutor*. At the primary level there are manifest references to concrete elements: 'space occupied by the figure' (Protocol 2), 'quantity of substance covering up the figure' (Protocol 3), 'to hatch the figure' (Protocol 4), 'to colour the figure with pencil' (Protocol 8). There is a precise use of efficient 'ostensives' (hatching, colour). Things change when the school level increases: already in the statements addressed to eight grade students the language is more formal, which does not mean necessarily more precise, nor more oriented to generalisation and abstraction. Students do not add useful information, they only paraphrase the statements addressed to younger students eliminating ostensives and concrete ideas and adding negligible details. For example, in Protocol 2 the expression addressed to the fourth-fifth grade student 'space occupied by a figure', which evokes a physical situation, is substituted by 'portion of plane contained at the interior' when addressing to the eight grade student. The loss of spontaneity is evidenced in Protocol 8 where we observe a real escalation: - the ostensive for the fourth-fifth grade (coloured pencil), - the semi-intuitive for the eight grade ('what is contained in the segments') - the formal expression ('It is called'). It is curious to observe that to use the words 'It is called' was advocated by Smith (1911) in his famous treatise on mathematics teaching to 'mark the statement at once as a definition' (p.158), that is to say to distinguish it from a theorem. Protocol 8 evidences a different behaviour according to the age of the interlocutor: with the youngest the student tries to make understandable his message using the 'common sense', with the oldest and with the teacher he is only interested in conveying the idea that he is defining something, no matter if what he is saying is understandable or not. Another aspect of the evolution in the way of communicating according to the school level is the relationship general/generic/particular. In Protocol 8 the student adds the adjectives 'concave or convex' to the word 'figure' for mates and for the teacher. In Protocol 1 we find a similar behaviour: to the fourth-fifth grade student it is said 'portion of plane contained in *lines*', while for the older students the sentence becomes '[...] *segments or curved lines*' [emphasis is our].

As for Question Q.7 all students agree that there has to be a distinction between the languages used in the four different situations of Questions Q.3 to Q.6, but from their explanations we infer that the difference is in the form, not in the substance. Advancing in the age there is no gain in generality, abstraction or elegance, while

there is loss in meaning. The creativity seems inhibited along the years; the students' behaviour can be referred to the ritual and symbolic schemes discussed by Harel and Sowder (1996) in their studies on proof.

Question Q.8 adds further information *about the relationship between mathematical knowledge and common sense*. In Protocol 1 it is pointed out that in the common language the area is considered *only* [emphasis is our] as a measure. The student is not able to accept the right suggestions coming from the common sense, even more it seems that the common sense is perceived as something against mathematics.

Analysing our findings we feel that the students' capacity to grasp mathematical meaning through personal and autonomous elaboration of ideas has the asymptotic trend illustrated in the figure beside. At the age of the students examined (16 years) the upper bound seem to be already reached. In Protocol 2 we found the statement



'At the lyceum the teacher tells us where formulas come from', which is not the same as saying that the teacher explains how to find formulas. Freudenthal (1973) has criticised the teachers' practice of providing students with definitions given a priori instead of constructing them together with students. May we interpret the student's statement as a criticism in this direction?

Concept maps

At the moment of answering questions students did not know what 'concept map' means. We simply explained them that it is 'a graphical representation of domain material generated by the learner in which nodes are used to represent domain key concepts, and links between them denote the relationship between these concepts', see Jones (1998, p.161). To analyse the students' protocols we have considered:
- the number of issues reported in the concept map
- if arrows are single or double, how arrows go in and out at the nodes
- the presence and the kind of explanations of links
- the presence of the word definition or its derivative
- the type of iconic representations used (only words, words in boxes, etc.)

To single out the differences when using the verbal language or concept maps we have compared the number and the type of issues in the answers to Q.2 (that about the links with parts of mathematics) and in the concept map. Protocol 4 has the same number of issues (six, the highest among the eight students examined) and of the same type in both cases. The concept map of this protocol is very rich and it is the only in which links have a written explanation. In Protocol 7 there is the same number of issues (four) in Q.2 and in the concept map, but the issues mentioned are different. In Protocol 5 there is one issue in the answer to Q.2 and there is not the concept map. In the other protocols the number of issues appearing is higher in concept map than in Q.2.

About the different styles we can distinguish:

- those who simple translate into the iconic form their word statements (for example, Protocol 1)
- those having a linear pattern to approach the concept ((for example, Protocol 3). The arrows are single, at most one arrow arrives to and leaves from nodes. The focus is on the order of facts more than on links between them. The word definition appears among the nodes
- those who really add information making the links more explicit and finding new issues ((for example, Protocol 4).

PRELIMINARY CONCLUSIONS

The concept of area, through its epistemological and conceptual aspects and its character of being a cross-roads of different parts of mathematics has revealed itself suitable to show that students have in their mind a jungle of concept images, concept definitions, which are not completely under their control. Moreover for them the problem is not only to elaborate images which can flow into definitions mathematically acceptable, but to find means which may make them explicit, consistent and clear to others and to themselves. From our study we have obtained a number of indications for the classroom practice as well as topics which deserve attention for future research. A point we would like to stress as a first preliminary conclusion is the importance of making students to reflect not only on their way of thinking, but also on the way of representing their thoughts, so that to the Schoenfeld's expression 'thinking about thinking' we may add its paraphrase 'thinking about representing'.

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A NUMERACY ASSESSMENT FRAMEWORK FOR THE INTERNATIONAL LIFE SKILLS SURVEY

Iddo Gal

University of Haifa, Israel

Abstract. *This paper describes a conceptual framework for defining and assessing numeracy of adults as part of the planned International Life Skills Survey. The paper reviews some differences between assessment of mathematical knowledge of adults and younger students, discusses the notion of numerate behavior, presents a definition used to guide construction of assessment tasks, and outlines some dilemmas involved in conducting a large-scale assessment of adults' numeracy.*

Background

This paper describes a conceptual framework developed for assessing numeracy of adults as part of the International Life Skills Survey (ILSS) planned for the year 2001. This comparative survey is being jointly developed by Statistics Canada and by the United States' National Center for Education Statistics (NCES), in cooperation with the Organisation for Economic Cooperation and Development (OECD). This paper is based on a report prepared for the ILSS project by the Numeracy Working Group[1], comprised of individuals from Israel, Australia, Holland, United States, and Canada.

The ILSS project is a follow-up to the International Adult Literacy Survey (IALS), the world's first large scale comparative assessment of adult literacy. In the first phase of IALS, almost 21,000 adults from 7 countries, including the U.S., Canada, Germany, the Netherlands, and others, were tested based on survey methodology that combined household survey research and methods of educational testing. Key reports from the IALS were published starting in 1996 (e.g., Statistics Canada and OECD, 1996). Using a similar approach involving home interviews, ILSS will test nationally representative samples of adults aged 16 and over in multiple countries. Tasks will

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assess performance in several skill domains, including *Numeracy* as well as *Literacy*, *Problem Solving,* and *Practical Cognition*; other variables will be assessed via a background questionnaire. The domain of Numeracy is planned as an elaboration on the domain of Quantitative Literacy previously included in IALS and in prior studies. .

Feasibility studies involving sample tasks from the Numeracy scale have started and results are expected in summer 1999. At this point, this paper is presented to share with researchers and educators interested in mathematics education the conceptualization of Numeracy planned for the ILSS, and to outline some of the issues involved in the development of a scale for assessing Numeracy of adults.

Introduction

Numeracy is becoming a growing concern for diverse education sectors, following its apparent low profile for many years. As countries increasingly attend to topics such as improving workplace efficiency and quality processes, to resulting continuous learning needs, and to civic participation (European Commission, 1996), it is seen as vital that nations will have information about their citizens' numeracy, among other skills, if they want to plan effective education and lifelong learning opportunities.

The concept of numeracy is specifically related to the dialogue about the goals and especially *outcomes* and *impact* of school mathematics education. More educators now encourage links between knowledge gained in the mathematics classroom and students' ability to handle real-life situations that require mathematical or statistical knowledge and skills (Gal, 1997; Packer, 1997). However, while numeracy may be a key skill area, its conceptual boundaries, cognitive underpinnings, and assessment, have not received much scholarly attention so far.

With the above in mind, this paper is organized in five parts. Part 1 contrasts some ideas behind assessments of school-based mathematical skills and of adults knowledge. Part 2 outlines key knowledge bases used to inform our conception of adult numeracy. Part 3 presents a working definition of numerate behavior to be used to guide item development for the ILSS. In Part 4 we comment on several decisions that underlie scale development, such as task range, scoring, and difficulty of items. Lastly, a Summary section outlines resulting challenges in conducting a credible assessment of adult's numeracy in a large-scale context. (Due to space limitations, the discussion of all topics in this paper is understandably brief).

1. On large scale assessments of mathematical knowledge

The IALS, following on a framework established in prior studies in the U.S. and Canada, made use of three literacy scales, Prose Literacy, Document Literacy, and Quantitative Literacy, to operationalize its conception of literacy. Quantitative Literacy (QL) was defined as: *The knowledge and skills required to apply arithmetic operations, either alone or sequentially, to numbers embedded in printed materials,*

such as balancing a cheque book, figuring out a tip, completing an order form, or determining the amount of interest on a loan from an advertisement.

Some key distinguishing features of all three IALS scales were: (a) Respondents have to deal with information embedded in *text*. (b) Tasks involve genuine or realistic stimulus materials such as those found in newspapers or official forms and documents. (c) Tasks require performance that is "realistic", i.e., what may be expected of a person in real life. (d) Use of "correct/incorrect" scoring scheme. These features, especially the emphasis on the embedding of all tasks in text, including of Quantitative Literacy tasks, were an outgrowth of IALS' mission to assess dynamic and functional perspective of literacy as a purposeful skill area in adults' life.

In contrast, large-scale assessments of mathematical skills aimed at school-age student populations have taken quite different approaches. A review of large-scale assessments of mathematical skills is beyond the scope of this paper (see Robitaille & Travers, 1992). Yet, we reiterate here that large-scale assessments made in a *schooling* context usually start from the assumption that students should have learned symbol-manipulation competencies (and hopefully understood underlying "big ideas" along the way). Such assessments (legitimately) use a certain degree of formalization of math symbols, may employ tasks which are quite contrived or relatively devoid of a realistic context, and expect some memorization of formulas and notations. This is true, for example, of the mathematical tests included in the National Assessment of Educational Progress in the U.S., of the recent TIMMS study, or even of many existing nationally-recognized tests used to qualify knowledge of *adults*, such as the national vocational qualifications system in the U.K, or the GED in the United States.

The above, of course, is not a criticism of such assessment initiatives, but just a reminder that designers of *all* assessments make conscious decisions regarding what it means to "know math" or "be able to do math", in light of their overall mission and the information needs of the end-users of the assessment. School-based assessments concentrate on how students understand, use and apply mathematical skills and mathematize problems which are related to their *formal* mathematics curriculum. An emphasis on realism of tasks is secondary, given that most students have limited world experience. Though word problems are used, large-scale assessments are not explicitly interested in performance on text-rich tasks. (Yet, many quantitative situations adults face involve mathematical or statistical information embedded in diverse and sometimes complex or terse texts).

The notion of numeracy, however, implies a bridge that links mathematics and the real world. Our goal was to develop a conceptual framework of "numeracy" that is couched in assumptions about how *adults* "know" and "do" math in the real world, using not only their formal knowledge (of mathematics, of literacy, and so forth), to the degree it exists, but also other, experience-based knowledge.

2. Perspectives on adult numeracy

Our thinking about the nature and scope of adult numeracy, and about approaches to its assessment, is informed by prior work and publications in five interrelated areas:

1. General developments in continuing and vocational education in many countries, such as in the U.S., U.K, Denmark, and Australia, showing that educational systems now recognize the need to employ skills frameworks that include numeracy.

2. Work on the nature of adult *literacy* and on the different life functions/needs served by adults' literacy and numeracy skills. In one project (Kindler, Kenrick, Marr, Tout, & Wignall, 1996), for example, numeracy is organized according to its different purposes: *Numeracy for practical purposes*, *Numeracy for interpreting society*, *Numeracy for personal organisation*, and *Numeracy for knowledge*.

3. Recent thinking about the goals of learning mathematics and about "good" assessment in this domain. Influential perspectives were, e.g., those presented following the Realistic Mathematics Education movement in the Netherlands, emphasizing the need to develop (and assess) use of formal and informal reasoning strategies (Groenestijn, 1998), the curriculum and assessment frameworks of the National Council of Teachers of Mathematics (NCTM, 1995), or suggestions from those interested in school-work links (e.g., Packer, 1997).

4. Research on the nature of adult's mathematical thinking, mathematical practices, the cognitive processes underlying how people cope with out-of-school quantitative situations (Nunes, 1992), and the impact of attitudes and beliefs on performance.

5. Current thinking about adult numeracy, both at the conceptual level as well as in terms of needed skills and needed curriculum. Various views in this area have emerged from diverse workers and groups. Two key examples are:

- Work as part of the Numeracy Project at the National Center for Adult Literacy (NCAL) in the U.S. has began to formulate the processes that may comprise numerate behavior, based on an assumption that "numeracy" refers to the capacity for effective *management* of quantitative situations (Gal, 1997). It has been proposed that adults manage situations, using both generative and interpretive skills and supporting knowledge and dispositions, and that adults do not necessarily cope with problems in ways that can be classified as right or wrong, in contrast to how students solve word problems, even if these are supposed to simulate real-life situations. Cumming, Gal, & Ginsburg (1998) have argued that several aspects of numerate behavior are not reflected in how tests aimed for adults are created, scored, and interpreted.
- Work conducted by the Adult Numeracy Network (Curry, Schmidt, & Waldron, 1996) in the U.S. consolidated perspectives from the NCTM Curriculum Standards (NCTM, 1989), the U.S. Secretary's of Labor Commission on Achieving Necessary Skills (Packer, 1997), and results of

interviews with adult learners, numeracy teachers, and employers. Seven broad themes emerged, consistent with those offered by NCTM and using similar titles, but with numerous adaptations to adults: relevance/connections; problem solving/reasoning/decision-making; communication; number and number sense; data; geometry: spatial sense and measurement; algebra: patterns and function.

3. A working definition of numerate behavior

Based on the above and other related literatures, we have sought a view of numeracy that acknowledges the diverse purposes served by adults' mathematical (and statistical) knowledge, that encompasses the different suggestions regarding the skills adults need to effectively function in home, work, community, and other contexts, and that takes into account the cognitive, metacognitive, and dispositional processes that support or affect adults' numeracy.

However, one cannot assess numeracy, but *behavior* (broadly defined). We have thus chosen to focus on *numerate behavior*, which is revealed in the response to mathematical information that may be represented in a range of ways and forms. The nature of a person's responses to mathematical situations critically depends on the activation of various enabling knowledge bases, practices, and processes.

Table 1 (next page) presents our working definition of numerate behavior. It will be used to guide development of items for a Numeracy Scale for the ILSS. The definition in Table 1 distinguishes five facets, each with several components. (Space limitations prevent the presentation of further explanations of the logic behind the choice of these facets, and sample items. Further information is available upon request).

4. Scale development, scoring, and background data issues

Range of tasks. An item pool is being developed to span different combinations of the components in each numeracy facet. (Not all tasks will be given to all respondents: the ILSS will use a "task-spiraling" design, as in TIMSS, where each respondent will be given a subset of the tasks in the item pool for each of the scales). Regarding the fifth facet, "Enabling knowledge," the main component we will utilize is the first, 'Mathematical knowledge and understanding'. This is so the content of the tasks can be understood by *users* of the assessment results in terms of common school based mathematics topics (i.e., whole numbers, basic operations; percents, decimals and fractions; measurement; geometry; algebra; statistics and probability, etc.).

Contexts. All tasks will be derived from real life situations and embedded in a real life context. In the ILSS there will be no context-free tasks, which do appear in many school-based math surveys. Stimulus material will be chosen to have different levels of embeddedness in text, from text-rich to almost text-free. (As the assessment is based on a home interview format, verbal instructions will be pre-arranged for some items).

Table 1: Numerate behavior and its five facets

Numerate behavior involves...

...Managing a situation or solving a problem in a real context...
(contexts include: everyday life; work; societal; further learning)

...By responding...
(responses to quantitative situations can involve: identifying; interpreting; acting upon; communicating about)

...To mathematical information...
(this information may involve: numbers; statistical data; measurements; money; time; shape; direction; pattern and relationships)

...That is represented in multiple ways...
(the actor in a given situation may encounter: objects; pictures; numbers; symbols; formulae; diagrams & maps; graphs; tables; math information in text) (separately or in some combination, possibly with surrounding text)

...And requires activation of a range of enabling knowledge, behaviors, and processes.
(people's thinking about and actual behavior in response to quantitative situations is supported or influenced by: mathematical knowledge and understanding; mathematical problem solving skills; literacy skills; beliefs and attitudes; background world knowledge).

Scoring. Many math surveys use a "correct/incorrect" scoring scheme. In ILSS, selected numeracy tasks will use a 3-level "correct--partial credit--incorrect" scheme, to accommodate answers to some interpretive questions, as well as for cases where adults adopt reasonable strategies but reach incorrect answers.

Difficulty. To ensure a distribution of items at different difficulty levels, the complexity of items will be pre-estimated on the basis of five general factors gleaned from prior assessments of adult literacy or of mathematical skills: (1) complexity of Mathematical information / data; (2) Type of operation / skill; (3) Expected number of operations; (4) Plausibility of distractors (including in text); (5) Type of match / problem transparency. These factors can determine, separately and in interaction, the difficulty level of most numeracy tasks. For some tasks, such as those that are more interpretive in nature, other factors that affect complexity will also be considered.

Background variables. To shed a broader light on factors related to the distribution of numeracy skills across the adult population, additional data will be gathered through a background questionnaire, regarding three topics: school mathematics experience; numeracy practices (e.g., use of calculators, getting help from others, activity

structures); and dispositions (e.g., anxiety, confidence, interest). This is in addition to information about demographic variables and about *literacy* practices at home and at work that will be collected to support interpretation of results from all the ILSS scales.

5. Summary

Key motivations for conducting the ILSS are: to inform policymakers and educators regarding levels (distributions) of various skills, including of numeracy; to explore factors associated with observed skill levels (e.g., literacy); and to examine links between numeracy (or other skills) and important social variables, such as earnings, labor-force participation, unemployment, or health-related behaviors. However, the inclusion of a Numeracy scale in the ILSS also offers a significant opportunity to develop a new conceptual framework of adult numeracy, which should be of interest to educators and researchers interested in the development and application of mathematical knowledge in purposeful contexts.

Overall, numeracy is a multifaceted and sometimes slippery construct. Our basic premise is that numeracy is the bridge that links mathematical knowledge, whether acquired via formal or informal learning, with functional and information-processing demands encountered in the real world. An evaluation of a person's numeracy far from being a trivial matter, as it has to take into account task and situational demands, type of mathematical information available, the way in which that information is represented, prior practices, individual dispositions, cultural norms, and more.

Numerate behavior obviously includes the ability to calculate or manipulate symbols but is far from being limited to it. In a large-scale survey context, assessment of numerate behavior can be accomplished through tasks couched in realistic non-school settings, with limited usage of formal notations (unlike school-based assessments), and with significant presence of text-rich tasks (given their ubiquity, at least in industrial countries), as well as of some tasks where opinions rather than computation are called for (e.g., when interpreting statistical messages). Yet, while the scale we envision may cover a broad mathematical terrain, it may still fall short of encompassing the full scope of numerate behavior, due to pragmatic considerations. Some aspects of people's numeracy skills, such as those pertaining to problem-solving strategies, or to interpretive responses and their underlying reasoning processes, cannot be fully reliably and validly assessed with the methodology presently available in the ILSS.

Full assessment of adults' numerate behavior requires further work on the conceptualization of some the facets and components of adult numeracy, as well as grappling with a host of pragmatic challenges, such as translation to different languages that will retain task characteristics, training of interviewers regarding follow-up questions or scoring of partial responses, and more. It is hoped that this brief report will facilitate a dialogue on these and other issues raised in this paper among researchers and educators interested in mathematics education.

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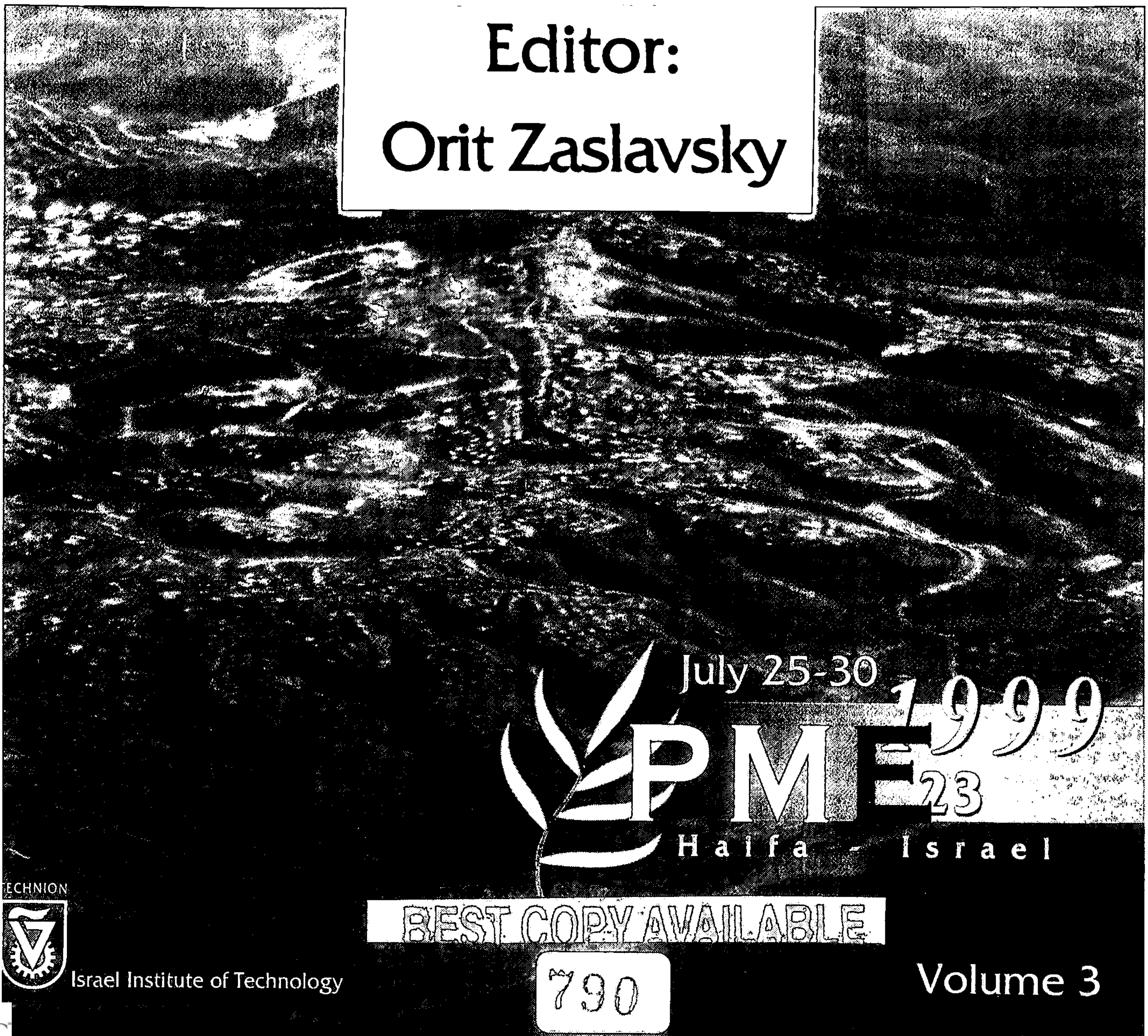

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Editor:
Orit Zaslavsky
Department of Education in Technology and Science
Technion - Israel Institute of Technology
Haifa 32000
Israel

Fax: +972 4 832 5445
E-mail: orit@tx.technion.ac.il

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RESEARCH REPORTS

Continued from Vol. 2

"What can we all say?" Dynamic geometry in a whole-class zone of proximal development.

John Gardiner, Brian Hudson, Hilary Povey
Sheffield Hallam University,UK

This paper first considers aspects of the literature relevant to class and group teaching in a social context. Ideas of socio-mathematical norms and argumentation, on the significance of local communities of practice and on the development of a whole-class ZPD are examined. These ideas have been used to influence classroom approaches to the use of dynamic geometry (Cabri II on the TI 92 with 11-14 year old pupils in the UK) and analysis of classroom observation is presented. Conclusions are drawn about the interaction of these ideas with the technology and how the alignment of mathematical meaning-making might be promoted.

Introduction

There are bodies of current research which consider, from different perspectives, the dynamics of social meaning-making in classrooms. Cobb and Yackel (1996), Winbourne and Watson (1998), and Hedegaard (1990) and Lerman (1998) all have viewpoints which can be used to inform an analysis of classroom interaction. This paper reports on the development and use of classroom material using dynamic geometry on the TI92 in lower secondary classrooms (age 11-14) in the UK. Classroom dialogue from lessons taught by the researcher was transcribed from audio recordings. This dialogue is analysed from a socio-cultural perspective, making reference to the viewpoints referred to above, and seeking to illuminate the ways in which students make mathematical meaning in areas such as construction and proof.

Literature and Theoretical Background

Vygotsky(1962) proposed a social background to learning and formulated the Genetic Law of Cultural Development, with learning moving from the social to the personal. He took up the idea of the Zone of Proximal Development as the area where interaction between the individual and the social leads to development. Lerman (1998) says of the ZPD 'it provides the framework, in the form of a symbolic space ,......for the realisation of Vygotsky's central principle of development.'(p71) Of particular interest here is a definition of the ZPD which includes the classroom as a whole, in this case incorporating the teacher, the pupils and the technology Hedegaard (1990) has reported in terms of the development of a whole-class ZPD rather than the consideration of an individual's learning:

This activity, in principle, is designed to develop a zone of proximal development for the class as a whole, where each child acquires personal knowledge through the

activities shared between the teacher and the children and among the children themselves (p 361).

Hedegaard reports in the same paper a motivational shift in children's focus, from an interest in the concrete to interest in the derivation of principles which can be applied to the concrete. Lerman (1998) takes the discussion further.

The ZPD is the classroom's, not the child's. In another sense it is the researcher's: it is the tool for analysis of the learning interactions in the classroom (and elsewhere) (p 71).

Insights into factors which might influence meaning-making in a whole-class or group ZPD can be drawn from the literature and indicate socio-cultural vectors which may operate for meaning-making. These include Local Communities of Practice, Socio-mathematical Norms and Choice of Materials.

Local Communities of Practice and Telos

Drawing on work by Lave and Wenger (1991) and Lave (1993), Winbourne and Watson (1998) have used the idea of 'local communities of (mathematical) practice'. They identify features of a local community of (mathematical) practice :

- pupils see themselves as functioning mathematically within the lesson;
- within the lesson there is public recognition of competence;
- learners see themselves as working together towards the achievement of a common understanding;
- there are shared ways of behaving, language, habits, values and tool-use;
- the shape of the lesson is dependent upon the active participation of the students;
- Learners and teachers see themselves as engaged in the same activity.'(p 183)

They examine classroom interaction in terms of such a community and go on to discuss the idea of 'telos', of the meaning-making of the whole class being aligned in directions generated by social interaction. They see telos as a unification of small scale 'becomings' by which many learners join a community of practice. They see:

a link between our notion of LCP and the situated abstraction of Noss and Hoyles (1996). Just as they claim the computer provides domains which support students' abstraction, so we claim LCPs support students' growing image of themselves as someone who is legitimately engaged in mathematical practice, as someone, in other words, who is becoming a mathematician. (p183)

Socio-mathematical Norms and Argumentation

This approach is echoed in the work of Cobb and Yackel (1996), who have analysed mathematics classrooms in terms of the negotiation and maintenance of social and socio-mathematical norms. Social norms include

- insistence on explanation of answers
- respecting the contribution of others
- making clear agreement as well as disagreement.

Socio-mathematical norms would include

- some notion of what constitutes a valid, complete solution
- agreement on the worth of alternative solutions
- negotiated agreement between teacher and students on the mutual acceptability of solutions.

Social norms will exist in all classrooms, and will bear a direct relationship to the society in which the classroom is situated. Because social norms will affect the negotiation of socio-mathematical norms, Apple (1992) has argued that the classroom is firmly situated in the wider context of the practices of school and society. Yackel and Cobb (1996) discuss the influence of socio-mathematical norms on argumentation in the classroom. They draw on the ideas of Toulmin (1969) as developed by Krummheuer (1995), seeing argumentation as made up of conclusion, data, warrant and backing. Yackel (1998) says of argumentation:

it clarifies the relationship between the individual and the collective, in this case between the explanations and justifications that individual children give in specific instances and the classroom mathematical practices that become taken-as-shared. As mathematical practices become taken-as-shared in the classroom, they are beyond justification and, hence, what is required as warrant and backing evolve. Similarly, the types of rationales that are given as data, warrants and backing for explanations and justifications contribute to the development of what is taken-as-shared by the classroom community, that is to the mathematical practices in the classroom. (p210)

Thus argumentation is seen as a social, rather than a logical process, a means of establishing that which is held in common about the topic in question and moving forward the 'held in common' by classroom interaction. Voigt (1995) discusses the reflexivity between learning and interaction and speaks of this reflexivity contributing to a classroom microculture which in turn affects the meaning-making which is taking place.

Choice of Material
Lave and Wenger (1991, pp102,103) address the issue of the transparency of a resource, and this is further examined by Adler (1998, pp8-11). A resource used in a mathematics classroom can be so visible to students that it obscures the mathematics and prevents meaning making. At the same time some visibility is necessary. We want the resource to be visible in the sense that it should direct the gaze of students, so enabling their meaning-making.

Invisibility of mediating technologies is necessary for allowing focus on, and thus supporting the visibility of, the subject matter. Conversely, visibility of the significance of the technology is necessary for allowing its unproblematic- invisible - use. This interplay of conflict and synergy is central to all aspects of learning in practice: it makes the design of supportive artifacts a matter of providing a good balance between these two interacting requirements.(Lave and Wenger, 1991 p103)

Clearly the familiarity of students with technology such as the TI 92 governs its use, in a way which is informed by arguments such as this. As they become more familiar with the software the teacher will be able to introduce the use of more complicated functions without losing transparency.
It is proposed here that these approaches, of a whole class ZPD, of a recognition of local communities of practice, and of negotiated socio-mathematical norms and argumentation have much to offer in looking at how technology, appropriately transparent, can be used in the classroom. In this study such approaches are used, in particular, to analyse social meaning-making in the area of construction and proof using Cabri with the TI 92 hand-held computer with lower school (11-14 years) pupils.

Methodology and Data Collection
A qualitative and ethnographic approach to research has been adopted, with case studies used to provide instances of rich incidents for subsequent analysis. These were subjected to microethnographic interpretive procedures (Erickson, 1986 and Voigt, 1990) Classroom interaction between teacher/researcher and individuals in whole class and group situations was audio recorded and the transcriptions of these recordings analysed. In addition, field notes of memorable incidents were recorded.

Each student had a TI 92 hand-held computer and used the dynamic geometry environment Cabri as available on this machine. An overhead projector version was available for demonstration by pupils and the teacher to the whole class. The following examples were an attempt to set up possibilities for whole class meaning-making with the minimum of previous knowledge of the TI92. The pattern followed was for the class to generate and discuss a simple dynamic image, and to record the result in exercise books as a diagram after the dynamic image had been appreciated. The hand-held nature of the TI92 is particularly suitable for pair discussion and, indeed, for consigning to a corner of the desk when work on paper is preferred.

Collection and Discussion of Data

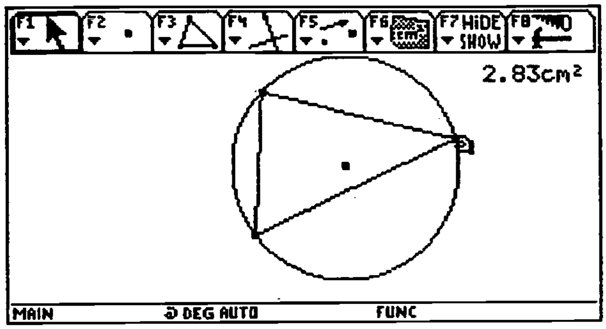

Fig 1

1. The class was asked to draw a circle and a triangle with its vertices on the circle, then to measure the area of the triangle (Fig 1).They were then asked to investigate the effect of dragging **one** of the vertices, and to look for the maximum area of the triangle. In jointly exploring the same screen in this way, but each on their own machine, a telos is created and students are aligned in the domain provided by the technology.

The following dialogue ensued.[1]

JG	What area have you got?	
Class	General response	*There was no restriction on the original diagram, a wide range of areas was possible.*
JG	Why do we all get different answers?	
Alison	Because we all used different circles	*Use of 'we' suggests the possibility of an LCP*
Barry	And different points	
JG	Look at mine while I move the point. Tell me when it will be greatest. What can we all say about our diagrams?	
Barry	It's across from the centre	*Later discussion showed that Barry appreciated the co-linearity of the mid point of one side of the triangle, the centre of the circle and the other vertex. Leanne had realised that the triangle was isosceles*
JG	Yes, good. Anyone else?	
Leanne	It's in the middle ·	

[1] Throughout this paper the teacher/researcher is JG and pseudonyms are used for pupils.

Here the technology could be said to be driving along the LCP. Spontaneous concepts are developed by the participants by looking at the dynamic image, which can then be used by the teacher to interact with scientific concepts (Gardiner and Hudson, 1998), so that that which is 'taken as shared' is moved forward.

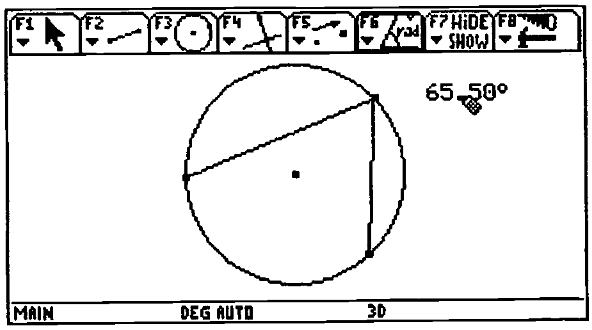

Fig 2

2. Another exercise which is available after only the briefest of introductions to the technology is based on a diagram such as Figure 2. Here pupils were asked to define and measure an angle in a circle as shown and to investigate the effect of dragging any one of the defining points along the circumference of the circle.[2] Transcription of classroom audio recordings resulted in the following dialogue.

JG	Does anyone want to tell me what they have found?		
Sonia	As you move this down it stays the same angle until you reach this point, then it changes to a completely different angle and stays the same.	*In this version of Cabri, if the vertex is moved round the circle until it passes one of the other points, the angle in the other segment, the supplement of the first, is measured*	*'conclusion'*
Nigel	Oh Yeah *(Wonderingly)*		*Drawn into a community of practice*
JG	Will you come and show us		
Sonia	It might not work you know… it might just be because of the shape of this one	*In order to demonstrate the OHP version of the machine had to be used.*	
Tom	It will work.. I got it to work		*Supporting the community of practice*
JG	Watch while she drags this. Watch the angle. Moving up …..angle getting bigger	*Dragging one of the non-vertex points*	

[2] An idea suggested by Geoff Wake of Manchester University

Nigel	If you change the middle one, watch the middle one, it stays the same and after a certain point it changes		*Now a more assured member of the practice*
JG	What's going to happen now?	*Dragging the vertex point*	
Chorus	Stays the same		
Nigel	Until you pass the point, then it will stay the same again		
JG	Look at the angle, it stays at 52.77 degrees, doesn't matter that the line goes through the centre, stays at 52.77. Now changes to..........127.23 *(to Sonia at front)*Can you make it flip between those two angles? *(to class)*What can you tell me about those two angles?		
Anne	Does it add up to 180?		*'conclusion'*
David	Ooo *(realising)*		
JG	Check that those results are true for your diagram		

Referring to criteria mentioned earlier for an LCP (Winbourne and Watson. 1998), here pupils can be said to be sharing tool use and purpose by being aligned in the task and their use of the technology. They are functioning and participating mathematically and recognising the competence of others. There is also, in this dialogue, a sense of telos, in which the pupils are aligned by the technology in a way which drives forward the meaning-making of the community.

Socio-Mathematical Norms and Argumentaion

In the passage quoted above there is evidence of two 'conclusions' (Yackel, 1998 p210) being reached (as indicated), without oral evidence of warrant and backing. However it appears that, in this dynamic geometry environment, warrant and backing are supplied by the shared experience of data generated by the technology.

Conclusion

This research has indicated how, with a background of individual and class development within a Zone of Proximal Development, the ideas of local communities of (mathematical) practice, telos, socio-mathematical norms and argumentation can be used to indicate how mathematical meaning making in the classroom might be analysed. In particular it demonstrates the benefit of suitably transparent use of technology in promoting alignment of pupil becomings within a whole-class ZPD.

Thanks are due to pupils and staff at Hope Valley College, Hope, Derbyshire

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BRINGING THE VOICE OF PLATO IN THE CLASSROOM TO DETECT AND OVERCOME CONCEPTUAL MISTAKES

Rossella Garuti
Scuola Media "Focherini", Carpi
I. M. A. - C. N. R., Genova

Paolo Boero
Dipartimento Matematica
Università di Genova

Giampaolo Chiappini
I. M. A. - C. N. R.
Genova

Abstract: The capacity of detecting conceptual mistakes and overcoming them by general explanation is important in the approach to theoretical knowledge, and its development in students calls for the teacher's intervention. Our working hypothesis is that the "voices and echoes game" can function as an appropriate methodology to this end. In order to explore this perspective in depth, a teaching experiment was performed in six classes (grades V and VII). This report provides a partial account of this complex experiment, presents some results and highlights some open research questions.

1. Introduction

Since 1995, the problem of approaching theoretical knowledge in compulsory school has been of major concern for our research group. We have produced an innovative methodology, the "voices and echoes game" (VEG), which is mainly based on Vygotskian elaboration concerning common and scientific concepts and Baktin's idea of "voice". This has been used as a guideline to plan and analyse teaching experiments intended to mediate crucial aspects of theoretical knowledge (see Boero et al., 1997, 1998; Garuti, 1997; for a brief account, see Subsection 2.1).

The research reported in this paper focuses on one aspect of the mastery of theoretical knowledge not yet considered in preceding papers and that is especially relevant to mathematics education: the capacity to detect conceptual mistakes and overcome them by general explanation. The development of this capacity in students calls for the teacher's intervention; our working hypothesis is that the VEG is an appropriate methodology for achieving such development (see Subsection 2.2). In order to explore this perspective in depth, a teaching experiment was performed in six classes (grades V and VII). The object of the experiment was a well known piece of the Plato's "Meno", that concerning the problem of doubling the area of a given square by constructing a suitable square (this means overcoming the mistake which consists of doubling the side length). This report provides a partial account of this complex experiment (see Section 3), presents some results (see Section 4) and puts some open research questions into evidence (see.Section 5).

2. Theoretical background

The purpose of this section is to provide essential background information, as well as (in Subsection 2.2.) some development of the theoretical framework related to the issue dealt with in this study.

2.1. About the VEG

What is the VEG? Some verbal and non-verbal expressions (especially those produced by scientists of the past) represent in a rich and communicative way important steps in the evolution of mathematics and science. Referring to Bachtin (1968) and Wertsch (1991), we called these expressions *'voices'*. Performing suitable tasks proposed by the teacher, the student may try to make connections between the voice and his/her own interpretations, conceptions, experiences and

personal senses (Leont'ev, 1978) and produce an *'echo'*, i.e. a link with the voice made explicit through a discourse. The 'echo' was an original idea through which we intended to develop our new educational methodology. What we have called the *VEG* is an educational situation aimed at activating students to produce echoes through specific tasks, for instance: *"How might.... have interpreted the fact that...?"*

Students' echoes: students may produce echoes of different types (depending on the tasks and personal adaptation to them). In Boero et al. (1997), ***individual and colletive echoes*** were classified. In this report we will focus particularly on *individual resonance echoes.* In this case the student appropriates the voice as a way of reconsidering and representing his/her experience; the distinctive sign is the ability to change linguistic register or level by seeking to select and explore pertinent elements *('deepening'),* and finding examples, situations, etc. which actualize and multiply the voice appropriately *('multiplication').*

What are the aims of the VEG? Our general, *initial* hypothesis on this issue was that the VEG might broaden the students' cultural horizon, embracing some elements of theoretical knowledge that are difficult to construct in a constructivist approach and difficult to mediate through a traditional approach (see Boero & al, 1997). The need to exploit the potentialities that emerged in the first series of teaching experiments led us to try to *characterize better the elements of theoretical knowledge* to be mediated through the VEG (cognitive strategies, methodological requirements, speech genre, etc.), in order to organize and analyse better their interiorization by students (see Boero & al, 1998)

2.2. Conceptual mistakes and the VEG

The research reported in this paper concerns another important potential of the VEG, namely the possibility of intervening in aspects of the student's mastery of theoretical knowledge - those related to detecting conceptual mistakes and overcoming them by general explanation.

The Vygotskian elaboration about consciousness as a condition for accessing scientific concepts, clearly pointed out by Vygotskij in his seminal work about "common concepts" and "scientific concepts", seems to be useful to frame this complex operation. According to Vygotskij (1990, chap. VI), consciousness is related to mastery of scientific concepts for different reasons: "scientific" concepts are not isolated (and consciousness is needed to control connections and inner coherence of the system); "scientific" concepts are explicit (and consciouness is needed to manage explicitation and especially the relationship between mediating signs and meaning); "scientific" concepts are in dialectic relationship with common ones (and consciousness is needed to be aware of the borders between them). During an activity in which students participate effectively in examining their conceptual mistakes, all these aspects where consciousness intervenes can come into play: contradictions with known properties are frequently a motive the teacher advocates for helping students recognize a conceptual mistake; explicitation of some concepts is needed in order to point out ambiguities that may be the root of mistakes; in many cases the teacher must point out that common intuition is a possible source of mistakes.

But how can productive classroom activities concerning students' conceptual mistakes be organised? According to Bachelard (1977) many conceptual mistakes come from ancient knowledge that is appropriate in earlier situations but which is no longer suitable. The teacher must take the responsibility for selecting and proposing appropriate tasks (those which lead to crisis of the ancient knowledge) and for helping the student to overcome his/her mistakes. The teacher's role is central for other conceptual mistakes as well: for instance, those related to misunderstandings or ambiguities. What's more, the student must be aware of the role played by the teacher and his own role as a condition for being able to reproduce by himself, in the future, the sequence of actions needed to detect and overcome conceptual mistakes (cf. Brousseau, 1997).

Our working hypothesis was that the VEG could intervene as an appropriate educational methodology for attaining both the aims pointed out in preceding analyses: to develop students' consciousness about the functioning of theoretical knowledge when conceptual mistakes come into play; and to promote awareness of the teacher's and student's roles during classroom activities concening conceptual mistakes. Indeed, detecting and overcoming conceptual mistakes plays a crucial role in the evolution of mathematics and science. It is therefore natural that the history of mathematics and, more generally, the history of culture should offer "dialogical voices" that speak about this issue (exchange of letters, imaginery debates, etc). The production of "echoes" of well chosen "dialogical voices" during suitable tasks could lead students to participate consciously in the process of detecting and overcoming conceptual mistakes as a preliminary step towards interiorization.

The teaching experiment reported in this paper was planned and performed in order to test and develop our working hypothesis.

3. Method

3.1. The choice of voice

Plato's "Meno" presents some crucial aspects of Plato's theory about learning (how the "learner" can reach truth) and teaching (how the "teacher" can help the "learner" to reach truth). Plato's general, underlying hypothesis is that forgotten truth can be restored ("recollection") through an effort by the "learner" led by the "teacher" and motivated by the fact that "*now not only is he ignorant[...] but he will be quite glad to look for it*". The crucial mediational tool is the "socratic dialogue", i. e. a dialogue intended to provoke crisis and then allow it to be overcome. In this framework, the excerpt concerning doubling the area of a given square is crucial as a practical demonstration:

Phase A) Socrates asks Meno's slave to solve the problem of doubling the area of the square by constructing a suitable square; the slave's answer (side of double length) is opposed by Socrates through direct, visual evidence (based on the drawing of the situation).

Phase B) Then the slave is encouraged to find a solution by himself - but he only manages to understand that the correct side length must be smaller than three halves of the original length. Socrates' comment is that from this moment on, the slave can learn: "*Nor indeed does he know it* [the solution] *now, but then he thought he knew it* [...] *Now however he does feel perplexed. Not only he does not know the answer; he doesn't even think he knows*".

Phase C) Socrates interactively guides the slave towards the right solution (achieved through a construction based on the diagonal of the original square).

We may remark that in Phase A) the slave's answer is similar to those usually produced by young students when they tackle the same problem - this fact offers an opportunity to involve students strongly in the VEG! And we may recognize in Phases A), B), C) a sequence of activities not dissimilar from some present-day views about how to guide students towards taking into charge and overcoming their conceptual mistakes. This is true especially from a Vygotskian perspective, where the teacher takes a strong mediating role in the evolution of students' culture.

We chose to ignore "recollection theory" in classroom activities. This entails a violation of the authenticity of the historical source. But our aim was to lead students to grasp that a general explanation must be reached in order to definitively overcome a mistake. And, in general, compromises of this kind appear unavoidable if we want to exploit historical sources in the classroom (cf. Fauvel, 1991).

3.2. The choice of classes

Six classes (114 students) took part in this teaching experiment: five fifth-grade classes and two seventh-grade classes.These classes belonged to different school settings (four primary school classes and two junior high school classes) and to different educational contexts (in particular, three fifth-grade classes were following the Modena Group Project on "mathematical discussion", one was following the Genoa Group Project). Their socio-cultural backgrounds were extremely different. This set of classes was chosen in order to reveal "invariant" elements and significant conditions for the productivity of the methodology.

As concerns the mathematical background, the students were able to measure lengths and construct squares; they had met the concept of area of a plane surface in the preceding months and knew how to calculate the area of a given square.

3.3. Teaching sequence planning and observations

The teaching sequence can briefly be described as follows:

i) students are briefly informed about the whole activity to be performed; then they individually try to solve the same problem posed by Socrates to the slave. The aim of this activity is to involve students in the problem dealt with in the voice;

ii) students approach the voice under the teacher's guidance: firstly, they read and try to understand (with the help of the teacher) the three phases of the dialogue; then they read the whole dialogue aloud (some students playing the different characters); finally, they discuss (Disc. I) the content and the aim of the whole dialogue, trying to understand (under the teacher's guidance) the function of the three phases. After negotiation with students, a wall poster is put up summarising the three phases in concise terms. This suggests the structure of the following echo;

iii) the teacher presents the students with some, possible mistakes that could become the object of a dialogue similar to Plato's, and they are invited to propose other mistakes. The aim is to negotiate and agree on a mistake that is appropriate for the echo (i.e. a relatively frequent student mistake that is recognized as a mistake by students and can be exhaustively explained through a discussion guided by the teacher). Here is a sample of the 5 mistakes that were chosen in the 6 classes:

"By dividing an integer number by another number, one always gets a number smaller than the dividend" (the two seventh-grade classes: see Annexe for two examples of echoes).

"By multiplying an integer number by another number, one always gets a number bigger than the first number" (one V-grade class).
"By multiplying tenths by tenths, one gets tenths" (another V-grade class).

iv) students discuss (Disc. II) about the chosen mistake, trying to detect (under the teacher's guidance) good reasons explaining why it is a mistake, then trying to find partial solutions, and finally arriving to a general explanation. The aim of this discussion is to create the common base of mathematical knowledge needed to construct the echo, and prepare its three phases;

v) students individually try to produce an echo, i.e. a "socratic dialogue" about the chosen mistake;

vi) students compare and discuss (under the teachers' guidance) some individual productions.

All the individual productions and recordings of the Disc. I and II are available; for the other discussions the teachers took notes.

3.4. Analysis of students' behaviours

In line with the educational aims of the teaching experiment we drew up the following guidelines for the analysis of students' protocols:

I. How the student keeps to the roles of Socrates and the slave in each phase of the dialogue. This point is related to the aim of developing awareness about the teacher's and the student's roles in the activities concerning conceptual mistakes;

II. How the student appropriates the roles of the phases of the dialogue in detecting and overcoming the mistake. This point is related to the aim of promoting consciousness about the mechanisms of detecting and overcoming mistakes;

III. How the targeted mathematical content (the knowledge allowing the students to overcome the mistake) is appropriated by the student: are the choice and presentation of counter-examples appropriate? Is the guiding of the slave performed through general, theoretical considerations about the knowledge in play?

4. Some Results

This section reports a selection of results we consider to be of interest.

Here we will consider only the 102 students from grade V to VII who took part in the whole activity. Only 6 students completely fail their echo (do not produce a dialogue, or mixed up Plato's original dialogue with the new situation).

Roles in the echo: Among the other 96, 10 show serious difficulties in keeping to the roles of Socrates and the slave in Phase I. Appropriate "deepenings" (including original expressions intended to highlight the mistake and provoke the slave's crisis) and "multiplications" (including choice of appropriate counter-examples not presented in Disc. II) are found in almost all the other students' texts.

Detailed comparison between fifth- and seventh-graders is inappropriate, as different mistakes were tackled in different classes. However, the percentages related to success in keeping to the roles in Phase I do not differ much.

It is not easy to detect Phase 2 in the students' protocols: students were not asked to separate the three phases. And for some of the mistakes chosen it is objectively difficult to create a specific Phase-II dialogue!

At least 67 (out of the 96 students) have serious difficulties keeping to the roles of Socrates and the slave in the last phase of the echo. In many cases the quality of the interaction between Socrates and the slave suddenly changes from

Socrates' questioning in order to make the slave understand, to Socrates' presenting some formulas or procedures in order to avoid the mistake, (see end of TEXT 1) with the slave reduced to a passive role of listener. In the same cases the quality of deepenings and multiplications falls: the expressions become "assertive", not "explanatory", and the examples (if any) stick closely to those discussed in Disc. II.

Consciousness about how to detect and overcome mistakes: We obtained good overall results about the consciousness of the fact that appropriate counter-examples can reveal the mistake. This kind of consciousness was attained by practically all the students (86) who kept to the roles in the first phase of the dialogue, as shown by the "multiplications" and "deepenings" in their echoes.

At least 50 students out of these try to give a general "explanation" of the mistake or find a general "rule" in the third phase of the dialogue, showing to be aware of the necessity of doing it.

Mathematical content: We must distinguish between: **I)** consciousness about the fact that a statement is false; **II)** consciousness about the reasons that may provoke the mistake; and **III)** consciousness about the theoretical reasons "why it is false" and how to overcome the mistake in general (i.e. the connection with systematic mathematical knowledge that can frame the mistake and the correct solution). As remarked above, the first level of consciousness seems to be reached by all the students who keep to the roles during the first phase of the dialogue. One half of these students reach the second level of consciousness (see TEXT 1 and TEXT 2).It is interesting to note that there is also an almost complete coincidence, across tasks and classes, between the students who are able to attain the third level of consciousness and those who are able to keep to the roles throughout the third phase of the dialogue (see TEXT 2). The breaking point in reaching the third level is well exemplified in TEXT 1: the student tries to explain why the result of the division is larger than the dividend if the divisor is smaller than one. An appropriate geometric example (similar to those considered during Disc. II) is provided. The interactive structure of the presentation is kept. Then the student tries to move to a general explanation. Some lines are written and then crossed out. At the end, a rather confused rule is provided and Socrates takes the role of he who "gives the rule". Compare with TEXT 2: here a real interaction is maintained in the last phase as well, and seems to be perfectly functional to the development of a complex inner discourse concerning the mathematical knowledge in play. Remarkably, "Socrates" considers both the operational side (how to divide an integer number by a fraction) and the explanatory side (why it is necessary to behave in such a way). The dialogue allows these two sides to be represented in a clear way.

5. Discussion

The above description of students' behaviours raises an interesting research question about the reasons why there is an almost complete coincidence between students who keep to their roles during the different phases of the discussion, and students who reach the different levels of consciousness about the knowledge in play. A possible interpretation refers to the dialogical nature of the acquisition of theoretical knowledge (cf. Brown, 1997, Chapter 2, for some hints in this direction) and could be summarised as follows: Plato's voice presents a model of

dialogical treatment of mistakes; at the beginning, by echoing this voice, students are forced (during both the discussions and the individual production of the echo) to make explicit the knowledge which originated the mistake. Indeed the need to keep to the roles can entail a shift to an inner questioning: *"What idea about the knowledge in play should the slave bear"*. This interpretation is justified by the fact that (as happens in TEXT 1 and in TEXT 2: see (°)) the *"idea about the knowledge in play"* is expressed in most cases by the slave and not by Socrates. In this way the transition to the second level of consciousness about the knowledge in play is realised. The passage to the third level is performed in two steps: first, by considering examples (with the dialogical function of bringing the slave to see how the appropriate knowledge could work, and the inner function of better understanding how it does work); and second, by shifting to theoretical framing of the examples. Here, the breaking point described at the end of Section 4 could be interpreted as follows: under the necessity of posing appropriate questions to the slave, some students are able to answer the inner question: *"How is the correct rule related to the meaning?"*- i. e. to the examples tackled by the slave. Indeed in these cases (see TEXT 2) Socrates keeps to his questioning role: the inner question is transformed into appropriate questions posed to the slave. The other students are not able to keep to the role of Socrates: perhaps because the shift to the inner question *"How is the good rule related to the meaning?"* is too difficult. Or, more probably, because it no longer concerns the slave in an immediate way and it is difficult to find appropriate questions for him, and so a traditional model of teaching prevails (it is the teacher who provides the rule!).

Further experiments (possibly with on-the-spot interviews with students who fail at the "breaking point") are needed to test the validity of this interpretation, whose research and didactical implications might be significant as concerns the potential of exploiting dialogical voices for the VEG.

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Annexe

Seventh graders. Mistake chosen: *"Dividing an integer number by another number, one always gets a number smaller than the dividend."* Two texts from the same classroom.

<u>SO</u>=Socrates, <u>SL</u>=Slave

TEXT 1 (mean level production)

<u>SO</u>:*Tell me, my boy, do you know the result of this division: 15:5?* <u>SL</u>: *It is simply 3, Socrates.* <u>SO</u>: *And now try to perform the following division: 15:3 What is the result?* <u>SL</u>: *Clearly 5,*

Socrates. SO*: Look at the results and try to tell me how they compare to the dividend.* SL*: They are smaller, Socrates.* SO*: In your opinion, does this happen for all divisions?* SL*: Yes. surely* SO*: Could you explain why?* SL*:(°)Naturally: this happens because if I divide one thing I do not have it, but only one piece of it, so the part is smaller in relationship to the whole.* SO*: Fine. Can you tell me the result of the following division: 15:1 ?* SL*: Fifteen, Socrates.* SO*: Look at this number and think. How does it compare with the dividend?* SL*: It is equal, Socrates.* SO*: So then a division does not always generate a result smaller than the dividend!* SL*: My Zeus! It is true.* SO*: Now try to perform the following division: 15:0.5. What is the result?* SL*: Thirty, Socrates.* SO*: But 30 is greater than the dividend 1.* SL*: It is true!*

II- SO*: Tell me, slave: which divisions generate a result that is smaller, equal to or bigger than the dividend?* SL*: Smaller: 15:3=5; equal: 15:1=15; bigger: 15:0.5=30*

SO*: How do the three divisors compare with one?* SL*: In the first division, greater; in the second, equal; in the third, smaller.* SO*: Is there any link?* SL*: There may be.* SO*: Could you say what it is?* SL*: For Zeus, no!* SO*: You see, Meno: before your slave was sure in answering, while now he finds himself in difficulties. Before he was convinced he knew, while now he does not know. But he knows his mistakes and will no longer fail.*

III - SO*: Slave, could you say how many times 0.5 is contained in 1?* SL*: Twice, Socrates.*

SO*: And 0.25?* SL*: Four times, Socrates.* SO*: Try to write the decimal number 0.5 as a fraction.* SL*: The fraction is 1/2.* SO*: But how much is 1/2 compared to 1?* SL*: It is one half.* SO*: Then if 1 is contained an integer number of times in another number, how many times one half will be contained?* SL*: Double, Socrates.* SO*: But in your opinion is it correct to divide a division by another division?* SL*: No.* SO*: Indeed you have now seen that one half of one is contained twice. It will be sufficient to turn over the division, i.e. to perform the multiplication of the given integer number by the denominator and we will got the result. What we have performed shows us how many times a decimal number can be contained in an integer number,* SO *we have got a ratio between them.* SL*: Now I understand my mistake. Bye, Meno is calling me.*

TEXT 2 (High level production)

SO*: Tell me, my boy, what is the result of 15:3?* SL*: Five.* SO*: Is it smaller than 15?* SL*: What a question! That is clear!* SO*: And yet, how much is it 20:5?* SL*: Obviously 4, Socrates* SO*: Then is it smaller than 20?* SL*: Exactly.* SO*: Then, how do you think that results of the divisions are?* SL*: I think that they are always smaller than the dividend.* SO*: Are you sure?* SL*:(°) Yes, because "to divide" means "to break in equal parts."*SO*: Now perform this division: 15:1.* SL*: Uhm, ...it makes 15.* SO*: But 15 is equal to the dividend.* SL*: It is true.* SO*: Why is it equal?* SL*: Because dividing by one is how to give an amount to one person, it remains equal.* SO*: So does your theory still work?* SL*: Not completely. Now I see that in some cases it does not work.* SO*: Are you still sure you are right?* SL*: Yes... Pehaps... No... Perhaps there is one case in which the result is larger...or perhaps not... My Zeus, I understand nothing! (five minutes elapse).* SO*: What is the result of 2:0.5?* SL*: These are difficult questions. I am no longer able to answer.* SO*: Take this square* (drawing)*and divide it into small squares!* SL*: This way?*(the drawing is divided into 16 pieces by drawing 3 horizontal and 3 vertical lines, all equally spaced) SO*: Yes, good. Now the unit is the small square* [drawing]. *How much is 0,5 compared to 1?* SL*: One half.* SO*: Now make one half of the small square.* SL*: Done.* SO*: Do the same for all the small squares.* SL*: Just a moment...Done.* SO*: How many halves?* SL*: 1,2,3... 32, Socrates.* SO*: How many unit squares, at the beginning?* SL*: Sixteen, Socrates.* SO*: Then you got a result greater than the starting number.* SL*: Uhm... Of course.* SO*: And how is one half written as a fraction?* SL*: Uhm... perhaps 1/2.* SO*: Good! Are you able now to divide a number by a fraction?* SL*: Yes. surely!* SO*: Then divide 2 by 1/2. How many times is 1/2 contained in 2?* SL*: According to the preceding rule, I must invert the fraction and then multiply. OK, it makes 4.* SO*: How can you represent this?* SL*: I'll try... Two squares..*[drawing] *One half twice for each* [drawing]. *It works: 4.* SO*: Good!* SL*: I understand: the division is not only "breaking into equal parts", but also seeing how many times a number is contanined in another!* SO*: Make an example by yourself!* SL*: 1:1/4* [he performs and illustrates it]

MALE AND FEMALE CALCULUS STUDENTS' USE OF VISUAL REPRESENTATIONS

Elizabeth Ann George

Ball State University

The frequency with which mathematically capable high school students utilized visual representations in their written solutions to applied calculus problems and the nature of the visual representations they created were analyzed in this study. Several gender differences with respect to visual representation use were identified. Females drew diagrams more often than did males and were more likely to create complex visual representations by modifying given diagrams. Males, who were more successful in solving these problems, tended to construct fewer and simpler diagrams.

Students encounter visual representations throughout their study of mathematics. Diagrams, figures, and graphs are frequently components of instructional explanations offered by teachers and in textbooks. In addition, many mathematical problems found in written curricular materials or on tests include an accompanying visual representation which students must interpret or may modify. In solving problems in which no visual representation is given, construction of a visual representation is often helpful, advantageous, or, moreover, an essential component of the solution process. There is great diversity in the types of mathematical problems whose solutions require or are facilitated by reasoning with visual representations and in the ways that students can use visual representations when solving these problems.

Reasoning with visual representations is particularly crucial in understanding the theories and applications of calculus. Understanding fundamental calculus concepts (e.g., limit, derivative, and integral) requires significant use of visualization, and the ability to successfully solve many problems using calculus is dependent on visual images, either in the form of diagrams or graphs (Zimmerman, 1991). Yet throughout their study of mathematics, students' demonstrate a reluctance to visualize (Eisenberg & Dreyfus, 1991), and this behavior is particularly disturbing when displayed by students studying calculus (Vinner, 1989).

Though reasoning with diagrammatic representations offers many potential benefits, students who use visual representations are not necessarily more successful in their mathematical studies. Presmeg (1985) observed that high school students who preferred to use visual methods in solving mathematical problems often experienced difficulties in learning mathematics, while students who were identified as higher achievers in mathematics classrooms were almost always nonvisualizers. This pattern may prevail in college-level mathematics courses as well. College

calculus students who indicated a stronger preference for using visual methods achieved lower scores in calculus, while those students who showed less preference for visual methods achieved higher scores (Galindo, 1995).

Differential mathematics achievement of male and female students has been studied at various age levels and in many areas of mathematics during the past twenty-five years (Fennema & Hart, 1994). Gender differences in mathematics learning appear to be more pronounced for older than for younger students, most notably on problem-solving tasks and application problems (Friedman, 1989). Significant gender differences in mathematical achievement have been identified on national assessment tests (i.e., NAEP), as high school males consistently outperformed their female classmates (Fennema & Carpenter, 1981; Silver et al., 1988). Observing that the magnitude of the difference between male and female students' performance increased in relation to the amount of mathematics studied, Meyer (1989) suggested that gender differences in mathematics achievement resulted from the best males performing at a higher level than the best females. When examining mathematical achievement in actual classroom settings, Seegers and Boekaerts (1996) found that tests reflecting classroom content also showed marked gender differences, with males outperforming females. They also observed that the differences increased when the items covered topics that were more complex.

Several studies have identified gender differences when investigating the relationship between mathematics achievement and the use of visual representations. When solving tasks where males were more successful than females, females reported a greater use of pictures, though the pictures that males drew were more accurate and contained more pictorial information than those drawn by females (Fennema and Tartre, 1985). A stronger relationship between high spatial visualization ability and mathematical problem-solving success was found for females than for males. Battista (1990) concluded that males were more successful than females in solving geometry problems and that spatial visualization and logical reasoning were important factors in geometry problem solving for both male and female students, but that they contributed in different ways to achievement, depending on gender. Gender differences in visualization skills and the calculus achievement of college students were also identified by Ferrini-Mundy (1987). Explicit training in spatial skills, particularly in visualizing a three-dimensional solid of revolution from a two-dimensional representation, positively affected female students' abilities to draw visual representations of the solids of revolution and improved their mathematical problem-solving performance in the applications of integral calculus. No such effect was identified for male students.

Further research regarding gender differences in mathematics learning has been advocated (Leder, 1992), particularly the need for studies that focus on young women who have been successful in their mathematical studies (Becker, 1991). This study, whose purpose includes an investigation of the ways that mathematically

capable male and female students reasoned with visual representations in solving applied calculus problems, aspires to provide further insight into gender similarities and differences in mathematical problem solving.

Specifically, this study investigated the frequency of visual representation use and the nature of the visual representations created in students' written solutions to five free-response problems presented on the 1996 BC Level Advanced Placement Calculus Examination. Gender differences in achievement favoring males were significant for each of these free-response problems, as well as for overall scores on this examination (Morgan, 1996). Therefore, differences in the frequency of visual representation use and in the nature of the diagrams, figures, and graphs created were analyzed for subgroups of students, based upon gender and problem-solving success.

Frequency of Visual Representation Use in Solving Calculus Problems

Randomly selected from the approximately 21,000 high school students who took the BC level Advanced Placement Calculus Examination in May 1996 were 600 students' written solutions to the free-response questions. This sample was partitioned into subgroups along two dimensions, gender and overall performance level on the Advanced Placement Calculus Examination, as shown in Table 1. For purposes of this study, the five levels of overall AP scores were collapsed into three performance levels. There was no significant difference in the gender distribution between the sample and the population.

Table 1

Number of Students Achieving at Each Performance Level

	Males	Females
High scorers (5 or 4)	251	97
Moderate scorers (3)	82	65
Low scorers (2 or 1)	56	49

Students' solutions to five free-response problems were coded with respect to visual representation use. Three of the free-response problems included a given diagram in the problem statement; therefore, visual representation use included modification of the given diagram or construction of a new diagram. Construction of a new diagram constituted visual representation use in the other two free-response problems whose problem statements did not include a given diagram. A visual representation use (VRU) score was computing for each student, measuring the number of free-response problems for which evidence of diagram use was identified in the written solutions. The distribution of VRU scores for the 600 students in the sample is shown in Table 2.

Table 2

Distribution of VRU Scores

	VRU score					
	5	4	3	2	1	0
Number of students	44	147	169	134	84	22
Percent of students	7.3	24.5	28.2	22.3	14.0	3.7

The mean VRU score for the sample was 2.78, and the mean VRU scores for various subgroups of students are shown in Table 3. Using a two-factor analysis of variance and an alpha level of .05, differences in the frequency of visual representation use associated with gender ($F(1,594) = 36.51$, $p < .001$) and with performance level ($F(2, 594) = 4.42$, $p = .012$) were significant. The interaction effect of gender and performance level was not significant.

Table 3

VRU Score Means and Standard Deviations for Subgroups of Students

	Males	Females	Total
High scorers	2.73 (1.24)	3.26 (1.16)	2.88 (1.24)
Moderate scorers	2.33 (1.28)	3.09 (1.20)	2.67 (1.29)
Low scorers	2.23 (1.31)	3.04 (1.15)	2.61 (1.30)
Total	2.57 (1.27)	3.16 (1.17)	2.78 (1.27)

Determination of whether the pattern of visual representation use identified from the aggregated VRU score held for each free-response problem prompted further investigation. Thus, a within-question analysis of the frequency of visual representation use was conducted and revealed that, when solving each of the five free-response problems, the written work produced by females included use of visual representations more often than did the written work completed by males (George, 1998). The differences in frequency of visual representation use between male and female students were statistically significant at an alpha level of .01 in three of the five free-response questions. Recall that for both this sample and the population of students who took the 1996 BC Level Advanced Placement Calculus Examination, males significantly outperformed females, as measured by mean scores achieved on the examination and mean scores achieved on each free-response problem. Therefore, female students demonstrated more evidence of visual representation use in their written solutions and were less successful in solving the

posed problems than were male students, who produced fewer visual representations and were more successful.

Descriptive Analysis of the Visual Representations Students Created

Gaining further understanding into the relationship between students' visual representation use and problem-solving success required that the wide variety of diagrams, figures, and graphs modified or constructed in their written solutions be described and compared. Therefore, a qualitative analysis was conducted to more closely examine the visual representations students produced in solving selected free-response problems. The written solutions of 180 of the 600 students were further analyzed with respect to visual representation use. Specifically, similarities and differences were identified in the ways visual representations were modified and constructed by 45 students in each of four subgroups (i.e., high- and low-scoring males and females). Results of the analysis of one of the free-response problems is presented below.

One application of differential calculus which depends heavily on reasoning with visual representations is the solving of related rates problems. A common feature of this type of problem is the implicit variable of time. In these problems, several piece of descriptive information are given: the relationships of quantities in general regardless of time, the relationship of quantities at a specific moment in time, and information about the direction and rate of change. The dynamic nature of such problem situations often must be represented in diagrams that are static. Initially, such diagrams are drawn and labeled to illustrate only one moment in time. Subsequently, the problem solver must impose motion on these drawn diagrams in deciding whether specific quantities remain constant or change over time.

The following related rates problem was presented on the 1996 BC level Advanced Placement Calculus Examination (College Board, 1995):

> An oil storage tank has the shape shown above, obtained by revolving the curve $y = \frac{9}{625}x^4$ 0 from $x = 0$ to $x = 5$ about the y-axis, where x and y are measured in feet. Oil weighing 50 pounds per cubic foot flowed into an initially empty tank at a constant rate of 8 cubic feet per minute. When the depth of the oil reached 6 feet, the flow stopped. Let h be the depth, in feet, of oil in the tank. How fast was the depth of oil in the tank increasing when $h = 4$? Indicate units of measure.

Although all the information necessary to construct a visual representation of this problem situation was stated in words, an accompanying diagram was given. The graph of the function was drawn on the coordinate plane and reflected over the y-axis. Perspective was added to illustrate the oil tank and its radius of 5 feet and height of 9 feet were explicitly labeled.

A frequency distribution of the ways in which visual representations were used by the four groups of students in solving this related rate problem is displayed in Figure 1. Overall, the written solutions of females indicated use of a visual

representation significantly more often than did the written solutions of males (68% and 41%, respectively). Eighty-eight given diagrams were modified and 20 new diagrams were constructed in students' solutions. Female students were significantly more likely to only modify the given diagram than were males (58% and 29%, respectively). More high-scorers than low-scorers both modified the given diagram and constructed a new diagram within their written solution.

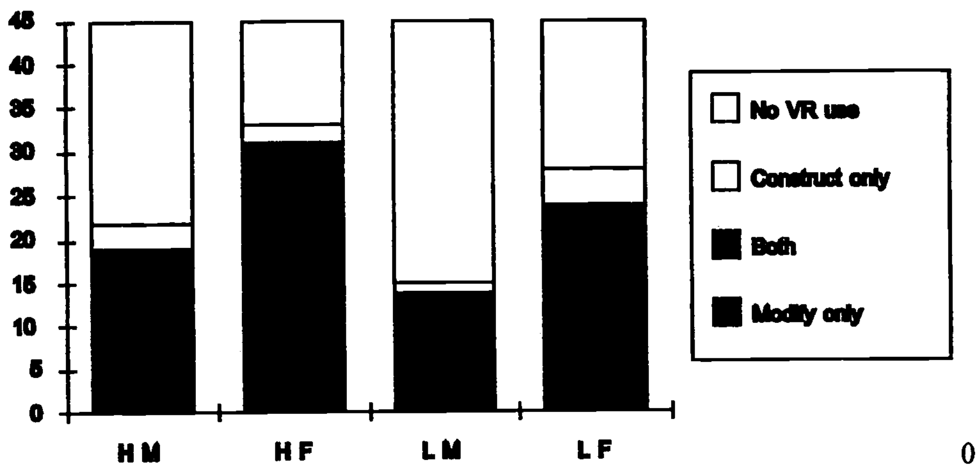

Figure 1. Frequency of visual representation modification and construction by high- and low-scoring male and female students in solving the oil tank problem.

A closer examination of the nature of the visual representations created by male and female students was accomplished by identifying the features which students highlighted when modifying the given diagram or which they included when constructing a new diagram. Four categories were identified in the oil tank problem to describe students' modification of the given visual representation: adding pictorial elaboration, labeling with numerical constants, labeling with variables, and marking of cross-sections. Overall, half of the students who modified the given diagram did so by simply adding pictorial elaboration and/or labeling with numerical constants; the other 50% produced more complex diagrams by labeling with variables and/or drawing a cross section. Females produced more complex diagrams than did males, as 56% (31 of 55) of the females and 39% (13 of 33) of the males who modified diagrams labeled the given visual representation with variables and/or drew cross sections. Of the four subgroups of students, high-scoring females produced the largest number of complex modified diagrams (61%); their solutions often included both labeling of variables and drawing of cross sections. While 50% of low-scoring females' solutions also contained complex modified diagrams, only 42% of high-scoring males and 36% of low-scoring males demonstrated this level of complexity in their modified diagrams.

Three categories were used to describe the new visual representations students constructed: a graph of the function on the x-y plane, isolated three-dimensional figures, and isolated cross-sections. Nearly half of the students

who only constructed a new diagram (9 of 20) produced a graph of the given function on the coordinate plane, then labeled and marked this graph in various ways. The others drew isolated figures, either three-dimensional solids or cross sections which were removed from the coordinate plane. Males tended to construct simpler diagrams than did females.

Similar analyses conducted on other free-response problems revealed that the nature of the visual representations used by males and females differed primarily in the complexity level of the diagrams they created (George, 1998). Males tended to focus more narrowly on the graph of the function or the geometric figure on which the problem situation was based. More females than males chose to only modified the given visual representations, often combining actions of pictorial elaboration, highlighting of cross sections, and algebraic labeling to create more complex visual representations. For all students there was greater diversity displayed in the ways they modified the given diagrams than in the ways they constructed new diagrams.

Results of this study show that mathematically capable high school students frequently used visual representations in their solutions to applied calculus problems and that there was great diversity in the visual representations they created. Recognition of the gender differences identified in this study can lead mathematics educators to make more explicit and informed decisions about visual representation use in curriculum materials, classroom instruction, and assessment both prior to and during the study of calculus, providing opportunities for all students to become more successful mathematical problem solvers.

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REAL PROBLEMS IN SCHOOL MATHEMATICS

Gialamas Vasilis
Karaliopoulou Margaret
Klaoudatos Nikos
Matrozos Dimitrios
Papastavridis Stavros
University of Athens
Department of Mathematics
Mitsaki 35-37
111 41 ATHENS. GREECE
tel +301-831-3904
fax +301-727-4282
email spapast@atlas.uoa.gr

Abstract

In this paper, we are presenting the results of a quantitative study concerning the effect of an innovative teaching model, based on Problem Solving approach, on the ability of 11th grade students in solving Real Problems and purely Mathematical Exercises. The research indicates that emphasis on this kind of teaching improves not only students' abilities in handling Real Problems themselves, but also it improves their abilities in dealing with pure mathematical ideas.

1.Introduction

The general goal of this research is to examine under which conditions, high school students can use their acquired mathematical knowledge in solving Real Problems. More precisely the research intends to study:

a) The effectiveness of Modelling Orientated Teaching, described below, as a general frame of learning mathematics, in the development of mathematical concepts and the ability of handling Real Problems.

b) The role of the Context of Real Problems at student's performance.

2. A theoretical framework

Questions concerning the adoption of applications and real problems have attracted the attention of many researchers the last years. Some of these questions are about the educational philosophy and goals that are served by mathematical applications. Others are about the way we will be able to incorporate mathematical applications in the schoolbooks and in everyday educational practice, in order to achieve specified goals.

Berry et al (1984), Berry et al (1986), Berry et al (1987), Blum, Niss, Huntley (1989), Blum et al (1989), Niss et al (1991), De Lange et al (1993), have been concerned with these questions.

Blum (1991), in order to accommodate Applications and Modelling in schoolbooks, put forward three aims of mathematical teaching, (see also Blum and Niss (1991)):

1. Pragmatic aims: Mathematics helps us describe, comprehend, explain and handle real situations.

2. Formative aims: Mathematics help us develop general skills, for example, the formulation of a mathematical model for a real situation, or attitudes, such us the desire for mental work.

3. Cultural aims: Mathematics as a reason for philosophical and scientific thought, as a science and as a part of the human history and civilization.

Blum and Niss (1989,1991) categorized the various approaches in organizing the teaching of Problem Solving, Modelling and Applications as follow:

1) The Separation Approach

2) The Two Compartment Approach

3) The Island Approach

4) The Mixing Approach

5) The mathematics Curriculum Integrated Approach

6) The Interdisciplinary Integrated Approach.

We are particularly interested in the Island Approach, since it is quite widespread and it is the approach used in throughout Greek high schools. According to this approach the book is organized in units, each one introducing the material in a definition-theorem-proof-corollary-exercise scheme, and then applications follow, directly connected with the material just presented. According to Blum and Niss (1989,1991), in this approach « the closer in time and content the relationship is between pure mathematics sections and subsequent sections concentrating on problem solving, modelling and applications, the more the latter sections tend to assume the character of being traditional exercises».

In an attempt to improve upon the Island Approach, Klaoudatos (1994) developed a teaching model, which he calls "Modelling Orientated Teaching". Klaoudatos' model consists of three stages:

1. Conceptual modelling:

In this part, mathematical modelling develops the conditions in which the need arises for the introduction of a concept. For this purpose we select the appropriate problems, the solution of which calls for the introduction of the concept we want to teach, while the didactical activities and in particular the didactical situations we design for the

occasion are aimed at developing a conjecture about the *hidden* concept.

2. Abstraction and formalization of mathematical concepts:

This part includes the development, the mathematical processing and the formation of a mathematical theory for the concept we want to teach about. This part is not a *substitute* for the traditional theory, as we know it, but attempts to ensure the independence of the concept from the specific contexts in which it was presented in part 1. In particular, the concept we have developed in part 1, must become a *mathematical object*, i.e. it must be viewed in mathematical context, regardless of any specific problem or individual. This aim is achieved by means of control and verification of the conjecture, as well as through deductions, generalizations and proofs. Consequently, the didactical activities are aimed at *institutionalizing* the knowledge.

3. Applied modelling:

Part three brings into focus all the possibilities of the modelling circle and it is here that the ideas and concepts we have taught are consolidated, reinforced and extended. In this part the concept or concepts are considered as already known and we use them in order to solve problems and applications.

Klaoudatos' model is an extension and elaboration of a teaching model proposed by de Lange (1987, 1989) within the realm of "Realistic Mathematics Education", in which the "abstraction and formalization of mathematical concepts" does not appear as a special part.

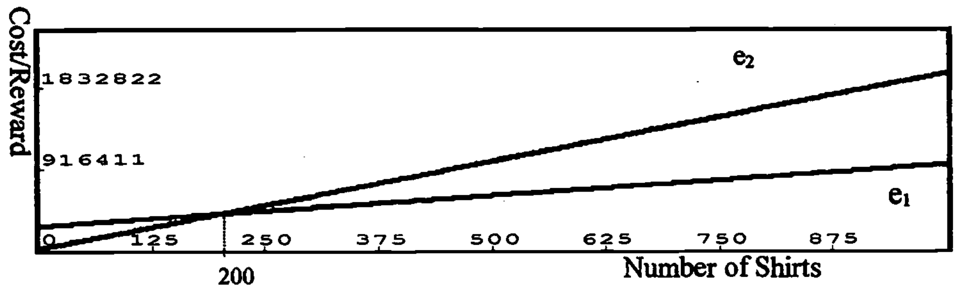
3.The methodology of the research

The research was designed for students of the 11th grade. It was realized at a senior high school in Athens during the school year 1997-98.Four sections participated, sections A, B, C, D (a total of 97 students). A teacher experienced in practicing the Modelling Oriented Teaching taught students of sections C and D (experimental group), by this teaching method, through the whole year. On the other hand, an experienced teacher who used the traditional Island Approach way of teaching taught students of sections A and B (control group). Both the assignments of students and teachers to the various sections were completely random. Throughout the previous academic year 1996-97, the same students were taught in a homogeneous fashion, following the Island Approach.

We gave two tests to the students. The tests were administered one month before the end of the school year 1997-98. Test 1 deals with "raw" Real Problems (R.P.), taken from real situations, described in layman's language, where Test 2 was composed of typical Mathematical Exercises (M.E.) stated in a formal mathematical language. Test 1 was given first and Test 2 followed one week apart. The questions of these two tests were in a strict one to one correspondence, corresponding questions were requiring exactly the same mathematical tools for their solution. Just the language used was different. Here are two examples:

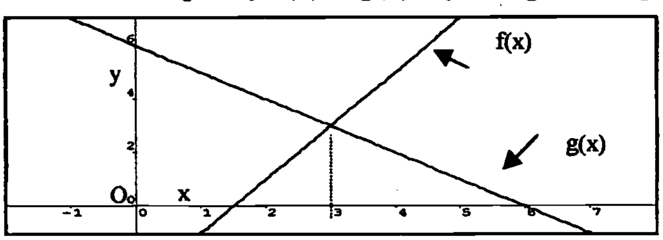
R.P.2 (the second question of test 1): The problem of handcraft

In the next diagram the lines e_1 and e_2 have been drawn. Line e_1 represents the cost of a handicraft in relation to the number of shirts produced and line e_2 represents the corresponding rewards. Find the number of shirts that the handicraft has to produce in order to start receiving a profit.



M.E.2 (The second question of test 2)

Solve the inequality f(x) > g(x), by using the diagram below.



R.P.5 (the fifth question of test 1): Children go to school

"It was a warm day. A student's team started walking to school from the usual meeting point. They were talking to each other loudly, on their way to school. When they realized that they were getting late, a student suggested to the others to hurry up. Then the students started walking faster. But the day was really warm and the conversation quite interesting, so after a while they started walking slowly again..."

Try to depict the above story in a diagram, in which the horizontal axis will represent the time t and the vertical axis the distance d.

M.E.5 (the fifth question of test 2)

The numbers a1, a2, a3 with a1< a2< a3, are representing the gradients of the lines, of the following diagram. Find the line corresponding to each number.

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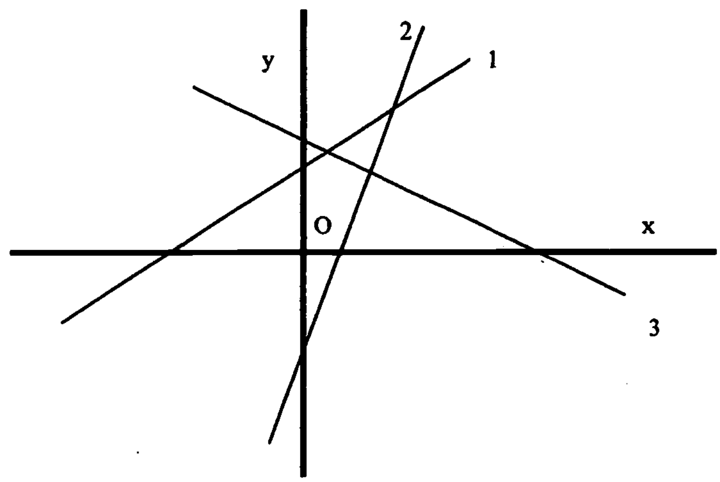

4. Some sample and results

The statistical analysis was based on linear regression, of the form $Y=\beta_0+\beta_1X_1+\beta_2X_2$, where Y is the dependent variable and X_1, X_2, are the independent variables.

The major dependent variables of the regression analysis we were interested in are:

TOT_R.P. is the cumulative grade of a student in Test 1

TOT_M.E. is the cumulative grade of a student in Test 2

The major covariate independent variable was

MATH_97, which is the grade that the student received in Mathematics in school in the year 1997, while

TREAT is the nominal independent variable taking the values 0 and 1 depending whether the student belonged in the control group or in the experimental group.

The main interest of the research lies in the coefficient of TREAT. If the coefficient of TREAT is positive and big, that means not just that the experimental group performed much better, but that the treatment produce a tangible increase of the students performance. The variable MATH_97 is the grade that student received at the end of the previous academic year in Mathematics, on the basis of his/her performance throughout the year. It is an objective homogeneous and very rigorous measure of student's performance throughout the school, and in our statistics MATH_97 serves as a measure of student's mathematical capabilities *before* our tests. We note that both the Mann-Whitney test and the Two-Sample Kolmogorov-Smirnov test, at 95% significance level, do not reject the hypothesis that the experimental group and the control group have the same mathematical capabilities *before* the administration of our tests. Nevertheless the Linear Regression model, by itself takes care of the fact that the control group and the experimental group, could have not the same mathematical capabilities *before* the tests. Because in the Linear Regression equation, the coefficient

of TREAT provides the *added value* to the student's mathematical capabilities through the problem solving teaching.

Linear regression of the dependent variable TOT_M.E. vis a vis the independent variables MATH_97 and TREAT, gives the coefficient of TREAT lying in the interval [2.423, 5.999], at 95% significance.

Linear regression of the dependent variable TOT_R.P. vis a vis the independent variables MATH_97 and TREAT, gives the coefficient of TREAT lying in the interval [2.448, 5.551], at 95% significance.

The above two regressions show that the experimental group performed significantly better, not only in the Real Problems Test 1, (something that may not constitute a surprise), but also in the Mathematical Exercises Test 2.

The above analysis provides strong indications that the Modelling Orientated Teaching offers a more complete teaching approach, in comparison with the Island Approach. In the Modelling Orientated Teaching, the student approaches the concept intuitively, via a real problem close to his experiences. Subsequently the concept is incorporated in his broader mathematical structure, and connected with other related concepts. This way the student becomes an active participant in the *mathematical discovery,* which results to an in depth understanding of the concept. Actually through interviews that we had with students, we realize that many good (in the sense of **MATH_97**) students of the control group, had difficulty in grasping the interplay between the real situation at hand and the underline mathematical structure. It was exactly in this domain that the experimental group was more at home.

5. The Context of the Real Problem

Up to now, the arrangement of having the questions of Test 1 and Test 2 in one to one correspondence, corresponding questions requiring the same mathematical tools for their solution, has played no role. It is our intention in a different paper to study conclusions connected with this very arrangement. For the purposes of this paper, we would like to point out a phenomenon that we observed, which in one hand stresses the superiority of the Modelling Orientated Teaching, on the other hand indicates a direction for further improvement of it.

The common sense would indicate that each question of Test 1 would be more difficult for the students to tackle than the **corresponding** question of Test 2, since the purely mathematical content was the same and the questions of Test 1 have the extra burden of a real situation *scenario*. Actually the grading of the tests indicated exactly that in most of the cases, **but not in all cases.** Among the notable exceptions were the questions R.P.2 and M.E.2, and R.P.5 and M.E.5, that were mentioned earlier in the paper, where students performed better in the Real Problem question instead of the Mathematical Exercise counterpart. Detailed tables are presented below.

Group ▶ Question ▼	Control	Experimental
R.P.2	49.0%	78.0%
M.E.2	19.6%	46.0%

Group ▶ Question ▼	Control	Experimental
R.P.5	36.4%	54.0%
M.E.5	35.3%	44.0%

(Percentage of correct answers in each group)

It seems to us that an interesting kind of conjecture is in order: **If the Context of the Real Problem is familiar to the students, that enhances their understanding of the mathematics of the situation.** This deserves further investigation.

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PEDAGOGY AND THE ROLE OF CONTEXT IN THE DEVELOPMENT OF AN INSTRUMENTAL DISPOSITION TOWARDS MATHEMATICS

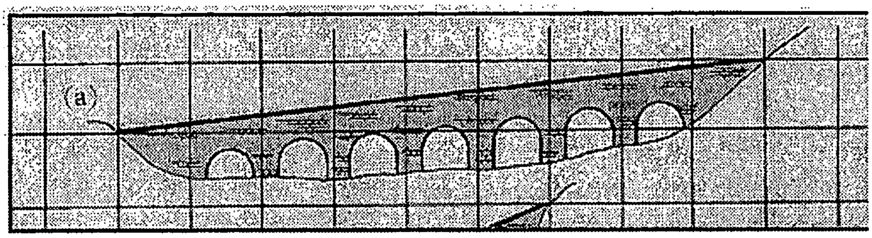
Simon Goodchild

College of St Mark and St John, Plymouth UK

A student's response to a routine task set in a 'real world' context raises questions about the origins of instrumental approaches to learning mathematics. In particular the unconvincing application of mathematics to contexts that contradict students' experience is conjectured as a possible root of a disposition towards mathematics that interferes with learning.

The Scene

Paul (15 years old, below average attainment) is working out the gradients of railway tracks; in each question the track is shown along a viaduct crossing a valley, the diagram is drawn on a one-centimetre grid.



(School Mathematics Project, 1986: 50)

When I asked Paul how he had worked out his answer he explained:

Paul I've got my ruler and then I've measured along there [along the track] or I could have counted the squares.

The first part of his response suggests that he is working out the sine of the angle of inclination rather than the tangent, but as he continues he reveals a misconception. When I asked Paul to measure both lengths he does indeed find them to be the same within the limits of accuracy of his ruler; applying Pythagoras' rule reveals the sides to be: 1 cm, 9 cm and 9.055 cm. The evidence appears to confirm Paul's misconception, but does he really believe it?

I Does that surprise you?

Paul Yeah, a bit because I would've thought that there ... but if you add that black one [the track] down on to there [horizontal line of the grid] they're longer.

I suggested that he tested the three other similar questions in the exercise, the triangles' sides are 3 cm, 7 cm, 7.615 cm; 6 cm, 8 cm, 10 cm; and 1 cm 14 cm, 14.035 cm. Paul observes the difference he expects for two but not for the last one.

To explore how Paul accommodates this discrepancy between his expectation and experience I try to get him to reflect on the evidence:

I Do you think that really is the same, that length there [the track in the first question] really is the same as that length [horizontal on the grid] there?

[pause, 18 seconds]

Paul It must be really.

I Why must it be?

Paul Because it's from there to that line there it's nine centimetres and you measure up there from there to there, it's nine centimetres.

In Piagetian terms I interpret Paul's statement as confirmation of his conception of the conservation of length. He has used his ruler to measure both line segments and this has revealed their lengths to be the same. Does this make sense to Paul, or is he experiencing 'disequilibrium'?

I Does it make sense?

Paul No. Them two there, if you done the same as done there, on there it's totally different it's that there smaller along the bottom than it is going up the hill.

I Does that bother you?

Paul I just think, figure out one way to work this out and then you go on to the next one and then you have to find another way to work it out.

From a constructivist perspective, in this episode Paul reveals a disposition towards mathematics (NCTM, 1989) that enables him to dismiss experiences that might result in cognitive conflict, reflection, accommodation and learning. He merely needs to find the new rule to work out each type of question. However, before continuing with the discussion about Paul's response and the possible roots of his 'disposition' I will outline the circumstances in which the conversation arose.

Methodological context

The conversation with Paul arose during a longer research programme in which I spent almost one complete school year attending every mathematics lesson of a class of 14-15 year old students. The students, described by the school as 'intermediate tier,' revealed a wide variation of attainment across the class; Paul was at the lower end of the attainment range of the class. The research adopted an ethnographic style of enquiry following Eisenhart (1988). During my time with the class I observed and audio-taped periods of whole class teaching and when students were engaged in individual activity or small group work I conversed with them about their work and tried to elicit evidence of the goals towards which they were working. Lessons followed a regular pattern of short teacher expositions and prolonged periods of individual or collaborative activity during which the students worked on exercises in the SMP 11-16 texts (School Mathematics Project, 1986) widely used in the UK. Key features of the students' experience were the teacher's attempts to familiarise novel tasks

(Doyle, 1986) and the fragmentation of mathematics into discrete, unconnected routines (Ernest, 1996). The students' response was to adopt an instrumental approach to their work (Skemp, 1976; Mellin-Olsen, 1981).

I spent substantial periods of time (10 - 20 minutes) observing, listening and conversing with students about their work; the time spent with individual students enabled me to expose their beliefs, attitudes, goals and understanding. The background and school experience of the students was remarkably similar to those at 'Amber Hill' in Boaler's study (Boaler, 1998), it is unsurprising, then, that the outcomes also matched those reported by Boaler, in her words, 'the students developed an inert, procedural knowledge' (1998: 59).

In the following discussion of Paul's responses to my probing I want to consider what associations there may be between his experience and disposition towards mathematics, especially I want to suggest that setting mathematics in a 'real-world' context may be a contributory factor.

Paul's beliefs about mathematics

Paul's response to my probing is remarkably reminiscent of 'Benny' described by Erlwanger in his seminal paper (1973).

> Benny believed that there were rules for different types of fractions, as illustrated by the following excerpt:
>
> B: In fractions we have 100 different kinds of rules …

Erlwanger goes on to remark:

> (Benny) has developed consistent methods for different operations which he can explain and justify to his own satisfaction. He does not alter his answers or his methods under pressures.
>
> … learning mathematics has become a "wild goose chase" in which he is chasing particular answers. Mathematics is not a rational and logical subject … (Erlwanger, 1973: 10, 12, 16)

Commenting on the influence of Erlwanger's paper Steffe and Kieren (1994) observe "In one ingenious stroke Erlwanger was able to falsify the behaviorisitic movement in the practice of mathematics teaching." (1994: 717). Erlwanger's paper may have been influential in guiding curriculum planners, text book writers and teachers, it does not however, appear have led to developments in learning. And although we might not describe Paul as having a 'behavioristic' approach to learning, in Mellin-Olsen's terms Paul displays characteristics of 'instrumentalism' (Mellin-Olsen, 1981).

I want to conjecture that Paul has acquired a disposition towards mathematics that makes cognitive conflict or unanticipated interruptions to the flow of activity (Raeithel, 1990) that might provoke reflection problematic. In the conversation with Paul the basis for conflict has been laid but he is able to dismiss the contradiction he perceives when he claims: "I just think, figure out one way to work this out and then you go on to the next one and then you have to find another way to work it out."

Possible sources of instrumentalism

Boaler (1998) suggests that it is the approaches adopted by the two schools in her study that account for differences between students' progress and attitudes towards mathematics. Additionally, Schoenfeld (1992) relates students' beliefs about mathematics to classroom practice. My own research supports this and I want to look more closely at a number of features of traditional approaches to mathematics, and one feature in particular, that may provide an explanation for Paul's disposition.

First, I want to consider what similarities there may be between Paul and Benny's experiences of mathematics. Benny's class was 'using Individually Prescribed Instruction' (Erlwanger, 1973: 7); Paul had followed an individualised scheme for the first two years of his secondary education, using the small booklets of the SMP 11-16 scheme (School Mathematics Project, 1986). Many teachers using this scheme continue to use an individual approach into and beyond the third year despite the scheme's transition to more traditional textbook format; I cannot, however, be certain that this was Paul's experience. The teacher of Paul's class approached the subject through short, whole class, introductory expositions with the major part of class time spent by students engaged in individual and collaborative activity as they worked at their own pace through the exercises. However, Boaler's research (Boaler, 1998) suggests that the earlier influence of an individualised learning scheme may not be a crucial factor in accounting for student performance beyond the immediate experience of the individual scheme.

Secondly, as noted above, a significant feature of the teacher's approach was to familiarise the work for the students. This action has been discussed by Doyle (1986). To facilitate students' activity in their tasks the teacher breaks them into small steps that provide little or no challenge. Students are left with the task of memorising the sequence of steps without necessarily understanding the orchestration or meaning in the context of the task. Doyle has explained how teachers are under pressure from pupils to familiarise tasks as a means of maintaining control. Woods' studies of schools confirms this, he observes "for pupils ... the most important attributes of good teachers are that they should ... be able to 'teach' and make you work and keep control." (Woods, 1990: 17).

Thirdly, and possibly related to the above is that students experience mathematics as a fragmented collection of discrete rules, routines and skills. An analysis of the mathematics experienced by the students in the class I studied revealed little sense of coherence between different topics studied; the only common themes that ran throughout the year were elementary number operations (supported by calculators) and measuring skills. Ernest (1996) attributes this fragmentation, in part, to the notion of the 'spiral curriculum'.

Fourthly, the regime of the National Curriculum (in the UK) and the pressure upon schools to raise standards which are narrowly defined in terms of test and examination results, has an impact upon students' experience of mathematics. Hoyles explains: "Regular assessment is also expected to enhance pupil

motivation through mastery of short term goals, and 'knowing clearly where you are and where you are going.'" (1990: 116). The effect of assessment on learning styles has been demonstrated in the research of Entwistle who remarks:

> … the type of questions given in a test can induce a surface approach to studying and that the factual overburdening of syllabuses and examinations may be responsible for the low level of understanding exhibited by students when prevented from reproducing answers by well-rehearsed methods. (Entwistle, 1981: 81).

The fifth feature of students' experience that may contribute to their instrumental approach to learning, which I will examine in more detail than the foregoing, is the contextualisation of mathematics. I want to argue that rather than making the mathematics meaningful or, possibly, demonstrating its application, context can often persuade students that the activities of the classroom have little to do with their everyday experience outside the classroom and this leads them to lay aside their expectations of rationality, meaning and consistency for the duration of mathematics lessons. The use of 'real world' contexts merely serves to convince some students that they are situated in a practice that is remote from their lives outside the classroom. And rather than providing a tool that might be called upon to provide a superior rationality, as Lave (1988) observed in her subjects, school mathematics is perceived by some students as a set of meaningless routines to be memorised.

Negative reactions to context

When questions are set in contexts in which students may have 'expert' knowledge the simplification of the context to make the mathematics accessible may help to convince the student that mathematics has little to do with 'real life' and does not make 'sense' because it does not fit with their understanding or experience. Consider, for example, another episode that arose from my time with Paul's class. A student was confronted by a question about the probability of a football team winning successive matches. The student was a keen soccer player and he could not engage meaningfully with the problem because it did not fit with his experience. The task suggested that the probability of winning successive matches remained constant irrespective of what happened in the matches. The student argued that this would not be the case because the result of the first match would influence the team's morale for the second match, also the comparative strengths of the teams would differ between matches. The question that, I guess, was intended to demonstrate an application of the mathematics studied served rather to convince the student that mathematics has little to do with his 'reality'.

Alternatively the contextualising may introduce complication and make a familiar situation inaccessible. An example of this is provided by Goodchild (1995) who illustrates his discussion using an episode with a student who is confronted with a question based on purchasing a quantity of fertiliser. The student is given the price of 4 kg, told that the cost is proportional to the amount, and asked to calculate the cost of 20 kg. Goodchild argues that the

student finds this question inaccessible in part because of its use of technical vocabulary, 'proportional,' thus setting the question in a classroom context in which the student has to know the 'special rule'. The context does not contribute towards the student's construction of a conception of proportionality. Instead, the language and classroom context separates the question and mathematical activity from a 'real' situation in which the student would have little difficulty.

Contextualising mathematics may convince the student that special rules are invented for the classroom. A fascinating example of this is provided by Brink (1991) when discussing the 'realistic mathematics programme.'

> All the children (aged 5-6) had recently travelled on a city bus; it had stopped repeatedly at bus-stops. In the Wiskobas project the bus is an important model for introducing addition and subtraction. We re-created the bus situation in the classroom: one pupil is the driver and wears a cap; the other children, standing in a line behind him, are the passengers.
>
> At the first stop all the children got off, which was not what the teacher expected. Some of the children thought they were on a school-trip and not on the city bus. Others thought they were on the city bus and were going to 'granny' but they failed to notice that the bus had stopped repeatedly. That did not matter to them on their real way with mother to see 'granny'. Others wondered whether the 'game' had finished at the first stop –
>
> Each child had his or her own particular idea which fitted into the play-situation of getting off. The surprise shown by the teacher however quickly put an end to the confusion. The children revised their ideas and quickly hopped on to the bus.

Brink remarks:

> With the help of such instructional manoeuvres (play acting and showing surprise) we can develop meaningful knowledge in children, in the sense of knowledge that fits into a situation as envisaged by radical constructivism. (Brink, 1991: 197)

The point I want to make here is that for the context to work in support of the didactics the teacher had to impose a special rule. In doing so the situation was not merely enacting a bus journey, it had become relocated as a 'classroom' activity. A similar situation became evident in the conversation with Paul from which the extracts above are drawn. I wanted Paul to reflect upon the meaning of change in height, as used in the formula presented in the text for calculating gradient: change in height/horizontal distance. Because he does not give me the answer anticipated I rephrase the question several times. Paul is consistent with his explanation, I reproduce just two, out of his five attempts:

> Instead of the going along the flat it'd just, it's going up the hill

> … the train's coming along, and along a flat surface and then it comes to a hill which it climbs but it needs more speed to get up the hill and if it's a steep hill it will need quite a lot of speed to get up to the top

I conjecture that Paul is interpreting change in height as 'changing' height. His experience of hills is partly dynamic, which is evident from his references to motion, and partly it arises from looking at road and track *surfaces* rather than the cross sections conveniently arranged in the mathematics text. The mathematics requires Paul to adopt an alternative meaning for change in height. He might be able to accept this meaning and remember it for the classroom context but it removes the context from his real, lived-in-world experience outside the classroom.

English and Halford (1995) draw attention to how stereotypical views can interfere with students' problem solving. In one example they explain how a female student attempted to solve a logic problem making the assumption that women do not play chess or golf. Again, if it is necessary for the teacher to contradict the stereotype *in support of the mathematics*, the possibility exists that students might listen to their teacher and follow the 'rules' within the classroom and, rather than adapting their expectations outside the classroom, they will develop a belief in two non-intersecting practices. In particular, the 'real life' practice of their lives outside the classroom and a 'classroom' practice where 'normal' beliefs and rules must be suspended and the teacher's rules followed and memorised.

I believe few would want to argue that instrumentalism has roots in traditional pedagogy but that contextualising mathematics may also be a root is a conjecture that may be contentious. Setting mathematics in contexts to reveal applications and facilitate meaning-making appears to be widely accepted without question. Admittedly my argument rests, largely, on the idiosyncratic responses of one student (as, we note, did Erlwanger's seminal and influential paper). However, I hope I have demonstrated that it is possible to interpret other evidence to support the argument; I believe the significance is such as to require further investigation.

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THE ENCULTURATION OF MATHEMATICIANS IN GRADUATE SCHOOL

A Research Report Proposal for PME 23

DANIEL L. GOROFF
HARVARD UNIVERSITY

Abstract" Earning a Ph.D. changes people's attitudes and values in ways that reverberate throughout the educational system, especially because the graduate students who become professors also end up teaching teachers. Many public policy issues, for example about the balance between research and teaching at research universities, also focus on graduate education. The many suggestions for reforming the Ph.D. therefore require careful analyses that discuss both philosophical underpinnings and empirical data. Recent studies expose some of the myths, realities, and mysteries of the enculturation process that turns graduate students into faculty.

1. Focus

One surprisingly effective way to understand complicated systems is to examine extreme cases. Analyzing behavior when parameters are pushed towards their limits can establish boundary conditions that help explain how the system functions under more typical circumstances. This paper argues that certain features of the highly complicated system of education in the United States can be understood as reflecting what does and does not go on in the special case of graduate education.

Graduate study represents the outer limits of educational activity along many dimensions. Compared to other degrees, earning a Ph.D. simultaneously involves more specialization and breadth, more time and pressure, more maturity and infantilization, more isolation and socialization, more research and teaching, more prestige and risk, more freedom and tradition, more intimacy and anonymity, more convention and originality, etc. By stretching people to their limits, these tensions of Ph.D. study end up powerfully enculturating those who survive to the norms and expectations of their discipline.

Both the process and content of graduate student enculturization have far reaching consequences. For example, graduate students become the professors who teach undergraduates in general, and who teach teachers in particular. Pedagogical practices picked up in graduate school can therefore propagate easily throughout the educational system. This is the argument that "teachers teach as they were taught," which may or may not be entirely true in

mathematics. Attitudes towards the history and philosophy of mathematics especially do seem to play a large role in teaching decisions in any case, and these attitudes do get passed on implicitly or explicitly. Most graduate students are enculturated to be Platonists on weekdays and Formalists on weekends. Both these philosophies are at odds, though, with the explicit Constructivism that is popular among K-12 teachers and teacher educators. This helps explain the rift many feel between mathematicians and mathematics educators.

Thinking about graduate student socialization not only reveals much about the internal workings of the educational system, but also about its treatment by outside policy makers. American research universities are often described as products of an uneasy marriage between the British college system and the German research institute system. The integration of research with education makes a great slogan, especially among government agencies seeking to protect budgets for either purpose (though not when the same agencies calculate indirect cost reimbursement rates). Yet no one seems able to articulate precisely what this slogan is supposed to mean in practice, and it is an especially important question concerning graduate programs. Whether Ph.D. candidates who teach and conduct research should be treated as employees or students when it comes to grants, taxes, and unionization rights are matters currently being thrashed out in American courts. At a time when consumerism, for-profit competition, and political expediency also pose threats to the traditional role and funding of research universities, a great deal may depend on how we describe what graduate study is supposed to accomplish.

2. Framework

Like much educational research, this project seeks to illuminate some current policy debates. Despite or perhaps because graduate school occupies an extremal position in the education system, many of the standard reform recommendations for K-12 seem designed to make students from kindergarten onwards behave more like Ph.D. candidates. For example, reformers often emphasize the importance to children of inquiry, group projects, constructing concepts, data gathering, standing on a par with teachers, and learning to think like scientists and mathematicians. Conversely, many of the standard reform recommendations for Ph.D. programs seem designed to treat those students more like school children. For example, reformers often emphasize the importance to graduate students of hands-on learning, of practical problem solving and real world experiences, of work on communications skills like how to write and speak, and of better supervision, mentoring, and help with decision making.

However well-intentioned, recommendations like these seem confusing and misdirected. Whereas it was once popular to discuss the stages that individuals and institutions pass through as they learn, the reforms as presented seem not to acknowledge the idea of progress over time. Such a stance is fundamentally uneducational, and particularly at odds with the cumulative nature of mathematics and science education. To help make sense of what might be valid in these recommendations, it is therefore necessary to reexamine what we mean by progress beyond just accumulating more content knowledge. For individuals, this project entails an investigation of epistemological development through schooling that locates kindergarten at one end of the spectrum and graduate programs at the other. For institutions, this project entails an investigation of historical development that locates current calls for reform within the context of previous successes and failures.

3. Literature

American graduate education is often touted as the best in the world. Especially in science and mathematics, professors often say that it is the one part of the educational system that works well and attracts the most able students from the world over. Yet prominent critics of the Ph.D. date back as far as the degree itself, including the article by William James entitled "The Ph.D. Octopus." There is no end of serious reports, panels, and studies filled with sober recommendations. Samples are listed in the bibliography below. The same themes run through nearly all of them: Ph.D. training is too narrow; it does not prepare people well to become faculty, let alone anything else; there is an overemphasis on research at the expense of teaching; there are not enough jobs; Ph.D. programs are exploitative, expensive, elitist, demeaning, and drawn-out; etc. In short, the critics maintain that Ph.D. programs enculturate graduate students too well in ways that society does not need. It is hard to tell if these critics are representative of academia, employers, or society, or if they are merely the most vocal, since those who are relatively content with graduate education may have little incentive to write about it.

Most of the statistical studies of Ph.D. education concern the supply and demand for academic jobs. As well, there is also a great deal of controversy about methods of ranking graduate programs, especially in science and mathematics. Research into broader questions about the attitudes, expectations, and experiences associated with graduate education, however, is just beginning to produce intriguing results. Maresi Nerad at UC Berkeley, for example, has conducted a comprehensive survey of Ph.D. recipients ten years after completing their degrees in seven fields including mathematics. Her statistics permit comparative analysis not only of their employment patterns, but also of

rich qualitative data about the strengths and weaknesses of their graduate experiences as seen with a decade of perspective. Another survey by Chris Golde at the University of Wisconsin asks current graduate students and faculty about their socialization with respect to teaching, ethics, and service. Moreover, Jody Nyquist at the University of Washington is collecting and synthesizing both written and interview data concerning Ph.D. reform in general. These results can be contrasted with a recent OECD study that places such efforts in an international context.

Background data concerning the attitudes of mathematicians in particular towards academic culture and priorities has been collected by the Joint Policy Board for Mathematics in their unusual "Recognition and Rewards" study. Shirley Malcom of AAAS has compiled powerful interviews and statistics concerning the graduate experience of traditionally underrepresented groups in science and mathematics. There are also several volumes that document and promote specific kinds of reform projects in mathematics departments, mainly about either the mentoring of graduate students or their training as teachers.

4. Position

At a time when there are so many forces for change acting on the American educational system, the one sector of it that seems most successful should proceed with caution. Especially because changes in graduate education resonate throughout the system, it is important to tease out myth from reality concerning Ph.D. programs and the ways they enculturate future faculty. For example, the integration of research and teaching on university campuses in the United States has clearly helped shape graduate education. Thus while reformers oppose one with the other too quickly and easily, it might be better to reexamine and reconcile research with teaching more carefully. Lee Shulman, for example, has pioneered one approach for accomplishing this by giving the "scholarship of teaching," as advocated by Ernest Boyer, a subject matter that Shulman calls "pedagogical content knowledge."

To make better decisions, graduate students need to understand issues like the balancing of research vs. teaching in mathematics programs, both in theory in terms of the purposes of higher education, and in practice in terms of career consequences. Although most reform efforts target the enculturation future faculty do or do not receive in graduate school, receiving institutions arguably have an even greater responsibility, and should more actively orient and mentor new hires. While many senior faculty perhaps are not yet well-equipped to take on this role, this situation is changing. For example, Project NExT and Project Kaleidoscope are two highly successful national efforts to provide such support

to young mathematics and science professors who are unlikely to receive it locally.

5. Implications

Investigating the enculturization of mathematicians in graduate school should help with clarifying and evaluating the many calls for Ph.D. reform in particular and higher education reform in general. Viewing the problems and successes of graduate programs through this particular lens not only highlights certain results and questions in educational research, but also suggests the need for further experiments, examinations, and explanations. Establishing a stronger sense of the direction and range of cognitive development in schooling through the Ph.D. could help organize better thinking about the entire educational system.

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IMAGES AND THEIR FRAMES OF REFERENCE:
A Perspective on Cognitive Development in Elementary Arithmetic

Eddie Gray
Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL, UK

Demetra Pitta
Psevdas Primary School
Larnaca
CYPRUS

This paper explores the relationship between children's numerical understanding and the frames of reference of their images. Embedded within psychological approaches that evoke imagery through verbal and visual cues, the study explores the different kinds of imagery identified from elementary school children's responses to such cues and links them to levels of numerical achievement. The results suggest that images with descriptive qualities manifest through specific and/or episodic references are common to both high and low achievers. However, the images of high achievers, in contrast to those of low achievers, display a spectrum of quality that have a more generic core. Such differences may have consequences for successive process–to-concept encapsulations.

INTRODUCTION

Our efforts to gain insight into why some succeed in mathematics and others fail has consciously taken a route that considers cognitive development. Whilst acknowledging the existence of a wide range of social and cultural influences on this development, (see for example Cobb, 1987; Gruszczyk-Kolczynska & Semadeni, 1988) our interest is manifest in seeking answers to the question "*What are children really doing in their heads?*"

This paper reports part of a wider study designed to investigate the ways in which different kinds of mental image may influence children's approaches to elementary arithmetic. It is not the purpose of the paper to become caught up in the format of mental image from a propositional or visual point of view. A guiding principle is that, irrespective of such format, there are other aspects of mental imagery that requires further discussion. These may prove to be important to our understanding of cognitive development and in particular to our understanding of divergence in numerical thinking.

Drawing upon work in cognitive psychology, the paper initially considers the existence of different kinds of image with particular reference to those identified by De Beni & Pazzagalia (1995). It continues by broadening the debate and illustrating how these different kinds of image may be associated with different levels of arithmetical achievement. Images children project when stimulated by single verbal and visual cues, in concrete and abstract form, suggest that those of children with lower levels of numerical achievement are of essentially of a descriptive kind. In contrast, those who are more successful project, in an 'integrated' way, a spectrum of the different kinds of mental images that reflect descriptive and relational characteristics.

THEORETICAL BACKGROUND

Setting the Scene

The concept of image has become less clear as more progress is made on the research front (Cooper, 1995) and this paper recognises that human cognition requires different representational constructs to describe it. Consequently in our context the term image may be seen as a mental reference which is a product of imaging in any modality whether it be visual, verbal, olfactory, auditory or kinesthetic.

Images are significant components of cognition and the notion of mental reference has particular relevance for the study of mathematics. Its association with a conception of 'thing' or 'object' draws our attention to a quality of abstraction that ranges from a mental analogue of a real object to a linguistic description or a symbol (see also Pavio 1986). Such a distinction would seem to be particularly relevant for cognitive development in elementary arithmetic which, it is suggested, is grounded in successive process-to-concept encapsulations (Tall (1995)

The issue for this paper is whether those who are less successful appear to have a disposition towards particular kinds of image that are qualitatively different to those projected by children who are more successful.

Different Kinds of Mental Representation

Thomas, Mulligan and Goldin (1996) have suggested that children's internal systems of representation of numbers go through a series of changes, from a semiotic one in which meaning is established through previously constructed representations, to an autonomous stage in which a new system of representation functions independently of its precursor. Pirie & Kieran (1994) indicate that a learners strong early attachments to particular dominant images can seriously influence the development of understanding The relationship between the understanding and imagery suggests that abstract imagery appears to dominate amongst relational thinkers, concrete and memory images amongst relational thinkers (Brown & Presmeg, 1993). In elementary arithmetic such differences may emerge because those who predominantly use procedures display less inclination to filter out information (Gray & Pitta, 1997). Relational thinkers appear to reject information or, to put it another way, are more able to select the information that is more relevant to a particular situation. This would suggest that the different images identified amongst children at extremes of numerical achievement have their roots in a qualitative abstraction governing the individuals active mental process of making sense of data through personal and/or impersonal involvement.

These forms of involvement have featured as some of the attributes which guide the classification of different kinds of imagery explored by Cornoldi, De Beni and Pra Baldi (1988) and De Beni & Pazzagalia (1995). The former suggest that images spontaneously evoked from a single verbal cue may be identified as *general*, *specific* and *autobiographical* in decreasing proportions. General images represent a concept

without any reference to a particular example or to specific characteristics of the item. For example, the cue 'table' may evoke the response "I can see a table". Reference to a single well-defined example of the concept without reference to a specific episode characterised specific images. Autobiographic images, seen to be special cases of the 'specific' category enlarged to include the involvement of the self-schema, were those which involved either the subject without a precise episodic reference or objects belonging to the subject.

De Beni & Pazzagalia questioned the meaning that may be given to the autobiographic image category. They suggested that there is a distinction between images referring to a single episode in the subject's life (*episodic–autobiographic*) and those that actually involve the subject without a precise episodic reference (*autobiographic images*).

METHOD

Within this study images are inferred from the words of subjects. Thus, two features governed the framework for the development of cues that formed the basis for an interviewee's response. The first is embedded in psychological approaches which evoke imagery through a verbal and visual stimulus (see, for example, Stillman & Kemp, 1996). The second is that the relationship between imagery and numerical achievement should focus on two issues:

- the existence of different kinds of image identified through distinct generation processes, and the likelihood that these can be grouped into categories.
- the relationship between an emphasis on one or more of these categories and the level of numerical achievement.

Visual and verbal cues were presented to a sample of 16 children representing the extremes of numerical achievement within each of four year groups of a primary school within the UK. Numerical achievement was measured by criterion based test results available in the school and a numerical component which formed part of a larger study of which this paper is a part. The final sample had 8 'high achievers' and 8 'low achievers', 2 of each drawn from children aged 8 to 11.

A modified version of the defining feature approach (see, for example, Roth & Bruce, 1995) was used to gain a sense of what it is children feel is important to communicate when faced with cues in verbal form. These included concrete words and conceptual labels that had more abstract meaning. The former, seven items which denoted things that could be perceived by one of the sense modalities and had shown they could evoke images more readily than other words (see Pavio, 1969) included the cues 'dog', 'table', 'dots', 'football'. The latter, more clearly associated with elementary arithmetic, had eleven items including 'five', 'thirty-three', 'half', 'three-quarters', 'three-eighths', 'naught point seven five', 'number' and 'fraction'.

Sixteen cues were presented visually, nine being visual representations of verbal cues The item bank was subdivided into two sections, pictures and icons, for example, a 'football', 'dots', a 'table', and symbols for example, 5, 99, 3÷4, 0.75.

Each verbal and visual cue was presented with the following instructions:

Verbal/visual:	What is the first thing that comes to mind when you hear the word (or see)...?
Verbal:	Talk for 30 seconds about what comes in your mind when you hear the word...
Visual:	Look at this, when I tell you close your eyes and put this in your mind. Talk to me for 30 seconds. Do it now.".

Children were interviewed over two separate occasions approximately 8 weeks apart. All interviews were video-recorded, linked to field notes and transcribed.

RESULTS

Classifying Responses

Children images were classified as *general* when responses indicated that they were not talking about a specific item. For example:

"It's a surface on metal or wooden sticks" (Y4+, verbal, 'table')[1]

"Part of" (Y6+, verbal, 'fraction')

The criteria for responses identified as *specific* were extended to allow for multiple examples which were qualitatively similar. For example:

"... a cheetah is one, a rabbit is one, a dog is one a cat, a Labrador, Dalmatian, owl, eagle, buzzard, etc." (Y3–, verbal, 'animal')

"Like one, two, three, four, five, six, and ten are numbers". (Y4–, verbal, 'number')

De Beni and Pazzaglia suggested that some images could be seen to be 'contextualised' since they had distinctive and relational characteristics. However, the way in which contextualised mental representation were identified as 'item specific' and 'relational', did not satisfy the clear distinctions observed in the responses of the subjects within the current study. They could be descriptive through association with a scene or a sequence of scenes or they could have a higher order quality more in tune with Skemp's (1976) notion of relational. After detailed analysis of responses it was decided that the notion of a contextual image would be better served if there were a distinction made between *episodic* and *generic* images.

Episodic kinds of image were associated with some scene or sequence of scenes and were most often narrated in continuous speech:

"Boys can kick it around and sometimes it can get lost over the field." (8–, verbal, 'ball')

"Number five. I think of a row of numbers and light shines on number five. A light goes along and stops over the number five." (Y5–, verbal, 'number')

1 The indicators at the end of each example provides the age of the child and their level of achievement, the phase of items and the item iteself. Thus (Y4+, verbal, 'table') indicates a 'High achieving' nine-year-old responding to the verbally presented item 'table'. Low achievers are denoted by the symbol –.

Other responses that implied the existence of a context were more fragmentary; more a collection of disconnected, seemingly arbitrary, *generic* statements. These originated from the same general concept that served as the basis for explicit relational connections. They were not descriptions of a sequential event with a clear beginning and end but more often a collection of statements that seem to have the potential to produce new ideas. Though they had a 'general' quality, the statements diverged to produce different ideas related to the item. For example:

> "Keeps you fit. An exciting game. Millions of fans. Important in every nation. Children and adults play it. Different types of football and balls." (Y4+, verbal, 'football')

> "... maths and writing. Seven... you could be doing some adding or times and the number seven might come up. Seven is also played in sport... seen on the back or shirt... has one digit... in a phone number." (Y5+, verbal, '7').

The 'autobiographic-episodic' category, which allowed for the "occurrence of a single episode in the subject's life connected to the concept" (De Beni and Pazzaglia, 1995 p. 1361), was noted infrequently but identified as follows:

> "My friend wasn't good at fractions and she had to take extra work home." (Y4+, verbal, 'fraction')

> "We have recently done reflections and they had lots of halves in them. We had to put our mirror down the side and see the rest of it. I saw lots of those." (Y4+, visual, 'half')

Though De Beni and Pazzaglia's approach requested subjects to construct 'good and vivid images' of 'high value nouns' given as cues, the current study involved reporting images of abstract nouns and of symbols or icons representing them. Such stimuli could evoke mental representations of a proceptual nature (Gray & Tall, 1994), for example:

> "It's divisible by nine" (Y6+, verbal, '99'),

> "3 parts out of 4, fraction, 0.75, more than half." (Y6+, visual '3/4')

Analysis of Results

Here we present results relating to the 30-second response that allowed children to contribute as much as they felt able to. Consequently, from each child, there may be a sequence of responses that embrace different kinds of image. All responses from each child were classified. Figures 1 and 2 display the different kinds of representations recorded as a proportion of the total number of child responses given (N). To provide a clearer sense of the more dominant mental representations classifications identified in less than 8% of instances are collated in the category 'Other'. This figure may be regarded as quite arbitrary, but careful consideration of the summarised results suggested that percentages up to this level frequently indicate more idiosyncratic and less common behaviour.

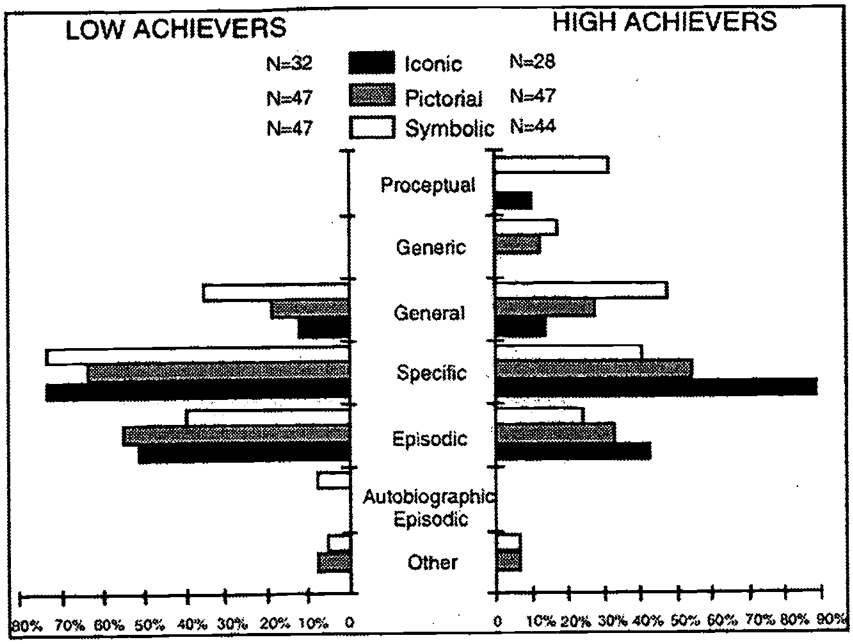

Figure 1: Occurrence of the different kinds of mental representation as a proportion of the number of items considered in the second response of the visual phase

Figure 1 provides a summary of responses to the visual items. The dominance of specific images may be clearly seen amongst the low achievers. These, together with the episodic form, are clearly dominant in reactions to each set of cues: the pictorial, iconic and symbolic.

Images from the numerical cues by high achiever's covered a wide spectrum of the different kinds. However, like the low achievers, high achievers appeared to have had difficulty detaching themselves from mental representations associated with specific and episodic content when cued by icons. These had easily distinguishable surface features but were less easy to name and connect with different experiences. It is suggested that these features militated against the abstraction of the intrinsic qualities that would have lead to the projection of generic and proceptual kinds of image.

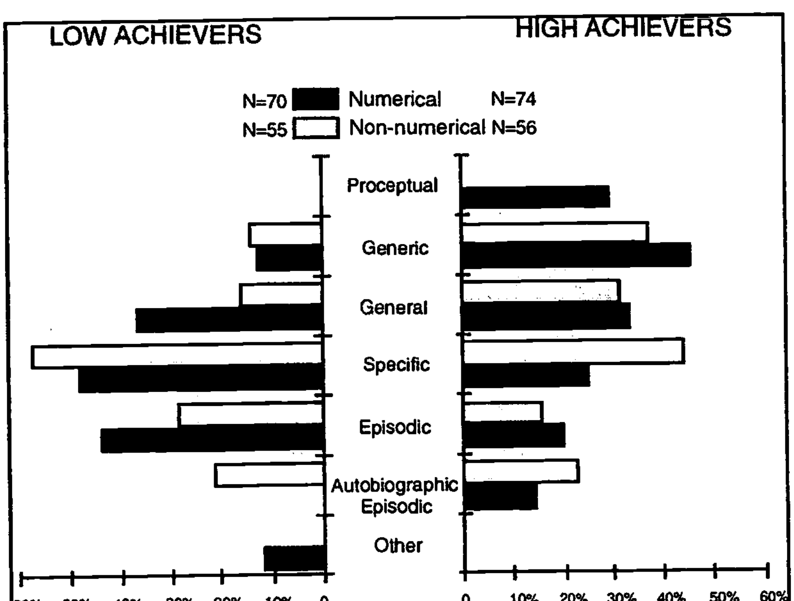

Figure 2: Occurrence of different kinds of mental representation as a percentage of the number of verbal items in the second response of the verbal phase.

Figure 2 summarises images associated with the verbal cues. The proceptual and generic images identified from high achievers when responding to these items suggest they have an ability to link the cues to different experiences so that intrinsic similarities may be abstracted. In contrast low achievers continue to project specific and episodic images but the symbols evoke a relatively high proportion of general ones.

DISCUSSION

This study supports the notion that different kinds of imagery may be identified amongst children of elementary school age. Additionally, it suggests that these may be identified through responses to both verbal and visual cues of either concrete or abstract nature and the evidence points to qualitative differences in the kinds of imagery projected by children at extremes of numerical achievement. Both groups projected imagery underpinned by descriptive qualities in that they are specific and/or episodic. However, whilst low achievers consistently project features embedded in such images, high achievers projected more relational imagery to display a spectrum of quality which, it is conjectured, has a more generic core.

Such differences, pivoting as they do around tendencies to project descriptive and/or relational images, are made apparent by the inclusion of generic and proceptual kinds of image. Perhaps it is no surprise to see that children who are selected on the basis of achievement, with the implications this may have for their proceptual/procedural interpretation of symbolism at an operational level (Gray & Tall, 1994), reflect such differences. However, the paper goes further than this. Not only do children at different levels of arithmetical achievement project qualitatively different images when prompted by numerical cues, they also project qualitatively different images of other conceptual ideas that are presented free of context through verbal and the visual cues.

It is hypothesised that the high achiever's reactions to the different phases may be accounted for by the processing differences that apply between the presentation of a visual stimulus and the presentation of a verbal stimulus. Though the invitation to consider the 'first thing that comes to mind' has not been considered in this paper, its analysis suggests that high achievers initially provide a general mental representation, often through naming or by giving a general comment about the item without any reference to other characteristics. It is conjectured that this needs to be done before a mental search to retrieve generic or proceptual qualities is carried out. When it was too difficult to project a general mental representation high achievers did not give a response but low achievers supplied a specific image associated with the surface characteristics or a specific example of the item.

Age differences have not featured in the analysis as presented. However, it does not appear to have much influence upon the quality of mental representations projected by low achievers. However, those of high achievers, being of a more generic nature, operate at a more relational level that seems to grow in complexity.

CONCLUSION

The results would seem to have important implications for our understanding of the way in which children view the development and use of numerical activity in the context of repeated process-to-concept encapsulations. The limited spectrum of mental representations projected by low achievers suggest they are either unable to, or simply choose not to, see through actions and objects to embrace more abstract qualities. It may even be that early teaching has influenced their focus of attention.

The development of elementary number requires an ability to concentrate the mind and give careful thought to an act or idea — to filter out irrelevancies and separate notions from their context. It involves the construction of relationships between and amongst objects and of the actions on them. It would seem that such a process might work to the advantage of the high achievers. Their disposition towards the formation of images that integrates descriptive and relational characteristics seems to ensure the construction of number concepts through the synthesis of pseudo-empirical and reflective abstraction Tall (1995). It is conjectured that this follows a very different cognitive development from that of children whose disposition towards descriptive images arises from their concentration upon empirical abstractions and direction, through teaching, towards the pseudo empirical.

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THE ROLE OF UNCERTAINTY IN CONSTRUCTING AND PROVING IN COMPUTERIZED ENVIRONMENT

Nurit Hadas & Rina Hershkowitz

The Weizmann Institute, Rehovot, Israel

The role of uncertainty in promoting the need to prove, in the sense of explaining and convincing in geometry, has been discussed by many researchers (e.f., Chazan, 1993; Dreyfus & Hadas, 1996; Goldenberg, Cuoco, & Mark, 1998). We demonstrated (Hadas & Hershkowitz, 1998) that uncertainty stems from a geometrical situation in which students cannot find any example confirming their intuitive conjecture, because such an example does not exist. In the present article we describe a case in which students are engaged in a construction task in a situation of uncertainty: the construction was possible, but was opposed to students' intuitive conjectures.

Background

Yerushalmy, Gordon & Chazan (1993) distinguish two kinds of geometrical problems in computerized learning environments: construction problems and conjecture problems. In both, one has also to consider the different roles of proof; i.e., as a tool to show the universality of a statement and as a tool to explain (Hanna, 1990). In many conjecture problems students feel that the universality of the conjectured attribute of the geometrical object is confirmed by the computerized environment. The dragging operation on a geometrical object enables students to apprehend a whole class of objects in which the conjectured attribute is invariant, and hence they are convinced that their conjecture will be always true (De Villiers, 1998). Therefore the only motivation driving students to prove is to explain. We have already described (Hadas & Hershkowitz, 1998) an example of a conjecture problem, in which the conjectured attribute does not exist, hence the dynamic tool does not play a convincing role. Such a problem is a good didactic opportunity to make the student aware of the role of explaining by proving, as a convincing tool.

In construction problems the universality and the explanation are interwoven. Schoenfeld (1986) has described two construction problems to support his claim that "empiricism is an essential component of the machinery of deduction, conversely, however deduction makes possible a discovery that is inaccessible to insight or empiricism" (p. 249). In common construction problems in computerized environments, students are asked to construct a figure whose existence is obvious. In the following, we will show students working on a problem in which they are asked to investigate by construction, the <u>existence</u> of a figure satisfying given conditions.

The problem - Congruent Triangles

*In this activity, we will investigate **if and when**, two triangles having several equal elements, are congruent.*

Task 1. Given a dynamic triangle ABC, build another triangle having two angles and the included side equal to two angles and the included side of ΔABC.

*Task 2. Is it possible to build a triangle with one side and two angles equal to those of a dynamic triangle ABC, but **not** congruent to ΔABC? If it is possible, build such a triangle; otherwise explain why .*

Task 3. Is it possible to build two non-congruent triangles, with five equal elements? Create a hypothesis.

Task 4. Is it possible to build two non-congruent triangles with six equal elements? If yes construct two such triangles, otherwise explain why .

Task 5. Is it possible to build two non-congruent triangles, with five equal elements ? If yes construct two such triangles, otherwise explain why .

Problem analysis: a. Task 1 prepares students for Task 2: they realize that congruence is an invariant attribute under the dragging operation. This means that when one changes the original ΔABC by dragging, the second congruent triangle changes accordingly. But, the second triangle cannot change when it is itself dragged by one of its vertices.

b. When students complete Task 2, they are asked to identify the equal elements, and to their surprise, discover 4 equal elements in the two non-congruent triangles.

c. The task of constructing a triangle non-congruent to ΔABC, with 5 elements equal to elements in ΔABC, is quite complex. We consequently decided to decompose it into three stages: a conjecture stage (Task 3), a discussion of the case of 6 equal elements (Task 4), and finally the construction of a triangle with 5 elements equal to elements of ΔABC, although non-congruent to it.

Problem Characteristics: (i) As in all other construction problems in such an environment, the computerized tool elicits empirical actions followed by a prompt feedback from the tool. But, for complete success, deductive considerations are needed. (ii) The problem provokes surprise followed by uncertainty. (iii) Students' construction processes depend on their ability to analyze the problem deductively, leading to the elaboration of an existence proof. The last two characteristics are discussed in detail in our analysis of some students engaged in this activity.

The Interviews

Pairs of Grade 10 students with ability in the upper half of the population, were interviewed while working on the problem. The interviews were videotaped and analyzed. The students were towards the end of a two year geometry course, and were familiar with ways to prove. In the following, we describe two representative pairs of students.

The first pair

In Task 1 a girl and a boy, Gili and Nadav, constructed a triangle non-congruent to ΔABC with two angles and the included side equal to the same elements in ΔABC. They discovered that the new triangle can be modified only by dragging the vertices of ΔABC, and explained this modification by invoking the relevant congruence theorem. They then struggled with Task 2 for 15 minutes. First they constructed the second triangle (see Figure 1) where: DE=AC; <D=<A; and <F=<B.

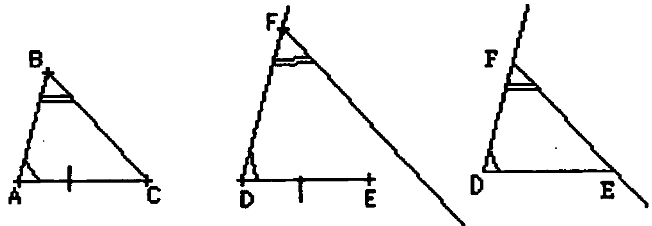

Figure1

The following excerpt expresses the dialectic process the pair underwent.

Nadav: *They were supposed to be non-congruent but they are* [congruent].

[F is a random point on the line DF, They drag F in such a way that FE remains the triangle side. When they change ΔABC, they had to adjust F again by dragging.]

Gili: *It's the same again.*

Nadav *Ah! It is like that because if this one* [pointing at <D and <A] *and that one* [pointing at <B and <F] *are equal, the third must be equal as well...If two angles are equal even if they are not at the two ends of the side, the third is equal too.*

I: *So what is your conclusion?*

Nadav: When *we have two* (equal) *angles and side, and they* [the triangles] *are not congruent, it is impossible.*

[Gili starts looking for a way to show that the construction is possible. The interviewer suggests analyzing all possibilities for locating the "second angle".]

Nadav: If *we copy it* [pointing at <B] *here* [pointing at a point on DF] *we get stuck!*

Gili: *If we copy <C here* [she points at a point on DF, and starts to construct].

[Nadav who was hesitating so far, starts watching Gili, and after she copies the second angle, exclaims:]

Nadav: *Ahh! you did it the opposite* [meaning not correspondingly].

[The pair complete the construction and check it by dragging. Nadav explains what they did explicitly. The discussion continues.]

Gili: *So, we succeeded?*

I: *Did you discuss the idea of correspondence between two triangles in the context of congruence in the class?*

Nadav: *No, we didn't. The teacher did and now I understand why.*

[To their surprise, the students find that the two non-congruent triangles have even 4 equal elements.]

In Task 3, the pair hypothesize that a triangle with 5 elements equal to elements of ΔABC, but not congruent to ΔABC, can be constructed in a way similar to that in Task 2. The interviewer does not let them try to construct such a triangle and asks them to move to Task 4. They discuss the issue of 6 equal elements in non-congruent triangles. Gili claims that it is impossible to construct such a triangle, but did not justify her answer. In contrast Nadav argues that it is possible and applies the strategy used successfully in Task 2, on a piece of paper (see Figure 2).

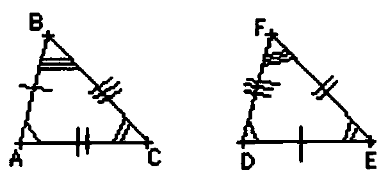

Figure 2

The interviewer asks them to analyze their drawing using their geometrical knowledge. Nadav answers immediately: *In fact there are 3 equal sides so the triangles are congruent, hence it is impossible with 6 equal elements.*

Their investigations on Task 5 are based on: (i) considerations they raised in the previous task, (ii) their hypothesis in Task 3 that it is possible to construct a triangle with 5 elements equal to elements of ΔABC, but not congruent to ΔABC. They try to apply similar methods to those they used in Task 2. First they plan the construction by erasing the "equal signs" of one pair of corresponding sides, on their drawing in Task 4 (Figure 2). They then try to construct ΔDEF according to this drawing. They succeed in this endeavor, but with 3 equal elements only. They use measuring and dragging operations to try to obtain all the 5 equal elements, but without success.

The students then discuss what they have done with the interviewer and conclude that it is difficult or even impossible to obtain the 5 equal elements in the two triangles by dragging, even if such a triangle exists. They initiate deductive

considerations as following: the triangles can have only two equal sides, so they must have 3 equal angles. This means that the triangles are similar, and therefore the ratio between corresponding sides in the two triangles must be constant. Following a suggestion by the interviewer, the students construct an example of two such triangles by choosing (with the help of the interviewer) particular lengths for two of the sides in each triangle (see Figure 3). They deduce that the invariant ratio between the corresponding sides in ΔABC and ΔDEF must be 1.5, and hence that AB must be 4 and DE, 13.5.

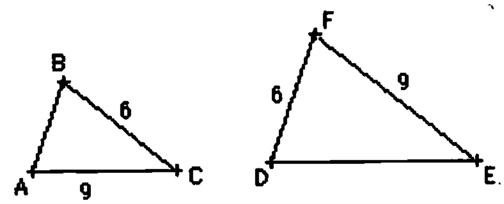

Figure 3

The pair then constructed the two triangles with the software. The following discussion took place.

Nadav: *But the angles should be equal as well.* [He thought for a while and added:] *but it is O.K.*

I: *Why are you sure that the angles should be equal?*

Nadav: *Because they* [the triangles] *are similar.*

<u>The second pair</u>

Two boys, Asaf and Asher answer Task 1 in the same way as the first pair. They also start Task 2 by constructing a second triangle (see Figure 1), in which DE=AC, <D=<A, and <F=<B. Once they realize that they obtain congruent triangles, they immediately conclude that they did it wrong, and replace the third equality by <F=<C. They then succeed in constructing the two non-congruent triangles. The following excerpt **came at** the end of their discussion concerning 4 equal elements:

I: *So, how many equal elements do you have in the two triangles?*

Asaf: *Three*

I: *No more?*

Asaf: *The third* [pair of] *angles cannot be equal because then they* [the two triangles] *will be congruent.*

Asher: *No because if the third* [pair of] *angles* [are equal] *they will be similar and not congruent.*

I: *Can you construct two triangles with two pairs of equal angles in which the third pair is not equal?*

Asaf: *Yes!*

Asher: *No!*

Asaf: *Wow! It's impossible!...Only the sides are not equal, because the sum of the angles in a triangle is 180°. Wow! four equal elements and the triangles are not congruent.*

In Task 3, they hypothesize that triangles with 5 equal elements are always congruent. This assumption is elaborated in Task 4, for 6 equal elements. The interviewer does not let them leave this issue too quickly, and brings the idea raised by the previous pair of students at this point by asking: *Maybe one can construct equal angles not opposite to the equal sides?* The students seem intrigued by this idea and draw a sketch similar to Figure 2, discuss it for a while, and conclude that because the triangles have 3 equal sides they must be congruent.

They come back to the issue of 5 equal elements in Task 5, and conclude that if such triangles do exist they must be similar, because the three angles are equal, and thus the ratio between corresponding sides must be constant. After a long dialogue between the two students, in which the interviewer suggests they focus on one special pair of triangles, they succeed in sketching one example, by calculating the ratio between the sides and applying it to calculate the third side in each triangle, - in the same way as the first pair (see Figure 3).

The interviewer checks their awareness of their own actions by asking: *What do you think about the angles, must they be equal as well?* This pair of students is unable at this point, to refer back to the similarity of the triangles in order to justify the equality of the angles, in spite of their intuitive feelings that this is true. So, they construct the two triangles with help of the software, measure the first pair of angles and find them equal. In the process, the similarity of the constructed triangles becomes visually obvious (see Figure 4).

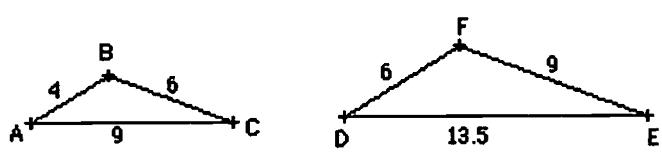

Figure 4

This visual clue leads the pair, after a while, to <u>deduce</u> the triangles' similarity,as we can see from Asaf's concluding remark: *If the ratio between the sides is fixed, then they* [the triangles] *are similar and then the angles are equal. But, they* [the triangles] *are not congruent. So, we did it!*

Discussion

The dialectic feeling of uncertainty.

As we mentioned above, one of the characteristics of this activity is the feeling of uncertainty concerning the existence of examples fulfilling the constraints. In Task 2, in contradiction to their conjecture, both pairs were surprised to find a way to construct the non-congruent triangle with the 2 angles and one side equal to those given, and even more, to discover that the two triangles have 4 equal elements. This way of copying an equal angle into ΔDEF in a non-corresponding position, is based on deductive geometrical reasoning. By using it to construct the required triangle and watching the result on the screen, students became confident that their construction was correct. This confidence led them to hypothesize that non-congruent triangles with 5 equal elements can be constructed. The students went even further to claim that when two triangles have 6 equal elements, they are not necessarily congruent, and they even tried to sketch such triangles. However, when they used a congruence theorem, they were confident that 6 equal elements always implies congruent triangles. This raised their suspicion in Task 5, and uncertainty characterized their first steps in this task. Only after they succeeded in designing their construction deductively did they become confident again that such triangles do exist, and insist on constructing them on the computer.

The deductive explanations.

Different kinds of deductive reasoning occurred in the various tasks. In Task 2, students constructed the ΔDEF by copying the two angles to two vertices which are not both on the given side, yet obtained a congruent triangle (see Figure 1). This action led them to a different construction, based on deductive considerations. This construction by itself is the deductive solution (an existence proof) to Task 2. The situation in Task 4 is similar to the one we have already described (Hadas & Hershkowitz, 1998), where students could not find any example confirming their intuitive conjecture, because such an example does not exist. Here a non-congruent triangle with 6 equal elements does not exist, so the only way to be convinced is to give deductive explanations. Task 5 has the same characteristics as Task 2; the construction is based on deductive considerations. And yet we saw that these deductive considerations are very fragile. Students needed to reflect on the whole process they underwent, before they became aware of the logical chain they themselves elaborated.

Deductive explanations were not the first weapon used by students in justifying their actions or their planned actions. This finding has been noticed by many researchers (e.f., Schoenfeld, 1986; Hoyles and Jones, 1998) Examples: (i) When asked, after accomplishing Task 2, how many equal elements the two non-congruent triangles have, the second pair spent a long time at arguing before being able to use deductive considerations, based on the sum of angles in a triangle. (ii) The two pairs had difficulties in seeing the logic chain on which the construction of the triangles in

Task 5 was based. It was even more difficult to recreate the opposite chain from the constructed triangles to conclude that the triangles must be similar, and hence the angles must be equal.

As described above, students gave deductive explanations of their actions, a fact that made them confident in what they did and in the ways they justified why certain constructions are impossible.

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Mathematics Education Research Project: Researching Teacher Development Through Action Research

Anjum Halai[1], University of Oxford, UK.

The report is on a one-year action research project where university-based researchers facilitated and studied the action research being conducted by a group of mathematics teachers from schools in Karachi Pakistan. The research design was qualitative in nature and used participant observation, field notes from meetings, and reflective journals maintained by all participants as tools for data collection.The study claims that action research promotes teachers' learning and professional growth by enabling them to take a critical look at their practice. However, also significant is the growth and learning of the university-researchers, as helping teachers to inquire itself requires critical reflection. The study has immense implications for inservice and preservice teacher education initiatives and raises some important questions regarding the structural changes that might be required in schools to allow them to become communities of inquiring practitioners

The mathematics research project looked at teacher development through action research. The research group comprised a team of eleven. Of this, six were teacher- researchers who taught mathematics in schools from the public and private sector in Karachi Pakistan. The remaining five were university-researchers who both researched and facilitated the inquiry that the teacher-researchers conducted into their practice. One of the five from the university coordinated the project. I worked as university- researcher with one mathematics teacher.

Action Research. The term 'action research' means different things to different people. Smith & Lytle (1993) defined teacher research as systematic, intentional inquiry by teachers. Carr & Kemmis (1986) saw action research as undertaken by participants in social situations to improve the rationality and justice of their practices, their understanding of these practices and the situation in which these practices are carried out. The concept of a reflective practitioner as defined by Schon (1983) and Stenhouse (1975) is consistent with the idea of teacher as researcher. Action research is seen as a vehicle to promote reflection and growth of the reflective practitioner entailing cycles of planning, implementing, observing and reflecting on ones own practice. Grundy (1987), describes three modes of action research i.e. technical action research which is effective but product directed. Practical action research which emphasizes the role of personal judgement in decisions to promote change and emancipatory action research which promotes critical consciousness towards change

[1] Anjum Halai is currently on secondment from the Aga Khan University, Karachi (Pakistan), pursuing her D.Phil in mathematics education from the University of Oxford, UK.

initiatives. Hence if the project is directed by the university-researcher and ownership of ideas is not taken up by the practitioner the research project will remain in technical mode. However, if the participants take over ownership,the project could become critical or emancipatory.

Similar to the project in Jaworski (1998), a central issue in this case was the interrelationship and interdependence between teachers' research and the study of the research. The university researchers provided a major support to the teacher-researcher in conducting the research and worked in dual roles of researcher and mentor. Given the nature of the project I see action research as defined by Pedretti (1996) as a form of professional development which encourages teachers to participate in cycles of planning, acting, observing and reflecting thereby creating possibilities for change and transformation and in which equally important are the transformations and growth experienced by the facilitator (mentor) in an action research context.

The Context. The school was a private school for girls in Karachi with a staff of about eighty teaching 1300 girls. The overall administration is in the hands of the principal. All the class room activity in this project was based in one particular section of Class Six (10-11yrs.) chosen because both the teacher-researcher and I were jointly responsible for teaching mathematics to it. By agreement I initially did most of the teaching and Zarina observed me. About two months later she indicated her wish to teach and be observed. Each provided feed back to the other.

<u>The universiy-researcher</u>. After teaching at this school for eight years, I participated in a degree course in teacher education at a local university. The programme selects and trains teacher educators, who then return to their schools to work jointly with the university and the home school that sponsored them: in university 6 months running courses for the visiting teachers, and in school for 6 months. In the school I taught mathematics to one class which was 1/3 of the total teaching load that teachers in school had. For continuity to the pupils, one other teacher (Zarina) was assigned to work with me in the same class so that when I left for my university work the second teacher could take over and continue smoothly.

<u>The teacher - researcher</u>. From all the mathematics teachers in school Zarina accepted my invitation to join the project. She had been teaching mathematics in the school for four years and 11 years elsewhere. She has a first degree in biology but no professional qualification. I got to know, Zarina better when she came to the university to attend the visiting teachers programme, which I was conducting along with a group of other colleagues.

The project team. The project team including the teacher-researchers, university-researchers and the coordinator met once a quarter over a period of one year to find out where each one was at, sharing learning, and raising issues. These meetings provided opportunities of on-going cross analysis of emerging data.

The Research Process. Over a period of one year Zarina and I met once a week for 90 minutes to plan lessons, share ideas and expertise including doing the mathematical activities that we wanted to carry out in the class. A regular feature of these meetings was to reflect on the previous week's happenings and discuss issues that had arisen.

Observation & post-observation conferences: After each classroom session a post-observation conference took place between Zarina and myself to promote reflection on key issues arising in the classroom.

Reflective journals: Writing and sharing journals raised an issue. If the journal was written for an audience then the entries might be coloured by considerations of what the reader might want to see in them. In order to let journal writing be an exercise in genuine reflection and yet allow others to learn from them Zarina and I decided to read, talk about and analyse portions of our journals that we wanted to share with each other in our pre- and post - observation conferences.

Analysis. For discussion of results, critical incidents have been selected from our work together because they throw light on how Zarina was thinking about planning for teaching, learning, mathematics and other emerging issues and how my own understanding developed of teacher learning and action research. To give a sense of progression these incidents have been described in sequence.

Action Research Focus. When I look at Zarina's initial journal entry I find her thinking about lesson planning in conjunction with planning in-groups.

> Planning lesson in-groups provides opportunities to learn from others and we learn many things, which we did not know before.
> Journal entry: 2-9-96

Later In a conversation she identified the focus of her action research by saying:

> Z_1: I want to know what a lesson plan is like.What should be in it.
> I want to plan using the ways shown in the VT programme.
> field notes: 28-9-96

I interpret from above that her action research focus is embedded in her recent experiences of professional development in the VT programme. Her journal entries indicate that she had realized the value of group work during her

experiences in the programme where working in groups provided opportunities to exchange ideas, work out meaning of new ideas or take an in-depth look at familiar concepts. So she links up the idea of planning with group work. The VT programme may have inititated the process of critical reflection on her practice but in action research that by engaging in cycles of planning, action and reflection that she takes overt and systematic action on that critical reflection. Does it mean that a springboard is needed to facilitate the action research process? Rudduck (1991) suggests that a pre-condition for teacher research may be that the teacher temporarily becomes a stranger in his or her own classroom. Becoming a stranger would allow them to look at 'ordinary' interactions and events with new eyes. My interpretation is that Zarina's experience of participating in the VT programme allowed her this opportunity to step back and look at her own practice with anew.

However, over the life of the project her question did not remain the same. I saw that the questioning stance enabled by the research process got her to look critically at other aspects of her practice. For example, she questioned the traditional examination which usually assesses pupils' procedural knowledge of mathematics but does not necessarily indicate pupil understanding of those procedures. Hence the examination paper was modified so that questions were not asked according to the set pattern.

An Example of Zarina's Learning.

The fraction lesson: This lesson on fractions had been planned together by Zarina and myself. However, I taught while Zarina observed. I asked the pupils to work in pairs on the problem given below and left them to choose their own ways of interpreting the problem and recording its solution. Squared paper and other manipulatives were made available to them.

Saima wants to divide three fourths of a cake into fifths so that she may send it to five of her friends who could not attend her birthday party. How much will each friend get?

A number of issues arose from this lesson. It was apparent that students were not sure which strategy to use and why. Some had got the solution through the routine rule 1/5 of 3/4=3/20 but could not explain the reasoning behind it. Naima had used 3/4÷1/5=15/4 and could also not explain the use of the strategy or what the answer 15/4 signified.

In a subsequent conference talking about her action research Zarina referred to this lesson and the following conversation took place:

I_5: Then what do you want to do?

Z_6: Show it (lesson plan) to you so that my lesson plan is not such that when I go to the class I get confused myself

I_7: i.e. you then want to teach yourself?

Z_8: and you then tell me

I_9: Tell what?

Z_{10}: feed back-what was right what was wrong-my language my questioning-

I_{11}: Explain confusion some more

Z_{12}: Like Naima's question today. Which I had also asked you yesterday (refers to the planning meeting)-whatever I am taking in the plan that I should also know and then what questions come out of it I should know the expected answers.

I_{13}: This confusion that Naima had what do you think about it?

Z_{14}: When you explained then I understood otherwise I also used to think like Naima---lesson plan should not be such that I do not know what answer to give the child-child will get confused but we should satisfy her-this question that Naima asked many should have asked-if children ask a question and we cannot satisfy them we are putting them in greater difficulty. Field notes: 1-10-96, Translated from Urdu

Her journal entry regarding her action research question read:

> In action research I want to learn to make a lesson plan which is correct i.e. its application is not difficult and neither is my plan such that the pupils are confused by it. Instead it should be clear and it should not require too much time to apply neither should it confuse pupil thinking. Zarina's journal: 3-10-96 Translated from Urdu

A number of questions arise as a result of the above conversation & journal entry. My interpretation is that Zarina is struggling with the question 'how does learning take place?' Hence when she says 'the child will get confused' she is talking about 'cognitive conflict'. She is saying that in the process of learning some confusion does arise and sees the teacher's role in facilitating learning by alleviating confusion. But she also regards confusion as a sign of weakness and so something to be avoided by the teachers. Hence, she insists on a perfect plan which would somehow help her avoid confusion. The question arises how does

Zarina anticipate creating cognitive conflict if she does not see a place for confusion in her teaching. Is she holding conflicting beliefs that confusion is necessary for learning and that confusion is something wrong and to be avoided. The above conversation with Zarina also raised questions regarding her perception of a teacher. Perhaps, it is her perception of the teacher's role as a source of all knowledge and authority, which makes her shy away from confusion. It could also be due to her feeling of inadequacy in dealing with the confusion that she tried to avoid it in her classroom.

In the above conversation, it was not clear what it was that Zarina did not know but had understood after our planning meeting. On asking, she said that like Naima (in the lesson above) she was also not clear about the rule of division of fractions. It was when we had worked through some of the mathematical activities in our planning meeting and discussed the reasons behind the rule that she became clear about the division rule in fractions. I found like Shulman(1987), that teachers' subject matter understanding plays a significant role in teaching mathematics. I was also convinced like Feiman Nemeser & Parker (1990) that doing mathematics with the mentee is a way of making subject matter a part of the conversation in learning to teach

<u>My learning</u>: I see that action research is suited to the purpose of helping teachers question their practice, understand it better and take steps to improve it. The reason as claimed by Smith & Lytle (1993) is that in teacher research the questions asked and the interpretive frames used to understand and improve the practice are owned by the teacher. For example, I saw Zarina identify lesson planning as an area of concern for her. Perhaps, she felt that a perfect plan is one that would help her improve her practice so that she could teach mathematics for understanding. However, during the course of her research I found her asking questions about her own understanding of mathematics. In the fractions lesson she learns the rule of division of fractions. I saw her questioning the process of pupils learning so that she thinks critically about the role of confusion in aiding or inhibiting learning. Assessment and its related issues and concerns also became part of her deliberations at the time of term examinations. Action research takes the teacher beyond the immediate focus of the research question so that the teacher focuses on almost all aspects of her practice. Carr & Kemmis (1986) claim that the questioning stance enabled by the action research made problematic the taken for granted aspects of the teacher's practice. To this end action research seems to be a tool which enables the teacher to look at her practice in a holistic way.

By breaking teacher isolation through creating opportunities of sharing ideas, issues concerns and celebrating success it helped build morale and self confidence of the teachers. For example, Zarina as I knew her in the VT programme and what I heard of her from other colleagues had always been a

retiring kind of an individual. But, during the course of the project Zarina participated in almost all the group meetings. She increasingly contributed her ideas whereas in the initial stage she had expressed her worry about speaking out in-group meetings.

The opportunity to work as a mentor allowed me to deepen my own understanding of the skills and attitudes required to be an effective mentor. I realized that there were certain behaviours, for example, being non-judgmental, encouraging, valuing ideas and respecting confidentiality. I also became sensitive to the impact my actions might or might not have for the teacher. For example, a constant question in my mind was will I create dependency if I do all that she wants in terms of providing new ideas, support, and resources. Grundy (1987) and Showers (1985) also indicate that issues of dependence and autonomy are central to the role of the facilitator or critical friend in teacher research. It would thwart the emancipatory spirit of action research if new forms of dependency are created by the facilitators in their concern for directing rather than facilitating action and reflection. This meant that I had to be very sensitive when deciding when and how to act on judgment formed. Research by Pedretti (1996), Carr & Kemmis (1986), and Jaworski (1998) also recognizes the need for the mentor to use among other things sound professional judgment to enhance the teacher inquiry.

Concluding reflections: For action research to be taken as model of teacher development for the whole school would require greater and more visible support from the principal including the building of a supportive infra-structure, with time for teachers to meet and work collaboratively with other teachers. In this case, the principal agreed to let us conduct action research but did not provide any release time to the teacher to meet and work with me. Smith & Lytle (1993) have indicated that participation in teacher research requires considerable effort for teachers to carve out opportunities to inquire and reflect on their own practice while fulfilling their other responsibilities in school. Zarina gave her own time during lunch breaks or after school. In informal conversations she indicated that she was learning so much that she felt it was time well spent. But this does not mean that other teachers would be in a position to give the kind of time that Zarina or I could.

An implication might be to introduce action research in all teacher education programmes whether pre- service or in- service. This would have the advantage of creating awareness regarding the procedure, value and implications of action research as tool for teacher development. Involving head teachers in action research projects would be one way to ensure support from them and bring about awareness about the kind of time and other resources required. Another implication would be greater involvement of head teachers in what goes on in class rooms and reduced status differential between head teachers and teachers.

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Using arguments from physics to promote understanding of mathematical proofs

Gila Hanna
Ontario Institute for Studies in Education
of the University of Toronto, Canada

Hans Niels Jahnke
Institut f? r Didaktik der Mathematik
Bielefeld, Germany

Abstract: *An important challenge faced by mathematics educators is to find effective ways of using proof in the classroom to promote mathematical understanding. We advocate the investigation of one very promising approach that has been insufficiently explored: The use of arguments from physics within mathematical proofs.*

The first premise of this paper is that proof must be part of any mathematics curriculum that aims, as it should, to reflect mathematics itself and the important role of proof within it. The second is that the most significant potential contribution of proof in the classroom is in the promotion of mathematical understanding, a role that it plays in mathematical practice as well (Thurston, 1994). Some educators, proceeding from these premises, have considered ways to make effective use of proof in teaching, and especially in the last twenty years there has been a significant reorientation towards intuition in the teaching of proof (D?rfler and Fischer, 1979). Wittmann and M? ller (1988) speak of "intuitive proof" ("inhaltlich-anschaulicher Beweis"), and Hanna (1990) and Dreyfus and Hadas (1996) draw on the distinction between explanatory and non-explanatory proofs.

These educators, however, have concentrated on the internal aspect of proof, focussing in the main on its function within mathematics (Hanna and Jahnke, 1993; 1996). This paper seeks to redress this imbalance somewhat by investigating proof primarily from the external viewpoint, with a focus on one of its important external aspects: the relationship between physics and mathematical proof. Jahnke (1978) and Winter (1983) have already argued that the usual opposition between "intuitive" and "deductive" is unacceptable, and that mathematical proof should not be seen as a turning away from observation and measurement, but rather as a guide to an intelligent exploration of phenomena. The specific question the paper poses is

twofold: What is the possible role of arguments from physics within mathematical proof, and how should this role be reflected in the classroom?

Previous scholarly work

The first part of this question has to do with mathematics itself. The close cooperation between mathematicians and theoretical physicists has led to a heightened awareness of the many benefits that mathematics derives from physics (Jaffe and Quinn, 1993). Jaffe (1997) points out that physics has traditionally been a source of important problems for mathematics, contributing in this way to its progress, and that in turn mathematical results have helped in the solution of difficult problems in physics.

In a paper on the phenomenology of proof, Rota (1997) maintains that the benefits of this close association are to be seen in mathematical proof in particular. Mathematicians often remain dissatisfied with proofs that, though they establish without a doubt that a theorem is true, provide no insight as to why it is true. Physical concepts and models can make an important contribution to understanding in such cases, and can even help mathematicians devise purely mathematical proofs of a more explanatory nature. In addition, however, an argument from physics may form an integral part of a mathematical proof.

The second part of our question relates to mathematics education. In approaching this issue, we have taken our cue in large part from two publications that deal directly with the role of arguments from physics in the classroom: Winter (1978) and Polya (1981). The ideas contained in these works would be a good point of departure for the suggested investigation. References to physical laws do appear in other educational publications, but only as remarks in passing. Castelnuovo (1971), for example, introduces projections and shadows when treating the notion of similarity. From a theoretical viewpoint, Struve (1990) discusses geometry as an empirical science in contrast to geometry as a theoretical system. In Bender and Schreiber (1985) one finds a different conception of the relation of empirical and theoretical geometry, based on the ideas of H. Dingler. The following paragraphs discuss work published on closely related issues.

Some recent publications describe various approaches to making proof meaningful in

the classroom with the help of empirical arguments. Dreyfus and Hadas (1996) showed that an empirical approach to teaching geometry with dynamic software can bring students to see that proof is required to explain results that are unexpected or counterintuitive. De Villiers (1995), Mariotti (1995), and Mason (1993) discuss several dynamic geometry constructions, illustrate problem-solving methods not possible with pencil and paper and advocate the use of dynamic software for fostering new insights into traditional geometry theorems. Greer (1996), as well, describes the use of empirical arguments for proving.

Among mathematics educators there are also those who advocate basing the teaching of mathematics upon its various applications. This movement includes suggestions for a closer relationship between mathematics and the other sciences (OECD, 1991), a theoretical framework for what is called Realistic Mathematics Education (RME), which advocates using reality as a source for mathematization (Freudenthal, 1983; Streefland, 1991), and other projects that in various ways seek to strengthen the role of applications in mathematics teaching. For the higher grades of school teaching, one must also take into consideration the publications of the ISTRON group (Blum, 1993). None of these proposals deals explicitly with the teaching of proof, however.

What is meant by "arguments from physics within mathematical proofs"?

To explain better the concept behind the proposed investigation, we would like to draw a clear dividing line between using arguments from physics within mathematical proofs and merely using physical representations or illustrations of mathematical concepts or theorems. An example of the latter is the representation of the laws for natural numbers by geometrical configurations of pebbles. The underlying idea we suggest, on the other hand, is to apply in a mathematical proof a complex law of physics as if it were a mathematical theorem. For this there are historical as well as educational examples.

To begin with the former, the application of physics to mathematics has a long history. When a purely mathematical proof of a theorem proves elusive or awkward, mathematicians have often found that the introduction of concepts and arguments from physics yields a straightforward proof. A famous example is Archimedes' use of the law of the

lever for determining volumes and areas (compare his work on "the method"). Another equally famous example, from the calculus of variations, is the so-called Dirichlet principle, which asserts the existence of certain minimal surfaces as solutions of certain boundary value problems. In the 19th century, Dirichlet and Riemann took this theorem as obvious for physical reasons. Weierstra? later criticized its use, however, forcing mathematicians to look for a purely mathematical proof of the principle. This was quite hard to achieve, but in the end the effort led to considerable progress in the calculus of variations (Monna, 1975).

As an example of the application of laws of physics to mathematical proofs in education, we might first mention the construction of the Fermat point of a triangle, where the most elegant method is to model the triangle by a physical system consisting of a perforated plate and weighted ropes (Polya, 1981). Some theorems of elementary geometry, such as the Varignon theorem that the midpoints of the sides of a quadrangle are the vertices of a parallelogram, can be proved most easily by applying the laws of the lever or the notion of centre of gravity. For these and numerous other examples see Winter (1978). A final example is the mean value theorem of differential calculus. If we interpret the derivative of a function as the velocity at a given instant, then the mean value theorem follows directly from the observation that a car going from A to B must have had, at least at one point, the mean velocity as its actual velocity.

Such applications of physics do much more than illustrate a theorem. By introducing productive concepts, they make possible a more satisfactory proof of the theorem, and one that, on the basis of an isomorphism between the mathematical and the physical constructs, is no less rigorous. (In this they differ from much of what has come to be referred to as "experimental mathematics," which in its essence consists of generalizations from instances.)

For the mathematician, indeed, the use of concepts and arguments from physics is primarily a way to achieve a more elegant proof. But such a proof may also be illuminating, in different ways. It may reveal the essential features of a complex mathematical structure, provide a proof that can be grasped in its entirety (we call this the holistic version), as opposed to an elaborate and almost incomprehensible mathematical argument, or point out more clearly the relevance of a theorem to other areas of mathematics or to other scientific disciplines.

These broader benefits are invaluable even to the practising mathematician, so they clearly have great potential for promoting understanding among students. Unfortunately this potential is not being exploited, however, because concepts and arguments from physics have not been integrated into the classroom teaching of proof to any great extent and certainly not in any organized way. This is not surprising, since there is no body of research work on this topic that might provide guidance and tools for teachers and curriculum developers.

Prerequisites for success in using arguments from physics

The educational aspect of our question actually comprises two tightly linked issues: How can the actual role of arguments from physics in mathematical practice best be reflected in the curriculum, and how can such arguments best be used to promote understanding?

To address the first issue, one would have to examine the epistemology of mathematics implied by much of present classroom practice and compare it with accounts of the nature of mathematics implied by the practice of mathematics itself or espoused by mathematicians and philosophers of mathematics.

Implicit differences of epistemology are important. For example, students are often taught that the angle sum theorem for triangles is true in general just because it has been proven mathematically. Ignoring the fact that measurements have shown this relationship to hold true for real triangles as well, this practice implies a very specific and limited view of the nature of mathematics and its relationship to the outside world. Students do not share this view, however, bringing to the classroom the belief that geometry has something to say about the triangles they find around them. In this they may unwittingly be closer than the curriculum to the broader view of the nature of mathematics held by most practising mathematicians. For this reason it should come as no surprise to educators when students are taken aback, misinterpreting the assertion that mathematical proof is sufficient in geometry to mean that empirical truth can be arrived at by pure deduction.

It would seem that educators themselves need to come to the classroom with a more satisfactory understanding of the nature of mathematics, one that encompasses its

relationship with the empirical sciences and everyday human experience. Of course the curriculum itself should be informed by the same understanding.

The second issue is the use of proof for the promotion of understanding. Students being introduced to mathematical proof come to the classroom with preconceived notions and complex epistemological uncertainty. Educators need to understand both much better than they do today. When confronted with the proof of a theorem, for example, students quite often say that they have understood the proof, but still ask for additional empirical testing. From a purely mathematical viewpoint such a request seems quite unreasonable, and teachers usually take it as an indication that the students did not really understand what a mathematical proof is. From the viewpoint of a theoretical physicist, however, the same request would seem quite natural; no physicist would accept a fact as true simply on the basis of a theoretical deduction. Thus a consideration of the role of mathematical proof in theoretical physics may well shed light on the way in which students view proof.

Keeping in mind the viewpoint of the theoretical physicist is useful when analysing how students approach proof when using dynamic geometry software such Cabri Geometry or the Sketchpad, which allow explorative work similar to experimental physics. Comparing students with theoretical physicists also promises to be of help in understanding how teachers might best cope with the questions that may be created in students' minds by the use of concepts and arguments from physics in mathematical proofs.

Broader educational aims

There are also broader educational reasons for studying the use of arguments from physics within mathematical proofs. We will sum up these up in four statements.

- As already mentioned, there is a trend in all Western countries away from using proof in the classroom. In our view this development will undercut the educational value of mathematics teaching and should be countered by fresh approaches to the teaching of proof.
- Of course the growing trend to experimental mathematics should be reflected by an

increased emphasis on experimental mathematics in the schools. But experimental mathematics in the schools should not be just "mathematics with computers." From an educational point of view, this would be a dangerous development. Rather, one should be guided in the use of experimental mathematics by the question of how it contributes to our understanding of the world around us.

- The holistic aspect which many arguments from physics can bring to mathematical proofs (see above) is an important part of mathematical competence that is frequently underestimated. Instead, there is a predominance of step-by-step procedures. Seeking good examples of instances where arguments from physics are useful to mathematics proofs will contribute to developing a way of teaching and learning mathematics which is more balanced in this regard.
- In Western countries physics is less and less a required subject. Physics is the discipline nearest to mathematics, however, and to maintain meaningful and interdisciplinary mathematics teaching it will therefore become necessary to include some elementary physics in the mathematics curriculum. In Germany there are already proposals in this direction, as a consequence of the country's poor results in the Third International Mathematics and Science Study (TIMSS), and the "Konferenz der Kultusminister" intends to decide upon a large scale project exploring new types of interdisciplinary teaching.

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GENERATING ADEQUATE MATHEMATICAL QUESTIONS ACCORDING TO TYPE OF PROBLEMS

Shirley H. Har-Zvi, Bar-Ilan University
Zemira R. Mevarech. Bar-Ilan University
Levi Rahmani, Tel-Aviv University

In recent years, mathematics educators, researchers, and philosophers have convincingly argued that students have to learn how to "do mathematics" and communicate their mathematical ideas to others, rather than memorizing formulas and applying procedures. It leads students to a deeper understanding of the mathematical processes. For this propose The students have to learn how to formulate and solve problems, look for patterns, make conjectures, examine constraints, make inferences from data, abstract, invent, explain, justify and so on (Stein, Grover & Henningsen, 1996).

The constructive theories of learning and teaching emphasize the importance of student-generated problems as a tool of developing deeper understanding. According to these theories, learning occurs when students construct connections between exiting knowledge and new knowledge (Witrock, 1976).

Generating problems is assumed to lead students to look for additional information, which in turn facilitates the constructions of these connections (King, 1989, 1991, 1994; Mevarech & Kramarski, 1997).

Mathematical problem generating refers to creating new problems, reformulating given problems (Polya, 1973), and generating questions based on a given set of quantitative information. It is usually part of inquiry processes.

Silver (1994) argues that problem generating (posing) is a feature of (a) Creative activity or exceptional talent (e.g., in mathematics- Kruteski, 1976; Ellerton, 1986; in others areas- Getzels & Csikkszentmihalyi, 1976; Mansfield & Buss, 1981;). (b) Inquiry oriented instruction (e.g., Brown & Walter, 1983 ; Schoenfeld, 1985). (c) Mathematical activity (e.g., Polya, 1973; Schoenfeld, 1985; Mevarech & Kramarski, 1977; Simon, 1973). (d) A window into students' mathematical understanding (e.g., Ellerton, 1986; Silver& cai, 1993). (e) Mathematics learning activity (Healy, 1993; Streeflend, 1991; Yerushalmi, Chazan, Gordon, 1993). (f) A Means of improving problem solving (e.g., Owen & Sweller, 1985; Silver &cai, 1993). (g) A Means of improving student disposition toward mathematics (Brown & Walter, 1983). (h) A Means for improving information processing and metacognition (e.g., in mathematics- Mevarech & Kramarski, 1997 ; Schoenfeld, 1985,1987; in other areas- Grasser & Person, 1994; King, 1989, 1991, 1994; Pressley & Pressley, 1985; Wong, 1985).

What is generating adequate mathematical questions?

Students are given a mathematical word "problem" in which the question sentence is missing and they have to generate questions that relate to the entire story "problem". Generating questions in this case may vary in the degree of their complexity as a function of the "problem" complexity: one question fits one-step "problems", a series of two or three questions fits two- or three-step "problems", respectively.

Rahmani (1994) developed software called "Pithagoras" (in Hebrew, English and Italian versions), in which students are taught how to generate adequate mathematical questions that are appropriate for various kinds of "problems". It was designed for pupils from 4^{th}-5^{th} grade to the level of elementary algebra problems.

The "Problems" were organized hierarchically, from one-step to multiple-step problems. Students could not move to a higher level on the hierarchy before they mastered the lower level.

The purpose of the software is to develop the ability of students to understand and solve mathematical problems, which are part of their school tasks and beyond.

The goals of the software

1. To develop the comprehension of a mathematical problem and mathematical reasoning.

This software follows closely the topics taught in elementary schools regular classes. The problems concerned with the underlying mathematical concepts such as the four basic arithmetic proportions.

The software emphasizes the comprehension of the linkage, similarities and differences, between the problems. The requirement for calculation is secondary to that of posing the adequate question according to type of problem.

The program leads the student to grasp the relationships between given items of information and ask questions about unknown items of information and arithmetic operations. This implies the ability to extract the substance from the story, the relations between quantitative values – numbers, fractions, etc. – and distinguish between unknown yet reachable information, and unknown and unreachable information.

2. To go beyond cognitive goals.

The software endeavors to increase students' motivation to confront with word-problems. A learner who experienced repeated failures to handle mathematical problems is encouraged to comprehend the problems while working at the computer. The learner is prone to run more "cognitive risks".

3. To develop the ability to categorize problems.

The program leads the student to categorize mathematical problems according to given information about quantitative relations and to the additional information that can be derived from it.

4. To lead the students from concrete to abstract.

The program progresses from rather concrete problems involving information on definite quantitative values, to more abstract problems, involving exclusive information on relations among quantities. Thus, the software advances from numerically presented problems to encourage computation to problems that require the student to determine whether two sets of data are quantitatively equal or not, while the quantities as such cannot be determined. Quite frequently students initially claim that such problems cannot be solved.

5. To distinguish between definite and possible solutions.

Students are trained to discover the framework in which solutions are possible and to understand which solutions are not possible.

6. To develop the ability to grasp an entire problem.

Frequently, students tend to grab a certain piece of information that happened to catch their attention, and embark on the wrong track to the solution, typically a series of casual trials and errors. The software promotes the habit of seeking for the whole available relevant information before taking the first step in the right direction and proceeding straight toward the solution.

7. To upgrade the comprehension of the text of a problem.

Students may fail to solve a problem because of misunderstanding a text, a word or a sentence or because of linguistic poverty and words with inappropriate and inaccurate meanings. The tendency of the software is not to avoid these items. On the contrary, such lexical items are incorporated into the problem story. Indeed, mathematical word problems are useful to enrich the student's vocabulary.

The Bar-Ilans' study

A study was conducted at Bar-Ilan University with 6th grade Israeli students. The hypothesis of the research was that training students to generate adequate question on the basis of given word "problems" in which the question sentence is missing would exert positive effects on students' ability to generate problems, solve problems, and on their attitudes toward the learning of mathematics. Moreover, there is reason to suppose that training students to generate questions in cooperative setting will be more effective than training them to generate questions individually.

Method

Participants were 122 sixth-grade students (63 boys and 59 girls) who studied in two Israeli public schools. Both schools are located in the center of the country and the students come from a similar background.

Treatments

The research was based on a 2X2 factorial design, as follows: about half of the participants were trained to generate mathematical questions based on "word problems" with missing questions, and then solve the problems. The other half was trained to solve the same problems in which the questions were provided. Under each treatment, about one third of the participants studied individually, and the others in pairs. Students did not receive any special training for enhancing cooperative behaviors.

The computer presented the problems under both conditions. The "Question Generation" condition: under this condition the students used the computer software- "Pithagoras", which was introduced before. Students were asked either to select a question (or a series of questions) from a set of questions or to generate a question (or a series of questions) based on all the quantitative information or the relational terms that were presented in the "problem" text. Students received immediate feedback from the computer telling them the extent in which their responses were correct. Then, students were asked to solve the problems.

The "Problem Solving" condition: Under this condition students were presented the same problems, organized in the same way as under the "Question Generation" condition. The two programs were identical except that in the "Question Generation" software the questions for each "problem" were missing. Thus, under the "Problem Solving" condition instead of generating questions, students were administered a "complete problem" in which the question sentence was included and were asked to solve the problems. The computer provided immediate feedback referring to the correctness of responses.

Measurement and Procedure

The battery of examinations and questionnaire included three measurements: Mathematical Word Problem Examination (MWPE), Mathematical Question Generation (MQG) Examination, and Mathematics Attitude Questionnaire (MAQ).

To get further information on the experiment, at the end of the study eighteen students were interviewed. Prior to the beginning of the study, all four groups were administered a battery of pretest. Then, each classroom was assigned to study as planned: about third of the students worked individually and the others cooperatively (in pairs). At the end of the study, all students were

re-administered the battery of examinations and questionnaire. The duration of the study was about three months.

Results and discussion

The findings of the present study showed that students' ability to generate problems prior to being exposed to the training was rather low. In particular, students faced difficulties in generating questions on the basis of two – and three-step problems. These findings are in line with previous studies that focused on questioning skills in non-mathematics classrooms (e.g., Mevarech & Susak, 1994). The findings further showed that training children to generate questions improved their questioning skills, specifically on higher cognitive questions that relate to two- and multiple-step problems. More information about the findings will be introduced in the conference.

These results are appropriate the constructivist theories of learning and instruction. Generating a series of questions on the basis of a given complex problem may lead students to focus on all the quantitative information provided in the "story problem", which in turn may facilitate the solution of such problems. An indirect support for this hypothesis comes from children's interviews.

Several students emphasized the facilitative roles of generating problems compared to solving problems. The following excerpt describes students' responses:

"Selecting and generating the most appropriate questions and thinking why a question is not appropriate increased the probability of solving the problem correctly". "Generating questions helped me to focus and take into consideration all the (quantitative) information mentioned in the story problem". "Generating questions helped me to understand the problem better. It helped me to understand complex problems".

Other students referred to the double roles of generating and solving problems:

"In generating problems I have to think more, because I have to think on both the questions and the answers (solutions)". "Being asked to find both the questions and the solutions facilitates understanding more than being asked only to write the solutions".

Some students mentioned the effects of the very nature of being exposed to a new way of learning mathematics:

"In the past we repeated again and again problem solving. That did not help. In generating questions, I succeeded better in understanding the problems".

Finally, several children had their own suggestions regarding question generation training: "Moving systematically from easy to complex problem (i.e.,

from one-step to multiple-step problems) facilitates learning". "Separating the "problem" and the "question" helped a lot".

The finding further showed that students' positive dispositions toward mathematics were rather stable for those who were trained to generate questions, but decreased for those who were not exposed to the training. This finding supports our hypothesis that being trained to generate questions enhances students' disposition toward mathematics, as seen in following excerpts: "Pithagoras can help children who are anxious in math classes because it teaches them how to generate questions, and that helps them to understand the problems. " Learning with Pithagoras made me like mathematics more because it helped me to understand mathematics". "Pithagoras strengthened my self-confidence because it taught me how to solve problems. I had to select and generate questions and that's easier (than solving problems). Also, the teacher sometimes loses her temper. Pithagoras never gets angry on me and therefore I am not anxious when I study with computer".

Comparing the responses of children who were or were not exposed to problem generation training indicated that "Pithagoras" helped them to (a) better understand mathematics (71% vs. 64% respectively); (b) overcome mathematics anxiety (100% vs. 70% respectively); and (c) strengthen their attitudes toward mathematics (29% vs. 22% respectively).

Contrary to our hypothesis, overall no significant differences were found between children who learned cooperatively versus those who learned individually. Also, none of the interactions between question generation training and learning environment (cooperative vs. individualized) were significant. This finding may be explained by Slavin (1996) as well as other researchers (e.g., Cohen & Lotan, 1995; Johnson & Johnson, 1994). Cooperative learning is effective under certain conditions, such as group reward, special training that focuses on cooperative behaviors, or special training regarding social cohesiveness of the small groups. Such training was not provided in the present research.

The effects of such training on the effectiveness of question generation and solving merits future research. It is also possible that, student with different learning styles and entry behaviors benefit differently from cooperative vs. individualized settings.

Summary and conclusions

The findings of the present research offer practical evidence that we can successfully promote higher cognitive processes regarding adequate mathematical question generation, word problem solving, and strengthening students' positive attitudes toward mathematics.

There is a need to continue this research for a longer period of time and with more students in order to be able to generalize the findings. Examine the effects of generative questions on the development of understanding of mathematical problems merits future research.

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CHILDREN'S MENTAL MULTIPLICATION AND DIVISION STRATEGIES

Ann M. Heirdsfield, Tom J. Cooper, Joanne Mulligan* and Calvin J. Irons

Centre for Mathematics and Science Education, QUT, Brisbane, Australia

*School of Education, Macquarie University, Sydney, Australia

This paper reports changes in children's mental computation solution strategies for multiplication and division applied word problems (involving 1, 2 and 3-digit numbers combinations). The study followed 95 Queensland children from Year 4 through to the end of Year 6. The children's responses showed a development from simple counting to use of derived or known facts for small number combinations, and from counting to quite complex and creative strategies to algorithmic procedures for large number combinations. There was some evidence of instructional effects in the increased use of the taught algorithms, the continued use of counting strategies. There was, at times, sustained use of wholistic.

Interest in mental computation as an important computational method for numbers of two or more digits is not new. However, its significance is now seen in terms of its contribution to number sense as a whole; for example, as a "*vehicle* for promoting thinking, conjecturing, and generalizing based on conceptual understanding rather than as a set of skills that serve as an *end* of instruction" (Reys & Barger, 1994, p. 31). To achieve this contribution, it appears necessary to develop proficiency in mental computation through the acquisition of self-developed or spontaneous strategies rather than memorisation of procedures (Kamii, Lewis, & Livingston, 1993; Reys & Barger, 1994). Mental computation in this form features in various models of computation (e.g., National Council of Teachers of Mathematics, 1989; Trafton, 1994), although usually in combination with written, and calculator methods.

There is research evidence that children can use self-developed strategies to efficiently and effectively solve mental multiplication and division problems of two or more digits, even before instruction (e.g., Anghileri, 1989; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Kouba, 1989; Mulligan & Mitchelmore, 1997). Even in studies of children's solution strategies for more difficult multiplication and division word problems (Murray, Olivier, & Human, 1994), some self-developed strategies have been used (e.g., repeated addition, decomposition and compensation for multiplication; and repeated subtraction, use of multiplication and partitioning for division). There is also evidence for the negative effect of traditional algorithm instruction on efficient mental strategies for multiplication and division examples. For example, Kamii et al. (1993) reported that 60% of third graders who had not been taught the traditional multiplication algorithm were able to mentally solve 13 x 11 (by thinking 13x10=130, 130+13=143); a problem which, in contrast, was only successfully mentally solved by 15% of fourth graders who had been taught the algorithm.

Research has also indicated that performance in mental multiplication and division problems is influenced by the semantic structure of the word problem, with some problems being more difficult than others (e.g., cartesian product multiplication was found to be poorly attempted compared with other types of multiplication - Mulligan, 1992). However, the solution strategy used did not always reflect the semantic structure, particularly as children progressed (Mulligan, 1992; Mulligan &

Mitchelmore, 1997; Murray et al., 1994); for instance, children rarely used sharing for partition division (Mulligan, 1992). Similar results have been found for subtraction (Heirdsfield & Cooper, 1996).

Many researchers have categorised children's solution strategies for multiplication and division word problems (e.g., Anghileri, 1989; Boero, Ferrari, & Ferroro, 1989; Bryant, Morgado, & Nunes, 1993; Carpenter et al., 1993; Clark & Kamii, 1996; Kouba, 1989; Mulligan, 1992; Mulligan & Mitchelmore, 1997). Most of this research has been limited to small number combinations and, therefore, has categorised strategies as counting types (Mulligan & Mitchelmore, 1996). Some research has focused on more complex number combinations, describing strategies in detail (Murray et al., 1994). However, little research has looked across all number combinations.

This paper reports on Years 4 to 6 children's responses to six multiplication and division word-problem tasks which formed part of an Australian Research Council funded five-year longitudinal study of Years 2 to 6 Queensland children's mental strategies for the four operations. Students were tracked over the three years from simple 1 by 1-digit to more difficult 2 by 2-digit multiplication and 1 by 1-digit to 3 by 2-digit division word problems. Further, the study traced strategy changes from pre-instruction in multiplication and division terminology and notation (for some children), through a period of number-fact instruction, and finally until children were taught the written standard algorithms for 2-digit by 2-digit multiplication and 2 and 3-digit by 1-digit division. It also differed from previous research in that the emphasis was not on the semantic structure of the problems; rather, the emphasis was on identifying strategy choice for simple semantic structure and increasingly difficult number combinations.

Method

Subjects. The subjects were 95 children from 14 schools (Independent, Catholic and State). The schools were representative of differing socioeconomic backgrounds. The children had been chosen, when in Year 2, by their teachers to comprise one third of each of above average, average, and below average ability. During the study on which this paper reports, the children progressed through Years 4, 5 and 6.

The Queensland mathematics syllabus advocates that children be introduced to the concepts of multiplication and division in Year 2, the multiplication symbol (up to 9x9=81) in Year 3, the standard written multiplication algorithm (2 by 1-digit) and the division symbol (up to 81÷9=9) in Year 4, and the standard written multiplication algorithm (2 by 2-digits) and the standard partition written division algorithm (2 by 1-digit) in Year 5. Although schools generally followed the Queensland syllabus, there were classes that had not been formally introduced to the concepts of multiplication and division by Year 4.

Instrument. The instrument used was Piaget's clinical interview technique. The tasks, reported in this paper, comprised three equal grouping multiplication word problems (5x8, 5x19, 25x19) and two partition (24÷4, 100÷5) and one quotition division (168÷21) word problems. The six tasks represented a cross section of the possible multiplication and division problems and were the most frequently attempted problems in the larger study. They involved contexts common to children (money, lollies, and children in classes). They were given in picture form (the child listened as the interviewer said the problem); no algorithmic exercises were presented. The numbers were chosen

and the pictures used in the hope that it would maximise the use of children's own invented strategies, and minimise the use of the traditional written algorithm.

Procedure. The students were interviewed in the second and fourth terms of Years 4, 5 and 6. They were withdrawn from the classroom and interviewed individually in a separate room. The interviews lasted for a maximum of 20 minutes and were videotaped. The word problems were presented visually in the form of pictures, and orally as the interviewer verbalised the task. Although all tasks were presented to all children, not all children were able to attempt every task. If the children attempted a task, further questions were asked to probe for the strategy they used.

Results

Strategy categories. The videotapes were viewed, children's responses were analysed for commonalities in relation to the procedures identified in the literature; and a list of initial strategies developed. Then, the responses of each child for each task in each interview were classified in terms of these strategies and recorded for each interview. Finally, the calculation strategies were considered carefully, and after discussion among the researchers, five strategy categories were identified for each of multiplication and division (see Table 1). All responses were then coded using these strategy categories, and the results were analysed for trends across the three years.

Table 1. *Mental multiplication and division strategy categories*

Category	Description	Examples
Multiplication		
Counting (CO)	Any form of counting strategy, skip counting forwards and backwards, repeated addition and subtraction, and halving and doubling strategies.	5x8: 5, 10, 15, ... 5x8: double 5, double 16, +8.
Basic fact (BF)	Using a known multiplication or division fact or a derived fact.	5x8: 10x8=80, so 5x8=40.
RL separated (RLS)	Numbers are separated into place values, then proceed right to left.	5x19: 5x9=45=40+5, 5x10=50, 50+40=90, 95.
LR separated (LRS)	Numbers are separated into place values, then proceed left to right.	5x19: 5x10=50, 5x9=45, 50+45=95.
Wholistic (WH)	Numbers are treated as wholes.	5x19: 5x20-5=100-5=95. 25x19: 4x25=100, 4x4=16, 4x100=400, add 3x25(75), so 475.
Division		
Counting (CO)	Any form of counting strategy, skip counting forwards and backwards, repeated addition and subtraction, and halving and doubling strategies.	24÷4: 4, 8, 12,... 24÷4: half of 24, half of 12.
Basic fact (BF)	Using a known division fact or a derived fact.	24÷4: 4x?=24. 6 24÷4: 5x4=20, so 6x4=24.
LR separated (LRS)	Numbers are separated into place values, then proceed left to right.	100÷5: 10÷5=2, 0÷5=0, 20.
RL separated (RLS)	Numbers are separated into place values, then proceed right to left.	100÷5:, 0÷5=0, 10÷5=2, 20.
Wholistic (WH)	Numbers are treated as wholes.	100÷5: 100÷10=10, 10x2=20. 168÷21: 5x21=100 ∴ 5x21=105, about 60 left, 3x20=60 ∴ 3x21=63, 63+105=168, ans. 5+3=8.

General trends within each task

The results for multiplication are presented in Table 2 and, for division, in Table 3. As would be expected, the percent of children attempting and correctly attempting the tasks increased across the six

interviews. Further, the multiplication tasks were easier for the children as evidenced by the higher percentage attempted, and attempted correctly, than the division tasks.

Table 2. *Multiplication responses for Interviews 1 to 6 (n=95)*

Question	Interview	% attempting (% correct)	% attempting (% correct)				
			CO	BF	RLS	LRS	WH
5x8	1 (Year 4)	92 (71)	54 (34)	38 (37)			
	2 (Year 4)	97 (87)	23 (17)	74 (71)			
	3 (Year 5)	99 (87)	22 (13)	77(75)			
	4 (Year 5)	99 (93)	17 (15)	82 (78)			
	5 (Year 6)	100 (96)	6 (5)	94 (91)			
	6 (Year 6)	100 (97)	3 (1)	97 (96)			
5x19	1	26 (18)	1 (0)		11 (7)	4 (2)	10 (8)
	2	68 (51)	6 (2)		37 (30)	12 (7)	14 (12)
	3	72 (51)	4 (1)		33 (26)	17 (7)	18 (16)
	4	86 (73)	16 (7)		44 (42)	6 (5)	20 (18)
	5	96 (85)	7 (4)		58 (51)	7 (7)	23 (23)
	6	98 (85)	4 (0)		56 (51)	11 (8)	27 (26)
19x25	1	2 (2)	1 (1)				1 (1)
	2	16 (5)	2 (0)		7 (0)		6 (5)
	3	28 (12)	7 (1)		8 (1)	2 (1)	11 (8)
	4	78 (32)	9 (3)		37 (8)	5 (1)	26 (19)
	5	79 (35)	9 (2)		28 (7)	4 (1)	37 (24)
	6	85 (45)	8 (2)		34 (13)	4 (1)	38 (29)

For task 5x8, *Counting* was the initial dominant strategy (included skip counting in fives and near doubles, e.g., double 8, double 16, add 8). However, by Interview 2, the *Basic fact* strategy was dominant and reasonably accurate.

A low of 26% attempted task 5x19 in Year 4, while 98% attempted it by the end of Year 6. From the end of Year 4 to the end of Year 6, the *RL separation* strategy was dominant, with the *Wholistic* strategy being used half as much (surprisingly due to the ease by which it applies to 5x19 (5x20-5). The *LR separation* strategy was used by a significant minority and some children persisted in using the *Counting* strategy into the last interview.

Task 19x25 was attempted by only two children in Interview 1. One child counted in 25s, the other used a *wholistic* strategy ("10x25=250, another 250, take 25"). Both solutions resulted in correct answers. From there, the number of children attempting a solution increased across the interviews, until 85% attempted the problem in the last interview. However, only about half the solutions were correct. Most errors resulted from the application of the *RL separation* strategy (which is not surprising considering the memory load needed to remember all the interim calculations). Strategies that were more successful in giving correct answers included *Counting* (counting in 25's and grouping in 100's), *Wholistic* (20x25=500, 500-25=475), and even *LR separation* (10x25=250, 9x25=225, using 8x25=200 as known, 250+225=475).

For task 24÷4, the dominant strategy for all interviews was *Basic fact*. Most children reported knowing, "twenty-four divided by four is six, because four sixes are twenty-four." The other strategy used was *Counting* (halving, doubling, repeated addition, skip counting and sharing). A very small minority of children solved the problem by sharing one at a time (reflecting the semantic structure of the problem), while another minority used halving accurately.

Table 3 *Division responses for Interviews 1 to 6 (n=95)*

Question	Interview	% attempting (% correct)	% attempting (% correct)				
			CO	BF	LRS	RLS	WH
24÷4	1	68 (57)	12 (5)	57 (52)			
	2	84 (77)	10 (7)	75 (70)			
	3	88 (82)	8 (6)	80 (76)			
	4	97 (91)	13 (11)	84 (80)			
	5	96 (95)	3 (2)	93 (93)			
	6	97 (96)	4 (4)	93 (92)			
100÷5	1	56 (48)	11 (18)	18 (18)	2 (2)		25 (23)
	2	72 (60)	8 (2)	26 (25)	6 (5)		31 (27)
	3	79 (73)	6 (5)	34 (32)	2 (2)		37 (34)
	4	90 (76)	12 (6)	40 (38)	2 (2)		36 (30)
	5	85 (81)	4 (4)	44 (42)	2 (2)		35 (33)
	6	94 (90)	1 (1)	56 (54)	5 (5)		33 (31)
168÷21	1	1 (1)	1 (1)				
	2	6 (3)	2 (0)		2 (1)	1 (1)	1 (1)
	3	11 (8)	5 (5)			1 (0)	4 (3)
	4	38 (26)	23 (13)		5 (5)	1 (1)	7 (6)
	5	47 (36)	21 (13)		4 (2)	5 (5)	17 (16)
	6	61 (51)	27 (21)	1 (1)	2 (1)	2 (2)	28 (25)

For task <u>100÷5</u>, accuracy levels were generally high, with *Wholistic* (ignoring the final zero, or using 4x25=100 or 10x10=100) and *Basic fact* strategies (knowing that 5x20=100 or 100÷5=20) popular throughout.

Like 19x25, <u>task 168÷21</u> was attempted by a lower number of children, with less accuracy, throughout the interviews (one child in Interview 1 through to 61% by Interview 6) and elicited a greater variety of strategies than the other four tasks. The *Counting* strategies included skip counting, repeated addition and doubling, and persisted across all interviews (only one child used repeated subtraction). The *Wholistic* strategies included trial and error for multiplication (e.g., "21 times something is 168. 7? No. I'll try 8. Yes.", "1 times something is 8, 8 times 2 is 16, so it's 8."), trial and error for division (e.g., "Something goes into 16, 2 times, and into 8 once. That's 8."), and partitioning (e.g., "about 100 and the rest, because I know 5 x 20 = 100"). By Interview 4 (end of Year 5), some children attempted to solve the problem using *LR separation.* Interestingly, a handful of these children said that they wouldn't be able to attempt 168÷21, because they had not been taught how to divide with 2-digit divisors; yet prior to this interview, no such excuse was made for the inability to solve the problem.

Discussion

Strategy use and preferences. Strategy use across the six interviews was influenced by number combinations and students' available strategies. Tasks that were basic facts (5x8 and 24÷4) tended to be solved initially by the *Count* strategy and, then, later by the *Basic fact* strategy. Tasks that involved more complex numbers were initially solved by a greater variety of strategies. Across the interviews, the strategy category preferences of the children moved increasingly to the more efficient strategies, specifically the *Separation* and *Wholistic* categories, except when the task was related to a basic fact (e.g., 100÷5). For multiplication with 2-digit numbers, the tasks were solved increasingly by *RL*

separation after Interview 2. For the division task with a 2-digit divisor (168÷21), *LR separation* began to be used, without success, in Interview 4.

There was little or no use of repeated subtraction or sharing one to one (contrary to recommendations for teaching division in Queensland). A sharing strategy was used by weaker students, generally unsuccessfully, seemingly because of the heavy load on working memory. The trial and error strategy (e.g., 4x?=24. 5?, check by skip count or doubles or basic fact; no, try 6.) was found to be more reliable and efficient (similar to Mulligan, 1992).

Instructional effects. During Years 4 and 5, the traditional written multiplication and division algorithms are introduced to children. Their procedures are similar to the *RL separation multiplication* and *LR separation division* strategy and, hence, should reinforce and reduce working memory load for these strategies. *Wholistic* strategies appear to be less complex mentally than separation strategies (requiring less working memory) because they do not require numbers in the different place-value positions to be remembered and operated on separately, as is required by separation strategies. The four tasks where numbers were 2-digits or more had numbers chosen so that the *Wholistic* strategies were applicable. For example, 5x19 is close to 5x20, as is 100÷5, while 19x25 is close to 20x25 (and involves 25 which is one-quarter of 100), and 168÷21 is close to 160÷20. Therefore, it seemed reasonable to predict that *Wholistic* should have been the most efficient mental strategy for these four tasks, that separation strategies should have been little used, and that the use of the *RL separation* strategy involved some component of instructional effect.

There is some evidence that there may be an instructional effect, at least for multiplication (similar to the findings of Cooper, Heirdsfield, & Irons, 1996, for addition and subtraction). There was a trend to the *RL separation* category in tasks 5x19 and 19x25, yet the use of the *Wholistic* strategy was a little more accurate (particularly for 19x25). For division, there was not the same strength of support for an instructional effect in the strategy trends. However, there was some extra support for an instructional effect in division in the comments of the children. In Interview 4 and with task 168÷21, some children would not attempt the task because they "had not been taught to do long division with two digit divisors". Previous to this, the children had been willing to "have a go" at many tasks they had not covered in their mathematics classes. It seemed that the teaching of the division algorithm had "coloured" their approach to arithmetic.

Conclusions

The findings of this study show children's changing accuracy and strategy preference for mental multiplication and division across three years during which they were introduced to written algorithms for these operations. The children improved in percentage attempting the tasks and accuracy. However, there was not the expected change to more sophisticated strategies. Children stayed with *Counting* and, where they could, *Basic fact* strategies, and there was some evidence of movement to strategies based on the written algorithms. There was growth in the use of the *Wholistic* strategies where it was appropriate, but not to the extent that might be predicted from the deliberate favouring of these strategies in the choice of numbers in the tasks. There was little use of strategies based on non-standard algorithmic procedures, which was different from addition and subtraction mental computation (Cooper et al., 1996)

In the world of computers and calculators, estimation appears to be a more useful human ability that correct written calculation. Estimation seems better served by trial and error strategies (one of the *Wholistic* strategies), particularly when it is used mentally (as it so often has to be in real world situations). This study shows that many children, by the end of Year 6, were able to use quite advanced *Wholistic* strategies for larger number combination multiplication and division. However, another (although less efficient) strategy for these larger number combinations was *Counting*. Considering the numbers involved, *Counting* was reasonably efficient, certainly more efficient for 168÷21, than *LR separation*.

There appears to be a need, in multiplication and division mental computation as well as estimation, for assistance to be given to children to use strategies different from those associated with traditional computation (e.g., trial and error and *Wholistic*, and, maybe, some forms of non-standard *separation*). This would seem to imply a reduction of emphasis on written algorithms for multiplication and division (even their removal from the syllabus), a growth in instruction time spent on arithmetical properties and alternative computational strategies, and a change to more child-centred and flexible approaches to teaching operations.

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A GOOD PUPIL'S BELIEFS ABOUT MATHEMATICS LEARNING ASSESSED BY REPERTORY GRID METHODOLOGY

Kirsti Hoskonen
University of Joensuu, Finland
Kirsti.Hoskonen@Helsinki.fi

Abstract: *What kind of beliefs pupils have on being good or poor at mathematics? What kind of characteristics does a good mathematics pupil have? How does the pupil act? What are the most important things in mathematics learning? Teachers are interested in responses to this kind of questions. In this case study, the ideas of one pupil about what makes a good mathematics pupil are assessed. One girl at the end of the lower secondary school was the informant. Her responses were used as elements in the repertory grid interview. Constructs were then elicited. She developed her own model how to act in studying mathematics. She thought that the most important things in learning mathematics were her own attitude towards studying and independent work she was ready to do.*

This research is concerned with a pupil's beliefs about what is needed to be a good mathematics pupil. Pupils have their own ways of coping with mathematics, and their own ideas about how to succeed in learning mathematics. The objective of this research is to assess one pupil's beliefs about her own mathematics learning. The emphasis if the study is to find out a student's beliefs based on his own ideas and expressed with her own words. The methodology is based on personal construct theory and utilised the repertory grid method (Kelly 1955).

Theoretical background

The methodology used to assess a pupil's ideas about mathematics learning is based on Kelly's personal construct theory. In his theory, a person is like a scientist who observes the world through transparent patterns built by himself. He makes assumptions, tests them and creates his own theory of the world around. With these patterns he wants to make sense of the word around. Kelly defines the central concept *construct* of his theory, as follows:

> "*Let us give the name constructs to these patterns that are tentatively tried on for size. They are ways of construing the world. They are what enables man, and lower animals too, to chart a course of behaviour, explicitly formulated or implicitly acted out, verbally expressed or utterly inarticulate, consistent with*

other courses of behaviour or inconsistent with them, intellectually reasoned or vegetatively sensed." (Kelly 1955, p. 9)

An other concept in Kelly's theory is an *element* which is defined in the following way: "*The things or events which abstracted by a construct are called elements*" (ibid, p. 137). When a person is asked to categorise elements he is thought to organise patterns. The categories are then called constructs. The constructs are bipolar so that every construct has its opposite, e.g. honest vs dishonest. Persons make sense out of their world by simultaneously noting similarities and differences. In addition to the theory, Kelly developed a method for assessing an individual's personal constructs. Elements and constructs which are core concepts in Kelly's theory are also important in the method, the repertory grid technique.

Kelly's constructs can be understood as beliefs (e.g. Jones 1980, Williams & al. 1977). Schoenfeld (1992, p. 358) describes belief as "an individual's understanding and feelings that shape the ways that the individual conceptualises and engages in mathematical behavior". One of the main components of beliefs is 'beliefs about mathematics learning'. It consists of 'beliefs about the nature of learning mathematics', 'beliefs about how learning should be organised', 'beliefs about what the role of the learner is', 'beliefs about what the degree of autonomy expected from pupils is', and 'beliefs about who sets the criteria for correctness' (e.g. Pehkonen 1995, p. 20). E.g. Martha Frank has interviewed four pupils of the seventh and eight grade. She reports as one of pupils' beliefs "the role of the mathematics pupil is to receive mathematical knowledge and to demonstrate that it has been received" (Frank 1988, p. 33), i.e. a pupil should sit and listen to his teacher's talk, and do his routine homework. Garofalo (1989, p. 503) describes three common beliefs in lower secondary schools. One of them is "only the mathematics to be tested is important and worth knowing". Another widely held belief is "doing mathematics is simply a matter of memorising and reproducing the fact, rule, procedures, and formulas" (Mtetwa & Garofalo 1989, p. 611). These both deal with the nature of learning mathematics. Understanding mathematics seems not to be important.

The repertory grid technique

The repertory grid technique is originally used in psychology. The informant is asked to produce a list of persons, who are important to him (e.g. mother, father, sister, etc.). The persons are then the elements in the method. After that he is given a triad of persons and asked to think in what way two of them were similar to each other and also different from the third. All the persons are included in one or more triads. The categories obtained from this task are the constructs. According to Kelly's theory the constructs are bipolar. A person makes sense of the world through categorising elements. Nowadays the method is used in many different domains, and the elements are objects or aspects of that domain.

The elements in columns and the constructs in rows form a matrix, a grid, which is important in analysing a person's ideas. In the method the person is asked to judge the matrix. Fransella & Bannister (1977) have described this process of judgement, as follows: "Behind each single act of judgement that person makes (consciously or unconsciously) lies his implicit theory about the realm of events within which he is making judgements. Repertory grid technique is, in its multitude of forms, a way of exploring the structure and content of such implicit theories." Kelly (1955) used factor analysis to analyse the grid.

The repertory grid methodology is a tool which has been used in mathematics education to assess teachers' beliefs about mathematics (Jones 1990, Lehrer & Franke 1992, Middleton 1995, Williams, Pack & Khisty 1997). With this methodology, however, there are some research about pupils' beliefs on mathematics (Chapman 1974, Thomas & Harri-Augstein 1985), but there seems to be none about pupils' beliefs on mathematics learning. Thomas & Harri-Augstein (1985) investigated college pupils' responses to certain 'command' words, as prove, define, etc., used in mathematics. Pupils responses were emotionally loaded constructs: engaging vs disinterest, reassuring vs despair, etc. In an other research (Chapman 1974) collage pupils of mathematics were asked to write down the one word which best typified their reaction to mathematics. About one hundred first year college pupils used constructs of which 70% were totally emotional. These pupils construe their mathematics by emotional constructs.

Data collection process

This is a part of a pilot study the aim of which was to clarify a pupil's beliefs on mathematics and its studying and learning. In this paper the focus will be a pupil's beliefs on mathematics learning. The informant in the study is Brita, a talented ninth-grader, who has done well in school and especially in mathematics. The author as her teacher considered her as suitable research object, since she seemed to have a clear and well-structured view of learned mathematics. Additionally, she was willing to co-operate and share her beliefs. In mathematics Brita was interested in open-ended problems, in problems which require a lot of thinking, and in project work. Right from the beginning she did her homework, nevertheless how demanding they were. She asked for help, if something was unclear for her, and in lessons she was ready to present her solution process and describe it to the whole class.

The method used for data collection was the repertory grid method. Its purpose was to find out Brita's beliefs based on her own ideas and expressed with her own words. The data collection began with an interview, where Brita was first asked to think about a pupil who is good at mathematics. Then the following questions were put forward: What do you think that it means to be good at mathematics? What is a good pupil like? How can you describe such a pupil? Her answers formed a list of a good pupil's characteristics, and these were called elements. She was also asked to

describe a pupil who was poor at mathematics. Such characteristics were elements, too. The elements were written on cards. When Brita had the cards on the table in front of her she was asked to check that they were the same ones she described. She was asked to chose the cards she thought could suit herself. She selected all the cards that she had told to be a good pupil's cards except one, 'active'. These elements were the attributes with which she seemed to make sense of the world around. The 20 elicited elements for a good mathematics pupil are given Table 1 (in certain groups described below).

The first interview continued and Brita was asked to sort the cards in any way which made sense for her. After she had finished the grouping, she had piles in front of her and she was asked to describe the similarities of the elements in each pile. To sort the cards was not an easy task for her: She began with 9 piles, but changed her mind in the case of some cards, and resulted with 7 piles. In Table 1 you can see Brita's 7 categories of similar elements. The names of the categories originated from Brita. Table 1 can be considered as Brita's first mental model for her thinking on studying mathematics.

Pupil's own attitude towards maths: enterprising, does eager the homework, is not afraid of the word maths, appreciates maths as a school subject, does not lose her enthusiasm to maths studies if she fails in tests, understands what maths is, finds that maths is interesting and challenging
Goal is to keep maths with in the future: receives a new matter easily, is interested in learning new things, intends to continue and deepen her maths studies after the comprehensive school
Studying in a group: studying in a group, is able to work in a group together with others, considers all pupils equal
Maths belongs to the leisure-time: is ready to think about maths in her leisure-time, tries to solve everyday problems by means of maths
Repetition and strengthening the former studied things: brush up old things; if she hasn't learnt a new item during the lesson, she takes care of learning it at home; asks the teacher for help if she has problems with learning new things
Adaptation to maths studies: is ready to learn maths in many different ways not only in the old customary way
Appreciation the studying: is not only "hunting" good marks, but studying maths is the most important thing

Table 1. Brita's first mental model consists of seven categories.

Further in the interview, Brita was asked to describe differences between the piles comparing them two by two. The similarities and differences were both called constructs. Since in the repertory grid technique constructs are considered bipolar, Brita was asked to determine the opposite of each construct. At the same time, she

should explain which of them, construct or its opposite, suited better for her and give reasons for her choice. In the case of all constructs, she was not able to give its opposite.

The construct / opposite pairs with Brita's words were:

- *"a pupil's own attitude towards maths / indifferent, maths hasn't much to do with oneself*
- *the goal is to keep maths with in the future / tries to get out of maths*
- *studying in a group*
- *maths belongs to the leisure-time / only at school*
- *repetition and strengthening the former studied things / without repetition go straight to a new thing*
- *adaptation to the maths studies / no adaptation to the studies*
- *appreciation the studying*
- *a pupil's qualities which influence the studying / one does not pay attention to a pupil's qualities*
- *discussion and debate at school /pondering at home or elsewhere*
- *tutored studying at school / independent studying*
- *studying on one's own initiative / strictly guided studies*
- *clearing up things to oneself / doesn't clear up things*
- *studying for future / only for present time*
- *different ways to learn / only one certain way to study*
- *behaving in a classroom*
- *relations between pupils*
- *an important part in studying maths*
- *how a pupil treats mathematical knowledge*
- *studying in the leisure time / studying only at school*
- *making things clear together in a group /making things clear independently"*

The elements and the constructs are set into a grid where elements are in rows and constructs are in columns. For data analysis, a form with the grid was given to Brita, and she was asked to rate the grid on the scale 1 - 2 - 3 - 4 - 5 (1 = the construct was not important to the element, 5 = the construct was very important to the element). Thus, all the elements are rated on all the constructs.

The main result

Since there was a big variety of information, the 20 x 20 matrix with loading from 1 to 5, factor analysis was selected to condense it. The completed grid was analysed factorially to determine relationships among constructs and to condense the information in a pupil's pattern. This is in coherence with the leading principle of factor analysis, as one can read e.g. in Cohen & Manion (1994, p. 330): "Factor analysis is a way of determining the nature of underlying patterns among a large number of variables". In realising the factor analysis, two constructs 'Studying for future' and 'Important part in studying maths' were not allowed to be with in the analysis, since all their ratings were the same (5). Because of no variance, they should be deleted in the factor analysis. When using the criteria "roots greater than one", the factor analysis resulted five factors.

Since aims of the study was to find out a student's beliefs from her point of view, Brita was asked to check the validity of the results. In the second interview, she was inquired, whether the five-factor solution was meaningful for her. The task was a difficult one for her, since it did not make any sense to her. Therefore she tried to improve the solution by making a regrouping, and resulted four categories of the constructs. Brita was asked to name the four categories. Furthermore, the ranking list of the categories was expected from her.

Attitude towards studying (1)
Pupil's own attitude towards maths, Appreciation the studying, Goal is to keep maths with in the future, Adaptation to the maths studies, Pupil's qualities which influence the studying
Independent studying (2)
Studying on one's own initiative, Clearing up things to oneself, Repetition and strengthening the former studied things, Different ways to learn
Skill to handle mathematical knowledge (3)
How a pupil treats mathematical knowledge, Studying in the leisure time, Maths belongs to the leisure-time
Studying with support (4)
Studying in a group, Making things clear together in a group, Relations between pupils, Discussion and debate at school, Tutored studying at school, Behaving in a classroom
Studying for future, Important part in studying maths

Table 2. Brita's second mental model for mathematics learning.

The author realised the factor analysis with four factor, and the solution was shown to Brita. Again, she was not satisfied, and wanted to make some changes. At the end

she named the categories and arranged them in the order of their importance. The results of Brita's evaluation of the four-factor solution is given in Table 2. This may be considered as a mental model for Brita's beliefs on mathematics.

Conclusions

In the repertory grid interviews, Brita has told something of her ideas how to learn mathematics, her beliefs about learning. She had to make her unconscious ideas conscious. Through these ideas she told her main goal: "to keep maths with in the future". To reach this goal she had to set some other goals and she told about the ways how to cope with mathematics e.g. that she was interested in learning new things. She thought that her own attitude towards the mathematics studying and her relation to mathematics were the most important constructs. She told "my own relation to mathematics is good". She appreciated the studying of mathematics and did eager the homework. Brita was not like the pupils in the study of Frank (1985). Independent studying was very important to her. She used to clear up the topics to herself, to brush up and strengthen the former studied topics. "I definitely stand for clearing up the things to myself. At the same time I repeat and learn the difficult things. If I don't clear the things up to myself I might later tumble over them". It was not enough for him only to sit, listen and do her homework. She wanted to understand what she was doing. She was eager to do mathematics in a different way as usual. "I am absolutely of the opinion that the teachers have to change their methods. Then studying is more meaningful."

Mathematics was not just computation for Brita. "The use of mathematical knowledge could be its application. If you have learned the thing, it is worth using and revising now and then." Mathematics was everywhere in her world. "You should take interest in maths in your leisure time, too. Mathematical applications could be good in every day problems." She wrote: "I like learning in the leisure time. Then a pupil has her own peace and a lot of time to study. The repetition for the test and doing the homework belong to one's studying in her leisure time. Studying in one's leisure time is an important part of all learning."

Studying in a group gave support to the members of the group. They could clear up the topics together, discuss and debate in their group. "I like to study both independently and in a group. You must get used to both of these. Different situations need different ways to study." The group of students discussed problems in order to understand them. "I stand for clearing up the things together. An independent work is more responsible and then you need to work more, since there is none who could give you advice." She thought that group work was good in a certain kind of a group. "They greatly influence on the learning if the pupils are 'on the same wave length' and have the same attitude to the studying. Therefore a good team spirit is also important."

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LINKING INFORMAL ARGUMENTATION WITH FORMAL PROOF THROUGH COMPUTER-INTEGRATED TEACHING EXPERIMENTS

Celia Hoyles and Lulu Healy[1]
Institute of Education, University of London, UK

Abstract: *In this paper, we present the results of two computer-integrated teaching experiments (one in algebra, one geometry) designed to help students connect formal proof with informal argumentation. The results are interpreted from the basis of findings from a large-scale survey of students' proof conceptions.*

Background: Research in mathematics education has consistently highlighted students' difficulties in engaging with formally-presented, analytical arguments and understanding how these differ from empirical evidence (Balacheff, 1988, Bell, 1976, Chazan, 1993 and more recently papers to PME 22, e.g., Arzarello et al, 1998, Furinghetti & Paola, 1998). The current National Curriculum for mathematics in England and Wales prescribes an approach to proving, partly as a response to these student difficulties, in which the introduction of formal proof is delayed until after students have progressed through stages of reasoning empirically and explaining their conjectures largely in the context of data-driven investigations (see Hoyles, 1997). This approach to introducing proof to school students has been the subject of considerable criticism. To provide systematic evidence in this debate, we started a research project in 1996 to describe how high-attaining students who have followed this curriculum conceptualise proving and proof in mathematics and to explore ways to address any difficulties through new teaching approaches.

The research comprised two phases: a paper and pencil survey of the conceptions of proving and proof held by 2,459 students aged 14/15 years at a high level of mathematics attainment, (about the top 20%), followed by two computer-integrated teaching experiments in geometry and in algebra. This paper is concerned with phase 2 and aims to present the principles underlying the design of the teaching experiments and the major findings from their evaluation. It will incorporate findings from phase 1 (see Healy & Hoyles, 1998) only in so far as they informed the design and analysis of the teaching experiments.

Methods: Our phase 1 analysis showed that students, even in this high attainment band, had a limited view of proof and this had a significant and negative influence on their competence in proving. Yet the majority of students recognised that a valid proof should be general and valued arguments they felt convinced and explained. Most students felt that formally-expressed arguments would receive the best marks, but few used deductive reasoning, with formal arguments occurring very rarely.

Our teaching experiments in algebra and geometry aimed to build on the evident strengths of our students in narrative explanations and help them develop a multi-faceted view of proof, which included verification, systematisation and deduction (see de Villiers, 1990), along with more formal presentation. Given that to construct on a computer[2] requires explicit attention to the processes used, we hypothesised that students would be better able to formalise explanations derived from computer-based activities.

1 Research funded by ESRC, Project Number R000236178
2 In algebra, we built a microworld in Microworlds Logo; in geometry a dynamic geometry system, Cabri.

Teaching experiments were piloted with 6 students in three schools (one mixed, one boys, one girls) and modifications made on the basis of feedback from students and teachers. These included more precise procedures for the collection of systematic written data from the students and the imposition of a common structure on both experiments; i.e. students were to construct mathematical objects on the computer, identify and describe the properties and relations that underpinned their constructions, use the computer resources to generate and test conjectures about further properties, and make informal explanations as to why they must hold. This was to be followed by a teacher-led introduction to writing formal proofs using paper and pencil examples, where students would be helped to organise the arguments generated during the computer activity into logical deductive chains in the appropriate formal language. Finally, the students were given a challenging computer construction to be explained and proved, followed by an opportunity to use the properties proved to explain why a subsequent construction was impossible (named here for reference the possible/impossible construction).

Each teaching experiment comprised 3 lessons and 3 homeworks and was conducted by the two authors. A total of 15 students, 5 from each school, undertook both experiments within a 5-month period[3]. The students were chosen by their teachers according to our criterion of high-attainment, but also so that the group experience would be beneficial both individually and collectively. All students completed the phase 1 survey before the teaching experiments and were interviewed immediately after them, when they were given the opportunity to review some of their survey responses as well as to reflect on the teaching experiments.

The mathematics teachers of the student group in each school completed the school survey[4] as well as the multiple-choice proof survey questions. They were also interviewed to provide further contextualising data on the school, class and mathematics curriculum followed, and the individual students involved in the experiments.

Data analysis: To cope with the complexity of all the data collected (i.e. students' and teachers' survey responses, worksheets completed by students during class and homework sessions, students' computer work, two sets of observation notes, transcripts of final interviews, school questionnaires and transcripts of teacher interviews), we adopted a cyclical and iterative process of analysis, during which we constructed student case histories, first by considering student and school variables, and then weaving in consideration of group, task and software. We also documented how students responded to the computer and how they viewed it as a tool to learn mathematics. Finally, we sought to take account in assessing progress of how the students interacted and worked together during the teaching sessions.

Results: Each individual student case history provides a rich research study in itself, but here we focus on general trends and school differences rather than individual responses and progress.

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3 We planned to have 3 pairs of students per school, but in each case one student did not attend every session.

4 The school survey was used to find out about a school, its curriculum and the mathematics teacher of the class selected to complete the proof survey.

Deductions from properties: As described earlier, in both geometry and algebra, the students were asked to undertake a possible/impossible construction, as summarised in Table 1 below.

Construction	Necessary Property (explained & proved)	New Construction
Geometry: Construct with Cabri a quadrilateral in which the angle bisectors of two adjacent angles cross at right angles. Write down its properties and prove them.	One pair of opposite sides *must be* parallel (i.e. trapezium).	Predict if you can construct a triangle in which two adjacent angle bisectors cross at right angles. Predict yes or no, try to do it, and then explain why your prediction was right or wrong.
Algebra: Construct 4 consecutive numbers in Expressor, write down any properties of their sum and prove them.	Sum of 4 consecutive numbers is even but *not divisible by 4*	Predict whether you can find 4 consecutive numbers that add up to 44. If yes, write them down, if no, explain why it cannot be done.

Table 1: Using properties to make predictions: possible/impossible constructions

All the students managed to complete successfully the first construction — albeit after time and help. In geometry, 11 students predicted they could construct the triangle. After trying to do this, all realised it was impossible and 13 could explain why, using the parallel property identified in the quadrilateral construction. In algebra, 6 students predicted it was possible to find 4 consecutive numbers adding up to 44. After trying to find these numbers, all then realised it was impossible, but only 6 referred to the necessary property they had proved about non-divisibility by 4, and 4 of these still did not regard this as adequate refutation and went through a calculation process, i.e. showed $4n + 6$ was not divisible by 4.

Overall, we found the students' responses to be surprisingly consistent, i.e. despite being utterly convinced of the necessity of the property in the initial construction, few students used it to deduce the impossibility of the new construction. Having noted this in respect to this particular task, we searched our observation data and found numerous instances of similar problems in making inferences from deduced properties, which will be shown in the presentation. Overall we conclude that the students were poor at making deductions from properties they had discovered, explained, and even proved.

Evaluation of visual arguments: In phase 1, we had analysed whether students assessed a visual argument in algebra and in geometry to be general or specific and found rather more thought them specific than general: 50%, 31% compared to 21%, 18%[5]. In analysing the responses of our 15 case study students, we found similar proportions. Clearly many students were unfamiliar with the power of visual representations and their potential to serve as generic examples, a conclusion supported by our process data: for example, the presence of a picture in a request for a geometry proof confused some students about whether it was general or not. Visual arguments were also frequently attributed lower status than other forms, and described as 'simple' or 'daft', even by those who acknowledged their explanatory power.

5 First percentage is algebra, second geometry.

Summarising the case studies: Apart from the general trends above, we found differences in progress across the schools, so report other findings categorised by school. First we present the school profiles, brief characterisations of each school, mathematics department, and teacher.

School A: School A was mixed-sex and comprehensive with mixed-ability teaching practised in the mathematics department. Students in Year 10 studied 3 hours of mathematics/week, and specialist interest in mathematics was encouraged by additional after-school activities, in extra sessions, and visits to outside school mathematics lectures. Proof was not taught as a topic until Year 11. The computer was rarely used in mathematics lessons. The teacher clearly valued different expressions of proofs, recognised students could construct good 'intuitive proofs' but had difficulties with formalising them. It was also apparent that the students were encouraged to take risks and challenge themselves mathematically.

School B: School B was a girls-only comprehensive where again mixed-ability teaching was practised in the mathematics department from entry, but there were few 'extra' mathematics sessions. Students in Year 10 were taught 2.75 hours of mathematics/week with proof addressed through investigations and coursework. The computer was not used regularly in mathematics lessons, although there were two machines in each classroom. The teacher thought of herself as an educator "not a mathematician" and fostered a collaborative and nurturing approach where students would not be faced with unnecessary challenge.

School C: School C was boys-only, with the boys set by attainment in mathematics from Year 8. 95% of the top set sat the higher GCSE paper. The students studied 3 hours of mathematics/week and proof was taught through investigations. The computer was not used in mathematics lessons, although there were well-resourced laboratories in the school. The teacher was a highly-qualified mathematician with no formal teaching qualifications, who described himself as computer-illiterate. He fostered a strong emphasis on exams in his department, and discouraged coursework as he felt the higher achievers spent too much time on this.

We then summarise the background, process and outcome profiles of the 15 students grouped by school. For the background profile (Table 2) we used five dimensions to capture the most salient aspects: KS3 test score[6]; views of proof and constructed proof score as assessed in the proof survey; and prior knowledge of algebra and geometry as assessed in the survey and from teacher interviews and classwork.

Though the process profile (Table 3) was harder to draw up, careful analysis of the case histories led us to draw out 3 dimensions, which though overlapping, could usefully be distinguished: reaction to the computer work as evidenced in observations and student interviews; nature of interactions with the computer and with peers; and modes of expression used in explanations and in proof (i.e. narrative, symbolic, visual).

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[6] Level 8 is highest grade attainable in Key Stage 3 (KS3) tests (about the top 1/2%). These tests are administered nationally at the end of Year 9 (ie age 13/14 year old students).

	School	Summary Description
KS3 Test	A B C	4 level 8; 1 level 7 5 level 7 3 level 8; 2 level 7
Views of proof	A B C	Only referred to proof in algebra Only described proof as verification Unclear about proof and what it was for
Constructed proof scores (range 0-3)	A B C	6 0's; 2 1's; 5 2's; 6 3's. Highest score for a formal proof 3 11 0's; 6 1's; 1 2; 2 3's. Highest score for a formal proof 0 3 0's; 9 1's; 2 2's; 6 3's. Highest score for a formal proof 1
Algebra	A B C	Mainly for expressing generality, and finding specific unknowns Very little algebra Some experience with algebra manipulation and finding specific unknowns, very little in expressing generality
Geometry	A B C	Very little geometry Very little geometry Knew a few geometry facts

Table 2 : Background profiles of school groups

	School	Summary Description
Reaction to computer work	A B C	Enjoyed the challenge of building and investigating computer constructions Found computer constructions difficult and took a long time to build them Focused on procedures "how to get it constructed" rather than structure of the constructions
Nature of interactions	A B C	Experimental and collaborative Insecure when unsure, helped each other Not very experimental, highly competitive
Modes of expression	A B C	Used many modes of expression flexibly Slow to adapt to new ways of working, dependent on others and teachers to validate methods and outcomes; used variety of forms of expression but made only fragile connections between them Liked to finish quickly using only one mode of expression; generally prioritised formal and rejected visual

Table 3: Process profiles of school groups

Finally, for the outcome profiles (Table 4), we distinguished five dimensions: competence in proof in algebra and geometry as evidenced in homeworks and final interviews, and reactions to the role of the computer and the teaching experiments as evidenced in students' written evaluations following each teaching session.

	School	Summary Description
Proof	A	Ended with multi-faceted views of proof, geometry as well as algebra
	B	Extended their views of proof to include explanation as well as verification
	C	Limited "internal" sense of proof; "external conviction" view still predominated
Algebra	A	Split between those who learnt algebra routines and those who connected algebraic to other forms of explanation
	B	Began to express algebraic relationships in context of microworld
	C	Learnt algorithmic approaches to constructing algebra expressions
Geometry	A	Learnt geometry facts, could separate explicitly givens from properties to be deduced, still some difficulties in constructing complete chains of argument
	B	Learnt geometry facts, began to distinguish givens from properties to be deduced, still experienced problems with local deductions
	C	Still found it difficult to organise geometry facts into logical steps within a formal proof
Evaluation of role of computer	A	Algebra: saw connection between programming and proof Geometry: helped to identify and see properties but not prove them
	B	Reaction same in algebra and geometry: i.e. helped to be accurate and locate properties; few links between constructing and proving
	C	Strong emotional reactions — hated or loved software; allowed work to be done more quickly and made it easy to construct things; few links between constructing and proving
Evaluation of teaching experiments	A	Writing formal proofs described as hard with specific problems identified — but enjoyable
	B	Writing formal proofs described as generally hard with no specific details
	C	Learning to use computer described as hard and proof mentioned infrequently

Table 4: Outcome profiles of school groups

School differences: The summary data shows that the teaching experiments were most successful in school A, had some success in school B and were least successful in school C. In school A, students made progress in algebra, geometry and most crucially developed a multi-faceted and connected sense of proof. This is illustrated in the case-history of one student,Tim from school A:

Tim enjoyed the computer work because 'it was different'. In his evaluations, he described the most enjoyable parts in algebra as 'programming', 'watching the programs work' and 'proving'. When asked he clearly saw a strong connection between proving and the computer work on algebra.

T I liked the programming stuff - that helped [to write proofs] because it sort of showed how it was constructed so... It helped prove because it showed you how they were made....how that construction was made step by step.

In geometry, this link between informal and formal proof was more tenuous. The computer work helped Tim 'see' relationships, but not to prove them formally, although it may have performed an important role in satisfying his need to be convinced.

T Well you could actually see if they were congruent - you could take however much you were allowed to take and actually make a triangle. If it was congruent then you could tell it was.

C Tell it how?
T Just by seeing....
C And did that help you write your formal proofs?
T Not really - not the formal stuff. ....Well it made it more enjoyable.

We conjecture that factors in the overall success in school A were that students were used to the experimental approach required by our activities, they had an adequate knowledge base to engage with the activities, and they were willing to share knowledge and help each other. In school B, the students started from a weaker knowledge base than anticipated in the activities, and although they were beginning to appropriate a broader conception for proof, they needed more time to consolidate this and gain confidence in experimenting with the software to learn algebra and geometry. This group of girls was much less secure — they were willing to share but lacked the competence and confidence to do so. Finally, in school C, the students viewed the activities more as learning to use the computer software than to explore the mathematics. They seemed more used to learning procedures and techniques than striving for conceptual understanding. As a 'typical' group of high-achieving boys, they were competitive and preferred to work on their own.

Individual variation: Clearly, individuals within a school group varied on all the dimensions of the three profiles, but in each school there was at least one individual (and rather more in School A) who adopted a flexible approach to proving which interweaved verification with seeking understanding and explanation. From our case histories, it was clear that interaction with the computer helped students to make and maintain these connections — they were able to reflect on the steps they had made in constructing their explanations and re-use these steps in deductive arguments. In these cases, internal conviction was achieved using a combination of empirical and anaytical methods, logical arguments were constructed and expressed in a variety of ways (including formal expression) while keeping narrative explanations in mind. We also found that in each group there was at least one student (and rather more in School C) who ended with a view of proof that prioritised 'external' over 'internal' conviction. These students moved quickly from informal argumentation to the production of a formal proof and in the process lost touch with the sense of the problem to be proved. We also found (particularly amongst students who had planned to drop mathematics after it was no longer compulsory), a tendency to be satisfied with their narrative arguments and remain unconvinced of the need for any formal proof. In both cases, we interpreted the responses to be a consequence of the mismatch between our goals, teaching style and experimental activities and the students' views of mathematics and of proof.

Our data also suggest that in part these individual variations can be accounted for by reference to: the adequacy of an individual's knowledge base as evidenced in the background profile — particularly crucial in a group where sharing/helping was not prevalent; a readiness to explore different ways of presenting mathematical ideas, an attitude to learning mathematics which included problem-solving and experimentation as appropriate learning strategies; and an approach to technology that did not preclude seeing computer interactions as relevant to appropriating mathematical ideas. Of course, fulfilling these conditions cannot guarantee success — the composition and dynamics of the group, and the nature of its interactions also serve as intervening influences — but not fulfilling them is likely to inhibit progress.

Conclusions: Our results show that the computer-integrated teaching experiments were largely successful in helping students widen their view of proof and in particular link informal argumentation to formal proof — a transition known to be problematic. Not all students however made the anticipated progress, pointing to the well-known complexity of the cognitive and metacognitive processes that need to be appropriated in learning to prove.

Our research also highlighted rather particular problems our students had with respect to local deductions and visual reasoning. Since these problems emerged in all 3 schools and in both algebra and geometry, we are led to conclude that these two processes are given rather little emphasis in the current curriculum for younger students — a conclusion supported by an analysis of the National Curriculum and by data from our teacher interviews. Our students are simply unused to making deductions and predictions — an unfamiliarity which our research shows inhibits their capacity to engage with the demanding complexities of proof at higher levels.

Finally our study identified considerable variability in responses between students and between students grouped by school — despite identical activities and teaching. We interpret this differential progress at the school level to be related to the match or mismatch between our expectations of the students' prior knowledge and their actual knowledge, and our goals and constructivist orientation and those of the mathematics department as mediated through the student group. There were also other variables that influenced success, such as attitude to computers and their role in learning mathematics among students and staff, and the gender mix and internal dynamics of the student group. Taken alongside evidence from previous research that more 'traditional' methods of teaching proof have variable success, our findings suggest that teaching can make a difference to students' competence in proving *but* the same activities and the same teaching approaches *inevitably* will not be equally effective in all schools or with all students.

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Alternative Assessment for Student Teachers in a Geometry and Teaching of Geometry Course

Bat-Sheva Ilany and Nurit Shmueli,
Beit Berl Teacher Training College, Israel

This work presents one of the assessment tools that we employed in the Geometry and Teaching of Geometry course, named "Three-Stage". This tool is composed of a group test, classroom discussion and individual test. With the help of the assessment tool we sought to achieve the following aims: examine the ability of the students to present and argue their solutions, and their ability to form links between various areas within the mathematical framework. The findings show that in general the tool reflects the students' knowledge. The tool examined understanding of mathematical concepts, methodological knowledge (such as ability to prepare activities for students), while expressing the students' skills and diversity and the processes of coping with working in groups.

With the appearance of the document Curriculum and Evaluation Standards for School Mathematics (NTCM, 1989, 1991), an effort was made to transform mathematical education and prepare it for the 21st century. This document proposed a new approach to the teaching of mathematics, ascribing special importance to independent and critical thinking, and recognizing that assessment and curricula standards could not be fully implemented without changing the manner in which mathematics is taught.

The document stressed development of ways of thinking and problem solving strategies, stressing the solution process (and not only the assessment of the finished product). These trends affected the nature of the contents in teaching mathematics. The changes in content and style of teaching mathematics necessitated corresponding changes in the assessment process as well (MSEB, 1993). As a result, a variety of methods and tools have been recently proposed for assessment of students in addition to, or instead of, the customary assessment by testing (Stenmark, 1991).

Clarke and Sullivan (Clarke, Sullivan, 1992) believe that performance tasks are the appropriate tool for guiding and assessing students in the area of unconventional problem solving. In performance tasks, each student is able to choose the strategy appropriate for the level at which he finds himself.

Since performance tasks make it possible to achieve different and original ways of solution, they make it possible to examine the student's mathematical ability, stressing the solution process rather than focusing on the final result. In the assessing the student's mathematical ability great importance is ascribed to **monitoring the solution process** and the thinking processes guiding the student. Tasks of this type are suitable to be used as an assessment tool in accordance with the new trends in mathematics instruction. Since we are striving to base teaching and learning in the classroom on the inquiry and discovery processes that take place in the course of solving problems, student assessment should also be based on tasks of this type.

The tool presented here was constructed for the purpose of assessing the "Geometry and the Teaching of Geometry" course. The course focused on acquaintance with geometric shapes, the relationship between them, their characteristics, as well as general didactic aspects. The course was given as part of a workshop while employing demonstration means relevant to the subjects being taught. Moreover, the

course incorporated various teaching means and didactic methods, and presented various studies and theories, such as the theory of Van Hiele. Teaching methods included work with heterogeneous groups emphasizing performance tasks requiring inquiry and discovery, mathematical discourse, both within the group and in the plenum.
We sought to adjust the assessment methods in the course to our teaching methods. The object was to examine the students' mathematical and methodological knowledge of geometry, to express the inquiry processes in the group and the individual knowledge and diversity among the students.

Methodology

The population and the framework: Thirty-five student teachers and elementary school and junior high school teachers attended the course. The course participants were a very heterogeneous group, composed of students who specialized in mathematics, those who did not specialize in mathematics, as well as teachers (both Arab and Jewish) who attended the course to complete degree requirements or as part of in-service teacher training.
Assessment tools: We employed for assessment the "Three-Stage" tool, developed by a Weizmann Institute team.
The tool is composed of three activity stages:
Stage 1 – The Group Test. Students worked on an activity in groups and write a joint report on the work performed by the group. Each group writes a single report and submits it to the teacher, without any intervention on the part of the teacher. The group report reflects the process that the group went through, including erroneous assumptions that were made. The students become accustomed to being aware of the process they are going through, to express themselves within the group, to explain to others and to conduct a mathematical discourse. Moreover, the group report can serve as a protective grade for weaker students.
Stage 2 – Classroom discussion, whose function is to summarize and consolidate the work of the various groups. The discussion is based on mathematical principles related to an activity prepared in advance by the teacher, as well as on the teacher's impression of the group activity and difficulties that arise in the course of the work of the various groups. The teacher's role in the discussion is to arrive together with the students at a generalization, a global outlook and greater depth of understanding.
Stage 3 – Individual test, the stage at which the students complete individual questionnaires and submit an individual report. The questionnaire contains questions related to the joint work carried out by the group at Stage 1. Each student is examined at this stage as to the extent to which he is capable of answering questions based on the group work. The personal report has two central aims:

1. To examine the students' personal knowledge of the subject matter worked on in the group, but without help from the group. Information is thus obtained on what the students learned in the group work and in the subsequent discussion conducted by the teacher. Students who can achieve deeper and broader understanding by themselves, can express such ability in their individual work.
2. Creating motivation for self-involvement of each group member in the group work, with each student knowing that he will be personally questioned on the work.

It is important to complete Stages 1 and 2 during the same lesson or in consecutive lessons. Stage 3 is to be carried out in another lesson, after the teacher has had the opportunity to study the reports of Stage 1.

For assessment of the course, we constructed a Three-Stage tool appropriate for the subject covered in the course. Chosen for inclusion in the Three-Stage tool were didactic and mathematical tasks in the subject matter dealt with in the course. In the group test we included questions that examined mastery of mathematical concepts, awareness of typical errors, familiarity with various demonstration means and their integration in teaching. At the second stage, a discussion was conducted which was primarily a summary of the mathematical and didactic principles related to the group task, where we have chosen to focus on the advantages and disadvantages of each model and to discuss additional demonstration means which could be used (some already encountered during the course). At the third stage, the individual test, we gave questions based on the group test and the class discussion. The remaining questions were based on the group test tasks.

For examining the individual and the group tests we constructed specific criteria typical of each question, where each criterion is assigned a value, with each of the criteria detailed and explained in Tables 1 and 2. We defined various performance levels in performing the task, and each of the reports was given a "genergrade" (100 maximum) according to the performance level. The assessment of each student was based on a combination of his work in the group part (50%) and his individual work (50%). The course grade was based on the performance of several tasks, one of which was this assessment task.

Presented below are representative questions from the assessment tool (a variety of demonstration tools means was available to the students):

Group Test

<u>Question 1</u>

Two teachers meet in the teachers' room. One holds straws and pipe cleaners, the other a worksheet.

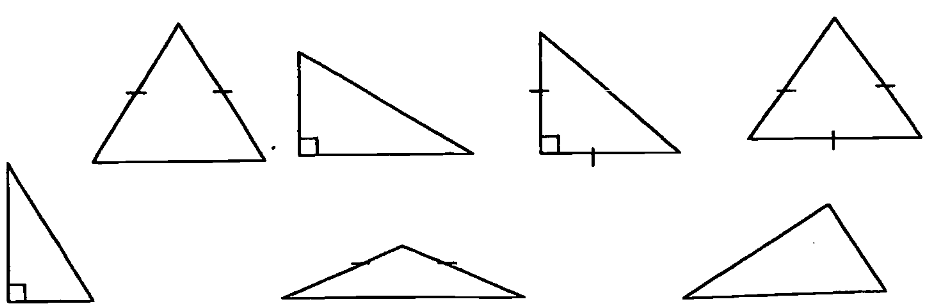
Teacher A: I understand that both of us are dealing with triangles.

Teacher B: Right, here is a worksheet suitable for this subject.

Teacher A: I prefer to use straws and pipe cleaners.

1. Propose an activity suitable for the "triangles" subject by using straws and pipe cleaners (list each of the activity stages, aim of activity, what it examines, etc.)
2. Study Teacher B's worksheet. Add your comments, propose activities and discuss both proposals: straws and worksheet. What are the advantages and disadvantages of each method. Explain.

Student worksheet:



Question 2

1. Record typical pupil errors regarding the concept of the diagonal.
2. Compose test questions whose aim is to examine the pupils' knowledge in respect of each typical error.

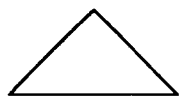
Note alongside each question which typical error it examines, and why.

Class Discussion (following the group test): A discussion was conducted on the advantages and disadvantages of using the worksheet and the straws. This point was chosen for the class discussion as representing a central issue dealt with by the students in the group report.

Individual Test:

Question 1

1. Propose an activity suitable for the "triangles" subject not using either straws, pipe cleaners or a worksheet, but rather another aid. Specify for each stage the purpose of the activity, what it examines, etc.
2. Assign two names to the following triangle (the students had various accessories on the table before them, which they could use to examine the characteristics of the given triangle):



3. Which of the two names was given by the majority of the class, and why?

Question 2

Construct a page of examples and non-examples of diagonals. Bring at least 5 examples and 5 non-examples, argue and explain.

Findings

To assess the work, we constructed criteria for checking the individual and the group tests. The following tables list the criteria and the overall score. Scores are also listed for two group tests (good and weak test scores), and two individual tests (good and weak test scores).

An example of criteria constructed for Question 2 in the group test (the criteria for Question 1 in this test are similar to the criteria for Question 1 in the individual test):

Criterion 1: List of Errors

Level	Details	Total Score	Good Score	Weak Score
1	No list of errors or incorrect list	0		
2	Partial list (less than five errors)	3	3	3
3	Complete list (at least five errors)	5		

Criterion 2: Understanding Concept of Diagonal

Level	Details	Total Score	Good Score	Weak Score
1	No understanding	0		
2	"Partial" understanding	3		3
3	Full understanding	5	5	

Criterion 3: Sample Questions

Level	Details	Total Score	Good Score	Weak Score
1	Samples unsatisfactory	0		
2	Samples satisfactory – no explanations	2		2
3	Samples satisfactory – partial explanations	3		
4	Samples satisfactory with explanations	5	5	

Criterion 4: State the error examined by each example

Level	Details	Total Score	Good Score	Weak Score
1	No	0		0
2	Yes	5	5	

Analysis of the entire "good test score" shows that the students made correct use of the various demonstration models and used them in the proper place and manner. They were able to construct different and varied activities for the pupil, activities that link inquiry and discovery and require using the demonstration means. They were aware of the difficulties encountered by pupils in the subject being studied, aware of the typical errors made by pupils in this subject, were familiar with their origin and knew how to deal with them. They had mastery of the subject being studied (mathematical as well as didactic), possessed good communication ability and used mathematical language while reasoning and explaining.

From the analysis of the entire "weak test score", we found that students did not use the demonstration models correctly. They failed to construct different and varied activities for pupils and did not relate to the aims of the activity. Their communication ability is faulty and they do not sufficiently reason and explain. Their understanding of concepts is unsatisfactory and they are insufficiently familiar with the difficulties encountered by students in the subjects being studied.

Example of criteria constructed for Question 1 of the individual test (Section 1)

Criterion 1: Understanding Concepts: Using mathematical concepts relevant to the activity, namely the student mentions a concept and shows that he understands it.

Level	Details	Total Score	Good Score	Weak Score
1	No use of mathematical concepts and/or incorrect use of concepts	0		0
2	Partial use of concepts	3		
3	Correct and full use of concepts	4	4	

Criterion 2: Manner of Using Model (demonstration means)

Level	Details	Total Score	Good Score	Weak Score
1	No use of model	0		
2	Attempt made to use model	1		1
3	Correct use of model	3		
4	Correct full and sophisticated use of model	4	4	

Criterion 3: Communication

Level	Details	Total Score	Good Score	Weak Score
1	Constructed unsuitable activity	1		
2	Activity not detailed	2		2
3	Activity detailed but not fully	3		
4	Activity fully detailed	4	4	

Criterion 4: Manner of Presenting Task

Level	Details	Total Score	Good Score	Weak Score
1	Unclear presentation of task	0		
2	Certain parts of presentation are clear	2		2
3	Presentation clear but details lacking	3		
4	Presentation clear, detailed and prepared in an organized manner	4	4	

Criterion 5: Relating to Activity Aims

Level	Details	Total Score	Good Score	Weak Score
1	Aims of activity were not detailed or no relating to aims	0		0
2	Activity partly relates to aims	3		
3	Activity fully relates to aims	4	4	

Criterion 6: Originality
Original work was awarded 2 points.

Analysis of the entire "good test score" (grade 100) showed that the student:
Is familiar with various demonstration models, knows how to use these in the right place and manner and is capable of constructing different and varied activities for the pupils, which link inquiry and discovery and require using different demonstration means. The student is aware of the difficulties encountered by the pupils in the subjects being learned, the typical errors made by pupils in this subject, the source of these errors and how to deal with them. The student has mastered the subjects being learned (from the mathematical and didactic aspects), possesses good communication skills and uses mathematical language with reasoning and explanations. To summarize, it is evident that this student has deepened and expanded her knowledge, while extracting the most from her ability in the individual test, beyond the group test (in which she received a grade of 89).
From the analysis of the entire "weak test score" (grade 55), we found that the student is insufficiently aware of the pupils' typical errors in the subject of "diagonals in polygons". Her work lacks reasoning and explanations, even though she was asked to provide these. Moreover, she is unfamiliar with the different triangle types.

Analysis

At the first stage (the group test), the students worked diligently, in cooperation, and conducted mathematical discourse. They enjoyed this very much and one of the students said: "It was really fun, even though I'm certain I didn't get 100, but I don't care, because I learned a great deal and enjoyed myself very much." The students noted that they prefer working in groups rather than individually "because it is much more interesting and productive."

Analyzing the results, we found that the score obtained by 14 students in the group test was higher than their score in the individual repor(by more than two points). The score of 10 students in the group test was lower (by more than two points) than the score in the individual test, while 9 students had a similar score in both.

In the group test, students at different levels worked together, mutually supporting and enriching each other, and it appears that therefore weaker students scored higher in the group test and lower in the individual test. The group score constituted a protective grade for the weaker students. Examination of additional work submitted in the course and the impressions gained in the classroom of this group of students whose group score was higher than their individual test, indicated that it is the lower grade in the individual test that reflects their knowledge.

As to the students whose individual scores were higher than their scores in the group test, it can be assumed that they learned from the group test and the class discussion.

We found special cases such as: a student who attained 90 in the group test and 52 in the individual test. Acquaintance with this student in the classroom and with her other work showed that she has not mastered the material sufficiently. However, in the group test she was together with students who have mastered the material well, and it can be assumed that they dominated the group test, thus the gap between the grades. On the other hand, another student received 65 on the group test and 98 on the individual test. Upon examining the composition of her group, it turned out that it included two students who have not mastered the course material. From an interview with the student, it turned out that she is lacking confidence regarding her knowledge and was thus unable to dominate the group.

Summary and Discussion of Results

It should be noted that the use of the Three-Stage tool achieved several objectives:

By examining the test results we learned about the central points in our work in the course, such as:

Importance of the mathematical discourse. Importance of working in groups, which develops skills of cooperation as well as supporting and helping others (beyond the mathematical discourse). Importance of employing the various demonstrations means. The various mathematical and didactic aspects.

The need to change the assessment methods in accordance with the teaching methods.

This method of assessment provided the students with an example of how to adapt the assessment method to the teaching method: Throughout the course, the students raised the question of how they would be assessed in light of the new trends. The question that was asked repeatedly was: "Since we are working in groups, will the test be a group test?"

Reflecting the students' knowledge: We can see from the results that the tool generally reflects the students' knowledge. The tool examined the understanding of mathematical concepts, methodological knowledge (such as the ability to prepare activities for pupils), understanding the task, originality and creativity, organizing the work and communication. The students fully cooperated in the group work, held discussions, made assumptions, and learning took place with mutual support. The tool expressed the skills and the diversity among the students and the processes of coping and working in groups.

Additional advantages in the group segment: Reduced test anxiety, support for the weaker students and creating motivation toward cooperation.

The individual test reflected and balanced the results of the group test and related differentially to each student in the group. Namely, information was provided on the knowledge of the various students by their level, by what they learned working in the group, and following the discussion conducted by the teacher. It was furthermore possible to see that some students deepened and broadened their knowledge of this subject by themselves, fully utilizing their ability in the individual work.

The individual test pointed out the knowledge gaps between the student teachers and the teachers. For example, it was possible to detect among the student teachers typical errors that occurred among pupils, such as in Question 1 of the individual report, in which only the student teachers assigned the incorrect name to the triangle, "acute angle triangle" or "equilateral triangle". In the question where they were asked to point out typical pupil errors regarding the concept of the diagonal, only teachers were able to link their answer to the thinking levels of Van Hiele.

To summarize, the "Three-Stage" is the appropriate tool for the new assessment and teaching approaches. It reflects the students' knowledge and expresses the skills and diversity among students, the processes of coping with working in groups, to originality, creativity and mathematical communication.

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CHARACTERIZING UNDERGRADUATE MATHEMATICS TEACHING

BARBARA JAWORSKI, UNIVERSITY OF OXFORD
ELENA NARDI, UNIVERSITY OF EAST ANGLIA
STEPHEN HEGEDUS, UNIVERSITY OF OXFORD

The Undergraduate Mathematics Teaching Project[1] (UMTP) is a one-year study aiming to characterize, and identify issues related to, mathematics teaching in undergraduate tutorials. It builds on earlier research into mathematics learning in undergraduate tutorials and involves a research collaboration between mathematics educators and mathematicians. From participant observation, semi-structured interviewing, and group discussion, it develops a set of qualitative data which is analyzed through repeated critical scrutiny to distil characteristics and issues of the teaching experienced which might be seen as germane to a wider variety of settings. Research is ongoing, but issues and indications from analysis of pilot-study data are included.

Rationale

The broad aim of the Undergraduate Mathematics Teaching Project (UMTP) is to explore, in a collaboration between mathematics educators and university mathematics teachers, current thinking and practices in mathematics teaching at first year undergraduate level. It seeks to elicit relationships between the enacted teaching, the mathematics being taught, the aims and objectives for students' learning, and the perceptions of those teaching (the tutors) and those observing (the researchers). In doing so it will begin to provide a knowledge base on which to make decisions affecting practice in university mathematics teaching and illuminate an under-explored area of influence on mathematics teaching more widely. The objectives of the study are:

1. To identify practices and processes in the teaching of mathematics at first year undergraduate level, and the thinking and beliefs which underpin them.
2. To examine such practices and processes in a collaboration with university mathematics tutors and develop a pedagogical discourse based on collaborative reflection by both researchers and practitioners.
3. To inform university tutors about the nature and implications of existing practices and processes, and the potential of alternatives. The research will explore the impact of these implications on current thinking and beliefs.

The research develops from two previous studies. The first explored undergraduates' mathematical learning difficulties in first year tutorials (Nardi, 1996) – in particular students' appreciation of abstraction and formalism. The

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second study followed from the first: data, in the form of transcripts of tutor-student dialogues, and analyses were presented to the tutors (whose tutorials were observed in the first study) to explore their related thinking and reactions (Nardi, 1998). The first study provided a rich account of students' difficulties with mathematical abstraction in a range of topics. The second provided insights into tutors' thinking about their students' learning, and, although very small in scale, was encouraging as an indication of the value of further exploration into the teaching of mathematics at this level. Both studies provided strong evidence of the potential of tutorials as a source of rich data, allowing insights into learning and teaching processes and practices.

Background and theoretical perspectives

The UMTP study should be seen in relation to key issues in mathematics education currently in the UK. The number of students opting for mathematically-oriented studies at university is decreasing and recruitment of good mathematics graduates to mathematics teaching at school levels is at an all-time low. Profound changes in secondary education, pedagogy and curriculum have contributed to an increased gap between secondary and tertiary mathematics teaching approaches and to a debate as to the preparedness of undergraduates for university study in mathematics. One response to these changes has been to modify university mathematics curricula to adjust to current needs of students. However, universities are now being required to be accountable for the quality of their teaching and there is an emerging realization that reform should be focusing on teaching. The above imply that there is a need for a revision of the underlying principles as well as the practices in the teaching of mathematics at university level. The UMTP study aims to explore critically the nature of undergraduate teaching and its potential for the future of mathematical learning at a variety of levels.

The research is embedded in a growing theoretical area which focuses on the development of knowledge in advanced mathematics, and the difficulties students face in dealing with mathematical abstraction. The work in the area of Advanced Mathematical Thinking (e.g. Tall, (1991) and Sierpinska, (1994)) is highly relevant and the work of Nardi, quoted above, fits into this tradition. The current study seeks to relate these theoretical perspectives to issues in teaching.

There is an extensive, curriculum-based, literature in this area, mainly in North America, which seeks to relate undergraduate learning to methods of teaching. For example, alternative approaches to calculus (e.g. Ferrini-Mundy & Graham (1991) and Selden & Selden (1993)) and linear algebra (e.g. Leron & Dubinsky 1995) reflect, in part, attempts to make these subjects more engaging and meaningful for the majority of students. However a general perception remains that the teaching of mathematics at the undergraduate level

has not to date made sufficient effort to deal with the backgrounds and needs of present day students. The research described here aims to go beyond particular practices, to seek more general awarenesses and understandings of the relationships between teaching and learning at undergraduate level. It draws also on research at other levels of mathematics teaching, seeking commonalities and differences. For example, it explores the use of the Teaching Triad, a construct used to characterize and analyze mathematics teaching at secondary level (Jaworski, 1994).

Project Initiation and Methodology

The UMPT is a one-year study (from October 1998) of a small sample – largely purposive and opportunistic – of first year undergraduate tutorials. It aims to study in depth the teaching of 5 university mathematics tutors from a list of volunteers willing to collaborate in the project. Over a period of one university term (the second of the year, January to March) data will be gathered by observing one or two consecutive one-hour tutorials per week for each tutor, and interviewing each tutor soon after each observation. Periodic meetings of researchers and tutors will be held to establish a community of practice within the project, to encourage open sharing and debate of practices and issues, and as an alternative source of data for validation purposes.

The UMTP methodology might be described as critically qualitative with a strong emphasis on participation, drawing on experience from previous research by Jaworski (1994, 1998). *Qualitative* data is sought to allow access to the complexities of tutors' epistemology and its relation to pedagogy. The *critical* nature of the research is in its questioning of processes and practices at all levels. Questions from researchers to tutors seek access to tutors' thinking underpinning observed actions. The earlier research suggested that such questions will lead to questioning by the tutors themselves of their own practices and associated theories and beliefs. Meetings between all the research participants are designed to encourage an airing of tensions and issues. As part of the analysis, feeding back into data collection, researchers will reflect critically on questions asked, to examine and expose their relation to underlying theoretical perspectives, and influences on tutors' responses.

The *participative* nature of the research is designed to develop a community of practice within which such questioning can take place. It is necessary to recognize overtly the differing perspectives and objectives of tutors and researchers (as identified in the aims of the project above), while fostering trust and mutual respect. The relationship is designed to be one of "clinical partnership" in Wagner's (1997) terms:

> The researcher is clearly the agent of enquiry, and practitioners are the people whose work is the focus of analysis and reform. But practitioners can also engage in the enquiry, at least by assisting their researcher colleagues, and

attention is given by both to the process of researcher-practitioner consultation itself. (p. 15)

Analysis *during* data-collection involves scrutiny and annotation of observation notes to produce observation-protocols and suggest questions for interviews. Analysis after data-collection involves scrutiny of recorded interviews and production of interview-protocols for use in extracting characteristics and issues. Protocols act as second-order data to inform subsequent analysis which involves a critical scrutiny of observations interviews and discussions to elicit relationships. Discussions in research group meetings are recorded and support the analysis of the interviews. Characterization involves seeking processes, practices and issues which might be seen to be germane to a larger number and wider variety of settings. Rigour is to be ensured through triangulation between alternative data sources, and a transparency of contextualization and critique (Delamont and Hamilton, 1984; Ball, 1990).

Indications and issues arising from pilot-study analysis

The pilot study focused on 7 tutors, which would reduce to 5 for the main study, once timetables in the second term were known. The purpose of the pilot was to try out observational and interview approaches, data gathering techniques and analytical procedures; to induct tutors into research practices; and to enable the researchers to adjust, critically, the mutuality of their perceptions of the research methodology. We shall offer some insights into the data emerging from one tutor and our preliminary analysis of it. In addition we shall identify, tentatively, an emerging issue, that of the nature and importance of interaction between tutor and students in the tutorial, with reference to data from a number of tutors.

Tutor-1: Data and Preliminary Analysis

Tutor 1 talked about his teaching strategies. For example, he used a (white)board to facilitate communication between tutor and students. He valued "teaching by example", offering reasons for providing his own examples. He emphasized the importance of "explanation skills". Developing students' confidence was one of his main aims.

In parts he was critical of his own teaching: for example, suggesting that he was "going too fast when student K was struggling", that he needed to be sure of basic understanding, that he had produced a "poor" diagram – it represented three vectors in the plane, but might have been seen as a set of three orthogonal axes in R^3. He said, "I should have been more careful …".

He recognized his emphasis on the importance of "the formalizing process", or "proof", saying "I'm the only source of rigour they have". He indicated a tension between "getting ideas right" and "giving them a feel for rigour".

Analysis involves critical scrutiny of the data, seeking links to other data items, and recognizing patterns which might be judged characteristic. In the pilot study, only two tutorials were observed from each tutor. Thus, pattern seeking is at a very early and elementary stage. Nevertheless there are analytical remarks to make about the above data. We provide, as exemplars here, three points which may be indicative of forthcoming characterization.

1. *Teaching strategies*: It is interesting that the tutor refers to various strategies he uses in his teaching. With regard to use of examples he says

> I like teaching by example, especially when you've got something abstract to say and certainly I think ... I consider it quite important. The reason I like to give my own examples sometimes is you quite often get the student who's done the ones on the sheets[2] and a student who hasn't done the ones on the sheet, so you don't want particularly to set the examples on the sheet[3].

'Use of strategies' might become a *characteristic*. It will be important in other data to look out for tutors' references to strategies they use. In this case, one example of a strategy is the tutor's use of 'his own examples' to provide alternative experience from that offered by the lecturer, and for discussion purposes in the tutorial. With this tutor it will be important to explore further his construction and use of such examples.

2. *Developing students' confidence*: The tutor's words were:

> I think the biggest thing to developing confidence is to make them not feel at all uneasy about being wrong. I mean ...you don't have to have a wonderful rapport but just enough of a ...never be patronizing and you know, just, er, OK, OK, tell them the question's hard, whether it is or not because that's sticking with them, tell them it's hard and [as] such it's, it's just a matter of er, getting your, your confidence to, to, to be wrong, you know is is a very good thing.

Some of the hesitation is left in here to indicate the tutor's struggle to articulate his perception of dealing with students' confidence. This statement points towards a 'sensitivity to students' on the part of the tutor – a teaching concern directed at the students' needs, in this case in the *affective* domain. It suggests that doing what he can to encourage confidence is going to help the student to tackle concepts more effectively. His criticism of his misleading diagram might be seen as sensitivity which is more *cognitively* focused: he suggested the diagram might have been construed by students in an alternative way to that intended, potentially leading to misunderstanding by the student.

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[2] He refers to the lectures which are open to all students, and the problem sheets, set by the lecturers, for students to tackle questions related to the lecture material.

[3] Quotations are edited slightly to take out hesitations and repetitions, unless these seem crucial to the sense with respect to the analytical point being made. Analysis, however, looks carefully at the whole data and considers the relevance and import of pausing, hesitation etc.

Sensitivity to Students (SS) is one of the elements of the Teaching Triad (Jaworski, 1994, Jaworski & Potari, 1998), the others being Management of Learning (ML) and Mathematical Challenge (MC). The teaching triad research suggests that SS is closely allied to MC which characterizes a teachers' interaction with the student in the domain of mathematics, attempting to encourage the student's cognitive development. The tutor's use of the diagram could be seen as a means of challenging students, which was unsuccessful because of the inadequacy of the diagram. Thus there are indications in this very early data of the inter-relationship of SS and MC leading to issues about teaching for students conceptual development.

3. *Ensuring Rigour*: A longer extract seems worth inclusion to indicate the potential of this data in early analysis:

T1 Yes. Well, I mean (pause). My role is to have them... I mean my role is an important one is that the, er, I am their only source of rigour that they have in the, really in the, in the, the first year. They'll, it, it's me explaining things and introducing style and er proof. But at the same time ideas I'd like to get er, it's it's a balance between getting ideas right, er, and giving them some feel for them but then also being able, being somehow being able to (pause) convince them that they feel they have to do er... concept is the same as the rigorous definition that might be on the board.

Ten lines omitted. Then:

I think, I think the important role of the tutor is to make sure the, try and give them handle on both things but that's often difficult. I mean one way is usally geometric and one way is analytic and a lot of people only think one way and not the other but er, at the same time I usually tell them as long as they can do it one way it's a good thing but it's always nice to have two ways of thinking about it. ...

Scrutiny of the transcript shows that this statement is one of Tutor 1's longest statements in the interview. Although very hesitant, perhaps unused to articulating his personal principles or theories, Tutor 1 presents a clear rationale for his emphasis on, and approaches to, rigour. This length and clarity are interpreted as indicating issues of significance to the tutor – in particular, perhaps, the issue of "getting ideas *right*" versus "giving [students] some *feel* for them" (our emphasis). In this case, the mathematics in question is linear algebra. It will be important to follow up further references to 'teaching for rigour' in subsequent data, and to question their relationship to particular topics, or their cross-topical generality. (c.f. Nardi, 1996).

An emerging issue, *the nature and importance of interaction*

Interaction in tutorials, unsurprisingly, is perceived and enacted in different ways by different tutors. Many students are typically silent unless asked to

speak, and so their uttering a word indicates interaction. In some cases the interaction is little more than a question from a student, followed by a monologue by the tutor. In others it involves a student in presenting ideas to the tutorial group, with comments and questions from other student(s) and tutor. Some tutors seem to see interaction as little more than an unfortunate necessity for discovering what is problematic, especially with weaker students. Others see it as a tool for empowering students' mathematical communication.

From the current data, we have identified elements of tutor 'inciting' interaction and 'empowering' interaction. In the first case, it is necessary to recognize a *need* for interaction. In the following quotation, a tutor recognizes that inappropriate imagery is a barrier to understanding, so that exposing this imagery – "you need to listen" – is essential for the tutor in judging appropriate support.

> I think they have just utterly different pictures in their heads sometimes. They have, they have completely the wrong picture in their head and you, you need to listen to what that is to push them in the right direction. So if there's the wrong picture in their head and you start talking about something based on the picture you have in your head or that you think they have in their head you, you just confuse them more. (Tutor 2)

The following quotation emphasizes the belief that interaction is a necessity for helping weaker students – "I very often ask them to explain it to me" – but unnecessary for "rather better" students where a monologue suffices.

> Um, I think it's actually more to do with the students ... because they are very weak. In trying to find out whether they understand something I very often ask them to explain it to me or say something about it. Because, er if I just say OK this is a Cauchy sequence and they nod I don't actually know whether they're just nodding to just to be polite ... Er, my Monday afternoon students are, are rather better and tutorials tend to be much more of a monologue. Um, I don't, I don't need to ask them what a Cauchy sequence is because they've demonstrated in their work that they know. (Tutor 3)

Using tutorials to *empower* students has emerged from some of the current data. For example, Tutor 1, above, spoke of the importance of building students' confidence, for example, of acknowledging that certain concepts are "hard", and of seeing "being wrong" in positive terms. Although these ideas were not expressed explicitly in terms of interaction, it might be believed that this tutor would engage his students in dialogue to expose "wrong" perceptions, to cut through hard mathematical concepts.

Another tutor (Tutor 4) talks explicitly of using dialogue for empowerment in students' learning "how to do mathematics" – "I want them to actually be doing it rather than me telling them how to do it".

I think I view it that the point of a tutorial is for them to learn how to do mathematics so I want them to actually be doing it rather than me telling them how to do it because I think they learn less by me telling them the answer to a dozen questions instead of, OK, how do I think of these solutions, how do you think of these solutions? How do you think of these solutions now and when you're back in your room without me prompting you how to do it?

In Conclusion

It is important to emphasize that we have as yet only a small set of data to analyze, so that the issues indicated above only just start to suggest characteristics which might be seen as more widely germane. However, the pilot has fulfilled its purpose in clarifying methodology and establishing initial relationships, quite apart from its promise regarding issues and characteristics. We look forward to presenting results from our main study at the conference.

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CAS, CALCULUS AND CLASSROOMS

Margaret Kendal and Kaye Stacey
Department of Science and Mathematics Education
The University of Melbourne

Abstract

Three teachers helped design and then taught an experimental program of introductory calculus in which students had full access to calculators with a computer algebra system (CAS) in the classroom, at home and during tests. Each class obtained similar mean scores on the test. However they made very different use of the CAS and performed very differently on items. One class frequently used the CAS. The second preferred by-hand algebraic techniques. The third group of students, with weaker algebraic skills, used CAS more selectively and demonstrated good understanding built from illustrating algebraic ideas graphically. The study demonstrates how teachers "privileging" impacted on student learning.

Introduction and Background

Computer algebra systems (CAS), incorporating graphical, numerical and symbolic algebra capabilities, have much to offer in the teaching of calculus. By reducing the obstacle of manipulative algebraic skills, they can release time in calculus courses to spend on concept development, problem solving and investigations (Hillel, 1993). In addition, the capacity to provide multiple representations and the ability to move freely between them makes CAS a teaching tool to enhance conceptual development. Several experimental studies now support these expectations. For example, Heid (1988), O'Callaghan (1998) and Repo (1994) have claimed that students using a CAS developed better conceptual understanding although they differ about whether this is accompanied by a loss of computational skills. There is also some research evidence to support the belief that students become better problem solvers while using a CAS (Heid, 1997). The availability of symbolic algebra on calculators, rather than on standard computers only, and the attendant reduction in unit price has stimulated even more interest in teaching calculus with CAS. Cappuccio (1996), for example, describes specific ways to teach calculus using the TI-92, a Texas Instrument calculator which incorporates a CAS similar to the desktop DERIVE.

The study reported in this paper is part of a larger study conducted with colleagues Barry McCrae and Gary Asp which explored the implications for state-wide assessment of teaching and learning calculus with CAS. In this paper we describe how three different classes of students used CAS in different ways while learning introductory calculus using the same teaching program. As such it is a study of the

type recommended by Penglase and Arnold (1996, p.79) "which directly attempt[s] to address the issues of graphics calculator use within particular learning environments."

The data reported here show how the outcomes for students in the three classes were quite different and we attempt to link these to the personal style and philosophy of each teacher. A wide variety of factors may affect the impact of an innovation such as new technology in the classroom. Amongst these are teacher-related socio-cultural factors such as attitude towards technology, prior experience teaching with technology and personal beliefs about the ways students should be taught (Thomas, Tyrrell, & Bullock, 1996). Of particular interest in this paper is Wertsch's concept of "privileging" which Berger (1998) describes as "the social setting and values which may elevate one form of mental functioning over another and in this way privilege a particular form of mental operation such as algebraic or graphical reasoning (p.19)."

The teaching trial, research methods and data collection

Early in 1998 a team including McCrae, Asp, the first author and three volunteer teachers designed a twenty-lesson introductory calculus program that aimed to use CAS to enhance conceptual understanding, connections between representations and appropriate use of the technology. Subsequently the teachers each taught the program to one class. In total there were 59 students in three Year 11 classes (students aged approximately 17 years) from two schools. The three classes had similar distributions of mathematical ability, but Class C had the weakest algebraic skills. All the students were familiar with the TI-83 graphical calculator. For this study, they were given a TI-92 graphics calculator for use during lessons, at home and for most testing. The three teachers were experienced in teaching mathematics with the TI-83 graphical calculator but inexperienced with the TI-92.

Students regularly completed questionnaires, challenging questions and log sheets to describe their feelings and progress with the calculus and technology. All students completed a written test (sample items below) and for each item they indicated whether they had used the calculator. Seventeen students were given a task-based interview by the first author, who also observed approximately half of the lessons and maintained a journal. The teachers wrote a brief reflective evaluation after each lesson and completed questionnaires and Likert items before and after the program.

Results

Table 1 shows the mean scores for each class on the written test (maximum possible score was 70). Although the absolute scores seem low, the results were very pleasing because of the difficulty of the items for Year 11 students. One third of the written test items were taken from state examination papers for Year 12 students and the study classes performed well in comparison to Year 12 classes.

Table 1 shows that the three classes performed similarly overall. However, the item-by-item analysis in Tables 2 and 3 shows differences. Table 2 indicates the relative amount of calculator use. For each class, it shows the number of items where that

class made the greatest or least percentage use of the calculator. For example, on 25 items, use was made of the calculator by a greater percentage of students in Class A than in Class B or C. Clearly Class A students used the calculator to a much greater extent than Classes B and C. Class B students used it least.

Table 1. Mean scores and standard deviation for total test by class.

	Class A (N=15*)	Class B (N=18*)	Class C (N=13*)
Mean	24.1	26.7	27.9
Standard Deviation	9.4	12.0	10.3

* *Not all students participated in all tests, so the numbers in the Tables vary.*

Table 2. Number of items where each class made greatest and least use of the calculator.

	Class A (N=15)	Class B (N=19)	Class C (N=16)
Greatest % calc use	25	3	5
Least % calc use	5	17	11

Table 3. Number of attempted items correct and incorrect by calculator use and by class.

	Class A (N=15)		Class B (N=19)		Class C (N=16)	
	calc	no calc	calc	no calc	calc	no calc
Correct	152	61	156	124	147	101
Incorrect	111	86	78	129	47	92
Total	263	147	234	253	194	193

Table 3 shows the total number of items attempted by students in each class broken down by calculator use or non-use and by success. This item-by-item analysis confirms the results in Table 2 that Class A students chose to use the CAS most frequently (64% of items attempted) and Class B least (48%), slightly less than Class C (50%). The Table 3 analysis shows more difference in overall success than does Table 1, with Class A correct on 52% of items attempted, Class B 57% correct and Class C 64% correct. This shows that there is no simplistic conclusion that greater use of CAS leads to better results. Class C achieved a higher success rate on items attempted both with and without the calculator. The percentages of items correct where students had used the calculator were 58% (Class A), 67% (Class B) and 75% (Class C) and without the calculator 41% (Class A), 49% (Class B) and 52% (Class C). This indicates that Class C made best strategic use of the CAS calculator, as will be explored in the next sections.

Conceptual and procedural errors
Despite the program's emphasis on conceptual development using technology, many students had difficulty with fundamental concepts, particularly rates of change and distinguishing between the gradients of secants and tangents. Test responses were analysed and errors were classified as conceptual and procedural. We classified as conceptual errors those which occurred when understanding was not demonstrated or the process to use was not correctly formulated. This is similar to Orton's (1983) definition of structural errors which "arose from some failure to appreciate the relationships involved in the problem or to grasp some principle essential to solution" from Donaldson in 1963. Procedural errors were algebraic, graphical, and numerical in origin or related to incorrect use of the calculator. Conceptual errors cannot be eliminated with CAS use; procedural errors may be avoided with CAS use. On a group of items attempted by all but a few students, the conceptual error rate per student was 7.3 for Class A, 5.7 for Class B and 4.9 for Class C. However, the procedural error rate was similar for all classes despite varying CAS use: 2.3 per student for Class A, 2.4 for Class B and 2.8 for Class C. In the CAS environment, procedural errors were less frequent than conceptual errors.

Success on specific item types
Many of the test items were grouped for analysis into "*core*", "*symbolic*" or "*options*" groups. There were approximately 5 items of each type. *Core* items were characterised by high conceptual and low procedural demands. In these items, using CAS is no advantage. One example is an item that gave the graph of a function and asked the student to select the graph of the derivative from five possible graphs. CAS would not be useful as no symbolic representation of the function was given.

Symbolic items have high demands on algebraic procedures and low conceptual demands. In these items, the symbolic manipulation capability of the calculators could be useful. One example on the written test was "Find dy/dx for $y = x^3(2x+1)^2$ giving your answer in factorised form". With CAS, an item like this is nearly trivial, but without it, remembering the differentiation rules and the algebraic manipulation can be difficult.

The *options* items are those that present the student with a choice between graphical and algebraic approaches. Using CAS may be advantageous. The following example item was accompanied by a diagram which gave a graph of the equation over a relevant domain:

> Given that a rider on the track of a super roller coaster follows the curve with equation $y = 1/720(x^3 + 20x^2 - 1200x)$, find the maximum height above the ground reached by the roller coaster.

This problem is an *options* item because it can be solved algebraically or graphically. The graphical route is conceptually easier than the algebraic pathway, which requires several conceptual formulation steps. With CAS the algebraic and graphing procedures are easy but not without it.

Taking the results for *core* items as indicating conceptual understanding shows that Class C students demonstrated the highest level of understanding (53.9% of approximately 80 attempts were successful) and Class A the least (40.0% of attempts were successful). We believe that the high score on these *core* items in Class C was due to the way the teacher demonstrated concepts in both graphical and algebraic terms. In total there were only 13 attempts to use the calculator (inappropriately) on these items, mostly from Classes A and C and they were mainly unsuccessful, as would be expected.

In contrast, on the *symbolic* items many students sensibly chose to use CAS. Class A had the highest CAS use and was most successful. Classes B and C both under-utilised CAS, making unnecessary mistakes in algebraic manipulation. In the interviews, students from Class A and B showed greater proficiency than students from Class C in using the algebraic facility on the calculator. For example, to find a gradient at a point, over half of the students interviewed from Classes A and B used a one-line instruction to differentiate the function and substitute the value. In contrast, nearly half of students from Class C attempted a graphical approach.

The *options* items could be solved algebraically or graphically. More Class B and C students used the calculators here than for the *symbolic* items. However, Class C students more often used the graphical facility in contrast to the algebraic route favoured by Class B students. This observation is supported by the student interviews. For example, on an *options* item used in the interviews, a graphical approach was selected by one out of five Class A students interviewed and by one out four Class B students but by 3 out of 5 Class C students. Confirming the previous result about choice of CAS, all 5 Class A students used the calculator, whereas half of the 4 Class B students worked by hand.

The differences between the behaviours of the classes that have been derived from the analysis above are summarised in Table 4. The terms used in the table express relative achievement only. For example, "higher" is relative to the other classes and is not an absolute judgement.

Table 4. Summary of the behaviours of the three classes in the study.

Behaviour	Class A	Class B	Class C
Use of calculator	most frequent	least frequent	frequent
Decision to use calculator	too frequent	discriminating	discriminating
Preferred approach	algebra by calculator	algebra by hand	graphical
Algebraic proficiency	moderate by hand	higher by hand	lower by hand
Graphical skills	lower	moderate	higher
Procedural competence	good	good	good
Conceptual understanding	lower	moderate	higher

Classroom influences

Teacher predictions

Prior to the teaching trial the teachers used a Likert scale to describe their own students' competence and probable reaction to using the new calculators. Table 5 shows their ratings. Again, the table entries express relative positions rather than absolute judgements. The table is highly consistent with the profile of each class that emerged by analysing the test results above. In particular, Class C has an orientation to graphical approaches and Class B to algebraic approaches.

Table 5. Teachers' predictions about class abilities and reactions.

Class characteristic	Class A	Class B	Class C
Algebraic competence	moderate	higher	moderate
Graphical competence	moderate	moderate	higher
Reaction to new technology	likely to succeed	will probably succeed	very likely to succeed

Teaching styles

During the teaching of the CAS program, about half of the lessons were observed by the first author and these observations lead to the descriptions summarised in the first half of Table 6. Table 6 also reports responses to three key indicators from a set of Likert-type items completed by the teachers (with additional written comments) at the end of the program. We summarise these observations as follows. Teacher A had a very positive attitude to technology, encouraged his students to use it as often as possible and gave priority to algebraic strategies. Students were taught efficient calculator procedures for standard tasks. Teacher B preferred the traditional algebraic approach using graphs when essential. He emphasised by-hand algebra, being wary that students might not otherwise develop adequate skills. Teacher C encouraged his students to use both algebraic and graphical methods and to explore connections between them. His explanations used the links between representations.

Discussion

The mean scores for each class on the written test were very similar and a casual observer would assume that learning outcomes for all students were similar. In fact this is not the case. During the teaching trial, the students in each class had very different cognitive experiences evidenced by the different ways the calculator was used. Each class acquired different conceptual understandings, a different set of competencies and different abilities to discern whether or not it would advantageous to use the various features of the calculator. These differences can be linked to the different teaching styles, personal philosophies and cognitive preferences of the teachers, even though they all helped plan the program and were thoroughly aware of

Table 6. Comparison of teacher characteristics

Characteristics	Teacher A	Teacher B	Teacher C
Classroom observations			
Teaching style	Direct instruction	Guided discovery	Guided discovery
Direction of lesson	Followed lesson plan	Controlled exploration	Open exploration
Attitude to using CAS	Enthusiastic	Reserved	Enthusiastic
Structured lesson around use of calculator	Mostly	Sometimes	Mostly
Used algebraic explanations	Very often	Very often	Often
Used graphical explanations	Sometimes	Sometimes	Often
Used both algebraic and graphical	Rarely	Rarely	Often
Teachers' own perceptions			
My usual teaching style suited the CAS Calculus project	Agree Θ	Disagree #	Agree *
There was little formal emphasis on by-hand skills and using pen and paper techniques	Agree	Disagree	Less emphasis, not little
Students enjoyed learning calculus while using CAS	Most yes, some enthusiastic	Some yes, some no	Most yes, a few no

Θ"I did not find it difficult to incorporate the changes" #"I found it difficult to monitor student work"

*"I have already used graphic calculator technology (especially with the view screen) for demonstration in classes."

the aims of the project and taught with the same guidelines. Teacher A privileged technological and algebraic approaches. Teacher B privileged conceptual understanding and by-hand algebraic approaches and Teacher C privileged graphical approaches and conceptual understanding built from illustrating algebraic ideas graphically. As a consequence, Class C's conceptual error rate was lowest. These students understood what to do in algebraic contexts so they could compensate for poor algebraic skills by appropriate use of the calculator and by substituting algebraic with graphical procedures whenever possible.

As we noted in the introduction, many authors have drawn attention to the great potential of CAS in providing multiple representations of mathematical concepts and objects. Multiple representations provide opportunities for students to employ different methods of problem solving and support students finding individual ways of understanding. However, the capacity to provide multiple representations will also lead to greater diversity of teaching methods. The three teachers in our study intended to teach the same material in the same way, according to the lesson guidelines that the whole team prepared. Yet the implementations of the lesson

guidelines varied significantly and the differences translated into substantial differences in how their students solved problems and what they understood.

This research raises two questions for our future research. Firstly, the teachers judged their students' abilities and attitudes to technology quite accurately. Did they adapt their teaching practices to take into account the abilities of their students or did they project their own mathematical preferences onto the students and teach in accordance with these? Secondly, the mean scores on the test were very similar, so is there a privileging or teaching style that should be recommended to teachers? Perhaps each particular privileging enhances development in certain directions and constrains it in others. What privileging (if any) constitutes the most advantageous learning environment for students?

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"Calculus in context": A study of undergraduate chemistry students' perceptions of integration

Phillip Kent and *Ian Stevenson,*
University of London, U. K.

ABSTRACT. We have carried out a formative evaluation study of a computer-based "mathematics laboratory" programme for chemistry undergraduates, where the intention is that students will develop their mathematical understanding as a natural part of doing chemistry. We report in this paper on one particular episode from the laboratory, when the students had to calculate the total energy of a chemical system by mobilising and interconnecting their knowledge of mathematics (integral calculus) and science (ideas of work and potential energy). By means of observations and interviews, we have analysed the "breakdowns" in the students' problem solving activity, and we discuss what this analysis reveals about the students perceptions of integration in context.

Introduction

The mathematical training of scientists and engineers has come under increasing scrutiny in the last decade. Whilst computer technology is allowing working scientists and engineers to make use of more and more sophisticated mathematical techniques (normally hidden away beneath convenient interfaces), there is growing evidence (in the UK and elsewhere) of a general decline in the mathematical preparedness of science and engineering undergraduates. Responses to the resulting pressure on the mathematical content of science and engineering degrees have taken several forms: one has been to simply reduce the mathematical content, and to rely on computer-based tools to do much of the mathematical computation involved in scientific work. However, there is a growing recognition that this approach is unsatisfactory for both mathematicians and engineers since it misses the notion of mathematics as "a precise analytic tool" (LMS 1995, p. 7). An alternative is to increase the mathematical content, but challenge the students to develop their understanding in the context of their subject specialism. However, a set of difficult questions emerge at the intersection of cognitive and epistemological domains: to what extent must the structure of mathematics be understood in order for it to be used effectively as a tool? Can mathematical procedures be learned effectively without appreciation of their place in the structure of mathematics? What is the impact of new mathematical technologies when they are used to highlight mathematics in the context of a science specialism rather than hide it?

In this paper we want to probe the connections between mathematics and science in the context of a computer-based "mathematics laboratory" programme for first year undergraduate chemistry students, which one of us (P.K.) has been involved in

developing at Imperial College (University of London) since 1994. (For a general history of this work, see Templer et al. 1998.) The laboratory makes use of a particularly powerful piece of mathematical software, *Mathematica* (Wolfram 1996), and in so doing plays down the teaching of the large body of mathematical methods which comprise the traditional mathematical training of a science or engineering student. The thinking behind the "maths lab" programme has been described by our colleagues as follows:

> We are trying to introduce mathematics as a natural, integral part of chemistry; *Mathematica*'s power gives the students the opportunity to do this in an authentic, uncontrived way, with something of the flavour of real research. ...
>
> As one might expect, students experience a mixture of emotions when presented with this sort of challenge. One of these is certainly shock. We have deliberately set out to stretch students, but not in a way that they would expect. Their anticipation is that they will learn the techniques and tricks which most perceive as being the proper realm of mathematical experience. Instead we present them with the sort of chemical problems found in research, and with the help of a powerful mathematical toolbox, ask them to investigate the chemistry using those tools. That is not to say that they are not learning any mathematics, far from it. The problems are designed in such a way that they have to understand the mathematical processes that are being performed and the limitations that these may impart on their results.
>
> (Ramsden & Templer 1998)

We report here on a formative evaluation of the "maths lab", focussing on one of the "chemical-mathematical" coursework assignments that required students to think explicitly about how mathematics is used in chemistry to *model* chemical processes.

A student assignment: Modelling the dynamics of ion collisions

In this assignment, the students study the different possible interactions between three ions (two chloride, Cl^-, and one sodium, Na^+), in a Newtonian model, where the ions are treated as charged point particles, and the equations of motion are solved numerically. Although this problem is somewhat simplified from a "real" research problem—it is restricted to one dimension, so the particles are moving on a straight line—the students are explicitly working with a sixth-order, nonlinear differential equation system. The solution of the equations is treated a "black box" operation by *Mathematica* (it would be very difficult to do otherwise, and fortunately it turned out that *Mathematica*'s code is highly reliable). However, the students are asked to check on the reliability of the numerical solutions (and hence, in part, the validity of the Newtonian model) by showing that the total energy of the ion system is (or is not) conserved. Calculating the energy required the students to

integrate force equations to find potential energies, and in the process they needed to mobilise and interconnect mathematical knowledge (integral calculus), physics knowledge (work and energy) and chemistry knowledge (ionic interactions). We wanted to analyse this process, as an entry point to a more extensive study of the cognitive and epistemological aspects of the "maths lab" approach.

Our study data consisted of (*i*) observations of six volunteer students in the second term of their first year as they carried out the ions activity, and (*ii*) follow-up interviews which examined the students' experiences of doing the activity. The observation data suggested that focussing on the students' understanding of integration in the context of energy conservation might be particularly rewarding. Integral calculus is an interesting example of an important mathematical idea that has both a complex structure and multiple uses. Our aim in analysing both data sets was to map out how integration was understood in this chemical context, and how that understanding might be related to integration as an "object of mathematical study".

Following Noss et al. (1998), we looked for (and in the interviews tried to provoke) "breakdowns" in the students' activities, in an attempt to reveal the potentially hidden connections between mathematics and the context. Breakdowns represent "the interrupted moment[s] of our habitual, standard, comfortable 'being-in-the-world'; breakdowns serve an extremely important cognitive function, revealing to us the nature of our practices and equipment, making them 'present-at-hand' to us, perhaps for the first time" (Winograd & Flores 1986, pp. 77-78). From this point of view, the present study represents the beginnings of work in an "academic" setting that is comparable with recent work on mathematics in "vocational" situations, such as banking and nursing (Noss et al. 1998, Pozzi et al. 1998).

Observations: Working with *Mathematica*

We observed three pairs of students altogether, usually for one two-hour session each week, over a period of three weeks. Our data collection consisted of video and/or audio recordings, computer "dribble files" (an automatic record of the students' inputs to *Mathematica*, from which we could later reconstruct both inputs and outputs) and the students' *Mathematica* working documents from each session.

The energy conservation task caused the major breakdowns in the students' understandings of what they were doing. It was intended as a 10 or 20 minute task, 30 at the most; the students had been given all the necessary ideas in a lecture course several months earlier, in addition to encountering the principles of force and potential energy in school physics and chemistry. But as it turned out, *all* the students that we observed, with mathematical backgrounds ranging from basic matriculation to pre-university specialists, spent around 2 hours on it. Whilst the kinetic energies were calculated in a couple of minutes, calculating the potential

energies generated significant confusions that took much effort for the students to resolve.

The task involved using the positions and velocities of the three particles as functions of time, as output by the numerical equation solver, to construct a function for the total energy as a function of time. The students also knew the expressions for the force between like-charged or oppositely-charged ions (i.e. Cl^- or Na^+), separated by a distance r:

$$F_{like} = -\frac{e^2}{4\pi\varepsilon_0 r^2}, \quad F_{opp} = \frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{(r_0)^8}{r^{10}}\right),$$

where r_0 is a constant.

In activity, the students are focussed on solving the given problems, and not, necessarily, on clarifying the definitions and relationships of the concepts that they are making use of. Thus, they are not necessarily thinking about mathematics, physics or chemistry with any clear separation. This explains, for example, why a student, highly qualified in mathematics, when faced with the need to write down an expression for potential energy, initially proposed, and then immediately rejected, the statement that the integral with respect to r of $\frac{1}{r^2} - \frac{(r_0)^8}{r^{10}}$ is $-\frac{1}{r} + \frac{(r_0)^8}{r^5}$, because "half of 10 is 5". Whereas this kind of breakdown is most likely a temporary mistaking of an integration "fact", other kinds of breakdowns may be far more subtle in nature.

For example, it is true that the total force on an ion is the sum of all the individual forces, each of which arises from pair-wise interaction with another ion; it is also true that "energy is the integral of force"; but it is not true, as several students tried to assert, that a "total potential" can be assigned to each ion. Such thinking leads to the construction of a total potential energy expression that contains only equal and opposite terms which sum, at all times, to zero. A characteristic of this kind of breakdown is the *lack of detail* in the knowledge that the students fall back on as they attempt to correct an error or find a solution. But the breakdown can also be a means to develop a correct understanding, and, for two of our student pairs, this zero result provided the stimulus to make the final, correct construction of the energy expressions.

Another reason for factual breakdowns in algebraic calculation may be that the task largely requires working in a graphical mode, where graphs tell a story about ion interactions. For example, Figure 1 shows the total kinetic energy of the 3-ion system against time for an interaction in which, at time zero, a single (Cl^-) ion is fired at a pair of ions that are bound into a molecule (NaCl); there is a collision during which there are large variations in energy, and then a molecular bond is re-established with a free ion travelling off to large distances (hence the repeating

pattern of energy variations as the molecule vibrates). A fact which all of our students missed, but which would have helped them considerably, is that if total energy is conserved then the total P.E. as a function of time must be the reflection of this K.E. graph in the straight line which represents the total energy. Instead of this, the students were faced with having to produce an elusive P.E. graph whose shape they had little idea about.

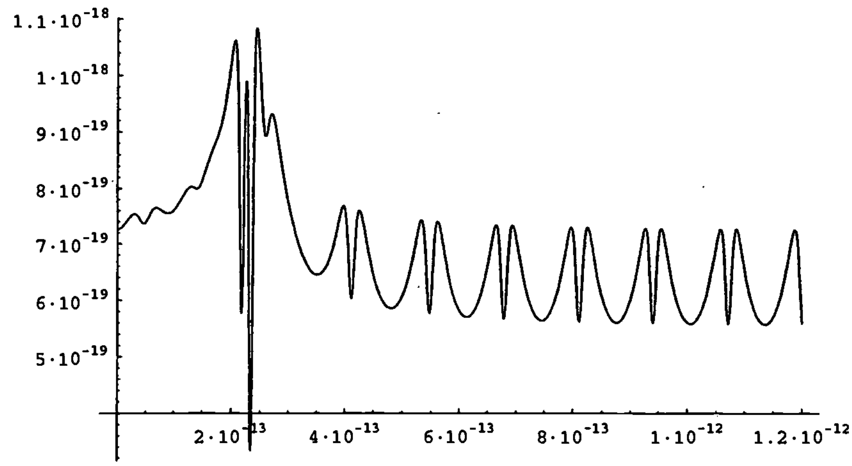

Figure 1: Student graph for the total kinetic energy of the system (in Joules), as a function of time (in seconds).

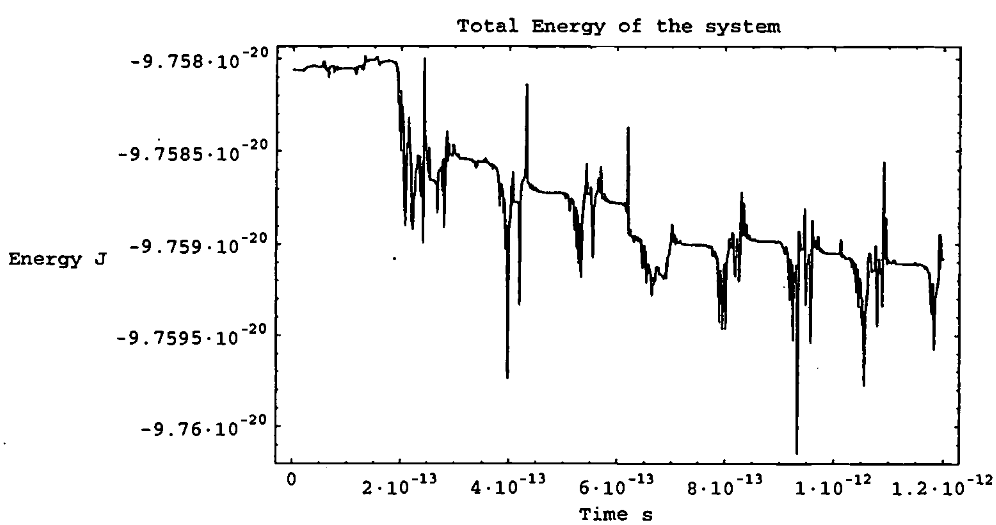

Figure 2: Student graph for the total energy of the system.

In fact the total energy graph is not quite a "straight line" after all—see Figure 2. The limitations of the numerical calculation, as well as the choices made by the graphing algorithm of the software, lead to a graph whose appearance is counter-

intuitive. It requires a critical reading with attention to scales and numerical magnitudes: the variations, apparently huge, are just ±0.01%!

Interviews

Some weeks after completing the ion collision assignment, each of the six students were interviewed for about one hour, and the interviews were audio taped and transcribed. The students were asked about their perceptions of the ion activity, its purpose, and what they felt they had gained from the experience in terms of their chemical or mathematical knowledge. The students were then asked to explain the ion activity to the interviewer (I.S., a non-chemist). Our interest was in how they viewed the "intersection" between mathematics and chemistry, which we hoped to trace out by probing their ideas about how energy and integration are connected.

All the students, as they talked about the energy conservation aspect of the activity, described the potential energy as the integral of force with respect to distance. In a sense, one might be surprised if they had not expressed this connection, since it represents a principle that forms part of the general background of "physical" discourse. The interesting question is what the students meant by this description, and how it related to the chemistry context. In the following interview extract, *A* (a male student with a strong mathematical background) articulates his understanding of the connection. Although the extract is continuous, it has been broken down into three parts to aid analysis.

In response to the question of why integration gives energy, *A* begins by connecting the ion interactions to energy:

Interviewer: Why integrate to give you the energy?

A: erm.... to change potential you're having to do work. If you have two ions that are bonded, simply because they have ... out of the nature of their charges, if you remove one you are doing work on it, and, hence you get the, um, energy.

Next, he connects energy with the area of a graph:

A: You have a force-distance graph, you're then, um, have the work done is the force times the distance... It's the area under that graph will be the, err, work done and hence the change of potential [energy].

Finally, when A is pressed, he connects area and integration:

I: So why is it the area?

A: Why the area? It's because you are multiplying the two axes together.

I: How then does that connect with integration?

A: Integration is a method of determining that area under a curve.

So what has *A* connected? In this sequence he articulates clearly an identification between energy and its representation as the area of a graph. First, *A* relates the

abstract concept of energy with work and its visual representation on a graph. Next he takes a feature of the graph—its area—to express the connection between force and distance, finally introducing integration as a mathematical(?) "method" for calculating the area.

When pressed, the other interviewees also made the connection between energy and the area of a graph, but varied in the degree of coherence and detail of their explanations about their notion of integration. One student tried to describe the process of approximating the area under a graph by the "trapezium rule where you just take loads of rectangles and stuff... eventually until you get to infinity, where you have an exact approximation, rather than an approximation, you get an exact answer". Another student—who had just been trying to revise the mathematics—spoke about limits as "the sort-of value that you are allowing the size of your delta's [displacements] to be". However, neither of these students could explain why the limiting process actually gave the exact value for the area in *every* case or why this might be an important question, from a mathematician's point of view.

Discussion

Most of the students seemed to have developed a working principle from their previous experience that "integrating the force gives the potential energy". This encapsulates a combination of scientific and mathematical knowledge: the change in potential energy is equal to the work done by a particle against a force, and the work done is the integral of the force with respect to distance. The principle "anchors" —to borrow Noss et al's (1998) term—the students, both in the sense of providing them with a secure approach to problems, and in fixing their knowledge in specific chemical contexts.

What can be said about this working principle at the intersection of mathematical and chemical knowledge? The process of trying to break down students' understandings revealed a connection that they made between energy and "area under a graph". It displays an understanding tied to visual images and qualitative arguments, but with varying degrees of analytic content, recalled mostly from somewhat distant school experience. The central role that energy plays in making the connections between chemical and mathematical knowledge suggests that integration—where it is seen as relevant—is understood in a purely functional, tool-like manner. One of the defining characteristics of tool-like behaviour is the transparency of the tool to the user—its "ready-to-handness"—with a consequence that tools become apparent as entities to the user *only* when they fail to be tools (Heidegger 1962, p. 98). The evidence from the observation and interviews suggest that, unless one provokes some kind of breakdown in the functionality of integration, the students will not focus their attention on integration as a concept. This is particularly significant if one is considering how mathematical ideas might be developed in the context of a scientific specialism.

As many others have noted previously (e.g. Harel & Papert 1991, Noss & Hoyles 1996), the demands for formal precision which a programming environment places on its user serve both to expose any fragility in understanding, and to support the building and conjecturing activities required for the (re)construction of concepts by learners. An important difference for our relatively sophisticated undergraduate students, compared with the younger students that have mostly been studied in the literature, may be that the learning experience seems to be largely one of anti-climax, a rather gruelling reorganisation of previous knowledge. We asked the students on several occasions what they felt that they had learned, and the typical response was that they didn't feel that they'd learnt any mathematics or chemistry except "in this particular area". We think that their perception is rather accurate, and that understanding the relationships between knowledge domains is precisely about establishing connections in a sequence of "particular areas".

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CHANGE PROCESSES IN ADULT PROPORTIONAL REASONING : STUDENT TEACHERS AND PRIMARY MATHEMATICS TEACHERS, AFTER EXPOSURE TO *RATIO AND PROPORTION* STUDY UNIT

Yaffa Keret

Levinsky College of Education

Abstract

The reported study examined change processes in knowledge and proportional reasoning among 87 student teachers and 20 teachers. A study unit on "ratio and proportion" was designed for this purpose, following a gradual developmental approach to the proportional scheme, based on the work of Fischbein (1975), Noelting (1980) and others. This approach, as opposed to Piaget and Inhelder's (1958), assumes that the proportional scheme does not mature of its own accord during adolescence. To elicit the maturation of the potential proportional scheme, a process of training is required. The present findings were obtained by means of individual tests, interviews and observations, and they led to the conclusion that since most learners had mastered the concept subsequent to studying the unit they could be considered "proportional reasoners".*

Introduction

Proportional reasoning is a person's ability to effectively use the proportional scheme. This ability plays a central role in the development of mathematical thinking while also being of practical importance in so far as it functions as an action plan that supports problem solving in a variety of fields, eg, mathematics, the sciences, and economics.

Much like any other type of mathematical activity, solving proportional problems requires three main knowledge components: intuitive, formal, and algorithmic knowledge (Fischbein, 1993). Acquisition of these components, and the ability to combine them, equips a student with the concept of proportion by means of which she or he will probably correctly solve ratio and proportion problems (Harel & Confrey, 1994; Thompson & Thompson, 1994).

Let us define a "proportional reasoner" as a person who makes intelligent use of the proportional scheme when solving ratio and proportion problems. This requires a three-step action program: (a) identification of the direct or the inverse proportional relation between the relevant factors - intuitive aspect ; (b) presenting the relation in the form of a mathematical model – formal aspect, and (c) use of this model in finding a quantitative solution to the problem - algorithmic aspect combined with the intuitive and formal aspects.

Piaget and Inhelder (1958) were the first to find that the proportional scheme develops in three stages. They also claimed that, like any other operational scheme, it suddenly develops during adolescence. This may lead to the conclusion that adults are

* This study is part of a doctoral dissertation, written under the supervision of the late Proof E. Fischbein at the School of Education, Tel Aviv University, Israel.

universally in the possession of the proportional scheme, which they spontaneously and intelligently use as an action plan for the solution of ratio and proportion problems. Numerous studies, however, have revealed that reality looks different: adults, at different levels of education, encounter difficulties when solving this type of problem (Fisher, 1988 ; Keret, 1994 ; Tourniaire & Pulos, 1985).

Studies on the development process of the proportional scheme subsequent to the work of Piaget and Inhelder have upheld the existence of the three stages, but contested these authors' developmental approach. Most of the criticism is directed to the types of tasks chosen, and their dependence on physical knowledge. Moreover, it has been argued that Piaget and Inhelder's relative lack of attention to judgement strategy and explanation led them to draw over-generalized conclusions, thereby leaving the developmental process insufficiently elucidated. .

Already in the seventies, Fischbein, Manzat, and Barbat (1975) delineated a gradual-continuous developmental process which depended on learning of the proportional scheme, mainly among children in the concrete stage. These findings were then confirmed by Noelting (1980a,b) and others, who gave children tasks that did not require physical knowledge (Tourniaire & Pulos, 1985). In the nineties, researchers set out to study young children's thinking to reveal patterns that develop prior to the acquisition of formal strategies and the proportion formula. Findings suggest that the strategy quality develops from additive thinking through qualitative to quantitative proportional reasoning. And that the same learning process that makes the development of these patterns possible, also assists in the development of proportional reasoning (Harel & Confrey, 1994; Lamon, 1993; Thompson & Thompson, 1994).

The above studies all share a developmental approach which, in contrast to Piaget and Inhelder, assumes that proportional reasoning develops gradually rather than making its sudden appearance in adolescence. As the child grows older, she or he acquires new action plans on a higher level of difficulty. Learning requires appropriate training - thus the potential proportional scheme comes to mature.

In the present study, this approach was examined with reference to adult participants. For this purpose, a study unit on ratio and proportion was designed which encouraged the development of proportional reasoning. We examined the processes of change in the knowledge of learners of this unit; whether, and to what extent, the study unit improved participants' ability to make intelligent use of the proportional scheme. That is to say, our question was : Did learning transform a dormant proportional scheme into an actualized one?

Methodology

Subjects

The study included 107 participants, subdivided into two groups : (a) 87 student teachers and teachers who were enrolled in two educational colleges, training to be primary mathematics teachers; 55 of these participants studied the study unit and 32

served as controls; (b) 20 primary school math teachers who all studied the subject as part of their inservice studies.

Instrument

1. A diagnostic test on ratio and proportion.

The test was administered both prior and subsequent to exposure to the study unit and was divided into two parts : The first part included 27 ratio and proportion problems of three general types : 13 proportion problems involving direct ratio, 9 with inverse ratio, and 5 problems referring to other "ratio" problems. Respondents were asked to solve each problem by means of more than one solving strategy, and to write down every stage in the solving process. An unexplained quantitative response was considered incorrect.

Problems referred to a variety of knowledge areas, referring to both daily situations and to more particular phenomena like balance and scale. These problems were content analyzed to represent problems involving proportional relations that were either obvious, or suggested or concealed, as well as problems including different types of numbers, i.e. integers, rational numbers, and percentages.

The second part included 10 open questions that required respondents to define and explicate the following concepts : ratio, proportion, the relation between two parts of a whole (e.g. the relation between profit and investment or the number of boys and girls in class), balance, speed, the difference/similarity between ratio, percent and division.

2. Individual, 20-30 min. interviews: these were conducted following exposure to the study unit among a sample of learners: 15 student teachers and teachers from the colleges and five practicing teachers.

Interviewees were asked to give verbal explanations to how they had gone about solving the problems in the first part of the diagnostic test. This made it possible to determine whether, and to what extent, they had been using the proportional scheme, i.e., whether they had been consciously aware of a direct/indirect proportional relation, and whether they could intelligently explain their choice of a mathematical model and its working. The interviews also offered an opportunity to investigate the source of typical errors.

3. A specially designed study unit on ratio and proportion, requiring eight weekly hours of teaching-learning.

The study unit was constructed on the basis of two main principles : (a) mastery of the intuitive, the formal, and the algorithmic components of knowledge in this topic and the ability to combine them, may assist to develop proportional reasoning, and (b) the importance of inclusive, extensive presentation of the topic that refers to both its mathematical and psychological-didactic aspects (Even & Tirosh,1995).

Learners were exposed to problems relating to different knowledge areas and to various solving strategies (preferably more than one strategy per problem) using

proportional reasoning. Learners were introduced to published research concerning proportional reasoning proposing different teaching techniques for conceptualization of ratio as a part of the development of mathematical thinking.

Results

Two types of results were yielded: (a) solutions to problems and verbal written statements, generated both before and after learning, and (b) interviewees' verbal statements, post learning.

The statements were content analyzed and the solutions were examined with reference to four categories :

1. Correctness of solution

Results generally indicate substantial improvement in ability to solve ratio and proportion problems following use of the study unit, while controls showed no change. Post-study unit, 86% succeeded to produce correct solutions. This is true for a variety of problems, relating to both standard and non-standard contexts, and including both obvious, submerged and concealed proportional relations.

We shall present a sample of problems (Table 1), which were solved correctly by only a small proportion of participants before exposure (not exceeding 30%, and sometimes less than 20%), and which, subsequent to learning, were answered correctly by a much larger percentage of participants (an increase of 50% or more).

Table 1: Distribution (percents) of a sample of problems and correct answers (pre/post course)

No	Type	Problem content	Pre	Post
1	Direct ratio problems	For each 2 turns of the wheels of a bike, its pedals turn 5 times. What is the speed of the wheels, if we know that the pedals are turning at 60 times per minute?	27	79*
2		The shortest, air-distance between two villages is 40 km. What would be the length of the straight line representing their distance on a map whose scale is 1: 200,000 ?	16	76*
3	Inverse ratio problems	If 6 workers can paint a hall in 7 hours, how long will the same hall take 4 workers? (All workers have the same output and they all work all of the time.)	30	92*
4		On an antique bike, the front wheel is large, with a circumference of 462 cm, while the smaller back wheel measures 132 cm. What distance has the bicycle covered if we know that the back wheel has made 30 turns less than the front wheel?	19	77*
5	Other "ratio" problems	The ratio between the areas of two circles is 1: 9. What is the radius of the larger one, if we know that the smaller one is 5 cm ?	10	73*
6		We have a chart whose scale is 1:400. If you photocopy the chart and reduce it by two, what will be the scale of the resulting chart?	15	64*

$*p < .01$

2.Solving strategy

Both teachers and student teachers, pre- and post-exposure, select from the same wide variety of solving strategies. All chosen strategies share a multiplicative orientation, as suits proportion problems (Tourniaire & Pulos, 1985). While the unit did not affect the variety of strategies in use, it did influence two other aspects:

(a) Post-learning, most participants succeeded to solve at least some of the problems by means of more than one strategy - 22% of the problems received more than one solution. This was especially pronounced for the direct ratio problems (41%). And (b) post-learning, the quality and correct usage of chosen strategy improved significantly (see Table 2). Even though 71% of the participants chose proportional strategies before the study unit, most of these were at the pre-formal stage (58%). Post-learning, 89% of correct answers used proportional strategies, with about half (43%) using proportion formula correctly.

Table 2: Distribution (percents) of correct use of strategies (pre/post course)

Chosen type of strategy in learners group			Pre course	Post course	Difference
Proportional Strategies	Pre-formal	Qualitative	40	31	- 9
		Quantitative	18	15	- 3
	Total of pre-formal strategies		58	46	- 12
	Formal	Proportion formula	13	43	+ 30
Sum total of proportional strategy choices			71	89	+ 18
Other strategies	Algebraic equations		12	6	- 6
	Other arithmetic operations		17	5	- 12
Sum total of other strategy choices			29	11	- 18

The change in the quality of strategies used (i.e., more correct applications of the proportion formula, especially as related to inverse ratio problems) suggested that this finding might serve as a useful indicator of proportional reasoning.

3. Extent of intelligent use of proportional scheme -proportional reasoning

Proportional reasoning is a mental process, which is put into operation when a person solves a proportional problem. We looked at our findings concerning solving strategies, correct solutions, and content analysis of written and spoken explanations to solutions to examine this process. All these data were needed because correct answers alone do not always provide unambiguous information about intelligent use of the proportional scheme. Correct technical use, for instance, of the formulas might lead to successful solution of direct ratio problems, despite the absence of proportional reasoning. The same cannot be said about inverse ratio and other types of ratio problems, where correct solutions are not so likely to ensue from technical ability alone.

To examine degree of proportional scheme use we examined the findings with reference to three measurable features that can be derived from our above definition of a 'proportional reasoner' : 1. ability to identify a proportional relation (intuitive and formal aspects interacting) - reflected in choice of solving strategy with a multiplicative rather than an additive orientation ; 2. ability to differentiate between inverse and direct ratio (intuitive and formal aspects interacting) - reflected in application of solving procedure fitting each type of relation, and 3. ability to find a quantitative solution (algorithmic aspect) - reflected in the appropriate algorithmic activity for particular problem.

As regards (1), the findings that indicate the absence of any attempt to employ an additive strategy suggest that all participants, both pre- and post-training, are able to identify a multiplicative relation, whether consciously or unconsciously. In regard of feature (3), after exposure to the study unit, 86% of the participants, on average, correctly solved most of the problems. We may therefore conclude that they have good algorithmic ability, which helps them find a correct, quantitative solution to the problem.

In the case of (2) - ability to distinguish type of ratio - it was more difficult to draw conclusions. We combined three types of findings : 1. The finding that after training, in 63% of correctly solved inverse ratio problems the participants used the proportion formula allows us to conclude that these participants correctly distinguished between the two types of ratio. They must have consciously used this ability when they referred to the proportion formula, because without this ability they most likely would have failed. Their high rate of success can be taken to indicate high correct use of this ability. 2. We used content analyses of statements accompanying correct answers illustrate that participants are aware of the need to check the type of ratio prior solving problems. Classification was done by answering the question what happens to one element when the other one is de/increased. Once conclusions have been drawn, participants appear to be using the fixed ratio model for the direct ratio problems and the fixed multiplicative model for the inverse ratio problems. And finally, 3. We used content analyses of spoken statements deriving from the interviews indicate the same, as it is presented in Table 3.

Table 3 : Statements related to problems in Table 1, after learning

No	Type	Spoken statements in response to the question: Please explain your considerations as you were solving this problem.
1	Direct ratio problems	"If we teach the notion of speed as a description of movement, students will have no problem understanding that, here, the greater the number of turns, the faster the bike will go- and therefore what we have here is direct ratio."
2		"Scale is really nothing but the description of a direct ratio between the distance on the map and distance in reality -that's also how you should present it to students: reduction or enlargement of the real world according to a given ratio."
3	Inverse ratio problems	"The larger the number of workers, the less time it will take them to finish the job. That means an inverse relation between number of workers and days of work necessary to complete the job."
4		"Here the product between the circumference of the wheel and the number of turns it makes is fixed, and the larger the circumference of the wheel, the smaller the number of turns - inverse ratio."
5	Other "ratio" problems	"Unlike the direct ratio between the radius of a circle and its circumference, here we need to get the ratio between radius and area. It looks like direct ratio (as the one increases, so does the other), but the factor changes, and therefore we must find a new proportion, which is $S_1 / S_2 = r_1^2 / r_2^2$."
6		" The more we reduce the chart, the larger- by the same factor (2)- will its representation per centimeter on the map become. So the new scale will be 1:800."

We can conclude that exposure to the unit made participants widely apply all three features of the ratio and proportion solving procedure. The findings suggest that most

participants had mastered the topic, and that they had combined intuitive, formal and algorithmic elements. Participants had acquired the concepts and were in command of proportional reasoning.

4. Analysis of typical errors

In 90% of the erroneous solutions obtained before exposure, the chosen strategy had been wrongly applied. This came about in three ways : (a) erroneous identification of type of ratio (45%) : applying procedures suitable for direct ratio problems to inverse or other types of ratio problems ; (b) language error (8%) : the fact that Hebrew writing moves from left to right may lead students to put the nominator in the place of the denominator; (c) wrong performance of procedure (37%) : wrongly isolating the unknown for solving the equation ; multiplication instead of division ; erroneous change of measures ; always dividing a larger number by a smaller one, etc. (This type of mistakes is typical especially for direct ratio problems.)

It should be noted that after training, very few errors were made (7.5% of all responses were wrong, and 7.5% of the participants did not respond at all).

Discussion and Conclusions

Even though the development of proportional reasoning is an issue of major importance, the topic of ratio and proportion receives scant attention in the colleges in Israel that prepare math teachers. The present findings suggest that inclusion of a study unit on the topic in teachers' training will go some way toward developing teachers who can use proportional reasoning.

This study unit should present the relevant concepts through a wide scope, and allow teachers to practice action strategies that involve intelligent use of the proportional scheme. Moreover, teachers should be exposed to problems referring to various, standard and non-standard, knowledge areas, and formulations should be of both the obvious, indirect and concealed types.

The present findings show that exposure to a study unit that was constructed on these principles caused most participants to acquire the three knowledge components and the ability to use them in combination. This in turn led to ratio and proportion conceptualization and to the maturation of the dormant, potential proportional scheme. This could be seen from the fact that most participants, after learning, made intelligent use of the proportional scheme as an efficient mental tool for the solving of proportion problems relating to various knowledge areas. They successfully identified direct or inverse ratio, expressed these by means of mathematical models, and made intelligent use of such models in problem solving. This is tantamount to concluding that they had become 'proportional reasoners'.

This conclusion has some significant educational implications : First, we can say that with the right type of learning, adult students' concepts of ratio and proportion and their ability to use the proportional scheme can be significantly extended. In this respect our results confirm the findings of Fischbein and others concerning children.

Second, it is important to include the topic of ratio and proportion - taught with reference to both mathematical as well as psychological-didactic aspects - in mathematics teacher training and inservice training. This is likely to raise participants' awareness of the importance of mathematical conceptualization of the topic, and of the development of proportional reasoning as part of the development of mathematical thinking.

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THE INTRICATE BALANCE BETWEEN ABSTRACT AND CONCRETE IN LINEAR ALGEBRA

Alkistis Klapsinou & Eddie Gray

Mathematics Education Research Centre
University of Warwick
Coventry CV4 7AL

Abstract

This paper focuses upon the strengths and weaknesses of disparate approaches to Linear Algebra. By identifying the theoretical distinctions between the abstraction-to-computation approach and the computation-to-abstraction approach, it presents examples of how one lecturer, recognising the cognitive obstacles associated with the nature of Linear Algebra, used the latter in the development of a first-year University course. Student reaction suggests that this laudable effort may be addressing procedural difficulties, but compounding conceptual ones, since the delivery of advanced mathematics material in a 'concrete' manner, can militate against the use of concept definitions and this may have broader implications for further mathematical development.

Introduction

Linear Algebra is one of the first courses of advanced mathematics at University level. Along with Analysis, it is intended to shift the students' way of thinking from school mathematics towards advanced mathematical thinking. It is probably the first 'real' mathematics course that students have to encounter, since it requires limited mathematical prerequisites, yet its theory is systematically built from the ground up (Tucker, 1993; Hillel & Sierpinska, 1994). In addition, Linear Algebra brings together methods and insights of geometry and algebra, and its wide range of applications in modern mathematics make it an essential component of all scientific courses (Tucker, 1993). Most importantly though, students have to become familiar with its main themes, such as vector spaces and linear maps, since they are central in the further development of pure mathematical theory (Tucker, 1993).

This paper intends to summarise the existing literature concerning the cognitive obstacles within Linear Algebra and some of the teaching methods employed to overcome these difficulties. Also, it considers how a particular Linear Algebra course was delivered, having those difficulties in mind, and what the impact was on a small group of high achieving students. It concludes by suggesting that there should be a balance between concrete and abstract approaches in Linear Algebra, however difficult to achieve, since, in some instances, in our effort to solve a problem we may create a new one.

Difficulties within Linear Algebra

During their pre-university courses, mathematics students will have met some components of the course, such as matrix arithmetic and solution of simultaneous linear equations (AEB GCE Syllabuses, 1999; NEAB GCE A/AS Syllabuses for 1998) . This, unfortunately, does not guarantee a smooth transition to the stark (?) Linear Algebra. On the contrary, as Hillel and Sierpinska (1994) argue, "both the teaching and learning of linear algebra at the university level is almost universally regarded as a frustrating experience" (p. 65).

Some of the reasons for the difficulties faced by the students are not confined to the content of Linear Algebra, but are a result of the transition from elementary to advanced mathematics.

> The move from elementary to advanced mathematical thinking involves a significant transition: that from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions. This transition requires a cognitive reconstruction which is seen during the university students; initial struggle with formal abstractions as they tackle the first year of university. It is the transition from the *coherence* of elementary mathematics to the *consequence* of advanced mathematics, based on abstract entities which the individual must construct through deductions from formal definitions. (Tall, 1991, p. 20)

Linear Algebra, though, has certain particularities which can also impede students' learning and understanding. The heart of these is that Linear Algebra was developed to unify, simplify and model already existing problem solutions, rather than to solve new problems (Harel &Trgalová, 1996). Students can solve many problems within a linear algebra course without using the relevant theory but by mere implication of direct manipulation techniques (Hillel, & Sierpinska, 1994).

Harel & Tall (1991) distinguish between three types of generalisations within advanced mathematics; *expansive, reconstructive* and *disjunctive generalisation*, and argue that the successive generalisations of vector sum and scalar multiples from $\mathbf{R}^2$ to $\mathbf{R}^3$ to $\mathbf{R}^n$ are an expansive generalisation for the students, since it involves "applying the same techniques to each coordinate in successively broader systems" (p. 39). The passage from $\mathbf{R}^n$ to the abstract concept of a vector space, on the other hand, requires a re-constructive generalisation.

> The learner is presented with a name for the concept ("the vector space V") and some of its properties (the axioms) and –usually guided by an expert–must follow a subtle and difficult process of construction and meaning of V and its properties by deduction from the axioms. This is further complicated in the learner's mind by the fact that the properties *to be deduced in V are known* to hold $\mathbf{R}^n$, causing the problem for the student that, although these properties are "obvious" in the (only) examples (s)he understands, judgement must be suspended on their truth in V until they are shown to follow by deduction from the axioms. (Harel & Tall, 1991; p. 39)

Even though the theory of Linear Algebra is universally applicable, when it comes to solving problems, there is a wide variety of algorithms for any certain task, with the restriction that different algorithms work in different settings. Thus, students are faced with the further difficulty of having to decide which is the most appropriate algorithm to tackle their problem. Carlson (1993), for example, notes that the procedure needed to find a basis for a vector space of row vectors is different than that to find a basis for a vector space of functions.

An additional disadvantage of the unifying character of Linear Algebra is the variety of representations that students have to get accustomed to. The word 'vector', for example, firstly introduced in the context of concrete $\mathbf{R}^n$ subspaces, can mean different things depending on the corresponding vector space. Hillel & Sierpinska (1994), argue that the initial representation of a vector as a string of numbers, becomes shaken when students realise that the representation of a vector depends on the choice of basis.

As linear algebra is one of the first undergraduate mathematical courses, students are required –probably for the first time– to deal with abstract concepts instead of numeric manipulations (Carlson, 1993). They have to start "thinking about the objects and operations of algebra not in terms of relations between particular matrices, vectors and operators but in terms of whole structures of such things: vector spaces over fields, algebra's, classes of linear operators, which can be transformed, represented in different ways, considered as isomorphic or not, etc." (p. 65).

This particular difficulty is not restricted to the context of Linear Algebra, but is common in almost all areas of mathematics. To understand a new notion in elementary mathematics students have to undergo a cognitive shift incorporating lengthy procedures in mathematical concepts. This conversion of actions or operations into what Piaget (1945) described as "thematised objects of thought or assimilation" (p. 49) was described by the term *encapsulation* (Dubinsky, 1991).

Cottrill, Dubinsky, Nicholls, Schwingendorf, Thomas & Vidakovic (1996) formulated the APOS theory, from the acronym of the words action, process, object and schema. *Actions* are physical or mental transformations of objects to obtain other objects. When these actions become intentional they are characterised as *processes* which may be encapsulated to form a new *object*. A coherent collection of these actions, processes and objects, linked in some way, is identified as a *schema*. A schema can be reflected upon and transformed and thus result in the formation of a new object.

The disadvantage of such an approach in Advanced Mathematics is that the students who are taught in this manner are not given a formal definition of the new object until the end –if then– of this whole learning process. Vinner (1991) argues that "it is hard to train a cognitive system to act against its nature and to force it to consult definitions either when forming a concept image or when working on a cognitive task" (p. 72). This situation can only get worse if the students' concept image has been built through actions and processes, without considering the concept definition.

The course

The Linear Algebra course under consideration took place in the Mathematics Department of a very demanding British University. The duration of the course was 30 hours, split into hourly sessions three times a week, for ten weeks in the second term. The lecturer provided the students with complete and explicit notes, so that they could concentrate on understanding the material, instead of keeping their own notes. There was also a recommended textbook, which was Anton's (1994) *Elementary Linear Algebra*. We should also note that the course was designed not only for Pure Mathematics students but also for students following combined degrees (Mathematics & Physics, Mathematics & Statistics, Mathematics, Operational Research, Statistics & Economics (MORSE)), a fact which explains the process-oriented nature of the course.

There seem to be two ways of sequencing the contents of Linear Algebra; the computation-to-abstraction approach and the abstraction-to-computation approach (Harel, 1987). The first approach suggests that matrix arithmetic and linear systems should precede vector spaces and linear transformations, in order to enable the students to develop the language and reasoning needed for understanding the more abstract material. The second approach starts with vector spaces and linear maps and then matrices and simultaneous linear equations are treated as applications of the former.

In this particular Linear Algebra course the approach chosen was the computation-to-abstraction one, because the lecturer felt it would be more beneficial to start with already familiar concepts and use them as building blocks for the development of the more abstract notions of vector spaces and linear transformations. Various introductory strategies were used in order to present the new material, such as 'abstraction' –introducing abstract ideas by initially illustrating them by specific examples (Harel, 1987)– and 'embodiment' (Dienes, 1960) –translating definitions and theorems in terms of given situations. The difference between these two processes lies in the timing of the presentation of the particular situation; in abstraction it comes before the concept is defined, whereas in embodiment, it follows the formal definition.

In order to demonstrate how these teaching techniques were employed, we have two extracts from the lecture notes; the first extract is a case of abstraction and the second of embodiment. These examples were chosen so as to reflect the nature of the delivery of the whole course.

<u>Abstraction</u>

Preparing for Eigenvalues and Eigenvectors

1. Draw a set of axes on a piece of paper.
2. Choose a vector from $\mathbf{R}^2$ and draw it on the axes.
3. Now multiply it be the matrix $A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ to get a new vector. Draw the new vector you get.
4. Repeat these three steps 3 or 4 times, choosing a new vector each time. Notice that A sends

vectors all over the place.

5. Now draw a new set of axes and plot the vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\-3\end{bmatrix}$.
6. Multiply each of these vectors by A and draw the result. Notice that these vectors stay on the line they started on. They have just been "stretched" in a positive or negative sense.
7. In these two cases,
$$\begin{bmatrix}2&2\\3&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}=4\begin{bmatrix}1\\1\end{bmatrix} \text{ and } \begin{bmatrix}2&2\\3&1\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix}=\begin{bmatrix}-2\\3\end{bmatrix}=-1\begin{bmatrix}2\\-3\end{bmatrix}$$
8. So, for certain vectors, multiplication be A just results in a "stretching" (which can include a change of direction) of the vector; in other words a scalar multiplication of the vector. Such a vector is called an **eigenvector** of A. The "amount of stretch" undergone by such a vector is called an **eigenvalue** of A.
9. So $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\-3\end{bmatrix}$ are eigenvectors of A, and 4 and -1 are corresponding eigenvalues of A.
10. In general, an eigenvector of A and its corresponding eigenvalue are related by the matrix equation
$$\mathbf{Ax}=\lambda\mathbf{x}$$
where **x** is the eigenvector and λ is the eigenvalue.

Embodiment

Definition We define the **adjoint matrix** of A, denoted adj A, to be the *transpose* of the cofactor matrix

Example $A=\begin{bmatrix}2&3&1\\4&1&2\\5&3&4\end{bmatrix}$

We replace each element of the matrix with its cofactor, to get the *cofactor* matrix, C.

$$C=\begin{bmatrix}\begin{vmatrix}1&2\\3&4\end{vmatrix} & -\begin{vmatrix}4&2\\5&4\end{vmatrix} & \begin{vmatrix}4&1\\5&3\end{vmatrix}\\ -\begin{vmatrix}3&1\\3&4\end{vmatrix} & \begin{vmatrix}2&1\\5&4\end{vmatrix} & -\begin{vmatrix}2&3\\5&3\end{vmatrix}\\ \begin{vmatrix}\,\end{vmatrix} & -\begin{vmatrix}\,\end{vmatrix} & \begin{vmatrix}\,\end{vmatrix}\end{bmatrix}=\begin{bmatrix}-2&-6&7\\-9&3&9\\5&0&-10\end{bmatrix}$$

We transpose the cofactor matrix to get the *adjoint* matrix, adj A.

$$\text{adj } A = C^T = \begin{bmatrix}-2&-9&5\\-6&3&0\\7&9&-10\end{bmatrix}$$

Another strategy used for the introduction of vector spaces, in particular, was consistent with the APOS theory (Cotrill *et al.*, 1996). Starting with vectors in $\mathbf{R}^n$, objects already familiar to the students, addition and scalar multiplication were defined, initially as actions on these vectors. When these actions were interiorised, along with their properties (the 10 vector space axioms), they became processes, which then were used to form the new object 'vector space'. These notions were later

extended to include vector spaces other than $\mathbf{R}^n$, resulting in the schema of a general vector space.

The students

The fact that this research is taking place in a highly regarded British Universities, as we noted earlier, means that the students who take part have a solid mathematical background, as indicated by the results in their A-level exams (University entrance requirements in Mathematics is 3 A's) and their initial 'mathematics techniques diagnostics test'. Our sample consisted of 8 high achieving Pure Mathematics students, seven of whom achieved first class grades in the course assessment (90% written examination, 10% awarded through weekly coursework).

The research was carried out by fortnightly videotaped group discussions (with two interviewers taking part), where the issues brought forward were associated both with the delivery of the course and the students' understanding of the new notions. Some issues were brought up by the students themselves, initiated by certain difficulties they faced when doing their coursework.

In the following extracts from the transcripts of the discussions, the talk revolves around the notions of vector space, linear dependence and independence, bases and spanning sets. Having under consideration that the definition of vector space was introduced in the APOS pattern (as described above), we can clearly see that the students' concept image of a vector space remains restricted to $\mathbf{R}^n$ spaces.

Int: Why are you always using the examples of $\mathbf{R}^2$, $\mathbf{R}^3$, ..., $\mathbf{R}^n$?

J: Because these are the easiest examples of vector spaces.

Int: But why are you associating the word 'vector' with vectors in space? We defined vector space as a set that satisfies those 10 axioms and any element of this vector space is a vector. So it does not have to be a vector in $\mathbf{R}^n$.

L: Because when we were first taught about vectors they were vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$. And that's what is still in my mind about what a vector is. I mean we have managed to extend it to $\mathbf{R}^n$ but not to other sets.

Int2: Does the previous image of vectors, the 2 or 3-dimensional image actually get in the way?

A: Yeah, actually it does.

When asked what the basis of a vector space is, no student gives the formal definition but they instead give an answer that applies only in a very specific context, in the same way that Carlson (1993) reports that "basis of a vector space" is for some students "the result of one specific algorithm applied in one specific context" (p. 29).

Int: What is the basis of a vector space?

J: Any two linear independent vectors.

Int: Can you have two linear independent vectors being a basis for $\mathbf{R}^4$?

J: Why not?

I: For a vector space of order n you need n linear independent vectors to span it.

A: Should the elements of a basis be all linear independent?

J: Yeah.

On the issue of the course delivery, the students seemed to have divergent views. Some appreciated the clarity and completeness of the setting out of the material, as well as the provision of algorithms for carrying out tasks, but for others the essence of mathematics lies in the challenge of having to struggle with theorems and proofs.

J: In specific cases there are tricks or rules you can use and it makes things easier than if you take the general approach to it. You have the general rule to cover yourself if you have any doubts.

Int: But why do you prefer to use these tricks?

J: Because it makes life easier. Would you use the definition of derivative in order to find the derivative every time?

Int: But is it not essential to know how you came up to this derivative, to know how to prove it?

I: It's not that essential to always carry in your head how to prove it, but to know where the results comes from and how it was derived. I think you'd be better off with knowing why it's true; that's what matters to me. If you just want to know the results you might as well study Physics!

L: Yeah, in applied maths we just had to use the tools, we didn't have to prove them.

Int: Do you think that the shift from school to University would be made easier if they gradually tried to change your way of thinking from the A-level to the more abstract?

J: There's something to be said about the 'shock treatment'. I think the sooner you get into the new way of thinking the better.

A: I think I'm learning the Analysis better because the lecturer threw us more or less straight in the deep end at the beginning of the course and we had to cope with it.

J: If you do the gradual thing then you might get the wrong impression. A year or something in the course you might realise "this isn't really for me". It's more honest to have the 'shock treatment'; this is what it's really like.

Conclusion

The position of Linear Algebra, as an initial course of advanced mathematics, implies that a bias towards either concrete or abstract approaches may cause difficulties. In this context, developing a Linear Algebra course based on the computation-to-abstraction approach does not imply that the students will not be introduced to both the concrete and the abstract aspects of the course. However, putting the emphasis on the 'concrete', may sway the students towards overlooking the 'abstract'. As a result, their concept images becomes enriched in processes but not in objects.

It is widely accepted that undergraduate students have to make a huge cognitive shift from school mathematics to advanced mathematical thinking. To this end, it is imperative that the balance between concrete and abstract is not only maintained but safeguarded in every possible manner.

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TEACHING MATHEMATICS USING EVERYDAY CONTEXTS: WHAT IF ACADEMIC MATHEMATICS IS LOST?

Hari Prasad Koirala
Eastern Connecticut State University

Mathematics education researchers argue that mathematics should be taught using everyday contexts so that the learning of mathematics can be meaningful to students. Although the learning of mathematics through everyday contexts is interesting for most students, many of them cannot make a leap from these contexts to academic mathematics. Because of this difficulty, teachers need to make a deliberate attempt to help students connect everyday and academic mathematics.

Introduction

During the past two decades, mathematics education researchers, who are especially interested in ethnomathematics, have explored the relationship between mathematics in and out of school (D'Ambrosio, 1985; Gerdes, 1996; Nunes, 1992). Out of school mathematics is usually carried out in everyday setting, which is very different from an academic setting of schools (Carraher, Carraher, & Schliemann, 1985; Saxe, 1991). While academic mathematics is still viewed as a culture and context free discipline, mathematics in everyday settings is determined by socio-cultural backgrounds of students. Ethnomathematicians, however, believe that mathematics both in and out of school must be based on socio-cultural practices of students. According to Gerdes, ethnomathematics researchers "emphasize and analyze the influences of socio-cultural factors on the teaching, learning and development of mathematics." (p. 917).

The psychology of mathematics education has been deeply influenced by the findings of the research carried out from a socio-cultural perspective. In the 1998 annual meeting of the International Group for the Psychology of Mathematics Education, one major theme of research was focused on mathematics in and out of school. The researchers in the meeting argued that the learning of mathematics becomes meaningful to students if their own cultural contexts are used in mathematics classrooms (Civil, 1998; Presmeg, 1998). If mathematical concepts, ideas, and skills are developed through students' everyday contexts, they may be more motivated to learn and develop a better understanding of mathematics.

The task of connecting students' everyday contexts to academic mathematics is not easy (Lave, 1988; Saxe, 1991; Walkerdine, 1990). These educators argue that students construct their everyday experiences in contexts different from the school context, and transferring ideas from one context to another is hard because the emergent goals are different. This may explain the discrepancy between school and

out-of-school mathematics experiences reported by numerous educators (Carraher, Carraher, & Schliemann, 1985; D'Ambrosio, 1985; Nunes, 1992; Saxe, 1991).

Although the above researchers have shown a discrepancy between school and out-of-school mathematics, their main focus was not to see whether students' mathematical abilities could be enhanced when taught using their own everyday contexts. Nor was their focus to investigate students' feelings and interests about mathematics when taught through such contexts. Instead, these researchers focused on how students' understanding of mathematics is embedded in their culture and personal experiences. More research is needed to demonstrate how students can be helped to learn academic mathematics using students' socio-cultural contexts. In this paper, I examine the influence of everyday social contexts in the teaching of mathematics to future elementary school teachers.

Methodology of the study

The data for this study were collected from a group of preservice teachers enrolled in a course entitled "Number Systems" taught by this investigator at Eastern Connecticut State University. The majority of the preservice teachers in this group did not have a sound mathematical background. Only about 10% the preservice teachers had taken some advanced mathematics courses such as calculus. For the majority of the preservice teachers this was their first college mathematics course. These preservice teachers did not have a good experience of learning mathematics in schools. They feared and even hated mathematics.

The purpose of the course was to teach academic mathematics using students' everyday contexts as far as possible. All students were required to keep journals throughout these courses describing their mathematical understandings and feelings. Students were evaluated based on various quizzes, class presentations, journals, midterm and final examinations. In all these evaluations, they were required to demonstrate their understanding of academic mathematics. Although various types of problems were asked to the preservice teachers the following problem called a "shopping problem" was used to evaluate preservice teachers' ability to understand academic mathematics using their everyday cultural contexts. The problem was modified from one of the textbooks usually used in the "Number Systems" course and represents the types of everyday socio-cultural contexts used in the class.

> Two friends are shopping together when they encounter a special "3 for 2" shoe sale. If they purchase two pairs of shoes at the regular price, a third pair (of lower or equal value) will be free. Neither friend wants three pairs of shoes, but Pat would like to buy a $56 and a $39 pair while Chris is interested in a $45 pair. If they buy the shoes together to take advantage of the sale, what is the fairest share for each to pay? (Adapted from Musser & Burger, 1997, p. 15)

The above problem was asked as a pilot problem to a group of preservice teachers in the previous year. The majority of the preservice teachers in that year did not provide a mathematical response to this problem. Their responses varied from "since Pat and Chris are friends they could divide the saving in any way they wanted," "I would not worry about the split but treat ourselves with a good lunch," to "just split the savings of $39 evenly". Because of these general responses obtained from preservice teachers in the first year, preservice teachers in the succeeding year were specifically asked to provide their *mathematical reasoning* to the problem. All 32 preservice teachers in the class wrote their responses in the blank sheet of paper provided by the investigator.

After the analysis of the written responses, a total of six preservice teachers were interviewed to explore more about their solutions. The selection of preservice teachers was purposive in order to include preservice teachers of various ability levels. Since the interviews were conducted after the completion of the course, final grades of the students were used to determine their ability levels. One preservice teacher was a high achiever who obtained *A* in the course. Four were middle achievers, who got *B*'s in the course. One was a low achiever, with a *C*. Each interview lasted approximately 20-30 minutes. During the interviews, the preservice teachers were shown their written work and asked why they chose their responses. They were told that they could change their responses if they wanted.

Data Analysis Procedures

Preservice teachers' written responses to the problem were categorized, coded, and tabulated to determine their frequencies. Each interview tape was first audiotaped and then transcribed. Preservice teachers' strategies of solving the problem were determined by analyzing their written responses. The responses to the interview were used to determine the reasons why preservice teachers chose their solutions in the written task. Interview transcripts were also used to determine whether or not preservice teachers were consistent in their thinking.

Results and Discussion

All 32 preservice teachers in the class agreed that they commonly encounter these kinds of sales in their everyday life in the United States. However, the majority of them did not provide an appropriate academic solution to the problem. Since they had studied ratio, proportion, and percent in the class, an appropriate mathematical solution to this problem would have been to use a method to determine the amount of savings to Pat and Chris on a proportional basis. The following two solutions are considered appropriate based on the teaching in the classroom:

(i) The total cost of three pairs of shoes is $56+$45+$39=$140. The cost for Pat is $56+$39=$95 and the cost for Chris is $45. Since there is a saving

(ii) of \$39 out of \$140, Pat should save $39\times\frac{95}{140}=\$26.46$ and Chris should save $39\times\frac{45}{140}=\$12.54$. So Pat should pay \$95-\$26.46=\$68.54 and Chris should pay \$45-\$12.54=\$32.46.

(iii) Pat and Chris save a total of \$39 out of \$140. So their percent saving is $\frac{39}{140}\times 100=27.85\%$. Hence, Pat should save 27.85% of \$95, which is \$26.46 and Chris should save 27.85% of \$45, which is \$12.54. So Pat should pay \$95-\$26.46=\$68.54 and Chris should pay \$45-\$12.54=\$32.46.

Not a single preservice teacher in the class gave one of the above two responses. Their solutions widely varied. Out of 32 respondents, eight said that "Pat should pay 2/3 of the price and Chris should pay 1/3. The total cost of the three pairs after the saving is \$56+\$45=\$101. Two thirds of \$101 is \$67 (rounded to the nearest dollar) and one third is \$34 (rounded to the nearest dollar). Hence Pat should pay \$67 and Chris should pay \$34." Here is a representative response from a preservice teacher:

> Chris and Pat would need to add up the cost of the two pairs of shoes that cost the most money because in a "3 for 2 sale" you pay the price of the two most expensive ones. The total of the two most expensive shoes would be \$56+\$45=\$101. Pat wants two pairs of shoes and Chris is only getting one pair, they need to divide \$101 by 3. The result when divided by 3 is \$33.66. Chris who is getting only one pair of shoes would pay 1/3 of the cost as \$33.66. Pat who is getting 2 pairs should pay 2/3 of the cost, which would be \$67.32. And everyone is happy. Chris saved \$11.34 off of the original \$45 and Pat saved \$27.68 off of the two pairs of shoes.

The above solution is clearly communicated. However, the shares are not distributed proportionally based on the cost of shoes.

Approximately the same proportion of the preservice teachers (seven out of 32) thought that both Pat and Chris should divide the savings of \$39 evenly. They said if there was no sale Pat would have paid \$56+\$39=\$95 and Chris would have paid \$45. Hence "Pat should pay \$95-\$19.50=\$75.50 and Chris should pay \$45-\$19.50=\$25.50." For example,

> The fairest thing would be to split the savings on the free pair in halves (\$39/2=\$19.50) and use the \$19.50 to subtract from the total price of the shoes they want. So the amount for Pat to pay is (\$56+\$39)-\$19.50=\$75.50 and for Chris to pay is \$45-\$19.50=\$25.50.

Nobody was worried that the sharing using the above method was unfavorable to Pat. He was saving only 20.5% of his original price of \$95 whereas

Chris was saving 43.3% of his price of $45. When this was brought to their attention in the interviews, they argued that the sharing was still fair because both of them were willing to spend their original cost if there was no sale. Moreover according to these respondents they should not be arguing about how to split this money because both of them are friends.

Five preservice teachers decided to split the saving of $39 into three parts and provide $26 to Pat and $13 to Chris. Hence for them the fairest shares would be that Pat pays $69 and Chris pays $32. Here is one response that exemplifies this method:

> Pat gets 2 pairs or 2/3 of the total shoes and Chris gets 1 pair or 1/3. Since the third (free) pair, which costs $39, is free they should divide it by 3 and get $13 per 1/3 of savings. Chris bought one pair so he should pay $45-$13=$32. Since Pat has two pairs of shoes, he should pay $95-$26=$69. Although Chris did not get a free pair he did save quite a bit of money.

The preservice teachers who split $39 in three equal parts were mathematically fair than preservice teachers who simply split the money evenly. According to this new method Pat was saving 27.4% of the original price and Chris was saving 28.9% of the original $45. The percentage was quite close because $95 is only little bit more than the double of $45. However if the difference between the cost of two pairs $56 and $45 was too high the percent savings would have been substantially different. This kind of complex proportional thinking was not demonstrated in any of the methods provided by the preservice teachers.

There were other preservice teachers who did not demonstrate any mathematical understandings. Three preservice teachers said that since both Pat and Chris were friends they could simply divide the total of $101 evenly and each pay $50.50. When asked why Chris should pay $50.50 when the original price of the shoes he wanted to buy was only $45, they could not give any mathematical reasoning and simply stated that if Chris is a really good friend then he should not mind paying extra $5.50 for Pat especially because of a good bargaining opportunity. One of these preservice teachers said, "since Chris does not want three pairs of shoes she should be willing to pay the extra $5.50 because of the bargain." Three other preservice teachers said that since Pat spent more money than Chris Pat should get the $39 pair free. These preservice teachers were clearly not thinking mathematically. They were using friendship as a reason. It appeared that some of the preservice teachers, who had a weak mathematical background, simply wanted to avoid high level thinking required to solve this problem.

It is interesting to note that some preservice teachers who had a good mathematical understanding of the problem thought that dividing the saving evenly is fair. Ayaz was the most capable mathematics student in the class. He was the only student who got a perfect *A* in the class. He reasoned that the money should be

split evenly between Pat and Chris. I provided a counter argument saying that the sharing between Pat and Chris can be considered mathematically fair only if their savings are proportional to their original costs. The participants in the study accepted my argument as an alternative to their solutions. However many of them were not willing to change their thinking. In the final interview Ayaz said that they can split the saving of $39 evenly. So Pat would pay $95-$19.50=$75.50 and Chris would pay $45-$19.50=$25.50. He stated that they can also split money proportionally such as $39 \times \frac{95}{140} = \$26.50$ for Pat and $39 \times \frac{45}{140} = \$12.50$ for Chris. Nevertheless, he insisted that splitting saving evenly was still fair. The following transcript between the researcher (Resh) and a preservice teacher (Ayaz) illustrates this issue:

Resh: Is splitting evenly a mathematical response or an everyday common sense kind of response?

Ayaz: I think it's a mathematical response. The common sense response would be spending more so that I should get a larger discount.

Resh: I was thinking that common sense response would be "Let's make it half-half." Why bother?

Ayaz: I can see that. It's an easier way to do. Nevertheless, Pat would be happy as long as she spends less than $95. I would not even bother to split the money. I would rather go out and treat ourselves with lunch. I think it's a philosophical and political question.

The above transcript indicates that Ayaz was comfortable with his earlier solution even though he understood how to split money proportionally. Other preservice teachers had a similar kind of argument: Debi, for example, determinined that Pat should pay $67.33 and Chris should pay $33.67. When asked if her method was fair, she argued that the method was fair from a real life perspective. Below is the excerpt from the interview with her:

Resh: Is that fair?

Debi: Pat is getting two pair of shoes and Chris is getting one pair of shoes. It should not matter. It is a split of free money. Doesn't matter how much money they are spending.

Resh: Don't you think their savings should be proportional to the amount of money they are spending?

Debi: May be from a mathematical perspective; but not from a shopping perspective, because they each have a choice of how much they want to spend. She happens to like $56 and $39 pair shoes. Well then that's what she should be willing to pay. If they were buying exactly the same pair of shoes, the splitting of money [on a proportional basis]

should have been considered. Since they have a choice they don't need to consider the split [proportionally].

Other preservice teachers such as Mona, Riva, Andy, and Hana were all satisfied with their way of solving this problem, which was not based on proportion of their cost prices. While preservice teachers' responses to the shopping problem were mostly based on the context of friendship, their subject matter knowledge of mathematics was not always academically strong. Many preservice teachers solved the shopping problem using a simplistic mathematical procedure even when the problem required a complex thinking. Their solutions did not involve complex mathematical thinking such as splitting the saving on a proportional basis. The majority of the students simply decided to split the saving based on the number of shoes purchased without considering their cost prices. No one in the class split the saving based on how much Pat and Chris would have spent if there were no sale. A few students did not demonstrate any mathematical understandings at all.

Conclusions

The results of this study indicate that the teaching of mathematics using students' everyday contexts does not necessarily enhance their understanding of academic mathematics. Instead of using academic mathematical concepts such as ratio, proportion, and percent, many preservice teachers solved the shopping problem based on a concept of simple division. When asked why they did not use a proportional reasoning the preservice teachers argued that they would not really set up a complex proportional procedure if they had to split the saving in a real life situation. They argued that since the problem appeared real they used a simple division, which many of them would actually use in their real lives.

The preservice teachers did have difficulty in using a proportional reasoning in this problem. Does it mean that we should avoid these kinds of problems in mathematics classrooms? If we do not use real life contexts like this, preservice teachers will see mathematics as a collection of isolated facts and skills to be memorized. It is therefore important to use these kinds of problems. Actually we need to use more of these problems and emphasize the fact that students are required to provide appropriate mathematical response to the problem based on what is taught in the class. In the shopping problem, for example, the use of simple division would have been fine if the responses were from elementary school students. However the responses to the problem from preservice teachers should include higher level mathematics of ratio, proportion, and percent. It appears that we must make our expectations clear to students and emphasize the fact that the purpose of using real life context in a mathematics class is to learn as much academic mathematics as possible. If this emphasis is not made students will simply bog down in contexts and not learn mathematics. Also, as Walkerdine (1990) argues, teachers should be aware that everyday practice of mathematics is discursively different from school practice and so the relation between everyday and school

practices "is far more complex than is suggested by the notion of doing mathematical examples in familiar contexts" (p. 54).

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Learning mathematics in heterogeneous as opposed to homogeneous classes: Attitudes of students of high, intermediate and low mathematical competence.

Bilha Kutscher The David Yellin Teachers College, Israel

Abstract

Seventh- and eighth-grade students, who studied mathematics in heterogeneous settings organized for small group work according to a 'cooperative seating plan', were examined as to their attitudes to studying mathematics in heterogeneous classes as opposed to homogeneous classes. The eighth graders concurrently studied part of their mathematics curriculum in homogeneous classes. All students believed that studying in these groups facilitated their learning. Most students favored learning in heterogeneous classes. However, the eighth-grade, low achievers were ambivalent: they favored heterogeneous classes provided their assessment grades were higher. In response, an evaluation model is proposed to answer both the learning and psychological needs of students of diverse abilities studying in heterogeneous classes.

Research has cast doubt whether tracking is the correct way of dealing with diversity in abilities in the classroom. Not only has research shown that learning in low tracks significantly reduces achievement (e.g. Gamoran & Mare, 1989) but it has also been shown that 'top track' mathematics students can achieve as well in heterogeneous classes as in the tracked classes (e.g. Linchevski & Kutscher, 1998). Theorists claim that tracking is a central source of social inequity (e.g. Braddock, 1990). All this suggests that, whenever possible, mixed-ability classes should be the preferred learning setting in school. Many researchers argue for the value of cooperative learning in groups as a means of promoting attitude, motivation and achievement (e.g. Slavin, 1996) and for cognitive growth (e.g Webb (1989)). On the other hand Cobb suggests that small-group interaction is more productive when the interactions are multivocal and when the conceptual possibilities between the participants are relatively small (Cobb, 1996). This implies homogeneous grouping that "clashes with a variety of other agendas... including issues of equity and diversity" (ibid p.125).

This paper offers a 'cooperative-learning seating plan' that may reconcile these two seemingly contradictory approaches: cooperative-learning in small heterogeneous settings among participants whose cognitive capabilities are similar. This study examined a) the attitude of students who concurrently studied part of their mathematics curriculum in heterogeneous classes, where this cooperative-learning seating plan was adopted, and part of their mathematics curriculum in homogeneous settings and b) the attitudes of students who studied mathematics only in heterogeneous settings with this cooperative-learning seating plan. The conjecture was that all levels of students would prefer learning in these heterogeneous classes to learning in their homogeneous ones. These outcomes were expected since the heterogeneous learning environment was designed using the theoretical considerations and previous research results reported above.

Study Design

The population for this case study was drawn from the eighth grade of a junior high school and from the seventh grade of a demographically similar junior high. The eighth grade in the first school is divided into heterogeneous 'homeroom' classes: The idea is that students should learn most of the school subjects - social studies, general science, language and the like – in their own familiar, heterogeneous-environment 'homeroom' classes. However, for mathematics three tracks are implemented - 'Low', 'Intermediate' and 'High' - so that each student learns mathematics in one of the tracks. This junior high school agreed to participate in an experiment: Part of the eighth-grade mathematics program, that was originally to have been taught only in the tracked classes, would now be taught in heterogeneous classes. Both the tracked mathematics classes and the heterogeneous mathematics classes would address topics that were part of the traditional syllabus. Each student would thus be assigned two separate mathematics environments: the tracked class and the heterogeneous class. All students studied mathematics both in their own tracked mathematics class, where they studied mathematics for four hours weekly throughout the year, as well in their heterogeneous, homeroom classes for two hours weekly, for the duration of one semester. The heterogeneous mathematics class studied according to the principles of TAP (Linchevski & Kutscher, 1998), a learning model developed for learning mathematics effectively in a heterogeneous setting. This model calls for alternating between heterogeneous and homogeneous settings so that each student in TAP is concurrently a member of two groups, a heterogeneous group and a homogeneous one. In this experiment the way to materialize this alternation was through the 'cooperative-learning seating plan', that was designed to promote productive interactions in accordance with theoretical considerations reported above.

Figure 1a

Seating arrangement of groups in a more diverse heterogeneous class

		*		
*	L	H	I	*
*	L	H	I	*
		*		

Figure 1b

Seating arrangement of groups in a less diverse heterogeneous class

*	*
H	I
H	I
*	*

L: Low-track students desk I: Intermediate-track students desk

H: High-track students *: Students seated relative to their desk

The cooperative-learning seating plan of the heterogeneous groups was configured both to encourage interaction in dyads between students with relatively similar cognitive capabilities (homogeneous settings) and at the same time to allow interactions, such as asking for help and getting explanations for solving tasks between students with more diverse conceptual possibilities (heterogeneous settings). Figure 1 illustrates two typical seating plans. Figure 1a depicts a seating arrangement that accommodates the needs of a more diverse heterogeneous class where the distribution of low, intermediate and high achievers might be similar. Students of relatively similar cognitive capabilities are seated in close proximity to allow productive, multivocal interaction (Cobb, 1996). Nevertheless, concurrently these seating arrangements allow less competent mathematical students to be

supported, when necessary, by more mathematically competent neighboring students. In this manner the heterogeneity of the groups is upheld while providing sufficient "homogeneity" for multivocal interactions. Figure 1b offers an example of a similar configuration for a less heterogeneous class where there might be few weak students and where the intermediate-track and high-track students are evenly distributed. All the 'experimental' mathematics teachers constructed the groups for their class according to the principles of the cooperative-learning seating-plan.

Students' attitudes and perceptions about studying in heterogeneous mathematics classes might be affected by their learning concurrently in a tracked mathematics class; thus a complementary study was designed. A demographically similar school agreed that one of its seventh-grade heterogeneous classes have its mathematics lessons conducted in the same experimental format as the aforementioned school: The students would learn according to the principles of TAP in their homeroom classes with the specially designed cooperative-learning groups.

The data collected in this study were observations collected by the researcher in the experimental classes and transcripts of semi–structured interviews carried out with students toward the end of the school year. These interviews were conducted with approximately half the students who studied in the experimental classes. The students were chosen to represent high, intermediate and low achievers of each of the high-, intermediate- and low-tracks from the experimental eighth-grade classes. These students were observed during classes and interviewed toward the end of their semestrial mathematics course described above. Concurrently a similar population of students from the experimental seventh grade, described by their mathematics teacher as being low-, intermediate- and high-achievers (henceforth, in both the seventh and eighth grade, called 'Lows', 'Intermediates' and 'Highs') were also observed and interviewed. This report concentrated on two factors that might contribute to a students' attitudes to learning in heterogeneous settings as opposed to homogeneous settings: a) Their views, beliefs and feelings about learning in the cooperative-learning heterogeneous groups and b) their beliefs about learning in a heterogeneous setting as opposed to in a homogeneous setting.

Results and Discussion - Students' attitudes to learning in heterogeneous settings as opposed to learning in homogeneous settings:

a) Results and Discussion - Learning in the cooperative-learning heterogeneous groups:

The immediate reaction of most of these students was that this group work facilitated their learning. For example:

(8: eighth grade; 7: seventh grade; H: Highs; I: Intermediate; L: Lows)

Moran (H, 8): There's cooperation (in the group), *one can ask friends in the group, we're helped by them, we do together...*

Ma'ayan (I,,8): It helps me!

Yossi (I, 7): The fact that I sit (in the group) *then if I need help, then they help me, those who are sitting with me in the group.*

Dudu (L,8): When I ask (the teacher) ... *the teacher can't be only with me; there are also other children...* (Asking a fellow student is less embarrassing than) *showing that I don't*

understand the material.. because I ask him quietly, alone. But if I ask the teacher aloud then the whole class listens.

Ran (L, 7): (He likes to study in the group because) *Yossi helps me sometimes when I need him and Ayelet* (the teacher) *helps me and so I improve little by little.*

Ran, a Low, spoke about being helped by the same Yossi, an Intermediate, who himself reported as being helped by others. It seems that the seating configuration did allow and encourage mutual support. A typical lesson portrayed students sitting in their groups, interacting with the study material and each other. Students could be seen asking for, receiving and bestowing assistance. Students of all levels seem to have enjoyed and to have appreciated the potential of the heterogeneous cooperative-learning group as a source of resource and support. Many of the students used the term 'fun' to describe their feelings about the group. When asked what they meant by 'fun' they elaborated this as mutual help. For example:

Yifat (H, 7): It's fun to help!... It doesn't mean that if I am strong I know everything. So sometimes they help me and I help them.

Apparently working in these groups evoked positive feelings. Further investigation disclosed that both affective and cognitive factors contributed to these feelings. All students had mentioned the mutual help in the groups. Following this the students were asked whether they felt they gained more when they helped others or when they, themselves, were assisted in their mathematics.

Nirit (H,8): I prefer helping because I think that the fact that I help someone else, in actual fact I am helping myself, this opens my eyes more and I see that maybe I also have to go over and do it some more. The fact that I help someone (means) *more practice for me.*

Uzi (H, 7): (He gains more) *When I explain, because that's how I learn. From their mistakes I learn.*

Kokhi (I,7): Truthfully, both. Because when they help me then I 'get' it and then I can help someone else, and when I already know it then I can help someone else and then I understand it better. The more you'll go over it, then you'll understand it better.

Ma'ayan (I,8): (He preferred) *Helping... because it's nice, a good feeling*

Ran(L,7)(Prefers helping) *because I feel I know many things.*

Interestingly all preferred helping, even the least competent students who, it would be reasonable to assume, of necessity received more help than what they gave. There seems to be a continuum in the reasons for this preference for giving rather than receiving help. The more competent students felt that explaining promoted understanding, intuitively sensing that "giving high-level elaboration to other members of the group is positively related to achievement' (Webb, 1989). They also mentioned affective factors, such as "a good feeling of helping a classmate" but it seemed clear to them that their gain was cognitive. The less competent mathematics students related their preferences more to affective factors, such as improving their social standing, self-esteem, 'good feelings' and the like. Generally, the latter students reported that they were more on the receiving end of the assistance. Perhaps more opportunities of high-level elaboration would highlight the cognitive-gain aspect also for these students and concurrently lead them to improved achievements.

b) Results and Discussion: Learning in a heterogeneous setting as opposed to in a homogeneous setting:

Lower self-esteem has been associated with learning in lower-ability groupings (e.g. Abadzi, 1984). Thus it would seem plausible that lower-track students would prefer learning in heterogeneous mathematics classes to learning in the lower-ability groupings. This indeed seemed the case with the Intermediates, but in the case of the eighth-grade Lows there was a surprising result.

The Lows: The eighth-grade Lows were ambivalent about learning in heterogeneous settings. They wanted to learn with 'everyone', but they were unhappy seeing others achieving high grades while they, themselves, were barely achieving passing grades. In the case of the students who were also learning mathematics in the tracked classes this feeling was probably exacerbated by the fact that they concurrently attained much higher grades in their homogeneous mathematics class. Yael's views disclosed ambivalence although she declared that she preferred tracking:

*Yael (L,8): Because everyone is on the same level. And there is no one who succeeds more than you... But if I were in Rola's (*her mathematics teacher in the heterogeneous class) *group I will get 60 let's say, and maybe I made an effort, I would feel that I'm getting nothing out of it.* (The price she thought was worthwhile paying for learning in a heterogeneous setting was a grade) *No lower than 65, the main thing is I will not fail. The main thing is that it* (the grade) *won't be lower than 55* (the passing grade).

What Yael seems to be saying is that the minimum requital for a diligent student is a passing grade. Following this she would prefer and recommend studying in heterogeneous groups:

Yael: Because also the children will help each other... Children will understand the material better and will have easier tests and they won't feel that they are the lowest. Let's say if there'll be a heterogeneous class it's preferable to being in the third (their lowest) *track, to feel that* (in the heterogeneous class) *they're in the middle, with all the* (students in the) *first, second and third tracks. It doesn't matter if someone is a little better, a little worse, the main thing is that they will help each other.*

The Highs: Here there was a surprising difference between Highs who learned only in the heterogeneous setting (seventh graders) and Highs who also learned in tracked mathematics classes (eighth graders). The latter students generally were happy to study in heterogeneous mathematics classes. Their reservation was that they sometimes found it tedious listening to repetitive explanations and that the pace was sometimes slow. But they even had reservations to this and put it in perspective. For example:

Rania (H,8): Sometimes they go too fast (in the high track) *... and then I don't understand.*

Itamar(H,8): There are topics (in the heterogeneous class) *that are also difficult for me and also I need another explanation, but some* (of the explanation) *is a bit boring...* (But the delay is worthwhile) *because here* (the good of) *friends are more important than to cover another half a page of exercises. It* (the delay) *is more worthwhile.*

However most of the High seventh graders were unable to imagine any disadvantages in learning mathematics in the high tracks. Even though they did enjoy and saw the benefits of learning with students less competent than they, they thought that studying in homogeneous

mathematics classes would prove the better option. For example, Efrat was eagerly awaiting the following year's mathematically tracked class:

Efrat (H,7): The moment we'll be in the same track the children will be on the same level as me, then (as opposed to the heterogeneous class) *she* (the teacher) *will certainly give all the class the same exercises that are suitable* (to all)... *If we had tracks we would be able to progress much, much faster.*

The Intermediates: Not surprisingly, the Intermediates in both grades were very happy learning mathematics in heterogeneous classes. Avishai (I,8) loved learning in the heterogeneous class. He described the class and its advantages both from the cognitive and from the social perspectives:

Avishai (I,8): She (the teacher), *like, integrates. She makes two groups and integrates, strong with weak and then everyone helps the other.. and then you learn more. What is learning more – you understand the material better... Like, let's say she* (a student) *lagged behind in the previous* (work) *sheet then I help her, and then I understood. And it could be also that I made a mistake* (in this particular worksheet) *and then I correct mistakes.* (He prefers learning) *In heterogeneous groups because tracking makes the students competitive. They say that those who are in the low* (track) *are stupid and they say all sorts of not nice things and this makes them competitive. They're always fighting about this.*

Table 1 summarizes the attitudes of the students to studying in heterogeneous classes as opposed to homogeneous ones.

Table 1: Distribution of students according to 'tracks' and attitudes to studying in heterogeneous classes as opposed to homogeneous classes

	H(8) (N=9)	I(8) (N=10)	L(8) (N=8)	H(7) (N=6)	I(7) (N=7)	L(7) (N=7)
For	8	10	0	2	7	7
Ambivalent	0	0	6	0	0	0
Against	1	0	2	4	0	0

In Table 1, it can be noted that out of 47 students, 34 (72%) favored learning in heterogeneous classes, 6 (13%) – only eighth-grade Lows – were ambivalent, and 7 (15%) were against learning in heterogeneous classes. These findings, together with the findings that students believed that the cooperative-learning groups facilitated their learning, indicate that most students felt that learning in heterogeneous classes could be beneficial to them. While there might be other variables that influenced these attitudes, an effort was made to control for them. For instance, the students interviewed were selected to represent as many groups and as many classes as possible to reduce any effect that was particular to the make-up of a particular group or of a particular teacher. As previously mentioned, the teachers constructed the groups not according to the students' social preferences but according to the principles of the cooperative-learning seating plan thus reducing the 'socializing' or 'best-friend' effect during the formation of the student's attitude. Also it was ascertained (via the mathematics coordinator) that the mathematics teachers who taught the eighth-grade tracked mathematics classes were all well-liked and well-respected mathematics teachers thus reducing the 'special teacher' effect. Thus, it may be said that the eighth-grade Highs, who had learned in homogeneous classes, had a more positive attitude toward learning in heterogeneous mathematics classes as opposed to learning in

homogeneous mathematics classes than the seventh-grade Highs who had studied only in heterogeneous settings. Both the seventh and eighth graders in the heterogeneous classes studied in a learning environment conducted along the same principles. It seems that having had the opportunity of concurrently studying mathematics both in the experimental heterogeneous class and in homogeneous tracked class, the eighth-grade Highs saw that tracking was not necessarily a better option. All Intermediates preferred the option of studying in heterogeneous settings. The eighth-grade Lows, however, were ambivalent: Apparently the anxiety of not achieving good grades seemed to put a damper on the Lows' attitude towards learning in heterogeneous classes.

Although it may be concluded that most students in the experimental classes favored heterogeneous classes, this study indicates certain soft points. In the case of the seventh-grade Highs, a follow-up study is planned to examine their attitudes to learning in a heterogeneous mathematics class when they learn in an eighth-grade homogeneous mathematics class. Only then may one deduce with more confidence whether their rosy evaluation of learning mathematics in homogeneous settings was well founded or wishful thinking. As for the ambivalent eighth-grade Lows, how can conditions be created for them to feel more comfortable learning in a heterogeneous mathematics class? One condition could be lowering frustration levels as a consequence of low achievements. Clearly, a necessary condition for students to be motivated to study mathematics is the formation of a positive correlation between their diligence when learning mathematics and their achievements in mathematics. Yael (L,8) wanted to study with "all" the children; however she did not want to be too low an achiever. Seemingly Yael had not been achieving very highly in the heterogeneous class. She obviously thought that the only way to gain better grades was through less challenging tests. However, an assessment with too little challenge would most likely be counterproductive to the Highs' feelings of satisfaction when learning in a heterogeneous class; to some extent assessment should reflect all levels of learning in the class. The question is how to reconcile these two ostensibly contradictory requirements: a test that will challenge the Highs and at the same time allow the Lows to achieve reasonable grades?

TAP assessments are constructed of two parts: a basic component (usually comprising about 70% of the test) that assesses the core material in the relevant topic, and a complementary component (completing the basic component to 100%) that assesses material connected to the topic but on a higher level than the basic component. Thus, although the assessment is composed of two parts, the student receives a cumulative grade out of 100. This assessment format offers all students the opportunity for success in the basic component, while usually the more accomplished mathematics students show competence in the complementary component. Research has indicated that Lows who study in TAP's classes gain higher achievements than Lows who study in the low tracks (Linchevski & Kutscher, 1998). Thus, achievement-wise it is clearly beneficial for the Lows to study in a heterogeneous class. However, for students, research results do not offer requital. Students do not compare themselves with theoretical instances such as, for example, learning with mathematics students in other classes who are or are not studying in heterogeneous or homogeneous groups. Comparing themselves to their classmates, it is

only natural for all students to hope for positive feedback in the form of commendable assessment grades in return for their diligence and effort in their mathematics class. An evaluation model should take these psychological factors into account and allow all students to keep their self-respect in the heterogeneous class vis-a-vis their achievements and their classmates.

Proposed Evaluation Model in Heterogeneous Classes

Assuming the TAP model of assessment, it is proposed that each component of the assessment be treated grade-wise as a separate unit, each unit receiving a maximum grade of 100%. Thus, students would receive back their test papers with two separate grades, each evaluated out of 100%, one for the basic component, the other for the advanced component. They would be informed that the basic component assesses the core material and as such forms the primary part of the assessment; thus reasonable competence in this component is required. Similarly, students would be informed that the second part of the assessment, although recorded by the teacher, evaluates more advanced material connected with this topic and thus no minimum grade is required on this component. This evaluation process might reduce the less-competent students' frustration. At the same time the more competent students could receive the acknowledgement they need. By emphasizing the importance of the 'basic' grade, all students – and notably the Lows – who have performed adequately on this component could feel that they had received due recognition for their efforts, thus avoiding frustration. By virtue of the presence of the 'advanced' grade on the evaluated test paper, the more competent students could feel both motivated to rise to the challenge of the advanced component in the assessment and also feel recognized for this effort. Conceivably, such an evaluation model could answer both the learning and psychological needs of the whole mathematical community learning in heterogeneous classes. Further research should be done to examine the effect of such an evaluation model on students' attitudes.

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TWO TEACHERS' BELIEFS AND PRACTICES WITH COMPUTER BASED EXPLORATORY MATHEMATICS IN THE CLASSROOM.

Chronis Kynigos* and Michael Argyris**

University of Athens and Computer Technology Institute*
University of Athens**

Abstract: *This paper describes two elementary teachers' beliefs and practices constructed after eight years of innovative practice involving one-hour-per-week computer-based maths classroom activity with small cooperating group of pupils. Participants were observed for 5 teaching periods, verbatim transcriptions were made from video recordings and semi-structured interviews were taken. Combined qualitative and quantitative analysis indicates that the teachers' actions may be influenced by their belief systems -not necessarily by one- as well as by wider cultural perspectives.*

Theoretical framework

In our effort to study the interrelations between teachers' espoused beliefs and their practice (Ernest, 1989, Lerman, 1992) we adopt a theoretical orientation which is influenced by two important paradigm shifts in the teaching and learning of mathematics: the appreciation of the teaching process and the formative role of the classroom to teaching practice. Perceptions of the teacher as implementers of prescribed pedagogy and curriculum have given way to those of the teacher as reflexive practitioner who's teaching is shaped by espoused beliefs and by practice itself (Hoyles, 1992). With respect to implementing innovation (as is the case in the present study), the teacher forms a personal pedagogy in the process of making sense of his/her environment as he/she acts upon it and is influenced both by his/her beliefs on teaching mathematics, the teachers role and the nature of mathematics itself and by the classroom culture and the wider culture (Olson 1989, Moreira and Noss 1995, Thompson 1992). The complicated relationship between teachers' beliefs and practices has captured the interest of many researchers revealing a disparity between espoused and enacted beliefs (Raymond 1997). To this end additional research is necessary to illuminate and map the nature of the relation between beliefs and beliefs-in-practice (Lerman 1992). A second paradigm shift has been from constructivist and interactionist towards socio-constructivist perceptions of learning mathematics. These have given new impetus to classroom research with particular emphasis on the interplay between mathematical learning and social interactions. The study of mathematical teaching and learning in classroom situations has been one of the important means to develop theoretical interpretations

of learning as a reorganization and construction of concepts in social settings (Cobb et al, 1996, Baruch and Swartz, 1997).

In our effort to further analyze elementary teachers' practices and beliefs about their role, mathematics, teaching of mathematics, we build upon previous research (Kynigos 1996) where teacher interventions were described by means of a) the aspect of the learning situation to which they referred to and b) the type of pupil activity they intended to encourage. Our approach has been to study the teaching process in the context of using exploratory software to mediate innovation, taking into account Hoyle's (1992) suggestion that computational environments inevitably perturb the dynamics of a classroom, make more apparent the mathematical beliefs and understandings of teachers and students providing a window, a magnifying glass even, on the interaction process.

This is a report on research into two teachers' practices and beliefs constructed after eight years of such innovative practice involving a one-hour-per-week computer -based mathematics classroom activity with small cooperating groups of pupils.

Research Setting

This study was part of the "YDEES"[1] project which involved ethnographic investigation in a variety of aspects of classroom practice in school based innovation programs involving the use of exploratory software for small group mathematical project work. The teachers in question were observed during a two month project, the object of which was for each group of pupils to find out quantitative geographical information on Europe and represent it in a series of bar charts. The mathematics on the teachers' agenda was not only representing data by means of a bar chart but also the process of constructing the different bars (rectangles) in a sequence and finding ways with which the process would not have to be repeated for each chart. They used a piece of software developed within the project called "Component Logo with a Variation tool" (Kynigos et al, 1997). The variation tool is in the form of a "slider" which allows the continuous change of the graphical representation of a figure created by means of a parametric procedure as the value of the variable changes. The teacher wanted the pupils to suggest that they construct a rectangle procedure with one variable for its "height" and then use the variation tool to create a "bar chart machine", i.e. a piece of software for creating bar charts.

Method

[1] The study was carried out in the framework of project YDEES: "Development of Popular Computational Tools for General Education: The Computer as Medium for Investigation, Expression and Communication for All in the School", General Secretariat for Research and Technology, #726, E.P.E.T. II, 1995-1998.

We employed an ethnographic approach (Hammersley & Attkinson, 1995) in order to a) to understand the teachers' beliefs regarding mathematics, the teaching of mathematics and their pedagogical role and b) to investigate their practices. More specifically, we wanted to investigate i) the nature of the role undertaken by each of the two teachers ii) the nature of the teachers' intervention regarding the mode of communication and the kind of activity they intended to encourage. To this end we built upon the previous works of Hoyles and Sutherland (1989), Farrell (1996) and Kynigos (1996), constructing a modified instrument for recording classroom events. First we studied observational data with the intention to trace the variety of roles and activities undertaken by teachers and pupils. Types of roles and interventions were allowed to emerge based on the data, instead of using data to test pre-existing hypotheses. To get a feeling for the balance of teacher actions in time, we then separated every teaching period in one-minute time segments and we studied the appearance frequency of each type of role and activity. Finally, we came back to the original text using the quantitative picture supportively in order to describe the practices of the two teachers. Each teacher was observed and videotaped for five class periods respectively. A remote microphone enabled transcription of all their utterances capturing responses of the group of pupils in which they intervened. Semi - structured interviews were subsequently carried out regarding their views on mathematics, teaching of mathematics, their pedagogical role and the role of the computer. Background data was also collected (i.e. observation notes and students written presentations of their work). Verbatim transcriptions of all audiorecordings and interviews were made.

Results

Kate has 27 years of teaching experience and Martin 26 years. In the last 8 years both of them have been working for one hour per week with their pupils organized in small groups in computer based classroom activities named "investigations". We present the results in two sections, one focusing on their beliefs as manifested in the interviews and the other describing aspects of their practice, i.e. the type of communication, their role and their practice when addressing the class as a whole.

a. Espoused beliefs

Martin enjoys doing mathematics and feels quite comfortable with the subject. *"I do like mathematics. And I think that I communicate this enthusiasm to the students. Many times before we start the daily curriculum I ask them what subject they want us to do and they select mathematics"*. He regards his role as *"a facilitator who provides students the opportunities to construct knowledge for themselves"*. Such statement could be regarded as consistent with the Problem-Solving view in the Ernest (1989) taxonomy. However, in some cases he articulates

views in which we may trace an impact of a Platonist view of mathematics. "*What I am trying to do is not to give answers but to guide students through the right questions to find the solution. I feel that I am responsible to guide children to reach the right solution*".

Kate, on the other hand, feels anxiety towards mathematics, a feeling that it has its origins in her school life as a student "*I had very bad experiences from mathematics as a student and now as a teacher I am most interested in removing any anxiety from students when they face a problem*". She regards mathematics as: "*facts and procedures for computing numerical expressions to find answers, something that you need in your everyday life*". A view that is most consistent with the Instrumentalist view of mathematics. This view doesn't restrain her to consider her role more as a counselor and a fellow investigator stating that "*although I don't feel quite comfortable using computers I enjoy this hour [she means the weekly hour mentioned above] as it gives us the opportunity to be students again. You see, we, the veteran teachers, tend to intervene too much. Using computers gives you the opportunity to think things different. Children don't need you to tell them what to do all the time. You have to permit them some autonomy*".

b. Teachers' practices

In order to describe teachers' practices we studied their verbal and nonverbal communicative behavior in the classroom characterizing their interventions, the roles they constructed and the activities they encouraged. We take two similar episodes, in the sense that they were points in time when the two respective pupil groups had not constructed a procedure with variable for the rectangle bar and both teachers intervened with an agenda for them to change course. We use these episodes to highlight a sample of the types of teacher interventions and then use the quantitative picture to discuss the full set of types and the balance of interventions in time.

b1. Teachers' interventions

Kate : Have you done the population of Athens?(means the construction of the appropriately sized bar on the chart)
S : Yes,
Kate: Right, Well done. Now you have to do what? Oh, wait a minute. Would it be better if you put all these commands in a procedure?
S : But it works this way Mrs.
Kate : Yes, but if you put all these commands in a procedure then you can do this thing [means the rectangle] as many times as you wish. It's better this way.
S : How?
Kate : Oh come on, you know that - Get me a pencil [She goes on modeling the move of the turtle on a piece of paper explaining the steps that pupils should follow].

In this episode Kate intervenes through her own initiative with a specific agenda to direct the pupils to construct the rectangle bar in a different way i.e. procedure with a variable so that they could change its length. Her argument is both

pragmatic and mathematical. Her mode of communication is exposition - explanation of they way they should do it.

In an analogous situation Martin reacts quite differently.

Martin : I can see that you are working without using variables
S : Yes. Does it matter?
Martin : What matters is that you don't take advantage of the software.
S : Well that doesn't really matter. It's still working.
Martin : Is it so difficult to use variables? I will help you if you want to.

[At this point a pupil from another group attracts his attention temporarily. Martin returns later to this particular group and observes their work silently without making any further comment although pupils insist on working without using variables. In the last lesson when every group presented its work he triggered a long discussion about the use of variables and set as an example the groups that used variables in their project].

Martin's unrequested intervention was in the form of a dialogue with the pupils aiming to explain the merits of using a procedure with variable for the bar. He did not, however, invest much in discussion and articulation of arguments. His argument was authoritative rather than mathematical or pragmatic, i.e. variables and the variation tool should be used because they were available rather than because the pupils could do such and such if they used them. After the pupils' resistance by means of a pragmatic argument, he withdrew without further instructional comments with the apparent intention that they reflect and make their decision.

The types of the two teachers interventions emerging from the data in this way are given in table 1 below.

Table 1 Characterization of teachers' interventions

The Initiative	Martin	Kate
requested	46%	50%
unrequested	54%	50%
To whom		
to the whole class	11%	13%
to a group	89%	88%
The mode of communication		
Verbal		
exposition/explanation	15%	42%
dialog	50%	48%
resource	17%	22%
Nonverbal		
silent observation	31%	12%
demonstration	4%	25%

The intention	Martin	Kate
nudge	29%	37%
discipline	1%	9%
directive	23%	36%
reflective	38%	22%
group dynamics	8%	21%
motives	9%	20%
procedural	12%	13%
supervision	39%	48%

As it is quite clear from the quantitative picture about half of the teachers interventions were of their own initiative and Kate tended to 'explain' and demonstrate things more than Martin who preferred in more cases to observe

silently. Martin's interventions are characterized by his apparent tendency to promote reflection. Instead of providing answers he prefers to return the question to the whole group giving them the responsibility to monitor their progress and make up their decisions. Kate seems to emphasize issues of communication and motivation within the group and tends to direct the pupils into clearly described actions.

In the interview, Martin seemed to have conflicting beliefs that a) reflection is at the heart of learning mathematics and b) that it is of primary importance that they solve the problem. The time-dependent quantitative analysis shows that while in the first hour 18% of his comments were directive and 45% intended pupil reflection (like, for instance, in the excerpt above), in the last hour the picture changed dramatically (34% and 35% respectfully).

This tendency is even more clear in the case of Kate. Starting with a high percentage (41%) of one-minute segments containing interventions for promoting reflection she concludes with a very low one (6%). In total her intervention is mostly characterized by her directive comments. In some cases she even takes control of the machine demonstrating what the pupils should do by doing it herself.

b2. Teachers' roles

A second interpretative perspective of the teachers comments is that of the social role (i.e. the norm or shared understanding about it) which they encouraged in their relations with their pupils. Looking again at the above episodes, Kate seems to have adopted an instructional role, since the taken-as-shared communication context was that she would describe, explain and demonstrate what she wanted the pupils to do. In this case we have a consistency between the nature of her intervention and the role she adopts. In the case of Martin things are different. In the episode, after observing what the pupils were doing, he intervenes with an intention for them to reflect on whether they should use variable or not and seems to accept their pragmatic argument. In this sense he adopts a managerial role but does not pursue discussion on the mathematics at hand, so he does not instruct or explain. He offers counseling but does not insist when the pupils show no eagerness.

Working in this way and studying the social interaction between teacher and students we came along the following table.

Table 2. Characterization of the teacher's role

	Martin	Kate		Martin	Kate
instructor	12%	43%	counselor	30%	40%
manager	40%	44%	resource	11%	12%
task setter	6%	4%	fellow investigator	6%	13%
explainer	9%	8%	silent observer	38%	10%

As a whole, during the five lessons under study, Martin seems to constrain his managerial role in benefit of the 'silent observer' one, allowing students to undertake an exploratory role. In line with his view about his role, it appears to be a conscious choice to give students opportunities to express their ideas, experiment with them, interpret the feedback they receive from the computer and finally construct their knowledge. Kate on the other hand, in contradiction with her statements, seems to construct a more directive role. She walks around the classroom, checking groups work and often suggests what they should do next.

b3. Classroom activities

So far we were focusing on the social interaction between the teacher and a group of students. A third perspective with which we analyzed the data was that of the social interaction between the teacher and the class as a whole. Looking back in the above mentioned episodes regarding the educational activities we have two dialogues between teacher and students while the rest of the pupils are allowed to talk to each other in their groups as they investigate the project at hand. Studying the educational activities which were trigged by the two teachers in the whole sequence of the five lessons we formulated the following table.

Table 3. Characterization of classroom activities

	Martin	Kate		Martin	Kate
exposition	7%	1%	discussion		
investigational work	84%	93%	student - student	87%	92%
			teacher - student	81%	94%

Both teachers seem to refrain themselves from frontal teaching and their discourse is not in antagonism with pupils' discourse as it is the case in frontal teaching. Both seem to encourage students to undertake an investigational role as this activity was the one most commonly presented in their classrooms. This is not surprising as, after eight years of practice in this particular school, it is embedded in its culture that during this weekly hour students should be allowed to work cooperatively in order to investigate a specific subject.

Conclusion

The results corroborate the view that espoused beliefs may be inconsistent with actions during classroom teaching practice. However, in both cases we find that more than one of what we may describe as a belief system (i.e. a coherent set of views on mathematics, teacher roles and mathematical learning) may coexist in teachers' descriptions. In one case, for instance, we had confidence with mathematics and appreciation of encouraging reflection in Martin who's

interventions were infrequent and often lacking in mathematical content. In the other, Kate expressed uneasiness with mathematics, but was rather directive and mathematically explicit in her interventions. Both teachers were influenced by timing, becoming more directive towards the end of the course. Studying teaching process in terms of communication mode, type of encouraged pupil activity, adopted roles and classroom activity was illuminative in the sense that it provided means to describe some of these inconsistencies. For instance, Martin may have had a coherent intent for pupil activity which he may simply have not related to his view about his role and to the way in which he communicated with the pupils. The relation between espoused and enacted beliefs is a complex one and is not necessarily consistent. Further research is needed to illuminate the nature of teaching practice in the process of developing focused theory on learning in classroom settings.

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BASELINE ASSESSMENT AND SCHOOL IMPROVEMENT: RESEARCH ON ATTAINMENT AND PROGRESS IN MATHEMATICS

LEONIDAS KYRIAKIDES
INSTITUTE OF EDUCATION, UNIVERSITY OF WARWICK, UK

ABSTRACT

This paper presents findings from research exploring the use of baseline assessment at entry to primary school to measure pupils' progress in Mathematics. Pupils' attainment in Mathematics when they entered the primary school and when they were at the end of year 2 were measured through external and internal methods of assessment. The predictive validity of baseline assessment for pupils' attainment at the end of year 2 was satisfactory. Pupil background factors were significantly related to pupils' attainment on the baseline assessment and to their attainment at the end of year 2. However, the baseline score was the most important factor in relation to pupils' progress. Both pupils who did not have any background in Mathematics when they entered primary school and those with a very good background made less progress than those who were typical in their age. Differences between schools' final results were reduced substantially when account was taken of their pupil intakes, but significant differences between schools remained supporting the conclusion that some schools are more effective than others in facilitating pupils' progress in Mathematics. Implications for the development of curriculum policy in Mathematics and areas for further research are drawn.

I) INTRODUCTION

There are four reasons why all school systems must have a strategy for finding out about pupils on entry (Blatchford & Cline 1992). First, baseline assessment may produce information about what children know and what they do not know in order to help teachers decide how to identify and meet children's learning needs and how to use their teaching time and their resources. An important implication of the identification of learning needs is that decisions about the next learning steps follow from this. A teaching plan, which is organised in such a way, might help teachers to plan class and individual programmes of work according to the different performance levels of the pupils. On the other hand, information gathered from baseline assessment can be used for summative purposes. However, there is widespread doubt that the summative and formative purposes can be achieved in a single set of assessment arrangements. (Brown 1991). Another important purpose of baseline assessment has to do with the fact that teachers may identify pupils with learning difficulties (Lewis 1995). Finally, baseline assessment as its name implies, provides the base for which pupils' subsequent educational progress can be measured.

Measures of the educational progress made by pupils in a school, relative to that made by similar pupils in other schools, have come to be called "value added" assessment and such measures are used in research into the effective teaching of Mathematics (Brown at al 1997).

The main purpose of the research presented in this paper is to explore the use of baseline assessment in Mathematics to measure Cypriot pupils' progress in the early years of primary school. The review of the purposes of baseline assessment helps us to see that this study can be linked with the following three aspects of curriculum policy. First, in Cyprus there is no policy in evaluating school effectiveness. However, the fact that in the Third International Mathematics and Science Study (TIMSS) substantial variation in Mathematics achievement between year 3 and year 4 Cypriot pupils was found revealed the importance of developing a policy on evaluating school effectiveness in Mathematics. Data derived from measuring pupils' progress may be more valid in exploring the effectiveness of a school unit than using outcome data only since variations in final test results of schools reflect partly the educational attainment of pupils when they entered the school (Fitz-Gibbon 1995). Thus, this study may contribute to evaluate school units in teaching Mathematics by adopting the technique of value-added assessment. Second, the fact that significant differences among the skills and knowledge of Cypriot school entrants have been identified (Kyriakides 1997) reveals the need for measuring the progress of different ability groups of pupils in order to explore the extent to which teachers respond to the learning needs of each group. Thus, the data of this study may provide implications for the importance of the formative purpose of baseline assessment. Finally, the measurement of pupils' progress may also help us to examine the extent to which the baseline assessment could be used to identify pupils "at-risk" of later educational failure. Although during the early 1980s considerable research was undertaken in European countries on screening instruments designed to identify pupils 'at risk' of later educational failure (Wolfendale & Bryans 1979) and significant positive correlations between attainment at the start of the school and subsequent attainment were reported, the research reduced in prominence because of the difficulties inherent in making predictions for individual pupils (Potton 1983). However, the findings of studies exploring the school effect on pupils' progress may help us to renew the interest of using baseline assessment to identify pupils with learning difficulties.

II) METHODOLOGY

Research data concerning pupils' skills and knowledge in Mathematics when they entered primary school constitutes of two elements. First, a teacher completed checklist including an assessment of social and emotional development and attainment in Mathematics was designed to build on good observational assessment Teachers were asked to rate pupils as "developing competence", "competent" or

"above average" for each section. The checklist is accompanied by a comprehensive teacher's handbook which provides guidance on each section. Watson (1997) argues that the development of criteria for informal assessment and the attention to documentation of unplanned observations are important but not enough to ensure teacher assessment is valid. Thus, a performance test was designed to assess knowledge and skills in Mathematics identified in the Cyprus' Pre-Primary Curriculum. Pupils were asked by the researcher to complete at least two different tasks related to the purposes of teaching Mathematics to pre-primary pupils. It has been shown that the baseline items are appropriate in difficulty for the typical Cypriot pupil entering primary school but are also able to allow the more able pupils to demonstrate their attainment (Kyriakides in press). In addition to their baseline attainment, information was collected on four further pupil background factors: pupil's age when s/he entered primary school, pupil's sex, and type and length of pre-primary school provision. Pupil's age was converted into months and expressed as deviations from the grand mean for all pupils. Finally, external and internal methods of assessment were used to measure pupils' attainment in Mathematics when they were at the end of year 2. Teachers were asked to complete a checklist for each pupil indicating whether the child had acquired each of the skills included in the Mathematics Curriculum of year 2, and a written test was administered by the researcher to assess pupils' knowledge and skills identified in the curriculum of year 2. Data were available for 1664 pupils who completed both baseline assessment in October 1996 and the written test for year 2 pupils in June 1998.

The stratified technique was used to select 48 schools out of the 242 Cypriot primary schools. All the first year pupils from each class of the school sample were chosen. It is important to note that the chi-square test did not reveal any statistically significant difference between the research sample and the population in terms of sex, type and place of school and size of class. It may be claimed that the pupils who took part in the research were representative of Cypriot pupils entering primary school in 1996 in terms of the above characteristics.

The reliability of the findings was measured by calculating the relevant values of Cronbach Alpha for the scales used to measure pupils' knowledge in Mathematics. The values of Cronbach Alpha for the scales used to measure pupils' responses in the performance test and in the written test were higher than 0.80. Similarly, the values of Cronbach Alpha for the scales used in teachers' checklists were higher than 0.82. Thus, we can be confident about the reliability of the measures used. Moreover, the use of two different methods (internal and external) of baseline assessment and assessment of pupils' attainment at the end of year 2 provide us with useful information about the internal validity of the research. Significant correlations ($p<.05$) between the way teachers assessed each skill of their pupils and the way the pupils responded to the assessment tasks of the relevant test have been identified. Although this finding does not necessarily imply that the validity of the research is

high since it is possible that they are both invalid, the use of both internal and external ways of assessment provides a basis for triangulation of data.

III) FINDINGS

A) Predictive validity of baseline assessment

Table 1 shows the multiple correlations between the baseline attainment and attainment at the end of year 2 of primary school. The multiple correlation of 0.63 between the average baseline assessment score and the average year 2 score provides a satisfactory starting point for value-added analysis. It is of interest that the combination of teacher-completed checklist and the performance test provided the best indicator of pupil's subsequent attainment, better than either type of assessment in isolation.

Table 1: Correlations between baseline attainment measures and attainment measures at the end of year 2

Outcome	Predictor/s	Multiple Correlation
Year 2 average test score	Performance Test	0.58
	Teacher Assessment	0.54
	Performance Test + Teacher Assessment	0.63
Year 2 written test score	Performance Test	0.56
	Teacher Assessment	0.53
	Performance Test + Teacher Assessment	0.62
Year 2 teacher's Assessment score	Performance Test	0.57
	Teacher Assessment	0.51
	Performance Test + Teacher Assessment	0.60

B) Pupil Factors

<u>Pupil factors and attainment when they entered primary school (average age 5.72 decimal years)</u>

The first two columns of Table 2 give the results for the analysis of baseline scores. The multilevel model compares the effect of each factor against a reference group which, in this case, is girls of average age with one or less year of public pre-primary education. All the measured pupil background factors but the type of pre-primary education were significantly related to pupil's attainment on the baseline assessment. Older pupils' attainment was significantly higher than younger pupils' attainment. Sex differences were marked with boys having significantly higher attainment than girls. Age and length of pre-primary education covaried, with older pupils experienced more pre-primary education than younger pupils. The confounding of these two factors might be overlooked when assessing the effects of pre-primary education. However, the multilevel model allows the influence of early education to

be assessed whilst simultaneously controlling for age and all other measured background factors. Thus, it was found that pupils with more than three years of pre-primary education and those with more than one but less than three years of pre-primary education, both had higher attainment than those with one or less than one year of pre-primary education.

Pupil factors and attainment when they were at the end of year 2 (average age 7.29)

The third and fourth columns give the results for the analysis of pupils' attainment at the end of year 2. Again younger pupils had lower scores than older pupils and boys had higher attainment than girls. Finally, pupils with three or more years of pre-primary education had higher attainment than those with one or less than one year of pre-primary education.

Table 2: Fixed effects of pupil background on baseline attainment, year 2 attainment and progress during the first two years of primary education

Factor	Baseline Attainment		Attainment at Year 2		Progress (from 5 – 7)	
	Estimate	Std. Error	Estimate	Std. Error	Estimate	Std. Error
Pupil level						
CONS (Intercept)	18.482*	1.913	2.012*	0.122	1.623*	0.238
Baseline Score	--	--	--	--	0.048*	0.001
Age**	0.838*	0.082	0.031*	0.003	0.004	0.003
Sex	3.121*	0.493	0.143*	0.021	0.052*	0.023
Pre Primary 2 Yrs	2.512*	0.576	0.057	0.029	- 0.019	0.021
Pre Primary 3 Yrs	5.931*	0.808	0.132*	0.038	- 0.028	0.027
Type Pre Primary	0.087	0.039	0.048	0.021	0.012	0.009
Group 1 (Risk)	--	--	--	--	- 0.042*	0.019
Group 2 (Gifted)	--	--	--	--	- 0.039*	0.020

*= Coefficients significant at p<.05, ** = Variable centred on grand mean

Pupil factors and progress during the first two years of primary school

The following observations arise from the figures of the last two columns of Table 2 which show the effect of pupils' baseline score on pupils' progress. First, baseline score was the most important factor in relation to pupils' progress. Second, boys made more progress than girls. This implies that the gender gap becomes even larger during the two years of primary education. Third, amount of pre-primary education was not significantly related to pupils' progress during the first two years of primary education. However, the positive effect of three or more years of pre-primary education on baseline and year 2 results, mentioned above, suggests that the benefits of early education persisted during the first two years of primary education.

Since a wide disparity in pupils' responses to the baseline tests was identified (Kyriakides in press), a further analysis was conducted in order to compare the progress of different ability groups of pupils. The following five homogeneous

groups of pupils according to the way in which they responded to the activities in the baseline tests derived from cluster analysis: a) "Pupils with identified needs for remedial help", b) "Pupils with needs for extended mathematical activity", c) "Pupils who are typical for their age", d) "Formal knowledge possessors", and e) "Skills possessors". Thus, the variable "group 1" compares the progress of pupils who did not have any background in Mathematics when they entered primary school (group 1) with the progress of pupils who are typical for their age (group 3). Similarly, the variable "group 2" compares the progress of pupils who had completed correctly almost all the activities in the baseline tests with the progress of pupils of group 3. The figures of the last two columns of Table 2 reveal that pupils who did not have any background in Mathematics when they entered primary school and those with a very good background, both made less progress than those who were typical in their age when they entered primary school. Thus, the ability gap between the group of pupils who did not have any background in Mathematics and the group of pupils who are typical for their age becomes larger during the two years of primary education.

C) Differences between schools

School effects on pupils' progress were explored through three multilevel regression models. The first model (null model) included only the intercept term and indicates "raw" differences between schools in their year 2 results. The second model explores the effect of adding information on pupils' background including their baseline score, sex, age and type and length of pre-primary school provision. The third model explores the effect of including variables at the school level, especially the average baseline score, the percentage of girls and the mean of the age of their pupils. These factors are all aggregated from the pupil level data. The analysis of the data revealed that knowledge about pupil's prior attainment and background explains a good deal of the pupil variation in year 2 results (34%), but very little of the school-level variation (4%). It was also found that baseline score was the most significant of the pupil background factor since it reduced the pupil variation in year 2 scores by 29%. Including the school compositional factor in the third model explained no more of the pupil-level variation but significantly reduced the school level variation by 38%. A substantial amount of the difference between schools in pupils' progress between baseline and the end of year 2 was explained by the overall composition of the school intake. However, substantial differences between schools remained. The third model revealed that about 10% of variation in pupils' scores was attributable to schools. Having controlled for both pupil factors and school contextual factors, the results show that there were still substantial differences in the performance of schools.

IV) Discussion

The evidence presented above can be discussed in terms of its implications for the development of curriculum policy in Cyprus. First, it is important to examine

assessment policy in terms of policy on classroom organisation. The fact that some school entrants had either achieved most of the aims of teaching Mathematics or had not achieved any one of them implies that it is not possible to organise teaching Mathematics without taking into account the different Mathematical background of school entrants. Spending most of teaching time working as a whole class, as is the case in Cyprus (Kyriakides 1996), is not an appropriate way of teaching Mathematics to first year pupils. It can be argued that baseline assessment provides teachers with information which could help them to respond to the learning needs of each pupil, or more realistically of groups of pupils organised by previous attainment However, Cypriot teachers did not systematically assess their pupils when they entered primary school (Kyraikides 1997). Moreover, Cypriot pupils with special needs (either for remedial help or for extended activities) made less progress than pupils who are typical in their age. This implies that Cypriot teachers respond mainly to the needs of pupils who are typical in their age. An issue that needs to be examined is whether the development of a policy on baseline assessment may encourage Cypriot teachers to give more thought to the best way in which to respond to individual learning needs.

Second, significant sex differences favouring boys have been identified. Thus, in terms of equal opportunities across the first two years of primary school the lower rate of progress of girls in Mathematics suggests that equal opportunities issues remain highly relevant in considering whether differences in the nature of primary school experiences of the two sexes may play a part. A conclusion by no means unique to Cyprus (Frempong 1998).

Third, the findings of this study reveal that the school a Cypriot pupil attends does make a difference to his/her educational progress, since schools with intakes of similar attainment and of similar composition can and do achieve different results. It can be claimed that even if 15% of pupils did not respond correctly to the tasks assessing pupils' attainment and are likely to turn out to have cognitive problems, the school effect on the progress of this group of pupils should be examined. Although baseline assessment can be used to identify pupils who need extra help, value-added analysis is also needed to provide information about school effectiveness in teaching Mathematics to pupils with learning difficulties.

It can be finally argued that while the value added analysis shows that there are differences between schools in the progress made by their pupils, further research is needed to identify what it is that schools do that makes this difference. Nevertheless, the analysis of baseline assessment in relation to year 2 results to provide indicators of variations in school effectiveness in Mathematics offers schools valuable additional information for the purposes of self-evaluation and review. The value-added feedback may be used to encourage schools to identify apparent areas of strengths and weakness to formulate provisional hypotheses about the factors which may have influenced pupil performance in specific areas of Mathematics. Thus, the importance of linking school effectiveness and school improvement work in

Mathematics should be raised. An approach of curriculum improvement in which the targets set would arise from school-focused needs, defined at the school level by the school staff, would be more likely to generate authentic reform and to raise standards in Mathematics.

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Learning pre-calculus with complex calculators: mediation and instrumental genesis

Jean-baptiste LAGRANGE
Institut universitaire de formation des maîtres
153, rue de St Malo 35043 Rennes Cedex FRANCE
Email: lagrange@univ-rennes1.fr

Abstract

University and older school students following scientific courses soon will use complex calculators with graphical, numerical and *symbolic capabilities. This paper is based on the experience of the design and the experimentation of an 11^{th} grade pre-calculus course. The first part is a study of this new context, stressing the role of mediation of the calculators and the development of schemes of use in an 'instrumental genesis'. From this study, the paper looks at tasks and techniques to help students to develop an appropriate instrumental genesis for algebra and functions, and to prepare for calculus. Then it focuses on the potential of the calculator for connecting enactive representations and theoretical calculus and on strategies to help students to experiment with symbolic concepts in calculus.*

Introduction

New hand held complex calculators offer, to some extent, a synthesis of computer software and calculators. Like computers they have the powerful applications: computer algebra systems, geometric software and spreadsheet. From calculators they inherit ergonomic characteristics (small, disposable) and numerical and graphical utilities important to the study of functions.

This paper is based on an experience of integration of these powerful calculators into the teaching of pre-calculus[1]. It provides reflection, based on theory and practice, on the changes that these calculators may bring into the teaching and learning of mathematics, and a search for efficient means to use them.

This experience was a continuation of an earlier research looking at the integration of DERIVE into the study of algebra and calculus in France. An important limitation of this DERIVE experiment was that students generally lacked the familiarity with this technology necessary to really use it to support their mathematical activities and learning. So, when complex calculators became available we saw the potential for easier student access to computer algebra technology which might affect their

1 This study was done by a team of five. Michele Artigue was the leader, Badre Defouad and the author participated with the teachers, Michele Duperrier and Guy Juge, to the definition of the sessions and did classroom observations and interviews. A report on the project can be obtained from DIDIREM Université Paris VII 75251 Paris Cedex 05, France. The project was founded by the French ministry of Education (DISTEN B 2).

everyday mathematical practices, and that we would be able to observe more substantial changes. For this reason, we did a pilot project, designing and experimenting lessons/activities in four classes of the ordinary French scientific upper secondary level (11^{th} grade) where every student had a TI-92[2]. This project forced us to conceptualise the changes in the mathematical activity of students using this complex calculator. We had also to think about the help that the various multilevel capabilities of this calculator can bring to teach a specific subject like pre-calculus.

The evolution of approaches to the use of computer technology in the learning of mathematics

From constructivist approaches to mediation

When computers became available many hopes were placed on the autonomous cognitive activity that a learner could develop when faced with specific tasks (Artigue, 1996). The general frame was a Piagetian approach: acting in adequately problematic settings, the learner meets insufficiency or inconsistency of his/her knowledge. Introducing computer environment could help to create settings of this kind. More recently Noss and Hoyles (1996) stressed that a computer application may operate as a linguistic tool, and they emphasise programming as a tool for expressing and articulating ideas. Therefore, Noss and Hoyles introduced mediation as a major role for the computer in the student's process of abstraction.

This idea of mediation is useful in our project because a purely constructivist view of the use of computers is insufficient to analyse the interaction between the user, his/her instrument and the objects in the settings. A constructivist view assumes that the computer settings will provide the means for a predictable and meaningful interaction. What actually happened when we observed the use of DERIVE was different: students' reactions and reflections did not have the meaning that the teacher expected because their perception of the feedback was influenced by the operation of the software (Lagrange, 1996).

The role of the instrumental schemes

Moreover, when the user is learning new capabilities, s/he does not distinguish the internal capabilities and the features of the interface. In this phase, the calculator is better seen as a complex instrument like those existing in the area of professional working, rather than as addition of a neutral interface and an internal algebraic language. For this reason, I use the theoretical approach of those instruments by psychologists like Verillon and Rabardel (1995). They stress that a human creation,

[2] At the time of the experiment, the relatively expensive Texas Instrument TI-92 was the only hand held calculator with symbolic capabilities. New student affordable calculators like the TI-89 and the Casio GRAPH 80 now offer symbolic capabilities.

an 'artefact', is not immediately an instrument. A human being who wants to use an artefact builds up his/her relation with the artefact in two directions: externally s/he develops uses of the artefact and internally, s/he builds 'instrument utilisation schemes' to control these uses. This process is the 'instrumental genesis'.

Schemes in calculus using a complex calculator

As an example of this approach, the figure 1 displays various schemes, calculator oriented or not, algebraic, graphic or symbolic that a user of a TI-92 can use to search for the variations of a function like $\frac{x^2+x+0.01}{x}$.

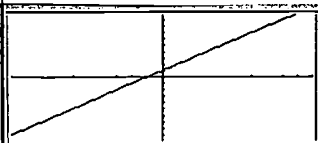
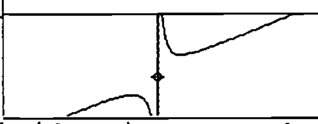
Nature of scheme	TI-92 output	Decisional dimension	Pragmatic dimension	Interpretative dimension
Graphic, TI-92		Graphing in the standard window is a good approach	Consider the graph of the function in the standard window	Function is increasing. Graph is a straight line
Algebraic, criticism	none	Graphical evidence must be compared to algebraic aspects	Consider the algebraic definition of the function	$f(x)$ is not a linear function
Analytic, TI-92	$\text{expand}\left(\frac{x^2+x+.01}{x}\right)$ $\quad x+\frac{1}{100\cdot x}+1$	Expanding an expression helps the interpretation of the graph	Consider another algebraic expression of the function	There is something special near x=0
Graphic, TI-92		Graphic display will confirm analytic ideas	Zoom in around x=0 and y=0 until something appears	There are two turning points near zero
Calculus, TI-92	$\frac{d}{dx}\left(\frac{x^2+x+.01}{x}\right)$ $\quad \frac{100\cdot x^2-1}{100\cdot x^2}$ $\text{zeros}\left(\frac{100\cdot x^2-1}{100\cdot x^2},x\right)$ $\quad \{-1/10 \;\; 1/10\}$	Use the derivative to find the turning points	Find the zeros of the derivative	Position of the turning points is confirmed

Figure 1: Schemes in a search for the variations of a function

The first scheme (graphing in the standard window) is prevalent among most students. In the initial stages of learning calculus very few students are able to produce critical interpretations like in the second scheme, even when they have the algebraic knowledge to do so. In contrast, more able students develop schemes where graphical action is linked with algebraic and analytic interpretation: they see, in the graph, properties that they anticipate from an algebraic analysis of the function.

Transforming the expression of the function like in the third scheme is not a spontaneous action to beginners. More advanced students choose the transformation randomly among the TI-92 capabilities rather than from rational reflection. Teaching can help to develop this reflection. Switching back to the graph window, as in the

fourth scheme, is quite natural. Able students anticipate immediately the required zooming, while others take considerable time over this decision. Students may have learnt the calculus approach in the fifth scheme as a method. As a scheme, it is often limited, because students are not able to activate it when the function is not a standard rational function considered in the teaching.

From this brief description of features of schemes appearing in a calculus task, and their apprehension by students, questions arise on the genesis of these schemes in an educational context. What is the link between the development of utilisation schemes by students and the development of their mathematical knowledge? How is teaching to be oriented to help the development of suitable schemes, their generalisation and their co-ordination?

An approach of teaching with instruments

Schemes for building knowledge

In Vergnaud's (1990) approach, schemes play a key role in conceptualisation: they organise the behaviour of a person in a class of problems and situations representative of a field of concepts, and they are a basis for knowledge in this field. When a person learns mathematics with an instrument, his/her schemes organise behaviours related to the use of the instrument as well as more general conducts. So the potential of complex calculators is to be searched in the many possible schemes that a user is able to build for its use. However, this potential will be effective only in situations where utilisation schemes are productive of adequate knowledge and mathematical meaning.

This is not all situations when we consider, for instance, the limit of a rational expression at a finite or infinite point. In an ordinary 'non computer' context, students may apply infinitesimal reasoning or methods, possibly rich in meanings. In contrast, with a calculator like the TI-92 or algebraic software like DERIVE, students are able to associate the idea of limit with a single scheme: pressing the 'limit' key of the calculator and reading the output on the screen. This scheme is effective for the task but, as Monaghan et al. (1994) observed, it may result in giving students a narrow understanding of the notion of limit.

So, depending on their co-operation with other schemes or meanings, schemes of use of the TI-92 or DERIVE are productive or not. Therefore, for the support of the technology to be effective teachers must control students' development of utilisation schemes and their co-ordination with the advancement of mathematical knowledge.

The role of tasks and techniques in the use of a complex calculator

Techniques and their relationship with the instrumental genesis are a key point in the use of technology to teach and learn calculus. Authors and teachers assume that the symbolic capabilities in this technology are means to lessen the stress on techniques

which, they consider, restrain students' reflection on concepts. This view was clearly present in teachers' expectations in the French DERIVE experiment, and I could establish the limits of this excessively conceptual approach (Lagrange, 1996). Of course, techniques without schemes are ineffective because they are not likely to evolve and cannot produce knowledge. However, classroom discussion on techniques is essential to help students to develop suitable schemes and enhance the reflective part of these[3].

Unlike the paper/pencil techniques, TI-92 techniques rationalise schemes of use of an instrument, and, according with Rabardel and Vérillon, these schemes develop in an instrumental genesis. A consequence is that the organisation of the tasks and associated techniques must comply with the constraints of that genesis and direct it in a productive way: schemes cannot develop arbitrary and not all combinations of schemes are able to produce mathematical meaning. Below we look more closely at these constraints and their implications in terms of tasks and techniques in the specific subject of pre-calculus.

Teaching pre-calculus with complex calculators

Tasks and techniques to develop an appropriate instrumental genesis for algebra and functions

At the beginning, the student tries to bring the schemes s/he built for his/her familiar graphing calculator into use. Tasks and techniques are thus to be organised to help him/her learn that, in the default mode, the TI-92 simplifies radicals and rational numbers symbolically and the difference between this 'exact' mode and the various decimal approximations. Then schemes of use of the algebraic capabilities are essential. A key point is the notion of equivalence of expressions and the need for awareness of the different equivalent forms of an expression. A student must learn to consciously use the items of the algebra menu, to decide whether expressions are equivalent as well as anticipate the output of a given transformation on a given expression. This work has to be continued when new expressions, like trigonometric functions, are introduced.

Like many calculators, the TI-92 offers a graphical window and a numerical table with a wide range of capabilities. Therefore, it may enhance early functional thinking because graphical and numerical schemes are essential for the growth of the function concept. As seen above, figure 1, notions like the variations of a function implies the co-ordination of algebraic and graphico-numerical utilisation schemes. A relevant task for developing these schemes is the study of functions whose properties are not obvious in a standard graph (for examples see Figure 1, and Trouche, 1994).

[3] For a more comprehensive study of the role of techniques in the use of CAS, in relation with Chevallard's (1992) theoretical approach, see (Lagrange, 1997).

Helping students to develop flexible links between calculus concept representations

The use of the TI-92 in calculus is quite simple: a menu entry for the symbolic calculation of limits and a key for the calculation of derivatives. So, as a difference with algebra and functions, no specific instrumental learning will be necessary. However, we saw above, with limits, that this use tends to produce symbolic manipulative schemes, likely to generate a narrow understanding of these concepts if they are alone. This implies a deep reflection on how teaching can help the development of other schemes.

Tall (1996) stresses that there is not a single way to teach pre-calculus, but "a spectrum of possible approaches ... from real-word calculus ... through the numeric, symbolic and graphic representations in elementary calculus, and on the to the formal ... approach of analysis". Like many others he emphasises the need for helping students to move flexibly from one representation to another and to establish a balance between representations

Technology tends to toss the traditional balance about. For instance, in France, every student in the secondary level has now a graphic calculator. This situation clearly changes the balance of the numeric and graphical representations and of the symbolic view of the concepts of calculus. Students often prefer experimenting from the graph rather than analysing with symbolic methods. Trouche (1994) noticed this behaviour and he highlighted the need for developing students schemes to control the graphs and numbers they obtain on their calculators. Now, with the TI-92 and others, calculators are graphico-numeric *and* symbolic. Little is known of how this new feature will affect the students' balance and flexibility between representations[4].

In the everyday use the calculator, higher complexity of the instrument may change students' flexibility. With common graphing calculators, graphical and numerical schemes are instrumental when analytic schemes are associated with pencil and paper practices. In contrast, with the TI-92, co-ordinating analytic and graphico-numerical schemes implies controlled switches between windows. In those switches, a clear view of the organisation in the calculator is essential and teaching has to help students to build this view.

The balance of representations was a key point in the design of teaching modules for pre-calculus courses. Our team assumed first that easier symbolic calculation enlarges the possibility of linking enactive representations and theoretical calculus,

[4] Ruthven (1997) reviews a number of researches into CAS in mathematics education. The prevalent topic is the comparison of student performances between CAS and non CAS students, so, little is known of the impact of CAS in the everyday teaching.

and also that teaching must avoid the danger of too close an association between concepts and symbolic manipulative schemes, wiping out other representations.

Enactive representations (Tall, 1996) exist in the prior differential knowledge of students. For instance, most students have a sense of the tangential behaviour of curves from their geometrical experience. It seems important to use this knowledge as a basis for the theoretical concept of derivative, because differentiation is an analytic answer to the question of the tangent line for a curve defined by a function. However, in the ordinary context of paper and pencil calculations, students cannot really question their enactive differential notions because they would have to consider, and give sense to, expressions which are beyond their abilities. In our experiment, using symbolic computation helped students to work with these expressions and to understand their meaning. A condition for this was the limitation of the cognitive complexity of the situation, in order that students could really reflect by themselves: an instrument like the TI-92 reduces time consuming calculations but not the difficulty of concepts. Also, students' development of algebraic instrumental schemes was essential to ensure the success of this work.

Our team was concerned that students may use the symbolic capabilities of the calculator for very simple limits or derivatives and see nothing more in those concepts than the manipulative aspects. As Monaghan et al. (1994) argue, symbolic computation may make manipulations effortless but tends to obscure other representations linked with infinitesimal approaches. For that reason, we preferred to introduce the TI-92 capabilities for limits and derivative only after students did considerable work on the concepts, linking enactive views with graphico-numerical approaches and symbolic forms.

For instance, in our experiment, the limit concept was introduced from an intuitive view that a function 'tends toward zero as x tends toward zero'. Students did a lot of graphic and numeric work, passing from '$f(x)$ is small when x is small' to '$f(x)$can be arbitrary small provided that x is small enough'. Then students had to study by the same means standard, as well as non regular, limits before the *limit* function of the TI-92 was introduced. After this it was time to consider the symbolic aspects of the concepts, namely the algebraic rules by which a person or a machine is able to obtain limits of expressions. Students could consider several examples on how the TI-92 computes limits and then learn to do those calculations by themselves. In this process the student had more self-reflection than in a formal approach where the teacher demonstrates the rules.

Conclusion

Comparing earlier approaches where mathematical knowledge is thought to be built from situations of personal interaction with the computer, the contribution of a complex calculator like the TI-92 appears different: the calculator acts as a mediator

for the action of students. In this mediation technology is by no means neutral: meeting new potentialities and constraints the students have to elaborate utilisation schemes, potentially rich in mathematical meanings. For this potential to be effective, attention is to be paid to the understanding that the students build in this genesis, and techniques of use of the calculator should be discussed in the classroom to improve students' schemes. In the learning of pre-calculus, teaching should help students to develop adequate algebraic knowledge and flexible connections of various representations of concepts. With respect to this aim, the analyse of the interaction of utilisation schemes is a key to design tasks and situations involving students' use of a complex calculator.

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STORING A 3D IMAGE IN THE WORKING MEMORY

Lea Latner and Nitsa Movshovitz-Hadar
Technion - Israel Institute of Technology, Haifa, ISRAEL

Abstract

This paper elaborates on some lessons learned from observing high school graduates while performing tasks in 3D geometry. We observed difficulties, which we were able to attribute to a change in the original mental image of a solid, created by students who were presented with its 2D perspective drawing. Evidently, the original mental image changed unwillingly while performing the task. In some cases it was replaced by a different 3D image, and in other cases, by a 2D image. In some cases the student was aware of the change as it occurred, while in other cases the change happened unnoticed. Following the description of such observations, an attempt is made to explain them, applying theories regarding the storage of information in the working memory.

Background

Israel Matriculation exams results indicate that the percentage of students choosing to answer questions in Solid Geometry is lower than the percentage of students who chose to answer questions in any other mathematical area. Moreover, of those who chose to tackle such a problem, the percentage of those who complete it successfully, is significantly lower than the percentage of students answering correctly questions in any other area. We found this phenomenon intriguing and set ourselves to study the difficulties students face while performing tasks in solid geometry and in other related areas requiring spatial ability.

Although a consensus as to the components of spatial ability, has not as yet been reached, most researchers (e.g. Clements & Lean 1981, Bishop 1983, 1989 Lohman 1979, Yakimanskaya 1991, Battista & Clements 1988) point at some or all of the following as major ones:

a. The ability to create a mental image of a solid in 3D space;
b. The ability to retain this mental image;
c. The ability to manipulate a mental image;
d. The ability to "see" the relations among the various parts of a solid.

This paper is focused on the first two components. Using evidence collected in a series of interviews it demonstrates some phenomena which can be attributed to difficulties in storing a mental image of a 3D solid, created from its 2D representation provided on paper, and to difficulties in keeping the mental image intact, while attempting to perform the task.

Theoretical Framework: Working memory

Cognitive psychologists distinguish between long term memory (LTM) and short term memory (STM). Since the beginning of the 20th century, studies of STM focused primarily on the number of items stored, items that are not necessarily related to one another. Those studies concluded that STM is limited to an average of seven items. Atkinson & Shivering were (in Loggia 1991) the first who came up with a more complex model to STM that included the distinction of working memory. According to their model the main function of the working memory is to store information while engaging in a thinking process such as planing, or problem solving. Consequently, one should regard the working memory as a calculating area with operational abilities. While performing a complex task that requires many calculations, a need arises not only to store numerous mid-way results, but also to organize them in the working memory in a way that relates each of them to the work in progress. There is a distinct difference in capacity and content between the working memory and the short-term memory. Not only can the working memory store five to seven separate and unconnected items, it must include also pointers that connect the different items to one another.

Carpenter & Just (1989) claim that there is a mutual relationship between the capacity of the working memory and the two functions it performs: (i) as a storage area; (ii) as a calculating area. When a need arises to use one of these functions in an extensive manner, it overrides the other function. If calculations are carried out more or less automatically, there is more room in the storage area.

The study, parts of which are described in this paper, was based upon a set of tasks that required both functions of the working memory.

The Method

The study was based upon semi-structured interviews with high school graduates, each lasting a total of 5-8 hours in 2-3 meetings. A special instrument was designed to guide the interview. It consists of 26 tasks, all of which required a certain degree of spatial ability. Each interviewee was required to perform the tasks while "thinking out loud". The process was recorded. (A few sample tasks appear in the next section.)

Performing each task, required creation of a mental image of a 3D object represented by a 2D perspective drawing on paper, and manipulating it. In other words, all tasks required retaining the mental image of the object throughout the performance. The 2D drawing remained available to the interviewee, throughout the time of performance.

All the interviewees took advanced level math in high school. Prior to the interview, each of them took ETS - a standard 3D exam, in order to determine their spatial abilities.

Observations

Three observations are documented below. They all indicate difficulties in maintaining the same mental image created by the interviewee when presented with the figure included in the task, throughout the period of time devoted to the task. Note that each task starts in a request to describe the configuration presented in the figure, followed by a request to operate upon it.

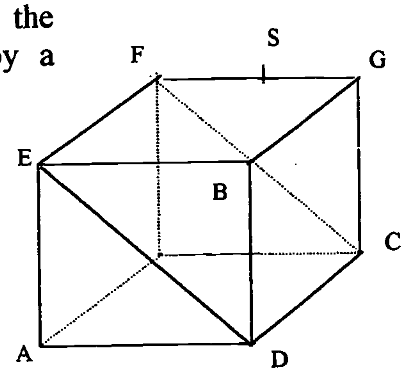

Figure 1

Observation no. 1
The task:

Consider a cube ABCDEFGH, and the plane EFCD (see Figure 1)

a. *Use your own words to describe the cube and the relative position of the plane.*
b. *S is the mid-point of FG. Find a symmetrical point to S, with respect to the plane FECD.*

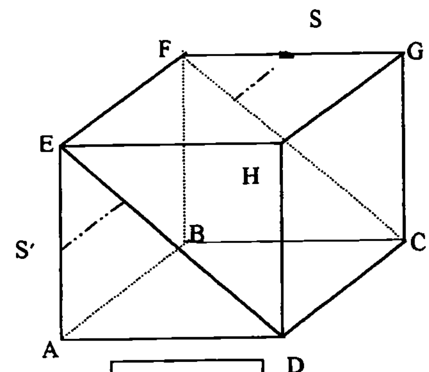

Figure 2

<u>Ilana's answers</u>

To part a: "It's a cube, it's a right cube, the plane crosses through the middle, inside the cube."

To part b: "Where do I drop the perpendicular to this plane? Perpendicular from S to the plane… oh…. I don't know where to drop it".

She drew the perpendicular from S to FC and extended it to the middle of EA. (Marked by------ in figure 2) and said: "I measure the distance by sight."

Observation no. 2
The task:

Consider a cube ABCDEFGH and the triangle GAD

a. *Using your own words describe the relative position of the triangle inside the cube.*
b. *Draw the angle between the triangle and the face EFGH.*

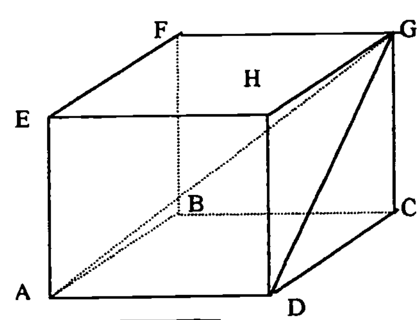

Figure 3

Hanna's answers

To part a: "The triangle (points to GAD) is slanted. No, actually it is not slanted. (She closes her eyes in an effort to visualize it, she uses her hands to draw in the air). "Yes, it is tilted, if it was from H it wouldn't be slanted."

To part b: "We need to drop a perpendicular to FG from triangle AGD. I don't know how to drop such a perpendicular. Wait …., we extend DH… and then, create the perpendicular (marked by –·–·-- in figure 4)…, DHEA is perpendicular to GHEF, I don't know, it can't be like that, I'm working in two dimensions….".

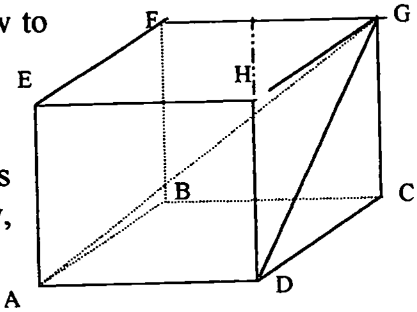

Figure 4

In the concluding part of the interview with Hanna, she tried to explain the phenomenon, as she perceived it: "When I concentrate more on the angles I switch to two dimensions. Because it is easier for me, because I understand the angles in two dimensions, the angles in 3D are different, they are not the same as in 2D."

Observation no. 3

The task:

Consider a right triangular prism ABCDEF, and a segment GA, where G is the mid-point of DF)

a. *Using your own words describe the prism and the relative position of GA.*

b. *What angle is created between GA and AB?*

Michael's answers

To part a: "Oops…I saw it a moment ago…oops, oops… I see it alright, but now it is different than I saw it earlier"

To part b: "GA with AB, oh, oh, oh…. Do you know what just happened to me? All of a sudden the prism turned outside-in. I'm trying to turn it back….how do you want me to see it, as a tent? No I didn't see it as a tent…I saw it… I can't get in to that now. I saw an oblique tent, but now I see a straight tent."

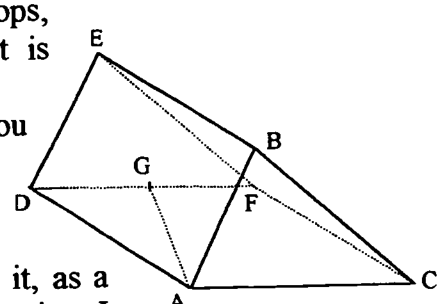

Figure 5

Observation analysis

Observation no. 1: The verbal description included in Ilana's answer to part a, indicates that she had created a 3D mental image of the cube and the plane inside it. In the first part of her answer to part b, the first mental image was still stored in her memory, while trying to drop the perpendicular from S to the plane. From her saying "how do I drop a perpendicular to the plane?" one can assume that she was familiar with the definition of symmetric points with respect to a plane. She knew she must drop a perpendicular. She also knew she must extend the perpendicular beyond the plane: "I measure the distance by sight", she said. However, she failed to execute it. A reasonable explanation for this gap is that the difficulty occurred because, while thinking about the process, the original 3D image of the object she created, was swapped with a 2D. Unaware of that switch, she created a new, flat image in which the lines FG and EA are on the same plane with the rectangle oblique plane. (See figure 6 as compared to Figure 2).

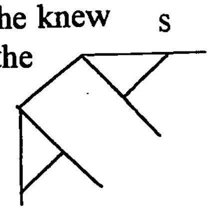

Figure 6

Observation no. 2: The first mental image created in Hanna's mind was most probably of a 3D object, as can be gathered from her reply to part a. However, due to her efforts to drop a perpendicular in the triangle plane, she lost her original 3D image and created a new 2D image of the object. In the new image DH, when extended, intersected with GF creating a straight angle: "Wait", she said, "... we'll continue DH and then create the perpendicular." At that point Hanna noticed she was working with a 2D image: "I was working in two dimensions", she noted, and reconstructed the cube image: "DHEA is vertical to GHEF". From that point Hanna managed to maintain a 3D image of the object. Unlike the interviewee in the first observation, in this case the interviewee was well aware of the fact that she was working in 2D, although she wasn't aware of the exact moment the image dimensional swap took place. Only after working a while in 2D did she realize it.

Observation no. 3: In the described event Michael was aware of the exact moment the image swapping took place, while examining the relations in the drawing. His awareness to the moment of the image swapping, enabled him to stop his work until he managed to concentrate and reconstruct his first 3D image. Due to the described circumstances this phenomenon is less serious than the previous one. Nevertheless, it may withhold the completion of the task due to the time necessary to concentrate and reconstruct the first image.

Discussion

The findings described above exemplify one phenomenon in three different levels, the phenomenon of change in the original mental image of a spatial configuration, while operating upon it. The 3D image stored in the working memory, goes through involuntary changes, while performing the task. In the first case the interviewee was not aware at all of the change, in the second case the interviewee was unaware of the exact moment the image swap occurs, although she became aware of it later on, while the third interviewee did notice it instantaneously. In the first two cases the change was from a 3D image to a 2D image and caused real difficulty in completing the task successfully. In the third case it was a matter of time and effort only, since the image twist was from one 3D to another one, as good as the first one.

The phenomenon described in the third observation is well known. Focusing continuously on some drawing, or attempting to examine the relations among different elements of a 3D object described in a 2D figure, may cause the first 3D image to modify. The back part of the object FD (see figure 7) "pops out" and creates the impression that the object turns inside-out or the other way around. This well-known phenomenon, especially related to cubes, is known in psychology literature as "Necker Cube phenomenon". This phenomenon occurs when the drawing of the object has a few stabled positions. One of the interpretations to this phenomenon is that the memory brain cells containing the object in one of its stabled forms wear down, and other brain cells take their place in storing the object, but this time in a different stabled form.

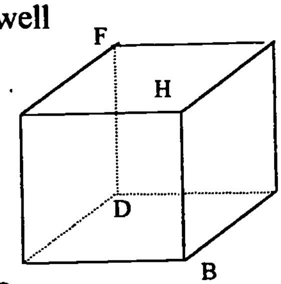

Figure 7

In our study we had many occurrences such as the third case. In all of them, except one, the original 3D image changed into another 3D image. In one case (which isn't mentioned in this paper) the 3D image changed involuntarily into a 2D image. This single event may indicate that the three observations described above are related to one another. The difference between the first two and the third one is whether or not the interviewee was aware to the moment the image swap occurred. Lack of awareness in the first two observations leads to unavoidable errors, while the in third one there was a temporary loss of concentration, not necessarily yielding an error. These findings are summarized in Figure 8.

In general, in many cases a 2D object requires a lot less room in the working memory than a 3D object both having the same 2D figure representation. For instance, a right angle in 2D drawing of a 2D configuration appears as such, while a right angle in a 3D configuration does not necessarily take the form of a right angle in a 2D drawing of that configuration. Consequently, it consumes more room in the working memory.

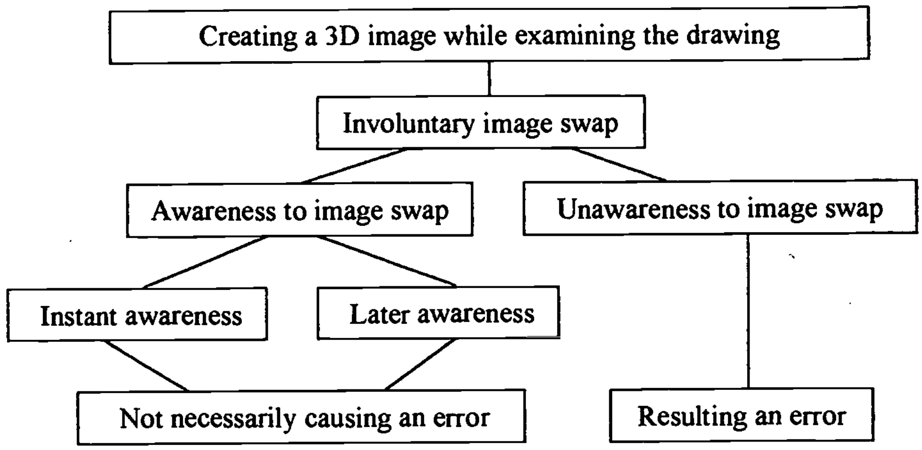

Figure 8

Observation no. 1, 2 may be explained as an overload in the working memory. As mentioned earlier, the working memory fulfills two functions, storing information and processing it. Each one of the two functions occupies memory space. Interviewees who spent a considerable amount of time in examining the relations within a drawing, had to store many details of the object's properties. We suggest that our first two observations can be explained by that storing and processing all the needed information, overloaded the interviewee's working memory. As a result some of the information, in particular information concerning the depth of the 3D image, was lost in the process. The 3D image changed into a 2D image as the 2D image occupies less space in the memory.

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EXPLORING VAN HIELE LEVELS OF UNDERSTANDING USING A RASCH ANALYSIS

Christine Lawrie
Centre for Cognition Research in Learning and Teaching
University of New England, Armidale, Australia

This paper reports a study designed to explore the difficulties associated with the development of appropriate test questions suitable for determining van Hiele levels of understanding in geometry. Initially, a written test based on the Mayberry interview schedule was designed and given to 60 pre-service primary teachers. Analysis of the results by concept and by level led to the identification of strengths as well as inconsistencies in the items. The initial results for the items were then processed using the QUEST software application of the Rasch partial credit model provided by Masters, and analysed in depth. The purpose of the Rasch analysis was to provide a quantitative edge to what had been primarily a qualitative procedure. This paper presents the findings of the Rasch analysis.

Many difficulties are associated with the designing of appropriate test items for determining the van Hiele levels of understanding in students, and with the assessment of the students' responses to the test items. To make an assessment, there needs to be available a reliable diagnostic instrument. At the University of New England, Australia, a written test based on the Mayberry (1981) interview schedule was designed. The items tested for understanding in seven geometric concepts. Each question was designed to the operational definition of one of the van Hiele levels. Examination of the initial assessment of student responses indicated inconsistencies in the results (Lawrie 1997, 1998). To confirm the initial results and to explore further the strengths and weaknesses of the test items, the results were analysed using an application of the Rasch model. In this article, an analysis of the Rasch model results is presented. Before doing this, a brief background to the important ideas underpinning the study is presented.

Background

The van Hiele Theory

In the 1950s, Pierre van Hiele and Dina van Hiele-Geldof completed companion PhDs which had evolved from the difficulties they had experienced as teachers of Geometry in secondary schools. Whereas Dina van Hiele-Geldof explored the teaching phases necessary in order to assist students to move from one level of understanding to the next, Pierre van Hiele's work developed the theory involving

five levels of insight. A brief description of the first four van Hiele levels, the ones commonly displayed by secondary students and most relevant to this study, is given:

Level 1 Perception is visual only. A figure is seen as a total entity and as a specific shape. Properties play no explicit part in the recognition of the shape.

Level 2 The figure is now identified by its geometric properties rather than by its overall shape. However, the properties are seen in isolation.

Level 3 The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised.

Level 4 Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions are used with understanding.

The van Hieles (van Hiele 1986) saw their levels as forming a hierarchy of growth. A student can only achieve understanding at a level if he/she has mastered the previous level(s) They also saw (i) the levels as discontinuous, i.e., students do not move through the levels smoothly, (ii) the need for a student to reach a 'crisis of thinking' before proceeding to a new level, and (iii) students at different levels speaking a 'different language' and having a different mental organisation.

The Rasch scaling model

The Rasch partial credit model was first introduced for the analysis of dichotomously-scored responses in 1960. When data are fitted to the model, person parameters can be freed from the item difficulties and item parameters can be estimated independently of the calibrating sample. Masters (1982, p. 163-166) used a maximum likelihood procedure to estimate the parameters in the model. Means and standard deviations of the infit (weighted) and outfit (unweighted) statistics allow a check on how well the data fit the proposed mode. The advantage of the partial credit model over other models is that the parameters in the model are separable. This allows on a single scale, a measure of item difficulty, and of an individual student's ability to achieve success.

The QUEST Interactive Test Analysis System (Adams & Khoo 1993) which is based on Master's procedure, was used to process the data. This software includes the most recent developments in Rasch measurement theory for the analysis of test and questionnaire responses as well as traditional analysis procedures. Three measures are available to check on the suitability of the tests as a measure of geometric understanding. First, the item consistency is used to estimate the extent to which items reflect the same underlying construct. Second, the fit statistics for question estimates and student estimates can check whether the data compared favourably with the model. Third, the infit mean square map shows the level of

parameter fit in the model for each question. The estimates produced by QUEST are used to create a number of statistics which then allow analysis of various features of student performance and of the hierarchy of level used to code the questions.

Design

To investigate the difficulties associated with the assessment of van Hiele levels of reasoning, a detailed study of the geometric understanding of 60 pre-service primary teachers was carried out at the University of New England, Australia. The study aimed, in part, to provide a written test based on the Mayberry (1981) interview schedule in which each item was designed to the operational definition of one of the van Hiele levels. The items tested for understanding in van Hiele Levels 1 to 4, in seven geometric concepts, square, right triangle, isosceles triangle, circle, parallel lines, congruency and similarity. To accommodate time restrictions, the seven concepts were divided between two test papers, Papers I and II. Initially the responses of students to the written test were assessed using Mayberry's pass/fail scoring method of evaluation. Follow-up interviews were conducted with students to validate the levels of thinking as determined in the written test. When collating the results, inconsistencies in the assignment of van Hiele levels for some students emerged (Lawrie 1997). Interviews did not appear to clarify these inconsistencies. The QUEST program (which utilises the Rasch measurement theory) was applied to the results to confirm the initial assessment, to confirm patterns which emerged in the results, and to determine whether there were any other patterns not yet identified, i.e., the Rasch analysis was applied to give a sharper quantitative edge to what had been primarily a qualitative procedure.

Initial Results

Table 1

Highest level achieved by the Australian students for each concept (% of sample)

Concept	No Level	Level 1	Level 2	Level 3	Level 4
Square	0	3	84	7	7
Right triangle	3	19	55	19	3
Isosceles triangle	7	27	43	20	3
Circle	0	13	19	52	16
Parallel lines	0	17	80	0	3
Congruency	0	32	35	3	29
Similarity	0	43	40	10	7

The results of the initial assessment of the students' levels of understanding are summarised above in Table 1. When students failed to identify concepts, their results were recorded as No Level. All results are given as percentages, with the horizontal sums being (approximately) 100%.

The results show that the majority of students (77%) were assessed as having mastery of no greater than Level 2 understanding, i.e., they were comfortable recognising concepts and listing the associated properties, but did not understand the relationships between the properties.

The results (7.8%) of a few of the students showed inconsistencies in that the results did not validate the level hierarchy. One particular issue related to the inconsistencies was that some of the items did not appear to be measuring the van Hiele level for which they had been designed.

Results using the Rasch model

The QUEST software application (Adams and Khoo 1993) which utilises the Rasch measurement theory was applied to the results. A comparison drawn by the program between the relative degrees of difficulty of the question parts demonstrated patterns which support the van Hiele level structure, and produced an estimate of geometric understanding for each student. The reliability of estimates for both items and cases, and the internal consistency parameters were all close to 1, showing that there was good separation between the items and between the cases, and that the items came from the same underlying construct, i.e., geometric understanding. All the infit and outfit mean squares for item and case estimates were close to one, indicating that the data fit the Rasch model well.

The statistics from the QUEST application indicated the following patterns. For the infit mean square maps, the items with the largest values are those which have been answered correctly by a greater than expected number of weaker students, and incorrectly answered by a greater than expected number of better students. Almost all the items with largest values were found to be testing for understanding of van Hiele Level 1. This suggests that while the questions were clear to the weaker students, the better students may have had difficulty in interpreting the thrust of the Level 1 questions. Second, the item estimate maps showed that Level 2 question parts requiring the identification and naming of properties of sides of figures generally had lower difficulty thresholds than those requiring demonstration of knowledge of angle properties. This indicates that students were more confident with, and had a better awareness of properties of sides than properties of angles.

The calibration of the likelihood of students to answer items correctly in the item estimate maps showed that the average student was able to give a correct response to questions testing for understanding of van Hiele Levels 1 and 2, together with a very few of the questions testing for understanding of Level 3. The item estimate maps indicated also that the difficulty thresholds of some items supported the earlier observation that those items were unsuitable for measuring reasoning at the thought levels specified by Mayberry. In particular, difficulty thresholds for some Level 3 and Level 4 items suggested that the questions do not have sufficient complexity to measure the level for which each had been designed.

Analysis

To facilitate a deeper analysis of the results, the question parts were grouped by concept within each van Hiele level, then plotted against their difficulty thresholds. Any clusters appearing in the resultant graphs could indicate patterns worthy of analysis. The graphs are given in Figures 1 and 2. Each graph shows an increasing degree of difficulty from lower left to upper right, corresponding approximately with van Hiele Levels 1 to 4. However, the change in the degree of difficulty in both figures is much greater between Levels 2 and 3 than between the other levels. There is little discernible difference, and quite a degree of overlap between the degrees of difficulty for the questions set to test Levels 1 and 2, and, again, for Levels 3 and 4. This suggests that the students found the questions testing for understanding at Levels 1 and 2 to be similarly easy, while the questions testing for Levels 3 and 4 were found to be similarly difficult. Such an interpretation supports the results of the initial analysis, that the majority of the students were unable to demonstrate understanding for van Hiele levels higher than Level 2. The overlap between the levels suggests also that not all questions are testing for the van Hiele level for which they have been designed.

Figures 1 and 2 indicate that there is a greater range in the difficulty thresholds in Level 3 than in the other van Hiele levels. The item analysis gives the range as from -3.40 to +4.02. First, there are several items which have negative difficulty thresholds similar to the Level 2 items. This feature is discussed below. Examination of the items with positive difficulty thresholds led to the emergence of a further pattern. When the phrasing of questions was more generalised, the difficulty threshold of the questions was greater. Correspondingly, a lower difficulty threshold was associated with questions which used short direct sentences and which referred to or included diagrams. Thus, the larger than expected range

in the difficulty thresholds for the Level 3 items appears to result from the particularly wide variety of question types.

Figure 1
Thresholds of Items for Each van Hiele Level - Paper I

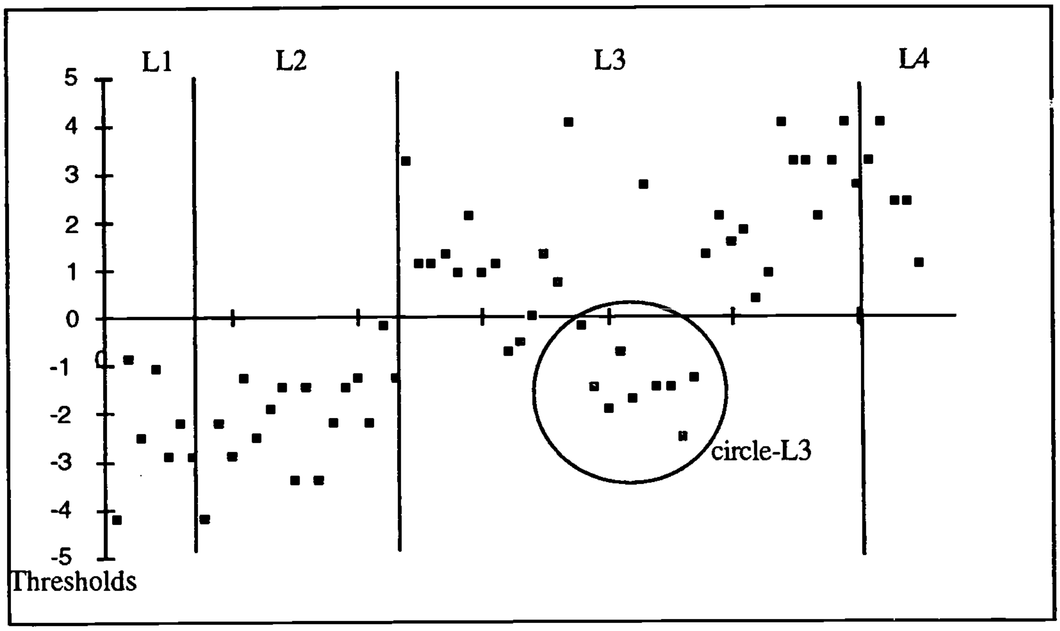
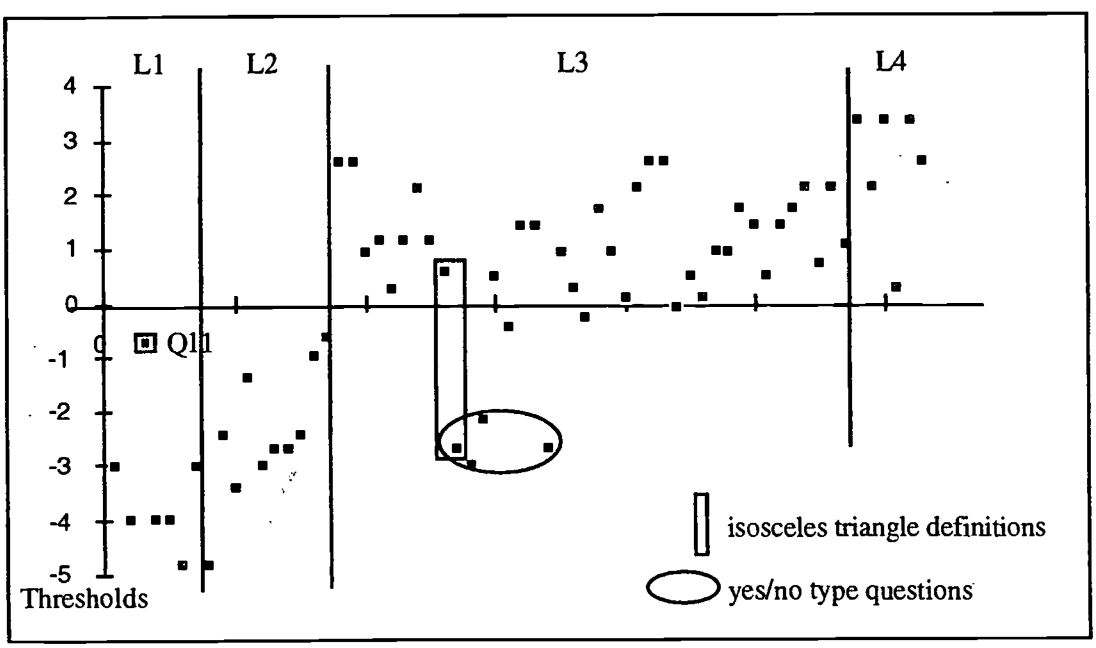


Figure 2
Thresholds of Items for Each van Hiele Level - Paper II



The four features highlighted in Figures 1 and 2 deserve comment. First, there is a cluster of nine items indicated by a circle in Figure 1. These question parts were all designed to measure for understanding of the circle at van Hiele Level 3. However, the graph indicates that the degree of difficulty of these nine question parts is similar to the degree of difficulty of the Level 2 items. This offers an explanation for the higher Level 3 success rate for the questions on the circle compared the results for other concepts, shown in Table 1. Second, Figure 2 shows a cluster of four items indicated by an ellipse. These question parts, which are across more than one concept, were also designed to test for Level 3 understanding. The four questions require either a simple yes/no response, or ask for recall of factual information. Their degree of difficulty is shown as being similar to that of the middle order Level 2 items, indicating that questions not requiring explanation of a student's answer do not necessarily provide insight into ability to reason at van Hiele Level 3.

Third, again in Figure 2, Q 11 is identified. The item was designed to test for Level 1 recognition of isosceles triangles. Its position in the graph and its threshold (-0.65) suggest that it is much more difficult than similar Level 1 questions. Examination of the item shows that a correct response requires identification of an equilateral triangle as an isosceles triangle, a Level 3 skill. Fourth, a rectangle in Figure 2 identifies two items showing noticeably different difficulty thresholds. Both questions test for awareness of necessary conditions in defining an isosceles triangle. However, Q 28 with the higher difficulty threshold (+1.41) probes for understanding of necessary conditions, while Q 29 (difficulty threshold of -2.72) asks for recall of a definition. This supports the above notion that questions requiring the recall of information without explanation of reasoning do not necessarily measure understanding greater than van Hiele Level 2.

Conclusion

The study employed a (relatively) new technique developed by ACER to provide a detailed quantitative analysis of the data. It involved use of the Rasch partial credit modelling process provided by Masters (1982). This allowed on a single scale, a measure of item difficulty, and of an individual student's ability to achieve success.

The purpose of the application of the Rasch model was to undertake a deeper analysis of the test items and the student responses than were available to Mayberry, and not previously undertaken in the area of geometry. Also, the analysis was designed to complement and extend the observations made in the initial results, by

giving a sharper quantitative edge to what had been primarily a qualitative procedure. Of significance are two features. These are:

1. The quantitative analysis confirmed the trends in the results which emerged in the initial analysis. These included confirmation that most students were able to demonstrate van Hiele Levels 1 and 2 understanding, but not higher levels of reasoning, and that not all questions were testing the level for which Mayberry had designed them.

2. Patterns in the responses which had not been perceived, or were not particularly clear in the initial analysis, emerged. Many of the Level 1 items showed the largest values in the infit mean square maps, indicating that higher performing students may have had difficulty in interpreting the thrust of these questions. A pattern made clearer in the QUEST analysis was that in the Level 3 questions, the more generalised the phrasing of, the greater was the degree of difficulty experienced by the students.

The focus of the analysis of the initial results on how students performed on a written test version of the Mayberry questions led to the identification of inconsistencies in the reasoning of some students. The QUEST analysis confirmed the occurrence of these inconsistencies.

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RETURNING TO UNIVERSITY: MATHEMATICS AND THE MATURE AGE STUDENT[1]

Gilah C. Leder and Helen J. Forgasz
La Trobe University, Australia

Abstract

In this paper we examine what motivates mature-age students to commence mainstream tertiary mathematics courses and subsequently to persist, modify or drop-out of their courses. Our explorations were shaped by the model of academic choice proposed by Eccles (Parsons) et al. (1985). Earlier findings from the study were presented at a previous PME conference.

Introduction

In many countries more students than ever before are accessing higher education. Yet, there is growing international concern about enrolment profiles and waning interest in the study of advanced mathematics (Jensen, Niss, & Wedege, 1998). The changing emphasis in Australia for entry into higher education from school leavers to first time participants, irrespective of age, was an important impetus for the present study.

What are the factors that motivate mature-age students[2] to commence mainstream tertiary mathematics courses and to make subsequent course-related decisions to persist, modify or drop-out of their courses? How effective are older students in utilising the opportunities for participation in tertiary mainstream mathematics courses and how do institutions respond to their specific needs? These questions formed the framework for a three year semi-longitudinal study. The study is now in its second year and comprised four phases:

1. determining relevant factors to explore (through an extensive literature review)
2. determining enrolment patterns of mature-age mathematics students
3. a large scale quantitative survey across five universities
4. a longitudinal qualitative study, to allow in-depth exploration of affective and related factors, of a smaller sample of mature age mathematics/science students.

Results from the first three phases, and the methods used, have been reported elsewhere (e.g., Brew, 1998; Forgasz, 1998a, 1998b; Forgasz & Leder, 1998; Leder 1998; Leder & Forgasz, 1998). In this paper we focus primarily on data gathered as part of the fourth phase.

The *model of academic choice* proposed by Eccles (Parsons) et al. (1985) served as an initial framework for probing the data. The model has previously been used to explain subject choice decisions of students and is assumed to be sensitive to gender differences. The authors contend that:

> choice is influenced most directly by the students' values (both the utility value of math for attaining future goals and the attainment or interest value

[1] The financial support of the Australian Research Council and the research assistance of Chris Brew are acknowledged with thanks.

[2] Students who are 21 or over on March 1 of the year in which University entry is sought.

of ongoing math activities) and the students' expectancies for success at math. These variables, in turn, are assumed to be influenced by students' goals, and their concepts of both their own academic abilities and the tasks demands. Individual differences on these attitudinal variables are assumed to result from students' perceptions of the beliefs of major socializers, the students' interpretation of their own history of academic performance, and the students' perception of appropriate behaviors and goals. (pp. 97-98)

Research aims

Whether the variables identified by Eccles et al (1985) likewise shaped the attitudes and decisions of the mature age sample in selecting and continuing with university mathematics is discussed in this paper. Concentrating on two students also allowed us to contrast the survey data gathered as part of phase 3 of the study with the much richer information yielded by the qualitative methods used in phase 4.

Literature review

Previous Australian research has indicated that gender, socio-economic status [SES] and home location (urban/rural) influence school completion rates (e.g., Ministerial Council on Education, Employment, Training and Youth Affairs [MCEETYA], 1994).

Students' attitudes to tertiary studies, decisions to enrol in higher education courses, and level of achievement attained appear to be influenced by a range of social, cultural and affective factors, the desire to pursue academic interests and to gain entry into an attractive career (McInnis, James, & McNaught, 1995; Ramsay, Tranter, Sumner & Barrett, 1996). Course related factors, such as difficulty, pressures, expectations, poor teaching, and boredom can also contribute to students' decisions to drop out of university (Abbott-Chapman et al., 1992). Impersonal and large class sizes, an ineffective tutorial system, lack of support, assistance and encouragement, and poor facilities were critical learning environment factors cited by withdrawing students. Pedagogical approaches, curricular content, the ethos of a mathematics department, and perceptions of discrimination have been given by mathematics students as reasons for their withdrawal (Forgasz, 1998a; Rogers, 1990; Taylor, 1990).

Australian studies (e.g., Hore & West, 1980) found that mature age students are typically from lower SES origins than younger students and that parental expectations for daughters to pursue tertiary education were often low. Forgasz (1996) and Pierce (1995) repoted mature age students to be highly motivated and success-oriented.

Overview of the project's instruments

Five data gathering tools were used: a survey questionnaire, interviews, regular e-mail (or snail-mail) correspondence, 'tag for a day', and the *Experience Sampling Method* (Csikszentmihalyi, Rathunde, & Whalen, 1993) or 'beeper-activated' schedule. A summary of the methods, with sample items, is shown on Table 1.

Table 1. Summary of data gathering methods

Method/Contents	Selected Sample items
Survey: computer	1a. Do you regularly speak a language other than English at home?

Method/Contents	Selected Sample items
scorable and open-ended items. Four main clusters: 1. biographical/ background details 2. enrolment issues 3. affective dimensions 4. perceptions of mathematics learning environment	(Yes/No) b. Parents' educational backgrounds (open) 2a. What degree/s are you studying? (open) b. Why are you studying mathematics? (check boxes from list provided) 3a. How good are you at mathematics? (1-5 rating) b. Do you enjoy university mathematics? (Yes/No/Sometimes) 4. Re: University mathematics: • well taught (5-point Likert: SA-SD) • lecturers approachable (SA-SD) • assessment is fair (SA-SD)
Interviews: semi-structured interview protocol (same issues as for survey)	• Tell me about your life pathway that has lead you to being enrolled in your current course. • Comment on university teaching: e.g., quality/ approachability of lecturers/tutors, level of support
E-mail/Snail-mail: monthly communications. Varied formats included open and closed items.	**May message**: How have you usually felt during lectures over the past month? (Mark with an "X" as many words that apply): interested; relaxed; worried; successful; confused; clever; happy; bored; rushed; panicky. Write one or more words of your own. **October message**: What are your reactions to your studies at university this semester? [content of courses, lectures, tutorials, assignments, pressures on your life etc.]
"Tag": spend time with student on campus. Observations recorded in field notes.	Observations of: • learning environment (to compare with student's views) • behaviour in lectures, with other students etc. Keep notes on conversation details
"Beeper": for six consecutive days, students were 'beeped' six times daily. They recorded what they were doing and feeling at these times on prepared *Experience Sampling Forms* [ESF]. Study, as well as other activities, were thus captured.	**As you were 'beeped':** • Where were you? .... • What were you doing? .... *On 5-point scales* (not at all - completely/very much) • Were you living up to your own expectations? • Was this activity important to you? • Were you satisfied with how you were doing? **Mood**: (semantic differential type; 5-point scale) irritable - cheerful; competitive - cooperative

The participants

Mature age students (99 students out of a total sample of 815) who completed the survey questionnaire were invited to participate in later stages of the study. Of those who volunteered, 26 were interviewed. Not all chose to participate in all qualitative data gathering phases. A summary of the sample sizes for each phase of the study to date is shown on Table 2.

Table 2. Summary of number of participants in each stage of the study to date

Survey	Interviews	E-mail	"Tag"	"Beeper"
99	26	19	9	21

The two students for whom we present data in this paper are Caitlin and Boyd (pseudonyms). Both were enrolled at the same university which, according to the large scale survey findings, was highly ranked on many of the tertiary mathematics learning environment variables tapped (see Forgasz & Leder, 1998).

Results and discussion: What the data say about Boyd and Caitlin

The survey data

Survey responses from Boyd and Caitlin revealed that they had similar English-speaking home backgrounds. They could be considered to come from low socio-economic [SES] circumstances, as indicated by their parents' educational levels (no tertiary education) and occupations (working class), and the school type (government) they attended. Neither Boyd nor Caitlin was in a permanent relationship and both lived in rental accommodation.

Both Boyd and Caitlin were enrolled as part-time students - Boyd in a mathematics and computer science diploma course and Caitlin in an Arts degree, majoring in mathematics. Neither received government financial assistance to study. Boyd had previously completed a degree course. At different times, Caitlin had begun, then dropped out of two university courses. Dissatisfied with her current situation, Caitlin wanted to enhance her career prospects and finally live up to family expectations. She chose mathematics because she had been successful at it in school. General curiosity and an interest in quantum mechanics (and later possibly in research) were Boyd's main reasons for studying mathematics. Both had enjoyed school mathematics but Boyd's school mathematics background was stronger than Caitlin's.

Both also enjoyed their mathematical studies at university. Boyd felt that he was excellent at mathematics and expected to achieve highly. Caitlin believed she was good at mathematics and also expected to excel. They did not think they were likely to drop out of a mathematics subject during the year. They found university mathematics to be fairly easy but challenging, believed they understood the work and were confident of passing. They considered that the subject was well taught, lectures were not boring, sufficient individual help was available, lecturers were approachable, and assessment and the work load were fair. Both indicated that they had neither perceived nor experienced any form of discrimination within the mathematics department.

From the survey data responses, Boyd and Caitlin appear to be similar. Much more was learnt about Boyd and Caitlin from the in-depth qualitative data.

Qualitative data sources

Interviews. Interviews were conducted early in the second year of the study. Summaries, including selected quotations, of the students' perceptions of themselves as mathematics students and of the learning environment are shown on Table 4.

Table 4. Summary of interview data for Caitlin and Boyd

Boyd	**Caitlin**
Past experiences and affective issues	
• failed maths "spectacularly" at school -	• excelled in mathematics at school

did not like it; returned to do grade 12 mathematics in 1996 - now has genuine interest in the subject. • 1st time at university: began BSc - transferred and completed BA • works in health-related job; felt he was burning out and also feared redundancy • studying now because of interest in quantum mechanics and philosophy; chose "mathematics rather than physics because I don't like laboratory classes... never get experiments to work". • finds 2nd year demanding; uncertain how far he will go	• parents: high expectations despite no-one in family with tertiary qualifications • 1st attempt at tertiary study: very poor, got involved in other things, dropped out • 2nd attempt: part-time: financial and personal crises, dropped out again • Now has different attitude - no parental pressure; "most important thing is to... get this degree after all the torment that it's caused me". "The greatest thing I've had to overcome is the confidence... to come back and give it another shot and succeed". • believes male mature age students have clearer vocational goals for studying mathematics than she has. Wants to work in a field that uses mathematics; thinks this unrealistic. Mixes mainly with male mature age students; sees female mature age students as sitting alone and fairly quiet. • derives pleasure out of "stretching my mind"
Perceptions of learning environment	
• noticed enormous gender imbalance: "This year is a less macho environment as the macho ones simply did not pass the first year". • expected staff to be intimidating but found mathematics tutors helpful.	• staff approachable, "only complaint... sometimes finding people talking down to me because I don't know as much about their field as they do".

E-mail correspondence. Both Caitlin and Boyd replied to all four e-mail/snail mail messages (May, July, September & October, 1998). Relevant responses to items probing their past experiences, tapping affective issues with respect to mathematics and views of the tertiary learning environment are summarised on Table 5.

Table 5. Summary of e-mail/snail mail correspondence from Caitlin and Boyd

Boyd	Caitlin
Affective factors	
May: Busy with university politics. Pain from old injury "disruptive enough to cause me to fall behind". "Enchanted" with one of his two mathematics subjects; "humoured" and "entertained" by lectures; "challenged" by tutorials. *July*: "Confident" in one mathematics subject; "worried" about the other. *Sept.*: "Apathetic & lethargic" about studies this semester. Has "a major motivational slump... towards studying the course material". Spending much time on mathematics unrelated to course: "In a way I can actually achieve my goals without even completing the course". *Oct.*: depressed and very stressed about	*From May & July*. Part-time job change: "I'm not sure whether this affected my exam performance but it was not an ideal way to study & I felt very stressed". "Not as well prepared [for examination] as I would have liked... [performed] very well... made two mistakes... very frustrating". *Sept*. Pleased with previous semester grade (A+); This semester even better: Enjoying studies, would like to be full-time: "I feel very confident and comfortable with the maths department and with my abilities". Work less stressful; has more money; "studying more... doing well". *Oct.*: Motivated "...by doing well... [and] by interest in the subject". "I am amazed at how good a student I am nowadays... still afraid of the fall...". Friends (other mature age students) who study mathematics

employment and family. Believes he failed one mathematics examination; did not take the other - opposite to initial intentions.	are "great for keeping up enthusiasm...". Expects to perform well, "doing better than expected".
Mathematics learning environment	
May: Lectures: "I think the lecturers in mathematics are very dedicated to the students". Tutorials: "... best if answers are explained in far greater detail... many mature age students have a sketchy background in maths and are unfamiliar with assumed knowledge"; comments on own inadequate school level background. *July*: "tutor could learn a few teaching skills... treats us [as would] a primary teacher". Assignments: insufficient worked examples in text book to "serve as a guide". Examination: "Felt sufficiently prepared] but "didn't perform as well as I thought for [one subject] but I was pleasantly surprised [for other]". Applied for special consideration on medical grounds. *Sept*: no relevant comments *Oct.*: spoke to mathematics lecturer before deciding not to sit for the examination.	*July*: Lectures: "I'd like... clear overview of what the subject is covering and quick summaries along the way". Had fruitful discussion with lecturers. Tutorials: "no improvement needed". Assignments: "... expectations & relevance clear and the time allowed reasonable... especially useful... Easy to be lazy... in tutorials". Exam.: Unfair - "I wanted the exam to test my understanding of topics, not just how to practice the methods of solving problems". *Sept.:* thinks "courses are administered well... feel very OK to approach staff... with questions". *Oct*: "always look forward to classes and enjoy diversity of format". High-point - tutorials: "I love them, working through questions... either alone of with someone else. It's great to speak maths, to iron out bad techniques, to see how my peers approach problems and get very personal direction". Low point - practice classes "[I] feel very isolated. Most of the class are young men and they tend to congregate and be very rowdy. I feel self-conscious, sitting alone and being so quiet. I wonder what they think of me...".

"Tag" and "beeper" data. Boyd and Caitlin were 'tagged' for about two hours on the days they were handed the beepers. Serendipitously, both were 'tagged' by the same researcher, on the same week day, at the same time of day, one week apart (Caitlin first). Both students were enrolled in the same second year mathematics unit. The researcher accompanied them to lectures. Field notes revealed that they behaved differently during the lecture. Caitlin concentrated and summarised the lecturer's notes. Boyd was less focussed and copied the notes verbatim. However, the researcher and both students concurred on the high quality of the lecturer.

Experience sampling forms.

Space constraints do not allow a detailed discussion of the data gathered in this way. Briefly, Caitlin's and Boyd's responses to the ESF prompts confirmed the impressions gathered from the interviews and their e-mail responses. For example, in his July e-mail message Boyd had commented on the lack of sufficient practice examples. When asked "What were you doing" when beeped, we learnt that Boyd used his travelling time on public transport to read a book about mathematicians and their work and to attempt some of the examples given. The beeper signal also caught him at a number of meetings of the environment collective, one aspect of his involvement with student politics. Caitlin told us in September that she felt confident and expected to do well in mathematics. When we compared Boyd's and Caitlin's descriptions of their mood for the times the beeper signals caught them during studying activities, Caitlin consistently revealed herself to be happier, more confident, more focussed, and clear. Interestingly, Boyd had found the 'beeper' activity very therapeutic and requested photocopies of

what he had written.

Concluding comments

The findings reported in this paper reveal that data gathered from a combination of quantitative and qualitative sources are both complementary and supplementary. From the survey data alone it could be concluded that Caitlin and Boyd were quite similar people. Even though the qualitative data were consistent with many of their survey responses, they also revealed ways in which the two individuals differed.

Our data suggest that the Eccles et al. (1985) *Model of academic choice* partially explains Caitlin's and Boyd's selection of mathematics as an academic pursuit. However, there were other factors associated with their personal lives and psychological well-being, as well as aspects of the learning environment, that appear not to taken into account by the model. These factors may be critical in the students' decisions to persist with mathematics into next year. As yet, we are very uncertain about Boyd's future at university and more generally.

The findings also raise a number of questions to be explored further:

- How do the lives of mature age students with and without families differ?
- Is mature age study only open to those who are financially secure?
- Do all mature age mathematics and science students return to study to compensate for educational mistakes made on leaving school?
- Do all mature age students cope with inadequacies in their learning environments?
- How critical is the role of the institution and staff of the mathematics departments in which students are enrolled?
- Do all mature age students share the same strong initial motivation to succeed?
- Is there a pattern of difference in the roles and expectations of male and female students?
- Are the perceptions of university courses and the daily routines and pressures that may affect studies the same or different for mature age students from overseas and from different ethnic backgrounds?

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ELEMENTARY SCHOOL TEACHERS' UNDERSTANDING OF KNOWLEDGE OF STUDENTS' COGNITION IN FRACTIONS

Yuh-Chyn Leu
National Taipei Teachers College, Taiwan, R.O.C

Abstract

This study was aimed to investigate elementary school teachers' understanding of knowledge of students' cognition in fractions. Instrument used was developed based on the theories of hierarchy of mathematics understanding in two stages. First, the results of elementary school students' fractions understanding was analyzed. Furthermore, the assessment instrument was developed based on these students' understanding.

136 subjects took the paper and pencil test. 33 were selected for interview. The findings: Elementary school teachers could be classified into three levels on their understandings of students' cognition in fractions. 18% of the elementary school teachers could be identified as excellent level.

Introduction

"Understanding the knowledge of students' cognition in mathematics" is one important component of the knowledge of mathematics teachers (Fennema et al, 1992; Markovits et al, 1994; Even et al, 1996). Research results supported the finding that having more or less of this type of knowledge effects teaching (Carpenter et al, 1989).

Fractions is closely connected to decimals, percentage, ratio and division. All of these concepts not only play important roles in mathematics but also occupy mass of the elementary mathematics materials. Students have many difficulties in learning fractions (Leu, 1991). Based on these reasons, this study wanted to investigate the elementary school teachers' understanding of the knowledge of students' cognition in fractions.

Because the coverage of the knowledge of students' fractions cognition is very broad, it's difficult to investigate all perspectives. This study only chose some important results of the theories of hierarchy of mathematics understanding, such as students' problem solving strategies, misconceptions and understanding levels etc. (Hart, 1981) as well as students' thinking tendency(Bruner, 1973). In this study, questionnaire and interview were used as research methods.

Sample and Assessment Instrument

Sample

A convenient sample of 136 elementary school teachers who took in-service summer school program in National Taipei Teachers College took the paper and pencil test in this study. Among these 136 subjects, 33 were selected according to the proportion of gender, teaching experiences and teaching grade from the 136 subject sample for semi-structured interview in one to one manner. It takes about 15 minutes to interview. To have the in-service training, these teachers had to pass a qualified exam. Based on their enthusiasm for learning and their capability, their performance on my research questions should be generally better than the average teachers.

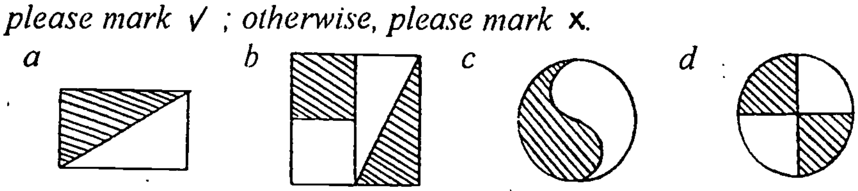
Assessment Instrument

The paper and pencil test and the interview questions have developed on the understanding of students' fractions learning. Therefore, the tasks were developed in two stages. At the first stage, the information of elementary school students' fraction understanding was established. Tasks for elementary students were either designed by the researcher or from Yong(1987). At the second stage, five tasks for teachers were developed based on this information from the first stage. Each task has two forms. Students' tasks were firstly presented. Then the elementary school teachers were required to answer the difficulties students might have, the problems-solving strategies students might take or the reasons why students made those mistakes...etc. How the tasks were designed will be illustrated by following two examples.

Task 1: types of thinking tendency

The younger the children, the more often they use the intuitive thinking to solve problems(Bruner, 1973). In task 1, besides evaluating whether the elementary school teachers know that students will use the intuitive thinking to judge about equal partition, this study also want to know whether the elementary school teachers do know the difficulties students might encounter when they learn equivalent fraction and the figures of same-size irregular-shapes.

In the following figures, judge which shaded areas of them are one half of the whole areas. If the answer is Yes, please mark ✓ ; otherwise, please mark ×.



(1)Please make a judgement about the order of difficulties

which might be encountered by the 2^{nd}, 3^{rd}, 4^{th} grade students.
The order of difficulties (Please mark with a, b, c, d)

Easier:____ easy:____ hard:____ harder:____

(2)Please write down the reason why you make such a judgement.

The 4 items in this task were answered by about 660 students in 2^{nd}, 3^{rd} and 4^{th} grades. The order of the difficulties of them is a, c, d, b. There is few percentage difference between the rates of correct answers in a (98%) and c (95%). The difference between c and d is around 30%; the difference between d and b is around 20%. a with c, d, and b should be belonged to different 3 understanding levels. To judge item c, according to the interview data, students compared the shaded area with the blank area, they used intuitive thinking to decide whether the shapes of figures are equal. Therefore, the rate of correct answer in c is pretty high. Although students can use intuitive thinking or analytic thinking to judge those 4 parts of d are equal, they thought the shaded area of d is 2/4 of the whole. They need to use analytic thinking to judge 2/4 equal to 1/2, so d is more difficult than c.

Task2: problem-solving strategies.

The students of Taiwan also have their own problem-solving strategies, but not for those who have learned the rules of algorithm. Those students still choose the rules of algorithm to solve problems unless they were encouraged to use alternative strategies. Students' performances in this task were from Yong(1987). They (around 260 students) had never learned the division of the fraction at school.

" Fraction divides fraction " is the mathematics material for the 6^{th} grade. The rate of correct answers for the two following questions were the results of the survey which randomly investigated students of 5^{th} grade in Taipei.

a. calculation:.$3/4 \div 1/8 =$ (rate of correct answer is 15%)
b. applied question: In a relay race, each contestant has to run 1/8 kilometers, if there is a relay race of 3/4 kilometers, how many contestants will be needed? (rate of correct answer is 43%)

(1) Do you think the students' rates of correct answers for these two questions are reasonable or not? Please write down your reason.

(2) In solving question of b, besides using $3/4 \div 1/8$, what were other

possible strategies you think that students might use? (strategies must be reasonable and can get correct answer)

In task2, this researcher told the elementary school teachers that "fraction divides fraction" is the mathematics material for the 6th grade and the students in the task were 5th grade. Teachers were reminded that the 5th grade students did not learn "fraction divides fraction" yet.

Result

This study also took task1 and task2 as examples to explore elementary school teachers' performances on understanding the knowledge of students' cognition in fractions. This study will show some of the teachers' answers as examples; T# is used to represent one subject's answer in the task.

Task1: types of thinking tendency

Question a appears frequently in learning materials as one half, so teachers know that it is the easiest for students.

- 27% of the teachers got order from easy to difficult as: a, c, d, b.
  T116: when dealing with questions a and c, students can use intuitive thinking to make a judgement. But d is 2/4 = 1/2, students must have the concept of simplification. b is much more difficult than d for students to explore.
- 38% of the teachers got order from easy to difficult as : a, d, c, b.
  T229: 1/2 is half of the whole area; so it is much easier to tell the figures of a, d and c. (d : The students will automatically switch the above shaded area with the below blank one; c: they will also switch the two small half circle to make up for the one large half circle.)
- 25% of the teachers got order from easy to difficult as: a, d, b, c.
  T417: a: students can immediately tell that 1/2 is half of the whole. d : divided into 4 equal parts, two of which are the shaded areas. It is also easier for students to tell 1/2. b: It can be divided into left and right two parts , the shaded area of each part is 1/2. c: students must draw the diameter of a circle, but most of them can't do this.
- 10% of teachers got other answers.

According to the above data, 27% of the teachers understood students' thinking tendency and the order of difficulty. From easy to hard, the order is : a: two equal parts , d: equivalent fraction, b: the figures of same-size irregular shapes. 63% of teachers don't understand students' thinking tendency but understand the order of difficulty. 10% of teachers neither know students'

thinking tendency nor understand the order of difficulty.

If the elementary school teachers don't know or don't accept that intuitive thinking is one of students' thinking types. It seems that:

- teachers' selection of instructional materials and sequences would not match with students' learning sequences.

  T414: for example, in order to divide a circle into a half, we must know what a diameter is, and the diameter must be across the center of the circle; then, we can get two equal halves of the circle. I don't think a 2nd grade child can answer such a question. Diameter and radius are teaching materials for the 4th grade students.

- elementary school teachers will adopt the way of analytic thinking to teach and the way they use to evaluate students' work might be based on absolute precision but not an visual and operational based precision.

  T414: such as dividing the chocolate bar, cutting it into a half, I doubt that student only did it by probable judgement. For our teacher to solve the same problem, we also need to measure the length, then, divide it by 2 and cut it.

Task2: problem-solving strategies.

- 14% of teachers were aware of the diversity of students' problem-solving strategies and can specify students' problem -solving strategies.

  T105: (a) The correct rates are reasonable. "How many contestants will be needed" is asked in the applied question. The answer can't be a fraction. Students will get an integer as answer by all means.

  (b) using $\frac{3}{4}=\frac{6}{8}$, $\frac{6}{8}$ *is six one-eights*

- 34% of teachers either were aware of the diversity of students' problem-solving strategies or can specify students' problem -solving strategies.

  T218: (a) The correct rates are reasonable. Students can reason out each question when solving the applied questions but couldn't solve calculated question if they forgot the rules of algorithm.

  (b) (no answer)

- 52% of teachers neither were aware of the diversity of students' problem-solving strategies nor can specify students' problem -solving strategies.

  T230: calculation is the foundation for the applied question;

only15% got correct answer in solving simple question of calculation. But in solving applied question, both adroit calculation and consideration are required. It seems not reasonable that 43% of the students got correct answer in solving such a complicated question.

Conclusion

According to elementary school teachers' performance on answering those 5 tasks, three levels of understanding of knowledge of students' cognition in fractions was found as table 1.

Table1: three levels of understanding of knowledge of students' cognition in fractions.

task	Knowledge of students' cognition in fractions	excellent	Mediocre	unsatisfied
1	Understanding students' thinking tendency and the order of difficulty of various figures	○ ○ ○ or	○ ◐ ◐ or	○ ● ● or
2	Be aware of and be able to specify students' problem-solving strategies.	(a)○ ○ ◐ or	◐ ◐ ◐ or	◐ ● ● or
3	Be able to evaluate students by considering the learning difficulties of fractions	○ ○ ●	◐ ◐ ●	● ● ●
Task 1~3 are paper and pencil questions.		10% (b)	73%	18%
4	Be able to explain the reason why students make certain misconceptions	○ ○	○ ◐ or	○ ● or
5	(c)Understanding that assembly ability will influence students' performance on solving equivalent fractions		◐ ◐	◐ ● or ● ●
Task 4~5 are interview questions.		18%	70%	12%

(a) The answers of each task can be stratified into three levels: excellent, mediocre, and unsatisfied.○, ◐ and ● are represented respectively as three levels.○○◐ means that teachers get one mediocre answer in task 1 or task2 or task 3 but get two excellent answers in the others. It isn't represented in the question order.

(b) 10% means the percentage of excellent performance in the paper and pencil tasks.

(c) Assembly ability means that students can divide unit into several parts, deal with each part correctly and then, assemble each part into the required fractions of unit. Students who lack such an ability can deal with each divided part but can not assemble each part into the required fractions of unit.

If elementary school teachers do not understand their students' learning of fractions, there could be some important effects on their teaching and students' learning.

1.Teachers' selection of teaching material and sequences do not match with students' learning.

Students could use intuitive thinking approach to solve problems. However, teachers think students tended to use analytical thinking approach

to solve problems just like themselves. So they select their teaching materials and arrange their teaching sequence according to the cognitive demand and task analysis hierarchy. These two approaches will not match well. For example, teachers think 4th graders can divide a circle to two equal halves only after they have learned the concepts of diameter and radius. But most students already can divide a circle to halves when they are in 2nd grade. They do it intuitively, i.e., through visual or by actually folding it into two equal halves.

2.Teaching and evaluation do not match with students' learning.

Teachers conduct their teaching according to the analytical thinking approach. However, students adopt intuitive thinking approach while they solve problem.

Teachers may think that they should teach students' the rules of algorithm, then ask students to apply those rules to solve applied problems. But what's accepted by students should be put on these mathematics contents in a meaningful and interesting context, then start from students' problem-solving strategies and gradually link them to more abstract, formal mathematics algorithm.

Teachers might take the criterion of absolute precision to teach and to evaluate students, but students might take a visual and operational based precision to learn or solve problems. For example: a teacher might use a ruler to measure the length of a stripe of paper; then, he or she will use the division to calculate the length of each part. In contract, a student might divide the paper stripe through intuitive observation.

3.Teachers can't identify students' misconception and will overestimate or underestimate the difficulties students encountered while solving problems.

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A BUDDHISTIC VALUE IN AN ELEMENTARY MATHEMATICS CLASSROOM

*Yuh-Chyn Leu **Yuh-Yin Wu **Chao-Jung Wu

*Department of Mathematics Education

**Department of Educational Psychology and Counseling

National Taipei Teachers' College, Taipei, Taiwan

Abstract

The purpose of this study is to investigate the value an elementary school mathematics teacher emphasized in her class. Participatory observation was adopted as the main research method. We found the mathematics teacher is a devout Buddhist. She pretty much emphasized the importance to reinstate the original enlightenment. We identified it as the core value in her teaching.

Introduction

In recent years, studies of teachers' beliefs showed an apparent inconsistency between what a teacher thought and what a teacher taught (Anne, 1997;Chin, 1995;Thompson, 1984). Then, we would contend that this discrepancy is precisely the reason why values rather than beliefs need to be studied.

According to Raths, L.E., Harmin, M., and Simon, S.B (1987), formation of value contains three steps: choosing, prizing and acting. Those steps collectively define valuing. Results of this valuing process are called values. In order to determine the deeper factor that supports underpinning teachers' preferred decisions and actions, this research tries to investigate a teacher's values shown in an elementary mathematics classroom by the definition of values from Rath et al.

Methodology

We mainly adopted case study approach to find out the values transmitted in a mathematics classroom. An elementary school teacher, Ms. Tsen, was selected among teachers we have interviewed due to her recognized teaching expertise. She has 21 years teaching experiences. Besides being a classroom teacher of 40 fifth grade students, she was a mentor advising a student-teacher during our observation period.

The following procedures were carried out in order to collect data: observing Ms. Tsen's mathematics class, interviewing her, observing the student-teacher's mathematics classes and joining Ms. Tsen and the student-teacher's advising meeting; interviewing the student-teacher to check Ms. Tsen's values.

According to Rath's theory, acting, especially repeatedly acting, is an important indicator for core values. Therefore, when analyzing data, we focused on what Ms. Tsen repeatedly acted in classroom and what she considered as important and valuable.

When we came to a conjecture of Ms. Tsen's core values she was told and asked to clarify.

Results

Buddhism is one of the major religions in Taiwan. Ms. Tsen is a devout believer in Buddhism. The belief of it produces a great impact on her viewpoint toward education. In this paper, we presented one value which she agreed upon: The purpose of education is to reinstate the original enlightenment (本覺). To achieve the goal, she required students to calm themselves down and start self-reflection on their own. Following are some events indicating the evidence of the existence of this value. In the following excerpts, "I" stands for the interviewer; "Tsen" for Ms. Tsen, "Ts" for the student-teacher; "S" for Ms. Tsen's students, "C.O." for classroom observation, and "Int." for interview.

1. By way of "calming down" to reinstate the original enlightenment.

In mathematics teaching, Ms. Tsen was used to ask her students to repeat mathematics questions or strategies, and require them to listen and look attentively

frequently. This action reflected the "acting" component of values.

Tsen: "... There are 6 soaps in a box; we assign 5 boxes to 6 persons. How many boxes can each person get? ... Lee, repeat the question I just asked."(11.7.97.C.O.)

I: " What's your purpose of requiring students to repeat the question?"

Tsen: "My purpose is to let each student know what we are doing in the classroom; on the other hand, I remind the students who are not concentrated in class to listen attentively." (11.11. 97.Int)

Tsen: "You don't need to tell me the answer for this question; just tell me what the difference between this question and the prior one is?... Listen carefully! Some students still play with pencils. Put it down."(11.7.97.C.O.)

Why does Ms. Tsen emphasize students' being concentrated in class learning? The answer was embedded in the following classroom observation.

Tsen: " I issue a warning to you against your not being concentrated. Ok, let's begin the next question: please use the method of filling the blank with the symbols of multiplication to write down the question. Some students are whispering. Once you calm down, you can solve this question."(97.11.28.C.O.)

Why does Ms. Tsen think "being concentrated" and "calming down" can enhance students' learning?

I: "In this forty-minute class, you have asked students to pay attention to what you said for more than 10 times. However, your students quite already paid attention on you, why you still keep asking them to be attentive to what you said?"

Tsen: " I found that once one is calm down, one can learn things much more efficiently. That is what I have learned from Buddhism. For example, I am able to integrate the subjects teaching pretty well. And the teaching pattern was less flexible than what I am doing now. I can notice the hazards that the students faced and try to

adjust my teaching style. (3.17.98. Int.)

I: " Besides requiring your students to be concentrated, what else would you ask them to do? Apparently, some students are very concentrated but they still can't get the point."

Tsen: "That is the point. (Researcher's note: Here we see the "prizing" component of values. When a person is concentrated, he is calm. Being able to calm down is an innate ability that can help people to observe things much more clearly. No matter what dumb or smart a person is, he/she can always show such ability."

I: "Those students who might be very concentrated in listening to the instructions, but... "

Tsen: " Right, a student probably can not understand some of the viewpoints. According to the terminology of Buddhism, it is a karmic obstruction (業障). Take "listening" as an example: for the same thing, some always look at the bright side of it but others always look at the dark side. It's regrettable that some persons always look at the dark side of events because they don't know the art of observing things. What I did is to teach my students how to observe a thing and how to detect everything around them. It's nothing but a root nature (根性) to which I pay close attention to. (Researcher's note: Here we see the "prizing" component of values.)... Generally speaking, Buddhism can be defined as one kind of education that can enlighten people's inborn wisdom and makes people understand the truths of the Universe and life. The basic purpose of Buddhism is to recover a person's original enlightenment, so Buddha means "enlightened being" (覺悟者). (12.2.97.Int.)

2. By the way of " self-reflection" to reinstate the original enlightenment

Ms. Tsen often warned students the mistakes occurred but seldom told them what their problems were.

Tsen: "This question requires you to write it down by the way of filling the blank. (Research's note: The correct answer is 13÷9=( )) Some of you write it down in this way; let's take the first reaction as an example... Come over here, S9, how did you write down this question by the way of filling the blank? "

S9: " 13÷9=1 and 4/9"

Tsen: "OK, 13÷9=1 and 4/9, if this is your answer, can you tell me what your problem is? "(11.14.97.C.O.)

When students made mistakes in the mathematics class, Ms. Tsen didn't tell her students what their problems were, only reminding them that they make mistakes. The reasons explained by Ms. Tsen were as follows:

I: "When a child made a mistake, apparently you would tell him he was wrong and ask him to reflect on what the mistake was. Why did you choose this way to ask him?"

Tsen: "When you make a mistake, you should be aware of the mistake by yourself. If a child couldn't find his own mistake, I would help him to discover it. This is for improving his nature of enlightenment (覺性). If he knows what his mistake is, he may find out how to debug it. ...During the process of discussion, I never tell them directly where the mistakes are and what should be improved."

I: "Will you first ask him what the mistake is when you found it?"

Tsen: "I won't tell him where his mistake is directly. This will put a damper on his awareness."

I: "What you emphasized is student's self-awareness. Do you think self-awareness is very important in learning?"

Tsen: "Self-awareness is very important not only in learning but also over the whole human life. (Researcher's note: Here we see the component of "prizing" in values.) That's your own feeling and awareness for yourself. Everyone has such ability but not every one can sense it. What I did is to guide my students to find their

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own mistakes out and restore the awareness. Therefore, it is not necessary for us to tell students where the problems are. No sooner have they become calm, than they will keep sober-minded and find the truths gradually." (11.18.97. Int.)

Ms. Tsen's requiring students to do self-reflection was also detected and found by the student-teacher.

Ts: "... Generally speaking, most teachers might punish or award their students for their performances. But I strongly feel that Ms. Tsen has done something more for her students. To have her students to do self-reflection, she asked several questions, such as:' Are your behaviors only for others to observe, not for yourself?' 'Do you know your behavior is right or wrong?'(3.26. 98.Int.)

Ms. Tsen's apprehension of Buddhism makes her choose such an education goal, reflection. In addition, following paragraphs can show us that this education goal is formed through "choosing", another one important component in the processes of valuing.

Tsen: "The core of elementary education is life education. The reason why there are so many problems in our society is because we put too much emphasis on subject learning and ignore life education. ... It's just for teaching convenience, so school subjects are classified into Language Art, Mathematics...etc.(Ms. Tsen thinks teaching implementation of each subject should origin from life education.)" (3.17.98 Int.)

Tsen: "Every thing is a Sura. I would require my students to observe deliberately the environments around them." (3.24.98 Int.)

Tsen: "... Traditional Confucianism only consult about this life of ours; in contrast, Buddhism clearly points out the past, the present and the future. My responsibility is to make my students reinstate their original enlightenment. I don't believe the knowledge we have learned in school are sufficiently to help us to break away from hardships... " (1.6.98. Int.)

According to Ms. Tsen, the purpose of education is to help students to break away from hardships and suffering. To reinstate student's original enlightenment can help reach this goal, which, however, can not be achieved only through subject learning. Therefore, life education should be more highly emphasized than subject learning.

Discussions

Buddhism is one of the major religions in Taiwan. Tzu-Chi is one of the important Buddhist organization. It publishes a series of books, such as: The world of Tzu Chi、The garden plot of Chin-Si (silent-thought)、Chin-Si-Yu (words for silent thought), etc. These books disseminate Buddhistic concepts and pass on the principles of how to get along with people and deal with things properly in society. In addition, in the northern area of Taiwan, there are 16 teacher sodalities. These sodalities in total have more than ten thousand members. They are from different levels of public and private schools, and various academic institutes. They weld the content of Chin-Si-Yu into their curriculum. Therefore, in Taiwan, it is not unusual that Buddhism influences teachers' teaching in elementary schools.

How do we define "values" and how do we claim that "this is Ms. Tsen's value?" These still remain disputable. We followed Rath et al.'s definition, even though his definition is more close to value clarification than value identification. We have identified the value of Ms. Tsen, to reinstate the original enlightenment, through the process of seeking the three components of values: choosing, prizing, and acting. Yet, it does not mean that all candidate values should be examined and pass the above mentioned criterion. Such kind of operational definition of values is adopted in this study in order to get started with the investigation of values. Better description of properties of values are needed.

We found Ms Tsen not only prized and cherished her emphasis of enlightenment, but also combined the thought with the trend of constructivistic teaching in mathematics. We were convinced that under such circumstance, it is easy and

comfortable for Ms. Tsen to express her thought in public. On the other hand, we were concerned that what if a teacher's core value is against the mainstream values in the society? Can we have chance to detect it with ease?

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CHILDREN'S CULTURAL ACTIVITIES AND THEIR PARTICIPATION

Pi-Jen Lin Wen-Huan Tsai
Hsin-Chu Teachers College, Taiwan
linpj@nhctc.edu.tw Tsai@nhctc.edu.tw

ABSTRACT

The study aimed to explore what children's cultural activities are and to understand the ways of their participation. Forty third graders selected from elementary schools of Taiwan were interviewed with four main questions, and their responses were classified. The socialization and mathematical skills needed in an activity and the amount of learning opportunities offered by adults were the indicators used in deciding the degree of children's participation. It is proposed that the conflict between cultural cognition and school mathematics might be a factor of school mathematics being disconnected from everyday mathematics.

Introduction

The purpose of this study was to investigate the mathematical concepts that occur in daily life, that children bring into school, and to explore the social contexts of such mathematics.

There has been a considerable number of studies that link "school mathematics" with "everyday mathematics" (Saxe, 1991; Petitto, 1982; Bishop & Abreu, 1991). The focus of school learning on formal mathematics is cited as one critical reason why children seem incapable of applying "school mathematics" to solve daily mathematics problems. Further reasons are weak connections between problem situations and inner representations (Hiebert & Carpenter, 1992), context-bound school mathematics (Bransford, et al., 1986), and the fact that school mathematics is learned away from social contexts. These are all possible explanations of the disconnection between "school mathematics" with "everyday mathematics".

Some researchers agree that the strategies of solving daily problems created by children are preliminary to learn school mathematics (Carraher et al., 1985; Bishop & Abreu, 1991). Madell (1985) claims that informal methods of solving daily problems are as often efficient as formal methods of solving school mathematics problems. Formal methods learned in school are initiated with informal methods of solving daily problems. If a concept to be learned seems relevant to the informal methods that have been used, children are more likely to learn it naturally. Thus to link knowledge gained from out-of-school with school mathematics, we should accept children's natural thinking. The design of "school mathematics" curricula should start with "everyday mathematics", since informal methods are originated with everyday mathematics.

According to Vygotsky's socio-historical perspective, the child is historical, social, and cultural. (Minick, 1989). Bruner (1990) emphasized that the cognition of human beings should focus on sense making of the world. He remarked that human

beings seem to have an innate tendency to make sense of the world in narrative forms and that narrative forms are distinct by cultures. This theory is supported by findings in our ongoing research project, where we have found children growing up in farm family learning the knowledge of farming and children living around harbor learning the knowledge of weaving fishnets and learning fishing skills. Similarly, planting mushrooms and cutting bamboo are viewed as natural skills for aboriginal children.

Brown et al. (1989) claim that learning situations should be embedded in authentic problem situations that makes sense to learners. Learning should naturally occur in real social contexts, but this does not mean that learning outcomes are universal to learners even in the same contexts. The assistance of adults, including parents, teachers, or peers in one important influence on learning. With Lave and Wenger's (1991) Legitimate Peripheral Participation (LPP) theory, apprentices learn to think, act, and interact in increasingly knowledgeable ways with people who do something well, by taking part in legitimate peripheral participation. Learning as participation treats the relationship among persons, their actions, and the world as a continuously evolving set of relations. The participation is at first peripheral but increases gradually in complexity to kernel participation. Situated learning theorists stress that the interpretation of learning behaviors should concentrate on the meaning and the content of learning in an authentic situation which learners interact with surrounding persons and the environment, rather than having focus on individual cognition.

In our research study, we are in the first year of a three year project, intended to explore children's authentic activities in daily life which have embedded mathematics contents. There are currently three research questions: a) What kind of cultural activities are children engaged in? b) What are structures of cultural activities? and c) How do children participate in cultural activities? The cultural activities that children participate in most frequently in the first year of the project will be the basis of the second year's study. The second year of the research project will focus on how to bring children's cultural activities involving mathematics concepts into classroom.

Method

This investigation was carried out in Hsin-Chu, a city of Taiwan. The participants were forty third graders in eight elementary schools, five children selected from each school.

To collect notes about children's cultural activities, each child was interviewed with four main questions: Who are in your family? What are they? What do you do after school or on weekend? Which festivals do you like best, why, and how do you celebrate the festivals? The two questions were to gather data on the interaction of the learner with their family, and the latter two questions were to collect information on children's out-of-school activities; especially, celebrations of conventional festivals.

Each subject was interviewed individually. Audiotapes of the interviews were

transcribed and coded. Four social contexts with mathematical activity embedded emerged: running a business, shopping alone, shopping with adults, and celebration of conventional festivals. Children had a distinct degree of participation in different contexts, and different involvement affected children's mathematical cognition and socialization significantly. The socialization and mathematical skills needed in an activity and the amount of learning opportunities offered by adults were the indicators used in deciding the degree of children's participation.

Results

Only some results appear below. We found that aboriginal children have a specific life style that typically includes activities such as planting mushrooms and jumping rope. They also have no cramming school, incomplete school homework, and no money for buying toys and computers. Doing assignments, biking, doing exercises and playing games in the arena of the temple are suburban children's regular out of school activities. Other activities include doing homework, attending cramming school, doing housework, playing with toys, and sports.

Some activities have mathematical structures. The structured activities which children were engaged in included pokers games, jump rope, chess, playhouse, hopscotch, and monopoly.

Each of these activities has mathematical content. When participating in a structured activity such as "Playhouse", children are becoming socialized but they also acquire mathematical knowledge. For instance, as Lily pretended to go shopping in a store with her cousin, she said:

> "... playing a cart game with my younger sister, using my aunt's room as a store. Two dolls, one is a man, the other is woman. They came to the door of the store to buy fish and a can and put it in the cart. After that they approached the cashier and paid. My sister served as a server, putting money in a drawer, changing money, and paying back the customer. They paid NT$500 for buying a can and fish NT$100 altogether, then the server gave them NT$400 change. They also bought pets in a pet store. A dog is NT$5000, A kitten is NT$3000, it is cheaper than the price of a dog. A bird is NT$7000, it is the most expensive. We made several pieces of rectangle paper and wrote the number on them as fake money for too dolls."

Fake money as currency, using room as a store, two dolls as shoppers, a younger sister to serve as a cashier, receiving and changing money—the role-play imitates a real shopping context. Lily's imagination and recognition of professional roles have been developing in the playhouse context. In play, Lily also learns to recognize the price per item, to compare the price of the items, to sum up, and to give change. Consequently, Lily's arithmetic algorithm of multiple units and comparing of big numbers have been developed naturally in the context of role-play, even though she

has not learned them in school yet.

We found that suburban and aboriginal children frequently help parents in buying general merchandise, such as salt, sugar, soy sauce, wine, cigarettes, etc. In order to verify the emergent mathematical goals in shopping context, they were asked to describe some of the routine activities they do.

We identified five forms of participation (no, partial, incomplete, complete, and kernel) that were unequally distributed in various contexts. For example, three learning processes were identified in the "shopping with parents" context — "no participation", "partial participation", and "complete participation".

Each level of participation was tabled against different activities. For example, children's "no participation" presented in four different contexts: shopping alone for parents, shopping with parents, family running a business, and Chinese New Year.

(1) No participation

We distinguished the structure of shopping activities in which children were engaged in shopping alone from that of shopping with adults. The structure of shopping in store can be described in three phases: preparing money to buy merchandise, buying merchandise, and checking the change. Children were involved in the three phases at different levels of engagement. For example, Ludan was asked to buy soy sauce and salt for her mother.

Case: Ludan, a policeman's daughter

Interviewer	Ludan
Were you able to help your mother to buy something?	Yes.
What did you buy?	Soy sauce and salt
How much does a bottle of soy sauce cost?	NT$50
How about salt?	NT$15
How much were you given by your mother?	NT$100
Before paying to sellers, did you count money?	My Mom told me I need to pay NT$65, the sellers will give me NT$35 change, and I need to check if it is correct.

Ludan did not need to solve any mathematics problems during the three phases, since her mother had prepared her for them. Thus we characterized Ludan's learning process as "no participation".

"No participation" was also identified in contexts of shopping with parents where children did not pay attention to the price of each item, or where a mathematical goal was not present in children's activity because they were bidden to follow their parents. Similarly in conjunction with a family business there were examples where children

had no participation. For instance, Weily, a son of a seafood restaurant owner seldom presents in the restaurant, so he does not know the price of each course. When he was asked to solve the problem: "Each bowl of fish soup costs NT$60, how much do 3 bowels of fish soup cost?", he had no confidence in answering and said "Likely NT$70". Weily had not been given any opportunity to learn the mathematical ideas embedded in the real context.

Susan enjoyed Chinese New Year, since she got lucky money as well, but she did not know the amount of money in all. Her mother was afraid of errors in calculating the sum. Susan said: "My mom helped me adding them up and she saved it in a bank". Susan was deprived of opportunities to learn mathematics in the natural context. Susan's learning process in this context was classified as "no participation".

(2) Partial participation

Most of children's learning in selling contexts was identified as "partial participation". Jen's home is next to a harbor, and her mother owns a seafood restaurant. Passing courses over and setting up bowls and chopsticks are Jen's jobs. She receives money from customers occasionally and is told by her mother the amount of change to give customers. She described helping in the restaurant passively. She said: "Mom told me which place was busier then I moved to there to help. If she told me she needed a hand here, then I came back here to help her." Jen's learning process was identified as a "partial participation", because the mathematical thinking was generally done by her mother. She did not know the price of each course, and her mother did not provide her with opportunities for adding customer's accounts. However, she had a chance to change money for customers, and can calculate change (e.g. NT$80 when NT$120 is deduced from NT$200, and NT$400 when NT$600 is subtracted from NT$1000).

(3) Incomplete participation

Running a store or producing farming products is a source of financial support for a family. Parents do not arrange the store intentionally to educate children, but children learn counting and changing money in the authentic contexts. We found that children whose families run a large businesses, such those selling ironwork and electric products, are seldom involved in their own store's activities. However, children whose families run small stores, such as those selling Chinese chewing gum (Ccg, a seed of a plant) or farming products, had more opportunities to do business.

Hong's father builds houses and his mother sells Ccg in a small store. He said that

he helps in selling Ccg in his mother's store every day after school. He knew the price of a package of Ccg and was able to count the sum NT$100 for two packages.

Mei is growing up on a farm family. Her parents produce and sell fruits in a farmers' market. The following episode describes her participation in a selling context. Note that both Mei and Hong (her older sister) learn mathematical ideas that are embedded in selling situations and interact with their customers actively.

Case: Mei (Note: NT$: Taiwanese dollars. US$1 = NT$33. kg = kilogram)

Interviewer	Interviewed
You just said you help selling fruits in a farmers' market?	Yes; otherwise, I worked in a mountain on weekend.
Did what?	Planting bamboo, cutting old branches of bamboo and firing them.
Were you able to help getting money from customers or did you tell customers the unit price?	NT$35 for each kg, NT$100 for each 3 kg
How much did you need to give the change when a customer gave you NT$100 for buying one kg?	NT$35 and paying NT$65 back to the customer.
Who received money from customers and who gave the change for customers?	Elder sister received money and I gave the change.
Before you gave the change, do you calculate the amount by yourself or were you told?	I did by myself. I was watching the weight needed for a customer. My elder sister received money first, followed by my calculation and my change. Consequently, I passed over the money to my older sister and paid back to the customer.
How much does it cost, if a customer need 4 kg	NT$135
If he paid NT$150, how much do you need to give as change?	NT$15

The difference between these two cases is the assistance received from Mei's elder sister. Hong was offered a complete learning process in selling Ccg. His mother did not interfere. Hong learned not only mathematical ideas but also the social ability in the real social context. Nevertheless, Hong is unable to manage the store without his mother's presence in the store simultaneously, so Hong's learning process in selling context was identified as an "incomplete participation" rather than a kernel participation. When interacting with customers, Mei's elder sister sponsored her social actions such as receiving money and paying money back to customers. Her knowledge of mathematics involved in selling fruits is developing, such as adding sum up, counting, and changing, but Mei's learning process was classified as an "incomplete participation".

(4) Complete participation

Case: Linda, a pharmacy's daughter

Interviewer	Linda
[Linda stated that she helped her mother buying merchandise]	
If you buy a bottle of soy sauce and a package of salt, how much do you need to pay?	NT$65
How much were you given?	NT$100. I get it by myself from mother's purse.
How did you know it is NT$100?	A bill is NT$100.
Before paying to sellers, did you count the money?	I gave it to sellers directly.
How did you know the change was correct?	I counted it and it was checked while I gave it to Mom.

Looking at Linda we verify that the structure of the activity is similar to Ludan's (described previously), but the interactions with their parents in phase one and three seem to be rather different. Linda coped independently with situations where she needed to use mathematics. Linda's learning process as "complete participation".

(5) Kernel participation

Children engaged in the context of shopping for themselves were classified as having "kernel participation". When buying something for himself without parents' involvement, a child needs to figure out which objects meet their needs and how to best buy according to the money they own. Children require higher social ability in the context of buying something for themselves than when shopping for parents. The emergent mathematical goal presented in both phase one and phase three.

With festivals, mathematical goals present significantly at Chinese New Year. Large amounts of currency such as NT$500 and NT$1000 are used and addition of multiple units is embedded in the social cultural context. For instance, Mary said that she got NT$7500 in a red envelope as lucky money from her relatives and parents' friends: NT$500 from Daddy, NT$500 from her grandfather, NT$500 from two aunts, NT$500 from two uncles, and her grand mother's NT$500. When counting, Mary uses a strategy of multiplying four times 500 and adding 500 and then 5000, the sum being NT$7500. In nature, the counting of money here did not involve social aspects so we identified Mary's interaction as "kernel participation".

Discussion and conclusion

It is clear from results of our project to date that mathematics is embedded in children's cultural activities. Children learned addition and subtraction of numbers (and other ideas such as recognition of geometrical shapes) in authentic contexts where children interact with surrounding people. The more involvement in activities, the better the opportunities for learning. The degree of children's participation in various contexts depends on the opportunities offered by parents or elder peers.

As previous literature described, weak connections between problem situations and inner representations, context-bound school mathematics, and school mathematics learning deviate from social contexts, and result in school mathematics that is disconnected from everyday mathematics. The conflict of cultural cognition with school mathematics presented in the study might also be a possible explanation. School mathematics does not recognize complexities such as the target of "less profit means more sales" — an essential strategy in the trade culture. Efficient strategies invented by children such as Mei for solving sale problems in order to give customers not only a correct but also a fast answer, are not recognized, leading to cognitive conflict between everyday experience and school mathematics.

The question of how to optimize children's cultural activities into their mathematics classroom learning is worth further investigation, and will be a focus in the next stage of our research.

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Making sense of informally learnt advanced mathematical concepts

Zlatan Magajna

University of Ljubljana, Slovenia
University of Leeds, UK

This paper considers the understandings of six vocational school students and two experienced constructors of splines (piecewise polynomial parametric curves used in Computer Aided Design (CAD) systems). All of the participants' knowledge of splines was learnt only in relation to CAD practice, i.e. informally, how to use the command for drawing splines. The constructors and some students related splines to their mathematical knowledge while other students, in making sense of splines, relied on the activity in which they occurred, machine design. Saxe's (1991) model is the basic vehicle for analysis but consideration of the role of intuition and expertise are also taken into account in the discussion.

Introduction

This paper is about the interplay between geometric reasoning in various practices. The post-modern view of mathematics has given rise to a conceptual distinction between mathematics and mathematical activities in various practices (Brown, 1994). Mathematics in no longer seen only as a decontextualized body of knowledge which is 'naturally transferable' to other contexts. New paradigms view mathematical knowledge as situated and linked to the various practices in which they occur. Whether the mathematical knowledge in various practices is the same or not obviously depends on the considered perspective on knowledge application (Ernest, 1998). In general, connecting the knowledge of various practices is no longer seen as something obvious, in fact, several researches (e.g. Lave, 1988, Nunes, Schliemann, Carraher, 1993, Saxe, 1991) have pointed to a marked discontinuity between the mathematics rooted in different practices. Yet even if the problem is considered from a situated standpoint one has to consider the interplay between mathematical knowledge linked to various practices. Saxe (1991), for example, in considering the mathematical practices of Brazilian candy sellers has observed some interplay between the abysmally divided school learnt mathematics and the mathematics of candy selling practice. Similar results have been recorded by Nunes, Schliemann, Carraher (1993).

The interplay between knowledge linked to various practices is not easy to observe and research. Most work in this area has been related to elementary topics (simple arithmetic and geometry). The absence of advanced topics is easy to explain. First, most advanced mathematical knowledge is very rarely used in everyday contexts. Second, most advanced mathematical knowledge is learnt first in mathematical

classes and is only later eventually linked to other contexts (e.g. other school subjects, workplace, etc.). The fact that a topic has been considered formally in mathematics classes obfuscates the observation of the process of generation of mathematics related meanings in a context.

Some important insights into the relation between school and out of school mathematics has been obtained via the meaning-making process of elementary mathematical concepts in a context by considering instances of mathematical knowledge acquired informally (not in institutionalised learning of mathematics, see Nunes, 1993; Saxe, 1991). I have researched a similar instance (see also Magajna & Monaghan, 1998) which involves more advanced mathematical ideas. I have found that Saxe's 4-parameter approach (i.e. interpreting the data in terms of activity structure, previous knowledge, social relations and artefacts used) explains well, though not completely, the meaning-making process in a context.

Splines - an example of informally learnt advanced mathematical knowledge

I begin with the observation that, at least apparently, cases of informally learnt advanced mathematical ideas are rare – students usually learn about relevant advanced mathematical ideas first formally, in the mathematics classes. Though they may use these ideas later, informally, in other settings (and may even re-learn them), such cases are never clear-cut and we can hardly speak of advanced mathematical concepts which are clearly informally learnt. Splines – mathematical objects encountered in CAD courses and drafting practices – are an interesting exception. Since splines are mathematical objects which are not widely known I shall explain this concept in very simple terms (see Boehm, Farin, Kahmann, 1984, for details).

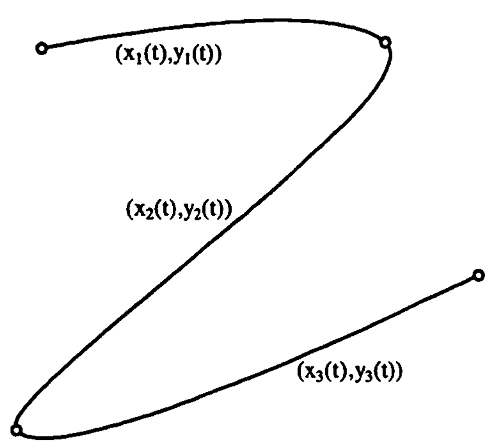

Figure 1. A simple interpolatory spline curve consisting of three polynomial 'pieces'.

Splines are curves used in most CAD systems to draw 'free' forms. On a computer a spline is usually defined by pointing to a sequence of points, called *control points*, along (resp. near) the imagined curve, and the computer draws a curve which can usually be interactively modified by moving the control points. This may appear as magic but there is a mathematical model behind this. The constructed curve is made of pieces, each piece being a parametric polynomial function, usually of 3rd degree. The end of each piece coincides with the start of the next piece not only in terms of the endpoints but also in terms of the derivatives (usually up to the second degree), which gives the impression of smoothness of the curve. There are many significantly different ways to relate the control points to a sequence of 'smoothly' connected pieces of parametric polynomial curves. In most applications a variant of

non-interpolatory splines are generated (i.e. the resulting curve does not interpolate the control points, see Fig. 2). Only in specific situations are constructions of interpolatory splines preferred. Note that in any case, given the control points, the construction of the polynomial pieces is not unique: thus, to convey the shape of a spline the whole piecewise polynomial structure has to be considered.

Figure 2 depicts a cubic spline defined by 4 control points (joined to a *control polygon,* for easier visualisation), which may serve as a prototypical example of non-interpolatory spline. I just list (without argumentation) some general features of non-interpolatory spline curves: 1. they start at the first control point and end at the last one; 2. the first and last line segments of the control polygon are tangent to the spline at the endpoints; 3. the construction of a spline from the control points is affinely independent - in practice this means that symmetric control polygons give rise to symmetric splines, that the mirrored image of a spline is the spline constructed from the mirrored control points, etc.

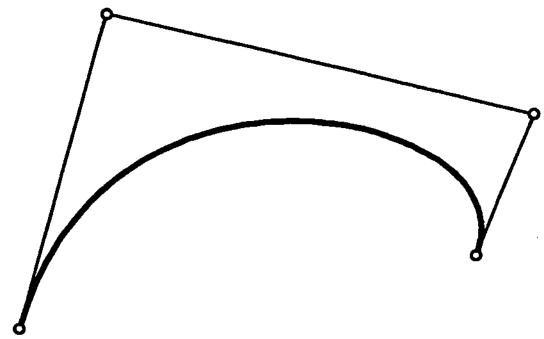

Figure 2. A non-interpolatory cubic spline defined by 4 control points joined in a control polygon

I consider splines as a case of informally learnt advanced mathematical knowledge for the following reasons:

- splines are rather complex mathematical objects, not included in vocational-school mathematics curriculum (in Slovenia) and well beyond the reach of vocational schools mathematics;
- mathematics teachers of vocational schools and students' parents usually do not know anything about splines;
- splines are common in (machining) CAD drafting, but the students and practitioners on CAD courses or elsewhere just learn how to use them, i.e. without any allusion to mathematics.

My main point of interest is how do users of CAD systems understand splines and where/what are the roots of their understandings.

The method

The aim of the study was to find cases of qualitatively different ways of making sense of mathematical objects that are used in a practice. Since this paper forms a self-contained part of a larger study concerning interrelations between geometric reasoning in workplace and school settings, the participants selected were those that took part in the larger study. Part of the research took place in a small factory in Slovenia that produces moulds for the glass industry. The work of six practitioners

working on CAD systems, all with vocational school training for machine technicians, was studied for three weeks using several techniques (interviews, observations, scheduled observations, document analysis, protocol analysis related to given problems). Among these two practitioners, who most often used splines in designing complex shapes, were selected for this study. Another part of the study took place in a vocational school for machine technicians in Slovenia. The participants were selected from a class of 22 students, aged 18, that recently finished a CAD-drafting course (lasting several months, two hours per week - a similar course was observed as part of the research). Six participants were selected on the basis of their attitude regarding the use of advanced mathematical ideas in CAD context (using a special test that served as a filter). As it turned out, two of the six selected students were inclined to use, in CAD context, the geometric ideas learnt in mathematics classes (they will be referred as GEO-oriented students), while four of them avoided this (and will be referred as CAD-oriented students).

Participants' ideas about splines were elicited using a semi-structured interview (Robson, 1993). The participants were interviewed in front of a computer while using the CAD software they normally used. They were asked to talk (for about 40 minutes) about splines in various contexts, to explain their ideas by making hand-drawings on paper, and, toward the end of the interview, to draw simple drawings using splines and to comment on them. The table below illustrates the kind of information which was sought in the interview and how it was obtained.

	tacit knowledge	**explicit knowledge**
knowledge about the spline command	observation during the interview	explicit questions about the spline command
relating splines to CAD; relating splines to mathematics	emerged during questions (e.g. 'How to convey the shape of a spline via the phone? Does the shape of a spline depend on the software used?)	'Did you learn something related to splines in mathematics/CAD class...' 'informal' talk at cooling down
knowledge about the shape and properties of splines (tangents, overall shape, smoothness, symmetry)	4 hand-made drawings given 4 control points each drawing spectacles' frames on the computer using only splines	asked to explain how to draw a spline given four control points explicit questions about the construction of symmetric glass frames
representation of a spline the 'structure' of a spline	'Does the shape of a spline (given control points) depend on the display, plotter, software, hardware?' 'Imagine you are looking at your mental image of a spline through a microscope. What do you see?' 'How would you convey the shape of a spline to another constructor by phone?' 'Do splines have length? What is meant by the length of a spline?' 'Assume the spline (or circle) command broke. Can you still draw splines (circular arcs) using just the command for drawing circular arcs (splines)?' 'Is it possible to draw with the spline command the shape of a bandy (and vice versa)?'	

All sentences of participants' answers were coded (often a statement was assigned several codes) and a large concept matrix with 50 entries for each of the eight participants was built. The matrix showed a marked distinction between the three groups (CAD-oriented and GEO-oriented students, experienced constructors) and rather consistent understandings of each participant. Below I present the main findings that emerged from this analysis.

Tacit and intuitive knowledge about splines

In working with splines all the participants relied to some extent on intuition (see Fishbein, 1987 for a treatment of the topic). It appeared that all the participants used intuition to complement their experience. They were very confident that splines have the properties they would like them to have. All CAD-oriented students, for example, have learnt only about non-interpolatory splines and have occasionally used only such types of splines - yet they convincingly, systematically and explicitly hand-drew all splines as interpolatory curves. All students also firmly, consistently and explicitly related the symmetry/mirroring of the control polygon to the symmetry/mirroring of the resulting spline. Only some students used this property in their work (e.g. when constructing a pair of spectacles).

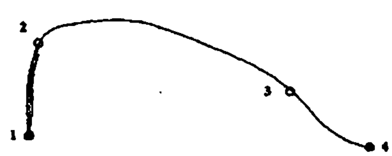

Figure 3. Andrej, a CAD-oriented student, drew the spline defined by four given control points as an interpolatory curve. Note the strange direction of the curve at point 4.

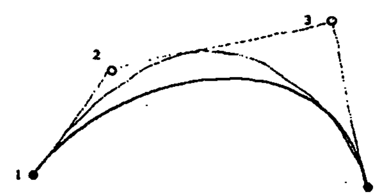

Figure 4. Branko, an experienced constructor, given four control points, drew the related spline (middle curve) quite accurately compared to the exact one (the lowest curve, added later for reference).

Experienced constructors were quite adept at drawing the shape of the spline by hand, given its 4 control points, but part of this knowledge was clearly tacit . One of them, for example, consistently drew the correct direction of the spline at the endpoints but did not mention this feature when he was asked to describe the shape of a spline curve. The experienced users also related the symmetries/mirroring of the spline and its control polygon and quite often used this in their work. However, they were much more cautious in using 'intuitively obvious properties'. Franci, an experienced practitioner, for example, when asked whether a mirrored couple of control polygons determines a mirrored couple of splines reasoned:

> **Franci:** *Ahm. I could try it now. I often try such things if I do not know how they work. Now. I would get the shape. It seems to me I would get the same*

shape. Except if it depends from which side you subdivide (i.e. of the orientation of the control polygon). For if you mirror this (the control polygon) it goes in the other direction. Does this matter? Anyway you subdivide the same curve (control polygon) but in different directions. I have not tried yet this.

Participants' ideas about the nature of splines

All the participants learnt how to use the spline command on CAD software in a CAD courses or from manuals. The command is rather easy to use: one just picks the control points and the computer draws a curve which can be interactively modified by moving the control points. Though the participants have not received any explanation about the properties or the structure of the spline curve - they had quite precise ideas about its the nature.

The CAD-oriented students (i.e. the students that avoid using advanced mathematics in CAD environment) understood splines as interpolatory curves which consists of a sequence of circular arcs (or line segments), each arc being tangent to the its adjacent arcs. Note that it is a standard practice in machine hand-drafting to approximate various curves with circular arcs (which can be drawn with a compass). As is easily seen the construction is not unique. Consequently, according to these students, the shape of the spline curve is not uniquely defined and depends on the software used. In conveying the shape of a spline, according to these students, one has to consider the data of all the circles of the particular construction. One of the four CAD-oriented students claimed that though there are many 'smooth' curves made of circular arcs and interpolating the given control points, the shape of the splines (given the control points) does not depend on the software used because all CAD system use a standardised way (standards are common in machining) of arranging the circular arcs.

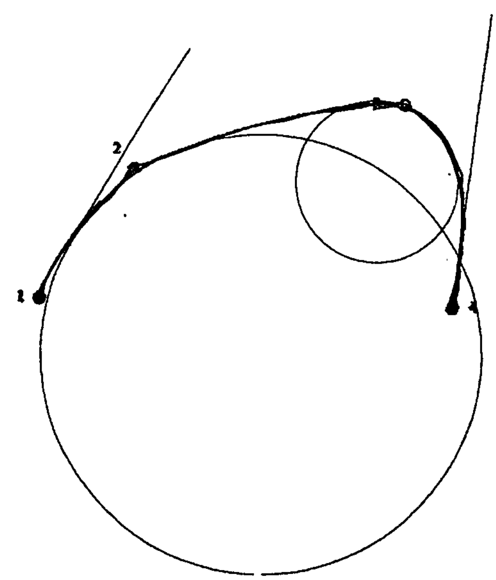

Figure 5. The analysis of a spline curve, drawn by Andrej, a CAD-oriented student, based on his explanation during the interview.

The GEO-oriented students consistently claimed that splines are mathematically defined curves (like parabolas or sine curves). They ignored how they are defined, but they were certain that there is a uniquely defined mathematical procedure to calculate the points on the spline curve. If they knew the procedure, they said, they could find out the properties of the curve, e.g. whether a spline can exactly match a

circular arc. Consequently, the shape of the spline curve depends only on the control points and not, in any way, on the software used. Thus, to convey the shape of a spline curve, according to them, it is certainly enough to just state the positions of the control points.

The expert constructors also viewed splines as mathematically defined objects, but they have developed interesting idiosyncratic ideas about them. Because of the mathematical nature of the splines, they claimed, the shape of a spline depends only on the position of the control points (and not on the software used). Nevertheless, if they had to convey the shape of the spline to another constructor by phone, though it depends only on control points, they would send as much data as possible about the curve, since they have to be absolutely sure that the shapes match.

The mathematical nature of splines was also reflected in the participants' interior representation of splines. The two GEO-oriented students and, to a great extent, both constructors viewed splines as mathematical objects. Their 'mental spline', for example, was infinitely thin and they did not bother about the irregularities of a displayed spline (due to resolution of the display). On the other hand the CAD-oriented students, though they obviously distinguished between the spline curve and its representation on the display, when they were talking about splines they often referred to their representation on the display (e.g. the shape of the spline, given the same control points, depends on the display). Apparently lacking a firm link between splines and mathematics they relied on visual appearance. A CAD-oriented students compared circles in CAD and circles in mathematics class:

> **MihaK:** *In CAD the circles are simplified, they are made of fewer segments, so that they take less place in memory. You can see sometimes how a circle is made of few straight segments, a hexagon or something like that. There is a command to increase the number of segments. In maths, I don't know, I think it is rather similar, you also have there the radius, the formula for the circumference is the same, the pi. I don't know, I would say it is much the same.*

Saxe's model

I view Saxe's 4 parameter model (Saxe, 1991) as a useful framework for understanding these observed phenomena. Splines, as objects in CAD activity, are not 'understood' by the students/practitioners who use them, they are simply told how and when to use them. I would say that the students/practitioners appropriate the form (how to use the spline command), attaching to it a non-cognitive function (e.g. denoting a section with a curved line). I have found that at least three of the four Saxe's parameters were involved in constructing the meaning of the spline curves. Some participants sought for the meaning of splines in the **activity structure** by relying the splines to standard approximation procedure of curves used in hand-drafting. What is most important, in my view, is that not all of the participants were

trying to make sense of the splines in the activity context. Some of them, the GEO-oriented ones, related splines to mathematical curves, i.e. to their **prior knowledge** of mathematics. Thus, though the procedure (form) for using splines was learned in the activity, the basic understandings of splines (their representations, structure and basic properties) are derived from the activity as well as from participants' prior knowledge of mathematics.

The role of **social relations**, due to the method used, was not observed, while the role of **artefacts** and **conventions** is quite obvious, e.g. the implementation of the command for drawing splines on CAD systems. Perhaps it should be stressed I have observed two additional factors that influenced the understanding of splines: *intuition* and *expertness*. Intuition apparently helps in making sense of splines, mostly at the initial phase. For most students splines were what they wanted them to be, and they had properties they would like them to have - and they found them as obvious. Expertness, on the other hand, gives at least some indication about what can be expected in a new situation. Both expert constructors intuitively 'knew' about certain properties of splines (e.g. that they depend only on control points, that a spline can be mirrored by mirroring the control polygon), yet - by experience - they knew one cannot rely on such knowledge and they claimed that such things have to be carefully tested and checked.

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AN ASPECT OF A LONG TERM RESEARCH ON ALGEBRA: THE SOLUTION OF VERBAL PROBLEMS

Nicolina A. MALARA - Department of Mathematics - University of Modena - Italy

We expose here some results of a research concerning the solution of verbal algebraic problems by 12/13-year-old pupils. This research was carried out within a wider study regarding an innovative approach to middle school algebra. It shows that if pupils are appropriately guided they can represent verbal relations in many ways, they can compose relations by substitution and get to the solution - without any specifically syntactic study - of complex problems at many unknowns and achieve the awareness of the need and importance of the autonomous study of algebraic expressions and equations.

Introduction

This report is about an activity concerning the solution of algebraic verbal problems and part of a wide research project of didactic innovation for the three grades of middle school. The research aimed at approaching algebra as a language for the production of thought (Malara & Iaderosa 1998a) and included syntactic and structural aspects of the discipline (Malara & Iaderosa 1998b), according to the theoretical model of algebra teaching/learning formulated by Arzarello & Al. (1995).

Our experience, together with the results of many classical researches in algebraic realm (Bell & Al. 1987, Chevallard 1989/90, Freudenthal 1974, Kuchemann 1981, Kieran 1992, Sfard 1991) make us believe that it is important to:

- teach the algebraic language in analogy to natural languages, trying not to face syntactical questions a priori and from an instructional point of view, but rather trying to suggest situations from which these questions arise as a natural consequence;
- make sure that the pupils get used to the plurality of representations of the same thing by creating didactic situations that make them aware of the fact that the choice of how to denote an object or express a relation influences the development of reasonings about them;
- highlight the passage from the procedural level to the structural level, so that the pupils acquire the awareness of the duplicity process-object, which is typical of the way mathematics develops and of the discipline itself.

From a general point of view, the working method we used in class was based on the systematic use of written verbalization, on the analysis of the working strategies enacted by the classmates so as to induce metacognition and on the collective discussion of the different results so that all conclusions drawn can be socially shared within the class.

In this research project, we do not study verbal algebraic problems separately from the rest, but we include it in such an activity that progressively develops all through the three years and concerns:

- the (manifold) translation into mathematical language of information expressed in the verbal language;
- the interpretation of formal writings and the recognition of their possible equivalence;
- the study of relations that are expressed by equalities between two terms and their transformation into equivalent ones;
- the introduction of the literal calculus starting along the study of problem-situations in general terms, and focusing on the role of the properties of the arithmetical operations.

Our methodological hypothesis

Traditionally, the study of algebraic verbal problems is started after the introduction of linear equations at one unknown; usually, considering the age of the pupils, the problems that are presented are in most cases solvable by intuition or by resorting to the appropriate graphical representations. However, as underlined also by Bernardz & Janvier (1996), this procedure doesn't allow the pupils to appreciate the goodness of the algebraic method.

Our methodological hypothesis starts the study of complex verbal algebraic problems at two or more unknowns even before equations are formally introduced, so that pupils can see why they should be studied as a mathematical object according to the historical evolution as well as to our ministerial syllabuses. This way the pupils get trained to the "principle of economy" that suggests the study of schemes representing a plurality of situations and that is so typically mathematical.

What we wanted to test with this hypothesis is the pupils' behaviour in front of:

- the formal translation of the relations expressed in the verbal text;
- the transformation and elaboration of the relations in order to get to a resolutory equation;
- the naive study of the equations for determining the values of the unknowns and solving a problem.

On selecting the problems that we wanted the pupils to solve, we considered the main difficulties about which the researchers have written, and in particular:

- the order in which the information appear in the text of the problem: the sequential nature of verbal language leads the pupils to a procedural reading, but if the order in which the data should be dealt with is different from the one in the text, it is very easy for them to make errors (Mac Gregor 1991);
- relations like "... is bigger than ...", "... is x times more than ...": they must be converted into terms of equality whereas pupils often do a translation that interferes with their proper elaboration (Mac Gregor & Stacey, 1993);

We can summarize the specific aims of the activities with problems as follows:

- to provoke and refine the pupils' ability to grasp and translate relations into terms of equality;
- to use hypothetical thinking in order to formalize information (if an object costs k Liras, then another object that costs twice as much costs ... Liras);

- to manipulate formulas by using the principle of substitution and the properties of equality appropriately for getting to an equation with a single unknown that has to be elaborated for determining the value of the unknown;
- to guide the pupils towards more and more refined syntactical questions through the solution of the problems.

Since we have limited room here, we shall only show the main results obtained with a group of 22 second-grade pupils with reference to a few algebraic verbal problems at more unknowns, underline by analysing the protocols the different difficulties and behaviours arisen and eventually we shall point out possible tips for overcoming them.

A few problems

In Table 1 you can see some of the problems at more unknowns on which we have worked. These problems usually have a rather simple text, that generally requires a formal translation that is not particularly difficult. The number of unknowns ranges from two to four and the relations can be additive, multiplicative, or both.

The didactic path developed with the pupils is focused on the control of the plurality of the possible writings of the same relation. For instance, on studying the relation "the segment $\overline{AB}$ *is greater than the segment* $\overline{CD}$ *of 4 (cm)*" the pupils usually express the sentence in the following ways $\overline{CD} = \overline{AB} - 4$; $\overline{AB} = \overline{CD} + 4$; $\overline{AB} - \overline{CD} = 4$ and they analyse and choose among them the most convenient writing as to the problem in exam. The substitution method is presented as "the game of changing", to be applied one or more times until they obtain an equation at one unknown which must be then elaborated. Thanks to the properties of the arithmetical operations, as far as the problems we have considered go, such equation can be lead to the structure ax+b=c, with a, b and c as natural or rational numbers.

Table 1

Algebraic verbal problems with more than one unknown done in VII grade
1. Consider a triangle ABC leaning on side AB of which the following information is given: the perimeter measures 60 cm; side BC is 5 cm longer than the side AC; side AB is 10 cm longer than side AC. Calculate the measures of its sides. Could the triangle be a rectangle triangle? If yes, which would be the right angle and which the hypotenuse?
2. Determine the two numbers of which the major one is such that it is by 50 bigger than the triple of the minor, and the sum of which is 110.
3. In a courtyard there are 224 animals between dogs and cats. The cats are 6 times as much as the dogs. Calculate how many are the cats and how many are the dogs in the courtyard.
4. Massimo goes to the pizzeria and spends 83.000 Italian Liras for a pizza, a pudding and a coke. The pudding costs 15.000 Liras less than the pizza. The pizza costs 400 Liras more than twice the coke. Calculate the cost of the pizza, of the sweet and of the coca-cola.
5. The perimeter of a rectangle trapezium measures 96 cm. The measure of the greater base is 20 cm more the smaller base. The difference between the measure of the greater base and the one of the oblique side is 13 cm. The difference between the measure of the oblique side and the one of the height is 10 cm. Calculate the measures of the two bases, the height and the oblique side.

We present this method in a collective activity of guided solution, in which the teacher acts as a model and shows the best way of facing each meaningful step of the process.
Apart from the number of unknowns and the kind of relation in question, the main difficulties concern: i) the translation of relations into terms of equalities and their transformation; ii) dealing with the resolutory equation; iii) choosing the unknown through which the others are to be expressed.
This is the order in which we faced them, but it is not strictly sequential.
The first problem presents two additive relations which express the relationship of two numbers as to a third one. It is very important to find the right translating formulas to get to the resolutory equation. To tell the truth, we usually start with easier problems than this, where the relations are more easily linkable together.
The second problem is even more complicated, because it contains an additive and a multiplicative relationship and also because you have to convert the expressions "is bigger than" and "is triple of" into terms of equality.
The third problem is quite famous (Mac Gregor, 1991). We usually get to it after dealing with other ones, with the specific aim of testing whether and to what extent the pupils, after they have practised with formal translation, have the difficulties that Mac Gregor has highlighted (difficulties with translating multiplicative relations that linguistically are expressed in the inverse order than in translation or confusing quantity with quality in using and interpreting letters).
The fourth problem, too, is faced after other problems. It contains an additive relation, a subtractive one, and an additive-multiplicative one. It is used for testing the pupil's ability to deal simultaneously with different pieces information to be translated, as well as their ability to elaborate the information obtained.
The fifth problem is a bit different: apart from the context, which is geometric, this problem contains four unknowns and it is linguistically complex. Of the four relations, two have the difference between two unknown quantities as subject.
You will find in the following paragraphs some protocols of the pupils' work which can help us understand what they were asked to face.

Analysis of the protocols

The protocols you see in Table 2 contain some difficulties the pupils had and the strategies that they enacted.
The first two protocols concern Problem 2. The first one shows a good ability in translating formally and elaborating the information, whereas the second contains a typical error for beginners: to indicate the two unknown numbers with the same letter. Although there seems to be a good "local" control of the relations, it lacks a global control of the situation and the ability to organize the information into a formal translation.
The protocols about Problem 3 testify the difficulties concerning the syntactic aspect of dealing with the resolutory equation. In the first one, Annalisa can't interpret and elaborate correctly the writing "C + Cx6 = 224", but she understands

that her simplifying it into Cx6 = 224 is wrong. (Just like the researches have demonstrated, this error was quite frequent during our activity).

In the second protocol, Chiara works in parallel with two different encodings of the problem and makes a good and immediate translation of the relations expressed in the text. Still, she makes a very rough mistake on transforming, which reveals her procedural reading of the formal encoding of a relation (the expression "dogs+ dogsx6 = 224" is transformed into "dogs·2·6 = 224") and shows difficulty in controlling the meaning in the correct reading and elaboration of the equations she obtains.

Table 2

Examples from the pupils' work

From the protocols on Problem 2

Silvia: A = Bx3 + 50 ; A + B = 110 ; B + (Bx3 + 50) = 110 ; Bx4 + 50 = 110 ; 110 - 50 = 60 ;
60 : 4 = 15 B = 15 ; 15x3 + 50 = 45 + 50 = 95 ; A = 95.

Anna: maj. 1st number 2nd number minor number
is bigger by triple 110:3=(X·3)+50=X
X + 50 → X·3 major number
sum 110

From the protocols on Problem 3

Annalisa: G = Cx6, C = G:6, C + (Cx6) = 224; 224:6 = it doesn't work

Chiara: dogs + cats = 224 ;

dogs x 6 = cats ;	cats: 6 = dogs;
dogs + dogsx6 = 224 ;	cats + cats: 6 = 224
dogs·2·6 = 224	cats·2:6 = 224
224:6 = 3	224x6 = 1344 ; 1344: = 672; cats 672; dogs 672:6 = 112.

From the protocols on Problem 4

Chiara B.: 8.300 = Pizza + sweet + coke ; pizza A , Coca = B , sweet = C. A = sweet + 1500/cokex2 + 400. B = pizza - 440:2 No. B = pizza:2 - 400 C = pizza - 500/cokex2 + 400 - 1500. 8300 = A + B + C = A + A:2 - 400 ; A·3:2 - 1900 = 8300 ; 8300 + 1900 = ; A + B + C = Bx2 + 400 + B + Bx2 + 400 - 1500 ; B·5 + 800 - 1500 = 8300 ; 9800 - 800 = 9000 ; 9000:5 = 1800 coke ; pizza = 1800x2 + 400 = 3600 + 400 = 4000; sweet = 4000 - 1500 = 2500.

Eleonora: P pizza, C = coke, D = sweet. P + C + D = 8300 ; D = P - 1500 £; P = D + 1500 C = P:2 + 400 NO CP = C + 400x2 NO C = Px2 + 400 ; P = C:2 - 400. The following part appears erased: P + P - 1500 + P;2 + 400 = 8300 3P - 1500:2 + 400 = 8300; 8300 - 400x2 + 1500:3 = 900x2 + 1500:3 1800 + 1500:3; 3300:3 = 100

From the protocols on Problem 5

Eleonora: 2p = 96 cm = AB + BC + CD + DA;
BC = AB-13 , AB = BC+13 ; BC = CH-10 ,
CH = BC+10 ; AB = DC+20 , DC = AB-20
AB + + AB - 13 + BC - 10 = AB + AB - 13 + AB - 20 + AB - 13 - 10 = 96 ; 4AB - 13 - 20 - 13 - 10 = 96 ; (96 + 13 + 20 + 13):4 = 119 + 20 + 13:4 = 139 + 13:4 = 152:4 = 38 ; AB = 38 ; CD = 38 - 20 = 18 ; BC = 38 - 13 = 25; DA = 25 - 10 = 15.

Alberto: AB - CD = 20 cm ; AB - BC = 13 cm ; BC - CH = 10 cm AB + BC + CD + DA AB-CH = 23 ; 96 = AB + AB - 20 + AB - 13 + AB - 23 ; 96+20+13+23 = 152 , 152:4=38=AB; AB = 38 cm; 38-13 = 25, 38-20 = 18 , 38-13 = 25; BC = 25 cm ; CD = 18 cm AD-CH = 15.

The protocol about Problem 4 reveals more difficulties in reading the formulas correctly so that they can be transformed. The young girl makes a mistake in explicating B from the relation A = B x 2 + 400 since she writes B = A:2 - 400 instead of (A-400):2. However, it is interesting to see that since she cannot elaborate the equation in unknown A in a way that she can manage, in the end she reconverts the relation of equality in function of the unknown B, which allows her to achieve the solution.

The two protocols regarding the last problem show two different ways of organizing the information for getting to the solution. Eleonora, who is not particularly brilliant, translates the information literally and gets to the fundamental equation at one unknown by operating subsequent changings. Alberto, on the other hand, is quite smart; he operates in a very original way. He expresses all the information in terms of difference between the unknowns and combining such differences he obtains a further relation of difference which he uses for eliminating one of the unknowns.

Generally speaking, as to the first problem, the "game of changings" turns out to be quite easy for almost all the pupils. As to the second one, on the contrary, the protocols are rather different. The best solutions revealed themselves to be a good model for the pupils who were a bit lost. As far as the third problem goes they generally give a good translation of the text. Only the 14% of the pupils make errors of this kind, but there is a widespread presence of mistakes and serious difficulties concerning the syntactic elaboration. One of the difficulties that we have detected in the pupils, for instance, is well known in didactic research (see Lopez Real 1998) and it is due to the fact that the pupils are working with natural numbers: they couldn't convert writings like g + g:6 into g + g/6, which is more expressive as to a correct transformation. The fourth problem was quite difficult for them owing to the variety of relations involved. The biggest obstacle seems to be managing the correct translation of the relation: "the pizza cost 400 liras more than twice the coke", which is tied to the correct interpretation of the linguistic expression "more than twice the". As to the fifth problem, beyond the number of relations involved, the pupils are negatively struck by the novelty that two of the four relations of the problem concern the difference between two of the unknowns.

The global analysis of the protocols tells that the solving strategy chosen by the pupils is "mixed": they begin by operating algebraically (they write down the single relations formally and by substitution they reach an equation with only one unknown), and once they get to the equation, since they lack appropriate algebraic knowledge, they study it "arithmetically" by dividing it into two steps; on the basis of the meaning of the equality of the two terms of the equation, they isolate and solve the numerical expression constituted by the known terms, then they lead the initial equation to one of the kind ax=b by summing the terms including the unknown and by substituting b with the result of the numerical expression they have elaborated. On manipulating the equation, the pupils have mainly syntactic difficulties, and in particular on doing the sum of the terms in x, especially when

some terms have fractionary coefficient, or when there's one term without numerical coefficient among the terms including the unknown (which makes them neglect it on doing the sum), or even when they have a bad management of the parentheses. Some protocols show very clearly that if by working on an unknown they reach an equation that they acknowledge to be difficult for them, they go back to the beginning and look for a different unknown which gives them a much easier equation. These protocols are rather useful for classroom discussions aimed at introducing 'forecast thinking' and 'choice economy'.

General considerations

This activity had a very strong impact on the pupils both for involvement and results achieved. Almost all of them understood that they were dealing with a general model for solving problems with unknown data, based on the translation into formulas of the information contained in the text, which allows to reach the solution thanks to the elaboration of the formulas obtained. As to their behaviour, we can divide them into two groups of the same size: the first one contains those pupils who adopted the algebraic method and were able to face syntactic elaboration on their own, according to the principle of economy; the second one contains the pupils who actually learned the principle of formal translation, but who never went beyond the models they had learned, which made them get stuck in front of new syntactic situations. Besides, a small group of very weak pupils didn't understand the spirit of the algebraic method and resorted to graphic solutions whenever they could. Nevertheless, we must say that with time almost all the pupils developed the skills for translating and organizing by substitution the relations contained in a text. Of the errors reported in research studies we didn't see neither the conjunction error that usually appears with additive and multiplicative relations (Mac Gregor & Stacey, 1994) nor any difficulty on passing from one to many unknowns (Radford, 1994).

As the protocols witness, the fact of having done algebraic problem solving before doing any algebra syntax implied of course that the pupils sometimes enacted wrong transformations - some of which had no justification, like those connected to the omission of the parentheses, or those which were due to lacks of thinking about the meanings of the writings.

Anyway, this method made the pupils approach syntactic questions in itself, outside any reference context, concentrating on the arithmetical properties. For example, on the basis of the difficulties arisen, at a certain point the teacher asked these questions: *If M is a number, what can I say about:* $M + 2M + M + M$; $M + 2M + 7M - 4M$; $-7M + 7M$? and concentrated on the following answers for discussion: $\underline{M + 2M + M + M}$ is: $5 \cdot M$ (Eleonora); $M \cdot 3 + 2 \cdot M$ (Jessica); $M + M + M + M + M$ (Silvia) ; $2 + 4 \cdot M$ (Cristina). $\underline{M + 2M + 7M - 4M}$ is : $(1+2+7-4) \cdot M = 6M$ (Alberto) ; $5 \cdot M$ Vincenzo ; $10 \cdot M + -4 \cdot M$ (Salvatore) ; $3 \cdot M + 7\,M - 4 \cdot M = 10 \cdot M + -4 \cdot M = 6M$ (Chiara). $\underline{-7M + 7M}$ is : $0 \cdot M^2$ or $M \cdot (-7 + 7) = M \cdot 0$ (Sara) ; $2 \cdot M + 7 - 7$ (Chiara) ; $0 + (M \cdot M)$ (Cristina).

Some of these answers show that for many pupils the meaning of those expressions that the teachers consider easy and by no way ambiguous is not at all straightforward. This means we should spend more time than usual with problems like these, which are so important for the control of the meaning of literal expressions. We therefore resort to very simple problems of this kind, which unfortunately are not contained in our textbooks and are often considered banal by teachers, but can be found in foreign projects (e.g. Harper 1987).
These results make us suggest that in the future we should carry out more studies on this aspect, face equation solving beside problem solving, presenting them like hypothetical equalities between two expressions in the sense of Herscoviz & Linchevski (1992). These equations should be solved without resorting to ready-made techniques, but rather in different ways according to the features of the equation (through reasoned trials, through the weigh-model and using the law of cancellation, resorting to inverse operators, etc.), also for reducing to the minimum the difficulties connected to the different structures of the numerical realms in which they might operate.

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Conjecturing and Proving in Problem-Solving situations

M.A. Mariotti & M. Maracci
Department of Mathematics - University of Pisa - Italy

Abstract *This paper reports on the first phase of a research project which aims to investigate conjecturing and proving processes in problem-solving situation. The analysis focuses on the relationship between the process of production of a conjecture and that of its justification: in particular, deductive proof is compared to argumentation supporting a conjecture. The paper discusses on the difficulties related to the passage from the production of a conjecture and the production of a proof.*

1.Introduction

Previous investigations (Shoenfeld, 1985) have provided evidence of the appearance of argumentative reasoning accompanying the solution of an open-ended problem, so as its relationship with the production of a proof (Boero, et al. 1995, 1996, Harel, 1998)

Despite the undeniable difference between "deductive organisation of thinking" and "argumentative organisation of thinking" (Duval, 1991), some aspects of argumentative activity were observed and described, in the production of conjectures so as of proofs (Boero et al. ,1996; Mariotti et al., 1997). The peculiarity of the teaching experiment, within which those aspects of continuity were observed put in question the generality of the observation and claimed for further investigations, in order to identify the relevance of the particular context and the role played by the specific 'didactic contract' set up in the classroom. Actually in the teaching experiments, which Boero et al. (1995) refer to, open-ended problems were commonly included within school activities with the explicit agreement that the solution should provide a conjecture as an "argumented choice"; the issue arises whether similar results could be found outside this special contract.

Despite the fact that problem solving activities have become very popular in mathematics education, so as in the school practice of some countries, this is not the case of Italy, where open-ended problem solving and investigation rarely appear among school activities, even less at the high school level. Consequently, a research project, still in progress, has been carried out with the aim of investigating on the solving processes of open-ended problems.

This paper reports on some results of that analysis, in particular the relationship between producing a conjecture and constructing a math proof of that conjecture.

The main hypothesis

Mathematical proof is a complex idea dealing with many different aspects, concerning logic, epistemology and cognition (Mariotti et al., 1997). The facets that proof can assume in different institutional contexts depend mainly on the function attributed to proving process: as far as school mathematics is concerned, the most relevant functions are explaining and convincing, while that of

systematising a result within a theoretical framework represents a secondary function (Bell, 1976; de Villers, 1991). That seems consistent with the conclusions recently reached by Hoyles and Healy (1998) As the authors say "Students are unlikely to use deductive reasoning when constructing their own proof". (p. 42) and "An argument felt to convince or explain is more likely to be selected as a student's own approach than one that is not, and the likelihood increases still further if it does both"(p. 30).
That can be summarized in the following Hypothesis.

Hyp. Proving is providing logically enchained arguments referring to a theory, but at the same time proving is providing an explication which can remove doubts about the truth of a statement.

This twofold meaning of proof, which is unavoidable and pedagogically consistent (Hanna, 1990 quoted by Harel 1998), may cause difficulties and misunderstanding, when open-ended problems are concerned.
Open-ended problems appear to provide situations to introduce students to theorems (Boero, Mariotti et. al., 1997): arguments are mobilised in order to produce and support a conjecture, but for the same reason the need of justifying a conjecture and removing uncertainty can cause the aims of explaining and convincing to prevail. As a consequence, the resulting argument may be very successful in explaining an answer, but may be completely inadequate as a proof of a conjecture. What Harel states (1998) could be adapted as follows: "The proof schemes held by an individual are inseparable from his or her sense of *what means to solve an open-ended problem* ".
In Italy at the high school level a common task concerns the proof of a given statement, but according to a well established didactic contract (Brousseau, 1986), a proof must be clearly required: specific expressions are used to formulate the task "Prove that ..."
On the contrary when open-ended problems are concerned, no clear and established norms may be found: the process of solution seems to require a justification, but that is usually conceived as the explication of one's own 'reasoning'. That means that the solution must be accompanied by an argument which explains and justifies it.

The experimental design
Seventeen subjects were selected in different Scientific High Schools: according to school evaluation, all of them were high-attaining XI and XII Grade students and generally speaking were considered brilliant students. Although they did not belong to the same class and not even to the same school they followed a deductive approach to Geometry. Four problems were selected and proposed to the students in an individual interview; during the solving process they were asked to think aloud and the whole problem-solving session was videotaped, then the transcripts of the interviews analysed (Maracci, 1998).

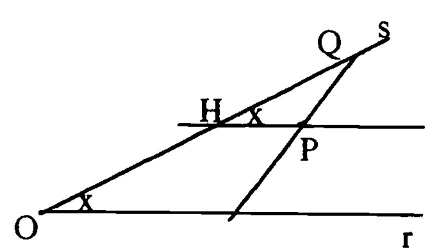
In this paper only one of the problems is considered; the text is the following.

Pb1. *A convex angle rOs is given , where the two rays r and s are not on the same straight line, and a point P internal to the angle, determine a line segment which has its ends on the arms of the angle and P as its midpoint.*

That is a construction problem; it presents some difficulties in the identification of the solution, but may be successfully approached by assuming the segment given and looking for some characterising properties.
The problem is not too difficult not too easy and for this reason provides a good test for our purpose. Students can undertake the solving process, but the solution does not come immediately: often successful, the solution comes after some investigation.

Argumentation supported by theorems and definitions
The analysis of the transcripts reveals a great variety in students' performances, but also interesting similarities.
It is difficult to separate the process of solution into different phases corresponding to an ideal sequence of producing, formulating and proving a conjecture. What occurs is more like to an intermingled combination of this phases. The process is clearly directed by the goal of attaining the solution together with the certitude of its correctness. The culture of theorems into which students have been introduced leads them to consider that for a reasoning to be correct it must refer to well known geometric properties and theorems.
The solving process results in a long and often tortuous discourse with arguments referring to the possibility of "applying theorems". But it is very difficult to isolate a statement which could be possibly recognized as the formumation of a conjecture, either in the hypothetical form "if ...then" or in any other form; similarly, it is very rare to find a final deductive argument which could be possibly recognised as a proof.
The following protocol can be considered an exemplar.

Gia 11th grade (Scientific high school) Pb1
Giacomo reads the text of the problem and after few attempts (drawing an angle and a tentative solution segment) seems to have an intuition
Gia. : let us draw the lines parallel to the arms of the angle and passing through P - *he draws the lines, then draws a new drawing (4) where he traces few segments passing through P; stressing one of them* -
Gia. Yeah, one can make the parallel to r pass through P - *he draws it on drawing 4* - at this point - *he marks the angles* - here, practically ... if I trace



the segment through P - *he stresses the segment* - which should be practically one half, I should find that because of the similarity ... this is one half of this ... and then also the ratio of the other sides, that is - *he put the labels Q and H* - OQ and OH must be 2.

Thus in order to find the segment one can do in that way - *he draws a new figure (5) and labelling* - given a point P one draws the parallel line to r - *he draws the line on figure 5* - after that - *he marks the intersection between the new line with the ray s* - one doubles ... I should obtain OH, let us consider a segment with the same length - *he marks the point Q on s* - and one make the segment pass and this is ... has the ends - *he marks the point R, intersection between r and the line QP* - on the arms of the angle and P is the midpoint.
I.: Have we finished that way?
Gia.: I think so; here we used the 'similarity' and we could use it because I had two triangles - *he comes back to drawing 4 and puts the label M on the end of the segment passing through P* - OQM and HQP similar because constituted by one equal angle and because I drew the parallel to the ray r - he points at it - I know, because of a theorem that I have already proved, a group of parallel lines which intersect a ray, divides it in ... a group of rays ... divides it into similar triangles; <u>thus from the similarity I just pass to this ratio of sides which between QM and QP must be 2, but, in order to be 2 it must be 2 also the ratio of QH and QO</u>, because we know that in two similar triangles the ratio must be constant <u>and then I come to the end</u> : given a point P one can draw the parallel line, in this case to r, but one could do the same with s, and consider the segment OH, double it in order to get another equal segment, pass the segment through Q and ...
I.: OK, I ask you only another question: if I were your teacher and she had given you that problem, at this point would you say "I finished" or would you do anything else?
Gia.: I would say that I finished, because I got the thesis that I wanted, I found the segment.

The production of the solution is accompanied by the explication of one's own 'reasoning '. The formal approach to which students have been exposed lead them to approach the problem 'theoretically', i.e. the segment must be "geometrically constructed" and for the argument to be correct it must refer to well known properties and theorems. Thus, since the beginning, Gia. refers to "tracing *parallels*" and to "apply *similarity* ". But the student seems not to feel the need of reorganising that reasoning in the form of a statement and its proof. The main objective of the argumentation remains that of explaining and convincing oneself and/or the hypothetical interlocutor, and the discourse does not assume the form of a Theorem (a statement and a proof within a theory, Mariotti at al. 1997).
After the question of the interviewer Gia takes up his argumentation and immediately points out that he used ("ho sfruttato") a geometric concept - similarity -; then he seems to undertake the first steps of a proof, although no statement has been clearly formulated and remains implicit in the description of the construction. What seems to me most interesting is that at the crucial point, when Gia is going to apply the theorem, he goes back to the indirect argument, used in the production of the solution: with the same words of his previous argument, he says:

<u>thus from the similarity I pass to this ratio of sides which between QM and QP must be 2, but, in order to be 2 it must be 2 also the ratio of QH and QO,</u>

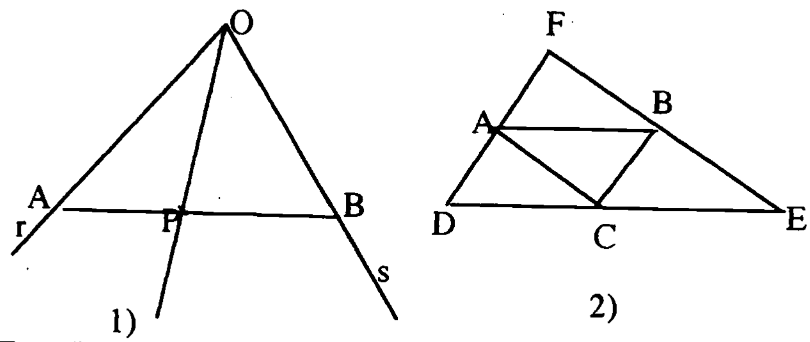
In other terms, the inversion between hypothesis and thesis, typical of the change from 'analysis' to 'synthesis' and required by a deductive systematisation of the argument, did not occur.

Gia's reasoning explains in a narrative form both the genesis of the solution and its justification; that reasoning satisfies the student because it fulfill his needs in that situation: it accomplishes both the function of convincing and explaining, and the 'didactic agreement' that the support of a conjecture must be provided by theorems already known. Thus, despite the pressure of the interviewer the structure of the argumentation does not change and it does not even change after the explicit reference to the didactic contract "If I were your teacher ..."
Let us consider another example which presents some similarities, but also an interesting difference, with the preceding one.

Fra. 11th grade (Scientific high school) Pb1

Fra. reads the text and draws a first diagram, that gives him the idea of a theorem about midpoints in a triangle. Confronting the first drawing and a new one representing the theorem, Fra comes to the solution: at this point, he summarises his reasoning running through his previous arguments and concludes:

1. **Fra.** the figure is not precise ... because of the fact that ... now the figure is not well done, anyway these points here - *in the drawing 1 he changes the labels of the points images of O in the symmetry of centre A and B, he points at D,E and P* - they should be collinear.



2. **I**: D, E and P should be ... and so they are?
3. **Fra**: They are, I mean they are and also they are parallel to this line here - *he points at AB* - thus I got the end, if I drew a good drawing.
4. **I**: sorry, you are saying that D, P and E are collinear, ...
5. **Fra**: Yes, they are, because of this reason ... if I have this point - *C on the drawing 2 and P on the drawing 1* - which would be practically the midpoint on this side here - *he uses the pen to point at the segment to determine, but without drawing it* - I can use the relationship holding between the midpoints - *he points at the drawing 1, PB and r* - ... being that parallel - *BC in drawing 2* - I draw this parallel - *PB in drawing 1* - in order to find this - *point B in drawing 2* - then drawing the other one - *PA in drawing 1* - and finding this point - *A in drawing 2* - I know that this - *OA in drawing 2* - is equal to half that side - *he points at AD in drawing 1* - I do the same - *he points at drawing 1* - I take DE and that's all.
6. **I** : That seems a good idea, but it could be that I am not convinced that D,P and E are collinear.
7. **Fra**: ... but ... the theorem ... - *he points at drawing 2* - ... this theorem says that if I draw the lines through the midpoints of a triangle, for instance B and C, all the segments, the length of the segment between the midpoints is equal to half ..., in all the triangles, any triangle ... I mean I find this relationship ... I mean ...
8. **I.** In our problem, we have the point P and the angle, we draw the parallel lines to the arms of the angle, we find the point A, B; let us do the symmetry ...
9. **Fra.**: let us call them O' and O" - *he changes the labels in drawing 1.*
10. **I.**: ... you construct O' and O"
11. **Fra.**: now O'O" passes through P

12. **I.:** I don't know that, I don't ... that O', P, O" are collinear, I don' see that so evident from that theorem, try to explain that better ... what does it mean that they are collinear?
13. **Fra.:** If I draw the straight line - *he points at a hypothetical straight line passing through O', P, O"* - they all are on the same straight line
14. **I.:** How can you prove it?
15. **Fra.** : One must prove that if I take the straight line O'O", P is on that line .
16. **I.** : ... It doesn't seem straightforward to me ... from that theorem
17. **Fra.** : ... let us suppose that P is not on that line, then there should be another point P', let us put it here - *he marks P,' separated from P', on O'O"* - such that ...
...
18. **Fra**: if I did a new drawing ... - *he draws the angle, the point P and the parallel lines* - P is here, I draw the parallel lines ... let be like that - *he draws a segment that does not pass through P* - it would exist a point P' which is the midpoint ...
...
19. **Fra.** : *the comes back on drawing 1* - then it should exist a point P' on O'O" ... a point P' which would be the midpoint of the side ...
...

Similarly to Gia., the first justification produced by Fra (5) is a summary of his previous arguments centred on the reference to the "midpoints theorem"; also the following attempt remains confused and vague. At this point the interviewer intervenes: he reformulates the solution, and directly asks to *prove that* P belongs to this segment and is its midpoint. After the direct request of proof Fra begins a proof by contradiction, which is then completed .
It is interesting to remark that in order to accomplish that proof Fra needs
1. to detach himself from the previous reasoning through the drawing of a new figure, where the statement to be proved is clearly expressed (18).
2. to come back to the first drawing (19), where the first conjecture was produced, in order to resume his reasoning and the relationships among the elements involved (an example of cognitive unity, Garuti et al., 1996) .
Similarly to the previous example (Gia), after the intuition of the solution, there is neither spontaneous reorganisation of the reasoning in a deductive form, nor the formulation of a clear conjecture. The student is able to provide a proof, only in response of a direct request (14) of the interviewer.

Discussion
Open-ended problems confirm themselves as catalysts of argumentative activity: the feeling of uncertainty that accompanies the solving process provide a good stimulus for the production of arguments. It is also clear that the need of a Geometric support for those arguments has overcome the status of a norm of the didactic contract and has evolved in an epistemic need.
Never the less, the main objective of students' argumentation remains that of convincing oneself and/or the hypothetical interlocutor, and the discourse does not assume the form of a theorem. In other terms, students seem unaware of the distiction between the process of constructing arguments, i. e. "proving", and the deductive systematisation of those arguments in a "proof" (Douek, 1998).
But in this case speaking of a cognitive rupture between these two aspects (Duval; 1991) would be misleading. Rather, our hypothesis about the presence of

different functions of proving, which coexist and guide the organisation of the argumentation, provides an interpretation for both the examples. In the former example (Gia.) there is a prevalence of one function on the other and the absence of the correct formulation of a deductive proof. In the latter example (Fra), a direct request of proving may cause the shift from the function of explaining and convincing to the function of validating a statement. The example of Fra shows that the necessity of reorganising the arguments in a proof can be understandable, although not immediate. Actually, some difficulties must be overcome: as the protocol shows it is after the mediation of the interviewer that the student understands what he was required to do. Spontaneously he would never think of doing that.

Although, proving as part of the solution of an open-ended problem seems similar to proving as response to a "proof task", between these two processes there is a great cognitive difference. As far as the proof task is concerned, the statement and the request of its proof are directly posed to the student from outside. On the opposite, when an open-ended problem is concerned, the production of a proof must be the response to a self-imposed task, encapsulated within the solving process, and strictly dependent on the production of a conjecture. Thus, within the solution of an open-ended problem the functions of explaining and convincing may completely prevail over the other. This effect may be amplified in the particular situation of the interview: the request of thinking aloud and making one's own reasoning explicit to the interlocutor naturally leads the student to shift to the functions of explaining and convincing.

The peculiarity of a construction task

As a final remark I want to focus on the particular task selected for the interview: a construction task.

When a construction task is concerned the solution requires to describe the constructing operations, and then to validate it in a Theorem (Mariotti et al., 1997). That means to formulate a statement, expressing the geometric relations among given and new elements, and prove it; but summarising in a 'hypothesis' all the relationships stated by construction may be too difficult to be done. That could explain why in our protocols it is almost impossible to find any formulation of a conjecture, and when it occurs the conjecture is never expressed in a hypothetical form. On the other hand, the absence of a statement may obstacle pupils to assimilate the situation to the familiar situation of a 'proof task' and hinder the production of a proof. On the contrary, the formulation of a clear statement could direct the students towards the idea of proving and facilitate the correct construction of a proof.

Conclusions and didactic implications

The pervious analysis shows of the continuity between the arguments used both in producing and justifying a conjecture and its potentialities, but makes also clear

that the relationship between the process of constructing a conjecture and that of constructing its proof presents a complexity which must not be underevaluated.
In particular, the delicate didactic problem concerning the relationship between the different functions of proving must be directly addressed because of its effect on the process of constructing a mathematical proof.
A suggestion comes from our analysis: teaching intervention could foster students to reconstruct their arguments into a Theorem, i. e. a statement and a proof within a theory. Moreover, according to our results, a basic point seems to concern the formulation of a conjecture in a clear and precise statement.
It is true that most of the solutions provided by our students appear very close to mathematicians' practice of doing mathematics, but the obvious difference between mathematicians and students cannot be forgotten. Mathematicians can afford (take the liberty of) informal arguments, in fact beneath an incomplete and informal discourse they maintain the epistemological control of what is going on, of what can and/or should be proved On the contrary, as far as students are concerned, the control of the different functions of proving is not expected and the balance between them is not immediate. As Hanna wrote: "what needs to be conveyed to students is the importance of careful reasoning and of building arguments that can be scrutinised and revised" (1989, p. 23).
As any characteristic feature of mathematicians attitude, the complementarity between different forms of argumentative activity, could be the result of a slow and complex appropriation process: an explicit educational goal.
That is not simply to comply with the standards of mathematical argumentation, but because, in our opinion, the appropriation of a productive relation between argumentation and proof may improve students' awareness of the functioning of mathematics.

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FORMING RELATIONSHIPS IN THREE DIMENSIONAL GEOMETRY THROUGH DYNAMIC ENVIRONMENTS

Christos Markopoulos and Despina Potari

University of Patras, Greece

This study explores how children build relationships between geometrical solids and their properties and between the solids themselves while working with dynamic three dimensional models. The data used in this paper comes from a teaching experiment where two pairs of children, one of grade four and one of grade six work on the models which have been developed by the researchers. Issues that emerge from the study are the role of constructed environment in bridging the gap between intuitive and formal school knowledge, in encouraging the development of relationships between the two and the three dimensions and in supporting interactions between "concrete" and "abstract".

A number of research studies have focused on children's thinking about three dimensional geometrical solids. However, most of them explore children's conceptions on solids' representations on the plane (Bishop, 1979; Cooper and Sweller, 1989; Mitchelmore, 1980; Piaget and Inhelder, 1956). Even in cases where the solids themselves are considered, they have been examined through their plane representations either in static or in dynamic environments (Battista and Clements, 1996; Ben-Chaim, Lappan and Houang, 1989; Chiappini and Lemut, 1992; Gutierrez, 1996). In this study, children's geometrical thinking is investigated through models of geometrical objects which through dynamic transformations can take different forms. We accept Bauersfeld's position (1995) that the materials and their properties are not "self-speaking" but we consider that the dynamical manipulation of them may help children to construct and extend dynamic images and develop their geometrical thinking. This analogy can be seen in the work of Shepard and Cooper (1982). Kaput (1992) supports that in the case of geometrical objects this process is characterized by the recognition of the invariance of these objects and this can be encouraged by the variation of the objects.

The focus of the study

In this paper, we present a part of a research study which focuses on children's geometrical thinking about geometrical solids and their properties. In particular, we explore the role that the dynamic transformations of geometrical solids can play in the creation of a learning environment which encourages the development of children's ability to focus on the properties of each solid and build relationships between these properties and, based on these relationships, between different solids. This development can be seen within the model of the van Hieles' levels of spatial thinking

or within Presmeg's (1986) and Johnson's (1987) view of the development of mathematical reasoning and understanding through the transition from concrete images to more abstract and flexible images. The dynamic transformation of a geometrical solid is considered as a process where the solid changes its form through the variation of some of its elements and the conservation of others. Similar transformations have been studied in the case of two dimensional geometry and especially through the use of computers (Laborde, 1993; Markopoulos and Potari, 1996). However, the case of three dimensional geometry requires the development of more complex relationships as the properties are related both through the linear properties of the solids (e.g. the relationships between the edges) and through the properties of two dimensional geometrical figures (eg. the relationships between the faces).

The above transformations have been seen in three different contexts. The first is defined through children's manipulations of physical materials of three dimensional geometrical models. The second involves children's interactions in a computer based environment which simulates such geometrical models and their transformations. The last one is formed through children's involvement in imaginative situations where dynamic transformations of solids take place. The possibilities and the constraints of these three environments make possible the appearance of different aspects of children's conceptions of the solids. These contexts vary in the type of transformations which they require through children's actions. It seems that through these contexts the children move from physical to visual and finally to mental actions. Here, we can probably see a development from concrete to abstract considerations of the solids. The data used in this paper concerns the first context.

In this paper, we focus mainly on cognitive aspects of the learning environment by working with pairs of children. However, the research is concerned with the process where the role of the transformations is explored in a mathematics classroom environment. The focus is not only on the development of children's geometrical thinking about solids but also on the wider environment, such as the teacher's actions, the cooperation between teacher and researcher, the adaptation of the materials in the mathematics curriculum, the whole interaction between the teacher and the children and between the children themselves, which frames this development.

The process of the study

To explore children's constructions and how these are developed, we have used a "constructivist teaching experiment" where the researcher acts as a teacher who models children's constructions. The researcher, reflecting on his/her interactions with students, tries to explore children's actions and construct models of their mathematical understanding which throughout the teaching sessions are tested and revised. Based on current interpretations of children's actions, the researcher - teacher makes decisions concerning situations to create, critical questions to ask, and the types of learning to encourage. (Cobb & Steffe, 1983)

In this experiment we worked with two pairs of 10 and 12 year-old children. Each pair worked in three 45-minute sessions per week for 4 weeks. The sessions were videotaped and transcribed. The two pairs were selected from the forth-grade and the sixth-grade classroom of the same school according to their responses to a given task. Children were given three different solids and asked to write a description of each solid which would be given to one of their classmates. The description should be as accurate as possible, so that their classmate could identify the solids. The three solids were: a prism, a solid consisting of a cube and a pyramid and a cylindrical can . From the analysis of their responses, we identified two groups in each classroom according to the way they referred to the properties of each solid. Each pair consisted of one person from each group. We also took into account the gender (one boy and one girl for each pair) as well as the teachers' opinion about the personal relationships between the children.

Description of the materials

These are models of solids which were developed having some of their elements varied dynamically. In this experiment three models are mainly used by the children. The first dynamic model is a cube formed only by its 12 edges whose length could vary. The children could also vary its angles. In these transformations the degree of variation is constrained by the fact that the lengths can double in size while the angles can vary by up to thirty degrees. The second model is a cuboid where the length of its edges remain the same while its height varies from an initial length to zero when the solid "becomes" plane. The third one is a rectangular or a cylindrical transparent plastic solid filled two thirds with salt. The children can study the various forms that the salt takes. All these transformations which can be performed on these models are characterized mainly by the variation of the form of the solid and not only by the variation of its position.

The tasks used were based on our interpretations of children's actions. They varied from the free exploration of the models to more goal directed tasks where children were asked to transform an initial solid into a different one and study this transformation, considering, for example, the properties that have been changed. However the tasks have evolved throughout the sessions. Children were encouraged to face extreme cases, to anticipate and to generalize. In our analysis below, we exemplify more clearly the role of the tasks in the whole learning environment.

Emerging issues

From the analysis of the transcribed videotaped teaching sessions we exemplify below some points that indicate children's thinking in three dimensional geometry and its development.

<u>Building relationships between the solid and its properties</u> The children were asked to transform a given solid into a different one and compare the two solids to identify similarities and differences. This task encouraged the use of different comparison strategies which fall into three categories:

- comparisons based on the form of the solids
- comparisons based on the variations of the initial solid caused directly by the children
- comparisons based on the subsequent changes caused through the transformation

Throughout the teaching experiment all these three categories emerged and there was a development in children's strategies from the first category to the third. In the first category, the children considered the solids as whole entities. For example, in the case of transforming a cube into a cuboid by changing one dimension one pair of children (12 years old) initially saw the cuboid " like a cube but a little longer". Later on they started to consider them as separate solids with different properties " It (the cuboid) has these equal and those equal (the opposite faces) but it is not like the cube which has all equal". By moving from one form of the solid (the cuboid) to the original one (the cube) and vice versa, they start to become conscious of the underlying changes in the dimensions of the solids "its length has changed while its width has remained the same (of the cube)". This is an example of the second category where the children consider the change in the properties which was caused through their own actions in this dynamic situation. The problem becomes more complex when the children are asked to make another cube, different from the original one. In this case all the three dimensions should be changed properly. The children initially have not developed a particular strategy but they try different cuboids by changing the dimensions and finally approximate the cube while they start to realize that for a cube " we need to have equal sides".

In the third category, the children form more abstract relationships as the changes in the properties are not visible, for example the changes in the surface area or in the volume. Even in cases where these changes can be seen, for example in the case of the angles, they have not been caused directly by the children but are the result of the transformation. Through the researcher's encouragement the children started to consider these types of changes. We give an example below of how the children conceived the change in the volume when they made an oblique cuboid from a cube, keeping the same sides. Initially the children think that the volume becomes bigger "it becomes slanted and takes up more space". Through the discussion we realised that the children meant that the new solid does not fit easily into a cubic frame. Later on when they were asked to justify their opinion, they could not see any point in doing so as they thought that it was obvious. When the researcher asked them to find a way to calculate the volume of the two solids, they started to recall the "formal" school method of multiplying the dimensions in the case of the cube. They extended this method to the case of the oblique solid. Then they started to believe that the two solids had the same volume as they had the same edges. The whole problem changed to a manipulation of numbers, the lengths of the dimensions, while their intuitive conceptions of volume disappeared and the problem was transformed into the comparison of the two solids in terms of their surface area. The integration of intuitive and school knowledge took place when the researcher intervened and asked the

children to imagine what would have happened to the solid if they continuously made the cube more slanted. This led the children to see the effect of the height both on the surface area and on the volume, and doubt their previous belief. As a result they explained by using the variation of the height, that the volume of the solid will become smaller.

The development of these three comparison strategies is associated with the evolution of the task during the experiment (see figure 1). Students were asked to transform an initial solid into a different one and study this transformation from three different perspectives. These perspectives are characterized by different manipulations of the transformation. One was the transformation into a different solid (the result of children's free experimentation with the materials) where the varying properties had to be identified. The second was the transformation into a specific solid (e.g. the transformation of a cube into a different cube) where the properties of the two solids had to be taken into account in order to get a satisfactory result . The third was the transformation into a different solid through a specific variation of a property (e.g. the solids to have different volumes) of the original solid. In the last two perspectives the children are forced to take into account the properties of the solids and build relationships. So, we see the appearance of the most advanced comparison strategies (the last two categories). On the other hand, the first perspective encouraged the use of more holistic comparison strategies, although this was not exclusive.

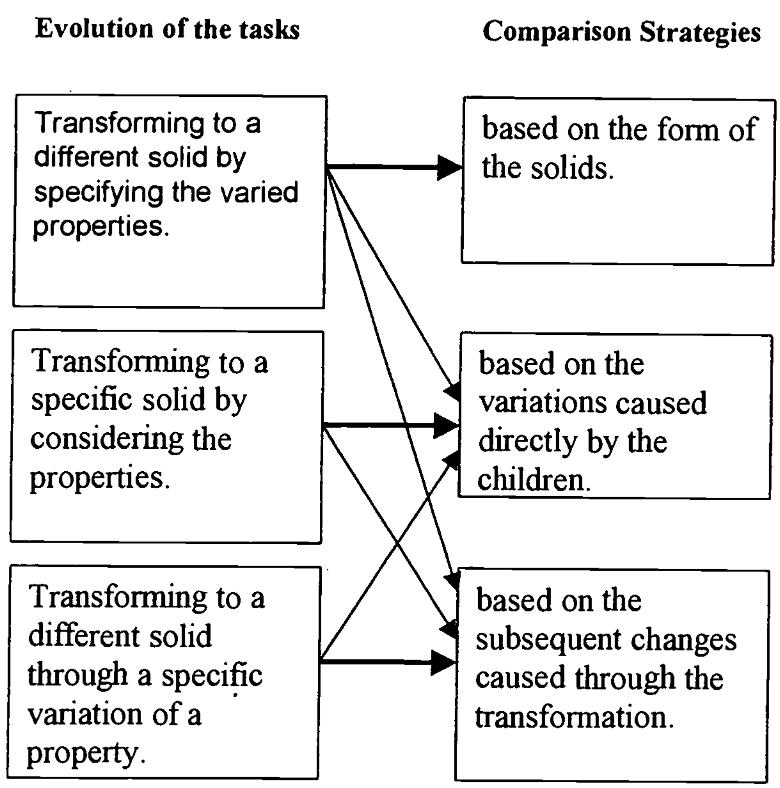

Figure 1. Associating the evolution of the tasks with the children's comparison strategies

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Forming relationships between the solids The dynamic transformations of the solids allowed the children to experience through their own actions, different forms of the same type of solid and possibly overcome the prototype phenomenon (Hershkowitz, 1989). Even the younger pair of children (10 years old) could relate the cube and the cuboid and developed arguments to express and justify their relationships. For example, these children, in their attempt to make a cube different from the original, make a cuboid by making one dimension of the cube bigger. They recognize it as a cuboid and they explain that this is different from the cube as "it does not have all the edges equal".

Concrete and abstract Although the tasks involved the physical manipulations of solids, in the whole process we see the "concrete" and the "abstract" to be interwoven. By this, we mean that the children, through the evolution of the tasks, could go beyond a specific physical action to a generalization. This development was not linear as the specific and the general continuously interact mainly through the children's attempts to justify their opinions. We see below an example of such an interaction. The pair of twelve year-old children had made different solids by moving the salt in the cylindrical object and they came to the conclusion that all these solids (made of salt) had the same volume. The problem is extended to an imaginative situation when the children have to decide what would happen to the solid if the salt were in a bigger cylindrical object. In this problem the children actually consider a mathematical problem which is to examine what happens in a cylinder when its volume remains the same but its dimensions change. The boy in this pair used his dynamic images which have possibly been created through his involvement in the experiment to face this problem: "if the perimeter (the circumference of the circle) was the same, then it would occupy exactly the same space but if it was much bigger (the circumference) it would become very tiny (showing with his fingers the height of that cylinder)".Here, the children come to a rather general mathematical conclusion by referring to the specific context of the initial concrete situation. The children were often challenged to make generalizations through considering "extreme" cases.

Intuitive and school knowledge By "intuitive" knowledge, we refer to what Fischbein and Grossman (1997) define, to global, direct estimation based on some initial information, on some mental operation. By "school" knowledge, we refer to what the children had been taught in their mathematics lessons. The dynamic features of the materials and the type of interaction which was developed encouraged the children to anticipate, estimate and form hypotheses. These hypotheses were mainly justified either from the variations observed or from those imagined and were of an intuitive nature. The oldest pair of children tended to recall their school mathematical knowledge in their justifications. On the contrary, the youngest pair of children remained on the intuitive level as their school knowledge in this area was very limited. The mathematical knowledge was used for recognition of the objects, for the formulation of definitions and rules and for computation. In some cases it supported the whole activity and it obtained a meaning for the children and it was developed

further. For example, the oldest pair of children used the process taught in school, the transformation of a parallelogram into a rectangle by cutting and pasting the right-angled triangle formed by its height and the one side, to make an analogy in three dimensional geometry. They tried to compare, through the same model, the volume of a cube with an oblique cuboid which had edges equal to the cube's by cutting and pasting a prism. On the other hand, there were some incidents where the children's persistance in using mathematics prevented them from seeing other alternatives, even in cases where they did not have any clue as to how to face a situation.

Concluding remarks

Goldenberg (1995) supports the view that the use of dynamic geometry suggests new styles of reasoning. He also poses a number of questions open for research concerning the role of this geometry in mathematics education. In this paper, we attempted to approach some of these questions and show the possibilities of the dynamic transformations of three dimensional solids in helping children to form relationships. The issues that emerged indicate that it is possible to create an environment where the children build and extend mathematical investigations by interacting with the teacher and the developed materials. In this environment, through the continuous deformation of a geometrical solid, the children seem to realize the role of the properties in the solid's form and are led to make generalizations. They develop the ability to predict and anticipate the result of their physical actions. Moreover, this environment encouraged the children to think intuitively, in some cases overcoming the "lack" of mathematical school knowledge and in other cases developing an alternative meaning for this knowledge.

It is expected that the complete results from this project will extend the features and the possibilities of such an environment by taking into account the complexity of the mathematics classroom.

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Concept Maps & Schematic Diagrams as Devices for Documenting the Growth of Mathematical Knowledge

Mercedes McGowen

William Rainey Harper College
Palatine, Illinois 60067-7398, USA
e-mail: mmcgowen@harper.cc.il.us

David Tall

Mathematics Education Research Centre
University of Warwick, UK
e-mail: david.tall@warwick.ac.uk

The major focus of this study is to trace the cognitive development of students throughout a mathematics course and to seek the qualitative differences between those of different levels of achievement. The aspect of the project described here concerns the use of ***concept maps*** *constructed by the students at intervals during the course. From these maps,* ***schematic diagrams*** *were constructed which strip the concept maps of detail and show only how they are successively built by keeping some old elements, reorganising, and introducing new elements. The more successful student added new elements to old in a structure that gradually increased in complexity and richness. The less successful had little constructive growth, building new maps on each occasion.*

Introduction

A *concept map* is a diagram representing the conceptual structure of a subject discipline as a graph in which nodes represent concepts and connections represent cognitive links between them. The use of concept maps in teaching and research has been widely used in science education (Novak, 1985, 1990; Moreira, 1979; Cliburn, 1990; Lambiotte and Dansereau, 1991; Wolfe & Lopez, 1993) and in mathematics education (Skemp, 1987; Laturno, 1993; Park & Travers, 1996; Lanier, 1997).

This study focuses on how concept maps develop over time. Students taking a sixteen-week algebra course using the function concept as an organising principle were asked to build concept maps of FUNCTION on three occasions at five-week intervals. In addition to qualitative analysis of the successive concept maps, we use a simple pictorial technique to document the changes.

Given a sequence of *concept maps*, a *schematic diagram* for the second and successive maps is an outline diagram for each distinguishing:

- items from the previous concept map remaining in the same position,
- items moved somewhere else, or recalled from an earlier map,
- new items.

Ausubel *et al* (1978) placed central emphasis on building meaningful new knowledge on relevant anchoring concepts familiar to the student. Using the schematic diagrams we investigate whether fundamental concepts persist in the development of successful students' concept maps and what happens to the less successful. This will be triangulated with other techniques of data collection and analysis. Given the extensive literature on the difference between those building a powerful conceptual structure and those remaining with inflexible procedures, we expect to find these differences reflected in the concept maps and schematic diagrams.

Concept maps and cognitive collages

The question as to whether a concept map actually *represents* the inner workings of the individual mind has long vexed the mathematics education community. Here we are not so much concerned with this issue as to how the individual *chooses* to represent his or her knowledge. It involves a wide range of technical, cognitive and aesthetic issues. Davis (1984, p. 54) used the term *cognitive collage* to describe the notion of an individual's conceptual framework in a given context. As one of us was for many years a professional artist and the other a practising musician, we warm to the rich inner meaning of the term "collage". For a child it may simply consist of a collection of pictures cut out of magazines and stuck on a piece of card, but for the artist it has a theme or inner sense that binds together distinct elements in a meaningful way. So it is with concept maps drawn by students. Some are seemingly arbitrary collections of items, others use all kinds of artistic and other devices to hold the ideas together. Figure 1, for instance, shows the first concept map of student MC drawn in the fourth week of term. The original is in colour, with colour coded inputs red and outputs representing links that are lost in a greyscale reproduction. By triangulating the development of these maps with students' written work and interview data we will explore how they provide a means of documenting cognitive growth over time.

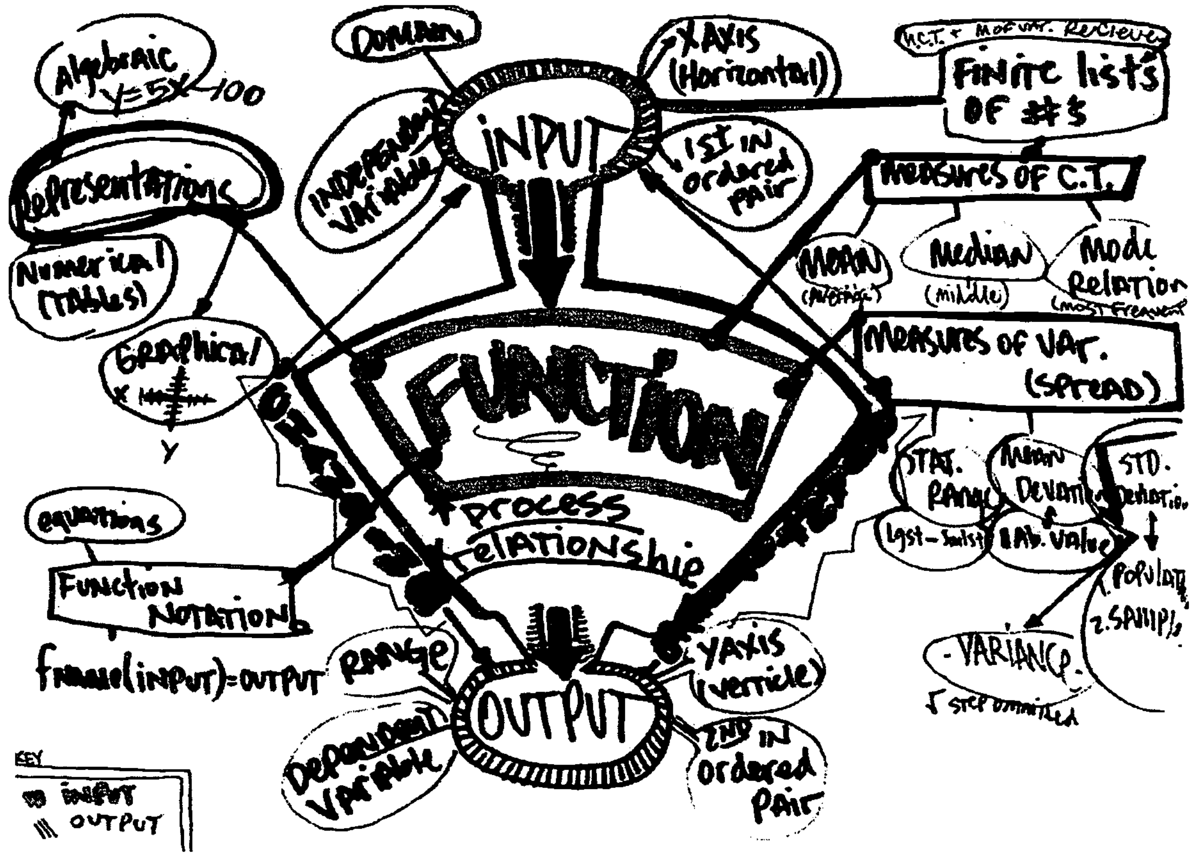

Figure 1: MC's first concept map after four weeks

Methodology

The subjects of the full study were twenty-six students enrolled at a suburban community college in a developmental Intermediate Algebra course. The curriculum used a process-oriented functional approach based on linear, quadratic and exponential functions supported by graphing calculators. Data was collected throughout the semester on every student including concept maps requested in weeks 4, 9 and 15 of the sixteen-week semester. Students were advised to use "post-it" notes to allow them to move items around before fixing the map. The maps were collected a week later, reviewed with each individual student to gain further information on the intended meaning, and then retained by the teacher so that at each stage the student was invited to draw a concept map anew.

Results of pre-and post-test questionnaires—together with results of the open-response final exam and departmental multiple-choice final exam—were used to rank the students. Two subgroups were selected, the four "most successful" and four "least successful" in the rankings, for more detailed profiling using follow-up interviews.

The concept maps of the eight selected students were analysed to document the processes by which they construct, organize, and reconstruct their knowledge. Schematic diagrams were constructed for the sequence of concept maps produced by each of them. The full analysis of the concept maps and schematic diagrams (McGowen, 1998) was triangulated with other data (Bannister et al, 1996 p. 147). Here we focus on two students, MC (in the most successful group) and SK (in the least successful).

Visual representations of students' cognitive collages

The second concept map of MC (figure 2) should be compared with the first (figure 1). Although the overall shapes change a little, the second is an expansion of the first. Some topics not studied in the interim (e.g. measures of central tendency and variability) remain unchanged, some are extended (representations, equations), and new items (finite differences) added.

The final concept map, created during week 15, was drawn on a very large piece of poster-board, too large to reproduce here. The topics included on the three maps followed the sequence of organization of the course and the connections shown are successively based on the main ideas of earlier maps. In his final interview, MC commented:

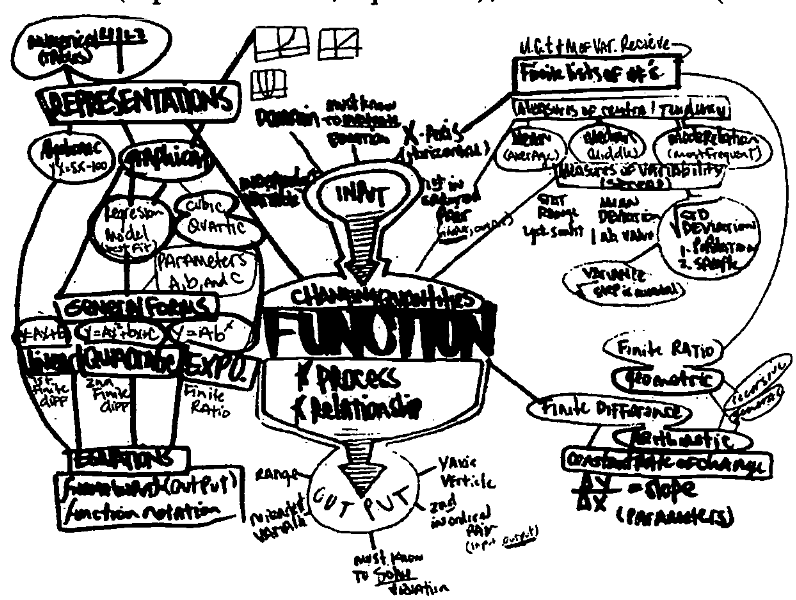

Figure 2: MC's second concept map, week 9

> While creating my [final] concept map on function, I was making strong connections in the area of representations. Specifically between algebraic models and the graphs they produce. I noticed how both can be used to determine the parameters, such as slope and the y-intercept. I also found a clear connection between the points on a graph and how they can be substituted into a general form to find a specific equation. Using the calculator to find an equation which best fits the graph is helpful in visualizing the connection between the two representations. I think it's interesting how we learned to find finite differences and finite ratios early on and then expanded on that knowledge to understand how to find appropriate algebraic models.

This final map is a rich collage, focusing on concepts and links between them, for instance, between graphic and algebraic representations, relating finding zeros in the first to factorising in the second.

The maps of SK provide a sharp contrast (figures 2, 3, 4). Week 4 includes definitions (in speech balloons). Week 9 is a bare skeleton with little in common with the earlier map. In week 15 the three basic function types (linear, quadratic, and exponential) become linked not to the central function concept but to *parameters.* The final map reveals procedural undertones by concentrating on routines (find slope, find constant common ratio, simplify, solve, evaluate, etc.).

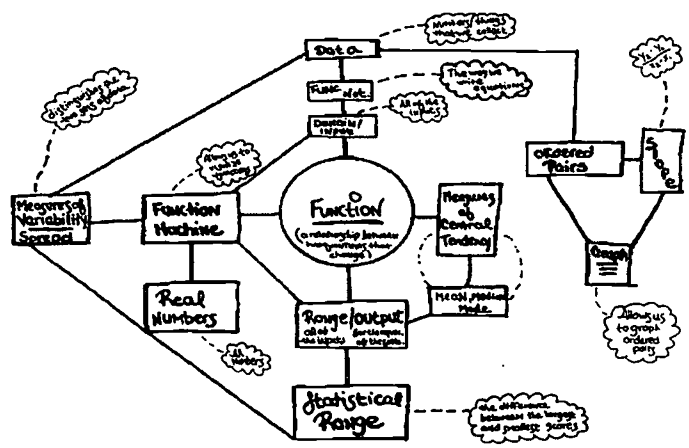

Figure 3: SK's concept map,week 4

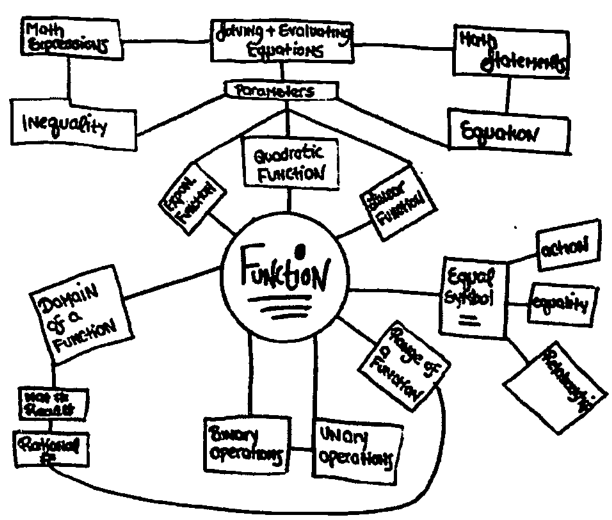

Figure 4: SK's concept map,week 9

Triangulating these concept maps with other data confirms that SK's knowledge is compartmentalized. She seems to have assembled some bits and pieces of knowledge appropriately, but others are missing, preventing her from building a cognitive collage with meaningful connections. When confronted with situations in which she is unclear what to do, she defaults to using remembered routines. She usually focuses initially on the numerical values stated in a problem. When confronted with a task for which she has no appropriate schema, she can only retrieve a previously learned routine. Her concept image of linear and exponential function on her week 15 concept map, for instance, is

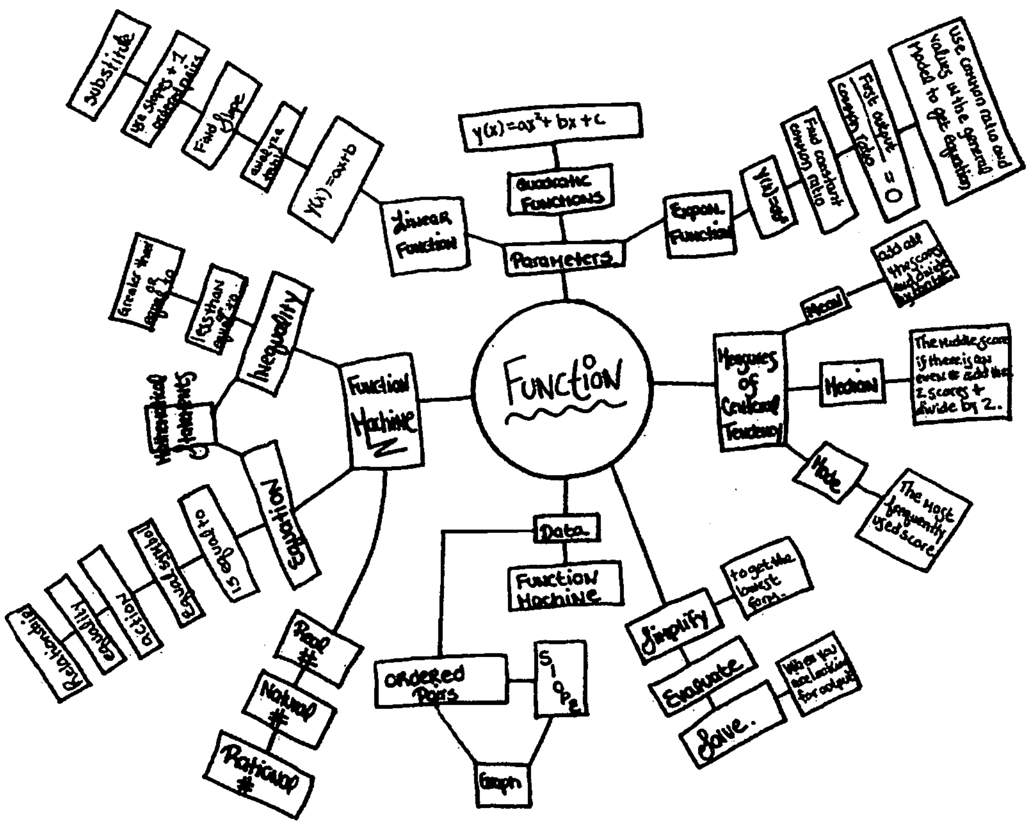

Figure 5: SK's concept map, week 15

limited to the computational procedures used to determine the parameters. Neither her classification schemes, nor her concept maps, reveal any interiority to these concepts or links to other concepts, to graphical representations or to alternative strategies for finding parameters. She demonstrated no ability to reverse a direct process in any context at any time in the semester. On at least two occasions, she retrieved and used two different approaches without realising her responses were inconsistent. She readily admits she is unable to distinguish between a linear, quadratic, and exponential function, even after sixteen weeks of investigation of these three function types. Confidence in the correctness of her answers decreased over the semester, and her attitude became increasingly negative.

The more successful student MC began with considerable lack of confidence in his algebraic skills. Despite this, he was able to select an appropriate alternative strategy when necessary, using the list, graphing, and table features of a graphing calculator. His ability to translate among representations is documented. His work suggests that he has formed mental connections linking the notions of zeros of the function, x-intercepts, general quadratic form and the specific algebraic model appropriate to the problem situation. He relates new knowledge to his previously acquired knowledge, building on the cognitive collage he has already constructed. The interview data indicates that he was able to deal with both direct and reverse processes, and recognizes them as two distinct

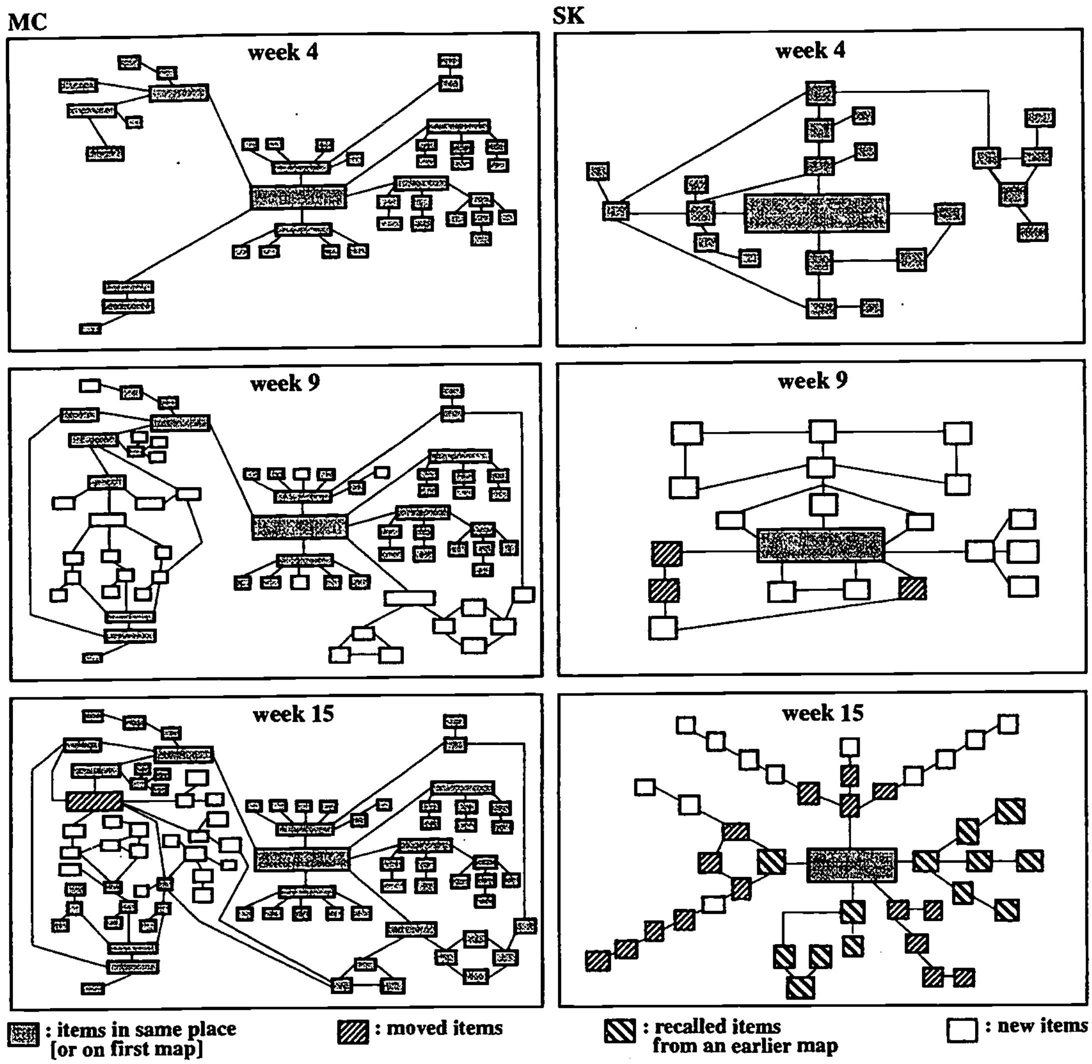

Figure 6: First concept map and successive schematic diagrams for MC and SK

but related processes. He was able to translate flexibly and consistently among various representational forms (tables, graphs, traditional symbolic forms, and functional forms). Confidence in the correctness of his answers increased over the course of the sixteen weeks. An examination of his work suggests that he initially focuses on the mathematical expression as an entity, then parses it as necessary, exhibiting the flexibility of process and concept necessary for more sophisticated study.

The use of schematic diagrams reveals these radically different developments (figure 6). It is immediately apparent that the basic structure of MC's first concept map is retained and extended in week 9 then further extended in week 15. However, the concept maps of SK seem to start almost anew each time, with few similarities and almost no basic structure that remains intact. Whilst MC builds from solid anchoring concepts, developing a strongly linked cognitive collage, SK builds on sand and each time the weak structure collapses only to be replaced by an equally transient structure.

Comparison with other students

Analysis of the other selected students reveals striking similarities among the schematic diagrams for each group. *Each student in the more successful group produced a sequence of concept diagrams whose schematic diagrams retained the basic structure of the first within a growing cognitive collage.* Each set of schematic diagrams for the least successful also exhibited a common characteristic: *a new structure replaced the previous structure in each subsequent map, with few, if any elements of the previous map retained in the new structure. No basic structure was retained throughout.*

Triangulating this information with other data reveals that the basic classification schemes used by both groups of extremes confirm the concept map and schematic diagram analyses. The more successful have processes of constructing, organizing, and restructuring knowledge that facilitate the building of increasingly complex cognitive structures with rich interiority. Their basic anchoring structures, are retained and remain relatively stable, providing a foundation on which to construct cognitive collages whose concept maps are enhanced by imaginative use of layout, colours, and shape.

The concept maps and schematic diagrams of the least successful reveal the fragmentary and sparse nature of their conceptual structures. No category appears on all three maps of any individual student, nor even was there a single category common to all four of these students. As new knowledge was acquired, new cognitive structures and new categories were formed, with few, if any, previous elements retained. Even those that were retained were reclassified and used in new categories with a different classification scheme.

Reflections

This study reports a wide divergence in the quality of thinking processes developed by remedial algebra students using graphing technology. High achievers can show a level of flexible thinking building rich cognitive collages on anchoring concepts that develop in sophistication and power. The lower achievers however reveal few stable concepts with cognitive collages that have few stable elements and leave the student with increasingly desperate efforts to use learned routines in inflexible and often inappropriate ways.

There remains the question of whether we are looking at these students through suitable lenses. Recent research offers new insights into the roles of perception and categorization (Rosch 1976), Lakoff (1987) which resonate with modern neuro-psychological theories of how the brain functions (e.g. Edelman, 1992) and how the evolution of the brain supports certain kinds of brain structure more than others (Dehaene, 1997). Within such wider frameworks we must ask "What if students like SK are organizing their knowledge according to a classification scheme which is not currently recognized or understood?" There exists the possibility that some students have different ways of knowing—ways of perceiving, categorizing, constructing, organizing, and restructuring knowledge—which those of us engaged in the teaching and learning of mathematics are unfamiliar with and have failed to consider. When one considers the significant improvement of the most successful students, the conundrum remains of how

and why students like SK—who claim to want to connect new knowledge to old—appear unable to integrate new knowledge into existing structures.

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REINFORCING BELIEFS ON MODELLING: IN-SERVICE TEACHER EDUCATION

Maria do Carmo Domite Mendonça
University of São Paulo, São Paulo, Brasil

The aim of this paper is a comprehensive discussion on my role as a teacher educator in a course for mathematics in-service teachers, containing some analysis on aspects of three teachers' teaching and on problems of their students. The course was at State University of Campinas (UNICAMP) for experienced and qualified in-service middle and secondary school mathematics teachers. In this paper I draw upon my own notes as an adviser related to the discussions during the teachers' coursework, a classroom experience using an innovative method centered in "mathematics modelling".

Introduction

In recent years much research has been reported on the use of mathematical modelling for the enhancement of mathematical learning. The racionale for teaching mathematics through mathematical modelling has been described by Bassanezi (1987, 1994), Blum (1990), D'Ambrosio (1986), Skovsmose (1994), among others.

According to Bassanezi (1994), "the study of problems and real situations with the use of mathematics as its language for their comprehension, simplification, and solution, aiming at a possible revision or modification of the object under study, is part of a process that has been named mathematical modelling". In terms of teaching, the use of modelling leads to the learning of mathematics content involving a real problem solution.

Bassanezi has suggested that one important variable involved in the mathematical modelling process is a situation, as its starting point, that belongs to the "real world". Blum (1990) has also suggested for teaching through mathematical modelling "an open situation which belongs to the *real world* and for the solution of which some mathematics might be helpful". By "real world", both, Bassanezi and Blum, refer to the world outside mathematics - in Blum's words: "by real world I mean the 'rest of the world' outside mathematics, i. e. school or university subjects or disciplines different from mathematics, or everyday life and the world around us".

Indeed, by assumimg such ideas for teaching mathematics, the issue of concern here is whether and how, for three in-service teachers, the researcher could influence and help them to use mathematical modelling as a method for teaching mathematics. It was a goal to begin working with situations that belong to the *real world* in order to develop basic and general foundations to problem posing and teaching through mathematics modelling.

Actually, if this research group had to choose a psychological motive for leading students to mathematical modelling, it would be ideas from Vygotsky, used also by Bruner, "if one is arguing about social 'reality' like democracy or equity (...) the reality is not the thing, not in the head, but in the act of arguing and negotiating

about the meaning of such concepts" (Bruner, 1989, p.122).

In order to analyse these researchers' role as an educator and the teachers attitudes, Ponte's ideas were followed, being that, even though the teacher absorbs understanding and information from an external force, he or she, decides on how to execute decisions – his/her movement/transformation is from the inside to the outside. (Ponte, 1998)

This paper is, then, a qualitative analysis on the author's role as a teacher educator in coursework for these in-service teachers. It also analyses the author's own learning from this experience and some aspects of the teachers' pedagogical practice through mathematical modelling.

Characteristics of the course: format and content

The coursework which is presented here is part of a larger program, entitled "Science, Arts and Pedagogic Practice", with duration of 360 hours (4 semesters), divided in eight classes of 45 hours (four of general themes and four of specific subject themes). All the participants in this program are required to take classes in each one of the following areas: Artistic Education, Physics and Mathematics. In each one of the semesters, the participants had to attend two classes, one from the general theme, and another related to the specific area they were involved in. The whole program was offered in a period of two afternoons a week, over two years. This first year the teachers attended required courses, and during the second year, they participated in *coursework* that consisted of *discuss and practice* form of inovation in their classrooms. The conditions and the description of the program coursework was given to the student-teachers on the first day.

Thirty students participated, in groups of 10, with an undergraduate degree in one of the following subjects: mathematics, science (physics or biology) and arts. All the participants in this program were experienced teachers of students aged 11 to 18. One of the objectives of this course was to provide a link among the public teachers' pedagogical practice, the academic education and the social/politcal questions about them. (A brief summary of the content of the courses appears in table 1)

At the begining of the second year, the student-teachers were introduced to a discussion/study about research in mathematics education, and moreover, about action research. The seconde year of the course stressed the importance of using cultural and social sources and personal experiences in the teaching of mathematics on one hand, and the steps the teacher has to follow/value in a mathematics education through the mathematical modelling process, on the other hand.

The results that I present in this paper correspond to the description/analysis on the changes a particular group of three in-service teachers - Gilberto, Maciel and Solange - went through during the course, and my own learning from this work.

Table 1

Summary of the General Disciplines. 1st sem/96 - "Conceptions and Organization of the Pedagogic Work": global vision of the subjects linked to educational conceptions and their relationships with the organization of the pedagogic work. 2nd sem/96 - "History and Production of Knowledge": historical production of pedagogic practice knowledge.   1st sem/97 - "Culture and Education": different manifestations and cultural practices related with school and education. 2nd sem/97 - "Science and Society": relationships between science and society and the implications of the scientific production.   ***Summary of the Specific Disciplines (Mathematics Education):*** - 1st sem/96 - "Problems of Mathematics Teaching/Learning 1": reflection on the school mathematical knowledge in its multiple dimensions: cognitive, episthemological, cultural, historical, social, semantics and politics.   2nd sem/96 - "Problems of Mathematics Teaching/Learning 2": study and reflection on the pedagogic work in mathematics starting from problems identified by the teachers in their classroom.   1st and 2nd sem/97 -"Teaching/Research Projects in Mathematics Education 1and 2": **the coursework.**

Method

The selection of a qualitative method was quite natural since this work is a type of case study which includes an analysis of personal discussions with the three student-teachers, as well as, an analysis of teaching through mathematical modelling and the existing teaching/learning results.

First of all, before actually starting the classroom experience through mathematics modelling, the student-teachers and I discussed the following four papers: a) "Modelling as a Teaching-Learning Strategy" (Bassanezi: 1994); b) "Applications and Modelling in Mathematics Teaching-a Review of Arguments and Instructional Teaching" (Blum: 1990); c) "Investigation: Where to Now?" (Lerman:1989) and, d) "*Problematização: Um Caminho a ser percorrido em Educação Matemática*" (Mendonça :1993)[1]. It is important to stress that the discussion on Blum's paper led the student-teachers and I to decide that the analysis of their mathematics teaching through modelling should be made taking into account the "pro and con arguments". Blum presents, from several investigations and inquiries in many different countries, the obstacles and potential roles that arises in everyday teaching through mathematical modelling – the author's "pro and com arguments" were discussed from the student's point of view, from the teacher's point of view and from the point of view of instruction.

The teachers were asked to register some of their actions related to the innovation proposed for their pedagogical practice in the coursework. The objective of the registration/diary was mainly to begin some reflections about themselves as a modelling teacher, and at the same time, as a critic of his/her educational practice. We decided that their diaries might not be shown to me, but naturally it was a very important resource since it was, in part, from these notes that I obteained my data and the student-teachers wrote their monographes. (Seen is in appendix I)

[1] The papers by Bassanezi and Lerman were read in Portuguese translations.

Having read the articles, the teachers decided to work through *generative theme*, as a starting point of the modelling process. In other words, the teachers would invite their students to choose a situation/story/context from social reality – a theme – and then, would help them to observe and investigate the facts underlying this theme in order to pose/choose the problems that would be interpreted in a mathematical model. Surely, the choice would be oriented, never imposed, by the teacher, and naturally, it is important that the students were involved in that process and felt motivated by the topics and problems raised. Indeed, the teachers Gilberto, Maciel and Solange were prepared to develop, among their students, the following steps: a) electing the generative theme; b) motivating a problematization/discussion about the theme; c) asking the students a historical study about the theme; d) helping the students to pose one or more problems d) using mathematical language for the comprehension of the problem, which means to obtain/establish the mathematical model, and, e) critically interpreting the obtained solution within the considered reality.

In a general way, the teachers were led to link their conceptions of teaching to the basic purposes, among others:

- mathematical knowledge is a set of relationships built in the interaction of the active person with the others, by means of intellectual/physic/emotional actions, on facts and objects of the enviromnent and,
- the starting point of the school work would not be a mathematical content, but a problem - especially, a problem that emerges from social reality.

Teachers' ways in the classroom

Some details of Solange's modelling experience (Group of Senior High School students)

Generative themes: *fotografia* e *camisinha* (photography and condoms). A historical study of both subjects was made. Two main problems posed using the *photography theme were*: 1) "The relationship between the opening of the camera lens diaphragm and the clearness of the photo" and, 2) "The time the photo would be submerged in the fixator and the quality of its clearness". (Observation: Solange stimulated the first question much more because she antecipated that in the second case, they would not have a way to measure the clearness). Some of the mathematical models developed: setting up points in a Cartesian diagram/graph of the given data/calculations of roots/curve sketching/adjusting curves. Two main problems posed on *condoms theme*: 1) "What types of preservatives are much more used?"and, 2) "The relation of its use and people's ages". Several statistical contents were studied and many hypotheses were tested with the use of computational software.

Some results of Solange's experience - Solange pointed out that from the questions emerged, great part of the program was developed and there was an active participation of the student in the teaching-learning process. Almost all problematization emerged from the students. Solange observed that she never clearly proposed questions. She noted that her students were naturally grouped according to a common interest - one group, for instance, studied the history about the chosen

theme, while the other group went to the research field to look for pieces of information which were, in general, collected through interviews or bibliography references. She said[2]: "my role was to make the process dynamic and to find a way of leading the students to solve their own problems, I mean to model the problem-situation. I actually worked, most of the time as a monitor... I would say that my work was to help them to interpret the collected data in mathematical language, that is to make the written mathematics meaningful". Solange concluded as follows: "The class was successful beyond my expectations... It nevertheless seemed that I, as a modelling teacher, merely scraped the surface of something much deeper... It is not that mathematical modelling can be, for instance, introduced in a High School classroom twice a week...the use of mathematical modelling, as a method of teaching is rather a philosophy of mathematics instruction, is related of higher order educational goals".

Some details of Gilberto's modelling experience (Group of 8th grade students)

- Generative theme: *futebol* (soccer). A historical study of *soccer was made*: the estudents studied the story/development of *soccer* with an expert, a doctoral student who has special knowledge about the theme. Problems posed by Gilberto leads to several mathematical models: area of different polygons, length of diagonal, trigonometric relationships, graph and formula of quadratic equation, among others.

Some results of Gilberto's experience - Gilberto pointed out that his greatest difficulty in adopting the modelling process in his classroom was to break the barrier posed by his 8th grade students, who possibly were influenced by the tradicional educational system, whose goal was to learn a sequence of prerequisits for the test they would have to attend at the end of the year. He said: "since my students showed a certain degree of resistence in posing a first question, I presented them a problem, a kind of story-problem related to the form and the measures of a playing field. I was afraid that the ready-made question could damage the process, but it didn't happen, they became extremely involved". His report reveals that this problem-situation, which was followed by some other ones, provided motivation to learn about the real situation - *soccer* - and about elementary geometry, measuring, linear and quadratic equations, elementary trigonometric relationships, proporcionality and graphs. As a final phase of this process, Gilberto tried to build a whole picture of the problem-situation, requiring a quadratic equation as a model, before getting to the analytical mathematical expression of this equation. Gilberto concluded as follows: "The theoretical readings, discussions and analysis were essencial elements in my development as a modelling teacher. My ideas changed... I would say that they were reinforced because I was already looking in this direction, but I didn't know how to apply them. I believe that the use of socio-cultural ideas of the students in a mathematics classroom is a very natural thing to do. However, for this result to be effective the undergraduate mathematical course must have courses according to modelling. It is like Blum says in his *pragmatic argument*: 'the ablility of students and teachers to master extra-mathematics situations does not result automatically

[2] All quotions from teachers come from interview data.

from learning pure mathematics, but can only be achieved by dealing in mathematics instruction, with such real situations'. I also would say that the work started in this course needs to be wide and deep. The mathematical courses for teachers do not discuss/teach how to integrate mathematics with social problems and other forms of knowledge. That was the first time I was invited to use activities that are close to students'socio-cultural context. I like my results in spite of thinking, there is still a big gap between modelling and my experience here".

Author's reflections

I would say that this paper has presented the argument that mathematical modelling using problems generating from "real situations" involved the students much more than traditional mathematical instruction, based only on subject content. In fact, these preliminary data suggested that in some aspects mathematical modelling achieved pleasing outcomes with respect to our student-teachers' learning/changes and my role in this process.

Indeed, little more than one semester classroom practice through mathematical modelling, was too short a time to evaluate these teachers. This realization came about because the three teachers commenced the work with a great deal of interest and motivation –they demonstrated a willingness to help their students to learn mathematics from within their reality. This instructive model allowed me to better see our learning and changes much more than was expected.

In order to follow the student-teachers involvement with this innovation, and to accomplish my analysis as their adviser, use of categories, such as: a) through my orientation, the degree of the teachers' perception growth in reference to mathematics modelling; b) the degree of possibility that the innovative method could actually be accomplished and, c) mathematical concepts and models worked with based on problems which emerges from real situations.

In the first few weeks of the coursework I assumed a radical position, following Freire's ideas that "it is only parting from questions that one must look for answers" (Freire, 1986, p.46). He asks, what would it mean if there are no questions, there will be no lessons, there will be nothing to teach. However, after a while I started wondering if the teachers were not *wasting their time* trying to use a problem posed by the group, given the fact that many mathematics topics they already knew how to teach (from a ready made way or formula) in an interesting and dynamic way which captured the attention of their students. But, it is interesting to note that once the teachers' interest increased, especially Solange and Gilberto's, when problem posing, from the facts of social reality became the heart of their mathematics instruction, my doubts quickly disappeared. After two weeks of classroom work through modelling, Solange spontaneously said "I can recognize that the pedagogical work stopped being that mechanical process that can take the learner just to learn an algorithm or a formula, but I developed, somehow, the attitude of maintaining a constant dialogue with my students, and among them, in order to motivate curiosity and problem posing, and... now I can face circunstancial questions from the part of

my students". Gilberto said: "I am finding better means of access to the didactical variables we need to study and the autonomous questioning from the students are not as poor as before.

During the coursework, I could note a great change in the teachers' ability to transform the problem between the real word and mathematics. This competence could be developed by means of suitable/successful processes, as discussed in "formative pro-argument", as quoted in Blum. According to him (and I could see this progress as well), the ability to translate from a practical/social context to the mathematical context, in which data would be presented by means of tables, graphs and equations, can only be advanced by means of experiences where the entire mathematical modelling process is covered.

It is also important to point out that the three teachers naturally noted/observed (what I was expecting!) that mathematical modelling, motivated by a problem that emerges from social reality, can guide the pedagogic work to a transdisciplinary cooperative teaching-learning process. They said something like: "the questions posed by the learners don't belong to a very defined area of studies and so, in order to find the means/ways for their solutions, we frequently transit into different areas".

The process of mathematical modelling with respect to intellectual/emotional/political students and teachers development, is a great challenge for both. That is, in order to teach the mathematics through modelling method, teacher and student should establish relations/connections with the world around them and the mathematical content, which demands a vast investigation out of the matter of mathematics. For instance, the *condoms* discussions made instruction *more open* and required classroom interaction unusual for the student-teachers and their students. Blum refers to this second aspect as one of the obstacles, from the teacher's point of view, to teach through modelling.

Based on these considerations, I intend to make some changes in the next experimental courses that I guide. First of all, I will attempt to make the register/practice diary more centralized, using much more time to examine the notes and to understand what set of variables could be the best for the teaching of mathematics through modelling. Second, the teachers should elaborate, as learners, alternative models from experimental data, before acting as a modelling teacher.

Finally, I would say that in spite of the process not being precise (a handicraft one) and the fact that this was the first experience teaching in this format, I found my own perspectives reinforced, which means that I reinforced patterns of beliefs about the use of social sources and personal experiences in making the learning/teaching of mathematics more effective and more meaningful.

Acknowledgment: The author would like to express her gratitude to Gilberto Chieus Jr., Maciel Gomes dos Santos and Solange Regina Pedroso for their helpful comments.

Note: The teacher Maciel was also involved in the course but he did a theoretical analysis of population growth phenomena and did not carried modelling classroom work)

Appendix I

Specialization Course: "Science, Art and Pedagogic Practice"

Task related to: the individual monograph / the classroom practice / pedagogic experience using a innovative method / teacher's development as a researcher / action research.

Dear student________

Please, from now on, I would like you to observe and register your difficulties and facilities in reference to: a) the teaching through modelling process in general; b) the students' actions and questions that show their involvement or not with the process of modelling and, c) the understanding, from the part of the students, of a mathematical fact. I ask you to make the registration in twice: 1) soon after each class, and 2) from your past classes/situations (either very interesting or very difficult).

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THE DEVELOPMENT OF ELEMENTARY SCHOOL CHILDREN'S IDEAS OF PRICES

Regina D. Möller
Institute of Mathematics
University of Koblenz/Landau, Germany

Abstract
The study reported in this paper investigates the competence of 6-10-year-old children with respect to putting prices on real-world goods. The methodological framework of the project is based on a qualitative and quantitative research approach and uses structured interviews with each student of the first four primary level classes. The results show competence for the "economic world" which could enhance the teaching and learning of applications in math classes.

Introduction
In his chapter "Towards a way of knowing" Bishop argues that a performance-orientated Curriculum with its emphasis on techniques, methods, skills, rules and algorithms is portraying mathematics as a "doing" subject and not as a reflective subject. In order for mathematics to educate the pupils he emphasises:

> *Surely what is needed now is more understanding and critical awareness of how, and when, to use these mathematical techniques, why they work, and how they are developed? This requires not only much greater thought, but also a different kind of thinking and therefore it requires a very different approach to the curriculum.*
>
> (Bishop 1995, 8)

Curriculum guidelines of many countries recognise the importance of applications of mathematics to everyday situations. Applications however are very often methodically used as a field of practice for previously introduced operations with different types of numbers and therefore occur at the end of a chapter unit. And although these applications of mathematics should manage to link the learning of mathematics with real world questions, children and elementary school teachers frequently seem to struggle with the applications.

This project was begun with the concern that little is known about elementary student's competence and strategies concerning real-world prices of goods. One of the first types of applied problems in elementary mathematics classrooms include price-value problems. Traditionally this type of problem (formerly called „regel de

tri“) belong to practical arithmetic serving the purpose to prepare for real life situations:

> *The solving of these problems was understood as a training for applying mathematics to real life and also for logical thinking.*
>
> Vollrath (1992, 229)

The didactical analysis of Kirsch (1969) clarified that the underlying mathematical structure of this type of problem is a functional relationship between two kinds of domains of magnitude (in German called Größenbereiche; Griesel 1973, 1997). Domains of magnitude are ordered semi-groups and some of them are isomorphic to (Q , +, <) which makes them a useful basis for elementary school mathematics concerning applications (Kirsch 1970, Steiner 1969). Kirsch argues that neither proportionality nor antiproportionality is a property of a real life situation and therefore this type of problem could not be used as an instrument for teaching logical thinking in math classes. However in the centre of real-world applications is the study of relations between magnitudes.

Kirsch (1969) suggests tables as a methodical tool to work with functional relationships. Nowadays already second grade pupils at primary level have to fill out tables in order to solve these price-value problems. A typical example looks like this:

Find out the prices for cold meat:

Gewicht	50 g	100 g	200 g	400 g	500 g	600 g	1 kg	1,5 kg
Preis		0,88 DM						

In solving these problems pupils use either an additive or a multiplicative approach. Both of these solving strategies are valid because they are properties of the underlying proportional relationship. Nevertheless the knowledge about this relationship is a subject matter of seventh grade mathematics classes.

In order to understand and solve this kind of problem children need competence in three ranges: Knowledge of the

- number system including decimal numbers (as prices)
- "money world", that is the money system (coins, bills)
- "economic world" (e.g. prices of goods, tasks of a bank)

While we are reasonably aware of students‘ abilities of the number system and of stages of knowledge of students concerning the money system (e.g. Burris 1983, Burti/Bombi 1981) we know little about the understanding of the "economic world" children have in industrialised countries. Because in contrast to the students Nunez et al. (1993) described in their study school children normally do not experience the "economic world" on a basis of everyday personally experienced buy-and-sell-situations. They have to rely more on observing experiences (e.g.

when parents buy groceries taking goods from long shelves in a supermarket) and of negotiations with other children (e.g. when they barter favourite toys). Piaget (1932/1973) therefore stresses the necessity of social mediation for economic situations.
Basic factors of prices of goods are production costs and properties like freshness with fruits and vegetables. Supply and demand are also important factors of trends of prices which can well be observed on markets. This kind of knowledge depends on processes of enculturation and of socialisation (Bishop 1991, p.37). Traditionally math classes inform with and through price-value problems about all subject matter linked with physical quantities like length, weight and time.

Basic ideas of functional relationships are the

- idea of one-to-one correspondence,
- idea of systematic change
- idea of observing a function as a mathematical object (Vollrath, 1989, p.8ff.)

These ideas emphasise the conception of a dependency between two (sets of) values together with methodical variations and an overall view of properties of a function. On primary school level the second idea is stressed because of the use of tables.

On the basis of the functional approach to price-value problems there are various problems:
Problem of fore-knowledge:
To solve the price-value problems the underlying assumptions of proportionality are neither openly assumed nor touched upon in class. Therefore the students rely with the methodical help of the offered tables on their intuitive strategies.
Problem of schematic solving:
Since it is not the explicit goal (in the German curriculum) to teach the proportional idea in early math classes, students rely more on a schematic solving procedure with these problems.
Problem of fixed assumptions:
Price-value problems with their representations of tables to use in order to find the solution give students the impression that prices of goods follow proportional rules as it is the case with the other physical quantities (Keitel 1979, 264). However prices of goods are not properties of goods like the length or the weight but assigned economic values by humans which can change according to peoples' necessities or opinions.
Problem of the concept of values:
Students bring to math classes a variety of value conceptions which is normally not touch upon explicitly when price-value problems are discussed. Although the system of coins and bills is normally discussed, the idea of linking their values to real life goods is generally not stressed upon. It is assumed that the solving of price-value problems give the pupils the appropriate ideas of concept of values. The task of reflecting upon economical facts linked to price-value problems are normally not undertaken in class.

Necessary knowledge to solve price-value problems belong to functional thinking which involve abilities like stating dependencies of domains of magnitude and/or making assumptions about the kind of dependency and/or finding the influence of changing one magnitude on the dependent other (Vollrath, 1992). These abilities and possible deficiencies become apparent in economical situations in which the student is asked assign prices to real-world goods.

The research questions underlying this study involve two aspects of research focussing on elementary students.

What kind of individual strategies and competence do students of the first four classes apply with respect to prices for real-world related goods?

How do they assign the prices? What reasons do students give for their prices?

Theoretical framework

Additionally to the theory of learning about functional relationships (Vollrath 1989) two further theoretical assumptions form the basis of our study: they are concerned with social constructivism as the underlying theory of learning and with problem solving as a part of mathematics (education).

1.Constructivism emphasises the individual's unique knowledge schemata (von Glasersfeld 1991; Davis et al. 1990) and also the role of interaction in the learning process (1995, 191) The inclusion of a social dimension (i.e. linguistic and cultural factors (the way one talks about money), teachers' conceptions (their opinion and experience about and with money) and their role in classes) in a constructivist theory of learning is referred to as social constructivism. The learning about the money system and its functions in economical situations depends highly on human interactions and influences the individual's learning (Bishop 1995, Seiler 1978).

2. For a long time problem solving has been regarded as an important part of mathematics. However the specific economic contexts have not been sufficiently emphasised with respect to open real-world related tasks in either (elementary) classroom practice or classroom research (Pehkonen 1991). Price-value problems belong to „Sachrechnen" (that is „real-world-related mathematical problem solving). Particular emphasis lies nowadays on real world contexts so that children's knowledge about real-life situations can aid them in their learning of mathematics.

Research Method

The structured interview approach (Bortz, Döring 1995) was chosen as the methodological framework of the study. Hence, the data collection and interpretation follow a strict procedure: each student of the elementary math classes was individually interviewed and had to respond to a series of questions.

The data collection and interpretation also followed a strict procedure: Each student out of the four classes of one elementary school was set aside from the usual math lesson and introduced to the specific set-up. All answers were taped

and transcribed. To be able to compare and recognise a development of qualifications in the answers of all students the first and second graders were joined together and the third and fourth graders. A Chi-square test for each of the questions produced significant results. The comparisons of the pupils' respective explanations gave rise to distinguish certain types of understanding of prices.

The Problems used in the Study:

Each student of all four classes (99 altogether) was given the idea to be an owner of a toy-shop in which he sold different kinds of wooden playthings of different shapes and colours (red, green, blue and yellow): cubes, quadrilaterals of different volumes, each two or three-times as big as one of the cubes and a flat quadrilateral, which had the volume of a cube, but was twice as long. In order to sell them the students were asked to assign prices to each of the objects. To make the investigation comparable, the starting point in assigning prices was given by the investigator: the price for the red cube is 1DM.

In the sequence of questions that followed the prices were asked of

1. each of the yellow, green and blue cubes
2. each of the small, big and flat quadrilaterals
3. two, three and five of the dices, of the small quadrilaterals and of the big quadrilaterals.
4. In the last question the student was informed that another shop sold the die for 1,50DM. Do you stick to your price?

The first question aimed at the relevance of colour stressed in psychological literature (Burti/Bombi 1981). The next question aimed at the ability to think proportionally with a number of pieces. The proportional aspect was also offered by the next question: What price do you give the quadrilaterals (the one being two and three times bigger than the die and the „flat" one)? Bags with three or four different kinds of these wooden pieces had to be taxed by the children looking for their abilities to proportionally appraise and allocate prices. The last question was used to assess the students' abilities to think about other peoples' motivation within economic situations.

Findings

The findings relate to:

(1) childrens' mathematical problem solving abilities,

(2) their learning of social conventions,

(3) their ability to apply the social knowledge to math class problems.

Owing to space restrictions this paper can only focus on some of the answers of the students. Examples of student answers as well as specific responses to the questions will be compared during the presentation.

With respect to childrens' problem solving ability the study suggests that primary school children have far more social knowledge of price-value problems than are asked for in math classes. They use much more "common sense" in assigning

prices and explaining their assessment than is normally asked for in usual textbook problems.

The results obtained indicate the following:

First and second graders did not clearly understand that the prices depend on the volume of the wooden toys. However the third and fourth graders understand this relationship well.

The ability to perceive a proportional relationship between the prices and the volumes of the various quadrilaterals grows with regard to age. However 50% of the pupils at the end of fourth grade still have difficulties with it.

The ability to perceive that the price of half a volume is half the price still cause difficulties for about 30% of the children at the end of fourth grade.

The understanding of the number system becomes obvious with the assigning of prices. The use of only natural numbers for the prices of the wooden toys was predominant with first graders.

Colour also plays a role when prices are put on to the toys. There are first and second graders who price the objects by their colours. For example, first grader's reasoning was: "because it is green and flat".

The power of perception increases with age: more pupils of second and third grade mention the specific form of the toys than the first graders.

The answer to the hypothetical question shows that some pupils have already a good understanding of prices. Some know that different shops offer the same goods for different prices. Others bear in mind the customer's motivation: "I keep to 1DM, then more people will come to my shop."

With respect to children's problem solving abilities the study suggests that pupils use far more criteria to solve real-world problems when the given problems are open and when "common sense" is necessary to their understanding. However, the levels of sophistication indicated by the pupils' explanations can vary quite drastically within one class.

As to the problem of fore-knowledge:

Although proportional relationships belong to the curriculum of seven graders' mathematics classes the pupils show economic competence in their assigning prices. They know about proportional price assignments with respect to a number of wooden toys and with respect to their volume. When growing older they are more capable of making the correct mathematical operations.

The problem of fixed assumptions:

Keitel (1979, S.264) was assuming pupils might think of prices as naturally given rules as within physical contexts. With respect to the flexible price assigning capabilities of the students with their knowledge of reductions and motivational incentives this observation looses its importance.

The problem of schematic solutions:

The solving of price-value problems might lead to schematic solutions on the part of the students. As to real-world situations like the one investigated in this study pupils use strategies and explanations which derive from their own experiences.

Problem of conception of values:

The students in this study use various categories like size, number of wooden toys to price their goods, but also as criteria for intended use, usefulness, moral and appreciation.

The problem of functional thinking:

The ability of proportionally assigning prices to real-world objects lying in front of them is present in elementary students. However the answers show significantly different results between first and second graders versus the third and fourth graders. The ability of proportionally assigning prices rises significantly with the second group, which has already been observed by Piaget. On the other hand the ability is already present in the first group. This result emphasises the existence of domains of subjective experiences (Bauersfeld 1983).

The ability of proportional assignment to a number of toys is present in third and fourth graders without any difficulties. They assign prices quickly without hesitation. First and second graders assign prices, in almost all cases the time with an additive and not a multiplicative strategy.

Generally the abilities of the proportional assignment of prices to volumina is less apparent than to numbers. The proportional assignment to different volumina seem to be difficult for third and fourth graders. In this study the students use the cube as a "measure" for the quadrilaterals. Either they "see" the cube into the quadrilateral and count how many times they could place the cube into the quadrilateral or they cut the quadrilateral into pieces of the size of the cube.

The consequences of this study suggest that the abilities and the competence of elementary students reach well into the "consumer world". The students carry into the classroom a variety of experiences and perceptions of real-world economic situations so that price-value problems can be estimated variously according to the students observations.

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RETEACHING FRACTIONS FOR UNDERSTANDING

Hanlie Murray, Alwyn Olivier and Therine de Beer
University of Stellenbosch, South Africa

This paper reports on the viability of a programme aimed at encouraging sixth grade students who have already been exposed to teaching practices leading to entrenched limiting constructions, to construct the concept of a fraction anew and to invent solution strategies for realistic problems involving fractions, in a school and classroom environment with serious practical and organisational problems.

Introduction

Much research has been done on the problems elementary school students experience with common fractions and on the design of teaching programmes for fractions at different grade levels (see Pitkethly and Hunting, 1996, for a review of the research).

An important issue is the effect that limiting constructions (D'Ambrosio & Mewborn, 1994) has on students' attempts to make sense of fractions. These include, for example, the influence of whole number schemes, which encourage the student to interpret the fraction symbol as two separate whole numbers, and limited part-whole contexts, where the student has had no or not sufficient experience of fractions as parts of collections of objects. Another issue is the possible adverse effect of rote procedures on students' attempts to construct meaningful algorithms for operations on fractions (Mack, 1990).

The above problems can be *prevented* by appropriate programmes for learning fractions in the lower elementary grades (e.g. Empson, 1995; Murray, Olivier & Human, 1996). However, when these limiting constructions are already firmly entrenched, it is to be expected that the task of encouraging students to develop strong and error-free conceptual and procedural knowledge about fractions will be much more difficult. Such attempts have already been made successfully (e.g. Bell, 1993; Kamii & Clark, 1995; Mack, 1990) in what we believe to be favourable learning environments.

In this paper we explore the possibility of implementing a programme for common fractions for Grade 6 students in less than favourable learning environments.

Theoretical framework

In line with our approach to the teaching and learning of whole number arithmetic (e.g. Murray, Olivier & Human, 1994, 1998), we believe that the teaching and learning of fractions should be based on eliciting and clarifying students' intuitions about fractions through posing realistic problems for which students have to invent their own solution strategies (cf. Empson, 1995; Kamii & Clark, 1995).

The following aspects are crucial in our approach to the teaching and learning of fractions:

Choice of problems. Knowledge about fractions involves knowledge about the concept of fractions, of which two subconstructs are the part-whole relationship between the fractional part and the unit, and the idea that the fractional part is that quantity which can be iterated a certain number of times to produce the unit. The unit may be a single object or a collection of objects. Fractions are also used in different ways and have different meanings, for example the part-whole mentioned above, but also a ratio, a quotient, a measure, etc.

If the problems posed in a teaching programme do not include, and students do not experience, these different subconstructs and meanings within a reasonable time, limiting constructions are formed (e.g. Murray, Olivier & Human, 1998). For the same reason, the fractions addressed should also immediately include thirds, fifths, etc., and not only halves and quarters, as is common in many teaching programmes. For example, in a previous study we found that Grade 1 students freely constructed appropriate different sized fractional parts in response to realistic problems, whereas many of the Grade 3 students in the same school who had only been exposed to halves and quarters during their teaching programmes for fractions, could not conceptualise thirds and/or could not recognise the difference between halves and thirds when they were trying to solve the same problems (Murray, Olivier & Human, 1996).

Social interaction. Social interaction creates opportunities for students to *talk* about their thinking, and this talk encourages reflection. "From the constructivist point of view, there can be no doubt that reflective ability is a major source of knowledge on all levels of mathematics ... To verbalise what one is doing ensures that one is examining it. And it is precisely during such examination of mental operating that insufficiencies, contradictions, or irrelevancies are likely to be spotted." Also, "... leading students to discuss *their* view of a problem and their own tentative approaches, raises their self-confidence and provides opportunities for them to reflect and to devise new and perhaps more viable conceptual strategies" (Von Glasersfeld, 1991, pp. xviii, xix).

We therefore believe that we should not only provide opportunities for students to build on their informal knowledge, but that students should be encouraged to make explicit and *become aware of* the nature of their own personal constructions (Bell, 1993).

Students' own representations. Students are expected to create their own representations of fractions. This is achieved by confronting students with sharing situations where a remainder also has to be shared out, for example sharing four chocolate bars equally among three friends (Murray, Olivier & Human, 1996). Prepartitioned materials are not used until later and the introduction of written symbols for fractions is delayed until the need for fractions and some conceptions of fractions have been developed by the students themselves.

This study

This study forms part of the Mathematics Learning and Teaching Initiative (MALATI) project aimed at curriculum and teacher development. The project teachers received student worksheets and teachers' guides which were studied during workshops, and were supported by regular classroom visits of project workers during the 1998 academic year. The student worksheets consist of two packs of activities, an introductory pack and a further pack.

The introductory activity pack of 33 worksheets is aimed at

- developing the fraction concept through sharing situations
- introducing realistic problem situations for operations involving fractions (e.g. division by a fraction)
- comparison of fractions
- equivalence of fractions
- introducing the fraction notation

The further pack attempts to make explicit students' informal procedures for the operations developed in the introductory pack.

Teachers were requested to firstly spread out the work over the year, and secondly to use the worksheets in the suggested sequence, with all the students in the class working on the same worksheet during a particular lesson period. The reason for this is that students are encouraged to solve a problem at whatever level they feel comfortable, and that students share their conceptualisations with the class to the benefit of all. Since our materials repeatedly pose problems with the same structures, it provides students with repeated opportunities to make sense of particular structures.

In this paper we limit ourselves to describing the effect of this intervention on a specific sixth grade class in one of the MALATI elementary project schools in a black township near Cape Town. The circumstances in the school and in the community do not support learning. Many students have transport problems to school and come from extremely unstable and impoverished homes. Absenteeism is not only high, but malnutrition and chronic health problems prevent some students from functioning optimally even when they are at school. The school organisation is poor; the lesson timetable is not followed and there are continual unscheduled interruptions in the form of staff meetings, celebrations and outings during school time. It is difficult to persuade teachers to attend workshops after school hours.

There were 42 students in the class. The teacher, Nolo, was eager to co-operate and although she started the year in a very traditional way (in spite of the initial workshop), she had after a few weeks established a culture of enquiry, argument and discussion among most of the students. However, even by the end of the year some students were unwilling to share their ideas through fear of being wrong.

By the end of the year, this class had only completed the introductory pack. This reflects their low level of knowledge at the beginning of the year as well as the above-mentioned practical and organisational problems during the year.

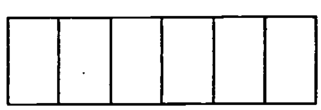
Results

We have available two sets of written tests and all the students' written work during the year as well as several videotaped classroom episodes.

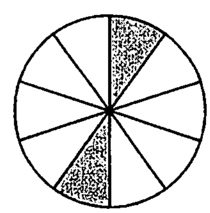
Test Set A comprises a pre-test completed in November 1997 (the end of the previous academic year) by all the sixth graders of the school under discussion, and a post-test completed in November 1998 by the sixth graders of Nolo's class. These are therefore not the same students, but students in the same grade in successive years in the same school.

Test Set B comprises a pre-test completed at the beginning of the 1998 academic year by Nolo's class and a post-test (a different one from Set A) completed by Nolo's class at the end of the academic year.

Test Set A. The items testing students' part-whole conceptions showed a very substantial gain in 1998. For example, these are the percentages of students correctly stating the fraction shaded in the following figures:



1997: 40%
1998: 90%



1997: 33%
1998: 89%

The following item testing the comparison of fractions also showed a substantial gain from 18% in 1997 to 31% in 1998:

Anwar and Amina each received R30 pocket money. Anwar spent $\frac{5}{8}$ of his pocket money and Amina spent $\frac{7}{10}$ of hers. Who spent more? Explain your answer.

Test Set B. The pretest at the beginning of the year revealed that students had very little knowledge of fractions. The post-test showed definite gains. For example, the success rate for the item "*Which is bigger, $\frac{3}{5}$ of a cake or $\frac{3}{4}$ of the cake?*" had a success rate of 4% in the pre-test, and a similar item had a success rate of 45% in the post-test. The success rate of the item "*$\frac{3}{5}$ of 10*" increased from 4% to 31%.

Both pre-tests identified the following main problem areas in students' understanding of fractions:

- a very weak understanding of the fraction concept
- strong interference from whole number schemes
- strong interference from rote algorithms

For example, this type of error occurred frequently as a response to *"Which is bigger, $\frac{3}{5}$ of a cake or $\frac{3}{4}$ of a cake?"*

$$\frac{3}{5} - \frac{3}{4} = \frac{0}{1}$$

Although both post-tests showed substantial gains, we felt the success rates to be low considering that students had solved similar and more difficult problems during their lesson periods.

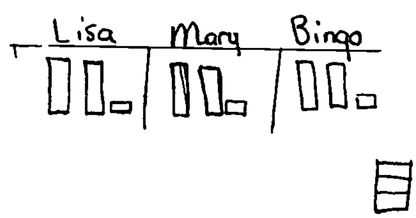
Students' written work. At first, there were strong signs of previous teaching. Mangaliso offered this incorrectly partitioned *apple* as solution to a problem involving *chocolate bars* (apples are frequently used to demonstrate fractions in many teaching programmes).

Because A Hole divided by three is a third

After a while, this type of response did not occur again in Nolo's class.

Most students produced their own representations of fractions in response to the initial sharing problems. For example, Thomboloxole solved the following problem as follows:

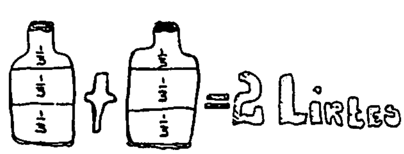
Lisa, Mary and Bingo have 7 bars of chocolate that they want to share equally among the three of them so that nothing is left. Help them to do it.



They also solved problems which prepared the way for operations with fractions successfully through their own representations of the physical situations. For example, Worksheet 9 poses an addition and a division-type problem. Zanele solved the two problems like this:

Peter and Anna prepare soft porridge for breakfast. For each bowl they use $\frac{1}{3}$ of a litre of milk.

- *If they make 6 bowls of porridge, how many litres of milk do they use?*



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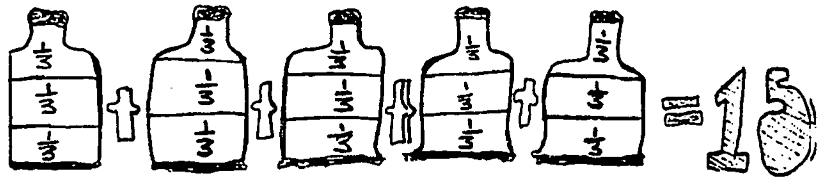
- *They have 5 litres of milk. How many bowls of porridge can they prepare?*

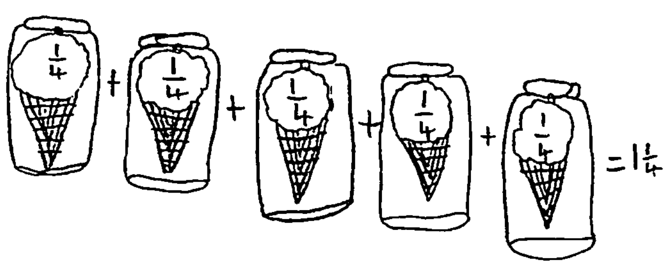


Although all the students worked on the same worksheet during a particular lesson, they functioned at different levels of abstraction. For example, Dumisani solved the previous problem like this:

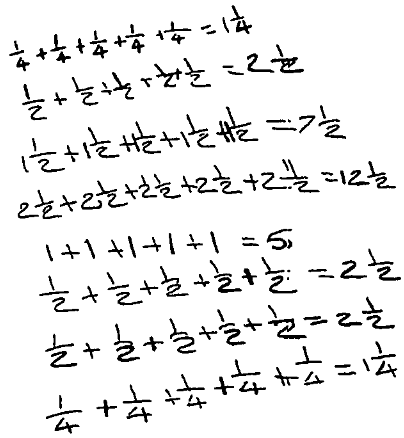
$$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+$$
$$\frac{1}{3}=15 \text{ bowls porridge}$$

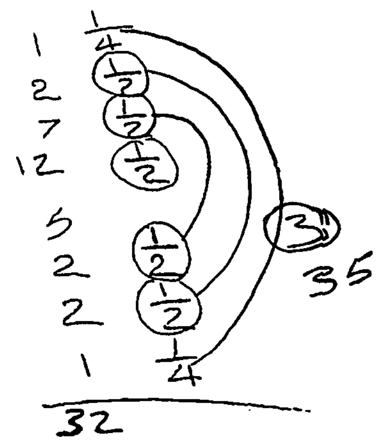
Students frequently used drawings which seemed inappropriate to us. After the initial realisation that drawings were acceptable as solution methods and as means of communication, some students came to believe that they *had* to make a drawing. For example, this drawing by Bonziwe as a solution to the following problem seems to be more decorative than functional:

How much ice-cream in total does Lisa have if there are 5 containers of ice-cream, each $\frac{1}{4}$ full?



Other classrooms in the same school produced a peculiar marriage of sense-making and rote learning. For example, Worksheet 8 lists the ingredients and their amounts in cups and teaspoons for one cake, and requires students to calculate how much of each ingredient would be needed for five cakes. Some students successfully calculated the quantities of each ingredient needed for five cakes, and then (correctly, but inappropriately) added up all the quantities in order to produce a *single* numerical answer. When challenged whether this was the *sensible* thing to do, the response was "no, but in mathematics we have to give only one answer to a sum". A solution such as this never appeared in Nolo's class (these are the different ingredients):





Discussion

The results show that in this class the intervention achieved decided success in addressing students' conception of fractions. Students' written worksheets show that inappropriate whole number schemes had disappeared, and that students were able to invent procedures to solve realistic problems involving fractions. It is also quite clear that in an approach like this, the traditional idea that division by a fraction is the most difficult problem type does not hold. Zanele's and Dumisani's solutions for the second problem of Worksheet 9 ($5 \div \frac{1}{3}$) illustrate this. We suspect that division may be the easiest of the four operations for which to invent informal operations as long as the context is sensible to the student (Mack, 1990).

On the negative side, many students could not manage realistic problems in the test situations although they had solved similar problems successfully as collaborative groups without the help of the teacher. It is possible that because many students missed worksheets through absenteeism, their understanding of a particular problem structure could not become stable.

We tried to help the teacher to identify students who were behind so that they could be given additional learning opportunities, but this proved to be organisationally unmanageable. We still believe that students solving problems collaboratively, combined with whole class discussions, is a better approach than individualised learning (compare Bell, 1993).

It is also possible that language caused a problem during the tests. English is the second language of these students, and in class the groups spent a significant amount of time talking about the problem situation before they started to solve it. Although the teacher explained the wordings of problems during the tests, this was probably not sufficient. It is possible that one or two written tests *in the course of the year* might have improved their performance in the final test. (We originally strongly rejected the idea of regular written tests because the then-existing school culture depended heavily on written evaluations.)

The question we posed at the beginning of the paper was whether it was possible to develop anew a stable conception of fraction in students who had already formed limiting constructions, using an approach which expects students to make sense of realistic problems and invent their own procedures in an atmosphere of discussion and argument, and whether this could be done under difficult conditions. This has proved to be possible.

An intervention in extremely adverse learning environments also has positive effects: An approach only reveals its essential aspects in difficult teaching and learning environments, whereas a competent teacher in a supportive environment unconsciously anticipates and copes with possible problems. As an example, we cite the case of the students who did indeed solve the problem where they had to calculate the quantities of ingredients needed for five cakes competently and sensibly, but then offered solutions which they considered to be mathematically correct but which made no sense at all. It has therefore become clear that encouraging students to build on their informal knowledge, and solve problems through their own inventions, without also changing their beliefs about the *nature* of mathematics, is not sufficient.

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Operations on "open phrases" and "open sentences" expressions-

Is it the same?!

Musicant Bracha
Beit Berl Teacher Training College, Israel

Abstract

It has been assumed, based on the daily practice of teachers, that students tend to apply to open phrases the properties and rules for open sentences (equations). The present research carried out on 9th and 10th grades confirmed that hypothesis. It was found that most of the students performed the transformations required for solving an equation. mechanically, without having in mind the corresponding formal justifications. The basic concept of equivalence with its properties seems to be totally absent.

Researches dealing with how students are performing operations on open phrases and open sentences apply differently, to models, which characterize students' mistakes in open phrases and in open sentences. The source of these mistakes is in abiding the rules in the wrong way, in generalizing of rules and errors in operating (Mutz, 1982). For example: researches dealing with open phrase indicate some cognitive obstacles in simplifying open phrases which are related to "incomplete nature" of algebraic expression, Tirosh, Even, Robinson (1994). That is to say, there is a tendency, among students, to simplify every open phrase until receiving a pattern of **ax** or **a**.

This research focused in two main questions:

1. Can the student identify permitted operations in open phrases or open sentences and what is the mathematical justification that a student gives to the operations he performed on those expressions?
2. Do students apply the permitted operations on "open sentences" to the operations on "open phrases" as well?

Methodology

Subjects:

The subjects were students enrolled in 9th and 10th grade (52 students in grade 9 and 53 in grade 10). In each grade, two levels of mathematical competence were

considered: group A, the high level and group B, the lower level. In all, the research population was consequently divided into four groups: 30 subjects in group 9A and 22 subjects in group 9B; 30 subjects in group 10A and 23 subjects in group 10B.

The levels of competence were established by the school itself and were expressed by the quantity and subject-matter of the mathematics taught to the respective students. The schools in which the research was conducted were situated in a region with a population of average and high socio-economic status.

Instruments:

a) A questionnaire was administered in which equations and open phrases appeared. The subjects had to estimate whether the operations performed (and indicated in the questionnaire) were mathematically correct. The central concept was that of equivalence. The subjects had to estimate whether the transformation was valid ones, that are whether the successive steps of successive transformation were correct and led to equivalent expression. After each question, the subjects were asked to justify their answers.

b) A number of subjects, different from those to which the questionnaire was administered, were interviewed with regard to the same questions.

Results:

A. <u>Operations on "open phrases" and "open sentences" expressions and the Justification of their performance.</u>

In performing operations on "open phrase" and "open sentence" expressions, we shift from one expression to another, repeating the process of replacing one expression by another, until we obtain an "open sentence" expression which is easy to solve, or a simpler "open phrase" expression which is easy to use. To the extent that all the shifts are reversible, we will obtain equivalent balanced "open sentence" expressions, and correspondingly, compatible "open phrase" expression. Therefore it is important for the student to be acquainted with the permitted operations.

<u>Operations in equation</u>

The reasons given by the subjects to justify their judgments in operations upon the "open sentences" expressions can be classified to the following categories: (Table 1)

a)Reasons relating to the operations maintaining equivalence, for example:

"Multiply both sides with the same number" or "you need to multiply both sides". This reason led to a correct judgment.

b) Reasons relating to the link between given numbers in the "open sentence" expression and the operations, for example:

1. In the equation $3x = 9$ a multiplying operation was performed both sides in the number 1/3. 14% of the subjects reasoned their judgement, regarding the number which multiplied the equation, when the number 3 written in the equation, "to multiply with a 1/3 is the same as dividing in 3". Therefore they came to the correct judgement. 10% from the subjects saw 1/3 as a "strange" number for the equation ("Where did the 1/3 come from?") and that's why the operations that were performed with this number were, in their opinion, incorrect.
2. In equations with fractions, the subjects reasoned: "eliminate the common denominator".

c)"Calculation" reason: some of the subjects ignored the performed operations and set their judgement only after they solved the given equations.

It can be explained by these reasons:

* The students do not have a deep understanding of the meaning of equivalence.
* The students tend to use ways they know well and feel secured with them.
* It is possible that the principle of equivalence is known to the students, but they feel a need to check every case. (Fischbein, 1982).

The wrong way in calculating led some of the subjects to an incorrect judgement, here they performed operations only in one side (especially true in equations with fractional multipliers), while calculating wrong or when there was no possibility in reaching a solution from the shape $x =$ number.

d) "Lack of information" reason, for example: "right way", "these is the way to solve". Such reasons (with no information) were given sometimes even when the judgement was correct.

Table 1: distribution of the judgements and reasons (in percentage) in the whole population (n=105)

examples of questions for multiple operation on equations	the judgments	The reasons				
		Referring to equivalent	link to numbers and operations	calculation	Lack of information	No explanation
$3x = 9$ $\Leftrightarrow$ $\frac{1}{3} \bullet 3x = \frac{1}{3} \bullet 9$ right or wrong, explain.....	correct 74 incorrect 16 no answer 10	19	24	21	16	20
$\frac{x}{2} + \frac{x}{3} = 10$ $\Leftrightarrow$ $3x + 2x = 60$ right or wrong, explain	correct 83 incorrect 13 no answer 4	18	44	9	2	27
$\frac{5}{3-x} + \frac{7}{3+x} = y$ $\Leftrightarrow$ $5(3+x) + 7(3-x) = y$ right or wrong, explain	correct 54 incorrect 34 no answer 12	35	33		4	28
$p(x) = \frac{2-x}{3} + \frac{2+x}{2} + 4$ $\Leftrightarrow$ $p(x) = 2(2-x) + 3(2+x) + 24$ right or wrong, explain	correct 39 incorrect 55 no answer 6	15	44	3	7	31

Additional findings were given in multiplying operations in equations with fractions:

a. The numbers of given addends with fractions in one side of the equation does not affect on the subjects' judgement and reasons. The explanation is in focusing the algorithm of eliminating the common denominator, performed by the subjects.

b. In "open sentence" expression that include fractions, the students tended to perform operations on one side only. As the given expression on one side became more "abstract", this "algorithmic behavior" grew stronger.

In the interviews where the subjects explained the incorrect judgement ("right") in the equation where the expression p(x) is written down: "it doesn't seem like an equation" or "it is a function therefore no operations are performed".

Operations in "open phrases" expressions

Table 2: distribution of the judgements and rationales (in percentage) in the whole population (n=105)

examples of questions for multiply operations on open phrases	the judgments	reasons					
		Referring to equivalence	Explanation regarding solving equations	links to numbers and operations	Lack of information	Turn to equation	No explanation
$3x + 9y = \frac{1}{3}(3x + 9y)$ right or wrong, explain........	correct 40 incorrect 49 no answer 11	11	26	8	17	2	36
$\frac{x}{2} + \frac{x}{3} = 3x + 2x$ right or wrong, explain	correct 29 incorrect 67 no answer 4	11	46		9	8	26
$\frac{5x}{3} + 1 = 5x + 3$ right or wrong, explain	correct 27 incorrect 64 no answer 9	8	36	5	16	4	31

This research indicates that most of the subjects assume that it is possible to perform a multiplying operation on a "open phrase" expression. The main reasons that the subjects gave, to explain their judgements in the multiplying operations performed on different "open phrases" expressions, can be classified in the following categories:

a. Reasons related to compatible "open phrase" expression. For example, in a multiplication operation performed on an "open phrase" expression, the student reasoned, "the expression changed after the operation was performed".

b. The reasons taken from the repertoire of explanations related to the solution algorithm of "open sentence" expression. For example: "eliminated the denominator", "multiplied it all in the required number", "multiplied diagonally". Such reasons led to incorrect judgement.

In interviews, made with some subjects, the following explanations were given to the interviewer's question: "What do you rely on when transferring from a given expression to the one you wrote after the simplification?"

* *"As I was taught, the goal is to eliminate the fraction, I work according to the rules".*
  One student explained: "The rules are to eliminate the fraction".
* *"According to the formula, according to the examples given in class, in any exercise*

with fractions one multiply the common denominator".

* *"I relied on the common denominator, I wanted to eliminate it". A student added: "This is how I was taught all through the years so it will be easier, more esthetic and more simplified".*
* *"I eliminated the denominator because I have already used it and I don't need it any more".*

c. Reasons related to the link between the number employed in performing the operation, and the given number in the "open phrase" expression. For example: "It is impossible to multiply a number that isn't in the expression".

d. Reason with lack of information - for example: "That's the way you solve it".

e. Transferring the "open phrase" expression to an equation.

B. Comparison between the performed operations on "open sentences" and "open phrases" expressions

Observe the following question:

"open sentence" expression

$$\frac{x}{2}+\frac{x}{3}=10$$
$$\Leftrightarrow$$
$$3x+2x=60$$

right or wrong, explain______

"open phrase" expression

$$\frac{x}{2}+\frac{x}{3}=3x+2x$$

right or wrong, explain_____

On this question 60% of the subjects gave an identical answer "right, right" to the operations they performed on the two expressions. 46% of the subjects gave identical reasons to their answers, the most noticeable one was "he eliminated the denominator". In order to understand the way of the subjects' thinking here are parts from some of the their interviews:

Interview a - a student from group 9A (T = teacher. S = student)

T: Given the expression: $\frac{x}{2}+\frac{x}{3}$ *can you simplify it?*

S: The student writes: $\frac{3x+2x}{6}=5x$

T: On what did you rely when you transferred from the first to the second expression?

S: One cannot simply add the two fractions. One has to find the common denominator.

T: On what did you rely when you eliminated the common denominator?

S: Because I have already used it and did not need it any more.

Interview b - a student from group 10A

T : Given the expression: $\frac{x}{2}+\frac{x}{3}=10$ *can you simplify it?*

S: The student writes: $\frac{x}{2}+\frac{x}{3}=10 \rightarrow 3x+2x=10$

T: On what did you rely when you operated the above transformation?

S: One has to find the common denominator.

T: Can you explain what you mean by common denominator?

S: Common denominator is something, which is common for a number of things.

T: On what did you rely when you looked for the common denominator?

S: One has to eliminate the common denominators.

T: Can you explain more?

S: No, it is the way to solve it and that is that.

From the comparison between the operations performed on "open phrases" and "open sentences", one can observe that most of the subjects perform identical operations in both types of expressions. The explanations, which follow the subjects' answers, are related to the algorithm's solution of the "open sentence". The identical elements in the "open phrases" and "open sentences" led to a hidden model of "a common solution" to both types of expressions.

Discussion

The "open phrase" and the "open sentence", as in many mathematical concepts have two aspects which complete one another: one is the formal-conceptual aspect which is represented by axioms, definitions, sentences and proofs, and the logarithm aspect which includes calculation knowledge and the understanding of the processes performed while calculating. The two aspects are combined in the mathematical operation. Skemp (1976) claims that learning the "rules not the reasons" enables the student to function in a very determined frame but without fully understanding the concepts, he wouldn't be able to deal with new assignments. This understanding enables the student to deal with any kind of information according to the changing variables.

The research indicates that only a small percentage of subjects use discretion that are related to operations which maintain equivalence. A great part of the subjects apply, in

their reasons, to specific numbers which were useful to them in order to perform the operation in the expression. Focusing on numerical hints causes them to perform some operations on "open phrase" permitted only in "open sentence" expressions.

As in "open phrase" (mainly with fractions) the subjects captured visual elements from the algorithm of solving "open sentence" expressions with fractions and implemented them in "open phrase" expressions. The justification to their operations came from the repertoire of explanations regarding the algorithm of solving open sentences.

Fischbein (1996) sees in the algorithm of equation solving a "Structural Scheme of Equation". In equations with fractions, students remember certain procedures of solving those equations. Open phrases with fractions are comprehended by the students as a structural scheme of equation that is why the student uses certain procedures of solving equations on "open phrase" expressions.

Summary

Teaching often focuses on the students' mastering procedures, and does not attempt to link formal knowledge with the procedures and situate these within a comprehensive framework. Such fragmented teaching may cause the student to perceive only the visual elements of the algorithm which will lead to faulty executions on "open sentence" and "open phrase" expressions. It is important to <u>create links between formal knowledge and the algorithm process</u>. in order to help the student understand the operations permitted on "open sentence" and "open phrase" expressions.

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USING SEMI-STRUCTURED INTERVIEWING TO TRIGGER UNIVERSITY MATHEMATICS TUTORS' REFLECTIONS ON THEIR TEACHING PRACTICES

ELENA NARDI, UNIVERSITY OF EAST ANGLIA, UK

Integrating the findings from a qualitative study of 20 first-year undergraduates' learning difficulties in their encounter with the abstractions of advanced mathematics within a tutorial-based pedagogy at Oxford (Nardi, 1996), a study of the tutors' responses to and interpretations of these difficulties was conducted. The study was intended as a bridging project between (Nardi, 1996) and a related ESRC-funded research project which started in October 1998 regarding current conceptualisations of teaching at university level as reflected in practice and issues arising from their relations to mathematics as a discipline. For the above purposes, samples of the data and the analysis from the initial study were presented to the tutors and discussed in semi-structured interviews. Here I demonstrate the tutors' reflective statements regarding their teaching practices as triggered by the interviewing process.

The study[1], on which the findings discussed in this paper originate from, is a small-scale follow-up of the author's doctorate (Nardi, 1996) and a precursor to the Undergraduate Mathematics Teaching Project (*UMTP*) currently being conducted at Oxford. In the following I briefly outline the doctorate (Project 1). Then I describe the aims, methodology and some of the findings of the bridging study (Project 2). Finally I outline briefly *UMTP* (Project 3).

PROJECT 1: A STUDY ON THE LEARNING OF MATHEMATICS AT UNIVERSITY LEVEL

The study[2] was a psychological study of first-year undergraduates' learning difficulties. For this purpose twenty first-year mathematics undergraduates were observed in their weekly tutorials for two terms. Tutorials were tape-recorded and fieldnotes kept during observation. The students were also interviewed at the end of each term of observation. The recordings of the observed tutorials and the interviews were transcribed and submitted to an analytical process of filtering out episodes that illuminated the novices' cognition. An analytical framework consisting of cognitive and sociocultural theories on learning was applied on sets of episodes within the mathematical areas of Foundational Analysis, Calculus, Topology, Linear Algebra and Group Theory. This topical analysis was followed by a cross-topical synthesis of themes that were found to characterise the novices' cognition. The findings were arranged in themes relating to the novices' difficulties regarding their image construction of new concepts as well as their adoption of formal mathematical practices.

[1] Research supported by the Harold Hyam Wingate Foundation in the UK.
[2] Research supported by the Economic and Social Research Council (ESRC) in the UK.

PROJECT 2: A STUDY ON UNIVERSITY MATHEMATICS TEACHERS' PERCEPTIONS OF THEIR FIRST-YEAR STUDENTS' LEARNING DIFFICULTIES

Project 2 is a follow-up study to the doctorate in which the tutors were invited to reflect and comment upon samples of data and analysis from the doctorate. In the following I describe this study as a bridge between the strictly psychological concerns of Project 1 and the directly pedagogical concerns of Project 3.

Aims. The primary aims were: to provide feedback to the tutors who participated in the doctorate and enrich its findings by including the participant tutors' point of view; to introduce a pedagogical dimension in the psychological discourse developed in the doctorate; and to inaugurate the collaboration between mathematicians and mathematics educators involved in *UMTP* in the development of discourse and methodology.

Methodology of Data Collection. For the above purposes, three tutors who participated in the doctorate were invited to participate in a series of semi-structured interviews. This choice resides theoretically in the methodological considerations, in particular regarding the interviewing of the students, in (Nardi, 1996) and in the literature regarding the teachers' reflections on their own pedagogical practices (see section on Project 3). Prior to the interviews the tutors were presented with samples of the data, transcribed extracts from the tutorials, and the analysis, presented in the doctorate. The samples were deliberately chosen to trigger tutors' reflection upon the students' learning processes, their own teaching actions as well as their response to the analysis in (Nardi 1996). The interviewees were informed of this agenda (see Fig.1) in a note covering the samples of data and analysis that were to be discussed.

An agenda for the Interview:

A Introduction: clarifications regarding the sample.
B Issues regarding learning:
- B1. a critical perspective on the sample of analysis
- B2. the interviewee's perspective on the sample of data

C Issues regarding teaching:
- C1. a critical perspective on the sample of analysis
- C2. the interviewee's perspective on the sample of data

Please read keeping in mind the interview agenda. The discussion can zoom in and out of the particular pieces of data and can address issues generally regarding your perception of your role as a tutor.

Fig. 1

Methodology of Data Analysis. The analysis of the interviews (Nardi, 1998 and Nardi, in preparation[3]) aimed at juxtaposing the analysis in the doctorate and the tutors' interpretations, as expressed in the interviews; and, moreover, at inaugurating reflection upon the tutors' teaching practices which is a fundamental aspect of the partnership currently being set-up in Project 3. The recordings of the interviews were transcribed and the contents of the transcripts were catalogued.

Subsequently the analytical perspectives on the data varied considerably. Some chronological considerations proved significant: when analysis was due to start, the provision of funding for Project 3 was confirmed by the ESRC. This altered the nature of Project 2 from being a self-contained follow-up to Project 1 to, additionally, being a prelude to Project 3. Three analytical perspectives were applied on the data. I will elaborate on the third one[4] by providing samples of the data and relevant analysis.

Analytical Perspective 3: a strong incorporation of methodological considerations. Did the interviews trigger the participants' reflection on their students' mathematical thinking and on their own teaching practices? If yes, how? If not, why? The answer to this question is affirmative[5]. A small set of categories emerged from the scrutiny of the transcripts:

R – S: The tutor describes standard practices, or standard difficulties observed in the students, or standard difficulties in the teaching.

R – D – CEP: The tutor defends their practice in the episode by challenging the critique in the sample on epistemological or pedagogical grounds.

R – D – U: The tutor defends their practice in the episode against the critique in the sample by undermining the representation of the events in the sample.

R – R: The tutor re-evaluates, or even regrets, their practice in the sample either by simply agreeing with the critique in the sample or by engaging in reflection on the students' learning processes.

In the following I present samples relating to the R-R category for each one of the three interviewees. In view of Project 3, in which the tutors are interviewed on a weekly basis, this category is particularly significant as the tutors' explicit self-evaluative reflections may have an immediate impact on their imminent teaching.

[3] By the time this research report is presented at the conference and if the paper has been accepted, the full reference will be available.

[4] Analytical Perspective 1: this emerged from the need for a transition from looking at episodes from the tutorials from a learning to a teaching point of view, and to immerse in the relevant literature. So I used the data as an empirical basis in which to embed my reading of standard texts on teachers' thinking - for instance, on the Teaching Triad (Jaworski, 1994).
Analytical Perspective 2: a domain oriented perspective. For this, I have isolated all domain-specific extracts and cited them along with findings in the literature and the thesis - for instance, on the difficulties of the students' enculturation into the necessity of proof as well as into specific proving techniques (a relevant presentation at the MAA conference in San Antonio, Texas was done by the third author).

[5] A number of methodological considerations were compiled and have influenced heavily the methodology on Project 3. These will be discussed elsewhere.

TUTOR 1: SELF-EVALUATIVE REFLECTIONS

Tutor 1's stronger self-evaluative statements were far lengthier and more elaborate about epistemological (regarding the logical coherence of his mathematical discourse to the students) than psychological (regarding his individual students' state of learning) or pedagogical (regarding his teaching decisions and actions) matters. For example he is spending a substantial amount of time during the interview on re-considering his appproach to proving what the derivative of x^n, for $n>0$ is[6]. In his concluding remarks however there is also a 'didactical reason' for his re-newed preference:

> Tutor 1: ... well, now my belief is that I would have preferred it for a didactical reason and justify my evaluation that it was a better proof on historical grounds partly. That it gets us... it is evident from the fact that we get closer back to the simple reasoning that it's the original proof, em ...So obviously I have got the wrong ... I have not presented things terribly well there.

Consistency is highly valued by Tutor 1, and, as he has always expressed a strong preference for 'more direct, simpler and for that reason more satisfactory' arguments, he possibly felt that in this case his consistency was partly let down. When confronted with a series of instances where his recommendations to the students about the use of pictures in order to acquire an intuitive understanding of an argument or a concept but not substitute for a rigorous proof, are rather lacking in consistency, he replies 'I try to be reasonably consistent about that. I may get it wrong from time to time...' and provides a lengthy and precise manifesto of what he believes the use of visualisation in the construction of mathematical meanings ought to be.

In the same vein of adhering to firmly established principles that guide his behaviour in the tutorial, the tutor is 'slightly embarrassed' that he has commented on a Calculus question that the students had great difficulty with, as an exceptionally hard one[7]:

> Tutor 1: [...] I believe that to be true but usually I try not to publish to the students evaluations of that kind, that this was a hard question or that wasn't. [...] I normally try not to do that.

Despite the fact that Tutor 1 was offered a set of concrete learning episodes to be discussed in the interview, he mostly preferred to use the episodes as a basis for exposition on the general principles that guide his tutorial teaching (his approach is in contrast with the approaches of Tutors 2 and 3). Among the three interviewees he is the one for whom stronger prompting was employed in order to explore his views on subtler psychological issues regarding his students' learning. Even so, when pressed to focus on specific learning incidents, he demonstrated a belief in the individuality of his students and in the need to adjust his teaching to this individuality:

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[6] In the tutorial he had expressed a preference for a proof based on the Binomial Theorem as a 'historically original' one. In the interview he says he should have based his preference on that it is a proof on First Principles.

[7] He grounded his comment on that 'it is harder to prove that something does not have a limit, than to prove that it does have a limit'.

Tutor 1: [...] Well, it's the picture that I've got in my mind and the picture I use for teaching but of course it's no damn good if you've got a student with a different type of psychology from mine. One has to be aware of that. [...] this particular em, metaphor, this particular picture, adaptation of the notion of a Venn diagram[8], em, needs to be abandoned or rethought through with this particular student, doesn't it? Quite probably abandoned.

In the following sections Tutors 2 and 3 engage in psychological and pedagogical discourse immediately and on their own initiative totally.

TUTOR 2: SELF-EVALUATIVE REFLECTIONS

In contrast to Tutor 1, Tutor 2 preferred immersing into the specific learning episodes she was provided with for reading before the interview — even though she would often conclude with a reference to her standard practices. Therefore a large part of the conversation was devoted to her reaching a new understanding of her students' utterances which at the time of the tutorial might have seemed to her as bizarre reactions. For instance, when reminded of the Linear Algebra episode[9] where all eight of her students reply $Tp(1)=2$, her initial assessment was that the students lacked knowledge of the relevant definitions. When I pointed out to her that in two of the tutorials the students had just demonstrated knowledge of these definitions, she responds that 'they still could not apply [them]'. She elaborates her reassessment:

Tutor 2: Well, polynomials ought to be easy but functions they have problems with, thinking about a function as being an entity in its own right. Whether that came into it...

Interviewer: You mean they cannot see a function as an object, to use it as an element of a set as opposed to a process.

Tutor 2: Yes, that's right, it's always been a process: you apply it to an element and another element comes out.

Apart from an insightful observation into her students' thinking, the above are also in resonance with findings on the learning of functions — the process/object duality of their nature is a well-trodden area of research in Advanced Mathematical Thinking (e.g. Dubinsky and Harel, 1992).The significance of the statements lies in that they come from a practitioner who is unaware of this literature and who, in the event of the interview, gains this awareness. This exchange highlights a convergence of perspectives and interests between mathematicians and mathematics educators on which the emphasis is largely missing in the field. Numerous similar exchanges took place in these interviews and will be reported elsewhere (Nardi, in preparation).

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[8] He refers to a misunderstanding regarding the use of an illustration of cosets in Group Theory as equal-sized parts of a square.
[9] T is the linear mapping $Tp(x)=p(x+1)$, for p a polynomial of degree 3. Therefore 1 is the constant polynomial mapped via T on itself, not 2. A similar incident occurred with the zero element.

Towards the end of the interview Tutor 2 offered an unprompted evaluation of the interviewing process. In this she seems to have valued mostly the discussion of alternative interpretations of her students' utterances:

> Tutor 2: So therefore I think there is other possibilities I hadn't had in mind because on the whole I don't actually know how much tutors think about their role as a tutor and what actually one is doing as opposed to, you know, what end result you would like to get out of it all. A lot of it I suppose is really done by what has worked in the past with our students. [A similar statement was offered by Tutor 3].

TUTOR 3: SELF-EVALUATIVE REFLECTIONS

As with Tutor 1, Tutor 3's initial persistent focus seemed to be on how epistemologically comprehensive was his introduction to the notion of spans and spanning sets in Linear Algebra. In the course of the interview however it transpires that his persistence was on trying to understand what initiated the discussion of the concept on the part of the student. When he hears that the definition of span used in the lectures was different to the one he used in the tutorials and that nobody had hinted at the equivalence of the two definitions, he responds that 'it seems in retrospect to be a great mistake' and that 'It's quite likely that I never discovered that of course which is a mistake'. Tutor 3 seemed to perceive the interview as an occasion for directly evaluating his teaching. In fact he had prepared a list of cases – based on the samples of data he was provided for reading before the interview – where he 'could have handled things differently'. He distinguished between 'errors that I am making here and I think I know why I am making them' and 'other ones that are pretty bad mistakes and it's not quite clear why I was making them'.

Given the limitations of space, I offer in the following his first mentioning in the interview of what he perceived as an example of each of the above mentioned types of 'mistake'[10]. Both 'mistakes' were discussed to great detail, in fact large parts of the interview were devoted to a line-by-line scrutiny of the transcripts:

> Tutor 3: Well, for example, [...] it seems to me to be a major error to go on talking about the theory of cosets without going immediately or fairly immediately to some concrete example. And I am amazed I am making a mistake like that. The other general mistake that I noted throughout both episodes is that I am rather keen on pushing for what I regard as the right understanding of concepts, or the right way of proving something, instead of pursuing what the undergraduate is thinking about. And I think I know why I make that mistake. It's quite a lot to do with the rather fast pace of the Oxford course, that there is a lot of pressure to try and get the student keeping up with the pace and then of course in the long run it's far better if they have a thorough understanding of the concepts and then we can accelerate the pace and catch up later. But I think that's a general mistake that I perceive in retrospect.

In the detailed scrutiny of the transcripts mentioned above, in which again a mathematician's and a mathematics educator's perspectives were actively co-ordinated, Tutor 3 seems to maintain a balance between strong self-evaluative comments and challenges to the analysis in (Nardi, 1996). In most cases his challenges regard a 'slightly more positive' assessment of the student's state of learning at the time. Nevertheless he

[10] The first refers to an episode from Group Theory and the second to one from Linear Algebra.

always concludes with re-evaluations such as 'in retrospect maybe I should have taken time to explain the difference', 'maybe I should have spent more time trying to find out what she really meant there' or 'one should be more patient and follow up what the student is thinking'.

SYNTHESIS AND IMPLICATIONS FOR PRACTICE

Whether the tutors in these interviews engaged in an explication or re-evaluation of their standard tutoring practices, in reflection on their students' thinking or in challenging the analysis in (Nardi, 1996), the aims of the interview, as outlined earlier here, were certainly fulfilled. This fulfilment can be partly seen in the extracts of the interviews exemplified in the previous three sections. In sum – and I offer these also as an evaluation of the methods used in Project 2:

- The tutors engaged in an articulation, justification and often reassessment of their teaching actions in the discussed episodes, or even more generally. Occasional inconsistencies in their practices were also highlighted.
- The tutors engaged in a scrutiny of the evidence on their students' thinking, a task for which any time is rarely allowed. This often amounted to their gaining an awareness of existing research literature , for instance, in specific areas of learning difficulties in undergraduate mathematics.
- Certain analytical themes from (Nardi, 1996) were enriched by the tutor/practitioner's point of view, e.g., in the extracts illustrated here, regarding the debatable value of using certain concrete graphical representations to introduce abstract mathematical concepts or the need to enhance a co-ordination of intuitive and formal practices for the novice advanced mathematics learner.

Implications for practice. During the interviews the tutors expressed their concern for a potential limitation of the study which related to their difficulty with recalling the specific learning episodes they were invited to discuss. All of them claimed that had the interviews taken place closer to the incidents, their recollections would be fresher, hence more reliable. It was agreed however that what was actually discussed in the interviews were the transcripts of the recorded events and the analysis in (Nardi, 1996). It was also agreed that had they been interviewed immediately after the event, there would have been no analysis to discuss. This would alter the nature of the data – actually this immediacy is exactly what Project 3 aims to achieve. These more immediate interviews would also benefit from a more balanced input from both the practitioner's and the researcher's point of view. As the tutors themselves acknowledged the impact of this exchange, in particular when taking place on a regular basis, on the tutors' perception and enactment of their role can be significant. These considerations have been built into the formation of the aims and the methodology of Project 3. In the concluding section I outline Project 3 which is currently in progress.

PROJECT 3: THE UNDERGRADUATE MATHEMATICS TEACHING PROJECT

UMTP is a one-year study which commenced in October, 1998. Its broad aim is to explore, in a clinical partnership (Wagner, 1997) with university mathematics teachers, current thinking and practices in mathematics teaching at first-year undergraduate level and to begin to provide a knowledge base on which to make decisions affecting teaching. *UMTP* develops theoretically from educational research at pre-university levels: in particular, the growing literature into teachers' thinking processes and personal theories and their links to pedagogy (e.g. Brown and McIntyre, 1993) and research on the correlation between teachers' perceptions of mathematical epistemology and cognitive development and their classroom practices (e.g. Jaworski, 1994). *UMTP* extends this literature to address pedagogical issues in university mathematics teaching, drawing on studies of advanced mathematics in epistemological and cognitive domains (e.g. Tall, 1991; Sierpinska, 1994). Another root of the study resides in a close collaboration between mathematicians and mathematics educators: *UMTP* addresses and seeks reconciliation between the differing emphases of these practitioners. The set of qualitative data developed in the study consists of tutorial observations, semi-structured interviews with the tutors and participant group discussions. The UMTP project team have submitted a Research Report to PME23 in which they discuss findings from the project (Jaworski, Nardi and Hegedus, submitted[11]).

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[11] By the time this research report is presented at the conference and if their paper has been accepted, the full reference will be available.

ADDRESSING STUDENTS' CONCEPTIONS OF COMMON FRACTIONS

Karen Newstead and Alwyn Olivier
Mathematics Learning and Teaching Initiative, South Africa

Grade 6 and 7 South African upper elementary school students' conceptions of and operations with common fractions were investigated before and after a year of exposure to materials which had been designed to challenge common limiting constructions identified in a previous study, and to a teaching approach of which reflection and interaction are essential components. There was a significant improvement in several of the items in the written tests, in line with the aims and extent of the materials used.

Introduction

This paper reports on the impact of an approach for the teaching of common fractions on Grade 6 and 7 students' conceptions of fractions. The approach was intended to challenge common limiting constructions (D'Ambrosio and Mewborn, 1994) and included materials that were designed to specifically address students' problems with fractions which had been identified in a previous study (Newstead and Murray, 1998) in a larger sample of which the current sample is a subset.

The approach and materials were used during the first year of implementation of the Mathematics Learning and Teaching Initiative (Malati) teacher and curriculum development project in schools, as a vehicle for introducing elementary school teachers to our approach. This approach requires a classroom culture as described below which originates from our theoretical orientation as reported in previous PME papers (e.g. Murray, Olivier & Human, 1996). Such an approach is based on the view that students construct their own mathematical knowledge irrespective of how they are taught. Cobb, Yackel and Wood (1992) state: "... we contend that students must necessarily construct their mathematical ways of knowing in any instructional setting whatsoever, including that of direct instruction," and "The central issue is not whether students are constructing, but the *quality and nature* of these constructions" (p. 28, our italics).

Based on the previous study (Newstead and Murray, 1998) and on the existing literature, and in line with the Malati philosophy, the fractions materials were designed according to the following basic principles:

- Students are introduced to fractions using sharing situations in which the number of objects to be shared exceeds the number of 'friends' and leaves a remainder which can also be further shared (e.g. Empson, 1995; Murray, Human & Olivier, 1996).
- Students are exposed to a wide variety of fractions at an early stage (not only halves and quarters) and to a variety of meanings of fractions, not only the fraction as part-of-a-whole where the whole is single discrete object, but also for example the fraction as part of a collection of objects, the fraction as a ratio, and the fraction as an operator.

- Students are encouraged to create their own representations of fractions; pre-partitioned manipulatives and geometric shapes do not facilitate the development of the necessary reasoning skills and may lead to limiting constructions (Kamii and Clark, 1995).
- The introduction of fraction names and written symbols is delayed until students have a stable conception of fractions. Written, higher order symbolization is not the result of natural learning, and students struggle to construct meaning for such representations of fractions in the absence of instruction which builds on their own informal knowledge (Mack, 1995).
- Similarly, students can and should make sense of operations with fractions in a problem context before being expected to make sense of them out of context (Piel and Green, 1994).
- The materials repeatedly pose problems with similar structures to provide students repeated opportunities to make sense of particular structures. Fractions are taught continuously throughout the year, once or twice a week rather than in a concentrated 'block' of time.
- A supporting classroom culture is required in which learning takes place via problem solving, discussion and challenge and in which errors and misconceptions are identified and resolved through interaction and reflection. Teachers do not demonstrate solution strategies, but expect students to construct and share their own strategies and thus to gradually develop more powerful strategies.

The Malati elementary school material and approach was implemented in 4 traditionally disadvantaged schools near Cape Town in 1998. In addition to student worksheets and comprehensive teacher notes, the teachers attended workshops and reflection sessions and were visited regularly in their classrooms by Malati project workers.

Methodology

To study the impact of the Malati materials and approach, research was conducted on both teacher and student change in two of the four elementary schools. The impact on Grade 6 and 7 students' learning was investigated using both written individual tests and observation of students interacting in groups to solve challenging problems. This paper reports on changes in students' responses to the common fractions items in the written tests only.

The written tests were designed by Malati researchers to 'cover' both the curriculum traditionally taught in the schools, and the intended Malati curriculum. Thus items in the Grade 6 and 7 tests differed in some cases according to the existing curriculum aims. All the Grade 6 and 7 students completed the three tests on the same day towards the end of the academic year in November 1997. For one of these tests, calculators were not allowed. The tests were administered by Malati project workers who encouraged the students to try their best, but assured the students that the results were not for school 'marks'. Exactly the same procedure was followed with the Grade 6 and 7 students in the same two schools in November 1998. The tests were coded by Malati project workers according to a coding schedule devised after an initial analysis of the first dataset. Out of every class of approximately 30 to 50 students, the first five complete tests were coded by two Malati

project workers who then compared and discussed any discrepancies before continuing to code individually. The data was input and the analysis carried out using SPSS for Windows.

Results

The response categories for the fractions items were collapsed for the purpose of a chi-squared analysis. In some cases, certain responses were coded as showing some understanding of the problem, in which case a third 'semi-correct' category is indicated. In these cases, the chi-squared analysis was conducted using the categories 'no correct response', 'semi-correct response' and 'correct response'.

The following tables show the success rate on the common fractions items. The items are numbered for convenience and not according to test item numbers. All the values in the cells are percentages of the total number of students who were tested. The number of students in each grade differ in 1997 and 1998 owing to the non-longitudinal research design, e.g. Grade 6 students in 1998 (who have been exposed to the intervention for a year) are compared to Grade 6 students in 1997 (who had not been exposed to the intervention).

Item 1 tested students' part-of-a-whole conception of fractions:

1. What fraction of the following figures is shaded? If it is not possible to say, explain why not.

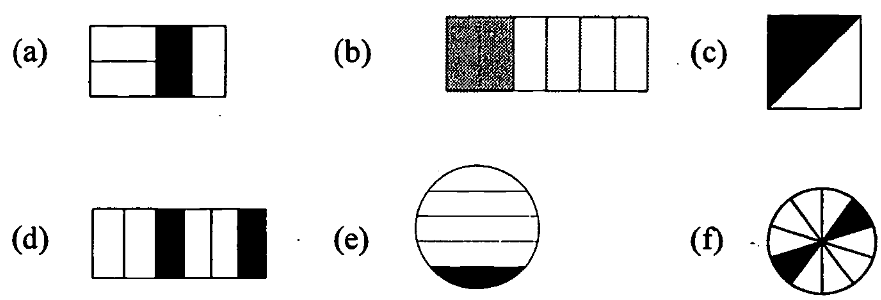


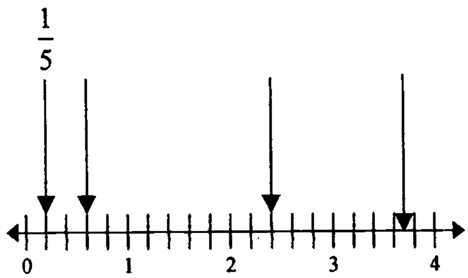
Table 1 shows the success rate on these items and significance of change in responses:

Item	Grade 6 1997 (N=208)	Grade 6 1998 (N=174)	Grade 6 Significance	Grade 7 1997 (N=191)	Grade 7 1998 (N=193)	Grade 7 Significance
1a	58	85	$p < 0,001$	71	78	$p > 0,05$
1b	52	83	$p < 0,001$	69	79	$p < 0,05$
1c	69	89	$p < 0,001$	87	90	$p > 0,05$
1d	53	79	$p < 0,001$	73	78	$p > 0,05$
1e	2	5	$p > 0,05$	4	12	$p < 0,01$
1f	48	77	$p < 0,001$	67	72	$p > 0,05$

Table 1: Success in identifying fractions using pre-partitioned shapes

Although rational number lines were not taught in the Malati curriculum, the following number line items were included to test students' conception of fractions as rational numbers:

2. *What numbers are shown by the arrows? The first one has been done for you.*



3. *Show the following numbers on the number line below. The number $\frac{3}{4}$ has been done for you.*

(a) $1\frac{1}{3}$ (b) $\frac{12}{6}$ (c) $\frac{2}{3}$

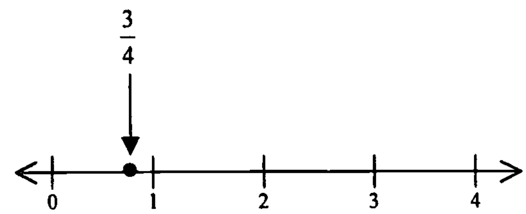


Table 2 below shows the success rate on these items and significance of change in responses. Blank cells in the table indicate that these particular items were not included in (in this case) the Grade 7 tests.

Item	Grade 6			Grade 7		
	1997	1998	Significance	1997	1998	Significance
2a	35	46	$p < 0{,}05$	33	47	$p < 0{,}01$
2b	16	32	$p < 0{,}001$	19	27	$p < 0{,}05$
2c	4	14	$p < 0{,}01$	14	17	$p > 0{,}05$
3a	17	35	$p < 0{,}001$			
3b	6	18	$p < 0{,}001$			
3c	9	17	$p < 0{,}05$			

Table 2: Success on items using rational number lines

Additional items that were included in the Grade 6 and 7 tests are shown in Table 3 and 4 respectively. For Item 5, a 'semi-correct' response indicates that students correctly chose Amina as having spent more, but did not supply a reason. For Item 6, a 'semi-correct' response indicates that students gave a response of 6 or 7 rather than the precise correct answer of $6\frac{1}{4}$. 'SC' indicates the frequency of semi-correct responses (where such a category was coded), while 'C' indicates the frequency of correct responses.

Item (Grade 6 test only)	1997		1998		Significance
	SC	C	SC	C	
4. Jackie spends $\frac{3}{4}$ of her pocket money and Piet spends $\frac{1}{2}$ of his pocket money. Could Piet have spent more money than Jackie? How?		3		2	p > 0,05
5. Anwar and Amina each received R30 pocket money. Anwar spent $\frac{5}{8}$ of his pocket money and Amina spent $\frac{7}{10}$ of hers. Who spent more? Explain your answer.	11	17	19	17	p > 0,05
6. Mrs Brown wants to cook porridge for 10 people. She normally uses 5 cups of oats for 8 people. How many cups of oats does she need for 10 people? Show your calculations.	16	<1	33	4	p < 0,001

Table 3: Success on comparison of fractions items and the use of fractions as a ratio

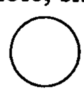
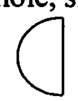
Item (Grade 7 test only)	1997	1998	Significance
7a. If the diagram below represents a whole, show by means of a suitable drawing how you would represent $\frac{7}{5}$. 	9	6	p > 0,05
7b. If the diagram below represents a whole, show by means of a suitable drawing how you would represent $\frac{1}{2}$. 	47	59	p < 0,01
8a. Four pizzas were bought: $\frac{1}{3}$ of the pizzas was eaten. Show this fraction by shading: 	13	24	p < 0,01
8b. How many pizzas were left?	7	17	p < 0,01
9. $\frac{1}{3}$ of a man's salary is R3200. What is his salary?	39	37	p > 0,05
10. After a party $\frac{1}{5}$ of a cake is left. The next day John eats $\frac{3}{4}$ of the leftover cake. What fraction of the cake is left then?	6	4	p > 0,05

Table 4: Success on items in context testing concepts of and operations with fractions

The following items tested operations with fractions out of context. Students were not permitted to use calculators for these items. For Items 11 and 12, a 'semi-correct' (SC) category was included for responses in which a suitable common denominator was used but the correct answer was not obtained. In the case of Item 17, a SC category was also included for responses of '30 min + 15 min' and '$\frac{1}{2}+\frac{1}{4}$'. Table 5 shows the frequencies in the various categories and significance of change in responses. In some cases, a chi-squared analysis could not be conducted as some of the cell frequencies were too low.

	Grade 6					Grade 7				
	1997		1998			1997		1998		
Item	SC	C	SC	C	Significance	SC	C	SC	C	Significance
11. $\frac{5}{8}+\frac{4}{5}$	1	<1	5	2	-	10	11	9	8	$p>0{,}05$
12. $3\frac{1}{4}+1\frac{4}{5}$	1	<1	6	2	-	8	11	2	8	$p>0{,}05$
13. $\frac{1}{6}\times 26$							5		9	$p>0{,}05$
14. $\frac{7}{8}\div\frac{1}{4}$		1		1	-		6		6	$p>0{,}05$
15. $\frac{3}{4}$ of $\frac{1}{5}$							19		9	$p<0{,}05$
16. $\frac{3}{4}$ of R120		11		16	$p>0{,}05$					
17. half of $1\frac{1}{2}$ hours	1	17	6	34	$p<0{,}001$					

Table 5: Success on context-free items testing operations with fractions

Discussion

Fraction as part-of-a-whole The results show a significant and substantial improvement in many of the items that reflect the students' conception of the fraction as part-of-a-whole, which is considered to be an essential foundation to the understanding of fractions. This improvement was most evident at the Grade 6 level in all Item 1 questions, except 1e (see Table 1). There was not a similar significant improvement in the Grade 7 responses, which could probably be ascribed to the relatively high success rate in 1997.

In Grade 7 there was a significant improvement in Item 1e which may indicate that some Grade 7 students are reflecting more successfully on the necessity of *equal* parts – In 1998, 78% of the Grade 7 students gave the answer as $\frac{1}{5}$, as opposed to only 44% in 1997. In Grade 6, the percentage of students who gave the answer as $\frac{1}{5}$ changed from 69% to 66%. However, the poor success rate in both grades on this item indicates that more emphasis needs to be given to the necessity of equal parts.

Fraction as rational number It is interesting to note the significant and substantial improvement on most of the questions of Items 2 and 3 that require students to make use of the rational number line. The Malati material does not make use of any such number lines, so this may be ascribed to an improvement in the ability to make sense of fractions as rational numbers. The exception to the significant improvement on these items is the Grade 7 response to Item 2c, although 'semi-correct' responses like $\frac{18\frac{1}{2}}{5}$ and $3\frac{3\frac{1}{2}}{5}$ were included as

correct responses. It is expected that students will respond to this item with greater success once they have been exposed to decimal fractions, which were not taught in 1998.

However, the lack of success on Items 4 and 5 indicates that the Grade 6 students were not able to use fractions as rational numbers within the problem solving context. According to Watson, Collis & Campbell's (1995) classification, this particular use of fractions within the problem solving context represents the most complex level of fraction items. Indeed, our data indicates that students need more experience with fractions as abstract rational numbers in a problem solving context.

Other meanings: Fraction as part of a collection, as operator and as ratio We had expected that the materials had addressed fractions as part of a collection of objects sufficiently. Indeed, there was a significant improvement in the responses to Item 8a. The disappointing success rate on Item 9 could be ascribed to the fact that the materials neglected to address fractions as operators sufficiently. The significant change in the responses to Item 6 (the fraction-as-a-ratio) was, on the other hand, unexpected as we had felt that this meaning of fractions was not sufficiently addressed in the materials. However, the significant change is in fact in the semi-correct responses and not in the correct responses, and could thus be ascribed to better reasoning and/or estimation, or simply to an increased willingness to attempt the problem (see below).

The role of the whole The lack of success on Item 4 may be attributed to the fact that comparing fractions of which the whole is not necessarily the same was not addressed at all in the existing material. The conception of the relationship between the fraction and the whole will need to be further addressed in 1999, as indicated by the lack of success on Item 7a. There was however increased success on Item 7b.

Operations with fractions The students did not show a significant improvement on the non-calculator items (Items 11 to 17) which involved context-free operations with fractions. This can be explained by the fact that thus far in the materials they have only been required to make sense of operations in the context of problems. Context-free operations with fractions are expected to be revisited and consolidated in further fraction materials. The Grade 6 students' significant improvement in Item 17 could be ascribed to the fact that this item is not really context-free although it was included in the non-calculator test.

It is of concern that the students did not achieve more success on Item 10 which concerns multiplication of fractions in context. However, there was a significant improvement in subtraction of fractions in context (Item 8b), which can be expected as students had much experience with such problems.

Willingness to try Significantly more Grade 6 students (at least $p<0,05$) attempted Items 1a, 1b, 1c, 1d, 4, 5, 6, 16 and 17 in 1998 than in 1997. Significantly more Grade 7 students (at least $p<0,05$) attempted Items 1b, 2b and 2c in 1998 than in 1997. The only item in which fewer Grade 6 students attempted a response in 1998 than in 1997 was Item 11, and

significantly fewer Grade 7 students attempted a response to Items 11 and 15 in 1998 than in 1997. Items 11 and 15 involved context-free operations with fractions, of which the learners had little experience.

Conclusion The items on which the students performed poorly in 1997, and on which there was no significant improvement during 1998, may indicate insufficiencies in our curriculum design principles and implementation. For example the role of the whole needs to be more thoroughly addressed. In spite of our attempts to address various meanings of fractions, more attention needs to be paid to some of these such as the fraction as operator. As intended according to our design principles, operations with fractions need to be covered out of context now. Another aspect of fractions which may not have been sufficiently addressed in our materials was the transition from unit fractions to related non-unitary fractions. This transition is not one which occurs naturally (Davis, Hunting & Pearn, 1993). This may account for the lack of success on several of the items.

However, in this study design principles to facilitate improved learning of fractions were developed based on research on students' understandings and limiting constructions. The resulting materials and classroom culture of reflection and discussion helped to facilitate a better basic understanding of the conception of fractions, and a greater willingness on the part of the student to try.

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THE EFFECTS OF A DIAGNOSTIC ASSESSMENT SYSTEM ON THE TEACHING OF MATHEMATICS IN THE PRIMARY SCHOOL

Steven Nisbet (Griffith University, Australia)
& Elizabeth Warren (Australian Catholic University)

Abstract

This paper reports that the introduction of a diagnostic assessment system in Year 2 has had a significant and positive impact on mathematics teaching and assessment in the primary school. Teachers reported using a wider variety of assessment techniques (notably individual interviews & observations), using assessment data for planning instruction, having a greater sense of accountability, and including more problem solving and hands-on activities in their teaching.

Introduction

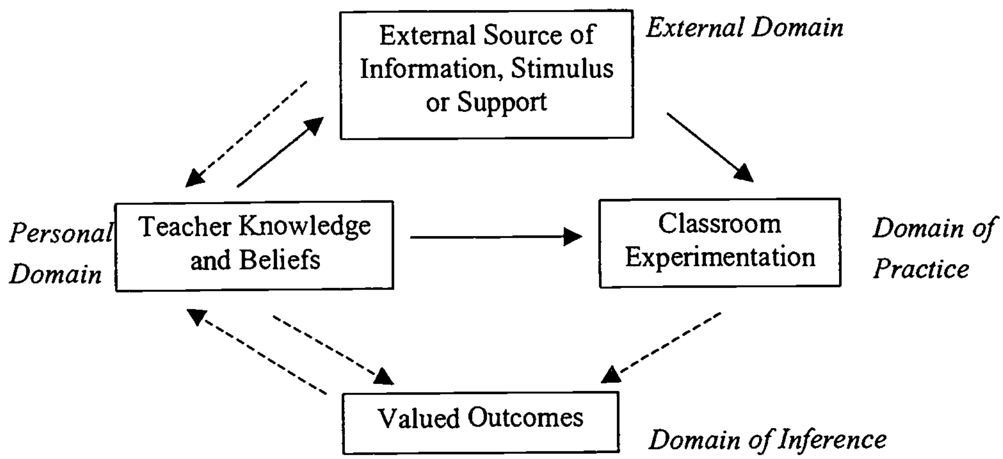
School teachers are continually faced with the challenge of implementing innovative ideas, some which are mandated by the school or the school system, and others which are promoted but not enforced. The literature on teacher change and professional development includes a sequence of models which, in turn, purport to describe the process and explain the success or otherwise of curriculum innovation.

The traditional model of implementing curriculum innovation assumes that teacher change is a simple linear process along the following lines. Staff development activities lead to changes in teachers' knowledge beliefs and attitudes, which, in turn, lead to changes in classroom teaching practices, the outcome of which is improved student learning outcomes (Clarke & Peter, 1993). Later models recognise that teacher change is a long term process (Fullan, 1982) and that the most significant changes in teacher attitudes and beliefs occur *after* teachers begin implementing a new practice successfully and see changes in student learning (Guskey, 1985). The professional development model of Clarke (1988) has refined the Guskey model by recognising the on-going and cyclical nature of professional development and teacher change. Later Clarke and Peter (1993) adapted the Guskey model further by broadening the original conceptual elements within the model. *Staff Development Activity* was broadened to include any external source of Information, Stimulus or Support, and was labelled as the *External Domain*. *Classroom Practice* became the *Domain of Practice* to include any Classroom Experimentation. *Student Learning* became the *Domain of* Inference to include any Valued Outcomes. Finally, *Teacher Beliefs* became the *Personal Domain* to include Teacher Knowledge and Beliefs (see Figure 1).

Reference to the above model of professional growth has the potential to illuminate the reasons why some innovations in mathematics education have succeeded and others have not. For instance, the importance of teachers' knowledge and beliefs in the cycle of professional growth was confirmed by Leonidas (1996) who found that the failure of a mathematics curriculum change in a centralized system was due to the

fact that teachers' perceptions of mathematics were inadequately considered at the adoption and implementation stages. Similarly, Philippou and Christou (1996) noted that if bright new ideas are to find their way into mathematics classrooms, it is imperative that change agents have a deeper understanding of classroom teachers' views, beliefs, conceptions and practices. Their study found that although teachers may be aware of and accept contemporary ideas (in this case about assessment), there can be a distance between their knowledge and intentions on the one hand, and their actual practice (in assessment) on the other.

Figure 1: The Clarke-Peter model of professional growth



Note on Figure 1: solid line = *enactive* mediating process; broken line = *reflective* mediating process. The mediating processes translate growth in one domain into another. The term *enactive* distinguishes the translation of a belief or a pedagogical model "into action" from simply "acting". Acting occurs in the Domain of Practice and each action represents the enactment of something a teacher knows, believes or has experienced (Clarke & Peter, 1993).

One particular example of an innovation which was not only successful at changing assessment practice at the target years but also produced a *ripple effect* through other school years was the system-wide adoption of the Victorian Certificate of Education (VCE) and its multi-component assessment scheme (Clarke, Stephens, & Wallbridge, 1993). It demonstrated that changed assessment practices in Years 11 and 12 mathematics had a strong impact on how mathematics was taught and assessed *throughout* the secondary school. The new assessment methods contained in the VCE included the use of multiple-choice skills test, an extended-answer analytic test, a ten-hour challenging problem, and a 20-hour investigative project. Such a wide range of assessment methods reflected the contemporary mathematics education literature, and would have been endorsed by the informed mathematics education community. In terms of the professional growth model (Clarke & Peter, 1993), the changes in the domain of practice must have produced valued outcomes for students as perceived by their teachers, and modified the beliefs and attitudes of practising teachers enough to continue the practice and to extend it to other grades in the school (the *ripple effect*).

This paper is concerned with the effect of a new state-wide system of assessment in lower primary mathematics on the teaching of mathematics throughout the primary school. In terms of the professional growth model (Clarke & Peter, 1993), the paper explores the nature of the enactive mediating process indicated by the arrow between the *External Domain* and the *Domain of Practice,* and the reflective mediating process indicated by the arrow between the *Domain of Practice* and the *Domain of Inference.* The process started in the *External Domain* with an imposed diagnostic assessment system, which lead to the teacher making changes in the mathematics classroom *(Domain of Practice)* and noting improvements in the *Domain of Inference* (valued outcomes).

The background to the adoption of the new system of diagnostic assessment in lower primary classes in the state of Queensland and a brief description of it now follows.

As a result of concerns about the levels of literacy and numeracy of children leaving school (Wiltshire, McMenniman, & Tolhurst, 1994), two state-wide initiatives have been undertaken recently. One is the *Year 2 Diagnostic Net* (the other being a *Year 6 Test*). Briefly, the Year 2 Diagnostic Net is a mandated process of monitoring and reporting on children's literacy and numeracy development in the early years of schooling. It identifies those children who are experiencing difficulties in literacy and numeracy and provides a framework for developing appropriate intervention programs for those children. The *Net* is a process in which teachers (i) observe and map all children's progress individually using *developmental continua* for aspects of literacy and numeracy, (ii) validate observations of children requiring additional assistance through specifically designed assessment tasks, (iii) provide appropriate learning support for children, and (iv) report to parents about these aspects of children's literacy and numeracy learning and development (Handbook for schools, 1996, p1). The numeracy section of the Net deals with patterns of simple objects, counting skills, and representing numbers from 0 to 10. This paper examines the effect of the introduction of the Year 2 Diagnostic Net on teaching and assessing mathematics in the primary school.

Methodology

In part of a survey, primary teachers were asked to indicate whether or not the introduction of the Year 2 Net had influenced their (1) teaching and (2) assessment of mathematics (by circling *yes* or *no* for each item). A blank space (20cm x 3.5cm) was provided below for respondents to explain how the Net had been influential. Fifteen hundred survey forms were sent to a random selection of primary schools representing different school systems (government & catholic), socio-economic areas (high & low) and geographic locations (metropolitan, provincial & rural). The return rate was 26% (n=387) and the resulting sample was representative of the different systems, socio-economic areas and geographic locations. The sample was also representative of school year levels.

Results

More teachers reported that the Year 2 Net had an influence on their assessment (42.2%) compared to an influence on teaching (28.3%) (see Table 1). However the majority of teachers influenced by the Net taught Years 1, 2 and 3: 70% of these teachers indicated that it influenced their assessment, and 43% said that it influenced their teaching. Teachers in other grades reported levels of influence lower than those for lower-primary teachers, reducing to 31.3% (assessment) and 20.8% (teaching) for Year 4 teachers and 15.2% (assessment) and 13.6% (teaching) for Year 7 teachers.

Table 1: Distribution of percentage of responses across grades on the question of *whether* the Year 2 Net influenced the teachers' assessment and teaching.

Year Level	Influenced assessment %			Influenced Teaching %		
	Yes	No	No response	Yes	No	No response
Preschool	25	58.3	16.7	25	66.7	8.3
Year 1	75.5	18.9	5.7	39.6	52.8	7.5
Year 2	80.7	15.8	3.5	50.9	45.6	3.5
Year 3	63.9	26.2	9.8	45.9	40.1	13.1
Year 4	31.3	54.2	14.6	20.8	64.6	14.6
Year 5	12.8	76.6	10.6	12.8	76.6	10.6
Year 6	11.9	61.9	26.2	9.5	64.3	28.6
Year 7	15.2	62.1	22.7	13.6	63.6	22.7
Averages	42.2%	44.5%	13.4%	28.3%	57.8%	13.9%

Only 175 (45%) of the respondents wrote an explanation in the space provided of *how* the Year 2 Net had influenced their teaching and assessment of mathematics. The explanations that were provided fell into four categories, with significantly more explanations given about *assessment practices* (n=73, 42%) than about *content taught* (n=37, 21%), *methods used for teaching* (n=38, 22%), and *comments about students* (n=27, 15%), $F(3, 18) = 4.15$, $p < .05$. More explanations were given by teachers of years 1 to 3 (n=138, 79%) than years 4 and 5 (n=31, 18%) and years 6 and 7 (n=6, 3%), $F(6, 18) = 9.24$, $p < .0001$.

(i) *Comments on assessment*: The comments on assessment were about different types of assessment tools the teachers now used or about different ways of implementing assessment procedures in the classroom setting (see Table 2).

Table 2: Distribution of comments on *changed assessment practices* (n = 73)

Category	Yr. 1	Yr. 2	Yr. 3	Yr. 4	Yr. 5	Yr. 6	Yr. 7	Total
Type of assessment								
Observation	4	5	3	1	2		1	16
Journal	1		3					4
Individual interview	5	6	1	2				14
Focussed assessment	3	1					1	5
Review sheets	1							1

Implementing assess.								
More systematic	1	5	4		1			11
More individually based	1		1		1			3
Greater array		1	2	4	2		1	10
Influenced recording	1	3	2		1			7
Influenced reporting		1	1					2

The majority of teachers who reported an increased use of observation and individual interviews taught in the lower primary grades. This increase should be seen as a positive effect of the introduction of the Year 2 Net. Many teachers reported that they assessed more systematically and used a greater array of assessment methods.

(ii) *Comments on content*: Many comments show that many teachers have embraced the Year 2 Net not only as an assessment device but also as indicator of what to teach (see Table 3), for instance more problem solving and activities with number patterns.

Table 3: Distribution of comments on *changes in content* (Total = 37)

Content	Yr. 1	Yr. 2	Yr. 3	Yr. 4	Yr. 5	Yr. 6	Yr. 7	Total
Number emphasis (patterns)		1	2	1				4
Problem solving		2	2	1				5
Investigations		1						1
Continuum (Net) is syllabus	1	11	6	2	2			22
Emphasis on money			1					1
Emphasis on different aspects		4						4

(iii) *Comments on teaching*: Many comments indicate a greater sense of accountability and the use of assessment data to inform planning (see Table 4). Others included the greater use of hands-on activities, and self-reflections of teaching and mathematics.

Table 4: Distribution of comments on *changes in teaching* (total = 38)

Teaching (method)	Yr1	Yr. 2	Yr. 3	Yr. 4	Yr. 5	Yr. 6	Yr. 7	Total
More hands on	4				2		1	5
Real world contexts	1						1	2
Teaching (approach)								
Integrated approach	1							1
More accountable		2	1	1	1		1	6
Informs planning	2	2	2	1	2			9
Ensure all children have knowledge of all areas	3	5						8
Teaching (Prof Dev)								
Changed my perceptions of mathematics	1	1						2
Identified my weaknesses	1	2						3
Network meetings inservice	2							2

(iv) *Comments about students*: These comments show that many teachers interact more with students on a one-to-one or small-group basis, and identify more children at risk (see Table 5). Such responses are consistent with those shown in Table 2 and should be seen also as a positive effect of the introduction of the Net.

Table 5: Distribution of comments on changes relating to *students* (total = 27)

Categories for *students*	Yr. 1	Yr. 2	Yr. 3	Yr. 4	Yr. 5	Yr. 6	Yr. 7	Total
More one to one	2	5	2	2				11
More small group interaction	1	2						3
Identify chn needing extension		1			1			2
Identify children at risk		4	1	1	2			8
Adapt to developmental levels	1	1						2
Plotting children's progress			1					1

Discussion

These results should be interpreted in the context of the methodology of self-reporting by teachers and the return rate of just over a quarter. Although the state-wide figures may not be as high as the sample figures, it can be argued that the introduction of the Year 2 Net nevertheless has had an impact on the teaching and assessment of mathematics in Queensland primary schools especially in the lower grades (Years 1, 2 & 3). The fact that 80.7% of the Year 2 teachers in the sample reported that it influenced their assessment methods and 50.9% reported that it influenced their teaching, indicates that the impact has been substantial. Even a worst-case scenario of a quarter of these percentages implies that the impact was not insignificant, with many teachers across the state improving their assessment and teaching skills.

In terms of the Clark-Peter model (1993) of professional growth, the External-Domain stimulus of a mandated assessment process has lead to changes within the Domain of Practice, namely, better assessment and teaching methods. The Year 2 Net has offered many lower-primary teachers a pedagogical model which they report being able to translate into action through the *enactive mediating process*. Hence the success of the Net may be explained in terms of teachers noting the outcomes of the assessment tools of the Net as being positive and *valued*.

The effect on the teaching and assessment of mathematics reported by teachers in grades other than Year 2 is another example of the *ripple effect* described by Clarke, Stephens and Wallbridge, (1993), this time in the primary school, with the effect in this case waning in year levels more distant from Year 2.

The impact of the introduction of the Year 2 Net has been positive in terms of *valued outcomes*. Why positive? The responses by teachers have largely referred to outcomes that have been *endorsed by the informed community* (Stephens, Clarke, & Pavlou, 1994). The mathematics education literature is replete with calls for broadening the range of assessment techniques used in mathematics classes (Clarke, Clarke & Lovitt, 1990; Beyer, 1993; Swan, 1993; Webb, 1993). Hence, it is

gratifying to note lower-primary teachers in this study reportedly using a greater array of assessment techniques, including observation and individual interviews. Similarly it is gratifying to note a reported increased emphasis on identifying children at risk and conducting more one-to-one and small group activities, in the light of public concern about standards of numeracy and the thrusts in the literature on intervention in early mathematics (Pearn, 1994; Steffe & Cobb, 1988; Wright, 1994). Given the sustained endorsement of problem solving and hands-on activities over the last twenty years (NCTM , 1979; Cockcroft, 1982; AMEP, 1985; NCTM, 1989, AEC, 1990), it is also satisfying to note more teachers reporting an increase in their use of such activities in the mathematics classroom.

The fact that many teachers have embraced the Year 2 Net not only as an assessment device but also as indicator of what to teach may imply a perceived inadequacy of the current syllabus statement (Department of Education, Queensland, 1987) and/or insufficient promotion of the syllabus and its accompanying resources. Nevertheless, it is a concern to educators when the assessment *tail wags* the curriculum *dog*.

However, other comments in the category of teaching indicate that there have been other positive outcomes of the introduction of the Year 2 Net. Firstly, some teachers report using the assessment data to inform planning - a positive outcome, given that a major purpose of assessment is planning instruction (Cole & Chan, 1994). Secondly, some teachers believe that they are more accountable, and that they ensure that their children "have knowledge of maths in all areas" – another positive result in the light of calls for achieving learning outcomes (Australian Education Council, 1994). Thirdly, it is gratifying to note some teachers being more self-reflective in relation to their own strengths and weaknesses and their perceptions of mathematics. This supports another facet of the Clarke-Peter (1993) model of professional development i.e. the personal domain, and is a cue for the next stage of this investigation.

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STUDY OF JUSTIFICATIONS MADE BY STUDENTS AT THE "PREPARATION STAGE" OF BADLY DEFINED PROBLEMS

Noda, A.; Hernández, J. and Socas, M.M.
University of La Laguna

SUMMARY

In this work we describe and analyze the behaviour of three final year Infant Education teacher training students when at the "preparation stage" (Bourne, et al., 1979) they have to solve six badly defined find problems in arithmetic, algebraic and geometric contexts. More specifically, we analyze the ways in which the students identify problems as either well defined or badly defined, the relationships they establish between the data and the aims of the problems, and the type of arguments they use in order to justify their actions in terms of validation and refutation.

INTRODUCTION

Attempts have been made to characterize the notion of problems from the viewpoint of Mathematics (Polya; 1957), Psychology (Newell and Simon, 1972; Chi and Glaser, 1986), as well as from that of Mathematical Education (Schoenfeld, 1985). Also, there have been many attempts to classify and group them (Simon, 1973; Butts, 1980; Charles and Lester, 1982; Frederiksen, 1984; Borasi, 1986; Pehkonen, 1991, 95).

As Greeno (1978) and Kilpatrick (1987) point out, it is very difficult to make precise classifications of problems because many problems possess common features in different categories.

We use the classification adopted by Polya (1945) who distinguishes find problems from prove problems, and adapt some elements found in the definition of problem space put forward by Newell and Simon (1972). In our work we have had to make a "local" characterization of well and badly defined find problems in order to identify the problems to be included in our research. One term we have had to introduce is that of the "Semiotic State" of a well or badly defined find problem; this term stands for the set of semiotic representations of the problem that the ideal solver would produce from the initial state E_0 up to the final state R. The notions of initial state, desired final state, operators, etc., are set out by Newell and Simon (1972, p.810).

It should be pointed out that while in their definition these authors refer to the various cognitive representations of the problem the solver makes on the basis of the task environment, we refer to a "formal competence model" and to the possible semiotic representations of the problem solution made by an ideal solver.

So, a find problem is determined by the tern <E,O,R> where **E** is the set of semiotic states, **O** the set of operators, and **R** the set of solutions; so, $\mathbf{E}=\{E_0,E_1,\ldots,E_n\}$, with $E_n \in R$, and E_0, the initial state, formed by the set of data given.

We characterize a <u>find problem as well defined</u> when in the previous tern $E_0 \neq \varnothing \wedge \exists (E_i)_{i\in\{0,1,\ldots,n\}} \wedge E_0 \subset E_i,\ \forall i \in \{1,\ldots,n\}$. { $(E_i)_{i\in\{0,1,\ldots,n\}}$ represents a succession of states included in E }.

That is, when we go through a solution process in which we can relate all the data to the solution by means of some type of analysis or synthesis, be it numerical, algebraic, or some other type.

So, if in this definition we again take the tern <E,O,R> and deny the conditions that characterize the well defined find problem, a <u>badly defined find problem</u> would be

characterized as $E_0=\varnothing \vee \nexists\ (E_i)_{i\in\{0,1,\ldots,n\}} \vee E_0 \not\subset Ei,\ \forall i\in\{1,2,\ldots,n\}$. That is, when there is no possible relationship between the data and the solution.

We can now classify badly defined find problems as: Type I (too few data) when $E_0=\varnothing$, Type II (too few data) when $E_0\neq\varnothing \wedge \nexists\ (E_i)_{i\in\{1,\ldots,n\}}$, Type III (too many data) when $E_0\neq\varnothing \wedge \exists(E_i)_{i\in\{1,\ldots,n\}} \wedge E_0 \not\subset Ei,\ \forall i\in\{1,2,\ldots,n\}$. (Noda, A.; Hernández, J. and Socas, M.M., 1997,1998).

EXPERIMENT DESCRIPTION

In this experiment we consider the models proposed by Dewey (1933) and Bourne et al. (1979) for problem solving and we basically analyze the "preparation stage", focusing especially on students' actions at this stage. In Dewey's terms, these actions can be specified as: identification of the problem situation, characterization of the problem, and analysis of means-ends, in other words, how solvers analyze and interpret the data initially available, the restrictions and how they identify the solution criteria.

Research questions.

Our aim is to study solvers' behaviour when they are faced with badly defined find problem in arithmetical, algebraic and geometrical contexts. To carry this out we ask ourselves the following questions: How do solvers identify the problem situations in terms of well or badly defined? How do they establish relationships between the data and the aims in these types of situations? And finally, how do the solvers justify their actions? Bearing these research questions in mind, we designed an analysis scheme to observe the real solvers at the preparation stage when working with well and badly defined find problems.

Methodology

Adopting the model established by Schoenfeld (1985), the scheme was designed as follows:

Reading: reading, silence, re-reading

L1. Have all conditions of the problem been noted?

L2. Has the aim of the problem been correctly noted?

L3. Does the environment (context) of the task affect the reading of the problem?

Analysis-Exploration: Understanding, reasoned actions

E1. Does the solver look for some relationship, either true or false, between the conditions and aim of the problem?

E2. Are the actions governed by the conditions of the problem (the data)?

E3. Are the actions governed by the aims of the problem (the question)?

Actions: Reformulation, introduction of elements, simplification of elements.

A1. Does the solver explicitly or implicitly recognize that the problem is well or badly defined? Does s/he justify this?

A2. Does s/he reformulate the problem? Does s/he transform the problem into a well or badly defined problem? Explicitly? Implicitly? How? Does s/he justify the transformation?

A3. If s/he does not transform the problem either explicitly or implicitly, does s/he set about It as if it were a well defined problem? Does s/he justify this?

Checking-Transition.

V1. Does the solver revise his or her actions? As a result of the process or of the result obtained? Does s/he justify this?

V2. Does s/he change his or her action plan? Does s/he justify this?

V3. Is the plan adopted suitable for the change made?

The justifications made by the solvers for their actions must be considered in terms of validation or refutation To this end, we establish, as in the case of well defined problems, three general types of resource (Calderón and León, 1996):

Internal cognitive resources (related to mathematics) For justifications made with regard to the mathematical relationships established by the solver or mathematical behaviour decided upon by the solver in the procedures followed.

External cognitive resources. For justifications made by using culturally established concepts to justify mathematics. The level of systematization in such justifications refers to beliefs and attitudes held about mathematics rather than to authentic processes of mathematical inference

Discursive resources. For interpretations made in a verbal or written form that exteriorize and show concepts and processes to do with well or badly defined problems or problem situations.

Subjects

In order to gather oral data, the experiment was carried out on three-year teacher training students specializing in Infant Education at the Centro Superior de Educación at La Laguna University (Tenerife, Spain). The three subjects were split into two groups: Group 1, made up of two students of average ability, and Group 2, made up of one student of high ability.

Instruments

Both groups were given a set of six badly defined problems (see Table 1): three badly defined Type-II problems with a lack of data (Al, B2 and B3), and three badly defined Type-III problems with too many data (A2, A3 and B 1). The problems were worked on in two one-hour sessions carried out on different days.

Context / Sessions	Arithmetic	Algebraic	Geometric
Session 1	(A1) M.D. Type II	(A2) M.D. Type III	(A3) M.D. Type III
Session 2	(B1) M.D. Type III	(B2) M.D. Type II	(B3) M.D. Type II

Table 1

The sessions were videotaped. Group 1 carried on their session by means of discussions between the two students in such a way that they alternately played the roles of interviewer and solver regarding the various problems, while in Group 2 a clinical interview was carried out.

RESULTS

Let us now look at the results obtained in the experiment in accordance with the analysis scheme we have designed.

By way of example, we shall describe the behaviour of the solvers when solving one of the six problems they worked on (A1), and include the transcription of the video recording of both groups of students. The parts underlined highlight the justifications made by the students.

***Al**: The menu in a bar reads as follows: Hot-dogs 100 pesetas; coffee 75 pesetas; fried potatoes 85 pesetas, Coca Cola 125 pesetas, toast 100 pesetas, and hamburgers 175 pesetas. If Luis does not like hot-dogs, how much did he pay for his breakfast?*

Group 1: Student 1 (S1) plays the role of interviewer while Student 2 (S2) that of the solver.

Sl: Reads out the problem.

S2: As S1 reads out the problem, S2 notes down the problem data (both the text as well as the numerical data in the form of a table with two columns: the left one for the text and the right one for the numerical data). When Sl finishes reading the problem, he asks: "What does the problem ask?"

Sl: "How much he paid for breakfast."

S2: Does not note down the question but immediately says: "Well, if he ate fried potatoes, toast, coffee ... (she pauses a little and then goes on) ... he can't have Coca Cola, he isn't going to mix Coca Cola and coffee. Look, if he had a hamburger and Coca Cola (writing on the board Hamburger and Coca Cola, and beneath this text the numerical data 125 + 175 = 300 pesetas, and doing the calculation in a slightly lower voice) he paid three hundred pesetas for breakfast.

Sl: "Eh! And how do you know he had a hamburger and Coca Cola and not fried potatoes, toast, coffee, Coca Cola and hamburger, but not the hot dogs because he doesn't like them? If the problem doesn't tell you what he ate, how can you know?"

S2: "That's stupid! Anyway, if he mixes coffee and Coca Cola, he'd be sick."

Sl: "Well, you might think that, but you don't know about Luis."

S2: "Well, let's say he had toast, coffee and fried potatoes."

Sl: "And why not everything except what he doesn't like, that's the hot dogs? The problem doesn't tell us anything else."

S2: "Well, in that case then, we'll say that for breakfast he spent (she begins to put all the numerical data on the board and does the calculation aloud 85+100+75+125+175=560 pesetas) five hundred and sixty pesetas.

Sl: "Alright?"

S2: Yes, though he could have had (she begins to say all the possible cases, including in each case two of the items on the menu) fried potatoes and coffee, or toast and coffee, or ..- anyway, whatever he pays depends on what he eats. Any solution is right."

Sl: (Nods, but at the same time shows he doesn't agree with the various solutions suggested by his classmate.)

Description of actions referred to the solver student.

The student looks for a relationship between the conditions and the aim of the problem, her actions being governed by the aim. She implicitly recognizes that the problem is badly defined and reformulates it, justifying this by means of refutation and by applying external cognitive resources based on common sense ("social logic"). Then, following a period of checking-transition, she modifies her actions, and identifies the problem as well defined with many possible solutions, justifying this action through validation and using discursive resources, such as description[1].

Group 2: Student (D), Interviewer (I).

D: Reads out the problem but does not write any notes on the board, and says: "it depends on what he eats, because if he chooses toast and coffee from the menu he'll pay a hundred pesetas plus seventy-five pesetas which would be one hundred and seventy-five pesetas, but if you take into account that he probably likes other things for breakfast, those amounts would be added up."

I: In that case, what would in fact the solution to the problem be?"

D: "Well, I'd answer that it'd depend on what he ate."

I: "O.K. Do you think, then, that there is something missing in the problem or that there's too much information, or do you think the problem's well expressed?"

D: "Something's missing in the text, for example. 'The cafe menu reads: Hot dogs 100 pesetas, coffee 75 pesetas, fried potatoes 85 pesetas, Coca Cola 125 pesetas, toast 100 pesetas, and

[1] Verbal explanation, by means of identifying, explaining or projecting the features of the situation.

hamburger 175 pesetas. If Luis had so much money, what could he eat?' Alternatively, 'The cafe menu reads: Hot dogs 100 pesetas, coffee 75 pesetas, fried potatoes 85 pesetas, Coca Cola 125 pesetas, toast 100 pesetas, and hamburger 175 pesetas. If Luis likes to have this and that, how much will he pay for his breakfast? But, the way it's formulated, we can't answer, because you don't know what he's had nor what he likes to have for breakfast."

Description of actions

The student looks for a relationship between the conditions and the aim of the problem, her actions being governed by the aim. She explicitly recognizes that the problem is badly defined and transforms it. She justifies her actions through refutation and uses internal cognitive resources based on counter-examples[2].

Here we briefly summarize the solvers' behaviour when working with the rest of the problems.

A.2: Some farmers stored hay for 4 days, but as the hay was of better quality than they had thought, they saved 100 kg per day, so that they had hay for 6 days. They spent a total of 1,200 pesetas. How many kilograms of hay did they store?

Description of actions of the solver student of the Group 1 (G1).

The student does not look for a relationship between the conditions and the aim of the problem. His actions are governed by the conditions. At first, he sets about the problem as if it were well defined. This action is validated though the use of external cognitive resources of a ritual kind[3]. Later, following a period of checking-transition, he changes actions and transforms the problem, justifying his actions by an authoritative (belief) refutation[4].

Description of Group 2 (G2) actions

At first, she sets about the problem as if it were well defined, justifying her actions through validation and using external cognitive resources based on authority. Later, after a checking-transition period, she looks for a relationship between the conditions and the aim of the problem, her actions being governed by the aim. She recognizes that the problem is badly defined and transforms it. Her actions are justified through refutation, using internal cognitive resources such as contradiction.

A3: How far is the school from the park if to get from one place to another the 60-cm diameter wheel of a bicycle travels 62 metres and it takes 6 hours.

Description of actions of the solver student of the Group 1 (G1).

The student does not look for a relationship between the conditions and the aim of the problem. Her actions are governed by the conditions (the data). At first, she sets about the problem as if it were well defined. The action is validated through the use of external cognitive resources of a ritual kind. Later, following a period of checking-transition, she transforms the problem, justifying her actions by refutation and using external cognitive resources based on authority (belief).

Description of Group 2 (G2) actions

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[2] She gives an example, taken from the same situation or from an analogous one, in order to justify the fact that the situation is badly defined or takes the same situation as an example.

[3] The subject tries to establish validity because of the way in which the situation is presented. The appearance rather than the coherence of the contents is more influential.

[4] The justification comes from a source that enjoys complete credibility: teacher, task (book), competent classmate, etc.

At first, she sets about the problem as if it were well defined and she puts forward her validation arguments by using empirical internal cognitive resources[5]. Later, after a checking-transition period, she looks for a relationship between the conditions and the aim of the problem, her actions being governed by the aim. She recognizes that the problem is badly defined and transforms it. Her actions are justified through refutation and she uses internal cognitive resources, in this case a counter-example.

***B1:** A woman raises a goat and 3 rabbits for 52 weeks. She buys 760 grams of fodder for one week. If the animals eat all the fodder except for 128 grams, how much fodder was left over at the end of the week?*

Description of actions of the solver student of the Group 1 (G1).

The student does not look for a relationship between the conditions and the aim of the problem. His actions are governed by the conditions. At first, he sets about the problem as if it were well defined, validating this through the use of cognitive resources of an authoritative kind. Later, following a period of checking-transition, he looks for a relationship between the conditions and the aim of the problem. He identifies the problem as a badly defined one, justifying his actions through refutation and using internal cognitive resources such as a counter-example.

Description of Group 2 (G2) actions

At first, she sets about the problem as if it were well defined. The action is validated through the use of external cognitive resources of an authoritative kind. Later, following a period of checking-transition, she looks for a relationship between the conditions and the aim of the problem, her actions being governed by the aim of the problem. Then, she recognizes that the problem is badly defined and transforms it. Her actions are justified by refutation and she uses internal cognitive resources, in this case a counter-example.

***B2:** In a farmyard there are hens and rabbits. If there is a total of 116 legs, how many hens and rabbits are there in the farmyard?*

Description of actions of the solver student of the Group 1 (G1).

The student does not look for a relationship between the conditions and the aim of the problem. Her actions are governed by the conditions, and so she sets about the problem as if it were well defined. This action is justified by means of validation through the use of analytical internal cognitive resources[6].

Description of Group 2 (G2) actions

The student does not look for a relationship between the conditions and the aim of the problem, her actions being governed by the conditions (the data). At first, she sets about the problem as if it were well defined. These actions are validated by using empirical internal cognitive resources.

***B3:** If the square in front of your house is rectangular and measures 90 in by 60 m, how far is it from your school to your house?*

Description of actions of the solver student of the Group 1 (G1).

The student looks for a relationship between the conditions and the aim of the problem, her actions being governed by the aim of the problem. She recognizes that the problem is badly defined, justifying her actions through refutation and uses discursive resources such as

[5] Justification is made on the basis of quantifiable experiences or physical facts.

[6] The student works within the field of mathematical demon[illegible]

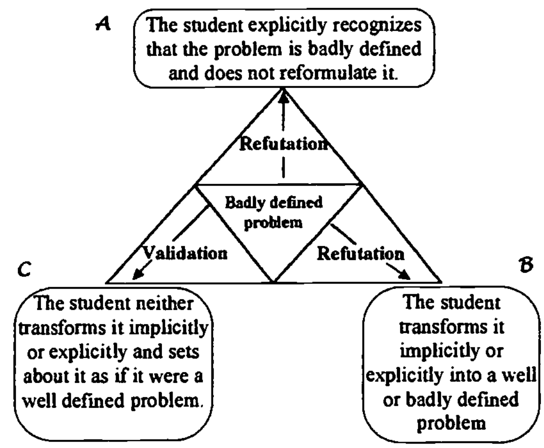
irony[7]. Later, following a period of checking-transition, she transforms the problem. These actions are justified through refutation, using internal cognitive resources such as the counter-example.

Description of Group 2 (G2) actions

The student looks for a relationship between the conditions and the aim of the problem, her actions being governed by the aim of the problem. She recognizes that the problem is badly defined and transforms it. These actions are justified through refutation, using discursive resources such as irony.

FINAL CONSIDERATIONS

- Regarding the solvers' behaviour, generally speaking, and except in the case of Group G2 when doing Problems AI and B3, we can see two clearly differentiated semiotic states which we call initial and final, and in which the students use clearly differentiated validation or refutation justifications. These types of behaviour regarding the "Identification of the badly defined problem in terms of well or badly defined" and "how they justify their actions" in terms of validation or refutation, can be represented in the following scheme, which also enables us to see the paths the students follow:



- Certain difficulties arise when identifying problems in terms of badly defined, where the influence of the context is notable. So, when we analyze the final semiotic state, identification as well defined (option C in our Scheme) only occurs in the algebraic context (both groups of students identify Problem B2 as a well defined problem). In the initial semiotic state these problems are considered to be well defined by both groups in the three contexts studied (Problems A2, A3 and B 1), all of these being Type III problems.
- With respect to the paths followed by the students when working on the problems through the various semiotic spaces, we can see that it is in Type III problems where they go from one vertex to another in the Scheme; in other words, they modify their action plans following a checking-transition period (in the three Type III problems).
- With regard to whether the actions are governed by the aim or the conditions, we can see that:
  - When the actions are governed by the aim and not the conditions, the students recognize that the problem is badly defined as such (option A or B in Scheme); on the other hand, when the actions are governed by the conditions, the students identify badly defined problems as well defined (option C in the scheme).
  - With regard to context, differences can be seen. Actions governed by the conditions of the problem are always undertaken in the algebraic context. However, regarding the

7 She refutes the validity of the situation by implying the opposite of what she literally says. This is a type of indirect justification.

types of problems, there is a large degree of balance between Type II problems (lacking data) and Type III problems (too many data).

- When we analyze justifications by means of either validation or refutation, we note different types of argumentation which, initially, can be categorized and classified as in the following table

Resources / Justifications	**Internal Cognitive**	**External Cognitive**	**Discursive**
Validation	Empirical Analytical	Authority (beliefs) Ritual (attitude)	Description Comparison
Refutation	Counter-example Contradiction	Authority (beliefs) Social Logic	Irony Ridicule

Table 2

In both validation and refutation types justification we note a certain degree of balance between external and internal cognitive resources. There are fewer justifications in discursive terms.

Finally, we should point out that of resources used in our analysis becomes a theoretical instrument that relates and typifies the students' justifications when they work on badly defined problems at this "preparation stage".

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THIS PATIENT SHOULD BE DEAD!

or

HOW CAN THE STUDY OF MATHEMATICS IN WORK ADVANCE OUR UNDERSTANDING OF MATHEMATICAL MEANING-MAKING IN GENERAL?

Richard Noss, Celia Hoyles and Stefano Pozzi
Institute of Education, University of London

Our data involves a detailed study of the ways in which a group of paediatric nurses think about the notion of average and variation. We describe some continuities and discontinuities between mathematical and nursing epistemologies, and draw some general conclusions about the ways in which more general mathematical meanings are constructed.

Over the past few years, we have been studying the ways in which mathematical meanings are constructed and mobilised in a variety of workplace settings[1]. Our overarching objective is to try to understand in the most general terms, how individuals make mathematical sense: that is, to try to exploit the richness and complexity of mathematical activity *in situ* in order to analyse mathematical activity *per se*. Our method has been to focus attention on the kinds of problems professionals actually solve, in order to try to make visible a range of activities that can usefully be described as mathematical, in terms of their participation — at least at some level — in the manipulation and interpretation of quantitative and spatial data and relationships (see Noss et al., 1998; Pozzi et al., 1998).

As our research progressed, we became aware that it was possible to make a provisional epistemological classification of some of the activities we were observing, and that it made sense to analyse them in terms of what they represented from our mathematical point of view. This analysis could then be synthesised and compared with the practitioner viewpoint. In this paper, we report the outcome of one such mathematical classification, consisting of the interpretation of statistical data on the part of a group of paediatric nurses.

There are numerous findings which appear to illustrate how people's intuitions often seem to stand in the way of developing statistical notions. Shaughnessy (1992) provides a comprehensive overview of this research. With regard to the mean and other measures of central tendency, there appears to be an incompleteness in what people believe, a perceived lack of balance between computational and conceptual

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[1] The workplaces studied involved investment bankers, airline pilots and (the subject of this paper) paediatric nurses. We acknowledge the support of the Economic and Social Research Council, UK, Grant No. RO22250004. The final report of the project *Towards a Mathematical Orientation* can be found at www.ioe.ac.uk/rnoss/tmo/.

understanding, and little understanding of why a mean should be calculated. As far as the understanding of variation is concerned, Shaughnessy notes that: 'people inappropriately believe there is no random variability in the "real world"' (p. 478). This is not only a problem for novices: for example, Batanero *et al* (1994) reported that a widespread misconception amongst university students was a deterministic conception of association; students do not admit exceptions to the existence of a relationship between variables and if they find any reject the idea that there is a relationship.

While providing a wealth of information, these studies can provide only a limited number of starting points for our research, as they fail to acknowledge the range of sense-making devices and strategies which people actually use in practice to describe, for example, randomness and variation. Neither do they identify important continuities with a view to understanding better how functional knowledge may be extended 'beyond its productive range of applications' (Smith, diSessa and Roschelle, 1993 p. 152).

It might be, of course, that there is a simple discontinuity between taught/school and intuitive/practical knowledge. Mokros and Russell (1995), in a study of fourth and eighth graders, concluded that some students 'were concerned with procedures, not with meaning. For them, average means a series of steps involving addition and division' (*ibid* p.35). The authors ascribe this procedural strategy to a failure of reification — and it is here that they locate the students' difficulties.

There is much in this argument that merits discussion, and we cannot do it justice here[2]. It does, at least, propose some rationale for students' misunderstandings. But if the reification hypothesis makes sense, we ought observe process-oriented strategies characterising everyday thinking about data, not just school students. In working situations, people mobilise all kinds of meanings for manipulating and making sense of data, and this ought to throw light on the construction of specifically mathematical meanings.

The Nursing Studies

The practice of nursing involves making sense of a wide range of quantitative information through the measurement, recording and interpretation of patient data.This clinical experience would suggest that nurses will have well-formed intuitions about a wide range of statistical ideas, although what these intuitions are and how they can be developed has not been researched in any depth. Moreover, analysis of texts and interviews with nurses suggest that only measurement and recording are regarded as obviously mathematical, while interpretation is mostly seen in terms of clinical knowledge alone. The question for research was this: what meanings do nurses construct as they interpret data within their day-to-day practice?

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[2] A full version of this paper is in press by Pozzi, Hoyles and Noss.

We gathered data on this issue in three ways: ethnographic observation on the ward, clinical interviews and questionnaires, and a short teaching experiment. We will outline our findings in an abbreviated and non-chronological form, in order to highlight the central issues, reserving detailed data for our presentation. We begin with an interesting insight from our teaching experiment.

The teaching experiment: We recruited for our teaching experiment 28 paediatric nurses who were enrolled on a research awareness course as part of their professional development. We aimed to explore the relationship between nurses' and 'mathematical' views of average and variation, in the context of teaching a range of statistical modelling ideas using data analysis software (we used *Tabletop*™)[3]. The teaching experiment involved a combination of computer-based hands-on work and whole-class presentations, during which different ways of analysing data were introduced, alongside discussion of the substantive nursing issues involved.

We asked the nurses to consider explicitly the relationship between age and blood pressure, and to write down or illustrate their hypotheses. Suppressing all details, a fascinating phenomenon occurred. While most quickly opted to present the data as a scattergram of blood pressure against age their reaction to this graph was as surprising as it was puzzling. Given their practical and theoretical knowledge regarding the relationship between BP and age, we believed that the group would be prone to a high degree of confirmation bias, i.e. finding a relationship of some description regardless of how it was represented on the computer. Instead, many found it difficult to see any relationship at all in the scattergram — and here was the surprise — decided there could be none after all.

It seemed clear that the nurses' judgements were influenced by three main factors. Firstly, the high variation in the blood pressure data was a hindrance to seeing any relationship and obscured the identification of any trend; i.e. variation and relationship were somehow antithetical. Secondly, the rise in the data, in as much as one could be seen, struck many as involving a slope too gentle to possibly indicate a relationship that was so important that it stood as an a priori assumption of nursing knowledge, i.e. important relationships should have impressive-looking gradients. Thirdly, the nurses *could* explain the variation on the basis of their practical knowledge. — i.e. they were able to reiterate substantive reasons for differences in blood pressure for particular age-groups (lifestyle, health and other physiological factors). Some of the extreme data points were of particular interest, not from the point of view of variation from a trend but because they indicated to the nurses that the data were unreliable. As one nurse put it, 'this patient should be dead!'.

These three factors together — in part predictable from the literature, in part more specific to nursing — meant that the nurses tended to focus on variation at the

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[3] *The Tabletop*™ (TERC Inc.) is a database with an appealing graphical interface, which allows students to overcome unfamiliarity with the software quickly.

expense of the relationship. At this point, most of the nurses had reached an impasse, so we prompted them to model the data using the other tools available. Many split the data into age groups, finding the mean blood pressure for each group — a familiar number to all the nurses. This introduction of central tendency allowed many nurses, to our surprise, to reaffirm what they believed but had previously rejected: i.e. that BP did indeed increase with age! The use of the average enabled them to push the variation in the data into the background, as they took on board the increments in the means from the youngest age group to the oldest.

What was it that changed in the nurses' thinking which allowed them to reach this reconciliation? Of course the tools mediated the ways in which the data were conceived — to divide the data into quartiles, to calculate the means — clearly played significant roles. This tool-mediation served as an essential aid in clarifying how the nurses' views developed, and just how their practical knowledge of the average BP of an age group could be effectively mobilised. But to illuminate the puzzle further, we needed to look more closely at the ways in which statistical concepts were actually employed in practice by the nurses on the ward. Accordingly, we now turn to the ethnographic study and the insights we derived from analysing statistically-based ideas as they arose *in situ.*

The interviews and the ethnography: We reconstructed our ethnographic data into a set of episodes each describing a nursing-mathematics activity. While the episodes painted a broad picture of routine practice, we also identified and analysed a number of *breakdown* episodes, which involved a rupture in the normal routines. On these occasions, activities which were normally characterised by unproblematic, routine action were replaced by conflict, disagreement and doubt, which resulted in spontaneous explanations and considerably more articulated (and therefore explicit) reasoning and problem-solving strategies. Although these breakdowns were rare, they provided a rich source of information about participants' views and the knowledge they mobilised in order to substantiate them. Apart from providing insights as they occurred, we reconstructed these episodes and used them as a basis for clinical interviews with a subset of nurses.

One such episode involved the interpretation of blood pressure data plotted on a standard nursing chart. We asked the nurses to find the (deliberately vaguely stated) 'average' of a set of blood pressure data plotted on a BP chart. Only one nurse used the mean[4]. One mentioned it while adopting another strategy, while reference to either the median or mode was never explicit — although both were used implicitly. The nurses used a variety of strategies, almost all of which would be judged 'wrong' on the straightforward criteria of general mathematical correctness, e.g. *'I looked at the chart and judged which was the middle range'* or *'At a glance, all the systolic pressures are on or around 110 mm Hg'.* But they made perfect sense and were

[4] To use any standard calculational technique would have been quite difficult, given the way the data were displayed.

correct as estimates of the child's blood pressure — a kind of everyday average. The graph presented the nurses with a familiar situation in which the *sense* of average was unambiguous, and for which the idea of *representativeness* posed no difficulty because of the naturally symmetrical distribution of hourly BP readings.

In order to make sense of the nurses' strategies, we need to give a more detailed account of the practice underlying them, derived from our ethnographic observations on the wards. When a nurse takes charge of a patient, she immediately establishes his or her *individual baseline*. For example, nurses will attempt to answer questions such as 'what is this patient's normal, stable blood pressure?' This means that nurses have a tendency to take less account of the *whole* body of data when looking at average, and instead focus on clusters of data around the same value — especially if they think the patient was stable or settled around the time of these readings. As all patients have their own individual physiology, their profile of vital signs when they are stable is in some sense unique to them. Establishing a patient's normal profile of vital signs thus enables a nurse to judge when data deviate from the normal. A further feature of this process is that nurses develop an awareness of non-critical factors that lead to unusual readings (e.g. instrument error, patient over-activity) in order to distinguish them from significant changes in the patient's condition.

The notion of an individual baseline is not the only organising idea in nurses' judgements of average, since they must also take account of population norms. As we have reported, nurses 'know' that blood pressure is related to age, and can usually spontaneously quote a number for the average BP for a given age. But we found that nurses will only use it when it makes sense in terms of clinical practice. For example, the nurses were sceptical of the meaning of an overall population average for a young child and some refused to acknowledge that it existed! Thus the variability of the measures meant there was no average — something which makes no sense mathematically but which makes sense for the nurses.

On one level, these are completely understandable responses. The average blood pressure *for an individual child* obviously depends on whether the child was premature, its size and so on. Yet these kinds of considerations are, of course, included in the notion of average — indeed, it is the variability in population data which makes the statistic necessary. Nurses have difficulties because of their work of considering an average *independently* of the individual as it is the *individual* who is the focus of her care. The fact that 'average' necessarily masks individuality — like any statistic — is therefore seen as problematic. For a nurse, it is the variation that is crucial, and she is prepared to accept the notion of average provided it is individually mediated.

Discussion

We begin with a straightforward assertion: no description of the nurses' view of average and variation is adequately characterised in terms of misconceptions of the official, mathematical definitions. The nurses' meanings were different from the

mathematician's, interwoven with meanings from their practice, and efficient and effective at work.

Nor does it make sense to describe the nurses' strategies as, for example, failing to have reified the notion of average. On the contrary, if reification means anything, it means that an individual has made sufficiently strong links between elements of a concept to ensure its functionality: and as we have seen, this certainly applied to the nurses' notion of average.

What, then, is the status of the nurses' knowledge? The notion of nursing average was, as we have seen, on the one hand a description of the normal state of a patient over time, and on the other, a data point for a population — a measure of location unrelated to variability. In relation to the first aspect, the average provides a baseline — for example, an individual's blood pressure when it is stable. Here the nursing average works like a mathematical average so it can be seen as another example of what we have termed a situated abstraction (Noss and Hoyles, 1996). It is well articulated and abstract, but abstracted within rather than away from the nursing setting. It retains crucial elements of the setting in the way it is conceptualised which define the limits of its general applicability. But as with a mathematical average, the nursing average is indeed representative of the data as a whole and is recognised as such by nurses, because of the small variability in the data and the approximately symmetrical distribution. Additionally, in this context, outliers really must be explained — as physical 'extraneous variables' or functions of a medical condition.

The population view of an average differs considerably. Contrasting data on an individual with the population from which the individual is drawn is a complex issue from a mathematical perspective. Yet our research showed that the readings taken by nurses, or data which are interpreted by them, take account of population norms in ways which are finely tuned to nursing practice.

When nurses are acting in their familiar situations of monitoring blood pressure over time, they coordinate these different meanings of average to produce functionality. They use the notion of average to reason about their patients' condition, to relate the state of their patient to others in a virtual and deliberately ill-defined population (e.g. an overweight nearly 7 year-old), to communicate information to each other and to doctors, and to make rounded judgements concerning what action to take. Thus, significant background factors in blood pressure readings (e.g. age, sex, obesity, smoking) have to be taken into account by nurses when they interpret daily BP readings, and consider the condition and treatment of patients. Variation *has* to be explained. Not surprisingly, this practice spilled over into the nurses' answers in the questionnaire for the average blood pressure of a particular age group.

How do these insights of 'statistics on the ward' help us understand the nurses' responses in the teaching experiment? Recall that they wanted to explain away variation and found variation and relationship antithetical, while we wanted to introduce them to a fundamental principle of statistical modelling: that data is made

up of an explanatory model and random or unexplained variation around this model. On first reading of the nurses' responses, it seemed that their 'intuitions' were falling foul of this important statistical principle, and were based on a deterministic conception of association. Like the university students in the study by Batanero *et al* (1994), it appeared that they were using the many exceptions (or variations) to the blood pressure-age relationship to reject the idea that there was a relationship at all. Or, to restate the problem in the terms used by Rubin and Rosebery (1990), there seemed to be a confusion between the variables in the model and extraneous variables, which meant that presence of the latter made conclusions about any relationships in the model less possible.

Yet our study indicates that the roots of the responses might be rather different: the 'transfer' of the successful strategies of dealing with data over time on the ward to the consideration of population datasets — which then had to be debugged. We use the word 'transfer' with due caution, as we are well aware of the pitfalls. Yet it seems that something like transfer is involved here. If the nurses' appreciation of average is situated — i.e. ward practice dictates that explanation is required for significant variation, while smooth plateaux in data suggest normality and stability and require no explanation — then we may well ask how this can connect with other 'situations', and in particular, mathematics, where the situation is more or less reversed.

In Noss and Hoyles (1996), we outline the idea of webbing, as a notion with which to make sense of the ways in which people connect together pieces of their conceptual and physical world to create understandings. The crucial idea is that individuals' sense of situation and the tools they have to hand, provides support for making meaning, and also the means for reconstructing these pieces in new ways (or developing new knowledge pieces).

We do not want to rehearse our earlier arguments, but rather propose that our current findings throw some light on the cul-de-sac in which the situated cognition perspective seems to leave mathematics. The point is that the nurses *had* made the idea of average an object, defined in relation to their connection with other objects, and their functionality within the cultural domain in which they are used. What we see in the teaching experiment is how the mathematical domain could be brought into contact with the nursing domain, and how different tools (in our case, including the technological tools of a particular database program) could catalyse this contact, by exploiting the fact that the nurses could make sense of the new (mathematical) world by using the one tool which they used in their day-to-day practice — the mean blood pressure for an age group. It was this notion that helped them to see the trend in the variability and recapture what they knew of a relationship between age and blood pressure.

Mathematical objects are defined in relation to other objects, typically other mathematical objects. For the learner or practitioner, there is not necessarily any privileged status for these objects — and as we have seen, in nursing epistemology,

the privileged elements of mathematical epistemology simply do not make sense compared with the notion of nursing average.

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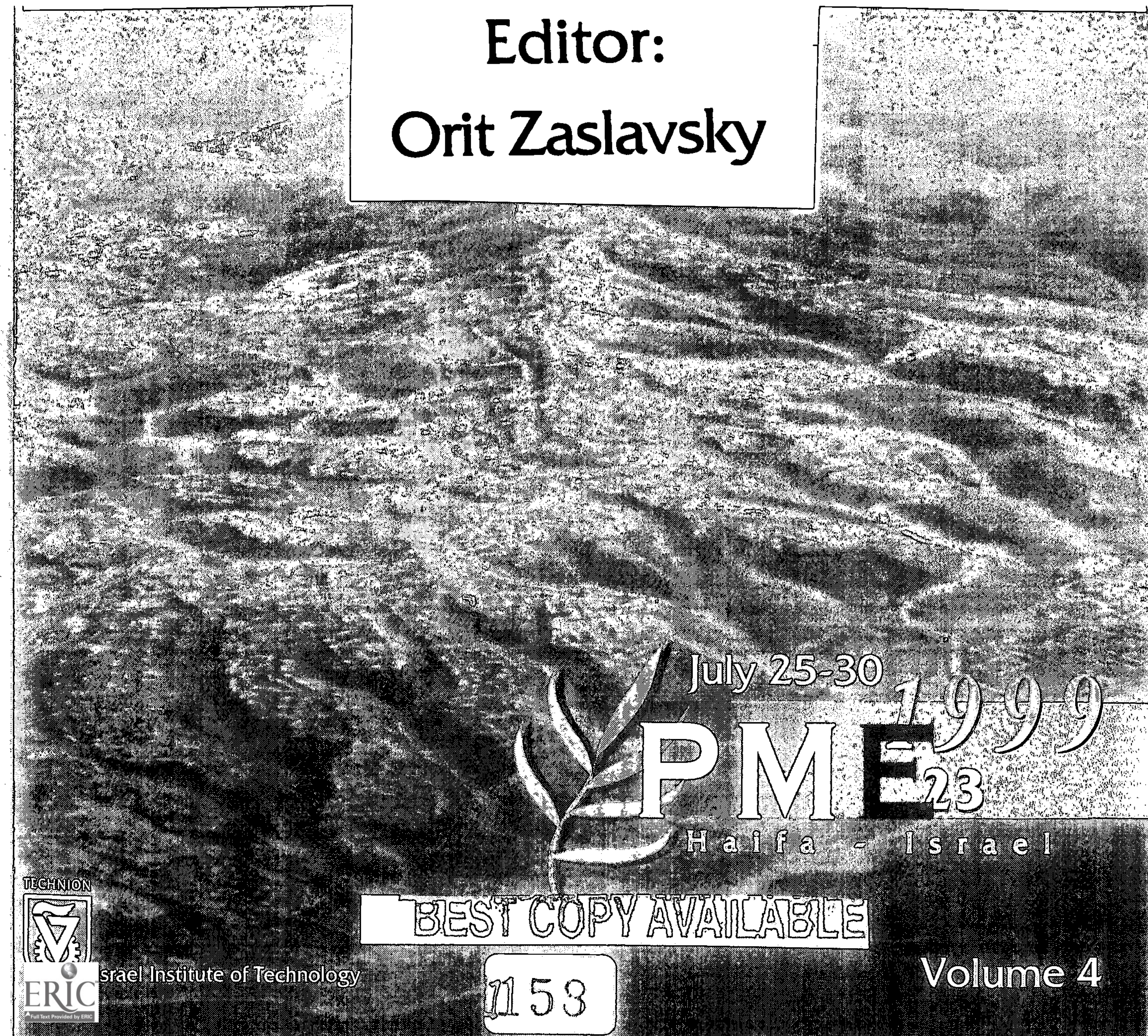

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Editor:
Orit Zaslavsky
Department of Education in Technology and Science
Technion - Israel Institute of Technology
Haifa 32000
Israel

Fax: +972 4 832 5445
E-mail: orit@tx.technion.ac.il

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VOLUME 4

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RESEARCH REPORTS
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THE RESEARCH OF IDEAS OF PROBABILITY IN THE ELEMENTARY LEVEL OF EDUCATION

Ana-María Ojeda S.
Cinvestav del IPN, México; University of Nottingham, U.K.

Abstract. *Ideas of probability have been investigated in Mexican elementary education. Two examples are given to illustrate the way in which epistemological aspects are considered in this research. Teaching experiments with 6-7 year old children suggest that pupils' interpretations of the tasks they were asked about may result in answers which do not inform on their idea of chance, since they tend not to focus on it. Additionally, by using questionnaires and clinical interviews with 10-15 year old children, it was found that correct performance in arithmetic does not assure that they can cope with questions about probability for which a quantification is required.*

Introduction

A research project interested in students' understanding of fundamental ideas of probability in the Mexican system of education has been carried out for five years. The project, which ranges over all the educational levels, considers epistemological, psychological and social aspects of the ideas of stochastics, in order to give an account of problems arising either from the teaching or learning of these ideas, as well as of their possible answers. The work we present here concerns only with the epistemological aspect, from the ontogenetical point of view, for the elementary mathematical education (primary and secondary school).

In this respect, one of the issues we have been investigating is the suitability of ways in which the ideas of probability have been introduced, nowadays apparently everywhere, in the educational elementary level. More specifically, we claim that the lack of consideration of factors influencing children's interpretation of the tasks proposed to them when ideas of probability are involved may mislead proposals for the curriculum of probability. We exemplify this with two of the studies which our project has undertaken; one concerns with the absence of the idea of chance in the Mexican syllabuses for the 6-7 year old pupils (Gurrola, 1998). The second study considers the role of arithmetic concepts in 10-15 year old pupils' understanding of probability (Perrusquía, 1998).

The problem and theoretical considerations

The delay of educational research in stochastics with respect to that research in other ideas of mathematics has led to call on results from this latter and from research on developmental psychology about the idea of chance (Piaget & Inhelder, 1975), for probability to be included in the mathematics curriculum. However, these views leave aside some aspects about children's interpretations of random situations, which seem to be important for the process of teaching.

Among the consequences of this, there is that probability is not considered in Mexico for kindergarten (5 year old children) nor for the first two grades of primary school, because after Piaget's model for the development of intelligence, children are uncapable to understand chance at stages previous to the stage of concrete operations (from 7 or 8 years of age). However, research in education with younger children (Falk *et al.*, 1980) suggests that the idea of probability can be introduced even for kindergarten pupils. On the other hand, children's performances when using other mathematical concepts to answer questions about probability, such as rational number, may not reflect their understanding of random situations, as probability demands a way of thinking that differs from the one used to face deterministic situations (Fischbein, 1975). Hence, Garfield and Ahlgren's allusion to the deficiencies in handling fractions as one of the difficulties for children to understand probability (1988) may result in delays for the introduction of probability or in waste of ways to profit on children's potential to approach this idea.

We are concerned here with the problem of characterizing children's understanding of the ideas of chance and probability specifically for educational purposes. More precisely, we are interested in having information from children's idea of chance and with what their quantifying of probability may reveal for considering their teaching in elementary education.

The problem has been investigated by referring to models of thinking proposed by other researchers. In particular, we refer to the origin of the idea of chance in children after Piaget & Inhelder's work for the study with young children without

instruction in probability. The study on the quantification of probability with older children was framed in Kieren's proposition of constructs for rational number (1988), in particular the constructs in the basic stage.

The studies and the methods used

The methods used in our research appeal to the aim pursued for education. Consequently, after having general data from the children who participated in our studies, interviews in depth with selected pupils took place.

The idea of chance in young children. Gurrola (1998) sought information about the convenience of introducing the idea of chance for 6-7 year old primary children. Accordingly, six children of this school age were asked about ideas of probability within a teaching experiment protocol (Glasersfeld, 1983). When using this method, a clinical interview is prolonged until evidence is obtained from the interviewed child making clear his/her understanding of the situation he/she is asked about.

A device was used to question children about random mixture similar to the one Piaget and Inhelder used (1975). It consisted of a rectangular tray which can be made to swing up and down by means of a fulcrum fixed at its base. Twelve equally sized marbles, six white and six green, can be arranged on both sides of a divider that the tray has at the middle of one of its sides, before letting them free by balancing the tray (see Figure 1), what we did with no intention for the marbles to mix to the least extent from each seesaw movement, as Piaget and Inhelder did. According to these authors, children of the age considered here do not anticipate the irreversibility involved in random mixture; instead, they try to find any kind of order on the grounds of common properties of the elements or of their original arrangement. As a result, they are not able to start understanding what chance is.

Nevertheless, among the six children we interviewed, four foresaw the mixture of the marbles when balancing the tray, that is, a different position from the one they had originally; the other two children proposed the marbles arriving at the same position they had at the start, but finished the teaching experiment stating that

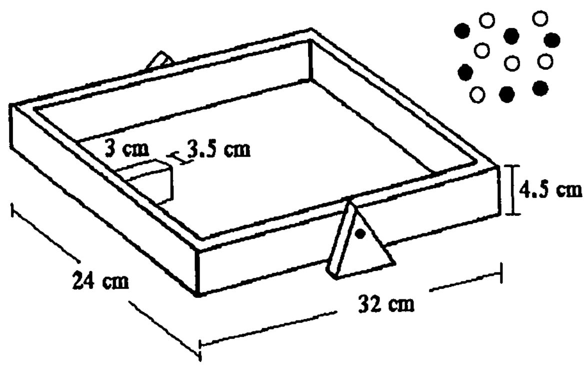


Figure 1

different positions should be expected. They realised that, as a consequence of the balancing of the tray, collisions between the marbles may well take place. Additionally, during the questioning, it arose that when interpreting the whole activity, some children may not focus on chance, but rather on trying to overcome it. That is, they may interpret the situation as a task to control the movement of the tray so to obtain, after each swinging, the marbles arranged as originally. This was the case of Almendra (7 years, 9 months).

In the first arrangement, the twelve marbles were shown to the girl separated by colour at both sides of the divider (in the following transcripts, *E* stands for interviewer, *A* for Almendra):

E *What do you think is going to happen if we balance the tray like this?*
A *One* [marble] *can go here, another here and another here* [she describes lines towards the opposite side of the tray but in the same direction]. *But without crossing here* [as she points to the middle of the tray extending an imaginary line stemming from the divider]: *these are green and these are white.*

After one movement of the tray, the girl sees that some marbles have changed their position. Before each of the subsequent trials, she arranged the marbles as they were presented to her at first. Apparently, she does not conceive the irreversibility of the mixture, but she states she does not know how to prevent the changing of the places between the marbles, and continues referring to the role of the divider:

A *This is meant to keep them* [the marbles] *at their place.*

Therefore, the divider was removed from the tray and the device was presented to her again showing the marbles mixed, i.e. not obviously arranged by colour (p. 99):

E *Do you think the greens will go back to their place and the whites to theirs?*
A *No.*
E *No? Why?*
A *Because the movement makes them change their places.*

She even realises that the collisions between the marbles contribute to the mixture.

The teaching experiment cases examined in Gurrola's research suggest that, in fact, young children could be faced with the idea of chance by posing to them appropriate situations and questions, so they could focus on features of that idea, such as variation in the occurrence of results and the possibility of different arrangements.

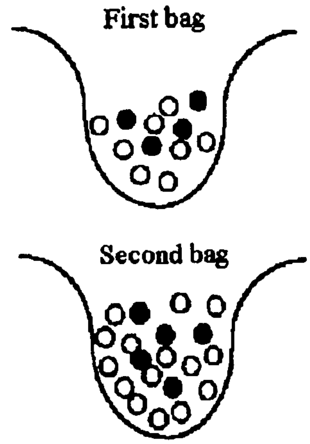
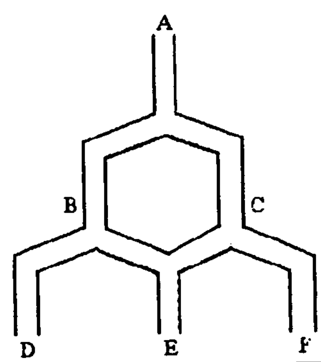
Arithmetical and probabilistic thinking. In order to analyse what children's use of numbers may reveal about their understanding of probability, Perrusquía (1998) aimed at obtaining information from 10-15 year old pupils (5^{th} and 6^{th} grades of primary school, 1^{st} and 2^{nd} grades of secondary school) when they assign fractional numbers to the likelihood of an event. After Kieren's work (1988), understanding rational numbers is shown when these numbers are used according to the different interpretations for which they are meant.

This qualitative oriented study had two phases. The first one consisted of a general inquiry involving 145 pupils who answered a questionnaire. The second phase was carried out with eight children chosen from the sample in the first phase.

The questionnaire used in the first phase was designed in an arithmetic version and in a probabilistic version, each of which included ten problems, two for each of the five basic interpretations of rational number (part-whole, measure, quotient, operator and ratio (Kieren, 1988)). The questions posed in the probabilistic version referred to the same situations posed in the arithmetic version but now involving chance and to number assignation to the outcome given in the answer. These questions required the child to identify events, to express the probability of events from a partition of the sample space, independent events, and to decide which of two random situations offered higher probability for a given event. Examples of the questions posed are shown in Table 1.

Pupils answered the arithmetic version of the questionnaire a week before they dealt with the probabilistic version.

Table I. Examples of questions posed in the two versions of the questionnaire.

Arithmetic version		Probability version
9. Some marbles are contained in two bags: in the first, there are twelve, eight are white and four are black. In the second bag, there are twenty marbles, fifteen are white and five are black. *What fraction of marbles in each bag are black?* _____ *Which bag has the biggest fraction of black marbles?* _________	(Ratio construct) 	*9. Some marbles are contained in two bags: in the first, there are twelve, eight are white and four are black. In the second bag, there are twenty marbles, fifteen are white and five are black.* *Without seeing, I want to draw a black marble. What bag should be chosen?* _________ *According to the bag you chose, what is the fraction of black marbles for that bag?*
10. Balls are let free in this maze. If one hundred and twenty are introduced at A, what fraction of the whole lot would come out through each of the exits D, E and F? *D* _____ *E* _____ *F* _____	(Operator construct) 	*10. Balls are let free in this maze. If one hundred and twenty are introduced at A, what exit, D, E or F do you think more balls will come out?* _________ *What fraction of the whole lot of balls do you think will come out through the exit you chose?* _________

The youngest pupils (27 children in the 5th grade of primary school) obtained the lowest rate of incorrect answers in the probability version (85%) of the ratio construct questions (the most abstract of the constructs we considered). For each of the school grades considered here, there was at least one pupil showing poor performance in fractions and a good performance in probability questions (e.g. Alberto, aged 10, 6.2% and 70.5%, respectively).

For the second phase of this research, the highest rates of correct answers were taken as the reference to choose two pupils from each school grade: one having the highest score in the arithmetic version of the questionnaire (four in all), the other in the probabilistic version (four in all). These eight subjects were interviewed using a protocol of semi-structured questions based on their answers given to the

probabilistic version.

According to the results, the children from the arithmetic group gave answers in the interview without referring to chance. Although their handling of fractions was correct, they showed a deterministic or linear approach to the questions posed. On the other hand, the pupils of the probability group accepted or even proposed the variation in the occurrence of possible events; they did not refer to a pattern to explain the probability they assigned to events. Only the oldest of these pupils (2^{nd} secondary grade) considered equiprobability. As an instance, we present some transcripts for question 10 (in Table 1):

Question 10. Probability group (pp. 127-128)

E *If we get 60 balls in A, which exit is more likely to have more balls come out?*
G *Twenty or thirty?*
E *60.*
G *Then there would be 30* [pointing at D], *saying twenty* [pointing at F] *and 10* [pointing at E]; *or thirty here* [D], *twenty this side* [F] *and ten here* [E]. *It could vary a lot.*

Question 10. Arithmetic group (p. 198)

E *If we get 60 balls in A, what exit is more likely to have more balls coming out?*
An *E*
E *What part of the sixty* [will go out through E]*?*
An *Thirty over sixty* [she writes "E 30/60"].
E *What part of the sixty will be out through the less likely exits?*
An [She writes "D 15/60" and "F 15/60"].

The pupils of the probability group did not show a consistent performance in the arithmetic version. However, in the probability version, three of these four children answered correctly the questions requiring the ratio construct (for instance, question 9 in Table 1):

Question 9. Probability group (pp. 172-173)

E *From what bag is it less likely to draw a black marble?*
D *Black ... from the second ... Cause ... in the first there are four and in the second five; then it increases but also there are fifteen whites in the second, and in the first there are eight; that difference of seven whites reduces chance.*

Question 9. Arithmetic group (pp. 200-201)

E *Let's consider two bags, the first having 6 whites and 12 blacks, and the second one white and 5 blacks. From what bag is it more likely to draw a black marble from?*
An *The first ... Because there are more blacks.*

E .....*If the first has 5 whites and 3 blacks , and the second three whites and 1 black?*
An *The first one ... Because there are more whites.*

The 'probability' pupils interpreted the random situations from a frequential approach more often than the children from the arithmetic group, or quantified by means of percentages.

Remarks

The results obtained suggest the convenience of introducing didactical activities for teaching probability at elementary education by giving priority to a frequential focus as a natural approach pupils have to chance. Preferences and interpretations young children make of the tasks proposed when these involve chance should not be neglected as the aim pursued may not be attained. However, an inquiry in depth is required to know about the results of children's understanding of random situations when the idea of chance is introduced into the classroom environment by means of a didactical activity. Finally, the fact that different basic interpretations of fractions can be required when facing questions about probability, offer an opportunity for the child to give sense to the use of these numbers by focusing on the study of chance.

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An Analysis of Individual Students' Views of Mathematics and its Uses: the Influence of Academic Teaching and Other Social Contexts

Tasos Patronis (Department of Mathematics, University of Patras, Greece)

This paper presents, analyses, interprets three long interviews with Greek students of mathematics and physics, as case studies of individual students' views of mathematics and its uses in various social contexts, with an attempt to investigate the influence of these contexts and in particular the influence of academic teaching on students thought and perspectives. As a framework of analysis of students' views, the interrelation is considered between the notions of a context representation and of a scenario, which leads to a classification of the views presented into the following scenarios: i) a scenario of «passiveness», in which the individual does not essentially act of all; ii) a scenario of «self-objectivization» in which the involvement of the self as a subject of reflection is ignored; and iii) a scenario of social construction of mathematical meaning.

Questions about the nature and applicability of mathematics are rarely discussed in school or in university courses. Students' perceptions of such questions may be considered as an important component of their beliefs about mathematics and science in general, and it is well known that such beliefs affect students' decisions and their mathematical performance as well (see e.g. Garofalo and Lester, 1985; Borasi, 1990).

In accordance with Paul Cobb's remarks on the close relationship between *contexts, goals and beliefs* in learning mathematics (Cobb, 1986), the question here arises, to what extent the students' epistemic views of mathematics and its uses depend: a) on the context of teaching in academic institutions and related ideology; and b) on broader contexts related to various social problems and to the overall goals and intentions of the individuals themselves.

The research of Ruthven and Coe (1994) had more or less similar aims to the present one. In that research a structured questionnaire was used, which made it possible (with the help of factorial analysis) to derive a classification of students' beliefs between certain central factors. However, as the authors themselves say, (their findings suggest that) «there is no simple systematic relationship between beliefs about the nature of mathematical knowledge and activity and about the teaching and learning of mathematics». This suggests that qualitative methods are necessary (in addition to quantitative ones), for investigations in the sociology and/or social psychology of mathematics education, in order that individual students' views are explored and analyzed in some more depth.

LOCATION, METHOD AND OBJECTIVE OF THE RESEARCH

A research with long and minimally structured, but taped interviews has been carried out, in Patras University, of which three individual cases will be presented here.

The compulsory core of content in the Mathematics Department of Patras University consists of 16 courses, 10 of which are traditional courses in Pure Mathematics (Algebra, Analysis and Geometry). These courses, as well as some «advanced» optional courses such as e.g. Topology, Measure Theory and Functional Analysis, begin by a set of axioms and abstract definitions. The so-called «axiomatic foundation» of real numbers gives students a

good taste of formalism from the very beginning of their studies. The situation in the Physics Department is similar in what concerns the teaching of mathematics and mathematical physics.

There are two interviews with students of mathematics and one with a student of physics. These subjects need not be considered as representative of any students' population, but simply as interesting cases of students with particular views and perspectives. It had been decided from the beginning that the research would be restricted to female subjects of about the same age (22-24 years) in order to have some uniformity and facilitate comparative analysis of the responses.

The main objective of these interviews was to perceive in what extent formal teaching in academic institutions as the above departments has an influence on the formation of students' views. Another related objective was to investigate the influence of other (non-institutional) social contexts. Thus, although discussion varied in each individual case, there was a focus in all interviews on two main questions:

- *Do you think that mathematical relations and operations express anything about the real world?*
- *Do you believe that mathematics should be used in building models for social needs?*

In addition, each student was asked about her total experience with mathematics during the school and university period, and about her future plans. In this way supplementary data was collected, which was necessary for putting the subjects' views in a broader perspective in connection to their overall interests.

THE SUBJECTS' STORIES

The method of presentation followed here is similar to that of Byrd (1982), who investigated some deep reasons for students' difficulty with mathematics. In order to understand better the views and representations of the subjects on such «charged» and subjective matters we need first to write in short a «story» for each individual case.

The subjects' names used below are not the real ones.

1. *Hypatia*, graduate of mathematics

Hypatia had graduated just before the interview. Hypatia's story is typical of many young people in Greece who come to study mathematics. Her father had not finished the primary school, and yet he was able to solve practical arithmetical problems that were difficult for her when she was a primary school pupil. Later she used equations in order to solve such problems, while it was always difficult for her to follow her father's way of thinking. When Hypatia entered the secondary school, she asked for a private tutor in mathematics, as she was not feeling «safe», because her parents were not able to help her. With her tutor's help she found school mathematics very easy.

Mathematics taught at the University has «enlarged» Hypatia's views and improved her «inner life». She believes that «freedom in thinking» is possible only in mathematics, which permits the scientist to «create new entities» and find new solutions to problems. But her possibilities for a job are now reduced to private tutoring, since most graduates of

mathematics remain unemployed in Greece. Hypatia is afraid that she could «humiliate» mathematics by connecting it with earning money.

Concerning mathematical relations and operations Hypatia thinks that these may not have a *direct* connection with reality:

> There are many mathematical operations, relations and formulas that directly express reality - and this can be perceived at a first glance. But there are other notions, as for example the negative numbers, which we cannot immediately say «express» something real, although we can think of sense (orientation) as a kind of physical reality expressed by those numbers.

Concerning model building for social needs, Hypatia wonders whether mathematics could help people eliminate classes and bring equality and justice in human society, but the idea of *using* mathematics seems to bother her. Mathematics is not an instrument for a rational organization of society; «rationalism exists within mathematics itself». When she was asked if mathematics would necessarily be useful just because of its rational character, Hypatia reflected a little. «It depends», she said, «on the way it is used; it may be good for society or it may not». In human society there is, according to Hypatia, a coexisting of the *rational* and the *irrational*, which makes doubtful the successful application of mathematical models. Perhaps, the «harmony» existing in mathematics should make possible a «harmonic solution» of social problems without conflict. But it is *in the philosophy of mathematics, not in its logistic or technical part*, that one must search for this spirit; thus «the vulgar, uncultivated men of power», who use mathematical techniques and calculations in their business, have nothing to do with true mathematics, since calculations and techniques «do not represent the nature of mathematics».

But how is it possible that a mathematical model, which is a rational and «harmonic» construction, lead sometimes to absurd conclusions when applied to human beings? Hypatia reflected on this question and after a while replied that

> ...there are unpredictable factors which are very serious, because when we try to comprise many things in large categories, some special cases escape, so that we lose them in generality - especially when we refer to mean values. For example, by using the normal distribution we miss all special interesting cases which are hidden at the extreme parts of the curve. This happens in particular in intelligence tests and in the evaluation of students' performance.

2. *Sophia*, final year student of mathematics

Sophia is the daughter of a shop keeper. Her experiences with mathematics in the primary school were «rather superficial, as is the case with a lot of children». However, the knowledge and abilities she developed in this «mechanistic way», as she said, were useful in her subsequent career and gave her a feeling of confidence. During the Gymnasium (lower secondary school from 13 to 15 years) she decided to be a «good pupil», and her mathematics teachers gave her a kind of psychological support.

At the moment of the interview Sophia considers mathematics in a totally different way. She tries «to get deeper» into everything she reads, and «to grasp the meaning, the applications and connections with reality».

> I try to analyze everything... But sometimes I am not satisfied with my explanations. For example, when I make the correspondence of real

numbers to concrete things, I wonder if this correspondence could be avoided. It is necessary to speak of all these things?

She explains the meaning of numbers and their operations on the basis of human communication.

> When we say to someone «yesterday I saw five men...», this expression gives him an image of what we have in our mind.

She says that in the case of number we have an immediate connection with the real world, while in other cases we have an «internal» act of generalization, which seeks to integrate many things to one in our mind. She mentions as an example of this kind of mathematical abstraction the *addition of equivalence classes*, but she does not seem able to produce a relevant example of such an algebraic construction (as for instance is the case of addition of vectors in the plane). Sophia concludes that the purpose of mathematical science is...

> to enlarge our view, to reduce the situation to a more general level, at which all things can be examined in a unique way and under a general point of view.

Concerning mathematical models in social contexts Sophia says:

> First, we could construct classes... not because I see it like a purely mathematical abstraction, but because one of the functions of human mind is to classify objects in certain categories; for instance the term «man» comprises certain beings with some known properties. When we have to deal with different things, e.g. a man, an animal, a chair etc., then we use another word to express all these things together; we may say «I have these things» or «I have these living organisms» (speaking of men and animals). What I mean by this is that we always have to do with classes of things. Thus mathematics interprets and analyses the way in which our mind functions.

Moreover,

> Generally speaking, the whole world can be considered as a set of classes. Even if primarily we have distinction and differentiation, we always can find similarities between different objects.

3. *Vangelio*, final year student of physics

Vangelio's parents were farmers, who had emigrated for some years to Germany. She went to a rural school but she chose her field of study very early. Possibly the main reason for this was her teacher of physics, from whom she says that she also learned mathematics.

She did not like practical arithmetic in primary school and she never memorized rules for solving problems; she always wanted to understand the task by herself. She never had a private tutor and disliked tricky and senseless exercises. When she entered the secondary school she had difficulties with equations and functions in mathematics courses, because she could not grasp the meaning of these notions from the irrelevant examples of the textbook and she refused to memorize rules and algorithms that she could not understand. Later, at the age of 15-16 years, she learned to solve equations of the first degree in the context of problems of physics:

> I learned from my teacher of physics how to solve equations very simply. I met these things in practical problems about velocity, time etc. Given two quantities how to find a third. I learned to solve first degree equations at the first class of the Lyceum (tenth grade).

Vangelio said that she liked much to read literature, especially novels «with a social perspective». In general she liked to learn, she was curious about scientific questions (especially in physics).

All this changed when Vangelio entered the university. The physics courses seemed senseless to her.

> I come to the first lesson of mechanics and I see only formulas. Derivatives here and there - while we hadn't comprehended the notion of a derivative in the Calculus. I couldn't understand the physical meaning of all these formulas. I left the classroom and I didn't come again.

Vangelio earns her living by private tutoring (like many graduates and final year students of physics and mathematics in Greece). Mathematics is not so interesting for her any more. Here is, also, her impression of mathematical research:

> (In mathematics) one finds some hole and digs in depth. I am not interested in this, I like to have a global image of things.

Vangelio thinks that the relation of mathematics to the real world is not immediate, but is made apparent by the study of natural sciences.

> When we look at the real world we don't immediately think of mathematical formulas with $x.y.$ etc. But... by studying mathematics, physics and other sciences like that, for a long time, one sees that the real world is indeed expressed in mathematical quantities... Sometimes I think of a mathematical relation in connection with real things, as e.g. in the case of the relation $s = v.t$ between velocity, time and distance in uniform motion; this happens in all such cases when simple notions are connected together by definitions.

A concept such as that of *limit* (which is needed for a rigorous definition of velocity)...

> is a human abstraction which corresponds to a real situation. I don't know whether this concept is «physics» by itself, but surely it *corresponds* to that situation. It is like two sets of which is mapped into the other.

Vangelio finds that it is not truly possible to predict human reactions by using a mathematical model, because there are too many parameters that may affect the result in a «chaotic» way. The same stimuli do not produce the same responses, she says, this depends on many other factors. But even if it was possible to predict human behavior by some special mathematical model, Vangelio should not accept such a perspective.

> It is terrible, it is fascism, to predict systematically human behavior by a mathematical model. It is disgusting. I don't agree at any point with such a trend of mathematization. Because, first, this trend is not at all accidental; (these models) are finally used for some people's benefit. If these models could be really applied, then a man of power would be able, by using them, to build the worst dictatorship, the worst totalitarian state against other people.

ANALYSIS AND INTERPRETATION

Context Representations and Scenarios

In this section a theoretical framework is introduced for the analysis and interpretation of the subjects' responses. This framework relates the idea of *context representation* to that of a *scenario*, which was introduced by Peacocke (1989) for the case of representation of a subject's perceptions and primary experiences.

As Evans (1994) says, «a full conception of the *context* needs to take account of the relations of power exercised (...) and the material and institutional resources». However, it is not possible, in our case, to analyze these relations and institutional resources without taking account of the individuals' own goals and perspectives, as these are influenced, regulated or prevented by the context. Thus we are led to introduce the idea of a «scenario» as dual, in some sense, to that of a «context representation». A *scenario* is a potential sequence of mental actions, which in general are viewed as permissible within a given context, according to a particular representation of this context. Conversely, a new *representation* of a given context may emerge through the mental actions of a scenario within this context. It should be emphasized that a scenario is a *potential* sequence of actions, so that two (or more) scenarios are possible of the same person acting within the same context.

A Scenario of «Passiveness»

Hypatia's view can be analyzed and interpreted in terms of the interrelation between scenarios and context representations. The individual stays in a continuous conflict under the influences of two contexts, which here are the context of teaching of pure mathematics in academic institutions, on the one hand, and the context of everyday life and practical needs, on the other. From the moment that Hypatia graduates and must earn her living, she is forced to solve the conflict in favour of the practical part. The only permissible scenario then is to leave the things as they are in both contexts and let everything be done without involving oneself at all. Such a decision can be justified by considering one of the contexts as «sacred» and the other as «evil» and refusing to «humiliate» the «sacred» one by «using» it for the purposes of the «evil».

In what concerns mathematical abstraction and generalization as a process *within* the context of academic institutions, *Sophia* also develops a «passive» scenario. Her representation of this context is similar to that of Hypatia, although her approach seems to be a little more «formal». But when Sophia says that the purpose of mathematics is «... to reduce the situation to a more general level, at which all things can be examined in a unique way and under a general point of view», she does not seem to understand the inner mathematical needs for such a generalization.

A Scenario of «Self-Objectivization»

I borrow the notion of «self-objectivization» from Habermas (1976). The community of researchers and students of a reductionist science cannot perceive itself as the subject of communicative action.

Sophia's view differs from that the *Hypatia* in two main aspects: First, although Sophia partly enters into a «passive» scenario, she does not identify mathematics with «freedom in thinking», as Hypatia does, but she reduces it to the act of generalization. Second, Sophia is much more specific than Hypatia is discussing the current meaning of numbers in everyday life, when she explains this meaning on the basis of «human communication».

By reducing the world to a set of classes and by identifying mathematical activity as generalization and classification of objects, which is «one of the functions of human mind», Sophia cannot perceive herself and other persons as acting within a social context.

Sophia does not separate mathematics from human and social reality. Mathematics can shape reality, but its function is like that of a «black box». She thinks of applying mathematics in this context by using *classes* for the description of similar things, and she considers the acts of classification, categorization and identification as natural functions of the human mind, thus objectivizing these acts in a similar way to that of early materialist (reductionist) philosophy.

A Scenario of Social Construction of Mathematical Meaning

In the last scenario which will be discussed in this paper, *Vangelio* operates in a reflective and constructive way on mathematics taught at school (and partly at the university). She consciously rejects formalism as a style of presentation and searches for the physical meaning of mathematics taught. The general education and social experience of Vangelio are part of her scenario.

Mathematical concepts are considered as «rooted» in a kind of physical «ground». For example, the meaning of the concept of *limit* in calculus is constructed by the human mind on the ground of the physics of motion and corresponds to *velocity*. While every mathematical concept is understood in this way (i.e. by a projection into a physical or social context or domain), the converse is not true, since not all of physical and social reality can or should be interpreted mathematically.

Compared with the previous scenarios, Vangelio's scenario is a non-passive and a non-reductionist one. Contrary to idealism, for which mathematics is exploring a reality limited into the mental sphere, this scenario develops an understanding of mathematical concepts outside the sphere of ideal mathematical abstractions. Furthermore there is a rupture with some prevailing context representations such as objectivization of mathematical activity and rationalization of human society according to science and mathematics.

CONCLUDING REMARKS

The present study shows that some students are exposed to a strong influence of the context of academic teaching of (pure) mathematics, while at the same time there is a lack of deeper understanding. *Sophia*'s scenario, in particular, illustrates the fact that

> *any de-personalisation of the mathematical process and reification of the product pushes mathematics back into the absolutist position by objectivising the 'truths'.*
>
> (Burton, 1995, p. 284)

Moreover, it shows that such a de-personalisation and de-contextualisation restricts the opportunities of the learners for critical reflection upon the present situation of *teaching* mathematics in university institutions.

There is a deep assumption about mathematical knowledge, which more or less penetrates the three scenarios presented in this paper, namely that mathematics has always to do with stable and predictable facts or events, while human society and human life belong to the realm of unstable and unpredictable. The teaching of mathematics - at least in Greece - has done nothing to challenge this assumption, and new mathematical areas such as non-linear dynamics, complexity theory and fuzzy sets and systems remain out of the university syllabus, with very few exceptions.

Even focusing on «real» problems and applications to physics and other sciences does not seem to be a radical alternative to de-contextualised teaching of mathematics, since the picture of the main stream of mathematical thought still remains distant and uninteresting to the individual (in Vangelio's words previously quoted in this study, mathematical research seems like «digging a hole in depth»). Accordingly, new links need to be established, *within the process of education*, between mathematical knowledge and other contexts of social and political activity, which will enable the learners to integrate mathematics among their own perspectives and interests.

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TO CHECK OR NOT TO CHECK?

(DOES TEACHING A CHECKING METHOD REDUCE THE INCIDENCE OF THE MULTIPLICATIVE REVERSAL ERROR?)

Duncan Pawley, University of NSW, Australia

Abstract

This study investigated the hypothesis that including a method of checking the answer in worked examples teaching translation from sentence to equation imposes excessive cognitive load, and thus interferes with learning. A measure of cognitive load (subjective) was used, finding that the level of difficulty of learning the materials was perceived as significantly higher by the "checking" groups than by the "no checking" groups. A third teaching method was then tested, introducing the checking method in the third week of the study, having used the "no checking" worked examples for the first two weeks. Groups using this "separate introduction of checking" method had significantly higher mean scores than both "checking" groups and "no-checking" groups.

Introduction

Bloedy-Vinner (1994, 1995 and 1996) identified defective knowledge of algebraic language as a cause of reversal errors. Following her work, results have been presented (Pawley and Cooper, 1997) showing that the use of worked examples emphasizing understanding of algebraic language had been effective in teaching translation from sentences to equations. This effectiveness was seen in the significant reduction in reversal errors made by students studying worked examples as compared with a control group using the "conventional" method.

This research had involved three experimental groups, all using worked examples, as well as the control group. In one of the experimental groups no checking method was included in the worked examples ("no-checking" group), in one the worked examples included a method of checking by comparison of quantities (Wollman, 1983), and in the third the worked examples included a method of checking by substitution of numbers for the variables.

The checking methods had been incorporated on the basis of previous research showing the importance of conscious checking in this kind of question. Davis (1980, p 192) suggested that an instructional program should make sure that the students "are aware of the likelihood of an incorrect choice, and form the habit of checking to see if they have in fact chosen correctly". Wollman (1983) notes that "experienced individuals consciously check their results" (p.170), and concludes that the inclusion of a check that the equation produced is correct is the "crucial step from a pedagogical point of view". These findings led to the expectation that in the test the groups studying worked examples which included checking methods would have higher mean scores than those studying worked examples which did not include

checking methods. This expectation was not supported. The conclusion of the study was "Taken singly, only the 'no-checking' method produced a significantly better result than the 'conventional' method" (Pawley and Cooper, 1997, p. 326). When the results were analyzed separately for grade 8 and grade 9, further divided into higher ability and lower ability levels, the only significant difference between any of the "checking" groups and the corresponding "no-checking" group was in favour of the "no-checking" group. The mean of the "no-checking" group was in most cases higher than that of either the "comparison" or the "substitution" group. (13 of 16 contrasts). (Pawley and Cooper, 1997, p 324, table 2)

The suggested reason for the lower mean scores of the "checking" groups was the imposition of a higher cognitive load by the inclusion of the checking method.

The concept of cognitive load is based on the fact that we humans have a limited working memory, though a very large long-term memory. We can only hold in working memory approximately seven elements. (Miller, 1956). These elements may be numbers, rules, mathematical operators, logical connections between other elements, etc. Cognitive load refers to the demands made on this limited working memory by the cognitive processing in progress.(Sweller, 1988). Cognitive capacity is required for learning, i.e., tasks such as recognizing a rule or pattern, formulating information into a schema, and storing it in long term memory. Overloading the cognitive capacity by requiring attention to too many elements at one time will prevent this learning taking place.

Thus the suggestion was that inclusion of the steps of the checking method in the worked examples increased the number of elements required to be attended to beyond the level at which the students were able to process the information. This seemed to be the only explanation of why teaching one group of students to check would actually make them do worse than another group of students of matched ability who were given identical instruction with the exception that they were not taught to check. If this explanation is correct, then teaching the checking method after the schema for translation had been acquired could not detract from students' performance. It would either make no difference to students' scores (if checking was unnecessary with the greater level of understanding obtained through the use of worked examples) or improve students' scores (if checking was still of importance).

Research Questions

The present study investigated two questions. The first was whether including the checking method in the worked examples imposed excessive cognitive load, hindering acquisition of the schema for translation. The second was whether the introduction of the checking method after the schema for translation had been better acquired would result improved performance.

Thus the two hypotheses were:

1. The perceived cognitive load for the "no-checking" group will be significantly less than for the "checking" group.
2. The mean score of the group taught the checking method subsequent to the translation schema (the "separate-checking" group) will be significantly higher than those of the "no-checking" and the "checking" groups.

Method

The subjects were all the students in grade 8 and grade 9 at a secondary school in Sydney. Each grade was divided into three matched ability groups, based on mathematics tests in the previous year. Each group was further divided into "higher" and "lower" ability levels.

All the groups had three acquisition sessions. The introductory material for all groups was identical. The worked examples studied by all groups in each acquisition session were identical except for the inclusion of the checking method in those studied by the appropriate groups.

The first of the matched ability groups was the "no-checking" group. This group used worked examples with no checking method for all three acquisition sessions.

The second of the matched ability groups was the "checking" group. This group used worked examples incorporating the "comparison" checking method for all three acquisition sessions.

The third of the matched ability groups was the "separate-checking" group, which for two sessions used materials the same as those used by the "no-checking" group, and in the third session materials the same as those used by the "checking" group in its third session. This allowed two sessions for acquiring the translation schema without a checking method.

To measure the cognitive load imposed in learning this material by studying the worked examples, a subjective measure of mental effort was used. This approach was taken following use of the method in experiments by Paas, (1992) and Paas and Van Merrienboer, (1994). They argued for the reliability of subjective measures of cognitive load (Paas, 1992, p. 429) on the basis of the high correlation between such subjective measures and other (objective) measures of cognitive load obtained by previous researchers. For example, Bratfisch, Borg and Dornic (1972) obtained a Spearman rank order correlation of 0.9 between objective and subjective measures of task difficulty. The method used in this study was that at the conclusion of the first acquisition session the students were asked to indicate, on a seven point scale, "how hard or easy this material was to learn", by putting a cross in the appropriate box. The scale ranged from "extremely easy" to "extremely hard". The responses were scored by giving a value of 1 to "extremely easy" and so on, up to a value of 7 for "extremely hard". One group of students had thus spent one session studying worked examples incorporating the checking method, and the two other groups had spent one

session studying worked examples with no checking method. The means of the responses for the "no-checking" group and the "checking" group were compared, separately for each grade and ability level, using ANOVA contrasts.

An immediate test was administered to all groups at the end of the third acquisition session. After two weeks, a delayed test was administered. These tests each consisted of two questions on equation formation. For both the immediate test and the delayed test, for each of grade 8 and grade 9 separately, the mean score of the "separate-checking" group was compared with the mean score of the "no-checking" group and also with the mean score of the "checking" group, using the Student's t-test. In addition, comparisons were made using ANOVA contrast tests between the mean score of the "separate-checking" group and the combined mean of the "no-checking" group and the "checking" group. This was done for both the immediate test and the delayed test, for each of the whole of grade 8, the whole of grade 9, and both the higher and lower ability levels of grade 8 and grade 9 separately.

One-tailed tests were used in all comparisons, since both hypotheses were directional.

Results

Table 1 shows the means of the responses for the "checking" and "no-checking" groups to the questions on difficulty of the materials, separately by grade and ability level. Table 2 shows the results of contrasts between the means of the different group/ability level combinations for grade 8 and for grade 9.

Grade	Group	Level	N	Mean	SD
8	No-checking	High	31	2.68	1.38
		Low	33	3.48	1.70
	Checking	High	16	3.50	1.10
		Low	16	3.63	1.45
9	No-checking	High	28	2.71	1.01
		Low	29	3.38	1.18
	Checking	High	15	3.27	1.10
		Low	14	3.50	1.16

Table 1: Mean scores for Perceived Difficulty (on 7 point scale).

For grade 8 at the higher ability level the "no-checking" group found the task significantly less difficult than did the "checking" group, ($p = 0.016$ on a one-tailed test). For grade 9 at the higher ability level this difference is not significant, but it is approaching significance ($p = 0.062$ on a one-tailed test) (Table 2). There is no significant difference between the "no-checking" group and the "checking" group at the lower ability level for either grade 8 or grade 9.

For both grade 8 and grade 9 the higher ability level "no-checking" group found the task significantly less difficult than did the lower ability level. (Grade 8, $p = 0.020$, Grade 9, $p = 0.013$ on a one-tailed test) (Table 2). However, in the case of the "checking" group there is no significant difference between the higher and lower ability levels in either grade.

Grade	Contrast	Value of Contrast	Value of t	p value (1 tailed)
8	High Ability, No-check vs. Check	-0.82	-2.230	0.016
	Low Ability, No-check vs. Check	-0.14	-0.299	0.383
	No-check, High vs. Low ability	-0.81	-2.096	0.020
	Check, High vs. Low ability	-0.13	-0.275	0.393
9	High Ability, No-check vs. Check	-0.55	-1.556	0 062
	Low Ability, No-check vs. Check	-0.12	-0.334	0.370
	No-check, High vs. Low ability	-0.67	-2.262	0.013
	Check, High vs. Low ability	-0.23	-0.566	0.287

Table 2: Results of ANOVA contrast tests, Comparing Scores for Perceived Difficulty.

Table 3 shows the mean scores of the different treatment groups in both immediate test and delayed test for grade 8 and grade 9, both ability levels combined and higher and lower ability levels separately. Table 4 shows the results of t-tests for comparisons between these means. (**Note:** In this paper the results of contrasts are reported separately for the higher and lower ability levels only where one of these contrasts is significant and the other is not. Otherwise the contrasts are reported for the grade as a whole, meaning both higher and lower levels are significant or both not significant).

For grade 9, the mean of the "separate checking" group was significantly greater than the mean of the "no-checking" group and that of the "checking" group, in both the immediate test and the delayed test. The mean score of the "separate-checking" group was therefore significantly higher than the mean of the "no-checking" and "checking" groups combined ($p = 0.000$ for immediate test, $p = 0.000$ for delayed test). (See Table 5 for results of ANOVA contrasts between these means).

However, for grade 8 none of these differences was significant. Rather than the "separate-checking" group performing better than the other groups, it was the worst (note the signs of all contrasts for grade 8 are opposite to those for grade 9). For the higher ability level, the "no-checking" group performed the best, though not significantly. The contrast for grade 8 higher ability level between the "no-checking" group and the mean of the "checking" and "separate-checking" groups had $p = 0.089$ on a 2-tailed test.

Grade	Treatment	Level	Immediate Test			Delayed Test		
			N	Mean	SD	N	Mean	SD
8	No-checking	All	30	1.63	0.76	31	1.65	0.66
		High	16	1.94	0.25	17	1.71	0.69
		Low	14	1.29	0.99	14	1.57	0.65
	Checking	All	31	1.61	0.72	34	1.47	0.83
		High	16	1.63	0.81	17	1.71	0.59
		Low	15	1.60	0.63	17	1.24	0.97
	Separate-checking	All	32	1.41	0.87	31	1.42	0.85
		High	14	1.71	0.73	14	1.64	0.63
		Low	18	1.17	0.92	17	1.24	0.97
	Total		93	1.55	0.79	96	1.51	0.78
9	No-checking	All	30	1.23	0.90	28	1.11	0.96
		High	16	1.56	0.81	15	1.47	0.92
		Low	14	0.86	0.86	13	0.69	0.85
	Checking	All	30	1.37	0.81	27	1.41	0.84
		High	16	1.25	0.86	15	1.67	0.72
		Low	14	1.50	0.76	12	1.08	0.90
	Separate-checking	All	30	1.87	0.35	27	1.74	0.45
		High	14	1.86	0.36	13	1.92	0.28
		Low	16	1.88	0.34	14	1.57	0.51
	Total		90	1.49	0.77	82	1.41	0.82

Table 3: Mean scores for Immediate Test and Delayed Test, for Grade 8 and Grade 9, different treatments and levels.

Grade	Contrast	Immediate Test			Delayed Test		
		Mean diff.	Value of t	p value (1 tailed)	Mean diff.	Value of t	p value (1 tailed)
8	No-check vs. Sep-check	0.23	1.090	0.140	0.23	1.170	0.124
	Check vs. Sep-check	0.21	1.028	0.154	0.05	0.247	0.403
9	No-check vs. Sep-check	-0.63	-3.606	0.000	-0.63	-3.167	0.002
	Check vs. Sep-check	-0.50	-3.114	0.002	-0.33	-1.814	0.039

Table 4: Results of t-tests, comparing mean scores for different treatments, for Grade 8 and Grade 9.

Grade	Test	Value of Contrast	Value of t	p-value (1-tailed)
8	Immediate	-0.17	-1.026	0.154
	Delayed	-0.12	-0.675	0.251
9	Immediate	0.57	4.602	0.000
	Delayed	0.52	3.714	0.000

Table 5: ANOVA Contrasts- Mean score of Separate-checking Group vs. Average of Means of No-checking and Checking Groups, for Grades 8 & 9.

Conclusions

Hypothesis 1 was supported for grade 8 higher ability level; the perceived cognitive load for the "no-checking" group was significantly less than for the "checking" group. For grade 9 higher ability level there was a strong trend in the expected direction, the difference falling just short of significance. This smaller difference for grade 9 than for grade 8 is due to the fact that the mean result for perceived level of difficulty for grade 9 was slightly lower than that for grade 8. That is, grade 9 found it slightly easier to learn from the "checking" worked examples than did grade 8, which is what we would expect.

The fact that there is no significant difference between the "no-checking" group and the "checking" group at the lower ability level for either grade 8 or grade 9 means that the lower ability students found the "no-checking" materials almost as hard as the "checking" materials. These students are having enough trouble still in trying to acquire the schema for translation; the extra steps of the checking method don't make it much harder for them than it is already. Note that this test for cognitive load was after one acquisition session. Previous studies have shown that three sessions are required for lower ability students to acquire the schema for translation, (Pawley, 1997), which is why three sessions were used for all groups in this study.

The fact that there is no significant difference between the higher and lower ability levels of the "checking" group means that the higher ability students found the material just as difficult as did the lower ability students.

Hypothesis 2 was supported for grade 9, the mean of the "separate checking" group being significantly greater than the mean of the "no-checking" group and that of the "checking" group, in both the immediate test and the delayed test.

That this was not the case for grade 8 may again be due to insufficient time (only 2 sessions) for the "separate checking" group to acquire the schema for translation before introduction of the checking method. The "no-checking" group had certainly acquired the schema by the end of the third session, with a mean of 1.94 out of 2 for the grade 8 higher ability level "no-checking" group. It appears that grade 8 required

the three sessions on the translation schema before introduction of the checking method.

In summary, there was support for both hypotheses, the first at the grade 8 level and the second at the grade 9 level. In both cases the results suggest explanations in terms of acquisition time required as to why the expected results were obtained by one grade but not the other.

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AN EXPLORATION OF THE INTERFACE BETWEEN VAN HIELE'S LEVELS 1 AND 2: INITIAL FINDINGS

John Pegg and Penelope Baker

Centre for Cognition Research in Learning and Teaching

University of New England

This paper is concerned with identifying a structure in the responses of students as they move from thinking at van Hiele's Level 1 to Level 2. This period in the development process is one in which students move from using global or overall features of a figure to having properties describe a figure. This development is important for students and marks the beginning of formal geometrical thinking. Three significant categories have been identified in this transition. First, specific features are identified. Second, attempts are made to qualify or quantify these features and, finally, the features take on an individual significance although they are not seen as determining the figure. To explicate the findings, a number of samples of student interviews are provided.

Introduction

The van Hiele theory (1986) continues to act as a catalyst for research into students' geometrical thinking. This is evident in the continuing numbers of papers, theses and book chapters which draw upon various aspects of the theory. Nevertheless, despite considerable empirical support (Clements & Battista, 1992) for various aspects of the theory there have been a number of sustained criticisms (Pegg & Davey, 1998). It appears that this conflict has resulted in some researchers rejecting the theory out of hand while others have attempted to move forward by taking specific issues and extending the theory in ways that maintain the original formulations. Four examples illustrate this latter approach. These are: the consideration of van Hiele's ideas within the SOLO model (Biggs & Collis, 1991; Pegg & Davey, 1998); the widening of the level descriptors to incorporate the original ideas but to allow for inclusion of coding of questions more typical of those currently asked in secondary schools (Pegg & Currie, 1998); deeper explorations of the nature of the levels and the properties associated with the level framework (Fuys, Geddes, & Tischler, 1985; Burger & Shaughnessy, 1986); and dividing van Hiele's Level 2 into two levels referred to as Level 2A and 2B (Pegg, 1997). It is the last of these issues which is relevant to this paper and provides the impetus for the research reported.

The splitting of van Hiele's Level 2 (in which figures are identified in terms of properties which are seen to be independent of one another) into two levels, Level 2A (in which students work with one property) and Level 2B (in which students work with a number of properties independently) is more than semantics. It represents an important psychological shift which is supported by empirical data which sees many younger students unable to move beyond applying a single property or 'concept' at a time, due to working memory constraints. It is only after some (considerable) time that students are able to simultaneously apply in sequence several properties.

The identification of Level 2A to student thinking is important. Its strategic position between intuitive/visual thinking at Level 1, where invariant features of a figure take no overt part in problem-solving activities, and Level 2 thinking, in which properties determine a figure, provides a

context for several research questions. Significantly for this paper is the question: Is there some identifiable cognitive structure to students' development of individual properties as students move from Level 1 to Level 2A? It is this question that is at the heart of this study.

Design

The study aimed to explore young students' responses to a series of questions concerning properties of figures on two occasions approximately 12 months apart. The sample comprised of twenty students. At the time of the first intervention 12 were nine years old, and 8 were twelve years old. The students were selected from two schools across each age group with equal numbers of males and females. The selection represents the bottom, middle and top 20% of the age cohort based upon their general mathematics performances on class tests. The nature of the intervention was such as to provide the students with a number of problem-solving contexts about which they could undertake an activity and then talk about the reasons for their actions. This technique seemed to be enjoyed by the students and the interviewer was free to probe for further information as was deemed important. However, care was exercised in asking additional questions so as not to prejudice further activities asked of the students. As was expected the twelve-month delay in re-testing was sufficient for the students to have no recall of the items or their earlier approaches. Due to space restrictions this paper reports on only one question and takes the interview data from the first intervention. A report on the second intervention and its relationship to the first for each child is currently under preparation. Significantly, the results of the second stage provide confirmatory support for the pattern suggested in this paper.

The task reported below was similar to that given to an older cohort of students and reported previously (Currie & Pegg, 1998). It was to have the students group and justify their groupings among seven different triangles, namely, equilateral, acute isosceles, obtuse isosceles, right isosceles, acute scalene, obtuse scalene, and right scalene. In this case, because of the young age and lack of geometrical experience of the students they were provided with a diagram of the seven triangles, each on a separate card, and were asked to sort the triangles into smaller groups. They were not given the names of the triangles and there was no attempt to have them provide names.

While the diagrams were designed to depict each type of triangle, they included no markings or labelling to indicate equality of sides or angle size. Individual cards for each type of triangle enabled the student to alter the orientation of the triangle when and where necessary. The student triangle groupings were used as a catalyst for discussion, involving prompts and probes, concerning the justification of the groupings. The investigation required the analysis of responses concerning the particular triangle groups formed, and the reasons for the identified groups.

Results

Of the twenty students interviewed four responded at Level 1 and three responded at Level 2A. For completeness, selected excerpts of their interviews are provided to indicate the boundaries of the groups of responses. No student in the sample tested was able to respond at Level 2B.

Level 1 responses

There were four students (all aged 9 years) whose responses were identified in this category. While the focus of attention was different in each case there was a clear attempt by each student to use aspects of the global appearance when grouping the triangles. These ideas ranged from using: fat and skinny by Emily; the position of the apex and how this position distorts triangles from some ideal template held by the student as in the cases of Patrick and David: and, to picking some important visual indicator of the triangle such as sharpness indicated in Joanna's response.

Int: Tell me about this group. (acute isosceles, right scalene and acute scalene)
Emily: Well they are sort of the long skinny type of triangle.
Int: And this group? (obtuse isosceles, obtuse scalene, right isosceles and equilateral)
Emily: They are the fatter triangles.

Int: Right so we have got the seven triangles. Why is that one on its own? (right scalene)
Patrick: Because it is not sort of like a triangle shape it is sort of like it is bent.
Int: And this group? (obtuse scalene, obtuse isosceles and acute scalene)
Patrick: This is the normal triangle except it is sort of like the point is not in the middle. This should be there like that.
Patrick: These ones here are normal triangles. (acute isosceles and equilateral)
Int: Can you tell me about the normal triangles?
Patrick: They're middle of the um, the point is facing the middle of the line.

Int: Tell me about this first group. (acute isosceles, obtuse isosceles and equilateral)
David: Oh, they are just straight triangles.
Int: What does that mean?
David: It just means that they um (pause) are just plain triangles without any changes to it.
David: Um these are triangles (scalene and right isosceles) that are pointing to the right hand side or the left.

Int: What is this group? (acute isosceles and equilateral)
Joanna: Those triangles are both going up in the same direction so I put them together.
Int: This group? (obtuse isosceles, right isosceles and acute scalene)
Joanna: These two are both pointing up and this one is kind of doing the same.
Int: And your last group? (right scalene and obtuse scalene)
Joanna: These are kind of the same?
Int: What is the same?
Joanna: They go sharp on the corner just here.

Overall, the responses in this category are consistent with findings of other research and the strong visual/intuitive nature of the students' thinking was evident.

Level 2A responses

Three students responded at Level 2A. Each had established three distinct classes of triangles, namely, equilateral, isosceles, and scalene. The decisions for these groupings were based primarily on the number of equal sides of the triangles. Interestingly, neither Sacha (age 12 years) nor Jessica (age 9 years) were able to name specific triangles. Cole (age 12 years), on the other hand had no trouble in using the more general names such as equilateral and scalene, although the word isosceles was not used. His response goes a little further than the other two students in that he mentions, under probing, equality of angles and, more importantly, he is willing to begin to consider links between isosceles and equilateral triangles based on them having similar properties.

Sacha: That group (equilateral) is a triangle and it has got three sides all the same length.
Int: What is it called?

Sacha: A triangle.
Sacha: This group (acute, obtuse, and right isosceles) all has a length at the bottom and its two sides coming to the point are the same.
Int: Your next group? (acute, obtuse, and right scalene)
Sacha: This group has um they are all different lengths. The sides are all different lengths.
Int: Do you think any of these different groups here can link together?
Sacha: No.

Jessica: This is the triangle (equilateral) that has roughly the same length sides.
Int: And this group. (acute, obtuse, and right isosceles)
Jessica: They are the triangles with the same length. The top two on each are the same length.
Int: And this group. (acute, obtuse, and right scalene)
Jessica: All the sides are different lengths.
Int: Do you think any of these groups belong together?
Jessica: These are sort of together because most of their sides are the same length.

Cole: Well they are equilateral ones and they have all three sides the same.
Int: Anything else they all have?
Cole: They have all the angles the same.
Cole: These ones are the scalene ones. They have all sides different and all the angles different.
Cole: These are, I can't think of what they are. They have two sides the same and one different, and two angles the same and one different.
Int: Do you think any of these groups go together?
Cole: No not really, they are all triangles.
Int: No links for any other reasons?
Cole: No.
Int: Do you think I could put these (isosceles) with these?
Cole: No not really, I suppose they do have two sides the same.
Int: Could you put them with that other group?
Cole: No they are separate.

The students responding at this level were able to group the triangles into three mutually exclusive classes. Each class was determined by a specific property. Under prompting the classes remained mutually exclusive although Cole alone indicated a possible link between the equilateral and isosceles triangles based on side properties.

The remaining 13 responses were seen to lie between these two boundary levels. There were three clearly identifiable groups. These groups appeared to have a logical development to them and provide a sequential pathway from Level 1 to Level 2A. For ease they are described as Categories 1 to 3.

Category 1

Only one student's, Hannah (age 9 years), response was identified in this category. Although there were examples to be found elsewhere, in these other cases, they were followed or accompanied by a more detailed response. The main characteristic of this response was a focus on the number of sides. In addition, there was much in common with the responses of the category described previously, i.e., the use of global imagery to support the description. This is shown in the response of Hannah.

Hannah: Well these (acute isosceles, obtuse isosceles, and equilateral) are like the triangles and these ones (right triangles, and scalene triangles) are like the sloped ones.
Int: Int: Tell me about the triangle group.
Hannah: Well they have three sides and a big triangle face there and that one goes up a long way.

Int: What do they all have in common so that they can go together?
Hannah: Well they all go up and they have this long bit there.
Int: Why do these ones go together?
Hannah: Well they go on a slope and I don't know what they are called but they have three sides like a triangle but they don't all go up to that point up the top.

Overall, the response in this category moved away from a sole reliance on a global representation of the figures, as would be expected of a Level 1 response, and started to focus on the number of sides. No attempt was made to consider the length of the sides in any way.

Category 2

This category was a logical extension of the previous response and six student's responses were identified. Four of the students, all aged 9 years, continued the focus on three sides, and, in some cases, three angles (which were referred to as corners), but, in addition, they attempted to bring in some notion of the length of the sides or the measure of the angles. However, significantly they were not able to use effectively ideas of equality (or evenness) to group figures consistently. In the case of Abbey, she was able to mention that one side is longer than another but was not able to group the triangles using this idea.

Abbey: They are the sort of triangles. (acute isosceles, obtuse isosceles and equilateral)
Int: What do you mean by that they are triangles?
Abbey: Because they just look all the same except for the width wise.
Int: What is the same?
Abbey: Well they just look the same sort of. They have sort of got three corners like that.
Abbey: They are the kind of weird sort of ones because one side is longer than another and I haven't seen that before.
Int: These two? (acute scalene and obtuse scalene)
Abbey: That one is sort of a weird one if you have it the wrong way there like that and these ones (right isosceles and right scalene) are sort of like part of a normal triangle if you have it like that. If you have them both like that they can go in there own group.
Int: What do you mean?
Abbey: If you have them up that way like that they are a bit like a triangle but they are not.
Int: In which way aren't they?
Abbey: Well because that one slopes that way and these ones go straight up. I could put one in with the weird and one in with the triangles.

Helen was similar but during the conversation concerning her grouping she only noticed the lengths of sides as a consequence of the groups she had formed. However, there was a lack of consistency in her explanations.

Int: So what does this group have in common?
Helen: They won't come straight up into an apex. They won't have even sides like that one. It has one big side and one short side.
Int: So what does this group have with its sides then?
Helen: They have equal sides.
Int: Are you sure? Have a look.
Helen: Um most of them do.
Int: Does that matter if they don't?
Helen: Um.
Int: Is the apex more important?
Helen: Um I am not sure.

Initially, Jack responded using global criteria typical of a Level 1 response but when probed started to focus on the size of the edges (sides).

Jack: That is the group (acute obtuse and right scalene) that are the more sloped and they are bigger.
Int: This group? (right isosceles, acute isosceles and equilateral)
Jack: This is the group with um they are the ones that are exactly the right way.
Int: What do you mean by that?
Jack: Like um um like they have got all the edges are the same size.
Int: And this one? (obtuse isosceles)
Jack: That is just one that has been pushed down.

Finally, Anthony begins by trying to use lengths of sides but is not consistent with the groupings he established. His attempts to describe the length of sides are drawn from specific examples within the group and he is unable to provide a broad generic description which encapsulates all triangles within the groups formed.

Anthony: They are all triangles.
Int: What does this group (acute, obtuse, and right isosceles and equilateral) have that relates them together?
Anthony: Some of them have all the edges the same size.
Int: What about the others?
Anthony: Um the bottom edge is smaller than the top edge.
Int: What about this next group? (right scalene)
Anthony: It is half of a triangle.
Int: And your last group? (obtuse and acute scalene)
Anthony: They are sort of triangles but one side on the edge is longer than the other.
Int: Are any the same on that?
Anthony: Um one is a lot longer.

The remaining two students, Alisha (age 12 years) and Simon (age 9 years), provided very similar responses to that described above in that they were still inconsistent in their justification of their groupings. However, they were able to move beyond the focus of individual figures to begin to talk in more general terms about the groupings they formed. They used examples related to specific figures only as a means to justify their more global decisions. For example, Simon responded:

Simon: These ones are all even sort of thing. (acute and obtuse isosceles and equilateral)
Int: What do you mean by that?
Simon: Um, they all um, they are the same on each side like on this side it is like 4 cm long and on this side it is another 4 cm long. Whereas on this group it is say 6 cm long, and that one is only about 3 cm long. And also on these it is half.
Int: What part of the triangle are you looking at?
Simon: I am looking at the top because the bottoms are all practically the same.

Here Simon has chosen to use specific lengths to justify his groupings even though the answer he came up with was incomplete.

Overall, these six responses in this category, while still relying heavily on visual cues, are characterised by the addition of attempts to use the length of sides to group triangles. However, there is often a vagueness in the students' comparison of lengths as they have difficulty in indicating spontaneously which sides are equal. In addition, the groupings lack stability as the guiding principle is not applied in a consistent manner across all triangles.

Category 3

There were six students who responded in this category. All students except Daniel (age 9 years) were 12 years of age. The most important characteristic of the responses used by this group is that

a single feature, such as equality of sides, is an appropriate criterion upon which to group triangles. As a result, all triangles with, say, two sides equal, such as isosceles and equilateral triangles, can be linked. This does not imply that the students see these properties as defining the figure but that given this feature certain triangles have this characteristic. Key aspects of the responses in this category include: consistency across groups formed in relation to the principle applied by the student; a strong reliance on visual cues; only one property or feature is utilised at any one time and, hence, the groups form spontaneously, as triangles are seen to have the appropriate chosen characteristic. Different grouping are possible depending on the feature or property chosen.

Two examples to illustrate this category are provided by Jason and James, both aged 12 years.

Jason: These (acute, obtuse, and right isosceles and equilateral) have either got um some of their lines are equal length with the others like those two are there.
Int: So they all have some lines of equal length to the others. Any other reason?
Jason: No right angles.
Int: What is this one? (right scalene)
Jason: Well it has a right angle and it could go with this group here because it doesn't have any sides that are equal.
Int: So why is that second group together? (acute, obtuse, and right scalene)
Jason: Because they all have unequal sides.
Int: Does that group have a name?
Jason: It does but I have forgotten it.

Int: Can you tell me why these (acute, obtuse, and right scalene) are all related together?
James: They are all triangles but each three sides of the shape are different lengths.
Int: Anything else?
James: Um nope.
Int: How come your next group are together? (acute and obtuse isosceles)
James: Because only two sides um ... but these are all wonky triangles (acute, obtuse, and right scalene) but these ones are still triangles but they are um.
Int: So what relates these two together?
James: They are still triangles. They have got three sides, and three corners, and that is all. These ones (equilateral and right isosceles) are just the same length, oh no they are not. These ones are almost all of the same length around the triangle.
James: Well if that one is all the sides are different and this one is almost the same, what is this group? (acute and obtuse isosceles)
Int: I just realised that this one is the same as this one. (added right isosceles to acute and obtuse isosceles)
Int: Are you sure?
James: Yep.
Int: Do they look the same?
James: (pause) In this shape there is only one odd side. In these shapes there is one odd side and on there are no odd sides on that shape and these here they are all odd.

Overall, students who respond in this category are able to form consistent groupings based upon the spontaneous identification of the chosen property. However, this contrasts with those responses coded as Level 2A as in this latter case the properties signify particular figures and are not simply a consequence or an associated characteristic of a figure.

Discussion and Conclusion

The sample interviewed provided an insight into the development from Level 1 to Level 2A. Three broad categories of responses were identified. These categories can be described as:

1. a move from a total reliance on global visual aspects to the realisation that there is significance in certain aspects of a figure, such as sides. However, in the early stages this results in students commenting on the number of these features;
2. an attempt to document more than one feature or, alternatively, to try and further clarify a particular feature. However, the attempts were inconclusive with either the student being unable to appropriately articulate the property or unable to move from specific figures and apply the notion across a group of figures; and
3. a successful attempt to group the triangles according to a single property. This grouping is not necessarily unique but it is consistent within the framework adopted by the student. However, the figures chosen within a grouping are simply those which are seen to possess the chosen observable property, hence, there is the possibility of different groupings of triangles being formed depending on the nature of the property selected.

The results of this study have offered an initial framework to describe a developmental path in student cognitive growth in moving from a visual basis for describing figures to one in which properties determine a figure. Due to the exploratory nature of the findings, as much transcript information as possible has been provided. Nevertheless, these results are tentative and require further investigation across different tasks and using other figures. Further research into this aspect of cognitive development is vital. The careful analysis of the process of students' acquisition of properties can offer new insights into teaching during this critical time in a young person's development of geometrical concepts. It is possible that future successes in students' growth will depend on how well a program of instruction addresses this transition phase.

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Pupils in lower secondary school solving open-ended problems in mathematics

Erkki Pehkonen & Jaana Vaulamo
University of Helsinki, Finland

Abstract
In spring 1998, eighteen pupils in the lower secondary school took part in research, the aim of which was to clarify how creatively pupils can solve open-ended problems. As a part of the research interview, two open-ended problems were given to the pupils in pairs to solve. Without any hints, all pupils ended with the same simple (addition-based) solution. After the first hint, only one pair of pupils came to the complete solution, and another pair got the right idea, but was not able to follow it. The rest, seven pairs needed, at least, two hints in order to solve the problem completely. The results bring clearly forth the meaning of creativity for problem solving. Some pupils seem to have weak ability to think creatively, i.e. to generate new ideas and ponder different alternatives.

The purpose of this paper is to clarify how pupils in lower secondary school can solve open-ended problems in mathematics. Especially, the focus will be on the role of creativity in problem solving. The paper pertains as a part to the research project "Teachers' conceptions on open-ended problems" which was financially supported by the Finnish Academy (project #162027).

Theoretical background

Conventional school teaching has been accused that it considers the action and the context where learning happens totally different and neutral concerning the topic to be learned. However, psychological studies show that learning is strongly situation-connected (e.g. Brown & al. 1989). Furthermore, the latest research indicates that learning of facts and learning of procedures seems to happen through different mechanisms (Bereiter & al. 1996).

This points out that in mathematics instruction, pupils should be offered different methods to learn on the one hand conceptual knowledge (as facts) and on the other hand procedural knowledge (such as using facts). Conventional school teaching suits very well for learning of facts, whereas learning of procedural knowledge demands other kind of instruction. One possible solution to the latter case is offered by open learning environments, since within them one can deal with real, existing problems, be active and learn in natural settings (e.g. Blumenfeld & al. 1991).

In search for mathematics instruction which is compactible with the constructivist view of learning (cf. Davis & al. 1990), one promising method seems to be the use of open-ended problems. They offer pupils opportunities to explore on their own, make conjectures, to prove their hypotheses, to communicate with their mates, etc

(cf. the papers in Pehkonen 1995 or 1997b) – thus they form open learning environments.

The idea to teach mathematics via problem solving leads naturally to the use of investigations (e.g. Klaoudatos 1999), i.e. a kind of open-ended problems. The ability to generate new ideas, to change flexible one's view point if one gets stuck in the solving procedure, to check obtained results (looking back) and compare them with the problem statements, and all the time to control that one works within the proper problem framework are essential components of successful problem solving (cf. Mason 1982, Schoenfeld 1985).

In problem solving activities, creativity has an important role (e.g. Silver 1997). But in the literature, there is no single definition for creativity that is generally accepted in research (e.g. Haylock 1987, 1997). The most common components of the definitions for creativity are divergent thinking and flexibility. For example, Haylock put forward that "creativity is the capacity to get ideas, especially original, inventive and novel ideas" (Haylock 1997, p. 68). In solving mathematical problems, an individual needs both logical thinking and creative thinking, and large mathematical knowledge with rigid logic is not alone enough (cf. Pehkonen 1997a, Solvang 1998). Therefore, pupils' possibilities to be successful in problem solving are also strongly dependent on their creative capacity.

The sample and methods

In spring 1998, seven teachers with nine teaching groups and from these 18 selected pupils took part in research, the aim of which was to clarify how creatively pupils can solve open-ended problems, i.e. what is the amount and the quality of ideas they produce during the solving procedure. The research was carried out in two lower secondary schools Apia and Tyry with 13-15 year-old pupils in the city of Valkeakoski in southern Finland, using observations and interviews. From each teaching group, a pair of pupils was chosen at random, in order to be interviewed. Additionally, one mathematics lesson from each teaching group was observed. Furthermore, the teachers' conceptions on open-ended problems were gathered with a questionnaire.

At the end of the interview, two open-ended problems were given to the pupil pairs to solve. The problems used were taken from the paper of Reusser & Stebler (1997, p. 312). The selected problems represent mathematically simple problems in everyday context with multiple answers, and the aim of using them was to find out pupils' creative capacity, not so much their mathematical skills. Since the problems selected were similar in their mathematical and creative level, one may suppose that the results obtained support each other.

Here we will concentrate on the results based on the solving procedure of the first open-ended problem, because of the space restriction. Some parts of the interviews and observations are used as background information.

Data gathering process

In the interview before problem solving, the pupil pairs were asked for their conceptions on open-ended problems. And directly in the beginning of the problem solving phase, they were told that following problems are open-ended.

Here we describe the procedure of solving the first open-ended problem which was as follows:

> *Carl has 5 friends and George has 6 friends. They decided to give a party together and they are asking all their friends. How many guests are coming?*

The pupils were asked to read the problem and to answer it, thinking aloud, if possible, in order their solutions and reasoning can be audiotaped. During the solution procedure, the pupil pairs were given only some hints, if they seemed to need some help. As the first hint, they were asked "Are there any other possibilities?" If the pupils have not yet produced the complete solution, they were given another hint: "What about if the boys have common friends?" As the third and last hint, the following completing question was given: "Could there be still more possibilities?"

During the solving procedure, the interviewer gave no comments on the quality of the answers. Her only intervention was to give the hints in the order described. But after the pupils have finished, after the hints, their solving of the problem, the interviewer worked the complete solution of the problem with the pupil pairs together through.

Two examples of the solution procedure

In the first place, we tried to find out, whether the pupils were satisfied with one answer or whether they found (or tried to find) more answers without a hint or after the hints. Additionally, the quality of ideas was noted. In the problem, pupils had to recognize that Carl and George might have some mutual friends, in which case there could be a varying number of persons attending the party.

As the first example, the solution procedure of a pair of seventh-graders, two boys (from Terry's teaching group, see Table 1) is presented. In the beginning of the problem solving phase of the interview, the boys began eagerly to study the given problem. As soon as they had read the question, one of the boys stated: "*I know already, there will be thirteen guests.*" And the other boy responded convincingly: "*Yes, so there are.*" Then the boys put more precisely that one gets 13, if one adds the guests (11) and the boys (2).

Hint #1: "*Are there any other possibilities?*"

After pondering a while with soft mumble, the first boy continued: "*If they have three common friends, then five plus one will be six, and plus boys.*" After a tangled explanation and common discussions, the boys reached a consensus: "*There will be eight guests.*"

Hint #2 was skipped, since the boys have already reached the idea of common friends.

Hint #3: "*Could there be still more possibilities?*"

After discussing a while, the boys passed as a resolution the earlier answer without any alternatives. "*There will be eight guests.*"

The second example is a pair of eight-graders (from Anne's teaching group, see Table 1). In the case of the eight-graders, there seems to be some scepticism against the question. Anne's pupils expressed openly their surprise after reading the question in the problem. One of them said: "*What ...? One cannot say anything on this. All might be possible, because you are not saying....*" And the other one continued: "*They will ask their all friends, and we don't know how many they are ... or we just know about.*" The pupils pondered also the aspect that the party might be arranged in Carl's or George' home, and then the host is not counted as a guest. After long discussions, the pupil pair concluded in consensus that the number of will be eleven.

Hint #1: "*Are there any other possibilities?*"

After a long-lasting silence, one of them answered first with a silent and unsure voice: "*No*". After a while, the other one repeats the same answer more sure and loud.

Hint #2: "*What about if the boys have common friends?*"

After this hint, they began to ponder together other possibilities for answers. At last, one of them states: "*Then there can be ten or less.*"

Hint #3: "*Could there be still more possibilities?*"

The third hint did not improve the given solution.

Results and discussion

Table 1 gives an overview on the development of the solution procedure in the open-ended problem. The first column gives the grade level. In the second column, one can find the pseudonames of the teachers: A for Apia and T for Tyry. Two of the teachers had two teaching groups: Annette and Teo. The third column indicates the interviewed pupils' last mathematics mark, grouped into three categories: A = marks are average level, A+ = marks are above average level, A- = marks are below average level. Results of all interviewed pupils (grades 7–9; 13–15 year-old pupils) are gathered in the four last columns. A dash in the box means that no new answer was given or that the hint was not offered, since the pupil pair has already reached the complete solution.

grade		marks	1. answer	after 1. hint (other?)	after 2. hint (common friends?)	after 3. hint (other?)
7	Terry's pupils	A	13	8 (3 common friends)	8	8
7	Ann's pupils	A	11	11 plus gatecrashers	6 or 10	5...10
7	Teo's pupils #1	A	11	6 (only guests)	6...11	-
7	Teo's pupils #2	A-	11	2 or 3 (it depends who comes)	10 or less	6...11
8	Anne's pupils	A	11	11	10 or less	-
8	Annette's pupils	A+	11 or 13 (who is a guest?)	6...11 (common friends)	-	-
9	Ted's pupils	A	11	11	6...13	-
9	Alfred's pupils	A	11 or 13 (who is a guest?)	13+ (friends of their friends)	5...13	-
9	Annette's pupils	A-	11	12 or 13 (who is a guest?)	5...10	-

Table 1. Development of the pupils' solutions for the open-ended problem (grades 7–9; 13–15 year-old pupils).

The described problem solving procedure of Terry's pupil pair was typical in the sense that the pupils needed hints to continue in their solving procedure. Characteristic for their thinking was the lack of flexibility – when they got an idea ("three common friends"), they were stuck in it, and were not able to variate the conditions, i.e. to get rid of the arbitrarily set preassumption ("three common friends").

Another example, the solution procedure of Anne's eight-graders showed low level creativity and anxiety in problem solving. Characteristic for the pupil pair was the small amount of speech during the solving procedure and their unwillingness to think aloud (if at all).

In contrast to these two examples, the solution procedure of Annette's eight-graders who were talented was very concise: On seeing the problem, they discussed briefly the question "Who is a guest?" and concluded the answer "*11 or 13*". After the first hint, one of them commented the idea that the boys might have mutual friends, and then they very soon resulted "*all between 6 and 11*".

Without any hints, all pupils ended with the similar simple (addition-based) solution. Some of the pupils discussed the question "Who is a guest?", and therefore, came to two different solutions (11 or 13). After the first hint, four pairs of the pupils began to generate alternative ideas, as gatecrashers or friends of their friends, but only two pairs came without the second hint to the idea of common

friends: Terry's (grade 7) and Annette's (grade 8) pupils. No pair of pupils did give a big variety of alternative ideas. Usually the pupils were satisfied in finding another possibility. All other pairs but one needed, at least, one more hint, in order to solve the problem completely. Two pupil pairs (Terry's and Anne's) were not able to find all the solutions.. Terry's pair developed a good idea ("three common friends"), but were not able to generalize the idea, in order to reach the complete solution. Their looking back stage was insufficient. Anne's pupils were satisfied with an about solution".

In the pupils interviewed, two pupil pairs have mathematics marks below average level (Teo's pair #2 and Annette's ninth-graders). Their solutions were in average on the same level as the other ones. Those pupils having mathematics marks above average level (Annette's eight-graders) were essentially better in solving this problem: With one hint they developed the complete solution.

There seemed to be no difference between grade levels in solving this problem. The result is understandable, since in such problems one does not need mathematical content knowledge but more creativity and thinking skills.

In their ability and capacity to generate new ideas, the pupils differed slightly from each other. Two pupil pairs (Annette's eight-graders and Alfred's pupils) begin to ponder the problem context from the very beginning (before the hints), but their considerations were only around the question: Who is a guest? After the first hint, four pupil pairs generated good ideas (as common friends, gatecrashers, friends of their friends), but only one was successful in solving the problem. All other needed the second (or third) hint. Summarizing, the pupils' ability to generate new ideas is weak, they need some hints in order to begin. But with hints most of them are able to produce enough ideas for solving the problem.

Background information

From the questionnaire and the interviews, we have gathered the teachers' and the pupils' conceptions on open-ended problems. Every teacher defines open-ended problems, using slightly different words, as problem tasks which have more than one solution. Almost each teacher has written in the questionnaire an example of an open-ended problem he has used. The teachers consider the use of open-ended problems as a method to foster pupils' thinking.

Before solving the problems during the interview, the pupils were asked for their knowledge on open-ended problems. Most of the pupils answered that they had no idea of this kind of problems. None of the pupils had seen such kind of problems in their mathematics text books. Only Alfred's pupils told that their teacher usually starts lessons by using similar questions. Also Teo's pupils (both pairs) stated that their teacher often asked questions with more solutions.

In this small sample, there seems to be no relationship between the teacher's use of open-ended problems in class (Teo and Alfred) and the pupils' capability to solve the problems (cf. Table 1).

Conclusion

Pupils seem to think rather schematically, although they were informed on the openness of the problems to be solved. Especially the seventh-graders "rushed" on the problem, in order to produce an answer in a hurry. In the case of the upper grade pupils, one can observe some hesitation (which might be interpreted to be thinking) in the beginning of the problem solving, but the results are mainly similar to those in grade 7. Probably, pupils lacked the skill to ponder between different alternatives, they were satisfied with the first solution which came into their mind. After the hints, almost all pupil pairs were able to solve the problems completely.

The word problems in Finnish mathematics text books are usually closed problems with one fixed answer (usually a "nice" number) and one way of solution. Since everyday problems which pupils will encounter later on in their future are not well-defined closed problems, pupils might have difficulties to transfer their mathematical knowledge. The use of open-ended problems in mathematics class, every now and then, might help them to use their mathematics skills in the future.

The results bring clearly forth the meaning of creativity for problem solving. Some pupils seem to have weak ability to think creatively, i.e. to generate new ideas and ponder different alternatives. Since creativity is a central component in problem solving and since problem solving can be considered as the core of mathematics, more emphases for practicing creativity should be given also in mathematics classes. For example, the teacher may use such open-ended problems in the classroom situation, in order to promote pupils' flexibility of thinking. And for this, the use of a proper set of hints might be helpful.

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GENDER DIFFERENCES IN PRIMARY PUPILS' MATHEMATICAL ARGUMENTATION

Leila Pehkonen
University of Helsinki, Finland
e-mail: Leila.Pehkonen@Helsinki.fi

Abstract: This paper deals with the gender differences in primary pupils' (N=185) mathematical argumentation. The theoretical framework is derived from the idea based on social constructivism that mathematical argumentation is regulated by normative aspects. The methodological approach is qualitative, but analysis is supported by some statistics. The results of this study suggest that there are gender differences in the conclusions and in the quality of argumentation.

Theoretical background

Gender differences is a widely discussed topic in the field of of mathematics education. Studies concerning primary school mathematics have found gender differences for example in the strategy use and in problem solving and calculation. In the first grade girls were more likely to count on fingers or use counters, i.e to use overt strategies; boys were more likely to use retrieval to solve addition and subtraction problems (Carr & Jessup 1997). Boys are found to be better at problem solving and girls at calculation (Marshall 1984) and girls make different errors than boys in problem solving (Marhall & Smith 1987). It has also been found that boys show greater self-confidence in mathematics than girls (Leder 1992). These and many other findings give us reason to explore the field of gender differences more thorougly.

This paper deals with the gender differences in mathematical argumentation. Argumentation in mathematics education has been studied i.a. by Krummheuer (1995). However, we do not know very well whether there are any gender differences concerning the quality of argumentation.

How do primary pupils give arguments in a mathematical situation? Argumentation is understood here as a chain of ideas proceeding from the premises to a conclusion. In a classroom, a common possibility for argumentation is, to show the rationality of one's own action when explaining a solution to a problem.

In the following, the theoretical framework is derived from the idea that mathematical argumentation is regulated by normative aspects (Yackel & Cobb 1996): What kind of arguments are justified, acceptable and valid? Thus the communicative nature of the argumentation is emphasized. It is assumed that the norms are produced in the classroom interaction processes. *Social norms* are general norms which regulate social activity, and they can be applied to any subject

matter. By *sociomathematical norm* Yackel and Cobb refer to the normative aspects of mathematical discussions which are specific to mathematical activity. A sociomathematical norm includes a shared understanding of what constitutes an acceptable and justified mathematical explanation in each community. An example of a classroom's sociomathematical norm is the understanding of what constitutes mathematically elegant or mathematically efficient solutions.

Data collection and analyses

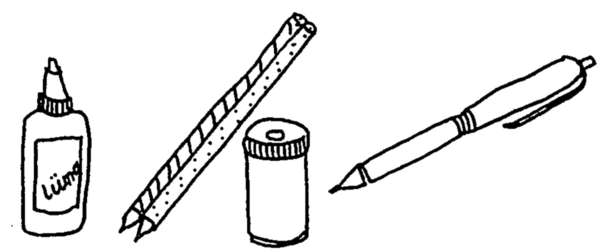
This paper deals with the quality of arguments primary schools pupils gave when they were faced with conflicting or confusing mathematical information. The data was collected by student teachers in nine different schools either in rural areas or in small towns. The total number of pupils was 185 of whom 98 were girls and 87 boys. Half of the pupils were fifth-graders, i.e. 11 years old, but fourth and sixth graders also participated. The following task was posed to pupils:

Pens 7.50

Pencils, 2 in a package 4.50

Glue 35 ml 9.30

Sharper 5.80



Accompanied by the illustration shown here, the following task was given to pupils in one classroom: **Jack buys six pencils. How much must he pay for them?** Arthur says that Jack must pay 27 marks. Lisa thinks that the correct answer is 10 marks and 50 pennies[1]. With which of the children could you agree? Give arguments for your answer.

The pupils were asked to write their answer and argumentation on a sheet of paper. The students teachers read the task aloud as many times as desired to ensure that a pupil's poor reading skills would not prevent him or her from accomplishing the task. The teachers were also asked to make sure that all pupils understood what they were expected to do. In our schools pupils pupils are not familiar with this kind of problems in which none of the suggested answers is correct. The task was mathematically simple, since the mathematic skills were not the main interest. The emphasis was put on the children's reasoning and argumentation process.

A richer and more relevant data could have been collected by recording classroom discussions, but this was not possible in this setting. However, "writing mathematics" is also important. Writing is one possible way to communicate. During the act of writing the writer clarifies and works out his/her thoughts and tries to make them understandable to others (Borasi & Siegel 1994; Shield &

[1] 1 Mark = 100 Pennies

Galbraith 1998). As the data is pupils' written works it is not possible to consider the development of argumentation, but nevertheless, it is possible to see what kind of explanations were used and acceptable in the classrooms' mathematical situations.

Both quantitative and qualitive methods were used in analysing. Statistical analyses were made to calculate the significance of differences. When analysing pupils' arguments I applied the method developed by Stephen Toulmin (1974), in which an argument is seen to consist of different elements. Data provide the starting point on which the conclusion is grounded. This process - from data to conclusion - is legitimised with the help of facts, which are called warrants. The warrants can be supported by some generally known facts, called backing. Krummheuer (1995) also used this method when he investigated the development of argumentation in a classroom.

The whole data (N=185) was analysed qualitatively to seek the patterns in argumentation. First, it was classified by the conclusion and thus three categories (+ blanco category) were formed. Then each of these categories were analysed by the argumentation. All the answers that followed the same-type chain of ideas were classified into the same category. In the beginning there were almost thirty categories, which include inter aliai the categories for work sheets with no answer to the posed question ("With which of the children could you agree?"), without any argumentation and blanco work sheets. In the course of process the categories were gradually reduced. The final analysis include four main categories and eight sub-categories. Each sub-category was described by the terms of Toulmin's model so that the each chain of argumentation was made visible.

Results and typical argumentations

The answers of 50 pupils (27%) were not included into the further analysis, because of the difficulties to follow their reasoning. Half of of them gave no answers at all, some gave answers and arguments without any reasonable logic, and some of students had made so many errors in calculations that it was impossible to them to draw any reasonable conclusions regarding the task. It was not expected that pupils should write their calculation on the work sheet - they were allowed to to do the calculation mentally if the preferred. The main issue was the argumentation. In the course of analysing it turned out that if pupils had made many errors in calculations, in most cases it was impossible to follow their reasonig or at least the intepretations had been very coincidental. So I decided to skip over those pupils. However, minor errors in calculation did not matter. There were exactly as many boys as girls among the skipped pupils. No gender differences were found ($\chi 2$ = 0.11 , df = 1) in this respect.

In the following I will discuss only the responses of those 135 pupils whose reasoning it was possible to follow. Of them 73 (54 %) were girls and 62 (46 %) boys. The task allows three possible conclusions to be arrived. Differences between

boys and girls are significant ($\chi^2 = 6.85$, df=2 $0.05>p>0.02$) in the conclusion. Table 1 shows the division of different conclusions among pupils.

Conclusion	*Girls*	*Boys*
I disagree with both	63% (46)	66% (41)
I agree with Liza	37% (27)	29% (18)
I agree with Arthur	0% (0)	5% (3)
Total	100% (73)	100% (62)

Table 1. The precentages of pupils' different conclusions.

Did the gender matter in the quality of argumentation? The main types of argumentation were divided in four categories which were labeled: 1) mathematical logic, 2) I know, 3) Lisa is nearer, and 4) Arthur is reasonable. First, the differences between argumentation and sex were analysed by $\chi 2$ -test. The result of this analysis suggested significant differences ($\chi 2 = 9.075$, df = 3, $0.05>p>0.02$). Then a more detailed analysis was made by comparing the differences between proportions.

Most pupils – 63% of the girls and 66% of the boys – stated they would disagree with both Lisa and Arthur. There were two main routes to arrive the conclusion "I disagree both Lisa and Arthur". Many pupils - and majority of them were girls - used arguments having the nature of mathematical reasoning: "*I disagree with both of them, because if the price for two pencils is 4,50 then for six pencils Jack must pay 3 x 4,50 (Jack needs 3 packages). So althogether 13,50.*" They started from the mathematical information and facts and proceeded step-by-step. The final conclusion was drawn by combining the data known from the solutions given by Lisa and Arthur with their own mathematical reasoning. 34% of girls and 26% of boys used this kind of mathematical reasoning in their argumentation. This difference is not statistically significant (z=1.01, p=0.08).

The less qualified arguments were based on pupil's own performance. This type of argumentation was more common among boys (40 %) than among girls (29 %) and the difference is significant (z = -1.39, p = 0.04). A typical argumentation was: "*I cannot agree either of them, because I got a different answer.*" Figure 1 displays an example of this kind of argumentation.

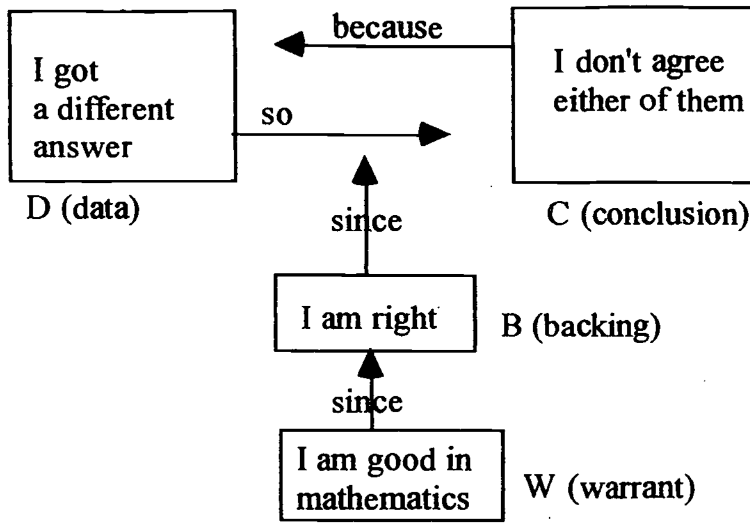


Figure 1. An example of argumentation

The distinctive aspect between the categories "mathematical reasoning" and "I know" was the quality of argumentation. In the first category the arguments had the basis in the mathematical reasoning: The pupils showed how they arrived the conclusion and proved it to be the correct one. It is the kind of reasoning that a mathematician does. In the second one the argumentation is grounded on the authority-based belief. A mathematician would never accept an argumentation: "It must be so, because I have calculated it and I know how to do it:"

32 % (=46 students) wrote they could agree with Lisa. However, majority of them stated that neither Lisa nor Arthur was right, but if they had to choose, they would side with Lisa, because her answer was nearer to the right one. This was the main argument. This conclusion was more common among girls than among boys, but the difference is not statistically significant ($z = 1.01$, $p = 0.08$). However, in this conclusion there were no differences in the quality of argumentation, all the arguments very were similar and so independent of gender. Only a few pupils - and all of them were boys - gave their support to Arthur, and the arguments given by them were very reasonable. They were sympathetic to Arthur and the error he had made: "*He did not notice that there were two pencils in the box. He has calculated the price for six boxes correctly.*"

The normative aspects of argumentation

I distinguished between two sociomathematical norms. The first sociomathematical norm (type i), which is based on the arguments of those students who agreed with Lisa, can be formulated as follows: "*The result which is closest to the right answer,*

is a better one." One third of the pupils used this explanation and regarded it valid and justified. This norm is interesting: Although Lisa's process was not in any way reasonable, pupils - however - though that her answer is better than Arthur's reasonable process but wrong answer. This gives us reason to believe that in classrooms' mathematical discussion products are more valuable than processes. Every pupil who applied this explanation carried out the calculations correctly and stated that, in fact, neither Lisa nor Arthur was right, but they were ready to give their support to Lisa.

The second sociomathematical norm, mathematical reasoning, (type ii) appears in some of those explanations where pupils stated they would disagree with both Lisa and Arthur. These pupils generated their own opinions and their arguments were based in the first place on mathematical reasoning. One third of the pupils applied this norm. This kind of sociomathematical norm gives us reason to believe that mathematical argumentation is practiced in some classrooms.

My hypothesis was that social norms could not be seen in pupils' written products. However this belief turned out to be wrong. I could find three social norms that regulated the nature of argumentation. There were five teaching groups with a great number of pupils who justified their decisions primarily on the basis of their own expertise or the lack of expertise of others (type I). A typical argument was: "*I disagree with both Lisa and Arthur because I got a different answer.*" or " ... *because both of them have calculated incorrectly*" (but in our language the meaning of "incorrectly" is stronger). In these teaching groups a valid argument seems to be the fact that the pupil has arrived at a different solution. Some of them have secured their backing by the fact, obviously well-known in their classroom: "*I am right, since I am good in mathematics.*"

We could say this norm is social, rather than sociomathematical, because it is grounded more on status or supposed expertise than on the efforts to show the mathematical basis. Furthermore, these explanations are often related to efforts to convince the teacher or the reader that the actor is innocent: "*I don't want to agree with either of them because both of them are wrong.*" This kind of explanation has actually nothing to do with mathematics, and it can be applied in any situation to explain the rationality, if it is accepted in the classroom.

The second social norm emerges only connected to the first one. This norm is very gender related. In all those teaching groups where - among boys as well as among girls - the social norm of one's own expertiness or other's inexpertiness was accepted, the norm of unsureness can be identified among some girls. The girls who have very well carried out the calculations and given very elegant mathematical arguments may state in the end: "*Naturally I may have made a mistake too*" or " *Maybe I am wrong*".

The third social norm was encountered in one teaching group. This norm includes the idea that you should understand other people. Regardless of what kind of

mathematical arguments the pupils in this group applied, they tried to understand the actions of both Lisa and Arthur: "*Arthur has made a little mistake, but it could have happened to anyone,*" or "*I don't know how Lisa got her answer, but it must be some kind of annoying and human mistake.*" The norm was independent of gender.

The quality of arguments was not dependent on pupils' age as I have shown in an earlier paper (Pehkonen 1998). Actually there was more variation across different teaching groups than across grades. Similar results have been reported by Yackel and Cobb (1996). The patterns of argumentation and norms could be very similar among the third graders than among the fifth-graders. But there were teaching groups of fifth-graders (or third-graders) each of whom had its own norms and ways to argumentate. And there were groups where it was not possible to distinguish any norm in the written data.

Some teaching groups were very able to give arguments and others were not. It seems obvious that some teachers have trained their pupils to argue and express their opinions. Argumentation can be seen primarily as a discourse technique (Krummheuer 1995, 238) which can be learnt.

Concluding remarks

Boys based their argumentation on their own expertise more often than girls and girls preferred mathematical reasoning more often than boys did. These findings are in line with results of previous research which indicate that girls are better at calculations and algorithms (Armstrong 1981; Marshall 1984). The findings of this study suggest us that girls are more able to use these skills in their argumentation to make them mathematically more elegant. Further research with more representative samples will be needed to comfirm these suggestions.

Gender influence on the normative aspects of argumentation could be seen in some teaching groups. Gender differences emerged in those teaching groups where the social norm was that acceptable mathematical accounts can be based on one's own supposed expertise or on the lack of others' expertise. In these - and only in these - teaching groups there were girls who were willing to suspect their solutions and reasoning - even when they were mathematically elegant and well grounded in logic. These findings support previous reseach which has pointed out gender differences in self-confidence. Boys are more confident (Leder 1992) and tend to dominate situations in class rooms (Carr & Jessup 1997).

The results of this study help us to get a more detailed picture of the nature of gender differences in mathematics education. The next question is what is the influence of teacher who belongs to the social setting of class room? What is the mechanism how teachers influence to the accepted, valid and justificated explanations in mathematic class rooms and so to the norms that emerge?

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OBSTACLES IN APPLYING A MATHEMATICAL MODEL: THE CASE OF THE MULTIPLICATIVE STRUCTURE

Irit Peled - University of Haifa
Leonid Levenberg, Ibby Mekhmandarov, Ruth Meron - Center for Educational Technology, Tel-Aviv
Alexander Ulitsin - Ministry of Education, Jerusalem

Third and fourth grade children, who started learning about multiplication in second grade, do not tend to use multiplication in multiplicative situations. This study looks at children's choice of mathematical models in different non contextual displays: equal groups, rectangular arrays, and a rod model (which they use in class). The findings show that many children do not perceive the displays as representing multiplicative situations. Even children who exhibit knowledge of multiplication facts do not apply their knowledge in these tasks. Instead, they use addition and counting strategies.

An important goal in teaching mathematical operations is the development of schemes, such as the additive scheme or the multiplicative scheme. These schemes, or structures can act as mathematical models of given situations. Usually the situation does not uniquely determine the structure, nor does it easily hint that a certain mathematical model can be used (although conventional school word problems might imply that it does). We teach children basic operations with the intention that they apply them in different situations. We also expect to see progress over time in the choice of a model, i.e. in the ability to apply more efficient models in the relevant cases.

Given a rectangular array of objects with the task of finding their total number, it is expected that a young child will count the objects one by one. An older child will count the elements in each row and then add up the rows. An even older child is expected to realize the size equivalence of the rows, i.e. the repeated addition structure and then add or multiply, or recognize the rectangular array structure, and multiply the number of rows by the number of columns.

The above description talks about different structures, which have been identified by several researchers. A lot of the research on multiplicative structures deals with the categorization of multiplicative situations (Vergnaud, 1983; Schwartz, 1988; Nesher, 1988) and with children's word problem solving (Fischbein et al., 1985; Kouba, 1989; and many other significant works reviewed by Greer, 1992).

Some of these researchers have focused on younger children's work. Carpenter et al. (1993) show that even kindergarten children can solve some types of multiplicative word problems. Kouba (1989) looks at solution strategies of children in grades 1-3, and is interested more in the nature (and quality) of their solutions than in the question whether they can answer a given problem correctly. Kouba uses equivalent set problems (later termed repeated addition problems or mapping rule problems) and finds that children use a variety of counting strategies. She also observes that the intuitive model that children seem to have for equivalent set problems is linked to the intuitive model for addition as both involve building sets and then putting them together.

Similarly, Mitchelmore and Mulligan (1996) show that during their second and third grades children use many different strategies in solving multiplicative problems. These strategies include quite a lot of addition and counting calculations. However, they also find that over time the strategies are chosen more efficiently.

These different research findings suggest that children do not necessarily use multiplication in solving multiplicative word problems. Many children use addition and some choose to use a long counting process.

It is often claimed that children are efficient and use a more effective tool or a shorter route once they possess it. Such a behavior is described by Woods, Resnick, and Groen (1975) in the case of choosing between solution strategies in subtraction (e.g. going two steps backwards in 9-2 while counting up from 7 to 9 in 9-7). This behavior is also evident in Siegler and Shrager strategy choice model (1984), where children choose to retrieve an addition fact rather than count, when they reach a reasonable degree of confidence.

If children tend to be efficient, why do they not use multiplication but instead do quite a lot of counting or adding? Several explanations can be suggested: They are not able to identify the structure of a multiplicative word problem as a multiplicative structure, or they do recognize a multiplicative structure but do not know the relevant multiplication facts. In this work we differentiate between these two obstacles by looking not only at children's solution strategies but also at the way they perceive different situations.

Most of the existing research, including the works described above, involves word problems. In solving word problems children are engaged in text interpretation, a stage which might contribute to the difficulty in identifying the efficient structure (although in some word problem types the verbal description contains clues for identifying a multiplicative structure). In this work we avoid this stage by looking at children's behavior in different non-contextual displays of objects. As we show

below, the use of a multiplicative structure is scarce even in these non-contextual tasks.

PROCEDURE

Fifty four third and fourth grade Israeli children were individually interviewed. The children were chosen from regular classes that use the same curriculum, called "One-two-and three". They were identified by their teacher as having some minor difficulties in mathematics. This curriculum usually introduced beginning ideas about multiplication at the end of first grade, and further develops the concept in second grade. A sequence of concrete models is used in order to represent multiplication: a "train" of equal rods, the same rods in a rectangular configuration, and eventually an array of dots. Using the array model the children discuss the number of rows and the number of dots in each row, and apply the model in different problem solving situations.

The purposes of the interview were: To find what mathematical model children use spontaneously in perceiving a given display, which has a multiplicative structure, to observe which strategy they apply spontaneously in calculating the amount represented in the display, and to find how they calculate multiplication facts in multiplication number problems. Eventually these findings were used to investigate the relation between knowing multiplication facts and recognizing and using multiplication in different situations.

In the course of the interview each child was presented with different displays, asked to describe what she sees, and then requested to find "How many there are". The order was: Equivalent sets, a "train" of equal rods, a rectangular array, and some contextual situations (involving eyes and fingers). Each display was presented several times with different numbers (4x5, 10x5, 4x2, 10x2). Here we present the results for the case of 4x5, (complete results to be presented in an extended article).

The request to describe what they see was intended for investigating how children perceive the display. The child was given a card with a dot configuration and asked to tell the interviewer, who supposedly could not see the card, what to draw.

Several additional tasks included: representing a given expression, such as 4x5, using rods, inventing a story problem to a given expression, and finding some multiplication facts, e.g. 4x5. The questions that mentioned multiplication came only at the end of the interview in order to avoid any hints about the choice of an operation in the different displays.

RESULTS

Children's conceptions were deduced from their display descriptions and explanations during the interview. In several cases a child was considered to be perceiving the display as a multiplicative structure even when an additive or a counting strategy was used in the calculation of the total amount of objects. In the following excerpt a third grader, Lorry, tries to figure out the total amount represented by a "train" consisting of four '5-rod'-s, as shown below:

yellow	yellow	yellow	yellow

I: What do you see?
L: Yellow rods of 5.
I: And what else? (no answer) How many rods?
L: 4.
I: How much is it?
L: Should I do 4 multiplied by 5 [note: In Hebrew 4X5 can be read as '4 multiplied by 5', or as '4 times 5'. Here she used the word 'multiplied'. Further on when she calculates the amount by addition, she uses 'times'].
I: Yes.
L: (Thinks for a while) 25.
I: How did you do it?
L: I did four times five, 5 plus 5 that's 10, and another 5 that's 20, and another 5 that's 25.

Table 1 presents the percentages of students who perceived the different displays of 4X5 as multiplicative situations. It also shows these percentages separately for students who could do a mental calculation of 4X5 (using either fact retrieval or repeated addition), and for those who could not do a mental calculation (e.g. had to use objects). It should be noted that the columns are not disjoint. A child who perceived a multiplicative structure in one display, could also see it in another display.

Table 1: Children exhibiting multiplicative display conceptions.

display / calculation	equal sets yes	equal sets no	rods yes	rods no	array yes	array no
mental n=32	7 (22%)	25 (78%)	5 (16%)	27 (84%)	10 (31%)	22 (69%)
non-mental n=22	0 (0%)	22 (100%)	2 (9%)	20 (91%)	2 (9%)	20 (91%)
all students n=54	7 (13%)	47 (87%)	7 (13%)	47 (87%)	12 (22%)	42 (78%)

The data in Table 2 shows the different calculation strategies in each of the displays for children who could figure out 4x5 mentally. This subgroup of children includes those who could potentially utilize their knowledge in the different displays.

Table 2: Strategy choice in different situations for children who could mentally multiply 4x5.

Strategy / Situation	multiplication	addition	counting	other	total
equal sets	6 (19%)	18 (56%)	6 (19%)	2 (6%)	32 (100%)
rods	4 (13%)	19 (59%)	-	9 (28%)	32 (100%)
rectangular array	8 (25%)	9 (28%)	9 (28%)	6 (19%)	32 (100%)
contextual cases	7 (22%)	20 (63%)	3 (9%)	2 (6%)	32 (100%)

As can be seen in Table 2, less than a quarter of the children who calculated 4x5 mentally, used multiplication in each of the different display calculations. Our data (not presented here) details this distribution separately for children who did the calculation of 4x5 by retrieval, and those who used addition. Most of the children who used multiplication in the display calculation were those who used it in the calculation of 4x5.

Children's answers and explanations contribute some interesting information on the way they perceive the given representations. The following are two of these examples:

1. Post hoc identification of a multiplicative structure:

Given a rectangular array of 4x5 X-s , Ron (a fourth grader) draws it correctly from memory. The task is followed by this dialog:

How many rows are there? 4
How many X-s are there in each row? 5
How many X-s altogether? (Ron thinks for quite a while) 20
How could you tell? I did 4x5.
Did you do it in your head? No. I counted the X-s.
So why did you say 4x5? Because it's 20.
But 2x10 is also 20. (Ron hesitates a moment) Ah! But here (in the array) we have 4 and also 5.
So why did you count earlier rather than do 4x5? Because I was in a hurry...

This dialog might initially suggest that Ron identified the multiplicative structure, or at least the repeated addition structure of the array. However, the time it took him to figure out the total amount, his own account on his counting, and his surprised discovery of the connection to the 4 and 5 lead to a different interpretation. Ron suggested the expression 4x5 after counting 20 X-s in the display. He might have chosen 4x5 because it is an expression that yields 20. It is probably during the discussion that he suddenly saw how the array dimensions related to the expression.

2. A selective application of multiplication:
Some children used addition or even counting for small amounts, while applying multiplication for larger amounts. Other children had different reasons for the choice of a strategy, such as: *I counted [even though I used multiplication in another situation] because I was in a hurry.... [and would have wasted time if I stopped to analyze the situation].*

DISCUSSION

Third and fourth grade children participating in this study showed very little use of multiplication strategies in non-contextual displays, while multiplication was the more efficient strategy. These findings could only partially be attributed to the fact that most of these children did not know the relevant multiplication facts. This was revealed by investigating the way children perceive the displays. Only a small proportion of children perceived the displays as multiplicative structures. Even among those who could figure out a multiplication fact mentally, only about a third identified the multiplicative structures. The conclusion that the blame does not lie in absent knowledge of facts is also manifested by the choice of strategies in figuring the amount in different displays. Less than a quarter of the children who could figure out the facts mentally used multiplication, a large proportion of them used addition, and some even counted.

The children in this study were identified by their teachers as having some difficulties. Yet the displays presented to them were familiar representations, the same ones through which multiplication was defined to them in first and second grades. If fact retrieval is not the main obstacle in applying multiplication, perhaps the difficulties involve the nature of the displays or the nature of instruction.

The identification of a multiplicative structure is quite complex. In equivalent set situations, for example, one has to be able to perceive all the given sets at the same time and recognize their equivalence. In deriving the number expression one of the factors is an intensive quantity, appearing not just in one set but in each of the sets. The second factor is

not even represented as a simple set. Rather, it is the number of the sets in the display. This complexity makes it difficult for children to identify and apply a multiplicative structure in a given situation.

The interviews disclose some of the display difficulties. In additional tasks, where children were asked to use manipulatives and represent an expression, such as 3x4, some of them tended to represent both numbers. Thus, for example, one of the children used rods and built a train consisting of a single 4-rod and three 3-rod -s, as follows:

(4)	(3)	(3)	(3)

When realizing that something was wrong because it does not measure 12, as expected, he changed it to one 3-rod and three 4-rod -s. He was frustrated upon realizing that it still does not measure 12. Another child represented this expression by building an array consisting of four rows, with three 3-rod -s in each row.

In the course of class instruction children are directed to those elements in the display which represent the multiplication factors. If we want them to develop the ability to look at a given display and choose an efficient mathematical model, we need to teach them to analyze situation structures and detect relevant features of these situations. Our instruction should include tasks that give them the opportunity to develop the ability to analyze and apply the mathematical models which are available to them.

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A SCHEMA-BASED MODEL FOR TEACHING PROBLEM SOLVING

George Philippou and Constantinos Christou
Department of Education, University of Cyprus, Nicosia 1678

In this report, we develop a model for teaching mathematical problem solving based on schema knowledge. The model makes extensive use of the graphical representation of problems to enhance students problem-solving schemas. Instruction proceeds through four successive phases: the recognition, interpretation, strategy development, and application. The design extends for about twenty-four teaching periods that spread over ten weeks. The model was field-tested in Grades five and six. The results presented in this report refer to students in Grade 5 and showed significant differences between pre-test and post-test scores in favor of the experimental group. This is an encouraging indication that the model is effective in improving students' problem solving ability.

Introduction and aims

In recent years problem solving has been constantly at the heart of mathematics learning and plays a central role in mathematics education. During the past decades, considerable progress has been made in understanding the mathematical and linguistic structure of several types of word problems and many of the factors that affect their difficulty were specified (De Corte & Verschaffel, 1987; Riley & Greeno, 1988; Nesher and Hershkovitz,, 1994). Research has developed three different types of "instructional models" for teaching word problems. Models of the first type emphasize explicit teaching of the distinct phases of problem solving and some general problem-solving strategies and heuristics (Polya, 1973). A second type of models focuses on the linguistic structure of the problems to make it explicit and less difficult, aiming at facilitating students to classify and connect the new to existing in the memory schemata (Rudnitsky, Etheredge, Freeman, & Gilbert, 1995). A third approach draws attention to problem posing i.e., to the process by which pupils are guided to develop understanding by actively working out given information to synthesize their own problems (NCTM, 1989). These efforts have done little to improve pupils' ability to solve mathematical problems (Lester, 1994); word problems continue to be the cause of frustration to teachers and students alike.

Though schemas knowledge is expected to enhance problem solving ability, Cooper, Baturo, and Dole (1998) found that instead of the expected schema based classification, proficient solvers of percent problems used computational proficiency. Marshall (1995) developed a comprehensive proposal for teaching and assessing problem solving, which is grounded on schema theory and the acquisition of basic schemas by the learners. Our goal was to develop and field-test an instructional approach of problem solving by extending Marshall's ideas to encompass in an integral form the basic characteristics of the three instructional models mentioned above. Furthermore, the proposed model incorporates some of the ideas of concept mapping which have been used successfully in language, arts, and social studies for years, as a way to teach students how to organize information in these subjects.

Theoretical background

From a developmental perspective, children's thinking about problem solving is part of a more general way of knowing. Of particular relevance to this research are the schemas proposed by Marshall (1995) that form a major foundation for problem solving in the primary grades.

"Schema" is a Greek word with multiple meanings most of which derive from Plato and Aristotle. The literal meanings of schema include form, figure, shape and diagram, while the metaphorical meanings refer to a means of organizing characteristics that might be used as frameworks unifying objects of the same structure. Marshall (1995) argues that the two main dimensions of the construct of schema focusing on memory and action might be reconciled. The schema is structured by our experience and it also structures it. Though we do not fully understand how memory works, schema is one means by which related information is retained in human mind. Schemas from the same domain are connected to each other and they are frequently displaced as networks of cohesive units. Thus, schema may be thought of as a storage and consequently retrieval mechanism.

Figure, the literal meaning of schema, is a geometrical representation of information. Figure understanding is an expression of problem thinking and, as such, it is a natural function of the human mind. It is a powerful visual technique, which provides a key to understanding the structure and the linguistic forms of a mathematical problem and helps students to construct understanding of mathematical problems, clarify their thinking, and justify their ideas.

The first steps toward a successful solution of a problem involves understanding the semantics of the problem, producing a mental representation, and relating this structure to existing schemata; a solution plan can only subsequently be formulated. Mayer (1987) breaks down the phase of problem representation into problem translation, which involves converting each sentence into an internal mental representation, and problem integration, which involves combining the information into a coherent unified structure. The final outcome of repeated similar problem experiences is the formation of a relevant mental schema, which subsequently becomes part of the solver's repertoire. Relating a new problem to an existing schema, the solver analyzes the features of a problem by transcribing the main ideas with the purpose to learn through organizing his thoughts. A problem schema could function as a means to describe and classify the elements of a story problem and hence conceptualize the relationships and connections among them. In other words, the problem schema serves as a teaching and learning activity.

The real value of mapping a problem onto a schema lies in the visual representation of the relationships among the elements of a word problem. Via this tool, the linguistic form of the problems is explicitly depicted and is visible to the students constructing the maps as well as to the teachers. A student-constructed

schema is the "hard copy" of his/her understanding of the syntax and relationships of the problem.

The instructional process leading to problem schemata constitutes a practical form of analyzing (conceiving) the particulars of the problem and next connecting (integrating) the parts into a holistic mental representation. Schema construction may be viewed as a natural function of the human mind, it is a process of transforming the script into a more tangible graphic form, which may provide a key to understanding the mathematical structure of the problem. Schema serves as a vehicle for developing problem solving abilities, offers pupils an opportunity to encounter various linguistic forms and situations. It helps pupils to construct deeper understanding of mathematical problems, clarify their thinking, and justify their ideas. It further provides teachers and students a useful tool to monitor progress of students' understanding of the process. The schema enables students to investigate their own solutions, and it allows them to feel successful at the end of the lesson, as long as they have tried valid strategies. Problem schemas may lead to rich discussions among students, as they explore relationships among the pieces of information comprising a problem, clarify misconceptions, and develop understanding in a meaningful way.

An overview of the intervention program.

A number of instructional programs emphasizing problem solving have been developed in recent years (Lester, 1994; Nesher and Hershkovitz,, 1994). For the most part these programs encouraged an active role for both students and teachers, promoted the learning of problem solving strategies, and drew attention to solving problems of different schemas. In addition to these characteristics, the present program had the following characteristics:

- It focused on the schemas and diagrams of arithmetic word problems based on the work reported by Marshall (1995).
- It emphasized extensive experience with word problems by asking students to identify the essential parts of a story, specifying the complete and incomplete elements of the problem.
- It emphasized the development of students' abilities to recognize, apply, select and create a schema.
- It incorporated a specific teaching strategy for problem solving by integrating the diagrammatic representation of a problem.

According to the main targeted abilities the problem solving experiences were organized in four distinct though overlapping phases: The identification or recognition phase, the elaboration or interpretation phase, the strategy planning phase, and the application or execution phase.

Phase 1: The identification of the main elements and features of a problem is the most important knowledge for schema activation; it is this understanding that contributes to the initial recognition of a situation or a story. The structure of each category of word problems involves some essential features that are repeated in all

similar situations and problems. Thus, pattern recognition occurs as a result of the simultaneous cognitive processing of many features (Marshall, 1995). The objective of this phase was to help students develop the basic identification knowledge. The students were expected to learn at this phase that there are four basic problem-situations (change, compare, group, and analogy schemas), and to identify the essential nature of each schema. To this end, during the first few lessons students were introduced to problems that underlie the four basic situations and were asked to discuss the salient features of each situation as well as the relationships among the elements of the problems. After all the situations had been introduced students were encouraged to identify problems and classify them into one of the four schemas.

Phase 2: The elaboration knowledge contains discussion of the main characteristics or features that are essential for the development of each one of the specific schemas. The students explain in their own words the details that are distinct to each schema. Through the elaboration knowledge, the students were expected to create relevant mental models about the problems. The identification and elaboration knowledge constitutes a framework that allows students to form a hypothesis about the schema into which a problem belongs, and tests it.

During the second phase teachers introduced the idea of representing the relationships of problems through a diagram aiming at helping students develop the elaboration knowledge. A schema diagram used in the study for the change problem is shown in Figure 1. Each schema can be represented in its own unique way. Thus, teachers spend about lessons on each schema discussing with students the three necessary components of each diagram. A broad description for each part of the diagram was undertaken so that students perceive what the shapes represent in each schema and what is the function of the arrays (see Marshall, 1995). Several examples were treated in the classroom, to help students understand the various situations that arise according to which part of the problem is the unknown. Both phase 1 and phase 2 dealt with one-step word problems to acquaint students with the structure of each schema

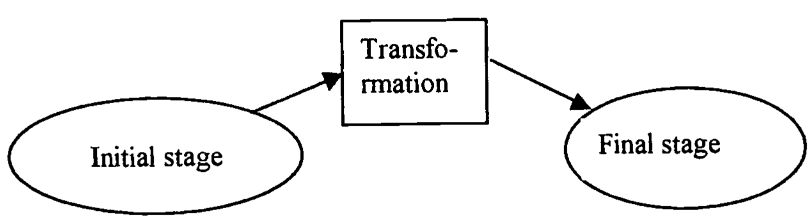

Figure 1: Diagrammatic form of the Change schema

Phase 3: The evaluation of the hypothesis is the result of the process of elaboration. The planning knowledge refers to ways through which students make decisions about the schemas that are appropriate for the solution of the problems at hand. Phase 3, which is the planning phase, involved students in two-step word problems, which are combination problems. Instruction in this phase focused on the need to identify the

two situations that compose each two-step problem and on the way each problem can be mapped. Given the presence of two situations in a single problem, the students are expected to acquire the necessary knowledge for selecting which one to examine first and to examine how two situations are related within a problem. Figure 2 illustrates the instruction for one two-step problem, which involves two change schemas. The two diagrams emphasize that one of the values from the first situation will of necessity be part of the second situation. This is graphically demonstrated by drawing both diagrams and linking the parts that contain the same value with an arrow. This is the most crucial phase, since the students must acquire the skills needed to formulate a plan of action (about twelve lessons were devoted to this phase).

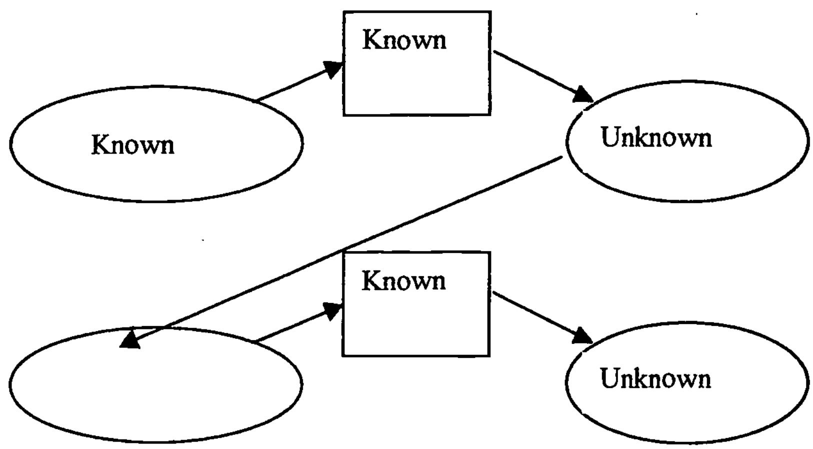

Figure 2: Diagrammatic form of the Change-Change schema

Phase 4: The execution or application knowledge consists of the techniques and skills that are necessary for carrying out the selected schemas. In the case of word problems, the execution knowledge is limited to the four arithmetic operations of addition, subtraction, multiplication and division. During this phase students developed their ability to execute the appropriate operations to give reasonable solutions to the problems.

Procedure

We used an experimental-control design. The subjects were 310 fifth graders from ten intact classes. Five classes of students comprised the experimental group, and the rest comprised the control group. The study took place during the first term of the 1997-98 school year. The intervention program was taught to the experimental classes during the regular mathematics sessions allocated in the weekly timetable. During the same period the control classes continued to follow the regular instruction on the same problems as those involved in the experimental program.

The teachers who taught the experimental classes had a short course at the University just to get acquainted with the basic constructs of schema theory, and the way they were expected to work with their students. All the classes were given the

same pre-test and the same post-test, which measured students' ability in solving one-step and two-step word problems. These tests contained items designed to measure learning as well as transfer of knowledge. The pre-test and the post-test comprised of two parts each, the first one consisting of ten one-step word problems and the second one consisting of twelve two-step word problems The subjects completed a retention test one month after the post-test. All writings of the pupils during these lessons were collected and will be analyzed in a later stage. However, the results that follow refer only to the twelve two-step problems of the pre and post-tests.

Results

For the analysis of the data a structural means model was used. The model is a representation of the two samples of students; the first sample is the experimental group, while the second is the control group. Figure 3 shows the path diagram used in the present study, where the pretest scores can be conceived as indicators of the latent variable F1. Ability at post-test was represented by the latent variable F2, which was also a function of the ability at the pre-test. The right part of the figure gives the constant V999, which was presumed to affect each of the twelve two-step problems in the pre and post-tests, reflecting their intercept (see Bentler, 1995). V999 also affects directly the factors F1 and F2, representing the intercepts of these factors. Factor loadings, factor regressions, and variable intercepts were constrained to be equal across experimental and control groups. Factor intercepts were fixed at 0 for identification in the control group, and were free to be estimated in the experimental group.

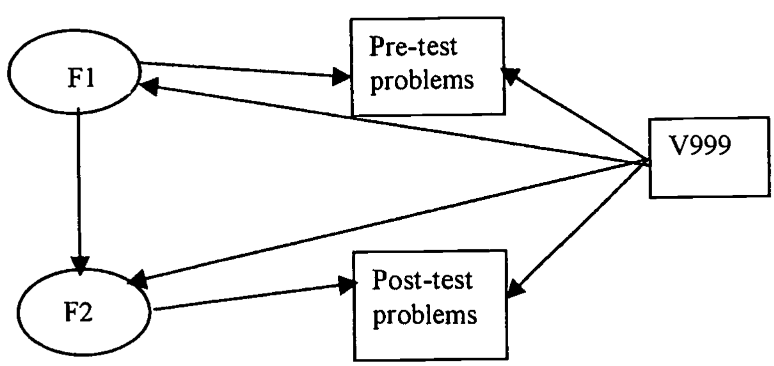

Figure 3: The Structural Means Model used in the Study

As reflected by the iterative summary, the solution converged quite smoothly, and the goodness-of-fit statistics showed the model to be a very good fit to the two-group data, as indicated by a CFI (Confirmatory Factor Index) of .934. To answer the primary question of whether the latent construct means for the two groups are significantly different, we turned to the construct equations, which are presented in Table 1. The parameters of interest in answering this question were the factor intercepts that represent the latent mean values. Because the control group had their

parameters fixed to zero, we concentrated solely on estimates for the experimental group (Table 1).

The regression of F2 (post-test scores) on F1 (pretest scores) was significant indicating that the experimental group did better than control group on the post-test (.971, $\underline{Z}$=10.036). The experimental group students were higher in pre-test to begin with, when .078 was compared to control's value fixed to zero. However, the difference between the two groups at the pre-test was found to be non statistically significant, ($\underline{Z}$=0.875) and thus we could conclude that both groups were initially equal in ability to solve problems at the beginning of the study. Thus, the experimental program produced a positive significant impact (.275) on students ability to solve word problems ($\underline{Z}$=2.124).

Table 1

Construct Equations with Standard Errors and Test Statistics

F1 =	.078*V999	+ 1.000 D1	
	.089	(SE)	
	.875	(Z -score)	
F2 =	.971*F1	+ .275*V999	+ 1.000 D2
	.097	.129 (SE)	
	10.036	2.124 (Z -scores)	

Conclusions

We began with the premise that schemas are basic mechanisms for learning and therefore, instruction ought to facilitate students develop strong schemas (Marshall, 1995). The proposed instructional model for problem solving was based on schema theory. Throughout the instructional phases students were encouraged to construct meaning and knowledge by analyzing and synthesizing the problem text, on the basis of schematic representation, and to develop the ability to work with various linguistic forms and become acquainted with the mathematical structure of relevant problems. Much of the process involves visual representation that enhances connectivity of the common characteristics of the problem. Thus, we may hypothesize that individual student and group activity with visual forms can anchor the schema development. The schematic representation of a problem could be used as a means of getting pupils make sense and communicate their understandings to make it open to inspection and scrutiny.

The results of the present study provide us an encouraging indication that the instructional model for problem solving based on schema theory, is effective in improving students' problem solving ability. The study provided evidence that the students in the experimental group acquired knowledge about problem solving in a rather short period of time. Another advantage of the model is the possibility to monitor student's abilities and weaknesses. By simply looking at the drawing

constructed by the student one can pinpoint misunderstandings about the mathematical relationships in the problems, possible misinterpretation of concepts, deficiencies in the problem representation or the strategic planning of a solution. For example, the teacher can identify the pupils who are unable to solve a problem of a specific structure when they use the inappropriate figure.

In light of the differences in performance observed between the experimental and control groups, pertinent research questions need to follow up. First, further analyses of the data will follow to provide us with insight of what categories of problems contributed most to the development of students' ability to solve problems. Second, which of the four phases was of greater importance in developing the problem schema. In addition, several points might be examined concerning involvement and relevant characteristics of participant teachers.

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Student constructions of formal theory: giving and extracting meaning

<u>Márcia Maria Fusaro Pinto</u>

Departamento de Matemática
Universidade Federal de Minas Gerais
Brazil
e-mail: marcia@mat.ufmg.br

David Tall

Mathematics Education Research Centre
University of Warwick
CV4 7AL, UK
e-mail: david.tall@warwick.ac.uk

*This paper describes an analysis of student constructions of formal theory in university mathematics. After a preliminary study to establish initial categories for consideration, a main study followed students through a twenty-week Real Analysis course, interviewing individuals at regular intervals to plot the growth of their knowledge construction. By focusing on the students constructions of definitions, arguments and images, two distinct modes of operation emerged—**giving meaning** to the definitions and resulting theory by building from earlier concept images, and **extracting meaning** from the formal definition through formal deduction. Both routes may be successful or unsuccessful in constructing the formal theory.*

Advanced mathematical thinking is so vast an enterprise that different individuals focus on different kinds of activities. One mathematician might focus on "thinking hard about a somewhat vague and uncertain situation, trying to guess what might be found out, and only then finally reaching definitions and the definitive theorems and proofs." Another may extend formal theory already developed by "getting and understanding the needed definitions, working with them to see what could be calculated and what might be true to finally come up with new 'structure theorems'," (MacLane, 1994, p. 190–191). The division of labour between those "guided by intuition" and those "preoccupied with logic" was noted by Poincaré (1913), citing Riemann as an intuitive thinker who "calls geometry to his aid" and Hermite as a logical thinker who "never evoked a sensuous image" in mathematical conversation (p, 212).

So how can we expect students to fully understand all the processes of advanced mathematical thinking when mathematicians themselves must specialise in only part of the total enterprise? This research project began with a preliminary study analysing written work and interviews with students to establish basic categories for analysis. It was founded on theory in the literature of advanced mathematical thinking (e.g. Tall, 1991, and subsequent developments). Few of the students concerned proved to have a grasp of the formal theory, exhibiting imagery already studied in the literature. The main study was designed to cover a wider spectrum of students, including highly successful ones. Students were interviewed at intervals on seven occasions through a twenty week first year course on Real Analysis. The methodology uses a form of theory construction following the style of Strauss (1987), Strauss & Corbin (1990). It begins by reviewing data and attempting to categorise it, re-evaluating the categorisations to fit the data collected until it falls into a natural structure that is grounded in the available data.

Preliminaries

A preliminary categorisation was considered in which students:

1. become acquainted with the definition,
2. use the definition to deduce results,
3. use the results in further theorems to build up systematic theories.

This may be summarised under the successive headings:

1. DEFINITIONS,
2. DEDUCTIONS,
3. SYSTEMATIC THEORY.

However, the cognitive processes proved to be more intimately interconnected. To truly understand the nature of a definition requires the use of deductions to construct its implications. There is therefore an important interplay of the form

DEFINITIONS <—> DEDUCTIONS.

To take account of this observation, Bills & Tall (1998) defined:

> A (mathematical) definition or theorem is said to be *formally operable* for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument.

In a preliminary study, Pinto (1998) analysed the final assessments of twenty student trainee teachers. Only *three* based any arguments on definitions and only *one* used a formal definition in an operable manner. The remainder gave informal justifications often based on a particular case. To take account of this spectrum, the heading DEDUCTIONS was modified to ARGUMENTS to include all types of justification, and the main study focused on DEFINITIONS, DEDUCTIONS and underlying MISCONCEPTIONS. The negative tone of the third category was later modified to focus on:

- DEFINITIONS,
- ARGUMENTS,
- IMAGES.

The first two headings were analysed in turn with each being related to underlying concept images as follows:

DEFINITIONS <—> ARGUMENTS

↕

IMAGES

The students chosen for the main study were selected using a test designed to provide a full spectrum of students following a first year pure mathematics course including potential high and low achievers. The students were interviewed on seven occasions throughout a twenty-week course. All interviews were transcribed from the tapes and significant episodes selected to be coded and organised into a classification system. The initial coding system followed the plan of themes highlighted by the exploratory study:

- DEFINITIONS given by each student were classified as descriptive, correct formal or distorted formal,
- ARGUMENTS were categorised as being based on concept images or based on the formal theory presented,
- IMAGERY, as evoked by the students, was classified as to whether it was apparently constructed from the formal theory or not.

Given the differences between the informal approaches of the students in the preliminary study and the desired formal theory, the responses initially were classified as follows:

Approach:	DEFINITIONS	ARGUMENTS	IMAGERY
informal	descriptive	based on concept image	not constructed from definition
formal	formal (correct or distorted)	based on formal theory	constructed from definition

Table 1

This analysis, however, was revised when two distinct approaches were found to occur:

- ***giving meaning to*** the concept definition from concept imagery,
- ***extracting meaning from*** the concept definition by making formal deductions.

Although reminiscent of the earlier-mentioned approaches of research mathematicians, they differ because students are given the definitions as starting points. However, there are certain parallels. *Giving* meaning involves using various personal clues to enrich the definition with examples often using visual images. *Extracting* meaning involves routinising the definition, perhaps by repetition, before using it as a basis for formal deduction. This led to a new categorisation (table 2) where giving meaning could lead to formal theory or fail by remaining image-based, while extracting meaning could be done either reflectively or mechanically, leading again to a spectrum of success or failure.

Approaches	Concept Construction			
Strategies	Characteristics	DEFINITIONS	ARGUMENTS	IMAGERY
Giving meaning (building from *informal* ideas)	1. Reconstructing old knowledge to give new 2. Interpreting new knowledge in terms of old	1. Formal: • correct • distorted 2. Descriptive: • general • prototype • specific	1. Based on thought experiments: • formally presented • image-based 2. Rote-learned	1. Reconstructed with the formal theory 2. Old images retained 3. New ideas added as extra information 4. Conflict between old and new
Extracting meaning (building from *formal* theory)	Routinising: • reflective • mechanical where either may remain compartmentalised or later be linked to old knowledge	Formal: • correct • distorted	Based on formal theory: • meaningful • rote-learned	Based on formal theory: • compartmentalised • linked

Table 2

Students building operable definitions through giving and extracting meaning

In the main study some individuals used both approaches at different times, but many showed a distinct preference for one approach. For instance of two highly successful students, Ross was categorised as an *extractor* of meaning and Chris, a *giver* of meaning.

In his first interview, Ross wrote down the definition as follows (Pinto, 1998):

A sequence (a_n) tends to limit L if, $\forall\ \varepsilon>0,\ \exists\ N(\varepsilon)\in\mathbb{N}$
s.t. $\forall\ n\geq N;\ |a_n - L| < \varepsilon$. (Ross, first Interview)

He explained that he coped by:

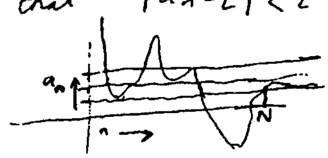
> "Just memorising it, well it's mostly that we have written it down quite a few times in lectures and then whenever I do a question I try to write down the definition and just by writing it down over and over again it get imprinted and then I remember it." (Ross, first interview)

Throughout the course he constantly attempted to prove results from formal definitions, seeing links with earlier established results until towards the end when he began to slip behind the pace of the lectures. At such times he might consider imagistic ideas but then always attempted to base his ideas on *extracting meaning* from the definition.

Chris, on the other hand, used imagery to support his thinking, drawing pictures to represent his main ideas. He wrote down the limit definition as he drew a picture, saying:

> "I don't memorise that [the definition of limit]. I think of this [picture] every time I work it out, and then you just get used to it. I can nearly write that straight down." (Chris, first interview)

~~If $a_n \to L$ then there~~ exists
For all $\varepsilon>0$, there exists $N\in\mathbb{N}$
such that $|a_n - L| < \varepsilon$ for all $n\geq N$



> "I think of it graphically ... you got a graph there and the function there, and I think that it's got the limit there ... and then _ once like that, and you can draw along and then all the ... points after N are inside of those bounds. ... When I first thought of this, it was hard to understand, so I thought of it like that's the n going across there and that's a_n. ... Err this shouldn't really be a graph, it should be points." (Chris, first interview)

The slip in drawing a curve revealed him concentrating on more important ideas and (temporarily) neglecting others. He always seemed to be negotiating with his ideas. For instance, he considered an alternative definition in which increasing N caused ε to become small before rejecting it and settling on the standard form. He seemed to enjoy the tension of challenge and was constantly *giving* meaning from his concept images whilst reconstructing them to take account of the formal theory.

Both students could use the definition of limit in an operable manner in different ways. For instance, when asked about "non-convergence", Ross wrote down the limit definition and negated the quantifiers, while Chris wrote down the definition immediately as if thinking the ideas through in a mental experiment. Ross practised and thought through his proofs formally, Chris wrote formal proofs linked to thought experiments.

Less successful students

Many students on the course had difficulty with definitions. Robin tried rote learning:

> "It's just *memorising* the exact form of it, making the actual idea sort of *understandable* ..."

Despite this, he could not write the definition of convergence accurately:

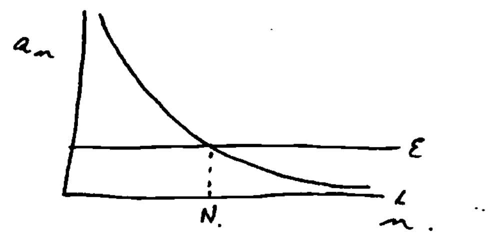
> A sequence (a_n) tends to a limit L ~~if~~ for $\varepsilon > 0$ if there exist $N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon$ provided $n \geq N$. (Ross, first interview)

He attempted to extract meaning from the definition, but was unable to remember it accurately, let alone make it operable. For instance, in one problem he set N=5 without mentioning any relationship between ε and N.

Colin was also unsuccessful with the definition, writing:

> If $a_n \to l$, then there exists $\varepsilon > 0$, such that $|a_n - l| < \varepsilon$ for all $n \geq N$, where N is a large positive integer. (Colin, first interview)

He often attempted to support his ideas using a diagram:



(Colin, first interview)

However, his pictures were highly specific and seemed to imprison him in their implied detail rather than provide the flexibility of thought available earlier to Chris. For instance, he denoted the limit by l yet wrote ε instead of l+ε, and considered the limit as a *lower bound* (a common concept image noted by Cornu, 1981, 1991). He explained:

> ".. umm, I sort of imagine the curve just coming down like this and dipping below a point which is ε ... and this would be N. So as soon as they dip below this point then ... the terms bigger than this [pointing from N to the right] tend to a certain limit, if you make this small enough [pointing to the value of ε]." (Robin, first interview)

Neither student could cope with non-convergence. Robin wrote:

> A sequence a_n does not tend to the limit L if for any $\varepsilon > 0$, there exists a positive integer N s.t. $|a_n - L| > \varepsilon$, whenever $n \geq N$ (Robin, second interview)

He leaves the original quantifiers unchanged, and only modifies the inner inequality $|a_n - L| < \varepsilon$ incorrectly to give $|a_n - L| > \varepsilon$. He is unable to treat the whole definition as a meaningful cognitive unit (Barnard & Tall, 1998), focusing instead only on the inner statement as something which he can attempt to handle.

Meanwhile Colin, said:

> "Umm ... I would just say there doesn't exist a positive integer because we can't work it out ... no ... you cannot find an integer N ...".

and wrote:

There exists a ~~point~~ term where $|a_n - L| \geq \varepsilon$ where $n \geq N$, where N is a positive integer. (Colin, second interview)

Both students used pictures and symbols in their work, giving meaning on some occasions and attempting to extract it from the definition on others. However, Robin's main preference was to extract meaning from definitions which were regrettably often erroneous, whilst Colin preferred to attempt to give meaning using concept images which were too limited to build the general concept.

Other unsuccessful approaches

Two other classes of students were considered—*users of mathematics*, including those studying physics, statistics, economics, etc and *future teachers of mathematics*. Some of the users of mathematics were successful, others were more interested in mathematics only for its use, having little interest in formal proof which appeared too complex, even alien. Rolf (an applied mathematics student) wrote the definition as:

$(a_n) \to L$ if

$|a_n - L| < \varepsilon$ $\quad \varepsilon \in \mathbb{Z}$ $\quad$ for $n \geq N$

$\varepsilon > 0$ (Rolf, first interview)

Cliff (a statistics student) wrote:

Let a_n be sequence where $\varepsilon > 0$. and N is a positive integer

$|a_n - L| < \varepsilon \quad \forall\, n \geq N$ (Cliff, first interview)

Both definitions are distorted and restricted to the inner statement, with total absence of the two external quantifiers and the functional relation between ε and N. Rolf saw the definition as a process, which he attempted to memorise, and use as a criterion to check if a sequence is convergent or not. He tended to try to *extract* meaning from it, but failed. Cliff seemed to think of it as a dynamic *description* of convergence which he imagined occurring in time as N increases and ε decreases. He attempted to *give* meaning but is unable to do it successfully. Given their inadequate definitions, neither student could define non-convergence. Both subsequently resorted to rehearsing routine computations requested in previous examinations and tried to rote-learn them to pass the course.

The student teachers in the main study all replicate the imagistic meanings of those in the preliminary study. They have dynamic images of convergence with terms getting "arbitrarily close" which have often been reported in the literature (Cornu, 1991). For instance, Laura evoked many personal images for the idea of a limit with built-in conflict:

> "The number where the sequence gets to, but never quite reaches."
>
> Let a_n be the sequence and L is the limit which it tends to. Then when some initial values are placed into the formula of the sequence the answers will never reach the value of L (negative or positive).
>
> "... oh, yes, I put 'never reach', and it *can* reach, and that will be the limit of it. ..."
>
> "... But it won't never get bigger than the limit. The limit is like the top number it can possibly reach. And I put never reach." (Laura, various sayings, first interview)

She was unable to write down the definition in any formal sense, although she had mental pictures that gave her imagistic meaning for some of the theorems. Any justifications she made involved attempting to *give* meaning using images. She was unsuccessful with the formal aspects of the theory as were all the other teacher-training students. Essentially, the idea of formal proof in analysis was alien to their day-by-day routine in teaching practice. As Laura explains:

> "I'm on another planet when it comes to Analysis. It seems just completely surreal to me. ... it sparks in a lot of people in the group ... a lot of people. I don't think there is anybody who understands it. And a lot of people are getting very frustrated, with it. I just want to throw books around the room and ... get up and leave." (Laura, first interview)

Summary

In this paper we began by noting that mathematicians use different cognitive techniques to generate new theorems. Some work with formal definitions, carefully extracting meaning from them by deducing from them and gaining a symbolic intuition for theorems that may be true and can then be proved. Some have a wider problem-solving approach, developing new concepts that may be useful before making appropriate definitions to form a basis for a formal theory.

Students learning mathematics have a different problem. They come from elementary mathematics, deeply ingrained in the computation of arithmetic and the symbol manipulation of algebra using standard algorithms to solve certain types of problems. The forms of proof at this level (often called "demonstration" or "justification") usually either involve algebra to give a symbolic description for a general arithmetic statement, or some kind of thought experiment focussing on a "typical" or "generic" case.

The transition from elementary mathematics to formal proof is a huge chasm for many students whose underlying concept image is unable to sustain the formalism. Many (including the majority of those in our sample preparing to teach mathematics) have informal images which dominate their thinking. Some remain entrenched with their old images and those that attempt to use the definition may only be able to cope with part of the structure, giving a personal definition that is not formally operable.

Success comes to those who achieve it in (at least) two ways, either *giving* meaning by working from the concept image, or *extracting* meaning by working formally with the definition. These two techniques can each be successful or unsuccessful. For the successful student, *giving meaning* involves constantly working on various images, reconstructing ideas so that they support the formal theory. The successful student who extracts meaning from the definition has a different task of building up a formal image based mainly on the proof activities themselves.

Those who fail to cope with formal proof but try to give meaning from their concept imagery may be able to imagine thought experiments which give generic proofs and an intuitive insight into some of the ideas, others may fail completely. And become extremely frustrated. Those who fail to extract meaning are unable to cope with the complexity of the definitions and be totally confused. A fall-back strategy to attempt to pass exams is to learn proofs by rote.

Teaching and learning formal proof remains an important component of theory building in advanced mathematical thinking. For future mathematicians it is essential. However, in using different approaches through giving or extracting meaning involves quite different sequences of construction. *Giving meaning* from concept images requires ongoing reconstruction of personal ideas throughout the course to focus on essential properties of the definition and to construct an integrated formal theory. *Extracting meaning* builds up ideas mainly from formal deductions with fewer links to other concept images and so avoids some possible conflicts at the time. However, this formal approach has its own difficulties and may end up with a formal theory unconnected to informal imagery. These different developments suggest that it may not always be possible to deal with different student approaches within a single teaching method.

The most serious finding is the negative effect caused by teaching formal proof in analysis has on future teachers which may have an implicit effect on their teaching of mathematics to the next generation.

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DISCONTINUITIES REGARDING THE SECONDARY / UNIVERSITY TRANSITION :THE NOTION OF DERIVATIVE AS A SPECIFIC CASE

Frédéric Praslon, Equipe DIDIREM, University Paris 7

ABSTRACT :

The mass phenomenon regarding university teaching today makes high school/college transitional problems bigger. We carry out a didactic analysis of these multiform problems in a specific mathematics field i.e. the analysis, and concerning a more precise topic, the derivative. A parallel study on the way each teaching institution deals with the derivative notion, and on student's personal relationship coming out from them, highlights the existence of two very distinct cultures. Finally, we will study the consideration of this institutional break and its management through adapted workshops.

I. INTRODUCTION

The mass phenomenon of university teaching as well as new requirements imposed to universities make the transition between high school and college a crucial issue today (ICMI, 1997). This transition involves very different facets, discontinuities and changes, and not only academic ones. For instance, one cannot deny that the evolution in students' social status and life plays an essential role. From an academic point of view, forms and contents of knowledge are affected by this transition, as well as assessment modes and relationships between teachers and students. In our research project, we address these transition issues from a didactic perspective, by focusing on one mathematical domain known as playing an essential role in the transition failure : Calculus. Moreover, in this mathematical domain, we focus on one specific area : the notion of derivative which, in France as in many countries, is the core notion of high school Calculus courses. Our aim is to understand the discontinuities and changes occurring within the high school/college transition regarding this notion and its environment, the way they are dealt with in standard university teaching practices with the corresponding cognitive effects on students. It is also to build didactic designs which could help to make both teachers and students more sensitive to these discontinuities and able to face them more effectively.

In this report, we will first present our theoretical frame and methodology. Then we will briefly describe the multidimensional grid we have built in order to analyse the institutional and personal relationships linked to the notion of derivative and its environment, before focusing on the analysis of one specific task proposed to

students. Finally, we will analyse the results obtained through this specific task when considering the more general outcomes of the research project.

II THEORETICAL FRAME AND METHODOLOGY

II.1 Theoretical frame

The research project relies on different theoretical frames. At a global level, it relies on the anthropological approach of didactic phenomena developed by Y.Chevallard (1991). This is specially useful for one major reason : we are here studying the transition between two different mathematical cultures : high-school culture and university culture. Chevallard's approach stresses the relativity of mathematical knowledge and the fact that the personal relationship such or such individual develops with respect to such or such mathematical object is shaped by the institutional values, relationships and norms the institutions he or she lives in, develop with respect to that object. These are conveyed by mathematical practices or "praxeologies" which, according to Chevallard, can be analysed in terms of tasks, techniques (the word bearing here a very general meaning), technologies which have to be understood as discourses with explanatory or justificative aims, and finally theories which can be seen as technologies of the technologies. In order to understand discontinuities and changes, we have thus to analyse the praxeologies relevant to both cultures concerning the derivative notion, in a very precise way. Another important point is the following one: Chevallard stresses that the knowledge development within one given didactic institution supposes what he calls the "routinisation" (routine process) of some tasks and official techniques which tend to become "naturalised", and then transparent to the actors of the institution.

At a more local level, we rely on different theoretical frames relevant to Advanced Mathematical Thinking (Tall, 1991), especially to what is now known as the "theories of reification" (Dubinsky, 1991), (Sfard, 1992) as we make the hypothesis that the transition between process and object views of mathematical concepts is a key point in the high school/college transition process in Calculus. We also rely on some epistemological distinctions introduced by A.Robert concerning the generalising, unifying, formalising dimensions of concepts or relationships to concepts at stake at university level and the cultural gap these epistemological characteristics necessarily introduce in the transition. We also rely on a recent synthesis of the same author where she tries to describe in a very detailed way the characteristics of mathematical practices at a university level and situates the latter considering high school mathematics culture (Robert, 1998).

When looking at Calculus and results obtained so far in this area more specifically, we refer to two articles recently published by M.Artigue (Artigue, 1996, 1998) which offer an integrated view of the different perspectives mentioned above, of their respective potential and limits as well as a synthesis of the results each of them was able to produce up to now.

II.2 Methodology

Relying on this theoretical frame, our methodology aims at investigating the discontinuities and lacks in the high school/college transition, both from an institutional and personal points of view. For this purpose, we have built a multidimensional grid allowing us to analyse in a detailed way the practices involving the notion of derivative, both at high school level and during the first university courses.

Regarding the institutional dimension, this multidimensional grid has then been used in order to analyse systematically the tasks proposed to students in most current French textbooks for grades 11 and 12, and Baccalauréat assessments for scientific sections. It has then been applied in the same way with worksheets used with students in their first semester at university in Calculus courses, in different universities.

As regards the personal dimension, students' relationships with the notion of derivative has first been investigated through a written test taken at the entrance at the university by students. The tasks elaborated for this test cannot be considered as familiar ones for these students. Taking into account the institutional analysis, they appear more as transitional tasks between the two mentioned above cultures. Secondly, we have designed and experimented specific workshop sessions where students, working in small groups, have to face mathematical problems representative from the main discontinuities previously identified (status and role of definitions, role of conjectures and counter-examples, work on classes of functions defined by general properties...). In the research project, these sessions have a diagnostic role, helping us to understand the difficulties met by students as well as their resources and evolution all along the academic year, but they also want to be the source of the engineering part of the research project mentioned above.

Validation is classically based on the triangulation of the results obtained through these different methodological approaches.

III. THE MULTIDIMENSIONAL GRID OF ANALYSIS

The analysis of the practices involving the notion of derivative is organised around a multidimensional grid of analysis of the tasks arising at least in one of the two cultures. Tasks are firstly classified according to themes, and for each theme the analysis is organised around five main components :

a) autonomy given to students in the resolution (eventual decomposition into subtasks, given hints...),
b) status of the notion of derivative in the task (according to the "tool" and "object" dimensions, the "process" and "object" dimensions...),
c) nature and context of the task (mere application of standard techniques, technical adaptations possibly required, level of generality, degree of reflexivity...)
d) settings and flexibility between settings required by the resolution of the task
e) semiotic registers present in the text of the task and/or required by its resolution.

For each of these components, both qualitative and quantitative data are collected. Moreover, when using this grid for analysing textbook tasks, for each theme (for example, the "finite increments inequality"), we define a rate of repetition "t" for the corresponding tasks. This rate allows us to measure what kind of tasks and techniques generates important and specific training in a given institution and identify important differences and lacks between tasks, even when they seem part of the two cultures.

IV. FOCUSING ON ONE SPECIFIC EXAMPLE

In this part of the report, as announced above, we focus on one specific task proposed to students in the entrance test.

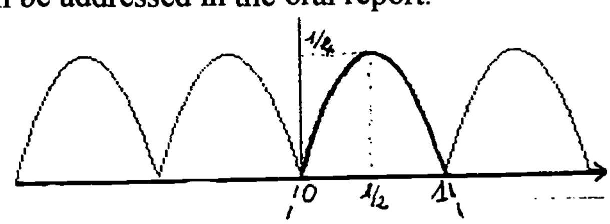
IV.1 Description and a priori analysis of the task

This task formally introduces the notion of symmetric derivative, and explores its relationships with the standard notion of derivative, firstly on one specific example : the periodic function (period 1) defined on $[0,1[$ by the expression $f(x)=x(1-x)$ then in general. More precisely, they are asked to analyse the continuity and differentiability of the function f and calculate (if they do exist) its derivative and symmetric derivatives at the following points : 0, ½ and ¼. Then they are asked to examine the three different following conjectures :

i) Every par function defined on R has a symmetric derivative at 0.
ii) Every par function defined on R has a derivative at 0.
iii) If one function, defined on R, has a derivative at x_0, then it necessarily has also a symmetric derivative at x_0, and $f_s'(x_0)= f'(x_0)$.

Note that students are also given the following graphical representation of the function f. In this text, due to space restrictions, we focus on the first part of the task, the second part will be addressed in the oral report.

Graphic scheme:



This task is far from being a familiar task at high school level : it deals with a function which is only implicitly defined through an algebraic expression, with a new notion formally introduced, it includes general conjectures... So, from students, it requires a kind of mathematical reflexivity and practice which goes beyond standard institutional relationships to the derivative. Nevertheless, this task can be considered at the interface between high school and university culture, at least for French scientific students : it doesn't require heavy technical work, it begins with the exploration of some particular case, which can help the transition to the more general part, and even provides a counter-example, students just have to identify as such. Moreover, the given graphical representation can help students to visualise the

properties of f : continuity, non differentiability for x=n with $n \in Z$, parity, and thus favour a more economical management of the necessary calculations in proofs. Let us add that one of the aims of this task is therefore to test the level of cognitive flexibility between the algebraic and graphical settings, in a situation where information in the two settings is simultaneously given.

Nevertheless, as mentioned above, this task presents serious difficulties and we would like to point out some of those ones relevant to the first subtask :

a) In order to study the differentiability of f, it is necessary to come back to the definition of the derivative, definition which, at high school level, has more a cultural status than an operational one. In order to succeed, students also have to overcome the obstacle associated with the possible amalgam between f and the polynomial function P defined by $P(x)=x(1-x)$. So they have to clearly distinguish f from its algebraic expression on one particular interval and consider it as a specific object. We can reasonably expect that a lot of students entering the university are not able to overcome this obstacle and will say that the function f is continuous and differentiable over R, as it is given by a polynomial expression on $[0,1[$ and periodical. In the following, we will label such arguments as those of level 0.

b) Students may be aware that there is a problem in 0 and try to develop a local analysis, but unaware of the necessity of distinguishing right and left limits, taking the same expression for negative and positive x, especially if there is a semiotic trap : the bracket is closed in 0 in the definition of f. We will generally label this kind of behaviour as a level 1 behaviour.

c) Students can be aware of the problem for integer values of x and define conditions for smooth connection, without using the notion of limit, still in a construction phase, for instance by checking if $f(0)=f(1)$ and $f'(0)=f'(1)$ by using periodicity and the given algebraic expression for f. No doubt that such a solution is not a correct one but it seems to us unreasonable to expect that, with the scope of experience they have, high school students can go beyond that point. We will label such a behaviour as a level 2 behaviour.

d) As regards the specific calculation of derivative values asked for, we can expect that students will correctly evaluate the derivative at ½ and ¼ , but also that a lot of them will use the same expression 1-2x for the derivative in 0, falling into the semiotic trap mentioned above. It will be interesting to notice if, in that case, they denote some contradictions with the characteristics of the graphical representation and, if so, how they manage them.

e) The calculation of the symmetric derivative values doesn't not obey the same pattern. Students may be blocked by the formal definition which is given there and unable to exploit it. Limit calculations at ½ and ¼ , if they are undertaken, are not beyond high school technical abilities, at 0 the result can be immediately obtained if parity is used, but if not the calculation becomes more complex.

f) The interpretation of the results obtained may raise specific difficulties : students are faced with a new notion they cannot easily interpret; it seems to coincide with the notion of derivative at ½ and ¼ , why such a discordance at 0 ? Students are not obliged to be specifically confident in their formal calculations. It is only through the second subtask that they are seriously required to adopt a reflexive attitude.

IV.2 Main results obtained :

As far as the continuity and derivability of f are concerned, about 1/3 of the students involved can be situated at level 0 (and in most cases they don't even invoke the periodicity of f in their argumentation), 1/3 undertakes a local analysis at points 0 or 1 (levels 1 and 2), 1/3 develops a mixed approach (level 0 for continuity and 1 or 2 for the derivative, or less often the opposite way). 16% clearly develop a strategy relevant to level 2. When there is some local analysis, the graphical point of view is only evoked by 20% among students and it then acts as the essential argument, never as a way of checking algebraic calculations. So the level of flexibility between the two settings appear to be poor, in this specific context at least.

1) a) f est le produit de deux fonctions continues sur $\mathbb{R}$:
la fonction f est donc continue sur $\mathbb{R}$
b) De même, f est le produit de deux fonctions dérivables sur $\mathbb{R}$:
elle est donc dérivable sur $\mathbb{R}$.

General procedure, level 0, expression of a typical high school "ritual":

As was anticipated, students massively succeed in calculating the derivatives at ½ and ¼ (95%), see more easily the situation as a problematic one in 1 than in 0, as the bracket is open in 1, and only use x(1-x) as an algebraic expression for f. Continuity, which is a very marginal notion in French high school syllabus, is not dealt with in the same way as differentiability, which is the notion at the core of high school Calculus courses. Most students feel the necessity to come back to the formal definition of the derivative when checking the differentiability in 0 or 1 and 60% of students distinguish between right and left limits in that case. No doubt that the institutional role of functions such as the absolute value function plays an essential role here. Nevertheless, the degree of familiarity with such functions is not enough to make what we could call "the angular point icon" something operational in that context and would allow them to overcome the difficulty generated by the limited validity of the given algebraic expression.

Visibly, students try to give some meaning to this unusual situation. This leads to a diversity of strategies, which even if they don't succeed are not at all deprived from interest and show that students have some mathematical resources, of course strongly shaped by their mathematical field of experience in terms of functions, a very limited one in terms of possible pathologies at the end of high school.

The same thing occurs with the notion of symmetric derivative. Unexpectedly, students in their great majority are not blocked by the proposed formal definition. The rate of success is even 75% for the calculations at ½ and ¼. The main error reveals an inadequate relationship with the notion of "undetermined form" which plays an important role in their relationship with the limit concept : obtaining the expression $\lim_{h \to 0} 0/2h$ for $f_s'(1/2)$ some students declare that the symmetric derivative cannot exist as 0/2h is an "undetermined form". But the calculation of the symmetric derivative at 0 is only correct for 3% of students; in that case, 88% of students calculate f(-h) by substituting –h to h in the given expression of f, and thus obtain the value : 1 for the derivative.

86% of students use the expression 1-2x in order to calculate f'(0), obtaining f'(0)=1. 26% of these had previously perceived the non differentiability of f in 0, by using the definition or graphically, but visibly there is a cut for them between the familiar and ritual calculation of derivatives, and the unfamiliar analysis of differentiability they were asked for in the first question. The following excerpt clearly illustrates this cut:

b). f est dérivable sur $[0;1[$ comme produit de fonctions dérivables.
$\forall x \in [0;1[$ $f'(x) = 1 - 2x$ $f'(0) = 1$ et $f'(1) = -1$. Or le nombre dérivé en un point ~~…~~ est le coefficient directeur de la tangente à la courbe en ce point
Donc comme f est périodique on a : $\forall x \in \mathbb{Z}$, f admet deux
demi-tangentes (point d'inflexion). conclusion: f n'est pas dérivable sur $\mathbb{R}$.

Finally, the comparison between the two notions in that particular case remains unproblematic for the majority of students as, due to the errors mentioned above, they have found the same value: 1 for the two derivatives at 0.

V. CONCLUDING REMARKS

This example is quite representative of the strong tendencies we observed in the test results and, beyond that, in the experimental part of the research. Results attest the attempts made by students in order to adapt their mathematical resources to the problematic and complex situations proposed to them in the test and in the workshops. These attempts most often lead to oversimplifications, favoured by their reduced field of experience. They also often lead to a disconnected treatment of continuity and differentiability issues which becomes more understandable if one takes into account the respective role of the two notions in the high school culture. They also appear strongly dependent on the context as well as on the form of the

questions themselves and, in spite of the obvious directions given in the syllabus, they are poorly operational at connecting their different settings of mathematical work (algebraic, graphical) and show little cognitive flexibility. Nevertheless, if we consider that the proposed tasks are really out of their high school culture, that they put at stake some important characteristics of the new relationships they will have to develop with respect to the notion of derivative and more generally with respect to Calculus at university, we have to consider these attempts, even clumsy ones, as very positive starting points, sources of adaptation, potential for a didactic work. It doesn't seem reasonable to think that the complex cognitive activities required here can be mastered at high school level, even if they deal with simple objects. They have to be seriously taken in charge in the high school/college transition, which cannot only be seen as a transition from some experimental and pragmatic Calculus towards a formal one. The institutional analysis we have developed in our research work clearly demonstrates that it is not presently the case in most university courses. There is some didactic gap and we hope that, modestly, our research work will contribute to better understand how it could be didactically managed.

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THE ROLE OF PICTORIAL IMAGES IN TRIGONOMETRY PROBLEMS

Lisa Pritchard and Adrian Simpson

Mathematics Education Research Centre, Institute of Education
University of Warwick, Coventry, CV4 7AL, UK

Year 10 pupils were involved in two task-based, semi-structured interviews after a short revision course on basic trigonometry. The tasks explored the pupils' use of pictorial images in solving traditional trigonometry word problems. This paper reports that a general pattern of use was discerned in which pupils moved from different types of diagram construction, to information extraction and finally to symbolic manipulation with little subsequent reference to the image of the trigonometric situation. The possible reasons for this unidirectional picture to symbol flow are discussed in relation to some fundamental difficulties in learning trigonometry.

Introduction

Trigonometry may be seen as the confluence of a number of streams of mathematical difficulty. It is a topic area in which students first meet functions which are not direct numerical manipulations of the functions' arguments: $\cos(30^{\circ})$ is not a direct manipulation of the number 30. It is one of the first areas where mathematical objects are given by definition: sine is given as the ratio of the lengths of the opposite and the hypotenuse in a particular type of right angled triangle. Curriculum designers see it as an area in which visual elements are naturally associated with word and symbolic problems: questions with and without attached diagrams are interspersed from the beginning of the topic. Problems often involve the abstraction of information from 'real world' situations: ships, buoys and lighthouses abound in trigonometric word problems.

All of these mathematical difficulties are compounded by the difficulties pupils have in moving flexibly between images of trigonometric situations and algebraic/numerical symbolism.

In this paper, we report on a research project which directly examined the roles pictorial images played for students in solving traditional, basic trigonometry problems.

The implicit assumption of the various UK mathematics schemes (and, indeed, of the UK national curriculum) is that trigonometry is a topic in which pupils can flexibly move between visual and symbolic ways of working. Despite earlier research (Blackett, 1990) suggesting the importance of a flexible approach to linking numerical and visual domains in trigonometry so as to give a better conceptual

understanding, most schemes follow the suggested pattern of development in national curriculum documents. Following a grounding in Pythagoras' theorem, the notion of similar right angled triangles are used to define sine, cosine and tangent functions in terms of ratios of appropriate side lengths. Pupils then find angles or lengths of sides in abstract situations in which the right-angled triangles are drawn for them. They progress to 'real world' word problems, such as finding heights of trees from a given angle of elevation and distance in which diagrams are sometimes given, but are expected to be constructed by pupils as an aid.

It is widely claimed that the use of diagrams or pictorial images is useful in helping pupils solve problems. It is claimed (Nickerson, Perkins and Smith, 1985) that

> Once a graph or diagram is drawn, the problem solver can bring perceptual processes to bear on it. Also, a visual representation of a problem can make apparent certain relations among parts that might otherwise go unnoticed.

It seems clear, then, that trigonometry is an excellent place to explore the ways in which pictorial images are used in the solution of problems.

Trigonometric problems

The research was an attempt to get a sense of some of the possible ways in which pupils might use pictorial representations as part of the process of solving trigonometry problems, having had only a basic introduction to it. Six pupils, two girls and four boys, were chosen from a top-set year 10 class who had just revised the basic concepts of trigonometry. They were tape-recorded solving some 'real world' word problems of the type (but not precise form) they were familiar with, during two different interviews. The questions which the data presented in this paper relate to were:

a. A ship is 10km due south of a lighthouse. It is sailing on a bearing of 60° towards a buoy which is due east of the lighthouse. How far is the lighthouse from the buoy?

b. A flagpole is hung horizontally, suspended from the wall by two ropes which are attached to the end of the flagpole. The angle between the ropes at the end is 60° and at the wall the ropes are 1m apart. How long are the ropes?

c. A 5m ladder rests against a wall, with the foot of the ladder 1.2m from the bottom of the wall. What angle does the ladder make with the ground and how far off the ground is the top?

d. A cuboid has dimensions of 5m by 12m by 15m. Calculate the angle made between the diagonals from the bottom of one corner to the top of the diagonally opposite corner and the 5m by 12m base.

The method of solution was left entirely to the students, no diagrams were provided, neither were the pupils directly encouraged to produce them, though such encouragement would have been given in an ordinary classroom situation.

The pupils' spoken protocols, their diagrams and their written work were analysed revealing three different useful categories for exploring their use of pictorial images:

- The creation of diagrams
- The use in solving problems
- The use in checking and making meaning

The Creation of Diagrams

For the typical 'real world' word problems, all of the pupils, in all of the problems drew a diagram. What stood out from this, however, was that there were qualitatively different ways in which the pupils used their mental imagery to produce them. The data showed three different ways in which they appeared to construct them: getting a whole mental image of the real scene and then effectively copying that image on to paper; getting a whole mental image and copying an abstraction on to paper or constructing the diagram piece by piece by taking one phrase from the question at a time.

Andrew, who seemed to be a strong visualiser throughout the tasks, talked of "thinking about the diagram in my head and what I would see if that happened" when drawing pictures. His also drew pictures with quite realistic properties - ladders that were an attempt to look like ladders, as seen in fig 1.

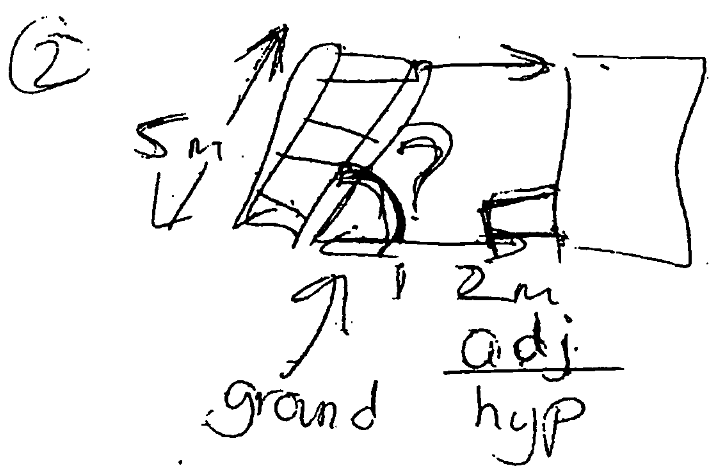

Figure 1

In Sophie's case, she spoke similarly about some form of complete mental image which she was struggling to manipulate (in this case, trying to draw the space diagonal of a cuboid for question d)

> "I can't see a diagonal from the bottom of one corner, it's confusing. I'm trying to get in my mind which corner which diagonal ... [the cuboid is] square, no, rect..., no it was square actually"

However, her diagrams, such as in fig 2 were more abstract than Andrew's.

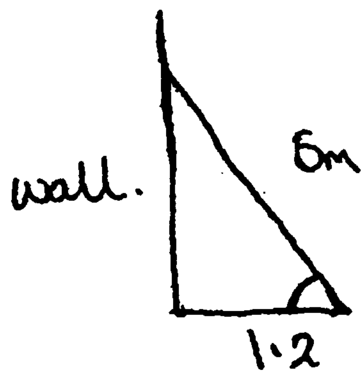

Figure 2

Most pupils, however, drew the diagrams by repeatedly taking the next piece information from the question and adding it to the drawing. They did not seem to construct a mental image and copy it - they directly constructed a drawing. Keith, for example, seems to take the information as instructions to be followed in a logical order. When asked how he drew parts of the diagram for question b he said

> Well I made it into sections, one thing at a time. I didn't just pinpoint 3 dots. I had to think about each thing at a time. I pinpointed the lighthouse and then thought that the ship's 10km south so I didn't do it in proportion. I drew 60^0 angle which I thought was about 60^0 and I drew a line straight down. I drew a line east until it met the other line and I drew the other dot there.

Uses in Solving Problems

The research showed only two main uses in solving problems for the pupils' diagrams: identification and extraction. They were used to sort out the information in the question, identifying the objects in relation to each other: as Sophie put it "as you're drawing it out you can see which one's opposite and which one's not". Once the drawing had all of the information arranged on it, the pupils then used it to extract

the appropriate information and choose which of the trigonometric ratios they would need to use.

This two stage process of identification and extraction was seen in all the pupils' solution strategies, but was put most succinctly by Sarah:

> Sketch out the triangle, label it and then work out what the question wants

All the pupils followed the same procedures for solving the problems. They recalled the mnemonic they had been taught to enable them to recall the basic ratio definitions ("sohcahtoa" for "sine = opposite/hypotenuse; cosine = adjacent/hypotenuse and tangent = opposite/adjacent"). The relevant information was substituted in to the 'formula' and the symbols were rearranged in an attempt to find a solution. As expected, it was when this attempt initially failed that the most interesting data was seen.

One problem was the reliance on memory – if a student could not remember the process, they could not proceed easily. In particular, when pupils were asked to find an angle in the problem (rather than the more familiar task of finding the length of a side) many could not recall the procedure they needed to follow after having reached, for example, $\cos(x) = 0.24$. Most, perhaps for want of any other interpretation to hand, thought of this as the product of 'cos' and 'x' being 0.24, even if, for some pupils, they knew this was not the case:

> Int: How would you find x?
>
> Sophie: 0.24 divided by cos
>
> ...
>
> Int: Does cos x mean cos times x?
>
> Sophie: cos is x is the cos. What cos is, like x belongs to cos.

This problem was quite common and highlights a further problem in the nature of trigonometry at this stage of schooling. In terms of the development of the function notion in the sense given by APOS theory (Breidenbach et al., 1991), the functions pupils will have been familiar with to this point will generally have been seen as processes: the direct numerical manipulation of their arguments. For many of the pupils, even in a good year 10 class, functions as objects may not yet have developed even in simple cases. It is reasonable to suggest that the trigonometry functions can not yet be seen even as a process: $\cos(30^\circ)$ is not a numerical manipulation of 30 in any obvious way. It is more likely to be seen as no more than an action: one based on procedure implicit in the definition and which is quite convoluted ('draw a right angled triangle with a 30° angle, measure the adjacent and the hypotenuse and find the ratios of their lengths'). In the pupil's nascent understanding of familiar functions, the inverse can often be associated with a reversed procedure. It seems much harder to get a sense of reversing the procedure above to find x given $\cos(x)$.

The use in checking and making meaning

The checking of solutions was not a natural procedure for any of the pupils. If they found an answer, they generally considered the task completed. As part of the process of gathering data, however, the pupils were explicitly asked if their solutions 'seemed reasonable'.

An interesting aspect of this data is that the checking followed the same flow as the solution, even if that flow was not complete. Some pupils did refer to their diagrams and the data from the diagram, whereas others checked only the symbolic/numerical calculations. None of the pupils seemed to put their solution back into their pictorial or mental image, to get a sense of whether the solution was of roughly the right size.

Discussion

Presmeg (1986) categorised solutions as visual if they involve

> imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are employed

and non-visual if they do not involve such imagery. The research reported here shows that there may be layers of complexity within that definition of a 'visual solution', at least in the context of these trigonometric problems. Even though no pupil used it throughout the solution of a problem different pupils used imagery to different extents.

In the research data presented here, we can see a general flow to the methods of the pupils' solutions: construction of a diagram, identification and extraction of data, choice of ratio and symbolic manipulation to a solution (or until stuck). The genuine use of visualisation in the sense of Solano and Presmeg (1995) seems only to play a part in the first aspect of this flow and only for those few pupils who constructed the diagram by first building a whole mental image and then copying it on to paper (either 'realistically' or more abstractly). For most of the pupils, there was much less use of visualisation even here: they transferred the information, a piece at a time, from the word problem to their diagram without, it seems, constructing a whole mental image. Even in checking, this general flow from (possible) visualisation to diagram to calculation was followed.

One of the problems with the construction of images in the case of trigonometry at this stage of schooling may be the reversal of the usual concept image/concept definition development (in the sense of Tall and Vinner, 1981). It has been argued that the majority of mathematics at school proceeds from getting a familiarity with concept images without any real reliance on concept definitions and that one of the difficulties with the transfer to advanced mathematical thinking is that this flow is reversed. In higher mathematics objects may be defined prior to any other aspects of a concept image being constructed. Probably for the first time, pupils are given definitions of the trigonometric ratios (often in shortened forms like 'sine is opposite

over hypotenuse') before they have formed any significant concept image of them. Indeed, these definitions do not lend themselves to the development of clear images: while the objects of the definition ('opposite side' and 'hypotenuse') are easily identified as part of a picture, the ratio itself is not. The objects within the definition and the object which is defined are quite different and are susceptible to quite different modes of reasoning - the former visual and the latter numeric/symbolic.

In this research we can see pupils who are able to move between these two modes of reasoning, but in one direction only and at only one point in the solutions to the problems given. They can extract from their diagram the information they need to begin the symbolic reasoning they use to handle the ratio itself.

This is quite at odds with the findings of Nunokawa (1994) where a mathematically more sophisticated student was asked to solve a problem and was seen to move more flexibly between pictorial and symbolic representations, modifying diagrams as the work on the problem progressed. Simpson and Tall (1998) distinguished between passive, organisational, conceptually generative and formally generative imagery. It appears that while Nunokawa's mathematically gifted student used diagrams in ways that fit all four of these categories, the pupils in this research are using their pictorial images in, at most, an organisational way. They are using the diagrams to organise and to allow them to extract the information they need for a symbolic calculation, but they do not use them to suggest solution strategies, hint at expected results or embody the formal definition of the mathematical objects under consideration.

It may be that the more sophisticated use of pictorial images comes from a more developed understanding of the topic. The student in Nunokawa's research had developed a much richer network of connections that constituted his understanding (in the sense of Hiebert and Carpenter, 1992) and it can be suggested his concept image and concept definition were more closely linked. With pupils beginning their encounters with trigonometry they inevitably can only draw on a sparse network of knowledge and with this, one of their first encounters with a defined object, they have dissociated concept images and definitions.

Clements and Battista (1992) suggest:

> At van Hiele Level 2 and higher, one's use of visual images is constrained by one's verbal/propositional knowledge. Images and transformations of images incorporate this knowledge and, as a result, might behave differently at different levels.

In this research we have seen that the verbal/propositional knowledge does indeed take precedence over the images and it may be that the paucity of such knowledge makes the images they produce such poor generators of subsequent thinking.

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THE RHETORIC OF GENERALIZATION
A Cultural, Semiotic Approach to Students' Processes of Symbolizing

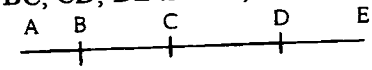
Luis Radford
Université Laurentienne
Ontario, Canada

Abstract: *Taking generalization as a cultural semiotic problem, that is, a problem about meaning co-construction occurring in the overlapping territories of writing and speech, this paper attempts to study generalization as a mathematical action unfolding in a classroom discursive Bakhtinian 'text' jointly written by teachers and students in the course of mediated activities. In the case that we shall consider here, what is at stake is the construction and the meaning, in a grade 8 classroom, of a new mathematical object —that of the general term of a sequence or pattern. We shall focus on the problem of how generalization finds expression in processes of sign use (particularly* sign understanding *and* sign production*).*

1. Introduction

This paper is part of an ongoing research program dealing with the students' processes of symbolizing in algebra[1]. By students' processes of symbolizing we mean the different ways in which students come to *understand, use* and *produce* signs. Our work is embedded in a post-Vygotskian semiotic theoretical framework that we elaborated elsewhere (Radford 1998, in print) in which signs are seen as psychological tools, symbolically loaded and intimately linked to the actions that the individuals carry out in their activities. Within this theoretical context, ways of symbolizing (Nemirovsky 1994) are not considered as acultural, pre-given processes. Instead, we consider them as instances of the general modes of signifying resulting of the juncture of sign-mediated activities of the individuals and the *Cultural Semiotic System* (e.g. beliefs, patterns of meaning-making; see Radford 1998) in which activities are subsumed. Mathematical generalizations as well as other mathematical activities are framed by specific and culturally accepted ways of symbolizing. This point can be made clearer if we consider sign use in the historical example of the study of numbers in Antiquity. While mathematicians in the Pythagorean tradition, legitimately used pebbles to investigate some properties of numbers, in Euclid's *Elements* not only the actual pebbles but also any iconic representation of them was completely dismissed and replaced by a referential, non-operational sign-segments/letters language couched in a deductive line of reasoning. The deductive Greek mathematical style was linked, as A. Szabó suggested, to the Eleatan distinction between true knowledge and appearance and the consequent rejection of the sensible world as carrier of knowledge. Moreover, and most important for our discussion, the Eleatan beliefs legitimized new ways of symbolizing which authorized certain rules of sign use and excluded others. How the Euclidean "mathematical generality" could be expressed depended on the Greek conception of the concrete and the abstract, the historical availability of the Phoenician letter-based alphabet adopted by the Greeks (a letter-based written language which was completely different from, for instance, the syllabic Akkadian cuneiform language of the Babylonian scribes) and on the accepted cultural normative dimension in which the use of signs was caught. Let us consider a short example. Proposition 21, Book IX of Euclid's *Elements* reads as follows:

If as many even numbers as we please be added together, the whole is even.
For let as many even numbers as we please, AB, BC, CD, DE, be added together; I say that the whole AE is even. For, since each of the numbers AB, BC, CD, DE is even, it has a half



part; [VII. Def. 6] so that the whole AE also has a half part.
But an even number is that which is divisible into two equal parts [id.]; therefore AE is even (Heath 1956, Vol. II, p. 413).

Euclid expresses here "generality" in natural language as a volitive potential action rendered by the comparative formula "as many even numbers as we please". And, within the Euclidean semiotics, the letters allow combinations (in fact assemblages, e.g. AB) which denote segments that stand for non-particular numbers. Interestingly enough, the proof was not recognized (either by Euclid or his later commentators) as lacking generality despite the fact that a drawn segment unavoidably has a particular length as well as the fact that it was actually based on only *four* numbers. As far as we know, the proof was considered completely general by the canons of Greek mathematical thought. As in contemporary classrooms, modes of symbolizing and expressing generality in Antiquity were shaped by the master's and students' beliefs about mathematics, and their mutual understanding and acceptance of legitimizing procedures about mathematical symbolization.

In this paper we shall deal with a problem which arises in the algebraic study of patterns, namely, that of generalization. Ordinarily, in such cases, because of curricular requirements (as is the case in the current Ontario curriculum for Junior High-School), generalization is expressed through the semiotics of the algebraic language. Of course, a great deal of experimental research has shown that the algebraic expression of generalization is very difficult for students who are still acquiring the mastering of the algebraic language (see e.g. Rico *et al.* 1996). In accordance with our theoretical framework (Radford 1998, in print), we will attempt to explore generalization as a semiotic problem, that is, a problem about meaning co-construction by teachers and students in the course of mediated activities. We shall focus on the construction and the meaning, in a grade 8 classroom, of a new mathematical object—that of the general term of a sequence or pattern. We are particularly interested in the problem of how generalization finds expression in processes of sign use. Since the general term cannot be ostensively pointed to as one can point to a door or to a desk, the semiotic construction of such a mathematical object acquires a particular didactic interest.

2. The methodology

In our research program we are accompanying for three years some 120 students and 6 teachers in the teaching and learning of algebra. This task includes the teachers', researcher's and assistants' joint elaboration of general and particular goals, the joint elaboration of teaching and learning settings, the video-taping of the lessons, discussions, and feedback. The teaching settings have been elaborated in such a way that the students (who are presently in Grade 8) work together in small groups; then the teacher conducts a general discussion allowing the students to expose, confront and discuss their different achieved solutions. In general terms, we are interested in investigating the students' processes of symbolizing in specific teaching settings about patterns on the one hand, and equations and inequations on the other. In this paper, however, we shall focus solely on the students' and teacher's co-constructive semiotic expressions of the "general term" of a sequence or pattern. The results that we shall present here come from an interpretative, descriptive protocol analysis (Fairclough 1995, Moerman 1988). Because of the length requirements of the article, we

shall limit ourselves to the protocol analysis of one of our student groups. The protocol analysis will attempt to disentangle texture forms underlying the process of sign use (particularly *sign understanding* and *sign production*) in terms of the conveyed meaning and of the classroom use of utterances genres (e.g. reading, confronting, requesting, informing). Our question can then be explicitly formulated in the following terms: how do students' and teachers' voices and writings find their way in the construction of the new object (from the student's perspective) of the general term of a pattern or sequence? How do teacher and students deal with the concrete and the abstract in pattern problems? In its most general terms, and taking the term 'rhetoric' as a mode of discourse or text making, the question is: How does the rhetoric of generalization take place in the classroom?

3. Results

The students were asked to work in groups to solve some problems about patterns. In previous activities they investigated some patterns and had to provide answers to questions like *a* and *b* shown below. Questions *c* and *d* required a new kind of symbolic understanding. For the sake of brevity, we will consider here only some excerpts of the episode concerning one 3-student group discussion of questions *c* and *d*. Let us nevertheless mention that, although questions *a* and *b* led to different understandings of how to investigate patterns, the students did not raise problems concerning issues on generalization. The students kept focused on concrete issues raised by those particular questions. The case for questions *c* and *d* was very different.

Observe the following pattern:

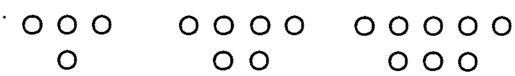


Fig. 1 Fig. 2 Fig. 3

a) How many circles would you have
 * in the bottom row of figure number 6?
 * in the top row of figure number 6?
 * in total in figure number 6?

b) How many circles would you have
 * in the bottom row of figure number 11?
 * in the top row of figure number 11?
 * in total in figure number 11?

c) How many circles would the top row of figure number "n" have?

d) How many circles would figure number "n" have in total? Explain your answer!

Time	*Line*	*dialog / remarks*
1:41	(21)	student 2: *(he writes the answer to the third part of question b while saying)* in total that comes to 24. Wow! This is easy! *(Now he reads question c)* How many circles would the top row of figure number ... What? OK. Somebody else!
1:59	(22)	student 1: *(reads the question.)* How many circles... [...] What does it mean?
2:07	(26)	student 2: I don't know. *(hitting the sheet with his pencil)*
2:13	(27)	student 1: What's figure n? / *(inaudible)*
2:22	(28)	student 1: (*talking to student 2*) Shut up! I'm going to kill you. ... n is what letter in the alphabet?
2:33	(29)	student 2: (*talking to student 1*) Ask the teacher.

In this passage the students are trying to make sense of the expression "figure number n" contained in question *c*. As we noticed elsewhere (Radford 1996), the general term of a geometric or arithmetic pattern cannot be explicitly expressed within the semiotic system (SS) of the objects of the pattern itself. Even to pose the question, it is necessary to go "out"

of the first SS (which will include, in the case of elementary school arithmetic, the basic "well-formed" expressions using the ten digits and certain signs like those required for the elementary numerical operations, equality and so on) and to rely on another richer SS (e.g. a meta-language). In the case of our text, we had recourse to the algebraic language to talk about the general term. As the engineering of the problem given to the students suggests, the idea of generalization that we decide to use resides in an experiential dynamics attempting to go beyond the concrete terms of arithmetic. As expected, the transcript indicates, however, that the students' understanding of the question remained circumscribed to their arithmetical experience. We reach here a nodal point in the development of the classroom activity whose unraveling will require the elaboration of new meanings. While student 2 bluntly abandons the quest for meaning (an action accompanied by exasperation as line 26 suggests), student 1 started a cardinal-arithmetic plan: to display the letters of the alphabet and to figure out what position n occupies in that order:

2:54 (32) student 1: How many circles would the top row of figure 14 have? n is fourteen.
3:00 (33) student 2: No it's not!
3:01 (34) student 1: Yeah it is!
3:02 (35) student 3: What is n? *(asking the teacher who coincidentally is walking by)*
3:04 (36) student 2: *(talking to the teacher)* What is n? We do not know.
3:08 (37) teacher: *(turning the page and reading the question aloud)* How many circles would the top row of figure number n have?
3:13 (38) student 3: What is n?
3:15 (39) student 1: n is fourteen because n is the fourteenth letter of the alphabet. Right?
3:20 (40) student 2: *(counting aloud the letters that student 1 wrote on the table previously)* one, two three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen.

Student 2 does not agree with student 1's interpretation about what n is and his arithmetical rephrasing of the question ("confronting" utterance in line 33). The arrival of the teacher serves as a potential way to overcome the conflict. The teacher, nevertheless, offers as an answer a re-reading of the question only. Interestingly (and probably because of the teacher's laconic answer) student 2 seems to change his mind and to agree in investigating student 1's idea; thus he starts counting the letters of the alphabet. Noticing that the students are taking an unexpected path that may take them away from the intended algebraic meaning, the teacher consents to explain a bit further:

3:28 (41) teacher: "n" is meant to be any number.
3:32 (42) student 2: OK.
3:33 (43) student 1: n is what?
3:35 (44) teacher: Any number *(in the meantime student 3 comes back to question a.)*
3:39 (45) student 1: I don't understand.
3:41 (46) teacher: You don't understand?
3:42 (47) student 1: No. [...]
3:44 (49) teacher: *(talking to student 3)* Do you understand what n is?
3:45 (50) student 3: Which one? *(pointing to the figures on the sheet)* this, this or this?
3:46 (51) teacher: It does not matter which one.

The rhetoric of generalization has now taken a different turn. The teacher launches the understanding of "n" as "*n being any number*". The students' reactions show that there is a tremendous difficulty in constructing this specific meaning. Student 3's answer (line 50) suggests that this difficulty is linked to a very specific semiotic problem that we will term as

the "multiple representation problem". Since sign-numbers, in the semiotics of arithmetic, generally refer to a single object (i.e. although in some contextual instances the number five may be represented as e.g. 6-1, nobody will agree that, in the base 10 arithmetic, two different 'basic' representations like "7" and "5" represent the same number) it appears very hard to conceive "n" (which by the way as sign has a similar 'basic' iconicity as "7" or "5") as representing more than one number (see line 50). This is corroborated by the following lines:

4:10 (53) <u>teacher</u>: *(talking to student 2 after a long period in which the students remained silent)* OK. There, do you have an idea what is n?
4:12 (54) <u>student 1</u>: Fourteen.
4:13 (55) <u>teacher</u>: It may be fourteen ...
4:14 (56) <u>student 2</u>: *(interrupting)* any number?
4:15 (57) <u>teacher</u>: *(continuing the explanation)* ... it may be 18, it may be 25...
4:18 (58) <u>student 1</u>: Oh! That can be any number?
4:19 (59) <u>student 2</u>: *(interrupting)* The number that we decide!
4:20 (60) <u>student 1</u>: OK then, *(taking the sheet)* OK, n can be ... uhh...
4:26 (61) <u>student 2</u>: Twelve.
4:27 (62) <u>student 1</u>: Yeah.
4:28 (63) <u>teacher</u>: But ... yeah. What were you going to write?
4:31 (64) <u>student 1</u>: 12.
4:32 (65) <u>student 2</u>: 12.

The teacher's attempt failed. The proposed meaning for n as being "any number" is interpreted as an arbitrary but *concrete* number ("informing" utterances, lines 61, 64-65). The teacher tries to give meaning to the expression conveying the generalization by re-investing the students' arithmetic point of view in a way which is still coherent with the global plan to introduce the general term in the context of the classroom setting. The teacher's voice hence acquires a specific tone made up of the pedagogical plan and the students' contextual voices. The Bakhtinian text in which generalization is being written appears to be heterogeneous in its meaning. Realizing that things had not turned out as expected, in the next line the teacher launches a rescue mission from where the wanted meaning could properly arise:

4:33 (66) <u>teacher</u>: And if you leave it to say any number. How can we find ... how can we find the number of circles for any term of the sequence *(making a sign with the hands as if going from one term to the next)*
4:51 (67) <u>student 2</u>: Figure n? There is no figure n!
4:54 (68) <u>student 1</u>: *(talking to student 2)* He just explained it! N is whatever you want it to be.
4:57 (69) <u>student 2</u>: *(talking when student 1 is still talking)* What is it?
5:01 (70) <u>student 1</u>: OK. Umm ... seven. *(writing on the sheet)*
5:10 (71) <u>student 2</u>: Not on top! It's seven circles *(taking the sheet and looking at the figures)*
5:13 (72) <u>student 1</u>: Yeah! And in the bottom is 5 circles!
5:21 (73) <u>student 2</u>: *(writes the answer and starts reading the next question)* How many circles would ... *(inaudible)* ... 12 circles *(writing the answer)*.

Speech does not unfold alone. Speech unfolds accompanied by other semiotic systems, for instance systems of gestures that we make with our hands and arms (see e.g. Leroi-Gorhan 1964). When we make gestures, the hands can be used to produce signs by e.g. sketching objects (Kendon 1993), while in certain cases concrete objects can be used as metaphors of absent objects (an instrumental strategy generally employed and which becomes a

cornerstone in the development of sign systems with deaf children). Gestures form a sign system with its own syntax and meaning that afford the production of texts. In previous activities, we frequently saw our students pointing to a concrete figure (the third figure of a pattern, for instance) to refer in fact to the 100th figure. And in the case of the episode that we are discussing here, the teacher makes an intensive use of gestures (line 66) to try to complement the sense of the expression "any number" —an expression whose even most forceful utterance cannot reach the students' understanding yet. Indeed, the students keep the arithmetic meanings for the relation between "n", "figure n", and "any number". We should note at this point that student 2 is clearly uncomfortable with their general understanding of "figure n". There is something that does not fit the modes of meaning generation as being used in their classroom culture. In the subjective understanding of student 2, the order of discourse (Foucault 1971), as legitimized by the discursive practices of the classroom cultural institution and instantiated here by the teacher's remarks seems to point to a different way to interpret "n". We may say that, for him, if n is meant to be any concrete intended number, as student 1 is proposing, then, according to the classroom culture, the teacher could have stated this clearly instead of using such a complicated phrasing. Student 2 is doubtful and this doubt appears as something very important for the future of the meaning negotiation process. Notice that the conflict between students 1 and 2 seen in lines 27-28, 32-34, arises differently here. In line 68, student 1 re-interprets the teacher's previous explanations as confirming his own arithmetic interpretation and challenges student 2 with an authoritative argument ("[the teacher] just explained it!"). Seeing this, the teacher decides to intervene again:

5:42 (74) <u>teacher</u>: So, uh... *(looking at the sheet)* Wait, wait, wait! But for any number.... There you did it for seven circles, but if seven... for any ...

5:52 (75) <u>student 2</u>: *(showing the sheet with his pencil)* You add 2 to the number on the bottom... subtract....oh no, you add 2 to the number on top. If it is seven, the number like this what I ... *(inaudible)*

As the dialog suggests, the opening towards a new understanding is not made possible through a discussion on a concrete example but through the *prise de conscience* of an action previously undertaken (in solving question *a* and *b* but also in many lines of the dialog presented here, e.g. lines 71, 72). The action is now formulated *not* as a *concrete action* within arithmetic (which would give as a result a concrete number, as in lines 71 and 72) but as a *potential action* in the metacode of natural language. As we can see, the new mathematical object is constructed with words: "You add 2 to the number on the bottom...". What we call "generality" is trapped here in the expression "the number on the bottom" —an expression that keeps all the sensuality of the figures in the space— and the operation of adding ("You add 2") to which is submitted this unutterable number ("the number on the bottom") within the elementary semiotic system of school arithmetic.

But what is it that finally made possible the negotiation of meaning? The answer resides not in the students' suddenly grasping the teacher's intentions but in the teacher's continuous (polite, encouraging but always clear) rejection of the students' solutions and the students' will to search for alternative understandings. The construction of the potential action with words is pushed further by the teacher in order to end up with a mathematical formula:

6:01 (76) <u>teacher</u>: OK. Could you put this in a formula?

6:04 (77) <u>student 3</u>: Uhhh ...

6:05 (78) <u>teacher</u>: ... using n.

6:06 (79) student 3: Uhh ... it's the term times two plus two.
6:10 (80) student 2: The term times two plus two?
6:12 (81) student 3: *(showing the figures with his pencil)* Uhhh ... 2 times 6 ... 2 times 3 is 6, plus two ...
6:21 (82) teacher: Could you say that again, please?
6:23 (83) student 3: Yeah. The term times two plus two *(student 2 writes the explanation)* / [...]
6:37 (86) student 1: *(reading the answer)* OK. The term times two plus two.

The students finish by writing: "n2 + 2 = ".

4 Concluding Remarks

The word "term" (which emerges as a sign representing the previous term "the number on the bottom") is not used correctly by the students from a mathematical point of view. There is a confusion between the term and its rank. Nevertheless, the attempted meaning was functionally clear. It is worth noticing that the word "term" comes first to be used by the students as a tool that allows a refinement in the construction of the object. The use of words seems to be similar to that of concrete tools in apprentices. At first the tool (in terms of the "specialist's norms") is used awkwardly and only later can one use it with progressive mastery.

The concept of general term appeared as a potential action bearing the concrete characteristics of actions previously carried out in the social plane undergoing an internalization (in the neo-Vygotsky's sense given in Radford 1998) *through* and *by* signs (in this case words and mathematical signs, whether iconic or arithmetic ones). Such a potential action —which seizes the actual form of the generalization— is the particular expression of concrete actions as afforded by the students' mediated activity (not only by speech but by writing and the related cultural artefacts allowing it, e.g. the sheet and the pencil, the latter functioning as a key instrument in deictic gestures, as in the crucial line 75) arising in the course of their reflections to solve the problem. The students' reflections and their understanding and production of signs are embedded in discursive schemes and discourse orders prevailing in the classroom according to its own culture.

As we have seen, the potential action making possible the overstepping of concrete arithmetic thinking and the reaching of generalization finds expression in the semiotics of the concrete actions and the mode of thinking thus produced. Contrary to the traditional idea, generalization is not something dealing with the abstract and its evacuation of the context but a different contextual semiotic expression of previous actions, which afford the potential action (for instance, giving sense and virtually existence to it). It is enlightening to remark at this point of our discussion that Euclid's proposition quoted in the introduction also bears this distinctive trait of generalization as a potential action that, figuratively speaking, still has the sent of the concrete Pythagorean actions from where it emerged. Generalization is not a mere act of abstraction from the concrete; indeed, generalization keeps a genetic connection to the concrete according to the mediated system of individuals' activities and the epistemic and symbolic structure of these. In turn, as paradoxical as it may seem, the generalizing potential action, even without being *there*, is already producing the concrete actions. Indeed, without being explicitly there, the potential action is already present, making possible that the sixth, seventh or any other term be investigated in the very same form. Beyond their synchronic temporal dimension, the concrete and the abstract bear a dialectical relation, in which they mutually condition each other within the limits traced by the historical and cultural rationality

of the individuals and the semiotic systems that the individuals are continually re- and co-creating.

As we saw, the web of possibilities from where generalization takes place is co-formed and revealed in the texture of the text that the teacher and the students deploy in their search for meaning. In the particular case studied here, we saw how meaning shifted from "figure n" to "n" to "any number" to "any arbitrary but concrete number" until they overcame the "multiple representation problem" and reached an algebraic "public" (Ernest 1998) standard meaning. We also saw that this was done by using different utterance genres ranging from reading ("R" e.g. line 22), requesting ("Q" e.g. line 36), confronting ("C" e.g. line 33), explaining ("E" e.g. line 44), acquiescing ("A" e.g. line 42), informing ("I", e.g. line 45)[2] . The students' production of signs in the formula was mediated by speech and its written form. After uttering the formula, the students wrote it in natural language as "the term x 2 + 2" and then as "~~n x 2~~ n2+2 = " . It is worthwhile to note that Vygotsky suggested that " [u]nderstanding written language is done through oral speech, but gradually this path is shortened, the intermediate link in the form of oral speech drops away and written language becomes a direct symbol just as understandable as oral speech." (1997, p. 142). The fate of the students' understanding of algebraic language seems to be the same, that is, it will be couched in speech (and the accompanying semiotic systems) and only later will it become a kind of autonomous semiotic action. Indeed, signs (like "n" in the students' formula), we would like to insist in closing this paper, are but the result of semiotic contractions of actions (concrete or intellectual as outer or inner speech) previously carried out in the social plane.

Notes:

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2. The number of occurrences of types of utterances are as follows (notice, however, that a same utterance may belong to more than one category depending on its pragmatic dimension).

R	Q	C	E	A	I
3	22	7	9	5	15

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UNDERSTANDING DATA TABULATION AND REPRESENTATION

Chris Reading
Centre for Cognition Research in Learning and Teaching
University of New England, Armidale, Australia

Statistics has received increased recognition in mathematics school curriculums in Australia partially due to the strand status assigned to statistics (within Chance and Data) in ***A National Statement in Mathematics for Australian Schools.*** *Consequently, research has focused on considering what 'statistical thinking' really means. To assist teachers to plan and assess the teaching of statistical concepts more needs to be known about students' statistical understanding. This paper takes up the theme by considering students' responses to two open ended tasks, one of which presents the data in table form and the other graphically. Both tasks require students to describe what they understand by the data representation. A developmental sequence of eight levels was identified and the responses to the two different data presentations were analysed. The SOLO Taxonomy was used as the theoretical framework to assist this process.*

Introduction

More statistical ideas are being incorporated into mathematical syllabuses across Australia but poor awareness of students' statistical understanding on the part of teachers (Watson, 1998) may well be a contributing factor to the poor treatment of statistical components in the curriculum. Truran (1997) identified lack of knowledge of statistical understanding as a concern when trainee teachers had difficulty in interpreting and applying concepts in the Chance and Data strand to create well structured sequences of lessons. In encouraging teachers to give students a chance to show what they *can* do statistically Shaughnessy (1997) stressed the need for research into students' thinking about chance and data.

A recent development in investigating student understanding has been the use of the SOLO Taxonomy (Biggs & Collis, 1982) as a framework, in both probability and data handling. SOLO levels have been used to classify student responses concerning uncertainty (Moritz, Watson & Collis, 1996), data representation (Chick & Watson, 1998), data reduction (Reading & Pegg, 1996) and data interpretation (Reading, 1998). This paper contributes by exploring students' responses to questions concerning the understanding of data tabulation and representation, using the SOLO Taxonomy as the theoretical framework.

The SOLO Taxonomy

Detailed descriptions of the SOLO Taxonomy can be found elsewhere (see for example, Biggs & Collis, 1991; Pegg, 1992). The model, which allows students' responses to be categorised, consists of five modes of functioning, with levels of achievement identifiable within each of these modes. The two modes relevant to the research being reported are the ikonic mode (making use of imaging and imagination) and the concrete symbolic mode (operating with second order symbol systems such as written language). Although these modes are similar to Piagetian stages, an important difference is that with the SOLO Taxonomy earlier modes are not seen to be replaced by subsequent modes and in fact are often being used to support growth in the later modes.

A series of levels have been identified within each of these modes, three of which are relevant to the this report. These are: *unistructural* - with focus on one aspect, *multistructural* - with focus on several unrelated aspects and *relational* - with focus on several aspects in which inter-relationships are identified. These three levels form a cycle of growth which recurs within modes and in different

modes. Within a mode the relational level response in one cycle is similar to, but not as concise as, the unistructural response in the next. A similar cycle of levels is identified in different modes but the nature of the element on which the cycle is based is different. This taxonomy is particularly useful because of the depth of analysis which can be achieved when interpreting students' responses.

Research Design

One hundred and eighty secondary students, selected randomly over gender (male, female), mathematical ability (low, middle, high) and academic years (7 to 12) were tested on a range of statistical questions. This paper reports on the responses to a two part question concerning the understanding of data tabulation and representation, an important step in the process of data analysis. The question was not testing the ability to arrange data into a table or a graph, but aimed at presenting students with some data and allowing them to describe what information they were able to gather from the representation. Part I of the question presented the data in a table, while in Part II the data presentation was graphical. The two parts were used in the question to investigate whether the form of data presentation influenced student understanding. The open-ended question allowed the student to respond with as much information as he or she felt was necessary.

Analysis of Responses to Part I

The question, as presented to students, is shown in Figure 1. Investigation of student responses showed that it was possible to divide the responses into a number of levels based on the statistical quality of the answer given. Three major groupings of the levels were identified based on the depth to which the response indicated the ability of the student to understand the representation of the data.

Part I Question

A class teacher wanted students to practice collecting data. One Year 8 student decided to collect data concerning the number of ice creams that she ate during a week for a seven week period. The table the student came up with is given below.

Week 1	3
Week 2	5
Week 3	7
Week 4	4
Week 5	2
Week 6	7
Week 7	5

What does the table tell you ?

Figure 1

First Group (No Data Use)

Responses in the first group dealt with only the requirements of the question and three broad levels, coded as 0, 1 and 2 were observed. These responses attempt to rationalise the requirements of the question but appear to make no use of the data when formulating the response.

Level 0 These responses indicate that the question has not been considered or the requirements were not clear. For example:

(7201) *Its like a graph.*

Level 1 These responses indicate that not all aspects of the question have been considered sufficiently to produce an answer. Usually, some key fact from the wording of the question is reproduced in the response. For example:

(7111) *The table tells us about the ice cream eating habits of a year 8 girl.*

Level 2 These responses indicate that all aspects of the question have been considered and a reasonable answer attempted but still no use is made of the data. For example:

(8213) *The table tells you in column one: what week it was and in column two: how many ice creams were eaten in that week.*

Second Group (Data Item Use)

Responses in the second group are concerned with attempting to understand the data, with three levels, coded as 3, 4 and 5, being observed. However, attempts to describe the data are produced in non-statistical terms.

Level 3 These responses indicate that, although students have considered the data, focus is directed back to the key facts in the question usually indicating in some way that weekly data is available. For example:

(11105) *It tells me how many ice creams she ate in 7 weeks, how many she ate each week.*

Level 4 These responses indicate an awareness that features of the data need to be mentioned in the answer. However, restricted experiences at data description result in the information in the table being quoted verbatim. For example:

(12207) *The table tells me that for week 1 the student ate 3 ice creams, in week 2 she ate 5 ice creams, week 3 she ate 7 ice creams etc.*

At this stage there is a divergence of the responses into two distinct paths which appear to develop at seemingly parallel rates. These are labelled:

Path A for responses which describe statistical features of the data

Path B for responses which make judgements about the data.

Level 5 These responses describe the data by making a simple observation. They suggest readiness to engage in data description but a lack of experience and appropriate tools to produce a statistically sophisticated response. For example:

Path A (11111) *Some weeks she ate more than other weeks.*

Path B (8108) *That the girl is very unhealthy.*

Third Group (Data Feature Use)

The final group of responses indicate a readiness to describe the information contained in the data in a more acceptable statistical form. Only two levels of responses, coded as 6 and 7, were identified. Both levels are split into A and B paths, with Level 7 also having some responses incorporating elements from both paths.

Level 6 These responses indicate the use of data from the table to make one detailed observation. They show more sophistication than those at Level 5, linking the observations to features of the data, rather than making broad statements. For example:

Path A (10109) *The table tells me that the amount of ice creams eaten varies from 2 - 7 over the 7 weeks.*

Path B (9201) *The table tells you that she likes ice cream for a couple of weeks then she gets back into them again.*

Level 7 These responses indicate a more in-depth understanding by presenting more than one observation related to the data. For example:

Path A (9112) *She ate the most ice creams in week 3 and 6. And the least in week 5. On average she ate 4.7 or 5 ice creams a week.*

Path B (12212) *The student likes ice cream or it is summer and she wants to keep cool.*

Some responses showed features of both Path A and Path B,

(10212) *The table tells you that the girl ate 33 ice creams in 7 weeks and that she must have liked ice creams.*

A third level in this group was not found but it is anticipated that such a level may contain responses which not only mention statistic(s) and judgement(s) but use the statistics as evidence for the judgements made.

The results, arranged by academic year, appear in Table 1 and a number of interesting points become apparent. First, there are only eleven students (18%) from the two senior years whose responses fall within the first group, compared to twenty five (42%) from Year 7 and 8. Second, there is a larger number of senior students compared to junior, noticeably twelve in Year 12, whose responses were coded as Level 7. Third, there is a large bulge in most years at Level 2. Fourth, there appears to be a larger number of responses in the last level of each of the first two groups (that is Levels 2 and 5), than in the previous two levels of the group. This is more noticeable in the junior years. Last, there appears to be a balance in the number of students whose responses reflect Path A and Path B.

Table 1
Response Level and Path by Academic Year for Part I Question

Level	**Year**												**Total**
	7		**8**		**9**		**10**		**11**		**12**		
0	1		0		0		0		0		0		**1**
1	4		1		0		1		0		2		**8**
2	4		5		10		11		6		3		**49**
3	14		3		3		2		7		2		**20**
4	0		5		0		3		0		2		**10**
	A	**B**	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**	
5	1	5	3	6	4	3	2	4	4	2	3	3	**40**
6	1	0	3	2	1	4	2	2	2	3	1	2	**23**
7	1	0	1	0	3	1	1	0	5	0	4	3	
	0		1		1		2		1		5		**29**
Total	**30**		**30**		**30**		**30**		**30**		**30**		**180**

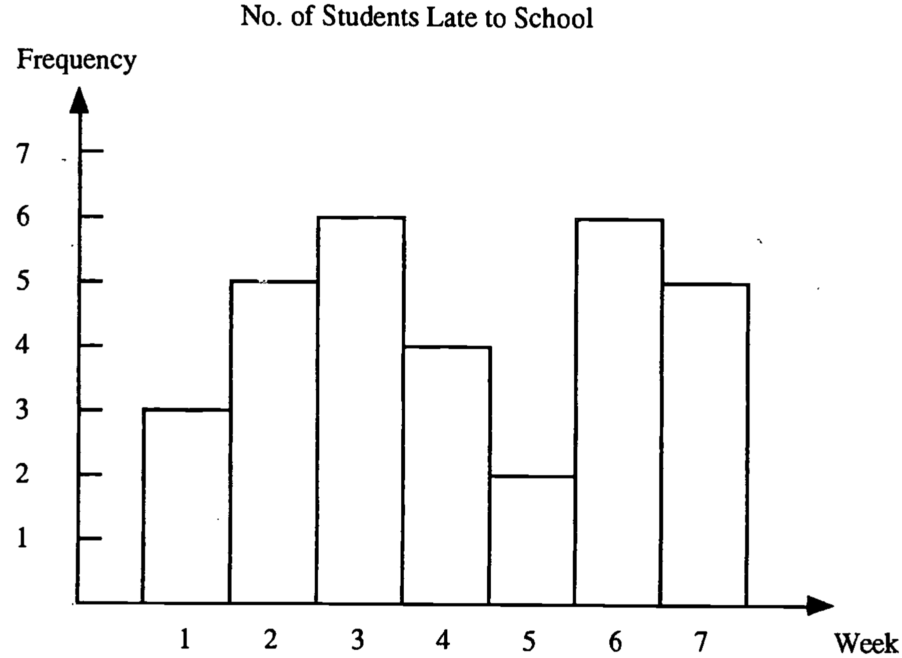
These results suggest that, for the process of understanding data presentation, the level of response improves progressively with academic year, although, the bulge at Level 2 suggests that many students have difficulty actually describing the data at all. Further, there appears to be no particular preference for descriptions using statistical features or judgemental observations.

Analysis of Responses to Part II

The second part, II, of the question is shown in Figure 2. Answering this question meant that students needed to be able to read the graph before describing the data. Examples of responses to Part II are not given because they are similar in form to those given for Part I and were coded into similar levels. The results, arranged by academic year, are presented in Table 2 and some interesting features emerge. First, while twenty one students (35%) from the two senior years gave responses in the first group, there were thirty four (57%) from the Years 7 and 8. Second, Level 7 contained thirteen Year 11 and 12 students but only three Year 7 and 8 students.

Part II Question

The deputy in the school kept a record of the number of students who were late to school each week. He decided it would be useful to draw a graph to illustrate the information. The graph is presented below.



What does the graph tell you ?

Figure 2

Third, there appears to be a larger number of responses in the last level of each of the first two groups (Levels 2 and 5). Last, there are almost twice as many students coded in Path A as Path B, with Path A more popular in all but Year 7 and Year 10.

Table 2
Response Level and Path by Academic Year for Part II

Level	Year 7		8		9		10		11		12		Total
0	0		0		0		0		0		0		**0**
1	8		7		1		0		2		3		**21**
2	11		8		11		14		12		4		**60**
3	2		1		3		4		4		1		**15**
4	0		0		0		1		0		2		**3**
	A	**B**	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**	
5	0	7	3	3	5	3	0	7	4	2	7	3	**44**
6	1	0	6	0	5	0	4	0	1	0	2	0	**19**
7	1	0	1	0	0	2	0	0	4	0	6	2	
	0		1		0		0		1		0		**18**
Total	**30**		**30**		**30**		**30**		**30**		**30**		**180**

These observations suggest a slight improvement in the quality of responses with increasing academic year and a preference for describing the data using statistical features rather than judgements. Difficulties were experienced making the step to actually describe the data and also in the step to describe the data in more statistical terms.

Comparison of Part I and Part II

The framework developed appears to be adequate for explaining students' understanding, as far as the basic description of data presentations is concerned. There is a slight upward shift in the trend of

the responses over the academic years as would be expected. All but one student felt that he or she understood the question sufficiently to attempt an answer and those few who had the most problems responding (Level 1), including misinterpreting the graph, were mostly in Years 7 and 8.

Testing the hypothesis that the level of response is independent of the part of the question yielded $\chi^2 = 13.53$ (6 d.f) which is significant ($p < 0.05$) and indicates that the level of a response is associated with the part of the question being answered. Many more responses than expected were coded at the lower levels in Part II, while Part I had more responses than expected in the uppermost levels. This suggests that students exhibit a higher level of understanding when the information is presented in a table rather than as a graph. Considering the number of responses that were graded into each path for Parts I and II, a test of independence resulted in $\chi^2 = 2.39$ (1 d.f.) which is not significant ($p > 0.10$) suggesting that the choice of path is not associated with the part of the question being answered. This means that the path, A or B, used by a student in Part I does not indicate which path will be used in Part II. For example, giving a judgemental response when describing tabulated data, does not necessarily mean a judgemental response will be given when describing graphed data.

Three noticeable trends emerge when comparing the results of the analysis of Part I and Part II. First, there are more Year 7 and 8 students in the first group than there are Year 11 and 12 students. Second, there are more senior students than junior students in Level 7. Third, there is a large bulge in the numbers at Level 2 in most years. So, irrespective of the form of data presentation there is a general trend for a slight increase in the level of performance of the students over the six academic years and an unexpectedly large number of students repeating key facts from the question and not using the data in any way in the description. The better quality responses from Years 11 and 12 could mean that by this stage most students are ready to describe the data in more detail, despite the fact that little statistical work is actually undertaken in their curriculum studies.

The comparison of results also highlights three differences. First, the number of Year 11 and 12 students in group one is much larger for Part II (21) than for Part I (11). Second, the overall number of responses at Level 7 is much larger for Part I, (29), than for Part II (18). Lastly, similar numbers of responses are categorised as Path A or Path B in all years for Part I but most years show a predominance of Path A type responses for Part II. These differences suggest that although there is an increase in level with academic year, the overall range of performance is better when data are presented in table form than as a graph. Also, that when presented with data in graphical form younger students are more likely to use judgements in their descriptions while older students are more likely to use statistics.

SOLO Taxonomy Framework

The SOLO Taxonomy is now used, along with the groups of levels described earlier, to develop a framework which can be used to assist with the interpretation of student responses. The first group of responses exhibit ikonic features, while the second and third groups represent two different cycles in the concrete symbolic (CS) mode.

Ikonic mode responses show no evidence of linking the required task with any sort of symbolic representation. Level 1 responses were coded as a mixture of unistructural (U) and multistructural (M) responses, while Level 2 responses were relational (R).

Second and third group responses have been able to link the concepts in the question to concrete experience. The answer, which suggests that the question has been understood, links directly to aspects of the data. These responses are in the CS mode, with two cycles, of U, M and R levels.

The first cycle involves appreciating that it is possible to describe data. The elements in the first cycle are the actual pieces of data themselves (data items). A relational response in the first cycle is not achieved until the student is able to consider all data items as a functioning set, and the data items as capable of being described in another form. The U, M and R levels in this cycle correspond to the Levels 3, 4 and 5 described earlier, with the R level showing the split into two parallel paths.

The second cycle involves showing an appreciation for the need to refer to the features or behaviour of the data as part of the description. The elements in the second cycle are the various features (or properties) of the data which a statistical description could include. A relational response in the second cycle is not achieved until the student is able to present both statistical facts and judgements, and also some relation between them. In this cycle, the U and M levels correspond to the Levels 6 and 7 as outlined earlier. These two levels still contain the separate A and B processing paths, while some M level responses show evidence of elements from both paths. No responses were observed at the R level.

In summary, the main feature which distinguishes the concrete-symbolic mode responses from those in the ikonic mode is the retrieval of facts from the recorded data. Ikonic mode responses go no further than recognising from the question the variables which are being measured. CS mode responses show that the data items have been considered. Within this mode, the first cycle responses suggest that the data items have been considered as separate items while the second cycle responses indicate an overview of the data, in the form of a statistic or judgement.

Conclusion

Three major findings have evolved as a result of this study. First, presenting data in graphical form alters the way students describe data, as compared to tabular presentation. The differences in approach to data description include an overall lower level of understanding and a greater likelihood to discuss statistics rather than make judgemental comments.

Next, the three broad grouping identified, namely, *No Data Use*, *Data Item Use* and *Data Feature Use*, assist in determining the stage a student has reached in understanding data representation and tabulation. *No Data Use* responses are dealing with aspects of the question and not the data, while in the other two groups use is made of the data, *Data Item Use* responses in a less statistically sophisticated fashion than the *Data Feature Use* responses. These groupings offer teachers a means to follow better student thinking when planning lesson sequences within the curriculum and assessing specific student outcomes.

Last, the groups of levels identified can be categorised as cycles of U-M-R levels, based on the SOLO taxonomy. The *No Data Use* group is a U-M-R cycle in the ikonic mode where the elements of focus are the facts in the question. The other two groups represent two U-M-R cycles in the CS mode. The elements of focus in the first cycle, the *Data Item Use* group, are the actual data items

while the focus in the second cycle, the *Data Feature Use* group, is on the various features (or properties) of the data. The identified levels in the CS mode are consistent with the U, M and R levels described in more general terms by Chick and Watson (1998, p. 156). Using this framework to assess responses, teachers can gain a greater awareness of students' understanding which will allow them to better prepare lessons based on the curriculum and to assess what students really know, understand and can do.

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NEEDING TO EXPLAIN:
THE MATHEMATICAL EMOTIONAL ORIENTATION

David A Reid

Acadia University

Explanations are accepted or rejected within a community on the basis of an emotional orientation. Examination and description of students' explanations in mathematics classes are used in this report to clarify the nature of the mathematical emotional orientation. The report also provides elements of a language for describing students' explanations as a contribution to the difficult task of conducting research into students' mathematical proving.

Proving in mathematics is a complex activity and research on students' learning to prove must employ a rich descriptive language to capture it. Lakatos (1978) and Polya (1968) offer language to describe two elements of proving: the formality of written proofs and the relationship between deductive reasoning and other types of reasoning. Classifying kinds of deductive and inductive reasoning has also been an area of much research (e.g., Bell 1976, Harel & Sowder 1996). Another element of proving, the formulation of reasoning, has been described by Mok (1997) and Reid (1995a) but further work is needed in this area. A further element, the need or purposes served by mathematical reasoning, has been receiving considerable attention (Balacheff 1991; Bell 1976; Hanna 1989; Lampert 1990; de Villiers 1991), especially the need to explain deductively in mathematics (Hanna 1989, 1995; de Villiers 1991, 1992; Reid 1995). This report elaborates on the language used to describe students' explaining, and discusses the central role deductive explaining plays in the "mathematical emotional orientation."

The mathematical emotional orientation

Maturana (1988a, 1988b) uses the phrase "emotional orientation" to describe the bodily predisposition that underlies individuals' decisions to accept some things as explanations and to reject others. An emotional orientation defines a domain of explanations, of which mathematics is one. Mingers (1995), in his discussion of Maturana's work, identifies three aspects of an emotional orientation:

> Each domain is constituted in three interlocking dimensions — the *criteria* for accepting explanations, different *operational coherencies* structuring such explanations, and the *actions* seen as legitimate (p.98, emphasis added).

In mathematics the *criteria* for accepting explanations include the use of deductive reasoning, a basis in agreed upon premises, and a formal style of presentation. There are many *operational coherencies* (shared experiences and assumptions) in mathematics, the most obvious of which is the language used to talk about it. There are also many *actions* that are seen as appropriate to mathematics (drawing diagrams, generalising statements, making conjectures, etc.).

The nature of the mathematical orientation may be clearer if we consider what is *not* mathematics. Explaining by reference to authority (as in the non-explanations discussed below) is not mathematics. Neither is focussing closely on procedural steps in mathematics classes, although this is the experience of many students. Finally, actions such as the use of abuse to establish authority are not legitimate in mathematics. On the other hand, feeling a need to explain conjectures, and preferring deductive reasoning as the means to do so, are a part of the mathematical orientation.

Ways of explaining

Most of the examples I provide in this report come from observations in grade 10 mathematics classes in which students were studying coordinate and Euclidean geometry. The students engaged in a series of activities, working in groups, and reported their conclusions to the class as a whole on a regular basis.

Behaviours which could be called "explaining" occurred in two kinds of contexts: activities in which there was an explicit demand to "explain" and contexts in which students engaged in what observers saw as explaining without being prompted to do so by a teacher or activity prompt.

In these two contexts several different modes of explaining were observed:

- Non-explanations;
- Explaining *how*;
- Explaining to someone else (in response to a question);
- Explaining to someone else (spontaneously);
- Explaining as part of social activity in a community where explaining is a social norm, i.e., part of the community's emotional orientation;
- Attempting to come to a personal understanding (explaining to oneself).

Individuals operating from a mathematical orientation are likely to use deductive reasoning in any of the last four modes of explaining listed here.

Non-explanations

Personal or institutional authority is a common mode of "explanation" in schools, especially for students and teachers who can't explain something and see that inability to explain as a negative reflection of themselves as people. In the following

In this report I will sometimes omit the word "emotional" from the phrase "mathematical emotional orientation" for brevity. This should not be taken as an indication that the role of emotions in defining a mathematical orientation is unimportant. Emotions are central to defining mathematics.

These examples are taken from an ongoing research project on the psychology of reasoning in school mathematics, funded by SSHRC grant # 410-98-0085, Acadia University and Memorial University of Newfoundland.

example Stu both refers to the teacher's authority to verify and falls back on his own authority when asked to explain.

1. Stu: [To teacher] That one and that are complementary right?
2. Teacher: Why?
3. Stu: Because.
4. Jill: None of them are complementary.
5. Stu: They are.

Abuse can be used to establish the authority to explain. In the following example Stu replies to Christy's questions with a question of his own, perhaps to help her reflect on her own mathematical activity, or perhaps to evade her question. When he and Jimmy do reply to her they provide only the procedure she is meant to use with an implicit reference to an outside authority ("you're supposed to"), followed by abuse from Stu. Stu's opinion that he is in a position to declare Christy "stupid" suggests that he is establishing authority over her.

1. Christy: How'd you get that?
2. Stu: How'd you get 6?
3. Christy: I don't know.
4. Jimmy & Stu: You're supposed to add them together.
5. Christy: Oh—
6. Stu: 3 minus 5 equals 8 ... Man you're — you're stupid.

Explaining *how*

Jimmy and Stu, in the previous transcript, offer Christy a procedural explanation, in response to her question ("You're supposed to add them together."). Many of the explanations offered in mathematics classes are not explanations of *why* something is the case (as would be expected from a mathematical orientation), but simply explanations of *how* something is calculated (which is consistent with what might be called the "school emotional orientation"). This happens not only when students ask each other how to do something (as in the previous example) but also when teachers ask students to explain. For example, this is Jill's response to the written prompt, "Which equation describes the graph on the left? Explain why."

In transcripts I use the following conventions: An em-dash (—) indicates a short pause. Several indicate a longer pause. Ellipses (...) indicate omissions (usually "um"s, etc.) to improve readability. Three asterisks (***) are used to indicate an omission of several lines of speech. A hyphen (-) ending a line indicates an interruption of speech at that point.

> The reason the equation I've marked describes the graph is because I took the points from the graph and made up a table of values then I did the equation with the table of values I made.

Jill does not explain why in any way that would satisfy a mathematician. Instead she describes what she did.

Explaining to someone else (in response to a question)

Another context for explaining, the first in which deductive reasoning is likely to occur (suggesting a mathematical orientation), is responding to another person's question. The other person might be a teacher or written prompt (like the one offered to Jill in the previous example) or, in this case, another student (CH is a research assistant).

1.	Melinda:	I have one.
2.	CH:	What do you have?
3.	Melinda:	Triangle ABC and BDC.
4.	Jill:	Why?
5.	CH:	They're congruent?
6.	Melinda:	'Cause they have a shared side and alternate angles.

Here Melinda uses notation specific to mathematics and makes implicit use of a shared experience of theorems and definitions in geometry to provide a deductive explanation.

Explaining to someone else (spontaneously) and as a social activity.

In the following transcript four grade 10 boys are trying to work out a generalisation concerning the sum of the interior and exterior angles of polygons. Their class has been studying Euclidean geometry (parallel lines and congruent triangles) for about a month, in a style similar to the exploratory methods described by Fawcett (1938). In this context the students in the class have come to adopt explaining as a major focus in their mathematical activity, and have come to value well formulated, deductive explanations. The following transcript offers a number of examples of spontaneous explanations, embedded in a social context that values explaining. (Bold indicates such explanations.) These explanations suggest that the boys were operating from a mathematical orientation at this time.

1.	Wane:	The exterior angles of them all — **because when there's more sides you can make more triangles.**
2.	Mick:	Which one is he talking about?
3.	Wane:	So it keeps going up by 180!
4.	Clark:	Here's what I was thinking. This one is 360. That triangle there and that one — exterior angles.
5.	Wane:	Oh yeah, that's true too.

6. Mick: That angle there is 360 and this angle here — Now they all equal 360, right?

7. Clark: You see, we got this thing here — a common vertex. That means we have to subtract some angles. 360 and 360 which is 720, right? Correct? Okay. Now we got to subtract some. Okay?

***

8. Mick: So what do we got?

9. Clark: We got four angles, right? Now — uhhhh — uhhhh — They're all counter-clockwise — Yeah they are. CCW.

10. Wane: No they aren't.

11. Mick & Clark: Yes they are because the way they goes up that way.

12. Clark: Okay. Now we got the four angles.

13. Wane: Four times 180.

14. Mick: But they're not all 180.

15. Clark: How do you know they're not 180?

16. Mick: **Because they're not all straight lines.** The angles aren't straight lines — I was just looking at that one.

17. Clark: **And they're all supplementary to an angle inside here,** right?

18. Mick: Oh, deadly!

19. Wane: And it has something to do with — if you know the interior angles — the sum of the interior angles — then they'll all wind up to be the same thing **because they're all supplementary to an angle.**

The behaviour of Mick, Clark, Jacob and Wane in the above transcript suggests that they have a mathematical orientation. The passages in bold show Wane, Clark and Mick responding to a felt need to explain their conjectures, which is an action in keeping with mathematical practice. Their explanations make reference to geometrical concepts, definitions and theorems that are a part of the shared experiences of their mathematical community. They propose explanations using deductive reasoning (an operational coherency that structures mathematical explanations) and they expect others to explain things deductively and with reference to the same definitions they use, indicating that at least two of their criteria for accepting explanations are consistent with a mathematical orientation.

Explaining to oneself.

The last mode (explaining to oneself) can be described in still greater detail, but in the interests of space I will limit myself here to one example, of two university undergraduates working on the Arithmagon problem (see Reid 1995a or 1995b for a more detailed analysis):

1. Stacey: What happens if you add the middle numbers together? —

2. Kerry: Well I guess we could, hmm.
3. Stacey: I just want to try something. If you take 27, 18, and 11. 2, 4, 5, 56. Right?
4. Kerry: Sure.
5. Stacey: And you have — So you add each of those twice, right? — Yeah you do. That's not going to help you either. That's what you end up doing right?
6. Kerry: What'd you do?
7. Stacey: You add A, B, C. Then you multiply them by 2. You get this answer. —
8. Kerry: Do you add?
9. Stacey: 22, and 34. Yup. Do you know what I mean?
10. Kerry: Sorry. So you add this and multiply by 2 so, like, the sum of this is 28 times 2. And it's 56. Good one. What's that mean?
11. Stacey: [laughing] Nothing.

In spite of the linguistic indications that Stacey was explaining something to Kerry ("Right?" "Do you know what I mean?") she is really explaining to no one but herself. Kerry, in spite of being involved in the same problem solving activity as Stacey, was unable to follow her explanation. It is clear however that Stacey *was* explaining something to herself, and elsewhere I provide a possible interpretation of the deductive reasoning involved in her thinking (Reid 1995b, Kieren, Gordon Calvert, Reid, & Simmt 1995) which suggests she was operating from a mathematical orientation.

Conclusion

Researching proving involves dealing with complexities. Many types of reasoning are involved, there are degrees of formality of written proofs and formulations of reasoning, and the needs which motivate proving are many and their importance is only beginning to be understood. The examples I offer in this paper are intended to help clarify one need to prove: the need to explain. In exploring the need to explain I also address the features that qualify explanations as mathematical explanations, which is an important part of defining a mathematical emotional orientation. The criteria for acceptance of explanations in mathematical communities include the use of deductive reasoning and reference to shared experiences of notation, definitions and established theorems. Observation of these characteristics in students' explanations suggests progress in their adoption of a mathematical orientation.

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MONITORING OF DYNAMICS OF STUDENTS' INTELLECTUAL GROWTH IN MPI-PROJECT

Sam Rososhek
Algebra Department
University of Tomsk – Russia

The system of tests, which allows to follow the dynamics of intellectual growth of students aged 10-15, has been worked out within the frames of MPI- project Accomplishment of test tasks corresponds to a certain level of organization of individual mental experience, which is considered as the level of intellectual growth. Student's accomplishment of every test allows to receive a multidimensional vector, characterizing the state of his/her individual mental experience at the moment of doing the test. Totality of such vectors for all the tests during the whole period of studies makes up the individual trajectory of a student's intellectual growth in multidimensional space. We propose to discuss the tests themselves as well as some results obtained in Tomsk schools.

The system of the tests, which allows to follow the dynamics of intellectual growth of students aged 10-15 has been worked out within the frames of MPI-project (MPI: "Mathematics. Psychology. Intelligence."), the head of the project is Prof. E.Gelfman. The project is directed at the development of students' individual cognitive experience. Fulfilling test tasks corresponds to a certain level of organization of individual mental experience of a child, in other words, a certain level of a child's intelligence. The theoretical foundation of the paper includes Kholodnaya's conception about intelligence as individual mental experience [1] and Weyl-Shafarevich conception about Algebra as the collection of coordinatizing quantities systems [2-4].

Let us have a look at basic pivotal lines, which cross the whole system of the tests and which correspond to certain components of individual mental experience of a child. They are: 1) a line of comparing numbers which by the end passes into a line of inequalities; 2) a line of equations; 3) a line of operations with numbers which by the end passes into a line of algebraic structures; 4) a line of word problems, which by the end passes into a line of mathematical modeling; 5) a line of search for regularities in a row of numbers which passes into a line of functions; 6) a line of visualization of abstract mathematical notions.

We shall dwell on some of these lines in a more detailed way. Let's begin with a line of comparisons. Here is a mid-year test for the 1st year of studies in MPI-project, which along with skills of comparing decimal fractions, checks the ability of transformation of one way of recording numbers into the other one.

Task 1 from a mid-year test for 10-11 year old students. Insert the omitted sign of comparison for the following numbers: "seven integers seventy nine thousandths" ... "seven integers eight hundredths. a) <; b) >; c) =".

A corresponding task which is given at the end of the 1 st year of studies (age 10-11). Insert the omitted sign of comparison for the following numbers:

"thirteen of tenths"... "one hundred and three hundredths". a) <; b) >; c) = ; d) another answer (point out, which one)".

Complication of task of the test which is given at the end of a school year is made simultaneously along several directions: first of all, if in a mid-year test it is presupposed that the right answer is given among the variants of the answers, it is not obligatory for the final test at the end of the school year. Secondly, besides transformation from verbal form of record of numbers into sign form, here transformations of a decimal fraction into a standard form is necessary and that requires comprehention of the notion of a decimal fraction, but not only formal habits of actions. It should be noted that a notion of a common fraction in MPI-project is given after a notion of a decimal fraction, that's why children are not

acquainted with formal transformation of fraction $\frac{13}{10}$ into decimal fraction 1.3.

Let's have a look at a corresponding task for the 2nd year of studies. Here is an example of a mid-year test for students aged 11-12. Insert the omitted sign of comparison into the following numerical expressions: "thirty one minus fifty"... "forty nine minus seventy. a) <; b) >; c) =".
A task which is given at the end of a school year to students aged 11-12.
Insert the omitted sign of comparison into the following numerical expressions: "minus one hundred eighty nine of twenty sixths"... "minus seven integers twenty six hundredths. a) <; b) >; c) =; d) the other answer (which one)".
Complication of a final test in comparison with a mid-year test for children aged 11-12 lies in the following: integers are compared in a mid-year test and in the final test we compare rational numbers (represented in different forms: common and decimal fractions).

Now let me give an example of a comparison task for students of the 3d year of studies, aged 12-13:
"Compare the values of numeric expressions and insert the omitted sign of compari-

son: $\left(0.04-\frac{3}{72}\right)\ldots\left(\frac{1}{6}-\frac{1}{8}-0.04\right)$. a) <; b) >; c) =".

This task is directed at exposing students' ability to make a generalized numeric reasoning. The majority of students who were doing this task tried to transform common fractions into decimal ones and they had to pass to approximate values of numbers and, as a result, couldn't get the correct answer. Some of the students transformed decimal fractions into common ones, and having done the required operations with the fractions, they compared the fractions, which they have got as the result. Maximum number of points was given to the students who, having transformed the numerical expressions into the form

$$\frac{1}{25}-\frac{1}{24} \text{ and } \frac{1}{24}-\frac{1}{25},$$

didn't start to calculate their concrete values, but compared a negative number to a positive one.
Another task from a test, which is given at the end of the school year to 12-13 year old children: "compare the numbers and insert the omitted sign of comparison: 2^{100} ... 1000^{10}. a) <; b) >; c) = ; d) another answer (which one)".
In this task a generalised numerical reasoning was in transformation of numbers into the form $(2^{10})^{10}$ and $(1000)^{10}$ with the consequent comparison of 2^{10} and 1000.

Let us pass over to the task of the 4th year of studies (children aged 13-14).
Here is an example of a mid-year task: "compare numbers and insert omitted sign of comparison:

$$\frac{\sqrt{2}}{3} \ldots \frac{\sqrt{3}}{4}.\ \text{a) <; b) >; c) =".}$$

A task which is given at the end of the 4th year of studies (13-14 years): "compare the numerical expressions and insert the omitted sign of comparison:

$\left(\sqrt{8}-\sqrt[3]{22}\right)\ldots\left(\sqrt{10}-\sqrt[3]{32}\right)$. a) <; b) >; c) = ; d) another answer (which one)".

If in a mid-year test a generalized numerical reasoning is given in the form of a numerical expression with consequent comparison of subradical expressions, then in a final test the reasoning is complicated by the usage of transformation of numerical expressions, which allow to establish that they have different signs.

In the tests for students of the 4th year of studies there appear numerical inequalities (inequalities and systems of two inequalities with one variable). Finally in the 5th year of studies we include quadratic inequalities and inequalities with a modulus in the test. Success in work with this material largely depends on success of work along the whole line of comparison task, beginning with the 1st year of studies.
Now let's consider the line of algebraic operations. Here is a task from a mid-year test for students, aged 10-11: "Calculate using the properties of arithmetic operations (point out, which ones): $92(57-39)+18\cdot 208 = \ldots$ a) 7100; b) 7200; c) 7300".
If the correct answer has been got without using properties of operations, one point was given. If the properties of operations have been made use of for getting the correct answer, but they were not named, two points were given and only usage of properties of operations for getting the correct answer with naming these properties was estimated by three points. Here is an example of a test, given at the end of a school year to 10-11 year old students: "Make calculations using the properties of arithmetic operations (point out, which ones): $1.71\cdot 0.18 + (3 - 2.982)2.29 = \ldots$ a) 7.2; b) 0.72; c) 0.072; d) the other answer (which one)".
Complication of a task of a final test in comparison with a mid-year test lies in the fact, that, firstly, there was the correct answer among the given answers in a mid-year test but in a final test it is not obligatory and, secondly, numerical domain with natural numbers is widened to decimal fractions.
In MPI-project we give two tasks to 2nd year students, aged 11-12. One task consists of using familiar properties of arithmetic operations and the 2nd task is aimed at

checking some of the properties for operations of subtraction and division in different numerical domains.
Here is a mid-year test for students, aged 11-12: "Make calculations using the properties of arithmetic operations (point out, which ones)
$39 \cdot (-2) - (14-16)(-111) = \ldots$ a) 300; b) –300; c) 144".
Another task from a mid-year test for 11-12 year old students: "we may say about property of commutativity for subtraction of whole numbers $a - b = b - a$ that it is ... a) always fulfilled; b) never fulfilled; c) fulfilled for some numbers and not fulfilled for the other numbers".
Tasks which are given at the end of the 2nd year (11-12 year old students):
1) "make calculations using properties of arithmetic operations (point out, which ones):

$$\left(6\frac{1}{4}\cdot(-32.4)-(-32.4)\cdot 8\frac{1}{8}-60.75\right)\cdot 3.4-15\frac{3}{5}=\ldots$$

a) –15.6; b) –199.2; c) –428.7 ; d) the answer is different (point out, which one)".

2) "we may say about property of associativity for division of rational numbers that it is: $a:(b:c)=(a:b):c$. a) always fulfilled; b) never fulfilled; c) fulfilled for some numbers and not fulfilled for the other numbers; d) the other answer (which one)".

Now, let's consider tasks of tests which are refered to the line of algebraic operations (third year of studies).
Tasks for a mid-year test for students aged 12-13:
I. "We may say about property of "right" distributivity for division of rational numbers relative to addition $(x+y):z=x:z+y:z$ that it is. a) always fulfilled; b) never fulfilled; c) fulfilled for some numbers and not fulfilled for the other numbers".
II. "We may say about property of "left" distributivity for division of rational numbers relative to addition $x:(y+z)=x:y+x:z$ that it is. a) always fulfilled; b) never fulfilled; c) fulfilled for some numbers and not fulfilled for the other numbers".
Tasks from the test, given by end of a school year to students aged 12-13:
I. "We shall define the operation on a set of whole numbers according to the following rule: for any pair of whole numbers m and n, let $m\Theta^2 n=(m-n)^2$. We shall call operation Θ^2 "quadratic difference" of numbers m and n. We may say about property of commutativity for quadratic difference of whole numbers $m\Theta^2 n = n\Theta^2 m$ that it is. a) always fulfilled; b) never fulfilled; c) fulfilled for some numbers and not fulfilled for the other numbers; d) another answer (which one)".
II. "We shall define an operation on a set of whole numbers according to the following rule: for any pair of whole numbers m and n let $m\Theta^3 n$ mean the following: $m\Theta^3 n=(m-n)^3$. We shall call this operation Θ^3 "cubic difference" of numbers m

and n. We may say about property of commutativity for cubic difference of whole numbers $m\Theta^3 n = n\Theta^3 m$ that it is: a) always fulfilled; b) never fulfilled; c) fulfilled for some numbers and not fulfilled for the other numbers; d) the answer is different (point out, which one)".

Now let's pass over to the test for the 4th year of studies. Here is an example of a mid-year test for students of the 8th form (13-14 year old):
"We shall say that natural number n is the order of permutation g if multiplication of permutations $\underbrace{g \cdot g \cdots g}_{n \text{ times}} = g^n$ is equal to identical permutation and that n is the smallest number with such property. Order of permutation $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$ is equal ... a) 6; b) 5; c) 3".

Here is an example of a final test for the 8th form students aged 13-14:
"Let us consider a set of A numbers in the form $x + y\sqrt{2}$ where x and y are whole numbers. Then, relative to addition of real numbers, this set is: a) noncommutative group; b) commutative group; c) not a group; d) the other answer (which one)".
Finally, in tasks for the fifth (concluding) year of studies, according to MPI-project, the lines of algebraic operations and visualization of abstract mathematical notions are absent as they are united on a new level in work according to computer program on visualization of algebraic structures (groups, rings and so on) and are tested separately.

In conclusion we shall give development of one more line — word problems, which by the end passes into mathematical modelling.
Here are examples of mid-year test for the 5th form (10-11 year old).
I. "What is the price of an ice-cream if the box with ice-cream costs 16 thousand roubles and there are 32 pieces of ice-cream? a) 5 thousand roubles; b) 2 thousand roubles; c) 500 roubles".
II. "Put down the expression for definition of the price of an ice-cream in roubles, if the price of a box of ice-cream is c roubles and the number of pieces of ice-cream in a box is k. a) $k \cdot c$; b) $c : k$; c) $k : c$".
It is interesting to note that while answering the proceeding task, many students give a wrong answer of 2 thousand roubles. But some of them, having got the correct answer to the second task, turn back to the 1st task and get the correct answer.

A task from the test given at the end of the 5th form (10-11 year old):
"Introduce additional information and define the distance which a boat covers in t hours if the speed of the boat is equal to u km/h and the speed of the river-stream is v km/h. a) $(u + v)t$; b) $(u - v)t$; c) $(u - v): t$; d) the other answer (which one)".
A task from a mid-year test for the 6th form (11-12 year old).

"At the beginning of the year a company issued k shares and by the end of the year got profit of n roubles. At a general meeting of shareholders it was decided that a part of the profit (namely, m roubles) should be used for development of the company and the part of profit, which is left, should be paid to shareholders. Then, shareholders will have per one share...a) $(n - m): k$; b) $n : k - m$; c) $k :(n - m)$" .
A task from the test, given by the end of the year in the 6th form (11-12 year old):
"During interurban bus route one bus stops every 40 minutes and the other bus-every 18 minutes. Both the buses began the movement simultaneously. What is the smallest period of time when both the buses reach their destinations? (It is presupposed that the time which they spend at the bus-stops may not be taken into account).

a) 3.5 hours; b) $5\frac{1}{3}$ hours; c) 6 hours; d) the other answer (which one)".

A task from the final test in the 6th form:
"A set of game device for TV-sets includes the cartridge, the price of which is n roubles, which makes 8% of the whole set price. What is the price of the game device without cartridge?

a) $\frac{8n}{100}-n$; b) $n-\frac{8n}{100}$; c) $\frac{100n}{8}-n$; d) the other answer (which one)".

A task from a mid-year test for the 7th form (12-13 year old):
Define an average speed of an automobile on the part of the road from point A to point B and backwards, if the distance from A to B is equal to S km and it took u hours to cover the distance from A to B and it took v hours to cover the distance from B to A.

a) $\frac{1}{2}(S : u + S : v)$; b) $2S : (u + v)$; c) $(2S : u) + (2S : v)$".

A task from a final test in the 7th form (12-13 year old):
"There are two sectors in the stadium with n seats all in all. Receipts from selling tickets to the match were: to the 1st sector — x roubles, to the 2nd sector — y roubles. How many seats are in the 2nd sector if all the tickets were sold at the same price and there remained no tickets unsold?

a) $y:\left(\frac{x+y}{n}\right)$; b) $y:\left(\frac{n}{x+y}\right)$; c) $\frac{x+y}{ny}$; d) the other answer (which one)".

We should note that the word problems given above were among the most difficult tasks for students of the 7th form.
Here are two variants of the tasks of final test for the 8th form (13-14 year old).
I. The area of a right triangle with cathetuses h_1 and h_2 is B and the area of a square, one side of which is the hypotenuse of this triangle, is A. Define h_1 and h_2.

a) $h_1 = \sqrt{A+2B} + \sqrt{A-2B}$, $h_2 = \sqrt{A+2B} - \sqrt{A-2B}$;

b) $h_1 = \sqrt{A+4B} + \sqrt{A-4B}$, $h_2 = \sqrt{A+4B} - \sqrt{A-4B}$;

c) $h_1 = \frac{1}{2}\left(\sqrt{A+4B} + \sqrt{A-4B}\right)$, $h_2 = \frac{1}{2}\left(\sqrt{A+4B} - \sqrt{A-4B}\right)$;

d) the other answer (which one)".

II. "A raft and a motor boat started simultaneously with the stream from city A to city B. While the raft was on the way, the motor boat, after arriving to city B turned backward and came to city A at the same time with as raft came to city B. In how many times is the speed of the motor boat larger than the speed of the raft? a) 1; b) 2; c) two decisions: $1-\sqrt{2}$ and $\sqrt{2}+1$; d) the other answer (which one)".

In word problems of the final year of studies according to MPI-project, the line of word problems is passing into the line of mathematical modelling. As this line is very important, I shall give corresponding tasks from final test for the 9th form (14-15 year old).

I. "A businessman made up his mind to buy shares of three companies. All in all he wanted to buy 12 shares for 61 dollars and not less than two shares of each of the companies. How many variants there exist on these conditions if the price of shares of these three companies is $7, 5 and 4? Solve the problem by means of making and analyzing a corresponding mathematical model. a) one variant: 3, 4, 5; b) two variants: 3, 4, 5 and 3, 3, 6; c) three variants: 3, 4, 5; 2, 6, 4 and 5,2,5; d) the other answer (which one)".

II. "In city N in the flour market there is the following situation (tabl.1):

Table 1: Flour market situation

Quantity of flour required during a week (*tons*)	Price per a ton of flour (*mln. roubles*)	Quantity of flour, supplied during a week (*tons*)
100	1.5	20
70	1.7	40
50	1.8	60
30	1.9	90
10	2	100

Construct a mathematical model of the given situation and give the answer to the question: what is the equilibrated price and the equilibrated quantity of flour for the given market? a) 1.7 mln roubles per a ton and 55 *t*; b) 1.8 mln roubles per a ton and 50 *t*; c) 1.7 mln roubles per a ton and 53 *t*; d) the other answer (which one)".

III. "In conditions of a standard model "foxes-rabbits" let's assume that at a certain period of time there were 120 rabbits and 50 foxes on an island. Find out short-term and long-term consequences of the administrative decision to shoot 10 foxes and 25 rabbits during the 1 st year. In particular, can this decision destroy the ecosystem of foxes-rabbits?"

Now we shall consider how, with the help of the tests, to define the individual trajectory of a student's intellectual growth during the period of studies. According to MPI-project, as it is seen from the examples given above, their complication from year to year (and from mid-year to final) demands more and more various in volume

and structure individual mental experience of a child. That's why we may suppose that the quantity of the points which a student gets for the tests of one line for the half or for the whole year, correspond to the level of development of one component of individual mental experience at a given moment. Then a set of points, which have been got for all the lines of the test system in a mid or final test is a vector in a certain multidimensional space. Totality of such vectors for all the tests during the whole period of studies makes up the individual trajectory of a student's intellectual growth. Preliminary results of the tests showed that, if maximum summary number of points is from 25 to 30 for different tests, there may be pointed out some ranges of summary number of points, which may be interpreted in the form of a certain current estimator of state of a child's individual mental experience.

If summary number of points for a test is less than 10, then current state of individual mental experience is considered unsatisfactory. If summary number of points is within range from 10 to 14, then current state of individual mental experience is considered satisfactory and for the range from 15 to 19 is considered good. Finally, excellent mark corresponds to the range of 20 points and larger. A key question: how should we evaluate different individual trajectories? This question requires a separate investigation, taking into account test materials in different schools. So far, on the basis of the results of preliminary tests we can move forward hypothesis: if in a certain test a definite range for summary number of points was achieved, then we may speak of intellectual growth, if in the following tests the summary number of points remains within the same range or higher. It should be noted that besides summary number of points, we should taken into account different components of the vector of points for each of the tests. In practice, the results of a test in any concrete group are recorded in the form of a matrix, in the columns of which we record different tasks of test and in the rows we record the students of this group. Following the results along different lines of a the tests, we may find out the dynamics of development of different components of individual mental experience.

In conclusion, a few words about psychological aspects of the given system of tests. Students who worked with these tests were asked to write reflections where they could express their thoughts and impressions about the search for decision or about the tasks themselves. Even, these reflections which we have now, contain such interesting psychological material that demands a separate discussion.

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STUDENT-GENERATED MULTIPLICATION WORD PROBLEMS

Denise W. Rowell
Karen S. Norwood, Ed.D.
North Carolina State University

This study examines sixth graders' attempts to generate multiplication word problems. Twenty-eight students were interviewed and were asked to write a story problem which could be solved by multiplication. Less than 40% of the students could write a problem which was multiplicative in context, yet most of the students used multiplication to solve the problem. In spite of the interviewer's attempt to promote disequilibrium, the students never understood that their problems were not multiplicative in context. This research report discusses the students' attempts to generate problems, their refusal to be put in disequilibrium, and their inability to recognize that context determines the operation.

Introduction

There has been a considerable amount of research done recently on children's ideas of multiplication. Multiplication is much more difficult than addition because it is a binary operation. (Anghileri, 1989; Clark & Kamii, 1996; Simon & Blume, 1994; Vergnaud, 1983) Addition is unary; it only requires thinking in one dimension. Multiplication, however, requires the "coordination of two dimensions". (Simon & Blume, 1994)

The difficulty that children encounter with multiplication is quite surprising considering that students tend to learn the facts and the algorithms at an early age. (Clark & Kamii, 1996; McIntosh, 1979; O'Brien & Casey, 1983) The problem is that there is more to multiplication than just facts and algorithms. In fact, there are two different aspects of multiplication: computational and contextual. (O'Brien & Casey, 1983) Many students can compute products without understanding multiplication in context. (Clark & Kamii, 1996; O'Brien & Casey, 1983) The focus of this paper is on children's *contextual* understanding of multiplication.

Several activities have been done to determine multiplicative reasoning in children. Clark and Kamii (1996) had students in grades 1 through 5 perform a Piagetian "fish task" to determine the ideas that children have about multiplication. They found that multiplicative thinking begins early in some students, but is slow to develop. Graeber & Tirosh (1990) had students give a definition of multiplication and found that most students define multiplication in terms of repeated addition. Dorwaldt (1989) and Anghileri (1989) found that many students do not understand the language of multiplication and thus have trouble understanding multiplication in context. In several studies (De Corte et al., 1994;

Graeber & Tirosh, 1990; McIntosh, 1979; O'Brien & Casey, 1983), students were asked to write multiplication word problems. O'Brien and Casey found that many of the students wrote problems that were additive in context, but used multiplication to solve them. De Corte et al. found that most students who wrote appropriate multiplication word problems used the repeated addition model. Graeber and Tirosh also found that the students who were successful in writing multiplicative problems used either the repeated addition model or the union of equivalent groups. McIntosh found that many younger students could not write a problems that was multiplicative in context, and that some of the students who did write multiplication problems wrote problems which were checks for division. (For example, "6 boys climbed up a tree and collected 18 acorns each boy had 3 acorns." (McIntosh, p. 14))

Children have an informal intuition about multiplication, but often they cannot match their informal knowledge with the formal instruction they receive in school. (Resnick, 1986) Even when teachers try to promote disequilibrium (Simon & Blume, 1994), students tend to rely on their formal ideas instead of their intuition (even when their formal ideas are erroneous).

Theoretical Framework

The theoretical framework for this study is based on a constructivist view of learning. "[C]onstructivism holds that all knowledge is constructed and that the instruments of construction include cognitive structures that are...themselves products of developmental construction." (Noddings, 1990, p. 7) In addition, most constructivist mathematics educators believe that children enter school with some informal knowledge of mathematics. (Baroody & Ginsburg, 1995; Resnick, 1986) The role of the mathematics teacher is to determine what knowledge they have constructed and provide them with the experiences that allow them to construct more logical or more efficient methods for doing mathematics.

In particular, the authors framed the study using the theories of Piaget, Vygotsky and Resnick. Like Steffe and Cobb (1988), we interpret children's cognitive development using Piaget's concept of schema, assimilation, and accommodation, and we agree with Vygotsky's concept of the zone of proximal development. However, in addition to this, we subscribe to Resnick's ideas about children's errors, as well as their intuition about mathematics.

According to Piaget, children adapt to the environment and organize their experiences using what Piaget calls *schema*, *assimilation*, *accommodation*, and *equilibration*. (Wadsworth, 1996) A child's *schema* is the cognitive or mental structure he uses to organize the environment. As he receives new bits of information, he tries to integrate that information into a previously existing schema. Piaget calls this process *assimilation*. When the child receives

information that will not assimilate into a previous schema, he is said to be in *disequilibrium*. This child has to create a new schema or modify an existing schema in order to make the new information fit. This process of modifying cognitive structures is *accommodation*. *Equilibrium* is reached when there is a balance between assimilation and accommodation. Finally, *equilibration* is the process of moving from disequilibrium to equilibrium.

In addition to the theory of Piaget, the authors agree with Vygotsky's zone of proximal development (ZPD). Vygotsky describes the ZPD as "the gap between the child's level of *actual development* determined by independent problem solving and her level of *potential development* determined by problem solving supported by an adult or through collaboration with more capable peers." (Dixon-Krauss, 1996, p. 15) It is through adult or peer collaboration that children can make constructions that they might otherwise have been unable to make.

The piece that seems to be missing in the theories of Piaget and Vygotsky is that they neglect to discuss the "gaps" in constructions. In other words, Piaget fails to describe what happens when children create new schema, or modify existing schema, incorrectly. Vygotsky does not discuss what happens when children are in their zone of proximal development, but fail to develop to their potential. Lauren Resnick addresses these issues. When students perform mathematical tasks, many of them make similar errors. (Resnick, 1986) Some constructivists call these errors "weak constructions". (Noddings, 1990) Resnick calls these errors "buggy algorithms". She claims, "Buggy algorithms are ... clear examples of inventions that are unsuccessful." (Resnick, 1985, p. 274)

Resnick also claims that children enter school with an intuition about mathematics. (Resnick, 1986; see also Carraher, Carraher, and Schliemann, 1985) However, the instruction they receive rarely makes use of this informal knowledge. Students have formal instruction in mathematics which is full of algorithms and is unrelated to everyday life. Typically, children who do well in mathematics are able to make the connection to real life; whereas, "weaker mathematics learners seem very prone to allow mathematical symbols to become dissociated from their quantity and situational referents. . . . [they] try to memorize and apply the rules that are taught, but do not attempt to relate these rules to what they know about mathematics at a more intuitive level." (Resnick, 1986, p. 191)

Methodology

The participants in the study were 28 sixth graders taken from an intact class. The students were enrolled in a small public school in central North Carolina. To collect data, we conducted clinical interviews with each student individually. The interviews were done in April of the school year, (after the

students had reviewed multiplication). Each interview was videotaped. The students were asked to complete seven different multiplication tasks, one of which was to write a story problem. Each student was given the following directions, both verbally and in written form:

Write a multiplication problem that follows these rules:

(1) The problem must end in a question.

(2) The question must be one that is possible to answer by multiplying.

(3) Solve your problem in as many ways as you can.

The interviews were conducted by one of three researchers. A pilot study was done with fifth graders to insure that the researchers were consistent in their questioning techniques. As a result of the pilot study, minor revisions were made with the interview protocol.

During the interview, the student had access to paper and pencil. Each question was read to the student by the interviewer. The interviewer then placed the written instructions in front of the student for reference. The interviews lasted from 20 to 45 minutes.

This paper focuses on the story problem task. After the student wrote a problem, the interviewer asked probing questions such as "Can you draw me a picture of your problem?" and "Can you think of another way to solve your problem?" In the cases where students wrote problems which were not multiplicative, the interviewers asked additional questions in an attempt to promote disequilibrium. In some of the interviews, after attempting to promote disequilibrium, the interviewers asked the students to write another word problem.

We coded the data using an adaptation of a coding scheme used by O'Brien and Casey. (1983) We chose to code our data as follows:

1. Wrote a problem involving multiplication
   a) Array model
   b) "Times" model
   c) Repeated addition
   d) Two-step problem (one step required multiplication)
2. Wrote a problem involving multiplication as a computation
3. Wrote a problem involving addition
4. Wrote a problem involving subtraction
5. Wrote a problem involving division
6. Wrote a problem involving comparison
7. Wrote a problem which was nonsensical

For each student who wrote a multiplicative word problem, the problems were classified according to the model of multiplication that the student used. Problems which involved multiplication as a computation were not considered to be multiplication problems. For instance, one student asked, "What times what equal 36?" Finally, a few of the problems were not story problems in that they did not ask a question or that they did not make sense. These problems were classified as nonsensical. For example, Todd wrote: "If a boy has 3 blocks on his desk and his friend ask him can he use his 3 bloks [blocks] to multiple [multiply] then his friend sied [said] what is the anser [answer]? 9"

Preliminary Results and Conclusions

We found that only 39% of the students were able to write story problems which were multiplicative in context. Of the students who were able to write a multiplication story problem, 90% used an *array* model of multiplication. In addition, there were 6 addition problems, 4 subtraction problems, 2 division, 1 comparison, and 3 nonsensical. (But in general, students who wrote these problems used multiplication to solve them.) Even when students used manipulatives to model their problems, they did not realize that the context of their story problem was not multiplication.

We also found that in spite of the interviewers' attempts to put the students in disequilibrium, the students refused to adjust their thinking to fit the circumstances. The following transcript is an example of an interviewer's attempt to promote disequilibrium.

Frank's Story Problem: "Their are 5 people playing basketball on one team and 5 other people playing on the other team. If you multiply 5x5= How many players will be in all?"

Interviewer: How would you draw a picture of this?

Frank: I would draw circles for the people -- 5 on one side and 5 on the other side and then make a multiplication [sign] [drew 5 circles, x 5 circles, then drew a 5x5 array of circles]

Interviewer: [pointing to row 1 of the array] I see 5 people on this team and [pointing to row 2] 5 people on the other team, but where did these people [pointing to rows 3,4, and 5] come from?

Frank: They came from...[long pause]... they came from when you times 5 times 5 and they came from that...from 5 times 5.

Interviewer: So why did you multiply?

Frank: So there would be enough people on the team...Like, just in case if 3 people got hurt, there would only be 2 people left and then you can put in 3 more people.

Interviewer: Let me ask you this: if you were watching a basketball game and ... they're playing each other and you see 5 on say the Chicago Bulls and 5 on ... you know. How many would be out there playing?

Frank: 5...I mean, 10

Interviewer: How did you get that?

Frank: How did I get 25?

Interviewer: No, how did you get 10?

Frank: Added 5 plus 5

Interviewer: What's the difference between your problem and my problem?

Frank: This is a multiplication problem [pointing to his problem] and this is addition [pointing to interviewer].

Interviewer: OK, why is this one [Frank's problem] multiplication?

Frank: [Long pause] 5, maybe 5 people wouldn't be enough to ... [long pause] play on the team or ... [long pause] because ... I don't know ... because ...

In spite of our efforts to put them in disequilibrium, the children did not understand that the context of the problem determined the operation to use. As McIntosh (1979) put it, "A great deal of children's difficulty with numbers stems from their not being able to see what the problem is about, so they simply extract any number in sight and perform some computation." (p. 15) In fact, one of the students who wrote an addition problem was given chips to model her problem. She saw that when the problem was modeled she got 5 (3 + 2) as the answer (instead of 6, which was her answer). She even claimed to understand that her problem was actually an addition problem. However, when we asked her to write another problem, she wrote another addition problem, but concluded the problem with "Use multiplication to show your answer."

This research report will address the children's inability to recognize that context determines the operation, their refusal to be put into disequilibrium, as well as the implications for teaching multiplication.

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THE CLINICAL INTERVIEW: CONDUCT AND INTERPRETATION

Tim Rowland

Institute of Education, University of London

The purpose of this paper is to illustrate the power and the process of the clinical interview as a tool for research into students' mathematical thinking. The case in point involves a 10-year-old solving a problem requiring division of an integer by a fraction. The interview unearths unexpected idiosyncratic methods and links with decimal fractions, leading the analyst to speculate about rational number constructs that might underpin such actions.

THE CLINICAL METHOD: PRINCIPLES AND PITFALLS

Piaget's post-doctoral studies in therapeutic psychoanalysis – he read Freud and attended Jung's lectures – inspired his approach to the study of children's thinking.

> This is the method of clinical examination, used by psychiatrists as a means of diagnosis ... [in which] the good practitioner lets himself be led, though always in control, and takes account of the whole of the mental context. (Piaget, 1929, pp. 7-8).

The clinical method, the basis of Piaget's work for half a century, has been adopted by researchers world wide. Ginsburg's work, for example, has drawn extensively on his own mathematical conversations with children.

> The primary method is the in-depth interview with children as they are in the process of grappling with various sorts of problems [...] Interviews like these ... are rare in mathematics education but essential to improving it. (Ginsburg, 1977, p. iv)

The clinical method is appropriate for the purposes of identifying (eliciting), describing and accounting for cognitive processes (Ginsburg *et al.*, 1983, pp. 11-13). The verbal clinical interview is characterised by the following features (*ibid.*, pp. 18-20): (a) the interviewer employs a *task or tasks* to channel the subject's activity (b) the interviewer's questions are *contingent* on the child's responses: the interviewer constantly makes instantaneous decisions about the direction of the interview (c) the procedure demands *reflection*: the interviewer asks the subject to reflect on what s/he has done and to articulate her/his thoughts (d) the contingent nature of the procedure enables the interviewer to *test hypotheses* that s/he has generated in this interview, or in earlier interviews.

The contingent interviewer is having continually to make rapid assessments of what 'witnesses' say, to probe without leading the witness, striving to create the conditions for the surface manifestation of the subject's thought.

Such mathematical conversations can provide teachers with effective assessment data, and the professional skills of teachers related to questioning ought to equip them particularly well to deploy the method for diagnostic purposes. The primary intent of the clinical interview as a research tool is not to teach the child but to enlighten the interviewer. The temptation for teachers to teach remains powerful, however. Piaget

himself remarked that "It is so hard not to talk too much when questioning a child, especially for a pedagogue!" (Piaget, 1929, pp. 8-9)

Doig and Hunting describe a programme for training teacher-clinicians to use interview methods of student assessment. They note that clinical approaches to assessment offer the advantage of interactive communication between the student ("data source") and the teacher ("data interpreter), but that teacher-clinicians have been observed to "fall back on ingrained methods such as telling students, or providing direct information rather than questioning" (Doig and Hunting, 1995, p. 285). At the previous PME, Markovitz and Even (1994) reported that junior high teachers are more prone to these teacher-habits than elementary teachers in clinical interview situations.

According to Hunting (1987, p. 145), it is as though the technique were "rediscovered" in the 1980s. The purpose of this paper is to make a contribution to the illumination of the process of the clinical interview – conduct and analysis. The bonus is speculative insight into the mind of a remarkable, young mathematical thinker.

CONSRUCTIVIST PERSPECTIVES ON THE CLINICAL METHOD

Continuing international interest in radical constructivism lends urgency to the need to make sense of the way individual learners have constructed their mathematical knowledge, and adds weight to the efficacy of the clinical interview as a means of achieving this objective. At the same time, radical constructivism forces a re-assessment of the analyst's interpretation of the clinical interview. An essential means of communication in the interview is language. From a constructivist point of view (von Glasersfeld, 1995), each individual associates words with past experiences. The 'meaning' of the word (for that individual) consists of a co-ordination of word and experience (or *re-presentation* of experience), so that each calls up the other. Von Glasersfeld calls this two-way (mental) use of language 'symbolic', arguing that symbolic communication is achieved by means of co-ordinations which are *compatible*: their being identical is neither necessary nor, for that matter, empirically verifiable. This compatibility is inherent in the notion of taken-to-be-shared meanings (Cobb, 1989).

From this perspective, the clinical analyst is not, in any direct sense, finding out 'what is going on in the head' of the subject, but fabricating a story, an account of the subject's inner world which is compatible with the analyst's own construction of the subject him/herself, of the subject-matter in hand, of the nature of knowledge and learning, and so on. Doig and Hunting (1995) and Hunting (1997) refer to the need for sound pedagogical content knowledge on the part of the clinical interviewer, as s/he constructs a theory of the student's understanding in the interview and tests the theory through contingent questioning. My 'reading' of a pupil's contribution in the account which follows is significantly informed by various investigations and reports on student's rational number constructs (much of it usefully summarised and integrated in Carpenter *et al.* 1993). Numerous related contributions to PME include Markovitz and Even (1994), Goldin and Passantino (1996) and Lamon (1996), each of which refer to clinical methodology.

A CLINICAL INTERVIEW: THE ANALYST'S ACCOUNT

The interview discussed in the remainder of this paper is one of a number which I conducted and transcribed over a period of some years (Rowland, 1999). My interest was in how pupils apply the framework of knowledge that they possess to the solution of non-routine problems which invite generalisation. The problem which inaugurates this interview can be represented as $100 \div \frac{3}{4}$. The problem is presented in terms of 'quotition', the separation into an unknown number of parts, each of a given size.

Susie was aged just 10. I had interviewed her a number of times, and judged that she would find a presentation such as "How many lots of three-quarters are there in a hundred" accessible but non-trivial.

The interview begins[1]:

1	TR:	Right, I want you, to start with Susie, to explain to me what you mean by three-quarters.
2	SL:	Well, if you have one thing, whole thing, and you cut it in half, and then the two bits in half again, and take away three of them, and take away one of them, the three left are three-quarters.
3	TR:	What about five-sevenths?
4	SL:	Well once again, if you cut a cake like that, [she draws] cut it into seven equal pieces, it has to be equal ...
10	SL:	It has to be equal, so if you take away two of those pieces you've got five left.

At this stage, as analyst/researcher, I am probing[2] to discover what kind of constructs are bound up in Susie's 'knowing' of rational number? How might her rational number scheme determine particular practical actions on her part? (Kieren, 1993, p. 57). Her contributions [2] and [10] are significant in this respect, and will be considered later. Following the preamble, I quickly got to the point:

11	TR:	Right. Right. Now the next thing I want you to think about is, how many lots of three-quarters are there in a hundred?
12	SL:	How many lots of three-quarters are there in a hundred.
13	TR:	Yes.
16	SL:	Em, [...] ... how many lots of three-quarters ... Yes ... I think ... have one hundred, add ... what was it? ... oohhh ... It's impossible to third it - a hundred, you need a third of a hundred. So that must be [writes] three ... thirty-three point three recurring. [Writes 33.3r, changes r to R] I'll put a capital R for that because that [r] means remainder. Ah, so if you add those two together, together, it should be one hundred and thirty-three point three recurring.

[1] The transcript 'turns' are numbered 1, 2, 3 ... and referenced as [*n*] in the commentary/analysis. Some turns have been omitted in order to prioritise other material in this paper.

[2] My commentary is mainly in the present tense, to convey the contemporaneous nature of interpretation and contingent response.

In effect [16], to find how many ¾ there are in 100, Susie adds to 100 a third of 100. I note in passing her preference for the infinite decimal representation of one-third. My attempt to find out why she believes (with good cause) that her procedure works does not meet with success.

17 TR: Right, [..] what I actually asked you, right, was how many lots of three quarters in a hundred. And what you've done is to take a hundred, and then a third of a hundred, and add it on. Now, can you explain to me how that tells you how many three-quarters there are in a hundred?

18 SL: I actually have worked that out quite a long time ago; worked that out, I did. It hasn't ... I did actually do some maths and worked it out that way [inaudible]. I did try doing it that way, and then tried doing it another way. It worked, but the other way I had was too difficult, so I just stuck to this way.

19 TR: When did you work out this other way?

20 SL: I can't remember ... well ...

I am still unclear about much of what she is saying in [18]. What exactly did she work out "quite a long time ago"?. What was this other way? Invert and multiply? Her teachers informed me that she had not been taught that method at school.

My next question is designed to find out whether Susie can adapt her method to division by other fractions, choose one that will readily allow such a method to be demonstrated.

23 TR: OK. What about, how many four-fifths in a hundred?

24 SL: Four-fifths. It's correct, isn't it? [refers to 133.3R written]

25 TR: Oh yes, that's correct.

26 SL: Em, that was three ... what was that again? That was three-quarters.

27 TR: That was three-quarters in a hundred, now we're talking about four-fifths.

28 SL: That must be a fourth, this must be a fourth of it ... mmm ... a fourth of a hundred [pause]

29 TR: A fourth of a hundred. OK, so what have you got?

30 SL: One hundred and twenty-five.

31 TR: OK. Describe what you do again. Just describe.

32 SL: I just had the hundred, and then I had one fourth of hundred, and added them together.

Much as I (the analyst) marvel at the ingenuity of her method, I also (the mathematics educator) realise its limitations. But does Susie? Indeed, she does. [34, 36]

33 TR: Right. What sort of ... can you give me another example of how many lots of something in a hundred, that you could do that way.

34 SL: Well, you could, there, it's a fact you can, could not do the same way with ... seven ... ninths.

35 TR: You couldn't do the same. Why not?

36 SL: Because it has to be one different. Like with this one [indicates ¾] it was one and that one [$^4/_5$]was one.

39 TR: One difference between which numbers?

40 SL: The number of them and the quarter. And it has to be the smaller one to the number.

Susie seems to be expressing the required general relationship between numerator and denominator (it is not clear whether or not she knows these terms) with reference to the particular example ¾ . In fact, "The number of them" [40] precisely captures the etymology of 'numerator'. I hesitate (the cough, [41]) before asking my next question, because I predict that Susie's answer will require her to divide 100 by 6, and (knowing her) that she will do it mentally. I was quite unprepared for her response:

41 TR: Right, [cough] could you, um, work out how many lots of six-sevenths there are in a hundred? [...]

44 SL: Now what's a sixth of a hundred? Mm [pause], yes, I know a third of a hundred is that [indicates 33.3R] [I a sixth will be half that. [ ] So what's half of three. Ah, one ...

49 TR: Can you work it out up here? [away from written 33.3 R]

50 SL: No, I'm working out this, so I can ... I will put it up there when I've worked it out. I need just to be close up there, which will help me work it out.

Susie wrote 15 beneath the 33, then over-wrote the 5 with a 6. Then she wrote 5 beneath the .3R; later she changed that 5 to 6 as well. She volunteers an explanation to me:

51 SL: So, do you know why I carried five into that? I had that one, and I had a half, so that I had that. Why I'm moving in this way, it's a bit different. In a normal adding sum you move the other way [i,e. right to left rather than the left to right here]

52 TR: I'm not clear what you're doing Susie, you're working out a half of thirty-three point three recurring ...

57 SL: I had five [from 15], and I have one of those [from 1.5], and then we carry the five .over to there. And it will continue happening. Six and carry five, six and carry five, so it must be [pause] sixteen point six recurring.

58 TR: Sixteen point six recurring. OK.

59 SL: Which is one hundred and sixteen point six recurring.

In such rare moments, this interviewer struggles to suppress his excitement. My question about dividing by six-sevenths had opened an oyster, to reveal a pearl. In effect, Susie writes that half of 33.333... is 15+1.5+0.15+0.015+ ... (half of each 3); adding the units, tenths, hundredths and so on she gives 16.6666... *"And it will continue happening"*. [57] Six and carry five, six and carry five. The at home-ness of this 10-year-old with infinite recurring decimals is an accidental (but characteristic) by-product of the main thrust of the contingent interview, a theme that I opt not to pursue further at this point.

I choose, instead, to see whether Susie is able to expand her schema to deal with fractions not of the form $^{n}/_{(n+1)}$. I thought it unlikely that she would, but had to test that judgement.

62 TR: OK. Now, you said that it wouldn't work for seven-ninths didn't you, this method. Right? Now, I'd just like you to write down five-sevenths, just here.

My contingent choice of $^{5}/_{7}$ in preference to $^{7}/_{9}$ is significant: I don't want any difficulty in dividing 100 by 7 to block Susie's method, if indeed she has one.

63 SL: I'm going to have to think though, very well. Um, I'll try ... [pause]. Ahh, of course ... [interrupted]

64 TR: You have a think while I push the door up.

65 SL: ... you can't ... I don't understand. It's definitely a hundred. So that means two ... Ahh, ahhh, you've got two left, and you need five each time. So if you have two hundred ... um ... divided by five. How many times does five go into two hundred? Well, it goes into one hundred twenty times ...

66 TR: Mm-hm

67 SL: Must go into forty times. So that's ... a hundred and forty. [...]

69 SL: That's a two difference.

The exclamation 'ahhh' in [65] seems to announce a moment of special insight. Susie proceeds to extend and articulate this generalised process. From this point, she consistently deploys 'you' as vague generaliser (Rowland, 1999):

89 SL: The next one, when you have two difference, you have to do two hundred divided by that, the number, and add a hundred to the equals.

90 TR: Right, add a hundred to the equals.

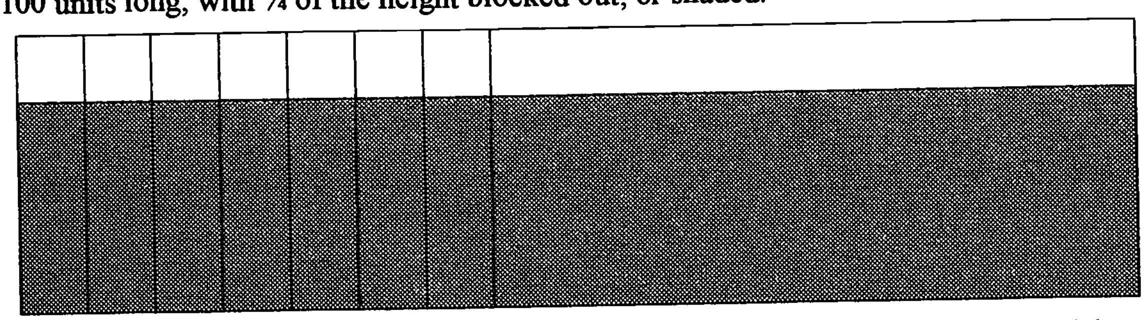
91 SL: And with three you have to do three hundred divided by whatever is the top number.

Susie's computational algorithm can be formulated as follows:

for $p < q$, $100 \div {}^{p}/_{q} = 100 + [100(q-p)] \div p$

DISCUSSION: PART-WHOLE OR PART-COMPLEMENT?

How does Susie conjure up these algorithms, progressively adapted for increasingly general application? I develop a hypothesis during the interview, as follows. I surmise that Susie takes a global view of the division problem $100 \div \frac{3}{4}$, in which she first imagines one three-quarter part of each of the 100 'things'. I don't know what her things are. My corresponding image would be a long thin rectangular strip, standing on a 'base' 100 units long, with $\frac{3}{4}$ of the height blocked out, or shaded.



Thus, 100 "lots of" $\frac{3}{4}$ accounts for $\frac{3}{4}$ of the whole. The remaining $\frac{1}{4}$ is one third of the shaded part. *Addition of one third of the part already accounted for will complete the whole.* The essence of the imagery is the perception of the fraction ($\frac{3}{4}$) in relation to the complement ($\frac{1}{4}$) which is one-third its size.

Pirie, Martin and Kieran (1994) asked three categories of students (school and university) a set of six (written) questions about fractions, including: How would you explain $\frac{3}{4}$? Pirie *et al.* describe four "major images" which emerged from the

questionnaires. (1) Division: a quantity divided by a quantity (2) Part of a whole (3) A number of some sort (4) A way of writing: a number over another number.

I suggest that Susie's image relates to (2)., but is crucially a part-part or *part-complement* image embedded in a part-whole construct[3]. I failed to elicit any direct confirmation of such imagery from Susie by contingent questioning, but there is indirect evidence to support (compatible with) my interpretive hypothesis, as follows.

- Describing a tenth in a classroom episode some months earlier (Rowland, 1992, p. 44), she had said:

  SL: If you have ten, and you take away nine ones, you have just the one left ... it's because you take away a ninth ... no, nine-tenths. So there's one-tenth left.

For Susie, a tenth is the remnant when nine-tenths is taken from the whole.

- At the beginning of this interview, she described three quarters [2] and five-sevenths [10] in part-complement terms in her drawings and in her choice of words.

2	SL:	Well, if you have one thing, whole thing, and you cut it [in four] and take away one of them, the three left are three-quarters.
3	TR:	What about five-sevenths?
10	SL:	it has to be equal, so if you take away two of those pieces you've got five left.

Susie constructs each fraction beginning first with the part-whole construct for the fraction *and* its complement, then removing its complement from the whole.

FINALE

To conclude the interview, I challenged Susie to adapt her method to divide by improper fractions, such as five-thirds. Again, as with $^5/_7$ earlier, I chose $^5/_3$ contingently to minimise for Susie any complications with whole number arithmetic. The challenge was speculative: the hypothesised imagery schema does not seem to extend to dealing with such cases, because the 'parts' to be removed are greater than the initial unit. She rose to the challenge however, this time *subtracting* a fifth of 200 from 100.

177	SL:	One hundred divided by five-thirds [sighs]. Right. See, this one [$^5/_7$] you added it. I can't understand that [$^5/_3$]. That one's harder. Ahh, ahh, oh yes, you would have twice as much for the difference, won't you. So that's ... so suppose we had five-thirds. Want five-thirds. You would have to take a hundred ...
178	TR:	Yes.
179	SL:	Take away ... um, a fifth of two hundred.
180	TR:	Right.
189	SL:	And then you have two hun ... one hundred would be ... that's forty. That equals forty. So take away forty equals ... sixty.

Perhaps she perceives five-thirds as having a complement of negative-two thirds within the whole, and hence the need to 'trim away' the excess two-fifths (from 100 lots of five

[3] The mistaken tendency of some children to denote the part as a proportion of its complement rather the whole is well-documented (e.g. Hart, 1981)

thirds). Alternatively, her thinking at this stage may be predominantly algebraic, encapsulating and making the appropriate adjustment to the earlier procedure.

I was convinced that Susie brought intuitive knowledge of fractions (Kieran, 1993), drawing on unusual but powerful fraction imagery, to create a global, gestalt overview of the problem and a highly idiosyncratic solution. She demonstrates expansive generalisation and readily perceives the possibility of reconstructive generalisation (Harel and Tall, 1991) as she confidently and flexibly adapts her method to solve a comprehensive class of related problems.

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DIVISION WITH REMAINDER

CHILDREN'S STRATEGIES IN REAL-WORLD CONTEXTS

Silke Ruwisch

Institute for Didactics of Mathematics

Justus-Liebig-Universität, Giessen, Germany

One-step division problems in the model of equal groups and equal measures can be partition or measurement problems. They can also be divided into problems without remainder, problems with a remainder which is not used for the solution, and problems with a remainder which raises the solution by one. Most studies concerning these aspects used word problems. This paper describes an empirical investigation about children working in the real-world context „juice punch". 12 second-graders – age 7 to 8 – and 28 third-graders – age 8 to 9 – were videotaped pairwise. The results of the qualitative analysis show, that it depends on the solving strategy, if the children had difficulties in dealing with the remainder.

Although division is not very difficult from a mathematical point of view, it is the most complex operation children have to learn at primary school (eg. Anghileri 1995, Burton 1992, Greer 1988; Harel & Confrey 1994; Schmidt & Weiser 1995):

- Multiplication and division are „second level operations", which are based on addition and subtraction.
- Division is taught as the inverse of multiplication, so the understanding of division requires the understanding of multiplication.
- There is a greater number of situational models with multiplicative structure than with additional structure.
- Within the models of equal groups and equal measures two aspects of division can be differentiated: measurement and partition.
- Division with remainder is an additional difficulty, because only integers are allowed as a result. No other operation requires this differentiation.

Division problems in the model of equal groups

Most division problems in contexts are founded in the model of equal groups and represent one of the following aspects: measurement or partition. In case of *measurement problems*, the divisor indicates the number of elements of each subunit, whereas the quotient gives the number of sets. From this point of view dividing means *grouping or repeated subtracting*. In *partitive problems*, the number of subsets is known, while the number of elements per set is to be found. The action „dividing" suggests a *sharing procedure* here.

Context-bounded division problems can also be divided into problems in which

- the remainder is zero,
- the remainder is non-zero but is not used for the solution, and
- the remainder is non-zero and raises the solution by one.

Therefore, the following six situations can be differentiated within the model of equal grouping (see Brown 1992, Burton 1992):

Division Type / Re-mainder	Measurement Problems	Partition Problems
Zero	Susan wants to pack up 16 plates into boxes. One box can hold 4 plates. How many boxes does Susan need to pack all the plates?	Susan wants to pack up 16 plates evenly into 4 boxes. How many plates will be hold from each box?
Non-zero but neglectable	Susan wants to pack up 18 plates into boxes. One box can hold 4 plates. How many boxes may Susan fill with plates?	Susan wants to pack up 18 plates evenly into 4 boxes. How many plates will be hold from each box?
Non-zero and raises the solution by one	Susan wants to pack up 18 plates into boxes. One box can hold 4 plates. How many boxes does Susan need to pack all the plates?	Susan wants to pack up 18 plates into 4 boxes. How many plates must be put into one box?

Table 1: Measurement and Partition Division Problems

Solving division problems

Although division is such a complex operation, young children without any formal instruction to the operation are already capable of solving one-step division word problems. The findings of different investigations (e.g. Brown 1992, Burton 1992, Kouba 1989, Mulligan & Mitchelmore 1996, Murray et. al. 1991) show:

- About half the children solve division problems by using manipulatives or fingers.
- Children use much more often grouping strategies than sharing one by one, although the problem may be a partitive one.
- Problems with remainder are more difficult than those without remainder.
- Even if division problems with remainders are calculated correctly, the interpretation of the result causes difficulties.

The Study

All investigations dealing with division problems in context used one-step word problems, which were told verbally. Qur question was, how children would solve problems in a realistic context, in which they had the possibility of using real materials. Especially we were interested in the following questions:

- Which strategies do children use to solve division problems in real contexts?
- How do children solve measurement problems with remainder in real contexts?
- How do children interprete remainders in real contexts?

To answer these questions we investigate the problem-solving strategies of German second- and third-graders (age 7-9) in real world situations. The main focus of this paper lays on the strategies of children dealing with the context „juice punch“.

Design of the situational context

Juice Punch
(60 glasses)

15 glasses orange juice

12 glasses peach juice

8 glasses pineapple juice

5 glasses banana juice

20 glasses sparkling water

Figure 1: Recipe for a juice punch

For our investigations we constructed three real-world settings which are similar to each other in their arithmetical structure but differ in the situational contexts: classroom party, juice punch, and doll's house (for details see Ruwisch 1998). The second context, juice punch, will be described in more detail now, because the examples used later are taken from it.

In the situational context „*juice punch*", the children were asked to determine the number of bottles which are necessary to realize the recipe given in figure 1. To simulate an everyday situation, the children had the possibility of using real materials for their solution: a punch bowl, a 0.1 l-glass to measure, and bottles of different sizes. These bottles contained the same amount of liquid as original bottles in a super-market – 0.2 l, 0.5 l, or 0.7 l. In contrast to a real situation, they were filled with water and labled with the type of fruit and the amount of liquid given in glasses. As in supermarkets, the bottles were sorted according to their size and the type of fruit (see figure 2).

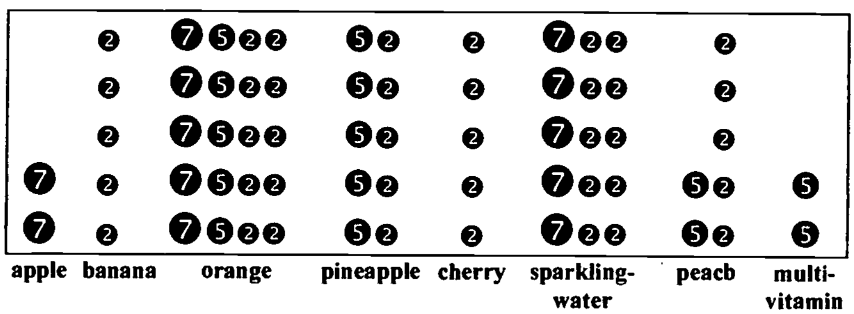

Figure 2: Tables with bottles of fruits and sparkling water

Data and Methodology

Subjects. The subjects of the whole study were 122 children from 7 different forms of German public primary schools, 40 of them have worked with the „juice punch". Thus, 12 second-graders – age 7 to 8 – and 28 third-graders – age 8 to 9 – participated in this part of the investigation. At the time of inquiry the second-graders had not been introduced to multiplication or division yet. The third-graders had already learnt the multiplication table and had been instructed to both types of division problems. They were also familiar with division with remainder, although the different situational interpretations are not taught explicitly in Germany.

Procedure. The children were withdrawn from the classroom and confronted pairwise with the materials in a separate room. They were given a short introduction into the situation (see figure 3). Then they had to work by themselves until they indicated to us that they had finished. On average the working-phase – which was videotaped – lasted 20 to 30 minutes.

„My niece asked me for a juice punch for her birthday party. Do you know, what that is?" ... (eventually explain) „10 children will come to the party. At home I looked for recipes for juice punches, and I found one which yields 60 of these glasses (show the glas). I have brought the recipe for you here (turn around the recipe). Can you read it out to me?" ...(let them read aloud by turns) „As you can see, the details of the recipe are given as glasses, and therefore I don't know, how many bottles I need. So that you can find this out for me, I have put juice-bottles here (point to the tables with the materials) which are filled with water. Can you fill in this shopping list (turn around the shopping list), how many bottles I have to buy for the juice punch in the supermarket?"... „If you are ready, please let me know."

Figure 3: Verbal instruction

Results

Analysis. The videotapes were transcribed in detail for each pair. We differentiated different forms of transcripts due to the degree of resolution (see Wollring 1994): a) coarse transcripts of sequences, b) verbal transcripts with middle resolution, and c) verbal and action transcripts with high resolution. The latter form of transcripts was used for the case-studies, this paper refers to. The transcripts were interpreted turn-by-turn in small groups of 1 or 2 researchers and 3 to 4 teacher students.

Children's strategies and action patterns. The children used three different solving processes to determine the number of bottles: 4 third-graders *estimated* the number, 4 second-graders and 10 third-graders *mixed the punch by measuring with the glass*, and 8 second-graders and 14 third-graders *calculated the numbers* (for details see Ruwisch 1999). Only the children of the last group came across the problem of division with remainder. Their difficulties, their searching and solving processes, and their discussions are described in detail now.

Children's strategies solving a measurement division problems with remainder. Treating the whole problem, the children had to solve 5 smaller problems. The number of bottles of every juice, except banana, could be solved without remainder. It was interesting to us, that most children tried to find out a solution without any juice left, although they were not asked to do so by the interviewers. So calculating children chose the corresponding size of the bottles and found the following solutions:

- 3 bottles of 5 glasses to get 15 glasses of orange juice,
- 2 bottles of 5 glasses and 1 bottle of 2 glasses to get 12 glasses of peach juice,
- 4 bottles of 2 glasses to get 8 glasses of pineapple juice, and
- 10 bottles of 2 glasses or 2 bottles of 7 glasses and 3 bottles of 2 glasses to get 20 glasses of sparkling water.

The 5 glasses of banana juice given in the recipe could not be realized with the given bottles of 2 glasses each without some amount of juice left at the end. So, here these children had to solve a measurement problem with remainder. We could observe three different reactions to this problem:

4 thirdgraders had no problems at all to find the solution. They did neither talk about banana juice as a special problem, nor had difficulties in finding the solution „3 bottles of 2 glasses", which all them wrote down at once on the shopping list. Although these children tended to calculate the minimal number of bottles necessary, it seems as if they did not even come across the problem of the remainder. Acting in a simulating real situation, it seemed to be clear to them, that they have to buy one more bottle.

8 second-graders and 10 third-graders did come across the problem. All of them were somewhat puzzled at the beginning, like the example of Max & Felix (3rd grade) shows:

F *(F. is still writing down the solution for the pineapple juice)*
M **Then?** *(looking at the recipe)* **Then banana.** *(turning around to the bottles)* **Banana, banana** *(finding bottles of banana juice, turning around to the recipe)* **Then we need, how much banana?** *(looking at the recipe)* **5 glasses.**
F **5?**
M *(turning around to the bottles, looking at a 7 glass bottle of apple juice)* **These are 7 glasses.** *(tapping the 7 glass bottle of apple juice)*
F *(turning around to the bottles)* **Yes, that's of course ... 5** *(looking at the bottles next to apple juice)*
M **2.**
F **Oh, damn, there are no bigs of this kind, that's mean.**
M **Oh yes, but they are 7ers** *(turning around to the recipe)*
F **Pardon, 5? How shall we manage that?**
(M. is looking into the recipe, F. is again turning around to the bottles.)
F **Are there also anywhere ones?** *(looking around the other given bottles)*
M **5, 5.**
F **Is** *(looking at the lables of the other bottles, looking at the bottles standing behind the others)* **there anywhere anything, dipdidipdidip**
M *(spreading out his arms)* **that's impossible.**
F **Ahm, what kind is this one** *(touching a bottle of multivitamin juice)*
M **Ahhh! ... Ha?**
F **Maybe of these?**
(Both are standing in front of the bottles of multivitamin juice)
M **There is all together. 5 glasses are in it. Well, it doesn't matter** *(shrugging his shoulders)*

All of these children knew that 5 glasses could not be reached by bottles of 2 glasses.

4 second- and 4 third-graders only stopped for a short moment, giving comments like „here are no 5s, how will that work? Okay, let's take 2 small ones and a half". These children also wrote down „2 and a half" as their result. Although they did not give

the solution „3“, they nevertheless interpreted the remainder in a correct way: Being asked by the interviewer at the end, how many bottles he or she would have to buy, the children answered „three, but there will be one glass left.“

The other 10 children had greater difficulties in finding a solution, which satisfied themselves. All of them showed a first puzzled reaction, looking for a quick solution, like Max & Felix did. Because they could not find one at once, they put this problem aside and started with another one. They only returned to the banana juice, when they had finished all the other parts of the recipe.

The process of searching for a solution was no linear approximation. The children tried several possibilities. But all processes can be characterized by expansion in some of the following different directions:

- spatial expansion: the children suggested, that bottles with 5 glasses of banana juice could be found between the other juices, under the tables, or at another place inside the room;
- offer expansion: The children looked for bottles with one glass of juice:

> F **I know! No, not really.** *(Turning to the bottles of banana juice)* **Are there anywhere ones, is here anywhere standing a 1** *(looking at the lables of all bottles of banana juice)* **please, let anywhere stand a 1, please, please,**
> M *(touching a bottle of 7 glasses of apple juice)* **we ought to choose a smaller bottle, that's not possible**
> F **Such a small picco** *(demonstrating with his thumb and fingers half the height of a 2-glass-bottle, turning around to the recipe)* **I don't get it** *(looking into the camera).*

- expansion of interpreting the problem posen: instead of buying bottles of one juice only, the children tried to use multivitamin juice:

> F **We don't need that. They want to pull our leg. That one** *(pointing to the bottles of multivitamin juice)* **we don't need as well. Or maybe, if we, how much do we need** *(looking into the recipe)* **5. or, if we from these, nooo, it's impossible, too, there we also have only big ones.**
> *(Both are standing in front of the bottles of multivitamin juice).*
> M **Aha, of every glas 1** *(pointing to the lable of the bottle of multivitamin, turning around to the shopping list)*, **then we should of these** *(pointing to the pineapple juice in the shopping list)* **1 less, and of these** *(pointing to the peach juice in the shopping list)* **1 less, and of these** *(pointing to the orange juice in the shopping list)* **1 less, and of these** *(pointing to the sprakling water in the shopping list)* **1 less,**
> F **Yes.**
> M **then it would come out even here** *(pointing to the banana juice in the shopping list).* **But, wait, does it work?**

- expansion of acceptable solutions: in the end all children found the solution „2 and a half“ and wrote it down.

F **5. .. Of these, these two** *(touching 2 bottles of 2 glasses)*
M **Then we do, shall we cut one in two?** *(looking at F.)*
(Both are loughing.)
F **Or half of it, then something is left. But, who did say, that it matters if something is left** *(going along the bottles, looking)* **Well, what's this here** *(touching a bottle of 5 glasses with pineapple juice)***?**
(Both simultaneously) **Hm, hm** *(shaking their heads).*

and some minutes later:

F **Ah, there is nothing else, Max.**
M **Yes** *(lets go the recipe and the bottles of peach juice and turns around to F.)*
F **As this solution.**
M **Pardon?**
F **3 point 2 glasses** *(taking the pencil, going to the shopping list and writing „3.2".)*
M **Hm, write it down. 3,? ah, 3 point**
F **2**
M **glasses.**
F **Stop!** *(writing „R 1".)*
M **What is?** *(reading, what F. is writing)* **R 1?**
F **Remainder one glas.**
M *(touching groaning at his head)* **Well, does not matter.**
F **Ready.**
M **Yes.**

Interpretation and Conclusions

The results show, that specific strategies are predestined for specific difficulties. Estimating will not give exact results, especially if the children are not used to do it. Measuring is a very slow solution strategy which cannot be used without having the bottles in front of you. So, calculating with the numbers given on the lables is one of the most effective and efficient strategies for the given problem. But: Children who used this strategy, tended to think very precisely, although the situational context did not require such a precise thinking. So, these children had to solve self-made problems, when they tried to give an exact number of bottles with no amount of juice left. Nevertheless, all children were able to find the solution, even if they were not very satisfied by it. Although some children wrote down „2 bottles and a half" or „3 bottles of 2 glasses, remainder 1 glass", all said, that 3 bottles had to be bought.

If the interpretation of the remainder as raising the solution by one is easier in a real-world context than in solving word-problems, cannot be said, because it could be, that the difficulty in both environments is to decide what should be written down, the solution or the interpretation of that solution in the situational context.

It also must be said, that the data which could be interpreted for this paper, was very small. It consisted of one measurement problem with remainder and 4 measurement

problems without remainder. Division with remainder had not the priority for the design of this study. Therefore, more investigations in real-world contexts have to be made, also encluding measurement problems, in which the remainder is not used, as well as encluding all kinds of partition problems.

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ARITHMETIC AND ALGEBRAIC PRACTISES: POSSIBLE BRIDGE BETWEEN THEM

Patricia Sadovsky
Universidad de Buenos Aires

Abstract
In this work we analyse strategies of students with no previous experience in algebraic work (11/12 years old) to tackle problems on Euclidean division where the notion of variable must be used. The results of the analysis let us make the hypothesis that solving problems which require, apropos of a particular concept, the use of the notion of variable, simultaneously contribute to a progress towards the conceptualisation of that content and also to enter the algebraic practises. These results are part of a broader diagnosis whose goal was to get data to characterise a possible articulation space between arithmetic and algebraic practises. Such a characterisation will orientate the design and implementation of didactic situations that will be tested in the next stage of our research.

The research questions

As we considered the many researches that deal with the passage from arithmetic to algebra (Bednarz,N and Janvier, B; 1996; Kieran,C; 1992; Mason,J; 1996; Panizza, Sadovsky, Sessa; 1996, 1997; Vergnaud,G et al; 1988), we made the hypothesis that it was possible to generate a didactic articulation space between the arithmetic and algebraic practises, so that the students could tackle much better the essential and inevitable rupture entailed by the entrance to the algebraic world.
(We are thinking of 11 to 12 years old scholars).

We talk about an articulation space naturally organised around a number of problems. But we also talk about rupture. Articulation and rupture are terms which seem to contradict one another. As it is well documented in the literature on this subject, there is an unavoidable rupture due to the fact that the entrance to algebraic world means to reject meanings and practises that have been build up throughout their scholastic education. Nevertheless, the possibility of a space of problems which the students can understand and for which the usual arithmetic procedures could work as base strategies that would not be enough, let us talk of articulation. Thus, there is articulation because the arithmetic knowledge they have is enough to tackle a set of problems though not to solve them, but there is rupture because there it exists the need to abandon certain practises.

Which algebraic capacities would this space include and which ones it would leave out? How could it be characterised? Under which didactic conditions could these articulation practises emerge in the classroom? How would this work modify the conditions to enter to the algebraic world? Since it would mostly be about the establishment of a practise: How would the unfolding of this articulation space impact

on the teaching of the different mathematics concepts involved in these problems? Up to what extent would the algebraic procedures unfolded with respect to a certain content be available when dealing with some other contents?

These questions were the ones that guided our research. Our methodological framework is the one of the engineering didactic (Artigue, M;1990). Our study has a stage which deals with the characterisation of an articulation space between the arithmetic and algebraic practises (at the moment we are working on this stage), a stage of designing and proving didactic situations and ultimately one of giving validity to the situations that have been designed.

While the mobilization of the notion of variable is a central element of the algebraic activity and represents and important rupture with regards of the arithmetic practises we have decided to structure this articulation space focusing on problems which from different mathematics contents and, apropos of different kind of tasks, demand implicitly or explicitly the mobilization of the notion of variable. On the other hand, we have considered that the distinction between sense and denotation (Drouhard, et al; 1995) in respect with the numerical algebraic expression could become a didactically valid starting point to elaborate problems that would urge the students to analyse the structure of the expressions and stablish relations among different expressions without appealing to the actual solution of the operations. Finally, taking into account the results offered by several researchers (Malara,N; 1998 Arcavi, A et al.; 1990) we have decided to include in this articulation space problems that would entail the students' inquiry, the pose of conjectures and the validation of arithmetic theorems.

Being our aim to better characterise the articulation space and to make hypothesis that nourish the elaboration of didactic sequences (1999 project), we devised a diagnosis test for students in grade 7(mean age11/12 years old) with no experience on algebraic work. The aim of the test was :

- to analyse the strategies unfolded by the students who lean on arithmetic practises when faced to problems which demand the mobilisation of some of the issues above mentioned.

- to study the fertility of the situations presented in order to propitiate an evolution towards algebraic practises and outline the possible evolutions .

The diagnosis was administered in three stages to Grade 7 students from three different schools. The children had to solve the problems alone, they were given pencil, paper and a calculator. After the problems had been solved we interviewed some of the students to inquire them about their productions. The interviews were audio-recorded.

The test consisted of two different kind of problems:

-problems for which the analysis of the structure of numerical algebraic expressions and the comparison among different expressions was a must.

-problems which could be modelized by means of a two variable equation (whether linear or not), some about rectangle areas and perimeters, others about realistic situations and a third group refered to the notion of Euclidean division.

We will only present here the results we obtained from the analysis of the third group of problems.

The problems

The problems on Euclidean division we presented to the students were:

1. Find a division operation whose dividend is 25, its quotient 8 and its remainder 12. How many are there? If you think there are less than three write them all down, and explain why there are no other ones. If you think there are more than three write down at least four of them and explain how other solutions could be found.
2. Find a division operation which dividend is 32 and its remainder 27.How many are there? If you think there are less than three write them all down, and explain why there are no other ones. If you think there are more than three write down at least four of them and explain how other solutions could be found.
3. If you divide 527 by 46 the quotient is 11 and the remainder 21.
a) Verify it.
b)Can you find any other number that when you divide it by 46 the remainder is 21? How many are there? If you think there are less than three write them all down, and explain why there no other ones. If you think there are more than three write down at least four possibilities and explain how other solutions could be found.
4. Find a division operation which quotient is 20 and its remainder 14. How many are there? If you think there are less than three write them all down, and explain why there are no other ones. If you think there are more than three write down at least four of them and explain how other solutions could be found.

The relation dividend= divisor x quotient + remainder, divisor>remaninder≥0, which from now on we will call Euclidian relation, is the core of all four problems.

What is the status of this relation for different students? For one specific student, does this status modify in the different problems? Up to what extent do children conceive the Euclidean relation independently of the division operation? Is this independence necessary to give solutions?, and to explain how they are obtained? Which transformations take place in the strategies when the notion of variable is at stake? For those students who implicitly mobilise the Euclidean relation to give examples (problems 2,3 and4): Is the explicitation of the relation a means to explain how other solutions were obtained?

Analysing some of the cases, we will try to answer the above mentioned questions.

Results

Broadly speaking the students perceive the Euclidean relation as subjected to the division operation and they frequently cannot disentangle it from the actual solution of the operation. Now, this way of perceiving the Euclidean relation perfectly adapts to

the first problem; and its limits can only be seen when facing problems where the mobilisation of the notion of variable is necessary. So speaking, **we think that these problems are simultaneously a chance to find the limits of the arithmetic practises and enrich the conception of Euclidean division.**

How does this dependence on the Euclidean relation with regard to the division operation manifest? What are its consequences? We will analyse it following Daniela's procedures.

Daniela finds the divisor to problem 1 using the Euclidean relation; she writes: $8 \cdot 25 + 12 = 212$. Then, to justify that there is only one possible division, she shifts the centration and says: " If they give you a result with a remainder and a divisor, there is only one possible solution. If we had divisor and quotient or divisor and remainder there would be more than three." Let us observe that to justify that there is only one possible division she mobilises one representation of the division as an operation in which given two numbers (dividend and divisor) we get another two (quotient and remainder). Having in mind this last representation, she anticipates aforehand in which cases she would get more than one possible solution. She mobilises once again the Euclidean relation when tackling problem 2. She gives several examples and explains that we could obtain more: "*multiplying 32 .X plus 27 .*" When she tackles problem 4, she points out: *" There is only one solution because you have a `given' quotient and remainder."* That is to say, before mobilising the Euclidean relation —as she had previously done in the case of problem 2— concentrated on the division operation, — and probably also on a certain representation of what the algorithms of the arithmetic operations broadly are— Daniela supposes that there is only one possible division operation, probably because she has been given the two numbers that are the results of an division. Taking into account the main importance of the rol of the focusing at its final stage to elaborate a proceeding (Inhelder,B y Caprona,D; 1992), we think that this first anticipation stops Daniela from refering to the Euclidean relation and stablish the analogous roles that unknowns and givens have in problems number 2 and number 4. (Let us realise that beyond the restrain on the divisor posed by problem 4, the formulas which model both problems are identical).

While Daniela was working, we could observe her commitment to problem number 4 and her hard attempts to get to a solution. There are traces in her sheet of paper that at a certain point, she proposes 945 as dividend and 46 as divisor and gets 20 as quotient and 25 as remainder. Nonetheless, she cannot get profit from this result. This would point out that she needs certain hypothesis with regards to the expected quantity of solutions, to make trials to move from an approximate solution to the desire one.

Let us analyse now the written representations she uses for problem number 4. After thinking hard she writes down:

$$\begin{array}{l|l} X & \underline{\ X\ } \\ 14 & 20 \end{array}$$

and then she writes down: $20 .X + 14 = Y$. After she has written this, she tries but cannot solve the equation as if the mere manipulation of it could make it emerge `the'

expected solution. That is to say, this representation has for her the statute of an equation in which X and Y are unknowns and not variables and , thus, she cannot conceive it as a formula which produces infinite solutions. (Should it be observed that the written equation has for Daniela an instrumental value —as it generally occurs with the production of a representation— though she is conceiving it wrongly.) Should it be noticed, that one of the aims of this didactic work, a propos of these issues, would be the possibility to make students conceive these expressions as formulas so that these formulas are a support that has both the functions of treating the information and of objectivation (Duval, R:1995). Quoting R. Duval, that they can make the conversion of register (in this case from operation register to formula register), conversion that as the author points out it is not cognitively neutral (Duval, R; op.cit.). The conversion of register should make it possible for the students to stablish the analogy between the quotient and the divisor in problems 2 and 4, analogy that cannot apparently be perceived when representing the division as an operation. In other words, the production of the formula would draw near two problems that when seen from the representation of the division as an operation they seem to be essentially different. We talk about analogy and not equivalence because these elements variation domains are not the same. So speaking, a didactic work with problems such as the ones we have proposed should think about the possibility of an interaction between the two registers we have been talking about. Why? We have found during our research some students who focus in the formula losing sight of the restrictions on the remainder. With respect to this issue let us compare Diego´s and Andrés´ works.

Andrés is one of the few students who`trusts´ in the Euclidean relation as well as in the operation. From the very beginning, he writes down the formula d= c.D+r and he uses it to provide solutions to problems 2,3 and 4. But he does not realise that in problem 4 he had to consider divisors greater than 14 and successively he gives the divisor the values 2,3,4,and 5. Having all his attention centred in the formula, he does not look for controls and does not notice his mistake. In other words, he trusts so much in the formula —thing most students do not do— that he does not validate his production.

Diego is a student that after working hard manages to solve problems 1,2 and 3. He uses the calculator and thanks to the many trials he makes with it he manages to put at stake the Euclidean relation in order to produce the demanded operations. Afterwards he makes the operations to validate them. This control is the one that allows him to realise he is doing something wrong when he gives divisors lower than 14 in the case of problem 4. These results make us think of the possibility of advancing in the conceptualisation of the Euclidean division, of giving the Euclidean relation both the statute of instrument and object, and, at the same time, of accepting the student´s entrance in algebraic procedures, asks for the jointed manipulation of the operation and the Eucledean relation controlling one another.

As we have already pointed out, a didactic work with problems such as the ones we proposed would let the students find the limits to the arithmetic practises, thus

generating the conditions for the entrance to the algebraic procedures. Since to face a problem like the second one, for example, it is necessary for the students to give any value whether to the dividend — if they use the trial and error strategy as to propose any operation with divisor 32 and then "move" towards the solution— or to the quotient to be multiplied by the divisor (32) and then added the remainder (27). The students resist a lot this practise of giving any value, they generally think that it is only possible to work with the numeric data given in the problem .

Hernán, for example, finds the correct dividend for the first problem and when we interview him he says that to obtain the dividend :" *I made 8 that is the quotient multiplied by 25 that is the divisor and added 12 to it that is the remainder.* " Then, he says that this solution is the only possible one because: *"with the operations I have made, the only possible solution is 212".*

When he tries to solve problem 2 he proposes 347 as divisor getting 10 as quotient and 27 as remainder. See that 347 is 32 x 10 + 27. When we asked him how he had obtained that number (347), he answered:" *I invented a number and added 27 to it.. It could have been any number.* " Though he says he has chosen any number, the number he has "invented" is 320 that is 10 times the divisor. It all happens as if Hernán somehow knew that number had to be "obtained" from the numeric data he had been given, but at the same time he would not be completely aware of the relation between the quotient 10 and the 320 that is ten times the divisor. He gives only this solution to this problem and does not says how many solutions it could have. On the other hand, as the number 320, which Hernán says he has invented, is an approximation to the dividend which apparently has for him no relation with the quotient, it could be said that he conceives the dividend as an "starting point" for the procedure he has to put at stake and not as the number he has to obtain to fulfil the requested conditions. We shall see here another trace of the arithmetic practise: "in relation with the division operation you start from the divisor and the dividend". On the third problem it is much clearer that the number that Hernán "invents" has somehow a relation with the data and that he takes as starting point the dividend. Let us see. This student carries out the verification proposed in part a). To do this he does the division operation —this is what most student did— and for part b0 he divides 5270 by 46 and as he obtains 26 as reamainder he proposes 5265 as divisor. When he finished the problems he says this to us :

I: How did you get the operation?
H: I added a zero to the 527 of the other operation (he refers to the divisor of part a)) and I got 26 as remainder so I started lowering the number until I got 5265.
I: Are there any other possible solutions?
H: Yes, if you invent other numbers. But I don't know which number to invent, I tried but I couldn't.

Throughout the procedures put at stake by Hernán we see that he mobilises the Euclidean relation for the first problem, but this same relation is not available when one of the elements of this relation must be determined by him. This does not allow

him the production of other solutions though he seems to intuitively perceive that they exist. To undergo the resistance to give values, what is necessary to face problems 2,3 and 4 would allow Hernán to advance in the conceptualisation of the Euclidean division.

At the very beginning, we wondered whether the explicitation of the Euclidean relation could be a means to explain how the solutions are achieved in each of the cases. On this issue, we would like to point out that a lot of students who immediately provided some results without showing the traces of how they had obtained them, they explicited the relation when they were asked to explain how the solutions were obtained. Taking into account that to make explicit the relations used it always means a conceptual advance a propos of the implicit put at stake, this result tells us about the potential didactic productivity of asking for an explanation about how they arrived to the solutions.

Conclusions

How do these results nourish the making up of didactic sequences that aim at the construction of an articulation space between the arithmetic and algebraic practises?

The results that arise from the analysis of these problems seems to confirm a general hypothesis that guides our project: tackling problems which require, a propos of a particular mathematics concept, the putting at stake of the notion of variable entails an opportunity to advance in the conceptualisation of such a content. The advance in these conceptualisations and the arise of algebraic practises would be completely imbricate processes. To this effect, the teacher's interventions in the didactic sequences should contemplate a twofold institutionalisation: regarding the mathematics concepts being worked and the practises being installed. The institutionalisation at algebraic practise level would run through different mathematics contents and it would be each time linked to the contexts in which such procedures were produced. Understanding these practises would have the characteristic of understanding "how it is to be done" , that is different to the mere conceptual understanding (Inhelder, B; Caprona,D; op.cit).Specially, in relation with the didactic implementation of the afore analysed problems it would be important to consider that there would be — on the side of the students — a variety of possible procedures all of which would enchain procedures that would allow them to evolve towards the conceptualisation of the Euclidean division in terms of the Euclidean relation. After the didactic work the students should stablish:

- The dependence of the dividend regarding the divisor, the quotient and the remainder
- The similarities and differences between the production of division operations when the divisor and the remainder are fixed and when the quotient and the remainder are fixed

- The identification of the quotient and the remainder from the Euclidean relation without making the operation.

At algebraic practise level the didactic work with these problems should let the students:

- To advance in the incorporation of the procedure by which a value is given to a variable to get the value the other one that depends on the first one .
- To stablish the instrumental value of the Euclidean formula as producer of special operations and representative of the many examples produced and the infinite solutions that the problems admit.

The fieldwork we are going to face from now on will most probably confirm some of these issues, will open up our spectrum to other possible ones that now we cannot loom and, as it always happens, will compel us to renounce to a lot of the elaborations that preceed the difficult, complex and challenging task of putting to test our work with the students.

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ON SOME UNDER-ESTIMATED PRINCIPLES OF TEACHING UNDERGRADUATE MATHEMATICS

Ildar Safuanov, Pedagogical Institute of Naberezhnye Chelny

The classical and contemporary experience in university mathematics teaching revealed the importance of such things as the strong individuality of a lecturer and especially the style (in oral lectures and in textbooks as well). Thus, the teaching has much in common with the art, and it seems useful to research teaching by means of art studies. Note the importance of such under-estimated principles as: 1) genetic approach: developing didactics from the method of science itself; 2) concentrism (teaching in a concentric way); 3) the principle of multiple effect.

1. Introduction.

As indicated Pototsky (1975), the undergraduate mathematics teaching is almost not investigated, the pedagogy of higher mathematics essentially is not present, and few monographs on problems of university mathematics education exist.

Though since 1975, when the Pototsky's book was written, the situation has changed in some extent (the researches of L.D.Kudryavtsev (1985), A.G. Mordkovich (1986), H. -C. Reichel (1992), D.Tall et al. (1991) devoted to various aspects of teaching of mathematical disciplines at universities and pedagogical institutes have appeared), the methods of teaching of undergraduate mathematics are investigated insufficiently (at least in the former U.S.S.R). Perhaps the Western situation is more prosperous so we will restrict our following considerations to the undergraduate mathematics teaching in Soviet and post-Soviet universities.

Kudryavtsev (1985, chapter 2.5) has put forward a problem of developing methodical principles of teaching of undergraduate mathematics. He complained of difficulty of the formulation of exact methodical principles.

The training of the university students essentially differs from the training of the school pupils. In universities such mechanisms of influence on students, as direct compulsion, authority of seniority by age are less significant; on the other hand, the university students are better prepared for the perception of a material through various channels. In the training of the university students there is less "didacticality" in a school sense and more art. Probably, partly for this reason the university teaching was less studied: the opinion existed, that the art cannot be learned, the art is not a subject of a scientific investigation, it can only be acquired on the basis of personal experience.

However, as L.D.Kudryavtsev (1985) and N.L.Gage (1964) indicated, any art has its rules, which can be explored scientifically, and any art can and should be learned (and taught): artists, and musicians, and writers study hard.

On the other hand, it might be more fruitful to investigate some aspects of mathematics education by methods developed in art studies. Indeed, mathematics teaching, unlike mathematics itself or, say, philosophy, is a completely applied field unable to develop independently of practice. It makes mathematics teaching really close to such artistic fields as theatre or literature. Investigating these applied fields must result in some practical recommendations. One can regard these fields not only as art but also as craftsmanship and their investigations even as "design science" [Wittmann (1995)]. Of course it is impossible to quickly develop the universal system of all required recommendations (similar, say, to the theatrical system of Stanislavsky). We attempt here only to indicate some (not ALL) UNDER-ESTIMATED (exactly as indicated in the title) principles that, in our view, could in some extent stimulate or refresh the efforts of the mathematics-educational community in elaborating the recommendations. We will touch only two aspects of mathematics teaching: 1) general subject-related directions in developing methods of teaching; 2) effect on students.

These two aspects do not cover the whole realm of the undergraduate mathematics education. We are not aiming at constructing a complete system of didactical principles for teaching university mathematics which would allow to increase all the results of teaching (students' knowledge, creativity, motivation, interest, future professional skills etc.). Moreover, we can not guarantee that the use of the principles we are discussing would improve students' knowledge of mathematics in traditional sense (say, amount of mathematical facts, proofs, problem-solving patterns understood by the individual and/or stored in her/his memory), and we do not invent non-traditional meanings of mathematical knowledge. Our aims are rather closer to those of (physics educator) R.Feynman (1963) who tried to construct his course in such way that most able students could keep and strengthen their enthusiasm. We also agree with his statement that "there is no benefit of teaching for somebody except those for whom there is almost no need of teaching". It is especially true in our Soviet and post-Soviet conditions. Lastly, we are fond of his intention, designing the course for the most active listener, to take into consideration also interests of less able fellows, so that they could "seize the essence".

In this paper, the first aspect is represented by *the principle of genetic approach to the methods of teaching.* This is a rather general principle, extracted from experiences and thoughts of generations of undergraduate mathematics educators. Of course, deep development of this principle requires the study of students' (higher mathematics) learning processes. We did not yet accomplished such deep studies, so we only survey various manifestations of this principle implied by several authors, and also present some examples inspired by this principle. In particular, we do not discuss in this paper such important and fashionable problem as understanding and constructing proofs by students.

We hope that scientific community, e.g., members of PME's Advanced Mathematical Thinking group can deeply and fruitfully contribute to the further

development of this principle from the point of view and in terms of the process of (higher mathematics) learning.

We draw more attention to the second aspect (effect on students) that, in our opinion, is greatly under-estimated. Here it is represented by two principles: of concentrism and of multiple effect.

We do not declare that these principles are quite new, or that *we* have discovered them. Many authors have expressed the ideas similar to these principles.

Thus, we will try to briefly describe three insufficiently investigated, but, in our view, rather important and global for mathematics education principles: the principle of the genetic approach to the methods of teaching mathematical disciplines, the principle of the concentrism and the principle of the multiple effect.

Last but not least, we would like to draw attention of mathematics educators to the importance of such also greatly under-estimated aspect as the *style* of teaching and writing textbooks. Shortage of space does not allow us to discuss this problem in details here, so we will do that elsewhere.

2. The principle of the genetic approach to the didactics.

The essence of the genetic approach is that the didactics of a scientific discipline should correspond to the method of the science itself. Many prominent mathematics educators stated the ideas, connected with this principle, e.g. F.A.W. Diesterweg (1962) and H. Poincare (1990). It is especially important for mathematics teacher preparation. As shows E.Ch. Wittman (1992), "a genuine integration of mathematics and education (during the preparation of the teachers) can be achieved only if educational and psychological relationships and processes *inherent* in good mathematics are elaborated and developed". Diesterweg wrote: " the method of teaching of each subject should correspond to its source or principle... Otherwise the method will be arbitrary, borrowed from the outside, not following from the nature of a subject, and rather being contradicting prescription ... Here alongside with the statement: " the Man represents a method " - a rule: " the Subject represents a method " is true. Diesterweg warns against the understanding genetic approach as historical. A student should be raised to the height of modern science. He should not learn all out-of-date, already rejected points of view and errors of the past. Rubinshtein (1989) wrote that there coexist dialectical *unity* and *distinction* between the ways of pupil's learning and the cognition processes of mankind. He argued that the subject's *logic* extracted during the process of the historical development of the cognition represents that unity.

Of course, it does not mean that there is no role for the history of mathematics in mathematical education. It is quite allowable to make students acquainted with the history of mathematics by giving special historical courses. Moreover, sometimes it

is useful to insert (not importunately and obtrusively) some historical facts in regular or advanced mathematical courses as it was done, say, by H.S.M.Coxeter (1966).

More precisely interprets the genetic approach to the didactics V.V.Davydov: " the substantial contents of a concept can be revealed only by finding out the conditions of its origination " (1972).

The most talented scientists - educators of the past skillfully used the genetic approach in teaching. The physicist M.Planck gave the best description of that: " I have never chosen shortest and most elegant proofs and reasoning, but always searched for those best corresponding, on my opinion, to the physical essence of a subject, because I did not want to show, how the known rules were open, nor how it is possible to prove them briefly today, but mainly the way, by which it was possible to open that theorem ... "

Much earlier G.W. Leibnitz (1880) expressed a similar idea: "I tried to write in such way that a learner could always see the inner foundation of things he is learning, that he could find the source of the discovery and, consequently, understand everything as if he invented that by himself".

Genetic approach can be used at various levels of teaching process: in designing syllabuses, in learning a topic or theme, in solving a problem. Accordingly, this approach may be used in various ways. One may find numerous examples of the genetic approach in classical and modern textbooks and in lectures of successful educators. For example, probably the worst way to teach number theory is a traditional for Soviet pedagogical universities approach: number theory is taught as application of the divisibility theory in rings of principal ideals. In our view, the genetic approach requires that most elements of number theory should be taught *before* abstract algebra because number-theoretical problems are interesting as they are for those at least minimally inclined to mathematics, and these elements can serve as good examples motivating many concepts of abstract algebra.

The next is an example of using genetic approach in studying a separate theme. Before studying linear congruences modulo *m*, we consider several problems where students have to check divisibility of numeric values of expressions into natural numbers. Together with students, we discover that remainders play the main role in solving these problems. Thus, the idea of introducing special relation between numbers with the same remainders arises etc.

A good example of using genetic approach in proving is provided by Davis, Hersh and Marchisotto (1995, pp. 307-313). G.Polya (1965) noted that in problem solving, the genetic approach implies the principle of *successive phases.*

However, one should not use the genetic approach as the absolutely strict rule but rather as the source of further interesting ideas [see Polya (1965, Ch. 14)]. For example, "pure" genetic approach would require that one should study not only the elements of number theory, but also theories of polynomials with one and several

variables before passing to theories of rings. It would be excessive waste of time and efforts. Motivated by number theory, students can learn divisibility theory in integer domains and then study polynomial theories as easy special cases.

Nor means the genetic approach that didactics of a discipline should be closed in boundaries of subject's contents. On the contrary, as noticed J.Dewey and Heintel [see Wittmann (1992)], taking subject matter's method fundamentally into account in building didactical models means using the method of highest order, scientific method and, therefore, breaking out the narrow boundaries of special disciplines. This means that one should also take into account all manifestations of human spirit's activities that are related to the scientific method and may help students to master the scientific method. In particular, genetic approach does not deny the importance of investigating the *effect on students* and the *artistic side of teaching*.

3. The principles of the concentrism and of the multiple effect

Many authors expressed the idea that teaching (not only of undergraduate mathematics) should be conducted in a concentric way. L.S. Vygotsky (1996, p. 87) wrote: " The rule is to completely avoid a repetition and to make teaching concentric, i.e. to arrange a subject in such way that it should be studied in maximally brief and simplified form at once in full volume. Then the teacher returns to the subject, but not for a simple repeating the past, but for studying the material once again in the deepened and extended form, with a variety of new facts, generalizations and conclusions, so that all things learnt by pupils earlier repeat again, but uncover their new sides, and new elements bound themselves with already familiar ones in such way that the interest arises by itself. In this respect both in a science and in life only new about the old can rouse our interest ".

Earlier Diesterweg (1962) suggested three rules close to the concentric way of teaching in the sense of Vygotsky: «1. Distribute a material in such a manner that at each stage a pupil would be in a position to guess or definitely expect the next stage. 2. Indicate at each stage some elements or parts of the following material and, not making essential breaks, cite certain elements from the future themes in order to excite inquisitiveness of pupils, not satisfying it, however, fully. 3. Distribute and arrange a material in such way that (where possible) at the following stage during studying new things the previous elements were repeated". He noticed that mathematics teaching could benefit from the concentric way.

We will try to specify here the traits of the concentric way of teaching mathematical disciplines at the undergraduate level. In our opinion, the main features are:

1) the *preparation* and, in particular, the *anticipation*;
2) the *repetition* at the higher or deeper level and the *increase*;
3) the *fundamentality* (the requirement of deep and strong study of the carefully selected foundations of a discipline);

The *preparation* is the extremely important element both in teaching and in various kinds of art. This element is well known by professional writers and theatrical directors. For example, A.P.Chekhov wrote: " If in the first act a gun hangs on the wall, in the last it must shoot ". A little bit modifying the classification of A.Zholkovsky and Yu. Scheglov (1977), one can consider three types of the preparation: a) the presentation; b) the anticipation;) the refusal.

Various ways of the presentation of the teaching material are investigated in detail in didactics of school mathematics (for example, before the study of a new material it is possible to activate the necessary knowledge from the previous themes or from related disciplines, to consider familiar examples etc.). Such ways can be studied also for teaching at the undergraduate level.

As an example of the element of the type "refusal" a problem way of studying a theme can serve (students are put before the fact of absence of the theory for the solution of the problem, and then the required is constructed in some way).

The most interesting and fruitful of these elements is, in our view, - the "anticipation". The indications to this element can be found at many classics of mathematics education. The above-cited rules of Diesterweg directly state this requirement. The demand to teach pupils to guess is put forward by G.Polya (1965) and Pototsky (1975). A guess and foreseeing by the students should be intensively used in teaching. Mordkovich (1986) also has described the elements of such method, comparing it with a spiral.

Concerning the repetition, note that one can speak not only about the repetition of those or other elements of a material, but also about repetitions of the relations between objects at various levels of a mathematical discourse. For example, the relations between objects in the theory of finite-dimensional vector spaces in many respects are repeated in the general theory of linear spaces, the relations between objects in analytical geometry on a plane in many respects are repeated in analytical geometry in a space; the relations between objects in the elementary number theory in many respects are repeated in the theory of polynomials. Even more interesting repetitions of the relations between objects one can find at the higher levels of abstraction, say, in abstract algebra. For example, categories and functors are in the relation similar to the relation between algebraic systems and homomorphisms; the composition of morphisms in a category is similar to the partial algebraic operation.

The *increase* means that elements repeated at later stages of study should be deepened and extended, equipped with interesting, impressing details.

For the efficiency of the concentric study, the anticipation should be based on very deep study of the fundamentals of a subject. The deep and slow study of the foundations requires the economical and thorough selection of the most necessary material. It is possible to name the requirement of deep and strong study of the carefully selected foundations of a discipline the principle of *fundamentality*.

We will name the approach to the teaching of a mathematical discipline combining the requirements of fundamentality, preparation and anticipation, repetition and increase the *principle of concentrism.*

The principle of concentrism, developed and used by many educators, is, in our view, closely connected to the *artistic side* of teaching. Indeed, elements of preparation, anticipation, repetition, increase are important for the composition of artistic works [musical pieces, theatrical plays, stories, paintings etc.] As we remarked in the ending of Section 2, it does not contradict to the genetic approach to methods of teaching. For example, G.Polya (1965) who supported the genetic approach, at the same time compared teaching to the theatrical art and even to music. Unfortunately, although it is widely admitted that teaching has much in common with art, the artistic side of teaching is under-estimated and almost not studied [cf., Koehler (1997)].

Another principle connected with the artistic side of teaching is the *principle of multiple effect* (on students and on the readers of textbooks). The essence of this principle is that the essential educational result can be achieved (in a lecture course or in a textbook, or even at a microlevel - in a separately taken lecture) not with the help of one means, but many, directed to one and the same purpose. For example, the following *means of expressiveness* may be used in teaching undergraduate mathematics:

1) The variation (passing through the various) – e.g., the explanation of a theoretical rule on a series of various examples, or, even more important, the consideration of a subject from all basic sides.
2) Splitting (material into smaller pieces);
3) The contrast - for example, the use of rather fruitful questions of a kind "How differs the concept (figure etc.) A from the concept (figure…) B. H.Poincare (1990) emphasized the importance of this element, speaking about mathematical definitions.

One may note that elements related to the principle of concentrism are connected with *metaphorical* relations (relations of *similarity*) and elements related to the principle of multiple effect are connected with *metonymical* relations (relations of *contiguity*). It would be interesting to deeply investigate the possibilities of combining metaphorical and metonymical relations in producing an educational effect [cf., R. Jacobson (1973), R.Barthes (1967), M. James et al. (1997)].

4. The conclusion.

Thus, for teaching mathematics at universities one can formulate the important principles indicated by classics of mathematics education, but poorly investigated and insufficiently used in practice: the principle of the genetic approach to teaching, the principle of concentrism and the principle of multiple effect. The *style of teaching* is extremely important, too. It is necessary to deeply study the role of above-discussed principles not only for classroom teaching, but also for the creation of the textbooks.

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FACTORS INFLUENCING STUDENT'S GENERALISATION THINKING PROCESSES

Marlene C. Sasman: *Mathematics Learning and Teaching Initiative (Malati), South Africa*
Alwyn Olivier: *Malati & University of Stellenbosch, South Africa*
Liora Linchevski: *Malati & Hebrew University of Jerusalem, Israel*

In this study we presented students with generalisation activities in which we varied the representation along several dimensions, namely the type of function, the nature of the numbers, the format of tables, and the structure of pictures. Our results show that varying these dimensions has little effect on children's thinking – as in our previous study, few children tried to find a functional relationship between the variables, but persisted with using the recursive relationship between function values, making many logical errors in the process.

INTRODUCTION

Number patterns, the relationship between variables and generalisation are considered important components of algebra curricula reform in many countries. These curricula often use generalised number patterns as an introduction to algebra. However, there is insufficient research that deals with the cognitive difficulties students encounter and the feasibility of such an approach. Much of the available research on students' thinking processes in generalisation reports on students' strategies in abstracting number patterns and formulating general relationships between the variables in the situation (e.g. Garcia-Cruz and Martinon, 1997; MacGregor and Stacey, 1993; Orton and Orton, 1994; Taplin, 1995).

In a previous study (Linchevski, Olivier, Sasman & Liebenberg, 1998) we presented grade 7 students with problems like the following:

(C3): Matches are used to build pictures like this:

Picture 1 Picture 2 Picture 3 Picture 4

The table shows how many matches are used for the different pictures. Complete the table.

Picture number	1	2	3	4	5		20		100		n
Number of matches	3	5	7	9							

Few students managed to construct a function rule to find function values. Rather, they focussed on recursion (e.g. $f(n + 1) = f(n) + 2$ in problem C_3 above), which led to many mistakes as they tried to find a manageable method to calculate larger function values. The most common, nearly universal mistake was to use the proportionality property that if $n_2 = k \times n_1$, then $f(n_2) = k \times f(n_1)$. For example, in problem C_3 above, from $f(5)^1 = 11$ they deduced that $f(20) = 4 \times 11 = 44$. Although this property applies only to functions of the type $f(n) = an$, students erroneously applied it to any function. It is possible that our use of "seductive numbers" in a sequence like $n = 5$, 20 and 100 *stimulated* the error (we regarded these numbers as seductive from a multiplicative point of view).

[1] Formal functional notation was not used in the interviews with students. It is used here for ease of communication.

We found that most students' generalisations and justification methods were invalid, because they are not aware of the role of the database in the process of generalisation and validation.. For example, in problem C3 above, they did not, and seemed unable, to verify their generalisations against the given data pairs (1 ; 3), (2 ; 5), (3 ; 7), (4 ; 9).

We also found that students worked nearly exclusively in the number context and did not use the structure of the pictures at all.

Based on the above we viewed the following as questions for further research:

- whether the use of non-seductive numbers will *prevent* students from making the multiplication error, also when they encounter seductive numbers in other problems
- whether the visual impact of the table, as for example shown in problem C_3 above, also contributed to the persistence of the proportional multiplication error
- whether pictorial representations in which the function rule is "transparent" will help students to use the structure of the pictures to more easily find function rules.

In this paper we report on some first findings on these three questions.

RESEARCH SETTING

The activities

We designed a series of eight generalisation activities in which we varied the representation of the activities. Four activities were formulated in terms of numbers only (in the form of a table of values), and four were formulated in terms of pictures only (in the form of a drawing of the situation). Each numerical representation had a corresponding pictorial representation. Two of the functions were linear functions of the form $f(n) = an + b$, and two functions were simple quadratic functions

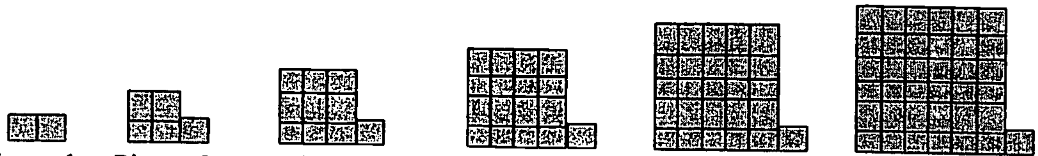
The numerical tables of values were presented in different formats: "continuous" (e.g. I_T below, in which input values for which the corresponding function values had to be calculated were included) and "non-continuous" (e.g. II_T, where the input values were not given, but were presented verbally by the interviewer). The tables were presented in both vertical and horizontal format.

The pictorial representations of the activities were chosen to be either "transparent", i.e. the function rule is embodied in the structure of the pictures (e.g. in I_P below), or "non-transparent", i.e. the function rule is not easily seen in the structure of the pictures (e.g. in III_P). As with tables, pictures were presented in both "continuous" and "non-continuous" format. All the drawings were presented to students in vertical format, but is here given horizontally due to space considerations.

The questions in each activity were basically the same, namely given the values of f(1), f(2), f(3), f(4), f(5) and f(6), we asked students to first find f(7) and f(8), and then the function values of certain further input values and to explain and justify their answers and strategies. These input values were both "seductive" (e.g. 20, 60) and "non-seductive" (e.g. 19, 59).

We supply below a selection of the activities. The subscript P indicates that the activity was presented in a pictorial representation and the subscript T indicates the problem was presented in the form of a table of values.

I_P: *Blocks are packed to form pictures that form a pattern as shown below:*

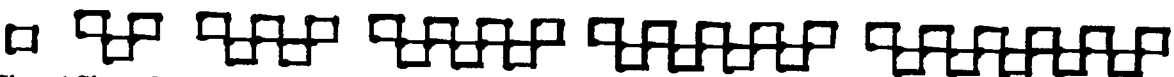


Picture 1 Picture 2 Picture 3 Picture 4 Picture 5 Picture 6

I_T: *Tiles are used to build pictures to form a pattern. The table below shows the number of tiles in a particular picture.*

Picture number	1	2	3	4	5	6	7	8	...	20	...	60	...	n
Number of tiles	2	5	10	17	26	37								

II_P: *Matches are used to build shapes. A different number of matches is used to build each shape.*



Shape1 Shape 2 Shape 3 Shape 4 Shape 5 Shape 6

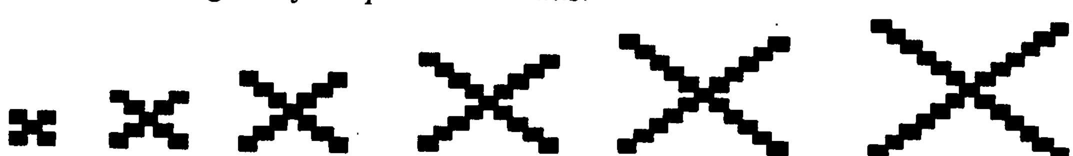
II_T: *Matches are used to build shapes to form a pattern. The table shows the number of matches used to build a particular shape.*

Shape number	Number of matches
1	4
2	12
3	20
4	28
5	36
6	44

III_P: *Cans are arranged to form pyramids like this:*

Pyramid1 Pyramid 2 Pyramid 3 Pyramid 4 Pyramid 5 Pyramid 6

IV_P: *Tiles are arranged to form pictures like this:*



Picture 1 Picture 2 Picture 3 Picture 4 Picture 5 Picture 6

Methodology

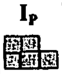
We interviewed ten grade 8 students at one of our project schools in a historically disadvantaged area of Cape Town before they had received any instruction on patterns, sequences or algebra. The students were selected by the teacher so that they were representative of the grade 8 class. Each student was interviewed three times in 45-minute

sessions. All interviews were videotaped and the tapes transcribed. The analysis will be used to design a teaching intervention to address the cognitive difficulties students have in the processes of generalisation.

RESULTS AND ANALYSIS

Most students had no difficulty finding f (7) and f (8) in any of the activities – they either found and used the function rule correctly, or used recursion correctly for these nearby values. However, in trying to find a manageable strategy for finding further-lying function values, students invented a variety of different strategies, both correct and incorrect. These strategies and their frequency are summarised in Table 1.

Table 1: Number of students using each strategy per activity

Activity number and representation format	Recursion	Proportional multiplication	Additive decomposition	Difference method	Extended recursion:	Function rule	Other
I_P Transparent picture, continuous  Seductive values: 20, 60, n Quadratic function: $f(n) = n^2 + 1$	2		1		2 wrong variations	5	
I_T Horizontal continuous table Seductive values: 20, 60, n Quadratic function: $f(n) = n^2 + 1$		4		1	1 wrong variation	3	1
II_P Transparent picture, continuous  Non-seductive values: 19, 59, n Linear function: $f(n) = 8n - 4$	1		1	3	1 2 wrong variations	2	
II_T Vertical non-continuous table Seductive values: 20, 60, n Linear function: $f(n) = 8n - 4$		4		3	1 2 wrong variations		
III_P Non-transparent picture, non-continuous  Non-seductive values: 23, 79, n Quadratic function: $f(n) = n^2$	3		1	1	2 wrong variations	2	1
III_T Vertical non-continuous table Seductive values (29, 87, n) Quadratic function: $f(n) = n^2$	1	1			1 wrong variation	6	1
IV_P Transparent picture, non-continuous  Seductive values: 20, 60, n Linear function: $f(n) = 4n + 1$	1	2		2	2 2 wrong variations	1	
IV_T Horizontal continuous table Non-seductive values: 23, 117, n Linear function: $f(n) = 4n + 1$	1		3	1	2 2 wrong variations	1	

The nature of the function

Finding function rules It is interesting to note from the data in Table 1 that more than half of the students found and used the function rules in activities I_P and II_T. These both represent simple quadratic functions. One could be tempted to conclude that students easily recognise such simple quadratic function rules. However, one immediately also notices the marked differences in students' responses for the same functions in the picture and the table contexts. In activity I_P the picture is transparent, but students find it much more difficult to recognise the same function rule from the equivalent table in activity I_T. In activity III on the other hand, students easily find the rule in the table, but not in the non-transparent picture.

It is clear that students found it much more difficult to formulate function rules for linear functions. From our interviews it seems that students try to construct simple multiplication (proportional) structures, but when it does not fit the database, they quickly give up and then invent all kinds of error-prone recursion strategies.

Recursion When students focus on recursion patterns, however, they find the constant difference between consecutive terms in linear functions much easier to handle than the changing difference in quadratic functions, leading to many errors. We describe these strategies and errors in the following sections.

Seductive vs. non-seductive numbers

The proportional multiplication error In our earlier work with grade 7 students we found a persistence with the erroneous proportional multiplication error. Also in this study six of the ten students interviewed used it at least once in the series of activities. For example:

Interviewer: How many tiles in Picure 20? (*in* IV_P)
Peter: OK, I am using 5 (*meaning* $n = 5; f(5) = 21$ *in the picture)* to get to 20. So 21 times 4 is 84, because 5 times 4 is 20.

This erroneous strategy was used only with what we call "seductive numbers". When students could easily find the function rule the nature of the input values was immaterial, i.e. they did not make the multiplication error, even for seductive numbers.

Extended recursion A few students managed to adapt their focus on recursion to a manageable strategy for finding further-lying function values. This extended recursion method is symbolised by $f(n) = (n - k)d + f(k)$, where d is the common difference between consecutive terms. Here is an example:

Interviewer: OK, Shape 59? (*How many matches in Shape 59 in* II_P*?*)
Hamid: So first I subtract 19 (*he had previously calculated* $f(19) = 148$) by 59 and then you get your answer of 40 and then I times it by 8 (*the common difference between terms*) and then I get my answer and then I add it by 148, that is Shape 19's answer.

Some students used this method also in the case of seductive numbers.

Students using this method often seemed to lose track of all the details. This was mostly because they worked verbally, and did not write down information or their strategy. For example, several students correctly calculated $(n - k) \times d$, but then did not add $f(k)$.

While the extended recursion method is correct for linear functions, many students also erroneously applied it to or adapted it for the quadratic functions. For example:

Interviewer: Ok, and then Picture 20? (*How many matches in Picture 20 in* I_T*?*)
Harold: I subtract 20 by 8 (*he had previously calculated* $f(8) = 65$) . . . I subtract 8 by 20, then I get 12 ... with that 12 I times by 2 is equal to 24 ... then I add 24 by 15, is equal to 39 then I add 39 to 65, is equal to 104.
Interviewer: Just explain the 15 please
Harold: That's the 15 I added by 50 ($f(7)$) to get 65 ($f(8)$).

Additive decomposition of input value The introduction of "non-seductive numbers" gave rise to other inappropriate strategies when students could not find a multiplicative relationship between the non-seductive numbers such as 19, 23, 59 and 117. Some

students used additive decomposition of the input value, symbolized by $f(n) = f(a) + f(b)$, where $n = a + b$. For example:

Interviewer: OK, in Picture 117, how many tiles? *(How many tiles in Picture 117 in activity IV_T)*
Errol: *(Writes 30 = 121, 40 = 161, 50 = 201 . . . 100 = 401, 117 = 470)* Picture 117 is 470.
Interviewer: Can you explain to me how you got that?
Errol: Uhm, as I followed on Picture 100, I had to end up at 401 and I added Picture 17 to Picture 100 which gives Picture 117 *(He had previously calculated Picture 17 as 69)*

The difference method The erroneous difference method, symbolised by $f(n) = n \times d$ was invoked with both "seductive" and "non-seductive" numbers. For example:

Interviewer: Ok, how many in Picture 23? (*How many tiles in Picture 23 in IV_T?*)
Linda: (*works on calculator*) . . . 92
Interviewer: Just explain please?
Linda: It will take too long to add 4 every time (*she previously found a constant difference of 4 between the terms of the sequence*). So I just said 23 times 4.

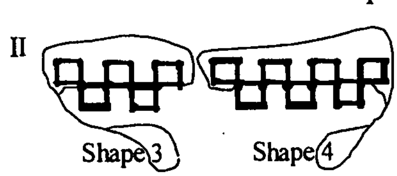
The visual impact of tables

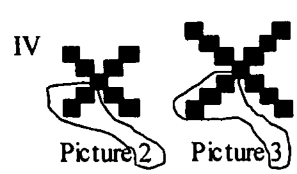
From Table 1 it is clear that the visual presentation of the numbers in a table format for the function did not impact on the errors students made. The table in activity I_T was horizontal and "continuous" whereas the table in II_T was vertical and "non-continuous". Four students made the proportional multiplication error in both these examples. One student committed the difference method error in I_T whilst 3 students committed the error in II_T. The way we presented the questions as "continuous" or "non-continuous" in the picture activities also did not effect students' strategies. This can probably be explained by the fact that when the input numbers were not presented in writing, students made their own "continuous" "tables", so the visual distraction remained.

"Transparent" vs. "non-transparent" pictures

In I_P five of the ten students successfully recognised the function rule from the structure of the transparent picture. In II_P most students recognised that 2 squares (8 matches) were being added but then converted to numerical mode, constructing their own "table" of values, e.g. " … 1 = 4, 2 = 12, 3 = 20, etc.". Only two students described the function rule from the structure of the pictures, namely as $(n + n - 1) \times 4$ and $n \times 4 + (n - 1) \times 4$ respectively. Only one student used the structure of the picture in IV_P to identify the function rule. No child could recognise the function rule of using the non-transparent picture as the database in III_P. Two students found the function rule once they reverted to the number context.

It seems that these students do not have the necessary know-how of *how* to use the structure of a picture to find a functional relationship. If one wants to find a function rule in a table, one necessarily takes some specific value of the independent variable (input number) and tries to construct a relationship between this input-output pair. In the case of pictures, few students seem to intentionally take a specific input number and try to see this number in the structure of the picture, as illustrated in the following diagram:





Of course, it further requires a rich number sense, e.g. in II to see a further relationship in the numbers (2 is one less than 3, and 3 is one less than 4) before one can formulate the function rule $[n + (n - 1)] \times 4$. In IV one must see the multiplication or equal addition structure before one can formulate the rule $4 \times n + 1$. A weak number sense will therefore also contribute to students' difficulties in using the structure of pictures to see the general in the particular required to formulate function rules.

Most students could see and use the structure of the pictures in a *recursive* way, e.g. in II students used the structure that 2 squares (8 matches) are added each time, and in IV they used the structure that 4 tiles are added to each successive picture. However, this did not help them to find the function rule, and students mostly then constructed a table of these values and then used the numbers in the table inductively. Of course, one could use the extended recursion method to use this recursive structure to formulate the function rules as $4 + 8(n - 1)$ and $5 + 4(n - 1)$ respectively.

Verification of strategies

Consider the following protocol:

Interviewer: Ok, Shape 19? (*How many matches in Shape 19 in II_p?*)

Peter: *(Peter successfully found f(7) and f(8) by counting the number of squares and then multiplying by 4 to get the number of matches. Now he starts making a systematic table of the number of squares in each Shape, using a recursive pattern:*

7	8	9	10	11	12	13	14	15	16
13	15	17	19	21	23	25	27	29	31

He then stops and goes back to looking at the pictures again.)

OK, I realised if I do this it is a bit of a hassle, so I looked at the pattern (*in his database*) and I figured the difference (*between f(n) and n*)

I took here (*pointing at f(5)*) . . . the difference between 9 (*f(5)*) and 5 (*n*) is 4 and by number 6 it is 5 . . . yes (*he checks again*) . . . 5. And by number 7 it is 6 and by number 8 it is 7. So I just tried it out. So I said to myself OK it is right and it will take too long to do it like this (*referring to his table of values*). So Shape 19 is 19 + 18, is 37, so 37 blocks times 4 gives you . . . 148 (*using the calculator*).

Clearly, Peter has constructed an efficient rule, which we can symbolise as $[n + (n - 1)] \times 4$, based on a sound analysis of patterns in the given and his extended database, and he verified that his pattern holds against the database several times. He was convinced and he could use the method with assurance. However, this style of working stands in stark contrast to most students' approach to such generalisations. While the students who used a function rule necessarily deduced the rule from the database, the other strategies reported in this paper are mostly not based on the database – students did not find the methods in the database, nor did they check it against the database. This applies to correct as well as to incorrect strategies. Students seem not to realise the need to validate their generalisations, and seem not to have the know-how of how to validate a generalisation against the database.

DISCUSSION

As in our previous study, students worked nearly exclusively in the number context and not with the pictures, favoured recursion methods, had difficulty in finding function rules and made many errors, including the proportional multiplication error. There is,

however, one marked difference, namely the variety of strategies used by the students in the present study in comparison to the previous study. Of course this could be attributable to the differences in the subjects, who are at a different grade level, and from a different socio-economic background. We would argue, however, that the difference is mainly attributable to the introduction of non-seductive numbers in our activities.

It is for this reason (the greater variety of strategies), that we plan to extensively use non-seductive numbers in our planned intervention. However, as is evident from the examples in this paper, the use of non-seductive numbers will probably not prevent the ubiquitous proportional multiplication error when students encounter seductive numbers, nor will it prevent the other erroneous strategies reported here. For that we believe we should address two more fundamental issues, viz.

First: The development of an awareness of the need to view any strategy as an hypothesis that should be validated against the database, and a focus on skills of how to do it. When one looks at the variety of strategies used by students, one can probably safely say that they have the ability and flexibility to find many patterns and relationships between numbers. The problem, however, is that most students are finding *random* relationships between the numbers without reference to the given database. We are struck not so much by the frequency and persistence of students' errors, but by their lack of an essential aspect of "mathematical culture", namely to view any strategy as an hypotheses that should be justified or verified against the given database. It seems that students lack simple *strategic knowledge*, e.g. to test an hypothesis against special cases.

Second: A more explicit study of the properties of different function types, and a comparison of such properties to become aware and make explicit which properties apply to which function types. All the erroneous strategies used by students are correct properties for some function, which students wrongly apply or overgeneralise to other functions.

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KEEP CHANGE CHANGE

Roberta Y. Schorr
Rutgers University-Newark,
New Jersey, USA

Alice S. Alston
Rutgers University-New Brunswick,
New Jersey, USA

This research presents and analyzes difficulties experienced by teachers who were attempting to find meaningful ways to help their students learn about addition and subtraction of integers. Results indicate that teachers can generate meaningful contexts to help their students understand addition, but have great difficulty doing the same for subtraction. The contexts that they generated appeared to be based on an understanding of subtraction that involved "taking away" one amount from another. No example of subtraction as the comparison of two quantities was given. As a result of their difficulties in finding meaningful contexts, they tended to prefer a more procedural approach to teaching subtraction, claiming that using an algorithm "is more meaningful than making up a story".

Introduction

The purpose of this paper is to present research findings related to our work with teachers in a long-term teacher development intervention. An underlying premise of this intervention is that teachers need appropriate experiences and materials from which to build new models of instruction. In addition, teachers must also be provided with opportunities to develop a deeper understanding of the mathematical concepts they are expected to teach and an increased awareness of the ways in which children learn (Schorr, Maher, & Davis, 1997; Lesh and Kelly, 1997; Janvier, 1996; Alston, Davis, Maher, and Schorr, 1995; Simon, 1995; Cobb, P., Wood. T., Yackel, E. and McNeal, B. 1993; 1986; Davis, 1984). To this end, in each workshop session, led jointly by the authors, teachers are presented with challenging problem tasks and asked to work together to produce solutions that represent important mathematical ideas. After sharing their own ideas and representations, they are encouraged to use these or similar tasks in their own classrooms. During classroom implementation, teachers are encouraged to recognize and analyze their students evolving ways of thinking about these mathematical ideas. Teachers are also encouraged to carefully study and assess student work, and to select particularly interesting products to share in subsequent workshop sessions. Studying these samples of student work together provides the opportunity to consider the development of these ideas in students and discuss the pedagogical implications of using this approach. As one might expect, this approach to thinking and doing mathematics is unfamiliar for many experienced teachers, and is not easy.

Experience and research have shown that all teachers possess beliefs about the way mathematics is learned (Thompson, 1985). These beliefs are generally acquired

prior to actual classroom experience, and held through years of teaching (Tabacknick and Zeichner, 1984). In addition to long held beliefs about mathematics instruction, many teachers may not possess adequate subject matter knowledge. Cohen and Ball (1990) point out "how can teachers teach a mathematics they never learned, in ways they never experienced?" (p. 233). The old models that teachers have for providing mathematics instruction are generally quite robust, and even teachers involved in long term interventions will fall back upon those old models, especially when under pressure to have students perform well on tests whose goals may be inconsistent with those of the teacher development intervention.

The purpose of this paper is to provide documentation, in a specific content domain, relating to some of the difficulties that teachers experience as they attempt to make the transition from teaching mathematics as a set of rules, procedures, and algorithms toward thinking about and teaching mathematics as a "collection of ideas and methods which a student builds up in his or her own head" (Davis, 1984, p. 92).

Background[1]

This research is one component of a multi-year teacher development intervention. that began with the purpose of supporting the effective classroom use of exemplary mathematics curriculum and meaningful assessments as an integral part of instruction. To this end, university researchers work directly with groups of teachers from four economically disadvantaged high schools in an inner city in New Jersey. The particular high schools were chosen because they have student achievement levels in mathematics that are amongst the lowest in the city. The school district is currently under the supervision and control of the state due to overall poor student performance.

Methods and Procedures

This paper is based on an analysis of the problem-solving activity that occurred during one full-day workshop session in the third year of the intervention. Ten teachers were present. Three of the teachers had been part of the project since year one (tier I) and three of the teachers had joined the project in its second year (tier II). The remaining four teachers had joined the project during the third year (tier III). The workshop under discussion was the second workshop of the academic year.

The format of this particular workshop, typical of each of the project sessions, began with discussion initiated by the teachers about issues that they consider relevant to the project. As a general practice, the researchers' role during these discussions is that of facilitator and observer, providing opportunities for them to learn about the knowledge, beliefs, and concerns of the participants. Based upon this, the

[1] This work was supported in its first year by a teacher enhancement grant from the National Science Foundation which was directed by Richard Lesh and Thomas Post. Beginning in the second year, support has come from the school district itself. The opinions expressed are not necessarily those of either sponsoring agency and no endorsement should be inferred.

researchers are able to make decisions about the direction of future workshop sessions.

The specific mathematical content of this paper, addition and subtraction of signed numbers, was introduced in the beginning discussion by one of the teachers. These particular ideas had not been considered in previous sessions, and, in fact, were not the topic selected by the researchers as the main mathematical idea to be explored during the session.

The data for this research includes the researchers' observation notes, problems and illustrations printed by the teachers on large pieces of chart paper as they were explaining their ideas to the group, and their individual written work completed during the session.

Results

On the occasion reported in this research, a spontaneous general discussion about operations with signed numbers occurred, following from a concern expressed by one of the teachers.

The following dialogue represents excerpts from that discussion as recorded in the field notes of the researchers with further documentation from the chart papers used in group discussion.

Kate[2] (tier III): *One of my students, Richard, is generally a very good student. I'm concerned because when he does examples like positive thirteen minus eight, he ends up with fifteen. I think that using calculators is the cause of his problem.*

Mark (tier I): *I think that signed numbers are a problem for many of my students as well.*

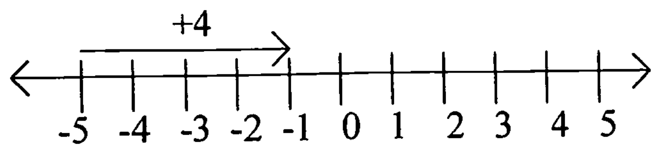
When other teachers also added that their students had difficulties with signed numbers, Mark wrote on a large piece of chart paper in front of the group:

$$-1-(-5) = -1+5 = 4$$

Mark: *See, the question is what do you add to negative five to get to negative one. There are lots of ways to do it. You can also think how many units must you move to the right on a number line.*

He illustrated his point by drawing a number line with an arrow directed from -5 to -1 and labeled it +4 (as below). Two additional examples were given and illustrated on a second number line, 1 + 3 and -3 + 2.

[2] Names have been changed to preserve the anonymity of the teachers and students.



Mark: .... *or you can always consider the negative number as money you owe, like you owe someone ten dollars and you pay six so you still owe someone four.*

Jack (tier II): *I like it when the teacher becomes a number and actually walks!*

Mike (tier III): *I like to think that the plus sign means "have" and the minus sign means "owe". So, if you have positive five minus three equals positive two, the positive sign in front of the five means have, the minus sign means owe, and the positive sign in front of the two means still have.*

He wrote the following on the large chart paper in front of the room:

"+" means "have"

"-" means "owe"

$+5 \quad -3 \quad = \quad +2$

have owe

Joan (tier I): *When you have to subtract a negative, you should use three words--keep, change, and change. So if you have negative three minus negative four, you keep the first sign, and change the other two.*

She wrote the following on the large chart paper in front of the room):

$-3 \quad - \quad (-4) =$

Keep Change Change

Jack: *I like that, because the more you try to give them a reality example, the more confused they get.*

Researcher: *Can you make up a story to go with Joan's problem if you wanted to?*

Jack: *Its hard, but you can.*

Researcher: *Go ahead, why don't you all try.*

Mark: *You can always do it with money, like I said before.*

After a general discussion, the researchers redirected the teachers' attention to the problem that Joan had written on the chart paper. They again challenged them to work individually or together to create meaningful situations or stories that would make sense out of each of four numerical expressions involving operations with signed numbers.

Researcher: *Well, make one up. While you are at it, try to make up problems for these as well. You can work together if you like.*

She added the following three problems to the piece of chart paper:

$$-3 - (+4) = \qquad +3 - 4 = \qquad -3 + 4 =$$

The teachers worked individually, occasionally conferring with neighbors. During that time, several teachers noted to the group how difficult it was to make up a real story to coincide with any of the problems listed above.

Susan (tier III) began the discussion. She said that she liked to use algebra tiles to help, and her example was based upon that. Her written example follows:

> *Sandy got squares for positive and negative numbers. -1 = □ in red color. 1 = □ in blue color. -(-1) = □ in blue color. She took 3 red squares, and then subtracted four in blue. How many squares in what color did she have?*
>
> $$-3 - (-4) = -3 + 4 = 1$$

When the other teachers responded that they did not understand the example and asked how this could be considered a "real situation", Susan explained that she was referring to the use of actual blue and red tiles, concretely, to solve the problem. She used the chart paper in an attempt to illustrate this, but immediately became confused herself, unable to remember how she had produced the blue (negative) and red (positive) squares necessary to concretely remove 4 red squares and end up with 1 positive blue square.

Harriet (tier 1) then suggested the following problem instead:

> *Sharifa had \$3 negative (out of pocket) and she gave Maria negative one times minus \$4. How much did they have together?*

Several other teachers responded that they thought this would be confusing for students as well. In the ensuing discussion, one of the researchers asked whether the teachers had been writing their stories for this problem to match the answer (positive one) that they already knew. All the teachers agreed that this was indeed the case and stated that, for subtraction of negative numbers, using a rule was preferable. Jack repeated his earlier remark, emphasizing the following:

> *The more you try to give them a reality example, the more confused it gets! "Keep, change, change" is more meaningful than making up a story.*

The written examples for the problems were collected from all of the teachers at the end of the session. Analyzing that data confirmed the difficulty that teachers had encountered in finding sensible situations for subtraction.

Ellen (tier 2) tried an example that documents how difficult developing a meaningful story had been for her. She wrote:

> *Sam has been borrowing money from some of his friends. He owes Tonero \$3. He owes Habebah \$4. He started to repay his debts.*

She first wrote the following annotated expression to show how the symbols connected to the words of her story:

reduce the debt

$$(-3) - ((-4))$$

for Tonero Habebah

On reflection, Ellen apparently changed her mind, and wrote this numerical expression on her paper to go with her story:

$$-3 + (-4)$$

Both Susan and Harriet also wrote sensible stories that were based on addition. It appears that Susan changed a subtraction problem to an addition one, as indicated by her symbolic representation.

Susan's work follows:

> *Pete was on the 3rd floor, then he took an elevator 4 floors down. On what floor is he now?*
>
> $$+3 - 4 = +3 + (-4) = -1$$

Harriet wrote the following story for the same problem:

> *Ericson owed \$4 to Alice. However he received \$3 from Mahmoud. How much money will he actually have out of his pocket?*

In analyzing the written work of the teachers, all but one of the papers included at least one example that was a sensible addition story using signed numbers. No teacher provided an example for subtraction of a negative number from another quantity that was realistic. Mark's example indicates that he may have been considering subtraction of a negative number as a "forgiven debt". He wrote:

> *A couple of weeks ago, James borrowed \$3 from Mary with the understanding of paying back the very next day. But James forgot to meet up with his obligation. Mary then got upset and demanded James to pay her money. James then replied. In fact you owe me one dollar. Do you remember when you borrowed \$4 from me 10 years ago?*

However, along with his story, he wrote the following equations as the appropriate numerical representation for what he had written:

$$-3 - (-4) = -3 + 4 = 1$$

Discussion and Conclusions

A number of examples generated throughout the session appeared to be based on an understanding of subtraction of whole numbers that involved "taking away" one

amount from another. In each case, the appropriate numerical expression involved addition, with the negative sign indicating either direction or a debt. No example of subtraction as the comparison of two quantities was given after Mark's reference to the number line, and even in this case his comment "what do you add to negative five to get to negative one" appears to indicate that he was thinking of addition.

The examples reported document the value of having teachers write meaningful stories both as a means of deepening their own understanding, and uncovering areas of difficulty. However, the results reported indicate that more attention should be given to developing meaning for subtraction as comparing a first quantity with a second, first with whole numbers, and then with integers.

In addition, the results indicate that the teachers still felt that it was helpful to offer students "real world" problems, especially those involving money, to help them understand addition of signed numbers, but that was not the case for subtraction. Jack's comment sums it up: *The more you try to give them a reality example, the more confused it gets! "Keep, change, change" is more meaningful than making up a story.* In fact, the teachers unanimously agreed that Joan's suggestion to use the procedure involving "keep, change, change" was the best way to approach subtraction of integers.

As indicated in the results, when the researcher asked the teachers whether or not they were generating their problems for subtraction with the answer already in mind, they all indicated that they indeed were. Further, their written work indicates that they consistently converted subtraction problems into addition problems, and their problem situations more closely reflected addition situations.

Another point worth noting has to do with the use of concrete materials in the teaching of mathematics. As indicated by the results, Susan had great difficulty in recreating a situation in which addition and subtraction of integers could be modeled with tiles. This occurred despite the fact that she had mentioned on more than one occasion during the discussion that using what she referred to as algebra tiles had been very helpful to her students. She not only had difficulty in representing how the tiles could be used, but also in generating the statement of a problem that could be used to precipitate a discussion involving them. In fact, her initial word problem using the squares (tiles) reveals some basic misconceptions about the difference between using concrete materials and generating a real situation. In this case, Susan thought that using the concrete materials was the real situation.

The results of this research indicate how difficult it is for teachers to provide meaningful problem-solving situations for their students in this particular content domain. Further, they indicate that at least in this circumstance, when teachers experience difficulty in providing a meaningful context for their students, they often resort back to a procedural approach.

A final point needs to be made. The teachers involved in this research are caring, committed teachers who have all successfully completed degrees in college mathematics. Each of them works hard to create meaningful instruction for students, and is eager to learn more about how to meet their needs. An important implication of this study is the need to encourage mathematics education researchers to constantly focus attention on the mathematical thinking of teachers, and to create contexts and environments in which teachers and researchers can openly work on these ideas together.

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WAYS OF TALKING IN A MULTILINGUAL MATHEMATICS CLASSROOM[1]

Mamokgethi Setati
University of the Witwatersrand, South Africa

In this paper I investigate the nature of talk in a multilingual mathematics classroom in South Africa. In particular, this paper explores how the teacher uses code-switching to facilitate communication of mathematical ideas. The paper draws from a wider study on language practices in multilingual mathematics classrooms in South Africa where multilingual teaching and learning is now encouraged. Current national policy defines eleven official languages and flexibility for schools in determining their language policy.

INTRODUCTION

It is well known that language (reading, writing and speaking) is important for thinking and learning, and that language is not only an issue in mathematics classrooms but in all classrooms. This issue, however, takes on a specific significance in multilingual mathematics classrooms. Learning and teaching mathematics in a multilingual classroom in which the language of learning is not the learners' main language is, undoubtedly, a complicated matter. Learning mathematics is similar to learning a language since mathematics, with its conceptual and abstracted form, has a very specific register. School mathematics also involves a range of discourses, ways of using and valuing language. These places additional demands on mathematics teachers and learners.

> *Mathematics teachers ... face different kinds of challenges in their multilingual classrooms from English language teachers. The latter have as their goal, fluency and accuracy in the new language - English. Mathematics teachers, in contrast, have a dual task. They face the major demand of continuously needing to teach both mathematics and English at the same time (Adler, Slonimnsky and Lelliot et. al., 1997).*

Learners on the other hand have to cope with the new language of mathematics (its specific register and discourses) as well as the new language in which mathematics is taught (English).It is therefore important to understand the different language practices that teachers in multilingual mathematics classrooms use to enable learners' meaningful communication of mathematical ideas, concepts, generalisations and thought processes. In this paper I will present the initial stages of an analysis of data collected for a wider study on language practices in intermediate multilingual mathematics classrooms (Grades 4 - 6) in South Africa. I will focus on one case and will explore the following questions:

- how does the teacher use the learners' main language to help learner access to mathematics?
- what languages do the learners use in mathematical discussion?
- how do learners use the language of the teacher, the language of mathematics and their own everyday language in the process of learning?

THEORETICAL FRAMEWORK

Two major areas of enquiry inform this study. The first relates to Vygotsky's theory of socio-cultural development. Development occurs in and through socially mediated activity and language plays a key role in mediation (Vygotsky, 1986). The presence of a more experienced other who embodies and models the intended outcome for the learner is crucial within this framework. In a mathematics class, the more experienced other can be the teacher and the intended outcome for

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the learner is mathematical excellence which includes the ability to talk "within and about mathematics" (Adler, 1998).

Pupils learning mathematics in school, in part, are attempting to acquire communicative competence in mathematical language. Learning to be able to articulate the meaning of certain concepts involves the development of a language that can best describe the concepts involved. This is especially pertinent to mathematics because mathematical talk is known for involving both specialised terms and different meanings attached to everyday words. Mathematical language comprises both informal and formal components. Informal language is the kind that learners use in everyday life to express their mathematical understanding. Formal mathematical language refers to the standard use of terminology (mathematics register) which is usually developed within formal settings like schools. In most mathematics classrooms both forms of language are used and these can be either in written or spoken form. "One difficulty facing all teachers, however, is how to encourage movement in their learners from the predominantly informal spoken language with which they are all pretty fluent, to the formal language that is frequently perceived to be the landmark of mathematical activity" (Pimm, 1991: 21).

Another useful way of categorising discourses in mathematical classrooms is to use Sfard's (1998) distinction between calculational and conceptual discourses. She defines calculational discourse as discussions in which the primary topic of conversation is any type of calculational process, and conceptual discourse as discussions in which reasons for calculating in particular ways also become explicit topics of conversations (Sfard, 1998: 46).

The second area of enquiry relates to the work that has been done concerning teaching and learning in multilingual classrooms. One of the significant findings in this area relates to the benefits that result from using code-switching[2] in teaching and learning mathematics (Setati, 1996; 1998, Adler, 1996). Other studies have shown that use of the learners' first language in teaching and learning provides the support needed while the learners continue to develop proficiency in the second language (Khisty, 1995; Adler, Slonimnsky and Lelliott et. al., 1997).

In this paper I look at ways in which the teacher models and uses different mathematical discourses and code-switching and how these enable the development of learners' mathematical linguistic abilities. I will argue that this case suggests a complex correlation between code-switching, using a range of mathematical discourses and pupils' ability to communicate mathematics.

METHODOLOGY AND RESEARCH DESIGN

This is a qualitative study (Cohen & Manion, 1994) that focuses on one carefully selected grade 4 teacher together with her class during their mathematics lessons. The study is both descriptive and exploratory (Bailey, 1978:38). It is descriptive in the sense that it describes in detail the way in which the teacher and learners use languages during the mathematics lessons. The study is also exploratory because it explains how the teacher uses languages to facilitate learners' access to mathematics.

Data was collected by means of teacher interviews, lesson observations and learner interviews. The teacher pre-observation interview was done before the lessons were observed and focussed on the

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[2]Code-switching is when an individual (more or less deliberately) alternates between two or more languages (Baker, 1994: 77).

preferred language practices of the teacher. Lessons were observed for a week and the last two lessons observed were video recorded. A reflective interview with the teacher after observation of lessons focussed on the critical incidents in the lessons observed and the teacher's understanding and rationale for the language practices used during the lessons. The pupil interview focussed on learners engaging in mathematical talk related to mathematics lessons observed.

To analyse the lesson transcripts and learner interview, four categories were used to understand ways of talking mathematics in this classroom: informal and formal calculational discourse and informal and formal conceptual discourse.

In the section that follows I first give a description of the context in which data was collected and then continue with the analysis.

THE CONTEXT

Ntombi, the teacher, teaches in a primary school (grades 1 - 8), west of Johannesburg in South Africa. She has been teaching for ten years and has a Senior Primary Teaching Diploma plus a three year university degree. Like her learners, she is a first language Tswana[3] speaker. However, in addition to Tswana she can speak three other languages (English, Afrikaans, S. Sotho).Her grade 4 class that was observed had 60 learners in total, 26 girls and 34 boys. They were all multilingual and could speak from two to four languages and this included English which is a second language for all the learners in the school. Compared to other learners in the wider study, these pupils were relatively fluent in English. While their level of fluency could not be compared to a first language speaker, they were able to communicate in English. The main language in the area and the school is Tswana and all the learners are fluent in it. The language of learning in the school is English[4] and its use is encouraged in the school.

A GENERAL DESCRIPTION OF LESSONS OBSERVED

Five consecutive lessons were observed in the same grade 4 class and they all focussed on multiplication. To introduce the first lesson Ntombi started by writing the word "multiplication" on the board and talking with the learners about what it means both in Tswana and in English. She proceeded to give them an example on the board:

$$\begin{array}{r} 14 \\ \underline{\times 16} \\ 84 \\ \underline{+14} \\ \underline{224} \end{array}$$	*This was elaborated procedurally: 6 times 4 is 24, write 4 carry 2. 6 times 1 is 6 plus 2 is 8. 1 times 4 is 4, 1 times 1 is 1. 4 plus zero is 4, 8 plus 4 is 12, write 2 carry 1 and 1 plus 1 is 2. Therefore the answer is 224.*

This was followed by group exercises and then class-work which were both similar to the example. During group work there was a lot of interaction mainly in Tswana between learners. During teaching, Ntombi communicated with learners in both English and Tswana and engaged them in mainly formal calculational discourses. These kinds of discourses were also observed among learners during group work.

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[3]Tswana is one of the eleven official languages in South Africa.

[4]According to the new language policy in South Africa schools have a right to choose their language of learning.

Lesson 2 started with checking and marking of home-work. Volunteers from different groups were called to the board to write their solutions. If the answer on the board was incorrect another volunteer was requested. The teacher identified those who had problems with the home work and did more examples with them, emphasising the procedures to follow, while the rest of the class continued with more multiplication problems. After working with the selected group she gave them an exercise to do as home work. Both the teacher and learners used English and Tswana interchangeably (code-switching) and formal calculational discourse was dominant during this lesson. The lesson ended with the whole class singing a song while they put away their mathematics books.

Similar to lesson 2, lesson 3 started with checking and marking of home work. She then worked with one group ('good group') on multiplication of three digit numbers by two digit numbers while the rest of the class was busy with corrections. Code-switching and formal calculational discourses were also dominant during this lesson.

After checking and marking homework in different groups during lesson 4, the teacher worked with the same group ('good') on a word sum while the rest of the class was very noisy and not involved. The word sum she did with the group was: "In Thusong primary school, there are 10 classes and in each class there are 19 learners. How many learners are there in Thusong?" After doing this example she started a song to get the learners' attention back. At the end of the song she wrote two different exercises on the board: one for the 'good group' and the other for everyone else. For the 'good group': "In KTS there are 15 classes. In every class there are 13 learners. How many learners are there in KTS school?", for the rest of the class: 301 x 15, 408 x 19, 485 x 15. During this lesson discourses became more informal and conceptual and code-switching continued to be a dominant practice.

In lesson 5, after checking and marking the home work, the teacher continued to work with the 'good group' on another word sum example: "In the SPCA[5] are 12 cages. In each cage are 12 dogs. How many dogs are there altogether?" The rest of the class was working on lesson 4's word sum. In handling the word sum with the 'good group', the teacher started by asking learners to read and then focused on the new words like "SPCA", "cage" asking them what they mean. Most of the learners' explanation of these words were in Tswana. This was followed by a discussion on what they were required to find in the word sum and how the solution can be found. After finding the solution she wrote two different exercises on the board for the learners to do as a class test. During this lesson the teacher engaged learners in both formal and informal calculational and conceptual discourses.

DESCRIPTION OF TALK IN AND ACROSS LESSONS[6]

During teaching, Ntombi focused mainly on formal mathematics language. Her classroom mathematical discourse moved across calculational and conceptual discourses. She taught procedures explicitly. Throughout lessons 1, 2, 3 she lead the learners in calculational processes used to solve problems. Her focus seemed to be on getting the learners to master the procedure and not on the reasons for using the procedure or on why the procedure works. Her talk was in terms of procedures where numbers are manipulated as objects that can be 'carried'. For instance,

[5]SPCA is an abbreviation which stands for Society for the Prevention of Cruelty to Animals.

[6]Due to limitations of space it was not possible to display transcript extracts in sufficient detail, examples have been selected.

"We carry down 1 and say 9 plus 3 is 12".What is interesting is that the teacher is not the only one who 'owns' this kind of talk. She models the talk and then gives learners an opportunity to practice it. This was evident during lessons and learners' interview where learners used the language of the teacher in most of their discussions. In the extract below the learner is working out the solution for 444 x 19.

P: Let us say 9 x 4 is 36.We write 6 and carry 3 then again we say 9 x 4 36 + 3 is 39 we write 9 and carry 3.We say 9 x 4 again is 36 plus 3, 39 and cover the units.We say 1 x 4 is 4. And again 1 x 4 is 4.We say again 1 x 4 is 4 and then we underline and then 6 + 0 is 6. 9 + 4 is 13 carry 1. 9 + 4 is 13 plus 1 is 14 carry 1. 3 + 1 plus 4 is 8. (Lesson 4)

In the above extract, the learner is imitating the 'teacher's language' of mathematics where numbers are referred to as objects that can be 'covered' and 'carried'.

While it can be argued that this kind of talk can and does occur in many mathematics classes, what actually makes a difference is the fact that in Ntombi's multilingual class this kind of talk is supported by the learners' main language. For instance if the teacher discovered that there was an error in the procedure she handled this in the learners' main language. For instance, in the extract below the teacher had asked one of the learners to work out 59 x 19 and according to the procedure she taught them they firstly needed to write this problem vertically. In trying to write it vertically the learner wrote the multiplication sign incorrectly between the 1 and 9 in 19. The extract shows how the teacher used Tswana to deal with this error in a non-threatening manner.

T: Alright, I must put it down, okay. And then we say 1 + 0. 3 +... Go na le phoso fa?[Is there a mistake here?]

P: No.

T: Nix, nix? Lebella sentle. [Really? Look carefully.] (The learner corrects the multiplication sign writes it at the correct place.) (Lesson 4)

This is not the only way in which the learners' main language plays a role. In fact to move across the discourses Ntombi used the learners' main language. The following episode which occurred during lesson 5 is a typical example of how Ntombi used the learners' main language to engage learners in informal conceptual discourse.

T: Eh, can you all read here?

P: In the SPCA are 12 cages, in each cage are 12 dogs. How many dogs are there altogether?

T: Now, first of all, what is this SPCA?

P: When your dog is ill...... (unclear)

T: Yes, sure.

P: Fa ntja ya gago e lwala go na le batho ba tlang ba tla go tsaya ntja ya gago a ba a isa ko spetlele fa ba bona e le botoka ba e busa.[If your dog is ill, there are people who will come and take it to the hospital and they bring it back when it is well.]

T: Ee, spetlele sa dintja akere?Ke ko diphologolo, di pets tsa mo ntlung di nnang teng akere?[Yes, it is a hospital for dogs, right? It is where pets are kept]

P: Ba kile ba tsaya ntja ya ko gae.[They once took my dog.]

T: Ba kile ba tsaya ya kwa lona? [They once took your dog?]

P: Le ya ko gae.[And mine too]

T: Ao! Ba e tlhatlhoba ka eng? La patela? [How do they examine it? Do you pay?]

P: No. Mahala. [Free.]

T: Ooh, ke mahala? [Oh it is free?] Go raa gore e a thusa ka gore seo se se tona ke gore re tshwanetse go tlhokomela diphologolo tsa rona akere?[This is helpful because the greatest thing is that we should take care of our pets] O tla bolella baba sa itseng, baagisani gore bathong ga ntjanyana ya gago e kare e a tlhotsa o tla e isa ko SPCA [You must tell your neighbours and all those who do not know that if their dogs are ill they can take them to the SPCA] (Lesson 5)

While it may seem as if the discussion that the teacher is having with the learners above is not important in a mathematics class, in this case it is. The problem that is being dealt with here talks about the SPCA and therefore the teacher uses this as an opportunity to educate the learners about the SPCA. The use of the learners' main language here enables active interaction with the teacher, for instance, learners are free to share their stories about the SPCA

T: Right. Jaanong ga re bala fa yare[when we read here it says] in the SPCA are 12 cages. Ke mang a ka re bolellang gore [who can tell us] what is a cage? E kare re bolela ka dilo tse re sa itseng gore di ko kae.[It seems that we are talking about things we do not know].
P: Ke ko ntja e dulang. [where dogs stay].
T: Ooh, a re ko ntja e dulang, gore ga o batla gore e ske ya latlhega o e tswalela mo caginyaneng.[She says that's where dogs stay, if you don't want your dog to get lost].
P: Ee.[Yes]
T: Jaanong di cages tse di di kae?[How many cages are there?]
P: 12.
T: 12?
P: Yes.
T: Ke rata go di kwala ga gore o mongwe o utlwa re re 12 mme ga re di bone akere? O bala le nna gone.[I want to draw them so that we can see them, right? Count with me.] (Draws 12 boxes on chart representing cages. Pupils count together with her)
T: Go raa gore re ko SPCA jaanong akere? [We are now at the SPCA]
P: Yes. (Lesson 5)

In the above extract the teacher is dealing with the word cage, which could be new to most second language learners. It is important to note that while the teacher engages learners in an informal talk about the new words in the problem, these words are explained in the learners' main language and not in English. The learners talk about what a cage is in Tswana. The teacher continued in the same manner to get the learners to interpret each of the sentences in the word sum.

In interpreting the sentence "In each cage are 12 dogs' the teacher made drawings of the cages and dogs inside and then moved on to what the question requires them to do.

T: Ee ke raa gore tla re baleng potso e.[Let's read the question.]
Ps: How many dogs are there altogether?
T: Go raa goreng? [What does it mean?] Ke batla go tlhaloganya seo pele.[I want to understand that first] Morero ke eo potso e re botsa gore dintja tso tsotlhe tse di mo dicaging di di kae. [Morero, there's a question, it says, how many dogs are there altogether in the cages.]Dintja tso tsotlhe di di kae?[How many dogs are there altogether?]Jaanong ke batla go itse gore karabo re a go e bona jang.[I would like to know how are we going to find the answer.]
P: We are going to write tens, hundreds, thousands and units. (Puts chart on the

board.) ... and we must underline, when we are through we say 12 times 12, we underline again when we are through we put the button here and we say 2 x 2...
(Learner goes on with the procedure in English until he gets the answer)
P: The answer is 144.
T: Go raa gore re na le dintja tse kae?[It means how many dogs do we have?]
P: 144. (Lesson 5)

It is interesting that in the above extract the teacher rephrases the question for the learners, a practice that she has not been doing since the beginning of the problem. On the other hand to deal with the teacher's question: "how are we going to find the answer" the learners move out of the informal talk, that they have been interacting with the teacher in, into the formal calculational discourse which they have learned.

What the above extracts show is that these learners are aware of the dominant culture of mathematics classrooms in which formal written mathematical language is valued and therefore when required to give an answer they draw on their knowledge of formal procedures. Another interesting factor is the fact that the formal calculational discourse happens in English and this is perhaps due to the fact that this discourse is acquired in English.

In the next extract the teacher tries to engage them more in conceptual discourse.

T: 144. Mara jaanong go tlile jang gore re tshwanetse gore re di timese ko gonne nna nka nne ka nagana gore mare why re sa re 12 plus 12?[But now, how did you know that you are supposed to multiply, why are we not saying 12 plus 12?]
Kenosi: Because re batla di answer tsa rona di be right. [Because we want our answers to be correct]
T: Oh, Kenosi o arabile are o batla go bona a tshwara dipalo tsa gage right ke moo a reng 12 x 12. [Kenosi has responded, he wants his answers to be correct.] O mongwe a ka reng? [What do the others say?]A ka re tlhalosetsa jang?[How else can you explain this?](A few pupils raise their hands and she point at one.)
T: O batla go leka?[Do you want to try?]Emella re utlwe, Ntsiki?[Stand up and try, Ntsiki]
Ntsiki: Bare ko SPCA go na le di 12 cages ene gape go na le dintja tse 12 bjanong ge re di bala dintja tse di di kae?[They say at the SPCA there are 12 cages and 12 dogs in each cage, so when you count the dogs in each cage what will you get?] (Lesson 5)

It is interesting that when the teacher asks them why they multiplied, the first reason she gets is that they want their answers to be correct. This is also very typical of most mathematics classrooms where it is important to know what the correct answer is and not why the answer is correct. On asking for alternative answers, Ntsiki used the teacher's drawing to explain how she would get the answer. Her response is also in Tswana.

It seems that in engaging learners mainly in formal calculational discourse the teacher communicated to learners what is valuable mathematics language. It is therefore not surprising that when the learners were engaged in conceptual discourse they quickly shifted back to the formal procedural discourse. Nevertheless, Ntombi's learners were exposed to and engaged in all four kinds of discourses. During the learners' interview learners could draw on all kinds of discourses. Mathematically, Ntombi's learners were able to engage in both calculational and conceptual discourses. They could carry out their procedures with ease and whenever they were required to

give reasons for some of the steps in their procedures they managed well. It is feasible to argue here that Ntombi's ways of talking enabled learners both mathematically and linguistically.

DISCUSSION

My analysis suggests an important correlation between code-switching, the kinds of mathematical discourses used and whether these enable or constrain learner access to communicating mathematics. As the analysis above shows, Ntombi uses a range of discourses in her teaching and these were reflected in the learners' communication of mathematics. What is very important to note, however, is that the movement between one discourse to another was facilitated by the use of the learners main language (Tswana). This is particularly important because while Ntombi's learners' are relatively fluent in English, it is not their first language and as the data shows some of the learners could not engage in calculational and conceptual discourses without using their main language, Tswana. It is therefore possible that if Ntombi did not allow them to use Tswana, the discourses could have remained formal and procedural. Obviously one cannot claim that use of code-switching enabled learners' communication of mathematics, however, the correlation between code-switching, mathematical discourses and whether and how they enable learners to communicate mathematics is a fruitful area to explore further.

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STUDENTS' USE OF DIAGRAMS IN STATISTICS

Pamela F. Shaw and Lynne Outhred,

Macquarie University, Australia

This paper summarises the preliminary findings of a study into why students in an introductory statistics course do or do not use diagrams in questions which could be done with or without the use of a diagram. It was found that students were reluctant to use diagrams but those who used diagrams were more successful. Very little relation was found between use of diagrams and previous maths studied.

INTRODUCTION

In mathematical problem solving two factors have been found to be important - a verbal or logical component and a visual component (Krutetski, 1976). The logical component includes translation of words into algebraic symbols while the visual component comprises diagrams, pictures and graphs. Moreover, students have been found to emphasise these components differently when solving problems. Presmeg (1986) used the term "visualisers" for students who prefer to use visual approaches for solving a problem which may be solved by either a visual or non-visual method.

In statistics it is important that students can utilise both visual and non-visual methods; visual (e.g., graphical methods) are necessary for an initial inspection of the data to determine distribution characteristics, such as normality, dispersion, and outliers, while non-visual methods are predominantly employed in hypothesis testing and calculating confidence intervals. However, students may be reluctant to use visual methods, such as graphing or drawing a diagram, because of an emphasis on algebraic methods in mathematics teaching. For example, Vinner(1989) found that tertiary students tended to prefer an algebraic proof rather than a diagrammatic proof, even when the latter, as stated by students was easier to follow. He felt that this preference was affected by the method of teaching where students gained the impression that symbolic solutions were more prestigious than diagrammatic solutions.

Visual representations may be included as part of a problem information or students may produce their own representations as part of the solution process. In research on primary and secondary students' interpretations of graphs that accompanied statistical problems, Curcio (1987, 1996) identified three levels of graph comprehension:

- literal reading from the graph;
- comparisons using the graph - reading between the data;
- reading beyond the data.

If students are at the literal and comparative levels in their knowledge of graphs, they may have difficulty solving statistical problems when they are only given the raw data. Reading and Pegg (1998) analysed the responses of Australian students in Years 7 to 12 to two data reduction questions, one of which presented the raw data

and the other a graphical representation. They found that students responded at a higher level when the data was presented in raw form rather than in graphical form. However, Roth and McGinn (1997) have questioned whether graphical ability is related to cognitive ability rather than to practice in using graphical methods in practical and social situations.

Students who represent problem information visually when they solve mathematical problems may be more likely to make a visual representation of statistical data (e.g., a stem and leaf plot or a graph). Shaw (1998) investigated whether the use of statistical displays was associated with the use of diagrams in non-mathematical and mathematical problems. The results of the study indicated a relationship between the use of statistical displays and the use of diagrams in mathematical, but not non-mathematical situations. Moreover, students who spontaneously drew diagrams for the statistical problems were more likely to gain higher marks in the final examination than students who did not.

In this paper we report some preliminary findings concerning the spontaneous use of diagrams by tertiary students as they solve statistical problems. The aims of the study were to:

- investigate the relationship between ability and the use of visual solution methods;
- determine if students who drew diagrams were more successful than those who did not.

METHODOLOGY

Students in introductory statistics courses at two Australian universities were given an assessment task comprising a number of statistical problems and one mathematical problem whose solutions would be facilitated by the use of an appropriate diagram. At both universities the course was a large (>500) service course for first year students. Both courses covered displaying and summarising data, distributions and sampling distributions, hypothesis testing of means for one and two samples, regression and categorical data. The problems required responses at Curcio's third level - extrapolation from and interpretation of the data. There were four versions of the task, the versions differing only in the order in which the problems were given. Students were asked to provide their University entrance score as an indicator of ability.

In addition, nine students from one of the universities were subsequently interviewed at the end of their course and the interviews audio-taped. These students completed the same problems but were asked to verbalise their solution. At the end of the interview they were asked about their use of diagrams.

The 88 students reported here were given the assessment task in a lecture midway through the course when they had studied hypothesis testing with one and two samples but had not yet received any instruction on regression.

Each student's response was given a diagram score of 0, 1 or 2 (0 = no diagram, 1 = partial diagram and 2 = correct diagram) and a score for the solution from 0 to 3

where 3 = correct solution. In addition, The students were asked to give the level of mathematics they had studied at secondary school and this was coded from 0 to 3 where 0 = no mathematics at Year 12 and 3 = 3 unit mathematics. (Students with the highest level of mathematics (4-unit) enrol in a different statistics unit.)

The assessment task

The four questions used in the assessment task are given below. The fourth question was a modification of a question used by Campbell et al (1995).

Problem 1 **Area under the normal curve**: The age of academic staff at Newport University are normally distributed with a mean of 38 years and a standard deviation of 5 years. What proportion of staff would be expected to be aged between 45 and 50 years?

Problem 2 **Linear relationship between two variables**: It has been claimed that as academic staff get older their tolerance of students decreases. A test of staff tolerance of students has been developed. Ages of a random sample of six staff at Newport University and their tolerance scores are listed below. Are the data likely to support this claim?

Name	Grey beard	Long legs	Mac	Boffin	Blondie	Shortie
Age	52	39	33	25	22	45
Tolerance	28	35	35	50	39	23

Problem 3 **Distribution of a single variable**: The employment history of a random sample of 30 academic staff at Newport University was obtained. Listed below are the number of years that they have worked at Newport University. As a person with statistical knowledge you have been asked to comment on this data.

1	7	5	2	5	6	4	5	5	6	2	2	2	6	6
3	5	6	9	5	5	1	3	4	7	5	2	2	7	6

Problem 4 **Linear algebra**: The blood alcohol readings of two lecturers from Newport University were recorded the morning after an accident. The readings were:

Alison	6 hours after accident: 5 units 8 hours after accident: 2 units	Brett	5 hours after accident: 7.5 units 9 hours after accident: 5.5 units

Assuming a linear relationship, when were their readings the same?

RESULTS

It is clear from Table 1 that for each problem (with the exception of Problem 1) the largest group of students are those who do not draw a diagram and who cannot solve the problem. Problem 1 would have been the most familiar to the students and in lectures and tutorials students would been told to draw diagrams for problems of this type. For Problems 1 and 3 those students who drew a diagram were far more likely to obtain a successful solution than those who did not. However, for Problems 2 and 4 successful students were approximately equally divided between those who drew

and those who did not, but very few students who drew a correct diagram could not solve the problem.

Problem 2 was given to these students before they had been given any instruction in regression and their responses reflect ideas developed at secondary school. The majority of the students (81%) did not draw a diagram and less than a quarter of these students obtained the correct answer whereas of the 19% of students who drew a diagram, most were successful.

Table 1 Students categorised by diagram usage and solution (%) for each problem (n=88)

	No solution or an inadequate solution (Score = 0 or 1)	Reasonable or correct solution (Score = 2 or 3)
Problem 1 Areas under the normal curve		
No diagram*	34	19
Correct diagram	13	34
Problem 2 Linear relationship between two variables		
No diagram*	63	18
Correct diagram	3	16
Problem 3 Distribution of a single variable		
No diagram*	60	6
Correct diagram	16	18
Problem 4 Linear algebra		
No diagram*	52	19
Correct diagram	8	21

* A partial diagram was included in the "No diagram" category.

In Problem 3 most students simply calculated measures of centre and spread without checking the form of the distribution. In fact, the data was bimodal with peaks at 2 and between 5 and 6.

Problem 4 was a mathematical question, rather than a statistical one. It could be answered graphically, algebraically or using ratios. Slightly less than half (45%) of the students satisfactorily answered this problem. Again there was a relationship between the use of a diagram and successful completion of the question.

Since students are in general more successful when they draw a diagram why do more students not apply this strategy to solve problems? Below we report on the information gained from the nine interviews as to why students draw diagrams. Four

of the students drew diagrams for three or four problems, the other five drew diagrams for at most two of the problems.

Why students drew diagrams

When students talked about why they drew diagrams their comments seemed to fall into two categories: those that appeared to be motivated by a conceptual approach to problem solution and those who appeared to be following procedures.

i. Conceptual basis

- Some students find that visualisation helps their concept formation. They think in pictures and get a general idea. Student 5 even considers statistical formulae as a form of picture.

I *Now, you've drawn a lot of diagrams. Why did you do that?*

S5 *Well, I think in pictures. I've always found it easier to do things like maths and science than English, because I have a lot of trouble with words, I often don't understand them, or I've felt I've understood what's been said, but I haven't. Whereas pictures, it's pretty easy to get the general idea, um, yeah, I've always found it easy with pictures. ... I just find it easy to work with pictures, but words, they just go straight out of my head. Like I say, I've always done badly in English and things like that because I don't do very well with words.*

I *So you prefer pictures to formulae, for instance.*

S5 *Much, yeah, but even formulae, they're like little tiny pictures,*

I *Right*

S5 *Um you know, like I can remember that the μ is the same as the biggest bit on the picture of the normal distribution, like I remember it as a picture, so that's how I, I just guess percentages by how much do you reckon that is compared to the rest of it. That sort of thing, that's why I'm not very good at remembering formulas and stuff, because they are sort of words, but I prefer them to the actual words, that's why the first thing I do is write down the μ and σ because they are better than reading them back in the paragraph we're been given.*

Another student uses diagrams as a way of reducing the information processing load during problem solving:

S7 *Why did I do that? So I can see what I am doing. Because in my head I can't have too many things at once, so I have to do everything on paper.*

ii. Procedural basis

- Some students drew diagrams because they had been told to do so in class. This was particularly evident in the question on areas under the normal curve. However, Student 6 had not noticed this emphasis until she came to revise for the examination and she was concerned because she did not feel confident in her ability to draw diagrams.

I *Well, in most of the examples that you've done, well you have drawn a lot of diagrams, why did you do that?*

S6 *Oh, they say to in class, basically, and like I just noticed going through some revision, it's often called for.*

I *What, drawing a diagram, in the revision?*

S6 *Um, yeah, they seem to refer to ... like with these it seems to make sense anyway, and with this stem and leaf plot, ... but like I didn't notice it throughout the year, throughout the semester, that we were being asked to do it much, except for here.*

I *Right*

S6 *And then I was going through the revision all the time, and I noticed that "oh, yeah, graph this " and I thought "Oh God" because my visual spatial is like really poor.*

Why students did not draw diagrams

i. Conceptual basis

Students gave several reasons for not drawing diagrams: they were not sure what to draw; they did not feel that they needed a diagram; diagrams had not been emphasised in teaching; or a previous diagram had not helped obtain a solution.

For the single variable question, Student 4 said that he had not drawn a diagram because "*I was only asked to comment on the data ... on a bunch of numbers*". However, when prompted on Problem 4 he had a sudden insight and went on to correctly solve the problem.

I *So you could see no relevance in drawing a diagram?*

S4 *No, what's the use of it. On question 4 there wasn't enough information to draw any linear relationship because like with Brett, I could of ... actually, actually ... I might be able to get something out of this ... you do, okay, blood alcohol units on the y axis, ...*

ii. Procedural basis

I *So why don't you draw diagrams?*

S8 *I do when I'm asked to do it for statistics.*

There were a number of concerns that the students raised that indicated they did not realise the value of diagrams in helping them solve the problems. For example, that diagrams were too time consuming to draw or that graph paper was not provided "*I guess if I had actual graph paper, I would have plotted those two points, draw a line and worked it out that way, put those lines on graph paper, but my freehand drawing isn't that good*". The other reason given was the emphasis on using formulae in the computer assessed tasks in the course.

Problems in drawing diagrams

Very few of the interviewees had any difficulty with Problem 1 (area under the normal curve). This was probably because the interview took place at the end of the term when students were studying for their final examination.

Although diagrams can be helpful, if students draw an incorrect diagram or cannot interpret the diagram they have drawn, then diagrams may be counter-productive. The main problems with diagrams that were apparent from the interviews were:

- Despite drawing a stem and leaf plot for Problem 3 some students identified the distribution as normal rather than bimodal;
- In the linear algebra problem (Problem 4) several students felt that they could not extrapolate beyond the range of the given data

S6: *Um, when were their readings the same? Well if their readings were the same, I would imagine they would have crossed over at some point.*

I *Okay. It seems a reasonable idea.*

S6: *So I'm just thinking I've chosen the wrong way to do it, or I've or whether I've plotted it incorrectly*

Relationship between visual solution methods and mathematical level

The mean diagram score for each level of mathematics studied is given in Table 2.

Table 2 Mean diagram score for each level of secondary school mathematics.

Mathematics level	N	Mean diagram score	Standard deviation
0	7	2.1	2.3
1	6	3.8	2.2
2	32	3.3	2.2
3	31	3.1	2.3

The results shown in Table 2 do not indicate that students with higher mathematics levels were more likely to draw diagrams. The students with no mathematics in Years 11 and 12 were less likely to draw diagrams than those who had studied some mathematics. However, students who had studied 2-unit or 3-unit mathematics had slightly lower scores than those who had studied the most basic mathematics. These results have to be treated with caution because of the small number of students whose results have been analysed at this stage. Nevertheless, the results shown in Table 2 may indicate that students who have studied more algebra may attempt algebraic solutions rather than graphical ones.

CONCLUSIONS

The preliminary results from this study show that for these four problems the majority of students did not draw diagrams, despite being encouraged to draw diagrams throughout the course. The problem for which drawing was most common

involved finding an area under the normal curve, a standard problem for which students had been taught in their course to draw a diagram as an integral step in the solution process. Despite this, only 47% of the group drew diagrams and for the other three questions less than a third of the students drew a visual solution. For all four problems it is clear that the students who draw a diagram as part of the solution process are far more successful than students who do not.

Why then do students not see that diagrams are a useful problem solving strategy? The main reason that came out of the interviews was that students were not sure that a diagram was worth the effort it took to draw it, rather than not knowing what to draw. Although when questioned, one student did not seem sure which diagram to draw, she went on to suggest drawing a histogram and the underlying reason seemed a lack of confidence in her drawing ability. Students seemed to want to calculate statistics, such as means and standard deviations without first obtaining a feel for the data. No student suggested a diagram might be an easier method of solving some types of problems.

We thought that students who had studied more mathematics at secondary school would be more likely to draw a diagram as these students were more successful on the assessment task. However, the preliminary analysis of the data did not seem to support this hypothesis. The results for the full sample may clarify this issue further.

Since students who draw diagrams are more successful than those who do not, a greater emphasis needs to be given to integrating the use of diagrams in the teaching of introductory statistics. Students were more likely to draw a diagram when they are clearly incorporated in the teaching. This result suggests that the construction and use of diagrams needs to be specifically integrated into course materials and assessment.

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CONCEPTUAL UNDERSTANDING OF CONVENTIONAL SIGNS: A STUDY WITHOUT MANIPULATIVES

Corina Silveira

The University of Southampton, United Kingdom

This paper focuses on the beginnings of the learning of the system of natural numbers seen as a sign system and as an object of social use. It presents data of one case study conducted as part of an on-going longitudinal teaching experiment. The broader investigation studies a group of 5-year-old children in a primary school in the South of England. The appropriation of the conventional system of numbers is seen as a gradual approximation mediated by the use of spoken and written numbers in a diversity of situations. An established model of children's conceptual organisation of the number sequence is used to analyse children's thinking. Work with the conventional system of spoken and written numbers without restriction of the series is reported and preliminary reflections on how children appropriate these systems are outlined.

Introduction

Understanding the system of natural numbers entails understanding of two cultural systems of external or physical signs: the system of number words and the system of written numbers[1]. Although there is a vast body of research focusing on the learning of the former (e.g. Steffe et al., 1983; Steffe and Cobb, 1988; Steffe, 1992; Fuson et al, 1982; Fuson 1988), much less is known about how young children appropriate the system of number scripts and further, how the use of the system might mediate this appropriation in the beginnings of cognitive development. Recently, it has been acknowledged that there is a paucity of investigations into how children acquire knowledge of written numbers (Wright, 1998) despite some exceptions (e.g. Lerner and Sadovsky, 1995; Sinclair and Scheuer 1993; Sinclair and Sinclair 1984). Related to the learning of written numeration is, of course, the understanding of place value. Investigations have shown that despite considerable pedagogical effort, English speaking children show "inadequate understanding" (see Fuson, 1990 for a review). Extensive research in this area seems not to focus on the fact that both written and spoken numeration are complex systems of signs available to young children's inspection and use without the graduations normally imposed by schools.

Theoretical framework and research literature

The study holds the view that children construct their own mathematical knowledge and there is no other way to access this knowledge except by building models of it (Steffe, 1991; Steffe and Cobb, 1983). With Cobb et al. (1992) we concur that this

[1] These two systems will be to often termed written and spoken numeration to highlight the relational nature of the ideas they embed.

construction takes place irrespective of the pedagogy used. However, the quality of this construction seems to be very much dependent on the situations an individual is called to participate in with more competent peers. Hence, these researchers establish the dual nature of mathematics learning, both an individual cognitive activity and a communal, social practice. These two aspects of mathematics learning seem to be intricately linked when young children are initiated in the learning of the first system of signs they might associate with "mathematics": number words and written numbers. On the one hand, this learning can be seen as a process of conceptual reorganisation (Steffe et al. 1983, Steffe and Cobb, 1988). On the other hand, following a Vygotskyan tradition, it can be seen as a process of appropriation of cultural inheritance. In this sense, the conventional system of natural numbers and its particular spoken and written signifying forms, are seen as "cultural tools" which serve as "mediators that relate the developing child to his or her cultural inheritance" (Cobb, 1995, p. 364). For example, children's construction of more abstract arithmetic units (Steffe et al, 1983) can be seen as taking place as they engage "in culturally organised practices in which cultural tools play a role" (Cobb, ibid.) "with the guidance of more skilled partners" (Vygotsky, cited in Cobb, ibid.).

The idea that cultural tools mediate cognitive development is held by cross-cultural investigations which argue that differences in the word number sequence of different languages promote differences in conceptualisation of the numeration system (see for example Miura et al., 1988; Fuson and Kwon, 1992; Bell, 1990).

It is in this sense that written and spoken numbers mediate children's conceptual organisation, in so far as these marks and spoken words *play a role* in situations which recreate cultural practices. We have developed the idea that children might be restricted from using and talking about number words and written numbers because curriculum designs present "in doses" the work on the series of spoken and written numbers[2]. For example, by working with spoken and written numbers up to 10, successively up to 100 and then extending the work beyond 1000 in the consecutive years of school[3]. This progression is a pedagogical choice fully endorsed by educational practitioners.

When the work with written numbers is extended beyond 10, instructional devices or manipulatives (e.g. Dienes blocks, Multilink) are usually employed in order to "bridge" the learning of numeration. Research has also put the rationale of the use of these devices under criticism (see for example Perry et al., 1994; Thompson, 1992 for reviews; also Baroody, 1989; Kammii, 1985; Holt 1982 for a critical view). For example, Perry et al. have argued that the "notion of manipulatives in mathematics education has focused on the use of concrete materials to 'produce' the ideas assuming that the learners themselves do not bring with them the ideas that can be

[2] This assertion is based on national curriculum designs both in the United Kingdom and Argentina.

[3] For example, in the UK numbers from 1to 10 are presented in Reception Year, up to 20 in Year 1.

manipulated" (Perry et al., ibid., p. 488). Baroody has stated that "it does not follow from Piagetian theory that children must actively manipulate something *concrete* and reflect on *physical* actions to construct meaning. It does suggest that they should actively manipulate *something familiar* and reflect on these physical *or mental* actions" (Baroody, ibid., p. 5, italics in the original). The use of concrete materials has been said to hold a representational view of the mind with the assumption that students "will inevitably construct the correct internal representation from the materials presented" (Cobb et al., 1992, p. 5) because the concept to be learned is inherent or materialised in the physical device. Contrary to this view Cobb et al. (ibid.) have termed these instructional materials as "pedagogical symbol systems" to emphasise their "symbolising role in individual and collective mathematical activity". Meira (1998) has argued that the "transparency view" should be replaced by the idea that transparency is not a property of the physical embodiments rather it is "created" (Meira, ibid.) or "experienced" (Cobb, ibid.) through specific forms of using the device in socio-cultural practices. He suggests that physical embodiments should be conceived as "conversational pieces" or the motive for engagement in conversation and argumentation. Even in the beginnings of learning, the "conversational pieces" can be the very object of learning: the written and spoken system of numbers. That is, instead of working with artificial "pedagogical symbol systems", children can use, frequent, talk, and reflect upon the written and spoken systems in all their complexity. In this sense, three main assumptions underlie the research reported here: 1. Children elaborate early ideas on written and spoken numerical signs (Lerner and Sadovsky, 1995; Sinclair and Scheuer 1993; Sinclair and Sinclair 1984), 2. If the standard series of spoken and written numbers is not presented in chunks, children can have opportunities to appropriate and master the system of numeration; and 3. Children use external representations and conventional signs to express their thinking (Cobb et al., 1992) and physical properties of symbols can be resources of reasoning (Greeno, 1991). These three assumptions have been the working hypotheses of the research

Background orientation and methodology

The aim of the research is to develop a model of children's appropriation of written and spoken numeration which is compatible with an established model of children's construction of units (Steffe et al., 1983; Steffe and Cobb, 1988). A longitudinal constructivist teaching experiment (Steffe, 1991; Steffe and Cobb, 1983) with selected case studies has been undertaken. Data presented in this paper corresponds to one of the on-going case studies which have been conducted with 6 children of an English Reception class (age 5). A fieldwork diary has been produced to record relevant information (for example what activities in relation to numeration children have been presented with). All 6 children were interviewed outside the classroom on a one-to-one basis and in pairs during the Summer Term –March/July 1998-. Task

based interviews were of two types: 1. Counting type assessment to ascertain children's stage of learning according to the model of construction of units set forth by Steffe and his colleagues (Steffe et al., ibid.). Tasks presented in these interviews resemble those used by these researchers and children were interviewed on a one-to one basis. 2. Semi-structured exploratory interviews to ascertain ways of knowing or appropriating written and spoken numbers. These latter interviews typically present a scenario as an "excuse" to talk about and write numbers, and reflect upon the numbers produced. These interviews were with pairs of children who were thought to be in the same stage of construction of arithmetic units. Interviews varied according to children's responses and engagement but always had the aim of prompting discussion on the written or spoken numbers. Of paramount importance is how these discussions may allow two kinds of inferences. First, if children "notice" any regularities in either or both spoken or written numbers and secondly, if children have started to reason about these regularities. Data presented below mainly stem from the latter type of interview. A brief description of the interview from which they have been extracted follows.

Cinemas scenario: A picture of a cinema auditorium was prepared on a piece of cardboard with 9 sections for the audience (64 cm x 45 cm). The sections were divided with black strips of cardboard. The screen was represented with a piece of black cardboard and it had a small picture of the film "101 Dalmatians". Children were told that this was a big cinema and that the owner wanted to know how many people could come and see the film. All children were asked to figure out how many people there would be in the cinema if there were 100 people in each of the sections and they were asked to write these numbers for the owner of the cinema (e.g. 100, 300, 900).

Findings and reflections: the case of Eamon

The episodes presented here belong to one case study. We present this case because we think it shows one form of appropriation of the conventional number signs which is discussed in the next section. It should be noted that Eamon is a counter of the initial number sequence stage[4]. Eamon gives every indication that he is remarkably interested in numerical signs. He told us he knows how to write infinity, *"the biggest number of all: an 8 turned sideways"*. He also volunteered some of his ideas on the written and spoken system of numbers when he said that *"the more zeros it is, the bigger the number"* and that a million is *"a one and six zeros"* . After counting in tens beyond 100, he said *"I know I can make it bigger with tens"*. In the following episode Eamon and Sam, are presented with the cinema scenario.[5]

[4] This means that his construction of the number sequence has undergone a significant change: a particular number word stands for the possible counting activity and refers to the individual number words of the segment from 1 up to that particular number word.

[5] In the transcripts "S" is Sam, "E" is Eamon, and "C" is the interviewer

C: " A hundred people can sit in this section and a hundred people *more* here, and a hundred people *more* here (pointing at the 3 middle sections). So how many people in total?" [E and S seem to be eager to answer, S stands up and sits down again]
S: "A hundred and thirty"
E: (looks at C) "Three hundred"
C: "Oh! Three hundred or a hundred and thirty? You have to decide now"
S: "It's a hundred and thirty" [E looks at C]
C: "What do you think Eamon?"
E: "Three hundred"

Eamon seems to use the hundred words as if they were entities to count. This is more clear in the following episode, where the children have agreed that only one hundred people are now in the cinema.

C: "What if only fifty people come back? Only fifty people, not a hundred, fifty people come back"
S: "It would be fifty hundred."
C: "Fifty hundred, do you think?"
S: "Yeah!"
E: (gestures as if he doesn't agree) "Fifty hundred?"
C: "Fifty hundred? What do you think Eamon…a hundred plus fifty…"
E: [about S's response] "That would it be more far than about thousands"
C: "So what if fifty more people come, how many would there be in total?"
E: "A hundred and fifty"

Eamon seems to suspect that *fifty hundred* means a lot more than a hundred plus fifty. This suggests that he is taking a *hundred* as a "thing" he can repeat. An entity that seems to mean one hundred individuals. After Sam is "convinced" that it is a hundred and fifty people, the following conversation takes place.

C: "That's good, and what if twenty more people come?"
E: "A hundred and twenty"
S: "Yeah"
C: "And what if…eighty more people come?"
S and E: "A hundred and eighty!"
C: "And what if eighty eight people come?"
E and S: "A hundred and eighty eight!"
C: "Very good! And what if a hundred more people come"
E: "Ugh…that must be...two hundred"

Eamon's particular use of number words transpires in another segment of the conversation.

C: "Now this is going to be a tough one for you. What if one million people come and see the film and tomorrow one million more come...how many all together?"

S: "..one..."
E and S: "Two million"
C: "How did you work that out so quickly?"
E: (smiling) "Because..."
S: (smiling) "We just know"
E: (subvocally) "We just know..."
C: "What if one million more come?"
E: "One million more is...three million"
C: "Three million?"
S: "Yeah..."
C: "And what if a thousand more come?"
E: "One thousand more...er...(shakes the pen in his hand), well...It's three million one thousand"

The above excerpt is not an isolated one. Unfortunately, Sam's contribution to the dialogue prevents Eamon to keep trying to justify his answer. Nevertheless, other segments of the discussion indicate Eamon's awareness of the organisation of the system of spoken numbers. The following episode occurs after the children have written 5000 and 9000.

C: "Which one is bigger...five thousand here or five hundred here?"
S and E: "Five thousand!"
C: "How do you know that?"
S: "Because it comes after a hundred!"
E: "Because first is one hundred, and then nine hundred and *then* one thousand, and then one thousand, and then two thousand, and then three thousand, and then four thousand, and then five thousand"
C: "Right...Oh, I see. And which one is bigger...three hundred...or two thousand?
E and S: "Two thousand!"
C: "How do you know?" [...]
E: "Because the same, three hundred and then...the last hundred is nine hundred and ninety nine...and then it's..."
S: "...a hundred..."
E: "...[it's] one thousand"

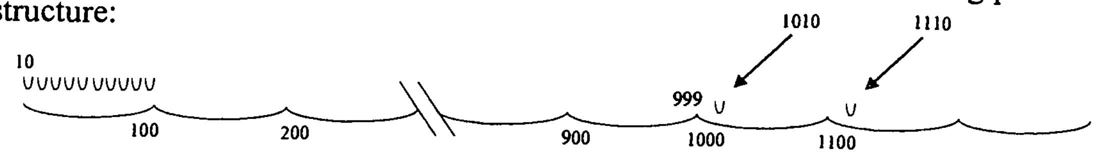
Eamon's explicit explanations on the organisation of the number words are present in other segments of the transcripts. In the transcript that follows, Eamon volunteers to count in tens *"up to two thousand and something"*. He counts in tens conventionally with no hesitation and he is interrupted at 500. Unfortunately for his companion, he wants to continue so it is proposed that he count from eight hundred.
E: "...800, 810, 820, 830, 840, 850, 860, 870, 880, eight hundred and ninety, one thousand, one thousand and ten, 1020 (clearly and conventionally), 1030, 1040,

1050, 1060, 1070, 1080, 1090,…(breaths in), t…one thousand one hundred, (looks at C), one thousand one hundred and ten, one thousand one..."

Eamon seems so eager to show what he *knows* that he misses out a whole interval (900-1000). This performance, it is believed, can no longer be thought of as a purely linguistic skill, that is, as another sequence in ones.

Discussion

Eamon has a remarkable command of the syntax of the conventional number words. He uses a multiplicative or an additive syntax in accordance with the appropriate semantics. For example, "two hundred and ten" for 200 plus 10 and "three million" for 2 million plus 1 million. His inferred organisation of the spoken sequence in tens suggests that he has reflected on these spoken signs and that "ten" has become intuitively important. Eamon's explanations seem to reflect the following piece of structure:



Based on the preliminary findings, we hypothesise that Eamon is forming a complex field of reference for signs such as ten and hundred. For example, he seems to understand that counting in tens *creates* bigger numbers *faster* and he knows somehow (implicitly or explicitly) that he has to count ten times an interval of a hundred to reach a thousand. This latter observation indicates that Eamon has developed an informal or tacit understanding of the role of ten in the organisation of our system of numeration. This, together with Eamon's intuitive understanding of the syntax of spoken numeration seems to suggest that his form of appropriation of the system entails a "relational understanding" (in the sense of Skemp, 1986) and because of this a "symbolic form of reference" (in the sense of Deacon, 1997).[6] This form of appropriation seems to contrast with others which are more "indicative" in nature (in the sense of Deacon).

This paper has attempted to focus on the intriguing role conventional signs play in the conceptual understanding of numeration. What seems to be evident is that Eamon is taking these conventional signs as material to think and to converse about. Because he is able to engage in discussion about written and spoken numbers, problematic situations can be posed upon these signs without the need to employ artificial devices. As Lerner and Sadovsky (1995) have pointed out: why do we have to create a new device in order to teach the real one? Further steps in the investigation will intend to take Eamon towards reasoning about these patterns or regularities of the physical signs.

[6] Deacon's argument is beyond the scope of this paper. However, this quote should clarify the point: "The symbolic use of tokens [physical signs] is constrained both by each token's use and by the use of other tokens with respect to which it is defined" (p. 100)

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ON FORMULATING THE TEACHER'S ROLE IN PROMOTING MATHEMATICS LEARNING

Martin A. Simon, Ron Tzur, Karen Heinz, Margaret Schwan Smith, and Margaret Kinzel, Penn State University

Abstract. *This work represents our efforts to articulate and explicate a conceptual framework for mathematics teaching in which we specify mechanisms of student learning and the role of teachers in promoting that learning. We discuss two mechanisms of learning and how they provide possible sites for teaching, the learner's experience of disequilibrium and reflection on activity-result relationships. Conceptualizing learning in terms of reflection on activity-result relationships is considered as a way of addressing the learning paradox.*

In this paper, we report theoretical work in progress. We choose to present this work at the PME Conference to promote discourse on the ideas presented. Further development of the ideas will be greatly enhanced through discourse among researchers with different cultural, theoretical, and experiential backgrounds.

The focus of this paper is development of a conceptual framework regarding mathematics teaching that builds on significant work on the learning of mathematics. We approach the need for conceptualizing mathematics teaching not only from our role as mathematics educators (teachers), but also from our role as mathematics teacher educators and researchers of mathematics teacher development. It is only through clearly articulating conceptions of mathematics teaching, that the goals of mathematics teacher development can be defined and the approaches to and results of mathematics teacher education analyzed and evaluated.

The ideas that we present are not a description of a "new" form of teaching. Rather, they are our attempt to make explicit (articulate) some principles of mathematics teaching by importing, synthesizing, and further elaborating ideas that exist in the literature. Although examples of teaching which embody these principles can be observed with increasing frequency, it is our experience, particularly in conversations among US mathematics educators, that these principles are often not accessible subjects of discourse.

The fundamental commitment that mathematics educators seem to share is to promote students' development of powerful mathematical ideas and ways of thinking (and participating in mathematical activities). Teaching is evaluated by its effectiveness in doing so. Currently, in a number of countries, mathematics educators are promoting and participating in alternatives to direct instruction (c.f., Brousseau, 1997; NCTM, 1991; Streefland, 1991). Teaching that is predominantly teacher telling and showing has been challenged because of the perceived ineffectiveness of its results. This need for alternative approaches to teaching in conjunction with recent research and theory development on mathematics knowing and learning have tilled the soil for the development of new conceptual frameworks with respect to mathematics teaching.

Getting off the Continuum: Articulating the Problem

Many teachers in the United States are engaging in teaching practices that seem to fall on a continuum bounded on one end by telling students the mathematics that they are to learn and at the other end asking students for the mathematics that they are to learn. Participating in the current reform effort has tended to encourage teachers to move away from telling, and they have gravitated towards asking the students for the mathematics. In some situations, asking students for the mathematics that they are to learn seems to work. It works when the students have the understandings necessary to take the next step on their own in response to the teacher's question. However, there is often a problem with asking students for mathematics that they have not yet conceptualized. At times, the problematic nature of this teaching strategy is masked by the presence in the classroom of students who are advanced and can take on the telling role formerly occupied by the teacher. In other cases, however, none of the students can answer the teacher's question and the teacher must adapt her approach. In such cases, we often see the teacher move back towards the middle of the continuum, asking leading questions and supplying hints. This becomes the compromise position, not telling, yet accomplishing the results of telling. Neither the use of leading questions and hints, nor telling by a student seems to accomplish the goals of the reform. From the observers' perspective, we would argue that there is a need for formulated (explicitly articulated) approaches to mathematics teaching that can allow teachers to get off of this continuum.

Reconceptualizing Teaching

Prior Work

Many researchers have contributed to the conceptualization of aspects of mathematics teaching (c.f., Ball, 1993; Cobb, Wood, & Yackel, 1993). However, few have tried to articulate theories of mathematics teaching. It is perhaps significant that in the PME classification of research reports and members' research interests, there is no category entitled "theories of mathematics teaching" (or "integrated theories of teaching and learning"). However, over the last 25 years, two significant programs of research have made important contributions to theories of mathematics teaching, the French Theory of Situations (c.f., Brousseau, 1997; Douady & Mercier, 1992) and the Dutch Realistic Mathematics Education (RME) (c.f., Streefland, 1990; Treffers & Goffree, (1985).[1]

Our work builds on these two programs. The ideas that we develop elaborate Brousseau's idea of "teaching as the devolution of a learning situation from the teacher to the student." Brousseau (1997) asserted:

> Teaching consists of inducing students to assimilate the projected learning by placing them in appropriate situations to which they will respond "spontaneously" by adaptations. ... The main objective of teaching is the functioning of knowledge as a free production of the student within her relationship with an adidactical *milieu*. (p. 229)

We will explore the adidactical *milieu* in relation to constructivist learning theory and how learner's free productions can contribute to conceptual

development. In RME, students' free productions are also emphasized and considered a starting point for "progressive schematization," a process through which students' spontaneous solutions are developed towards the goal of formal mathematics.

Working from Constructivist Theory

There is likely to be consensus that theories of teaching should be integrated with theories of learning, that is, they should reflect and build on relevant learning theory.[2] However, using a theory such as constructivism to think about teaching involves a non-trivial adaptation, from describing learning when it occurs to promoting learning when it might not occur without an appropriate pedagogical intervention. As Cobb (1994) wrote:

> Versions of constructivism do not constitute axiomatic foundations from which to deduce pedagogical principles. They can instead be thought of as general orienting frameworks within which to address pedagogical issues and develop instructional approaches. (p. 4)

We use constructivism in service of developing approaches to teaching that offer an alternative to the unsatisfactory continuum of teaching described above. Although, following Cobb and Yackel (1996), we coordinate social and cognitive analyses of learning, the focus in this paper is on the cognitive. The reader is referred to Simon (1997) for a discussion of teaching from a social perspective.

Disequilibrium. We begin our discussion of alternative teaching approaches by considering "disequilibrium," a key mechanism in describing learning from a constructivist perspective. Disequilibrium refers to a result of either a learner's experience of an event not fitting with her expectations or a perceived lack of fit among the conceptions she holds. Disequilibrium is thought of as a trigger for conceptual reorganization leading to learning. To what extent does this aspect of constructivist theory provide a possibility for the teacher to make an impact?

We can think about the teacher's role as including intentional actions aimed at provoking disequilibrium with respect to particular student conceptions. The teacher can attempt to provoke disequilibrium, but disequilibrium only results if the issue is significant to the students and they perceive a conflict. Provoking disequilibrium, when successful, would seem to constitute an alternative to telling or asking for the mathematics. In some cases, it does seem to promote learning. However, it is important to ask whether disequilibrium dependably leads to learning and to look more closely at situations in which the intended disequilibrium cannot be induced. In explicating these two issues, we focus on the relationship between assimilation and accommodation.

Piaget (Bringuier, 1980) asserted that "adaptation is a whole whose two poles can't be dissociated. Assimilation and Accommodation" (p. 44). We interpret this to mean that every assimilation of new experience requires some level of accommodation, modification of current schemes, and every accommodation requires an assimilation into a (modifiable and modifying) set of schemes. The latter is particularly important because the intention of a teacher who tries to provoke disequilibrium is to trigger an accommodation in the students' schemes. Based on this relationship between assimilation and

accommodation, we would argue that the necessary (but not sufficient) conditions for provoking disequilibrium, an intended conflict experienced by the learner, are that the learner has schemes that allow for a compatible interpretation of the situation (conflict) to that of the teacher <u>and</u> schemes which can be accommodated to construct an understanding compatible with that of the teacher. This is a complex way of saying that disequilibrium only takes place when the learner can understand the conflict and conceive of the possibility of a solution. Here we are not saying that she has a solution in mind, rather that she conceives of it as solvable. We will use a problem-solving situation analogically to illustrate this last point. Consider the well-known Prisoner and the Hats problem:

> Three prisoners, two sighted and one blind are put in the following situation: They are told that the warden has 3 red hats and 2 black hats. He places one hat on each of their heads. Two of them see the hats of the others, but not the hat on his own head. The warden says that whoever knows the color of his own hat shall be freed.
>
> The first sighted man looks at the others' hats and says that he cannot determine the color of his hat. The second sighted man does the same, whereupon the blind man says he knows the color of his hat.
>
> What color hat did he have and how did he know?

For the problem solver who anticipates that the second prisoner has an informational advantage and that the third must have a still greater advantage, the problem may provoke disequilibrium and that disequilibrium may lead to a solution. (We are using this to concretize certain points not to claim that a conceptual reorganization necessarily takes place.) For the problem solver who only considers what each prisoner sees, the problem has insufficient information, and he is likely to assume that it cannot be solved. This problem solver has no disequilibrium related to the mathematics, although he may experience a conflict from having received an "unsolvable" problem from the teacher.

We provided an example in which the learner experienced no disequilibrium with respect to the mathematics. Another possibility that can occur, if the necessary schemes are not in place, is that the learner experiences disequilibrium that does not fit with the conflict intended by the teacher. This situation can be difficult to perceive by the teacher or observer. In such cases it is not likely that the disequilibrium will lead to the advances intended by the teacher. Note, that the students will always find a way to lessen their disequilibrium, however, that process may not result in the intended mathematics learning or any mathematics learning at all.

Where does this discussion of disequilibrium leave us? We can think about learning as the coming to know regularities among comparable mathematical entities. When disequilibrium can orient reflection upon comparable entities, provoking disequilibrium can be an effective way of promoting learning. However, it provides only a partial alternative to the continuum that we discussed earlier. What is needed is a way to think about a teacher's role in fostering conceptual development when disequilibrium does not seem to be a viable option, i.e., when necessary schemes are not in place. In grounding this exploration in

constructivism, we continue to eschew the notion that more powerful concepts can be infused into learners and embrace the idea of promoting an internal process of construction. This quest however, puts us face-to-face with what has been called "the learning paradox" (Pascual-Leone, 1976), the need to explain how learners "get from a conceptually impoverished to a conceptually richer system by anything like a process of learning" (Fodor, 1980, p. 149 cited in Bereiter, 1985). Bereiter (1985) pointed out that based on the learning paradox,

> The distinction is between kinds of learning that can be accounted for on the basis of knowledge schemas that the learner already possesses and learning that involves new cognitive structure to which already existing schemas are subordinated. (p. 217)

Reflection on activity-result relationships. To begin to explain the construction of new cognitive structures, we refer to the child's development of a concept of manyness. Consider a child who has one-to-one correspondence and has learned number names from one to ten. The child learns to imitate his parents' behavior with a set of objects, touching one object at a time while simultaneously saying a (the next) number name. Through repeated experience carrying out this activity with different sets of objects, the child comes to know that the last number he says represents the manyness of the set of objects. This can be explained in the following way. The child reflects on the relationship between the activity and the results of that activity. (Note, we are not claiming reflection at a conscious level.) For example, early on, he may come to realize that when there is a large collection of objects, he gets to show off most of his sequence of number names, but when there are few objects, he only gets to say a few number names.

What is it that might be generalized from this example that can contribute to our search for alternative teaching approaches?[3] First, the child is constituting a "new cognitive structure." Second, the activity that leads to the development of that new structure is one that the child can carry out without and before the cognitive advance. Third, the activity, carried out by the child who holds certain conceptions, has specific affordances that can lead to a specific cognitive advance. Fourth, key to the cognitive change is the child's reflection on the activity-result relationship which leads him to identify regularities in that relationship. This is a process that takes place over the course of repeated experience which is not motivated by the desire to make a cognitive shift, but rather by the original goal of the child's activity (in this case to imitate and play with Mom or Dad).

Thinking about teaching. Based on the above discussion, we suggest that to promote the development of a new cognitive structure, a teacher's role is to engage students in an activity that they are capable of carrying out, independent of the teacher (*situation adidactique*), that can lead to identification of regularities contributing to the cognitive advance intended. To do so, the teacher needs to understand the conceptions of the students in order to anticipate activities in which they can engage and possible ways that they may reflect on activity-result relationships. As Hoyles (1991) suggested, "activities must connect with pupils conceptions at the outset."

Although our formulation of this approach to teaching is different, we believe that it is consistent with what in RME is called "progressive schematization" (Streefland, 1991) and with Brousseau's (1997) notion of adidactical situations. It is also consistent with what we would consider to be effective use of manipulatives, observable in the United States. For example, engaging first graders in solving realistic word problems using counters affords them the opportunity to identify regularities in their solution activities as the basis for developing arithmetic operations. (Of course this description of the process is incomplete.) Note, the children are capable of solving what we would consider addition, subtraction, multiplication, and division problems using their counting schemes, their knowledge of operations carried out outside of mathematics class (e.g. combining quantities, sharing cookies) and their ability to represent the items in the problem with counters. Also, the children engage in the problems to find the answer to the problems, not to construct more advanced cognitive structures.

This approach to teaching is what Bereiter (1985) calls "indirect" in that the cognitive advance cannot be directly brought about, rather specific experiences for the development of the cognitive structure are fostered by the teacher. Thus, based on understanding the students' mathematics, the teacher anticipates a developmental process in the context of particular learning activities, what Simon (1995) called a "hypothetical learning trajectory."

Tzur (in press) contributed an important distinction between a reflective level of knowing which takes place in the context of the activity (e.g., understanding that the last number pronounced in a count indicates the manyness of the group) and an anticipatory level of knowing which exists independent of the activity (e.g., understanding that when the number "4" is spoken it indicates the manyness of a countable collection). Tzur described how teachers can foster first the development of reflective knowing and subsequently the development of anticipatory knowing by selecting tasks and questions that invite different levels of reflection on activity-result relationships, an additional consideration in the generation of hypothetical learning trajectories.

Discussion

The theoretical work that we describe in this paper is part of an ongoing effort to articulate aspects of mathematics teaching that can be useful in a variety cf related domains: mathematics teaching, curriculum design, teacher education, and research on mathematics teaching and teacher development. In particular, we are attempting to articulate and explicate conceptual frameworks on teaching that specify the mechanisms of student learning and the role of teachers in promoting that learning. An assumption of this work is that teaching can be more scientific in responding to mechanisms of student learning than merely telling students the mathematics that they are expected to learn. Our claim is not that there are no examples of teaching that meet these specifications. Indeed, there are many rich examples of teaching that seem to be well-coordinated with the learning mechanisms that we have described. Rather, we are attempting to further the articulation and explication of and discourse on mathematics teaching.

Towards this end, we have described our growing understanding of the role of provoking disequilibrium and organizing instruction to foster reflection on particular activity-result relationships. Although the former seems to be useful, our current work emphasizes the latter, for as Bereiter (1985) pointed out, "certain kinds of learning really are problematic and . . . the learning paradox helps us see into the heart of the problem" (p. 221). That problem is how learners develop new and more complex cognitive structures. We have pointed to reflection on particular activity-result relationships as a mechanism for such learning, one that can be fostered by a knowledgeable teacher. For us, one of the indications that this mechanism may prove to be robust is that it seems to provide an explanation for much of the learning that we can observe in non-school settings. For example, the game, Checkers, can be played by any youngster who knows the rules for moving pieces and the goal of the game (to eliminate the opponent's pieces). However, it is predictable that youngsters who play the game over a considerable period of time, even without any coaching, will develop considerable insight into strategy for the game. We argue that this can be explained as a result of reflection on activity-result relationships.

In our current work, these ideas about teaching are proving generative at two related levels. First, as part of a conceptual framework on mathematics teaching, the ideas contribute to a conception of teaching that serves as a goal of our teacher development efforts. Second, we are exploring the use of these mechanisms of learning and teaching to inform our teacher education practice to contribute to our ability to promote more complex pedagogical structures. (See Simon, et al, 1998, for discussion of some of these pedagogical structures.)

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Footnotes

[1] We acknowledge that there are other important bodies of work in this area to which we do not have access. We hope to use the international meeting to learn about these.

[2] An integrated theory would also suggest that ideas of teaching affect ideas of learning in the context of classrooms.

[3] We use positivistic language here in describing what we consider to be our experience and not an independent reality.

THE MULTIPLE MOTIVES OF TEACHER ACTIVITY AND THE ROLES OF THE TEACHER'S SCHOOL MATHEMATICAL IMAGES

Jeppe Skott, the Royal Danish School of Educational Studies

A lot of the research has been done relating teachers' beliefs to their classroom practices. This paper proposes an alternative understanding of that relationship. It introduces certain moments of the teacher's decision making, Critical Incidents of Practice (CIPs)*, that are characterised by the simultaneous existence of multiple motives of his/her activity. These motives may be experienced as incompatible and lead the teacher into situations with apparent conflict between beliefs and practice. Rather than as examples of inconsistencies these may be conceived as situations in which the teacher's school mathematical priorities are dominated by other motives of his/her educational activity, motives that may not be immediately related to school mathematics. I suggest to use CIPs as focal points in pre-service teacher training.*

Over the last decade many studies have been made on teachers' beliefs and of their roles in mathematics classrooms. The results of those studies are by no means unanimous. Some claim that very direct relationships exist between the teachers' beliefs, their classroom practices, and the students' learning (Schoenfeld, 1992). Others argue that there is no such relationship and that none should be expected as beliefs are situated, and any attempt to look for priorities that are stable across different contexts is bound to be in vain (Hoyles, 1992; Lerman, 1994; 1996). Between these two extremes Ernest have used the espoused-enacted distinction and pointed to possible mediating factors and social constraints regulating the extent to which the teacher's priorities may manifest themselves in the classroom (Ernest, 1989; 1991). Bauersfeld has pointed to classrooms as jointly emerging realities (Bauersfeld, 1988) and Paul Cobb and his colleagues to the dialectic between beliefs and practice (Cobb, Wood & Yackel, 1990; Yackel & Cobb, 1996; McClain & Cobb, 1997), indicating that though the teacher is important, his or her beliefs are not the sole producers of the classroom environment, let alone the learning opportunities.

This paper aims to contribute to a further understanding of the relationships between on the one hand the teacher's images of mathematics and of its teaching and learning in schools (the *School Mathematical Images* or *SMIs*) and on the other on his or her classroom practices. It discusses the case of John, a 42 year old second career teacher, and two other novice teachers who were selected for the study because of the SMIs they presented partly in their responses to a questionnaire[i], partly because of how they elaborated on these responses in later interviews. Both in the questionnaire and in the interview John showed a strong sense of commitment to the current reform efforts in mathematics education[ii]. He consistently emphasised the importance of investigations and communication on the part of the students and argued that less emphasis should be given to the performance of routine tasks. Further he described the roles of the teacher as a facilitator of learning rather than an explicator of mathematical concepts and skills. Characterising a good teacher of mathematics John said in the questionnaire that he or she knows how to

"-inspire the children to discover connections in mathematics
-understand and accept the different solutions proposed by the students
-free himself from the textbook and introduce mathematics in many different ways".

John, the school, and the classroom

Already before he finished his pre-service training John took up teaching at a village school, where he was offered a permanent position upon graduation. It is a small school of 11 teachers and 130 students from pre-school to grade 7. Some years ago the school got a new headmistress who has influenced the atmosphere to the extent that some of the more conservative staff have left, while the rest are involved in different types of collaborative efforts in order to initiate changes compatible with the latest Danish educational law. In mathematics these changes are strongly inspired by the international reform efforts. John is pleased with these developments.

I attended John's grade 4 for in 2½ weeks in his third term after graduation, following his teaching of a chapter in the textbook on angles, parallel lines, and plane, geometrical figures. If possible we had a brief, general discussion about each lesson immediately afterwards, during which I also asked more specific questions based on my field notes. After the 2½ weeks I made a more comprehensive interview with John, asking him to comment on a number of clips from the video recordings made in his classroom.

The class consists of 14 girls and 6 boys, and though at times there are disciplinary and collaborative problems, John and the students have created an atmosphere in which the students' contribution to the interaction is clearly valued and in which they participate actively and make suggestions of their own. John uses a range of different approaches in order to maintain this atmosphere and to support the students' learning. Some of these have become conscious elements of his routine, while others seem to have developed less consciously. These approaches include

- making joint decisions during whole class instruction as to whether a student's suggestion is acceptable under the specified conditions (e.g. when the students are classifying quadrilaterals according to the number of sides of equal length: *How do you know that they are the same length? Does everybody agree?*).
- not accepting a question like 'I don't understand this', but encouraging or urging students to be more specific before he responds;
- insisting that students ask a peer before they ask him;
- encouraging students to modify or make up tasks of their own (e.g. when the students had made quadrilaterals on the geoboard each fulfilling certain criteria, he asked the them to modify the tasks to be relevant for triangles or pentagons);
- communication in everyday language prior to the introduction of standard mathematical terminology (e.g. when he asked the students to describe quadrilaterals some of which had parallel lines without introducing the term of parallel beforehand. Instead he mentions the term briefly afterwards and in a later lesson discusses it more thoroughly).

In all of this there is a fairly strong correspondence between John's SMIs and his interaction with the students. That seems to be the case both when he orchestrates a whole class discussion, and when he works with individuals or small groups of students. The latter may be exemplified with an interaction between John and one of the boys, Johannes. Johannes finds it difficult to solve a task of how to make a quadrilateral with four equal sides and no right angles on a geoboard. At first Johannes aggressively denies that he knows how to solve the task, but John is insistent and manages to make him come up with a first suggestion. Over the next 18 minutes John joins Johannes 6 times in order to push him, challenge him, and encourage him to make improved conjectures, and Johannes develops from claiming not to know what to do, over stating that the task is impossible to solve and claiming that he has solved it when in fact he has not, to coming up with a correct solution that he is able to justify. Throughout the entire period John avoids giving any direct instructions except the ones aimed at ensuring Johannes' initial involvement in the task ("Make a quadrilateral. You can always make a quadrilateral"). Instead he encourages and urges Johannes to move on when he first thinks he has solved the task: "Are they the same length, the four sides [pretends to be thinking aloud]? I think you are on to something here, but are they the same length? [Johannes shakes his head]. No, but you are on to something. You are on the right track. [John leaves him]." When Johannes finally succeeds, John – who has struggled almost as hard as Johannes – exclaims: "Yes! That's it. Bingo! Exactly." He waits at Johannes' table for eight long seconds building and sharing Johannes' triumph with him.

There are moments when John plays roles very different from the ones just described. An example of that occurs the next day, when one of the girls, Emily, claims to know how to solve the task that Johannes has struggled with in the previous lesson, and John invites her to show her solution to the class. He asks her to make her drawing on the chequered part of the blackboard, pretending it is geoboard. Emily, however, finds it difficult to present her results. She is initially incapable of using the blackboard as a geoboard, and having overcome that obstacle she gets stuck when trying to make the sides of the quadrilateral the same length. After she has made a few vain attempts, John asks her to move to the other side of the board to make a draft of her figure. She easily makes a free-hand drawing of a rhombus standing on a vertex. However, she gets stuck in her subsequent attempt to transfer the draft to the chequered board, and from that instant John takes over more and more, pointing, counting the squares, instructing her what to do, and finally making the drawing himself.

In John's interaction with a boy, Frederick, his comments also become very explicit. Frederick is working on a task of categorising ten different quadrilaterals according to whether they have two pairs, one pair or no pairs of parallel lines. The quadrilaterals, labelled from A to J, are to be written in a table that the students copy from the textbook. Frederick calls John and the following exchange takes place[iii]:

John: You have to find all the figures with parallel sides. Where the sides are parallel two and two [...] Those two, these two [points at two sides of a quadrilateral], they are parallel, they don't meet. ... If we extend them indefinitely, will they ever meet [places two rulers along the sides to emphasise the point]?
Frederick: No.
John: No. So at least it has two that are parallel. What if we look at the other two. Will they meet?
Frederick: No.
John: No. So it has two and two parallels, this one, doesn't it. So it belongs to the column you have made here [points at a column in Frederick's table].
Frederick: What am I to write, then?
John: You write this one into that column [points at the quadrilateral in the textbook and at the column in the workbook]. Above here you have to write that it is page 32 and task number 2 ... Then you write J in that column here ... right.
Frederick: [Short inaudible question].
John: Yes. Then let us look at the next one. [Frederick points at one]. This one. They don't meet either, any of them, so there are two here that are parallel and two here that are parallel [points]. So that belongs in the same column.
[Brief interaction between John and Andreas, the other boy at the same table]
Frederick: But what about the others? [Interrupts the interaction between John and Andreas. John does not respond. Johannes tries again] These two are parallel as well [John reacts. Frederick points at two sides of another quadrilateral].
John: Yes these two, but what about these? [points at the other two sides]
Frederick: No.
John: They aren't. So that one doesn't belong here, where they have to be parallel two and two. It only has two parallels, so that goes in the other column.

Multiple motives of teacher activity and *Critical Incidents of Practice*

John obviously provides very different types of support for the students' learning in the episodes referred to. He consistently avoids giving explicit instructions in the episode with Johannes; he initially tries to be unobtrusively supportive in the interaction with Emily, but when unsuccessful he takes over her attempt to initiate the others in her findings and becomes very direct in his explanations to her and the rest of the class; finally he is very didactic all the way through the episode with Frederick.

One interpretation of this is that John is inconsistent and only momentarily teaches in ways compatible with his SMIs. However, there are other interpretations, and in order to present one of them I shall briefly describe how two other teachers in the study reveal similar apparent contradictions between different elements of their practice. Christopher, a teacher from a Copenhagen suburb, sometimes deliberately refrains from providing direct instructions to the students and seeks to support their independent work, while in others he plays a much more direct role (Skott, 1999). Larry, another teacher with reformist intentions but working at a traditional private school, struggles to strike a balance between his own SMIs and what he considers to be the priorities of the school. It turns out that a key to understanding these classroom

interactions is the *Critical Incidents of Practice* or CIPs. These are defined as moments of the teacher's decision making (i) in which multiple and possibly conflicting motives of his activity evolve; (ii) that are (potentially) critical to his SMIs; and (iii) that are essential to the further development of the classroom interaction and to the learning opportunities.

In Christopher's case the motive of teaching mathematics is sometimes supplemented or replaced by other motives, for instance developing their self-confidence or solving organisational problems in the classroom. In these cases the main energising element of his activity may not be the facilitation of mathematical learning, but the fulfilment of more general educational aims or the management of problematic classroom situations. Larry often finds himself caught between the conflicting demands of the school culture and of the reform. These demands establish two dominant, but mutually competing motives of his activity: One of facilitating mathematical learning and one of complying with the perceived pressures of the school. Focusing on one or the other of these motives in turn, Larry gets involved in series of oscillating practices some which incorporate specific aspects of his reformist intentions, while others have virtually no resemblance with his SMIs. Both of these teachers, then, experience CIPs in which the existence of multiple motives of the their activity challenge the enactment of their SMIs. In these incidents their main focus may not be on facilitating mathematical learning. They may, so to speak, primarily be playing another game than that of teaching mathematics.

It is not surprising that the SMIs become less significant, when the object and motive of the teacher's activity is not one of teaching mathematics. In John's case the main challenge to the motive of facilitating mathematical learning – and therefore to the enactment of his SMIs - is his knowledge and understanding of the individual child. For example John commented on the episode with Johannes by saying:

> "It's difficult, but I try to make him get started on his own, because I know he can do it. The problem is that he doesn't think so himself, or he is afraid//in reality he is afraid//he is lost at that moment, because he feels good only when he is number one. You know, to him nothing is good enough, all that matters is being the best, that means finishing first. The moment Ian [the other boy at the table] puts the rubber bands on the geoboard he is ahead and Johannes gives up. It is very difficult for him to fight that. But that is his whole situation and the expectations of his parents and things. They are very ambitious. And they are very competitive. That's OK, but the problem for the kid is that he only thinks he is doing well, if he is number one. Nothing else matters."

The point here is that John is deeply involved in facilitating Johannes' mathematical learning in ways compatible with his school mathematical priorities, but also that he bases his interaction on the conviction that Johannes' self-esteem is vulnerable and strongly connected to criteria of success not immediately compatible with John's SMIs. It is just as important to John to convince Johannes that he is doing a good job and making worth-while efforts, even if he does not produce the first or only solution

to the task, as it is to provide him with the opportunity to construct concepts of parallel lines or rhombuses. In this episode, then, John tries and largely succeeds in striking a balance between two competing motives of his activity.

John's explanation of the interaction with Emily also includes much broader aspects than those related to her mathematical learning. John conceives her as both unrealistic with regard to her own competence and fragile when confronted with problems that challenge her self-perception. That is important

> "[…] when I react like this. I can feel that she is on thin ice, so I try to make her draw the draft. That is how I try to give//help her [...] Then at least she can make the draft. And then we can make the figure from her drawing between us. But I take over completely, I can see that."

In this episode John tries to be cautiously supportive, but Emily does not succeed. He then sees no alternative but to take over and make sure that they finish the task together using her example, if he is to avoid a blow to her self-confidence.

John's interaction with Frederick took place at the very end of the lesson, which partially explains why he becomes very explicit: He wants Frederick to finish the task before the bell rings. Also he commented that Frederick is normally good in maths, and that John expected him to be able to follow the explanations given. However, there are other situations with weaker students in which John plays a similar role of instructing them what to do. He says:

> "There are some children in here, some of the weak ones, with whom I've had to choose [...] especially with Louise, I've had to say to myself 'If only she acquires a system [of how to solve the tasks], then it doesn't matter if I've provided her with it, because at least she can follow what goes on'. I've chosen that. And she does [follow]. So up to now she is part of the team. She largely makes the same tasks as the rest, although she finds mathematics very difficult."

John, then, becomes explicit in a variety of different situations. With Frederick he finds the dual motives of supporting his learning and managing the classroom compatible. With Louise he has previously struggled to strike a balance between facilitating her mathematical learning and assuring that she remained part of the classroom community. Finding it impossible to reconcile the two he has settled for the latter and accepted that he needs to provide her with very direct support.

Conclusions

In the situations described above John's SMIs play very different roles. That clearly questions Schoenfeld's claim of a causal relationship between beliefs and practice. However, the interpretations made also challenge Hoyles' contention that beliefs are situated in the sense that each new context produces or encompasses a different set of beliefs. It is rather the motives of the teacher's activity that are contextually framed, and these motives in turn determine or influence the enactment of his mathematical or general educational priorities, sometimes urging him to enter another game than that of teaching mathematics.

This interpretation also questions previous studies relating beliefs to practice in another way. Sometimes rather condemning descriptions of teachers are made, when they are found not to be in accord with their professed reformist views. My argument is that what appear to be inconsistencies may rather be understood as situations in which the teacher's motive of facilitating mathematical learning is dominated by other and equally legitimate motives of for instance ensuring the student a place in the classroom community or developing his or her self-confidence. I am not saying that these different motives are necessarily incompatible. For instance in the episode with Emily, John may have accepted her draft as a solution to the task and asked the students in groups to develop ways of assuring that the side lengths were the same. Or in the case of Louise it may be important to define the *classroom community* differently from just having the students working on the same task. The point is not that there is no alternative to abandoning the SMIs in situations of multiple motives. The point is rather that in the critical incidents of practice the teacher often struggles to find such an alternative, although sometimes in vain.

This leads to a proposal for teacher education. There has been a growing tendency to focus more on the student teachers' practice in order to ease the transition from pre-service training to full time teaching and to ensure a high degree of consistency between the theoretical understandings and practical competence of the novice teacher. For the teachers in this study such a consistency has to a great extent been achieved. However, when they are challenged by the multiple motives of their activity during the critical incidents of their practice, they sometimes find it difficult to integrate their SMIs with broader educational aims or with managing the classroom. That may be because such an integration is sometimes impossible to obtain. But it may also be because they have not been confronted with situations that challenge the enactment of their SMIs in their pre-service training. The proposal, then, is to use CIPs – preferably from their own teaching experience – as focal points in discussions of the realisation of their SMIs.

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[i] The questionnaire contained open and closed items on mathematics and its teaching and learning and was sent to all student teachers doing mathematics at a Danish college immediately before their graduation in 1997. Among the respondents 11 were selected for interviews upon graduation, while John was interviewed one year later. Among the interviewees 4 were selected to be part of the continued study.

[ii] The notion of *a reform* is obviously problematic, but I refer to the trends envisaged for primary and lower secondary school, including a view of learning informed by constructivism, a view of teaching as facilitating learning, and a view of mathematics inspired by fallibilism and social constructivism. The Danish curriculum for primary/lower secondary school also reflects these trends.

[iii] In the transcripts I use the following notations: ...: a pause; //: the speaker interrupts her/himself or is interrupted; [...] : an omission on my part; [text] an explanation on my part.

"This is crazy. Differences of differences!" *On the flow of ideas in a mathematical conversation.*

Jesse Solomon
City on a Hill Public Charter School
Boston, USA

Ricardo Nemirovsky
TERC
Cambridge, USA

This paper describes the flow of a mathematical conversation in a high school mathematics classroom. The analysis attempts to elucidate how the discussion of a particular mathematical problem is inherently open-ended and to describe the complexity in managing the wide range of ideas and choices emerging from the interactions among students and the teacher. A major implication of our analysis is that whether an idea turns out to be mathematically correct is only one piece of what contributes to making it valuable and inspiring. Another implication is that the teacher's mathematical expertise plays out throughout multiple ongoing dilemmas that go much beyond the classic dichotomy between "telling vs. letting them discover it."

The nature of conversations in the mathematics classroom has become a prominent theme of mathematics reform. Some of the most central questions raised by the many initiatives to reform mathematics education focus on what counts as a productive conversation for mathematics learning and what are the roles of the teacher in fostering and enriching them. Chazan and Ball (1995) argued that the opposition between "telling or not telling" is an oversimplification. Often the interpretation of videotaped data ends up taking the form of a superficial judgment on whether the teacher made the correct move, whether the students "get it," or in general, whether what is seen is an example of good teaching. In jumping to these types of judgments we close our own understanding of what the participants are experiencing and what the documented utterances and activities meant for them.

Calvert (1998) argued against another common image that what counts in a mathematical conversation is exclusively the truth of the propositions that are being argued. Explanations, according to Calvert, are often attempts at understanding and they may affect the participants of a mathematical conversation in ways that have little to do with proving, convincing, and competing.

Other researchers (Cobb et al., 1993; Bordieu, 1983) have postulated that classroom conversations are regulated by social and mathematical norms, mostly implicit, which are developed over time through extended and ongoing interactive negotiations between the teacher and the students. Simon (1995) suggests that while the teacher plans lessons and designs tasks aiming at realizing a hypothetical learning trajectory in the students, in the actual interaction with the students his plans are constantly revised and subject to redefinition.

In this paper we build on Chazan and Ball's idea (1995) that "any discussion holds the potential for discrepant viewpoints" and that it is the teacher's role to "manage" these views. We argue that the discussion of any

mathematical problem can be by its nature open-ended. Instead of seeing the teacher's role as deciding whether to make a problem open-ended, we see a teacher's role as making a set of decisions about how to manage the open-endedness that exists in the interactions around a mathematical issue. In addition, we see the teacher managing the open-endedness by working on a sense of direction for him and for the class. This work is at times expressed through efforts to preserve certain voices which otherwise may vanish, to foster mutual listening, and to comment on the fruitfulness of the different contributions. The sense of direction is co-developed by the teacher and students; some students adopt a leadership role in this regard by commenting on the quality of the conversation itself or by suggesting changes in how the group talks to each other or what it talks about.

This episode took place in a classroom of tenth and eleventh graders in a second year integrated mathematics course at City On A Hill Charter School, a public high school in Boston, MA. City On A Hill admits students from across the city of Boston by lottery; approximately 75% of the student body are students-of-color and over half the students qualify for free or reduced lunch. The teacher is Jesse Solomon. The fifteen-minute discussion that followed was not part of the day's lesson plan. The episode began when a student raised a question about one of the previous night's homework problems. The problem gave the sequence 1, 8, 27, 64… and asked students to find and graph the next three terms of the sequence. Maria (all of the students' names are pseudonyms) asked the initial question:

Maria: And on [problem #] 18, I couldn't figure out the pattern, and I tried and I asked my mom and my brother, and no one could... (...) And, um, I just, I- this is the only one, I left, I had to leave the graph blank, cause I couldn't figure out what the next two would be. I figured- I did the differences, I did multiplication, I tried times 2 plus 4, times 2 plus 3. I tried that. I couldn't get it. My mom couldn't get it, and she told me just clean my room.

When Maria read the problem, I thought that it was trivial, that the 'answer' would quickly be evident for the class. However, the students immediately showed that for them the problem was far from obvious; that struggle made it interesting to pursue. [Bold type expresses first person commentaries by Solomon.]

Margaret made a first proposal:

Margaret: There is no sequence, I don't think. If there is no sequence, then can't you just guess the next two numbers? The only thing it has in common is that one is higher than the other.

Mr. Solomon: (...) Oh, just because it's higher? Okay. What do other people think?

When Margaret introduced her idea, she qualified it by stating, "There is no sequence." Because the only commonality she saw is that the numbers increased, she concluded that there is no sequence. Her comment suggests that proposing a number just because it is "higher" than the previous one was not a "good" solution–that it was not mathematically satisfying. The notion that there should be a rule to determine exactly the next number was an implicit part of the

background of cultural assumptions in this class. Solomon tacitly confirmed this assumption by restating Margaret's explanation with a "just:" ("Oh, just because it's higher?") and also by immediately asking the class for alternative ideas. At this point, Solomon took on the role of recording the students' ideas on the overhead projector.

Maria: Did anyone else get this one?
*Naomi: Well, 8*8 is 64.*
Daniel: [raising his hand briefly] Um... Hold on one second.

1, 8, 27, 64

Latisha: [whispering to Naomi] But where does the 27 come in?
Maria: But what does the 27 have to do with it?
Naomi: The 27 comes in from the 1.
Maria: 1 times 1 is 27?
Mr. Solomon: I just want you [to Maria and Naomi] to hold on one second; I don't want to lose it, I just want you to hold on one second [gesturing to David to say his idea]...

As other members of the class began to engage with the problem, we see the simultaneity of ideas and conversations that characterizes much of classroom discussion. Daniel asked for time to think, but while Solomon was trying to "hold" the class, waiting for Daniel's idea, Naomi expressed a relationship that she noticed, Latisha and Maria questioned the validity of Naomi's relationship and refuted Naomi's attempt to link 27 with 1.

Daniel then proposed an iterative linear rule ($(1 + 8)*3 = 27$) that used the first two terms to generate the third, but he could not see how to generalize it. Even though he concluded that this particular rule would not work, the class started to play with iterative rules involving addition and multiplication. At a lull, Solomon asked:

Mr. Solomon: How about if I show you the next number in the sequence?
Daniel: No!
Maria: Yeah! Yeah!

This offer of the next term elicited conflicting reactions. For some students (e.g. Daniel) adding the fifth term would diminish the merit of the solution they would eventually come up with. For others (e.g. Maria) it would help them solve a problem which, so far, had been almost intractable. Students are wrestling with their issues of confidence in their ability to find a solution, the value they attach to the independent finding of a solution, and a sense of how much effort is worthwhile to put into a problem.

Jamal: I got it! Haaah! I got it.
Latisha: Then, what is it?
[Solomon reiterated Latisha's request to Jamal. Jamal refusal to provide it, triggered an interaction with Maria.]
Maria: How are you going to be a genius and keep it to yourself? I mean...
Jamal: [after Solomon's request] 7. The difference between- the differences between, um each number [Molly: 7, 19, 37] is what it has to do with.

Note how Molly reacted immediately uttering the sequence of differences ["7,19,37"]; she had probably already taken the first differences.

Jamal (to Solomon): Go ahead, put 'em, write 'em down there. 21.

1	8	27	64
7	19	37	

Shawna: That's 19. [Jamal: That's 19].
Molly: 37. [Jamal: 37].
Maria: I see no......
Molly: There's, like, no pattern. That's what I did.
Jamal: Oh, I thought it was 7, 21 - see, that's why I messed up...all right.
Jamal: I thought it was 21. I thought it was 21.
Maria: [simultaneously] You don't got it. You don't got it.
Jamal: Yes I do.
Latisha: Okay, next.

Jamal admitted that he had been mistaken because he thought that the second term of the differences was 21 and not 19. The refutation of Jamal's idea was in part a repeal of his attitude, but his contribution did bring to the group the notion of playing with successive differences. Molly confirmed that she had tried this out but that no patterns came up. This approach, different from the previously dominant one of trying out addition/multiplication rules, would become a key idea for the ensuing conversation. Again, the group picked up on idea which at first glance had been rejected.

Mr. Solomon: Anybody else, anybody else want to try anything, before I put one more. Just say anything at all in the world.
Shawna: Well, no numbers, like, I mean, I forgot how you say it, it's a num- like, you know, 3 or 2 can't go into 7, 3 or 2 can't go into 19, can't go into 37.
Mr. Solomon: (...) What are those called?
Maria: A prime number!
Shawna: Yeah, prime, yeah.
Mr. Solomon: Okay, so the, the consecutive differences are prime, that's something you noticed.
Shawna: Yeah.
Mr. Solomon: Good. All right. Do you notice a pattern in those differences?

While approving Shawna's noticing of the difference being prime numbers, Solomon's question seemed to ask for a sequential property, implying that her answer was not 'sufficient.' Then Maria proposed a sequential property but she felt that there were not enough numbers to ascertain it:

Maria: Right, and then the 7, 9, 7, but we only have three numbers, so... I mean you can't really go much on -- I mean three differences. If you give us the next one, that's what I'm saying, if it has 9 as the, the last digit, then maybe we could figure something out. [A discussion ensues about whether Mr. Solomon should provide the next number or not. Jamal opposes it; Maria, Molly, and Mona are in favor]
Daniel: 'Cause, I was looking at the pattern of those, um differences right there [on the overhead]: 7, 19, 37. I said that it could be going like. See how, like, the last digit is like, 7, then 9, then 7.
Latisha: [quietly] I just said that.
Mr. Solomon: Yep. So what would you say the next dig-...
Daniel: It would be like 9. And, and, the, the number on the, um, at the beg- in front of the 7 would be like a zero, and it goes up one, and then it [Maria: Ooooo!] goes up two from there which would be 3, and then it will go up 3 from

there, which will be 6, and that's 69.
Maria: The next one would be 59! Cause look! 0, 1, 3, 5, 7
Daniel: I had 69.

Daniel proposed that the next difference was 69. His idea, triggered by Maria's previous pointing at the second digit sequence (7, 9, 7) was that the first digit increased by 1, by 2, by 3 and so on (so that it is 0(7), 1(9), 3(7), 6(9)) whereas Maria thought that the first digit could be the sequence of odd numbers after 0 (0(7), 1(9), 3(7), 5(9)). Daniel proposed 69 a few more times, but his idea was never picked by the class. Solomon then asked for all the guesses about the next term and wrote them on the transparency. Nadia proposed 121, Daniel 133, Shawna 113, and Maria 123.

Mr. Solomon: Well, I'll give you a hint. It's definitely not all four of them.
Class: Is it one of them?
Daniel: Nadia! I know why you said 121!
Mr. Solomon: Can someone make an argument for one of them?

At this point I was still thinking of this as a problem with only one answer. While I may have been willing to entertain different suggestions, I still had the 'right' answer in my head.

Naomi: I said 123 because, um... Oh yeah, because it was going, the um, the 10, in the 10's place of the difference, where it was 19, where the 1 and the 3, of 19 and 37, and then it was, like, plus 59...
Maria: Um hm, that's what I got, too.
Naomi: Because, you know, 1, 3,...
Mr. Solomon: Okay, that could work.

1		8		27		64		123
	7		19		37		59	

This is the first time that a proposed fifth number in the sequence is taken as a workable solution. Solomon asked other students to argue for the other proposed numbers.

Nadia: Okay, well, I didn't stop at finding the difference between the numbers given. I found the difference between the differences. Which I got 12, and then I got 16, from, wait, did I get 16?
Mr. Solomon: I think it's 18.
Students: [simultaneously] 18.
Nadia: 18. Okay wait. Then my, wait, my numbers might change then..6. 18. Wait a minute, wait a minute [writing]. [Pause] Wait a minute.

1		8		27		64		123
	7		19		37		59	
		12		18				

Maria: This is ridiculous! Differences from differences!
Mr. Solomon: No, this is actually...
Maria: No, it's probably right, but I'm saying that it's ridiculous that you're making us do something like this. [Laughter]

Even though the differences of differences idea came from Nadia, Maria still claims that Solomon is "making them do" this ridiculous thing. During this approach, Solomon tried to eliminate minor arithmetic mistakes that could get in the way of Nadia's idea.

Nadia: Okay, wait, wait, wait, let me come back, let me change that number.
Shawna: 37, and the next one, like, difference 37 and 59 is 22, right?
Mr. Solomon: Well, if, if we go with Naomi's thing of that being 59.

1	8	27	64	125
7	19	37	61	
12	18	24		
6	6			

Shawna: Oh.
Mr. Solomon: Well, no, I'm just... Right.
Nadia: All right, then I got 125.
Shawna: Well, let's go with that, let's go with that, and what's the difference between those two?
Mr. Solomon: So if we go with that 59, then the difference is 22.
Shawna: And, so there'd be a difference of 6 in all of those, right? No, no, no.
Mr. Solomon: Will you not give up! Look! Look what the problem is. It's not a difference of 6, it's a difference of 4, so...

I thought that Nadia's path could be fruitful, but she was not convinced of that, so that a mathematical error could have easily derailed her and this idea might have been lost. It felt like it was both a good and unstable idea that it was worth the intervention of saving it.

Jamal: This is crazy! This is crazy.
Maria: Differences of differences!
Nadia: It would be 125.
Mr. Solomon: (...) Well, let me, let me say one thing first. Um, given a sequence like this: 1, 8, 27, 64, there is not one correct answer. For example, you could define your sequence, as, you know, however you define this difference here [first difference], and come up with 123. That's not the answer I was thinking of, but that doesn't mean that it's not the right answer. ...
Nadia: So this is like an open-ended question.
Mr. Solomon: (...) the book, for example, really may have a specific answer, because this, this set of numbers (...) has sort of a mathematical suggestion. It is a very frequently seen set of numbers, which, when I show you, you're gonna say "Oh, duh!" so there is an answer that would come to mind quickly, but that does not take away from...

Although I did end up talking about differences and knew something about the use of differences in determining the kind of polynomial equation, they were not fresh in my mind and I was not confident that I knew how a constant third difference indicates a cubic function.

Mr. Solomon: (...)[pause] Let me say a couple things. Here's the sequence that this usually suggests, okay: 1^3 =1, 2^3 = 8, 3^3 = 27, ...
Maria: Oh, yeah, I knew it!
Jamal: I knew it!
Maria: I really did, I really did. We did learn this!
[Mr. Solomon explained to the students that finding a constant in the successive differences indicates the order of the polynomial.]
Jamal: Ooohhhh!
Mr. Solomon: So, what actually you just discovered is this method that people use to find out what kind of function describes their sequence. So, just by taking successive differences, if you kept doing this, and said, Oh! it's always 6, that would automatically tell you that you've found a 3rd degree function.
Jamal: So if this 4th one, if you had to take a 4th one it would be like a 4th power?
Mr. Solomon: That's a 4th degree function. And one of the things that we clearly

need to talk about is why that's true.
Jamal: Why you never told us that? 'Cause they have that on the SAT's.

Discussion

Often we talk about open-ended problems as if open-endedness were an intrinsic quality of certain mathematical problems, or as if we must choose between stating a problem as closed or open-ended. Was this problem, finding the next terms in the sequence 1, 8, 27, 64..., 'open-ended'? We find this dichotomy misleading; instead, we hold that it is the discussion surrounding a problem that holds the potential for open-endedness. The complicated set of interactions in this episode (as in any classroom mathematical discussion) are informed by the students' ideas about the problem, as well as by factors such as their cultural backgrounds and the social dynamics of a group of teenagers. Their contributions made the discussion open-ended. They were comfortable simultaneously trying out four ideas with four different answers.

At a few points throughout the episode, Solomon offered to tell the students the answer. Each time, at least one of them refused vehemently-they wanted to get it themselves. When he finally did explain the significance of constant differences, they did not feel like this piece of mathematics had been given to them. Rather, they had scratched away at some larger mathematical principle which Solomon placed in a broader context for them. Having explored and made progress on the problem and having developed a set of ideas about it, Solomon's telling was a confirmation of an idea that they felt some ownership over. In agreement with Chazan and Ball (1995), there was a mix between telling them and letting them discover it. The process of co-defining a direction–constant differences was not the topic planned for that day–led to the creation of a ground in which meaningful 'telling' was possible.

As varied contributions are generated by the group there is an interactive dynamic out of which an overall and changing direction emerges: certain ideas are picked up (e.g. Daniel: "I see that the $(1 + 8) * 3 = 27$"), others are abandoned or temporarily suspended, some participants shift to a leadership role (e.g. Maria: "Did anyone else get this one?), others seem to withdraw, the conversational atmosphere becomes competitive (e.g. Maria: "You don't got it. You don't got it." Jamal: "Yes I do."), productive, reflective (Nadia: "So this is like an open-ended question"), or agitated. In such an effervescent context, the teacher tries to contain certain forces and to stimulate others. A retrospective account can always embellish the interaction with clean rationales and formal chains of reasoning that led the teacher to make one or another "move;" however, the resulting image is unreal. No chain of reasoning can account for the teacher's instant recognition of imminent risks and possibilities and his ability to be responsive to a future that is yet to come. Solomon started the conversation expecting an easy recognition of the sequence of cubes and ended up discussing methods to recognize polynomials. While this was not the execution of a pre-conceived plan, it was not a random chain of events either. The evolving sense of direction was co-developed by Solomon and some of the students. Maria, for example, made many commentaries on both the mathematics and the quality of the conversation itself ("I mean you

can't really go much on three differences. If you give us the next one," "How are you going to be a genius and keep it to yourself?" "This is ridiculous! Differences from differences!"), or Latisha who, exasperated by the discussion with Jamal, asked the group to move on ("Okay, next."). Solomon and the students were trying to have the conversation 'go somewhere' while, at the same time, trying to understand what that "somewhere" could and should be. Even the final interaction of the episode suggested some possible future directions for this conversation: deeper understanding of the difference-of-differences method (Solomon: "And one of the things that we clearly need to talk about is why that's true"), or how to get better scores on a test (Jamal: "Why you never told us that? 'Cause they have that on the SAT's").

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THE DEVELOPMENT OF CRITERIA FOR PERFORMANCE INDICES IN THE ASSESSMENT OF STUDENTS' ABILITY TO ENGAGE CULTURAL COUNTING PRACTICES

Stephen Sproule
RADMASTE Centre, University of the Witwatersrand

This research explores grade seven students' ability to engage the counting strategies of various cultures. The purpose of the research is to develop performance indices for assessment of student performance. The performance indices have been created from the results of developmental research conducted with the students. The results of the research are presented as a scoring rubric and substantiated by student work.

Recently, Vithal and Skovsmose (1997) raised a number of concerns related to the implementation of ethnomathematics or critical mathematics within a school system or mathematics curriculum. The new South African mathematics curriculum, intended for gradual implementation over the next seven years, will include pedagogical attention to mathematics as a culturally situated activity. Thus a national curriculum will explicitly require teachers to highlight mathematics as a human endeavor by including activities that illustrate the mathematical practices of various cultures and focus directly on the cultural origins of the practice.

In the research reported in this paper I have developed various counting systems (as one aspect of a culture's mathematical practices) into classroom activities to introduce the cultural component of the new curriculum in a mathematics classroom. The purpose of these activities is the development of criteria for performance indices for teachers to use in their assessment of their students' ability to engage and operate within the counting practices of different cultures. To ensure that the performance indices are grounded in the practices of students this research answers the question: What mathematical approaches do students who are familiar with an English counting style apply and develop when they encounter number systems and counting strategies originating in other cultures and languages? The complete research agenda also focuses on the students' awareness and appreciation of the cultural nature of mathematics. This is not reported in the paper due to space constraints.

South Africa remains in educational transition. A new mathematics curriculum, currently in implementation, has adopted an outcome or performance based approach to education. In this paper I describe research that is related to a specific outcome of the new curriculum that requires students to understand mathematics as a human activity embedded in a socio-cultural context. The specific outcome states that students should, "demonstrate understanding of the historical development of mathematics in various social and cultural contexts" (Department of Education, p.4, 1997)

Although recognised by Presmeg (1998), and Vithal and Skovsmose (1997) as a limited approach to incorporating cultural mathematics practices into the curriculum, the Department of Education (1997) has adopted curriculum outcomes that are concomitant with a "histories of mathematics" approach. It appears that their intention is to provide students with a more global appreciation of mathematics as a human endeavor situated in a socio-cultural context. One aspect of the education process that will require change as a result of the changing curriculum expectations is the assessment strategies employed to evaluate student development. Clearly, traditional forms of assessment are not suitable for the stated curriculum outcome.

To improve students' abilities to count and work in various culture's counting systems and to provide teachers a means to assess their students' development I have employed an *educative assessment* framework (Wiggins, 1998). Student assessment has traditionally only served as a means to audit student knowledge. Delandshere and Petrosky (1998), amongst others, highlight the need for assessment to become an integral part of the learning process.

Assessment tasks offer a useful means to achieve an educative role for the assessment process. Using Wiggins's (1998) notion of educative assessment, the tasks illustrated in this paper may be considered *educative assessment tasks*. The intention of such tasks is not that students perform competently from the onset of the assessment (or learning) process but that students working through a selection of tasks develop competent performance through progressive engagement with the educative assessment tasks. In this process a "growing collection of evidence" of student learning is provided to the teacher for student auditing (Swan, 1993).

Within a performance oriented curriculum, auditing student work and providing meaningful feedback to students requires additional effort from the teacher. Supporting teacher efforts prior to and during the implementation of new assessment strategies is vital (Izard, 1993). One of the documented forms of assistance provided to teachers is the scoring rubric (e.g. Stenmark, 1989). These range from generic rubrics to content or process specific rubrics. Implicit in the purpose of my research is the need to assist teachers in the implementation of educative assessment tasks. Consequently, a qualitative scoring rubric illustrating levels of performance and the associated criteria has been developed and reported in table one.

Research methods

The use of developmental research in instructional developments and accompanying research is well documented by Gravemeijer (1994), is further explicated in practice by, amongst others, Cobb (1998) and resonates with research done by Presmeg (1998) on students' capacity to recognise mathematics in their own cultural contexts. I suggest that developmental research can also be successfully applied to the development of educative assessment tasks. Considering that educative assessment is as concerned with learning as auditing performance, the application of a developmental research methodology is as appropriate for the development of

assessment practices as it is for the development of instructional practices. Consequently, this research may be considered developmental research.

Participants

The school at which the research was conducted is situated in an upper-middle income neighbourhood of a large South African city. However the school draws students from all levels of the socio-economic stratum. Students from three of Ms. B's grade seven (approximately 12 years old) mathematics classes participated in the study. The students came from diverse cultural backgrounds representing traditional South African communities as well as recent immigrant populations. The students also demonstrated a wide range of mathematical ability while completing the tasks.

Assessment Tasks

The educative assessment tasks used in the research were created by applying an "anticipatory thought experiment" (Cobb, 1998) to cultural mathematics practices evidenced in Ascher (1991) and Zaslavsky (1996). Seven educative assessment tasks were used over 5 weeks with the participating students. Each task offered the student an opportunity to mathematically engage the counting system of a different culture. The scoring rubric presented in table one has been developed from student activity on all seven tasks. In this paper I illustrate the performance indices by drawing on student activity from the following three tasks:

1. *Yoruba Counting*: Students were provided the Yoruba words for 1 through 5, and seven sporadic examples ranging from 10 to 50. The equivalent English statements for the examples from 10 to 50 were also included. The Yoruba counting system operates on a 5 and 20 cycle. Students were required to use the examples to interpret the structure of the counting system, write the Yoruba word for 12, 16 and 35 and write the equivalent English statement for 55 and 100.
2. *Nahuatl Counting*: Students were provided the equivalent English statement for 12 examples of Nahuatl numbers ranging from 6 to 104. The Nahuatl counting system operates on a 5 and 20 cycle. Students are required to use the examples to interpret the structure of the counting system, write the equivalent English statement for 9, 13, 19, 38 and the numbers 50 to 60.
3. *Computer Numbers*: Students were provided with the postal code bar codes (figure one) used by the US Postal Service to assist the computerized mailing process. Students were required to write their own postal code using the bar codes, translate numeric postal codes to bar coded postal codes and vice-versa.

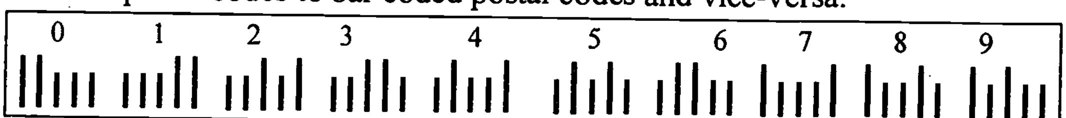

Figure 1. Bar codes for digits 0 to 9 used by the US Postal Service.

The other tasks included working with the Inca's quipu, creating a 5 cycle counting system, sharing the words for numbers from your language (other than English) and using hand gestures to represent numbers. More complete details, evidence and exemplars from student work will be provided during the presentation of the paper.

Data Collection

Data was collected from students' written responses on the worksheets and from recorded in-class didactic exchanges (Sproule, 1998). The written responses were used to organise the broader classification of student performance. The didactic exchanges were used to enhance and evidence the criteria for attainment of each performance index. Conclusions related to the performance indices and the associated criteria were triangulated using both data sources.

Results

Throughout the results section reference will be made to the summary of performance indices presented in table one. In keeping with the educative assessment framework the performance indices indicated in table one are directed at student development rather than a grade or mark. Following table one I evidence each performance index with examples of student work.

Table 1. Criteria for attainment of mathematics performance outcomes.

Performance indices	Criteria for attainment of mathematics performance index
Student's work exceeds curriculum expectations	The students is able to: • Identify and explore additional relationships and patterns in the number systems. • Offer informed additions or alternatives to the counting practices of a culture. The student does not impose these additions but is able to offer them as suggestions. • Identify similarities and differences between their own counting practices and the practices of other cultures. • Perform some of the four basic operations in unfamiliar counting cycles. • Create a multi-cycle counting system while engaged in one of the creative tasks (e.g. a 5 and 20 cycle system).
Student's work satisfies curriculum expectations	The student is able to: • Interpret the counting practices of the culture correctly and apply the practices consistently. • Recognise mathematical reasons for cultural counting practices. • Identify more than one possible solution when the information given about a culture's counting system is insufficient. • Describe the origins and progress of numerals in various cultures. • Translate numbers from other counting cycles to language of instruction and vice-versa. • Differentiate the properties of numerals in various counting systems. • Create all numerals in a 5 cycle counting system while engaged in one of the creative tasks.

Student is required to revise his/her effort.	The student: • Introduces additional features into the counting system that are beyond the features of the given information. • Demonstrates a careless inconsistency with the practices of a particular culture when counting or translating their counting system. • Creates a new word, without a cyclic nature, for every number while creating a cycle five counting system. Once a student has been requested to make revisions to their work their performance will then be classified as requiring additional learning experiences or satisfying the curriculum outcomes. • Only identifies the mathematical properties of individual counting systems. • Judges the relative size of the numbers based on the (syntax of the) words used within a language. • Made arithmetic errors in completion of the task.
Student requires additional learning experiences.	The student: • Develops procedures or counts in a manner inconsistent with the given information. • Develops an internally inconsistent counting strategy. This is most commonly demonstrated by an inconsistency between the numbers 1-10 and numbers larger than 10. • Reverts to a 10 cycle counting style while engaged in a counting system that is grounded on a different counting cycle. • Creates a non-cyclic counting system during one of the creative tasks.
Student requires attention to previous outcomes.	The student is unable to: • Express the structure of a cycle 10 number system. • Adopts an additive structure (e.g. Roman numerals) for a place value counting system.

Student's work exceeds the curriculum outcomes

David demonstrates performance that exceeds the expectations of the curriculum. He mathematically explored patterns in the postal bar codes.

I: Which other ones go together?

S: Zero and 1, 2 and 4, 3 and 6.

I: Does 2 and 4 go together?

S: No sorry 2 and 9, and 4 and 8.

I: And how about 7

S: 7 doesn't go with one.

I: They decided not to let 7 go with one. Which one does 5 go with?

S: None, it doesn't do with any either. Because 7, they can not because they have used all their

combinations up and if they do it exactly the same it will be the same as 7.

David identified that the bar codes for 0 and 1, 2 and 9, 3 and 6, and 4 and 8 were mirror images of each other (lines 2 and 4). Furthermore he recognised that the bar codes for 5 and 7 were mirror images of themselves and that all possible combinations were used in the construction of the 10 digits (line 8).

Student's work satisfies the curriculum expectations
The criteria outlined in table one for this performance index have been derived from two sources. The stated curriculum outcome formed the initial foundation of this index. Secondly, students' work that provided evidence of satisfying the curriculum outcome but that was not specifically stated in the language of the curriculum was included. For example, the ability to identify more than one possible solution when the information given about a culture's counting system is insufficient (table one).

I: Jabo, tell me what you've done there?

J: These, Nahuatl, of Mexico they like use five, a multiple of five then add-on other numbers. So you can see we wrote here **six means five plus 1**, and then 15 is a multiple of five and 20 is just 20 because it's a multiple of five, but when you want to **say 51 you go 2 times 20 then you add 10 plus 1**. 2 times 20 is 40 plus 10 is 50 plus 1

I: And how about 56.

J: Okay, 2 times 20, then you write plus, you could write plus 10 plus six. Or you could write 15 plus 1.

I: Which one do you think they do?

J: Plus 15 plus 1

Jabo consistently applied the examples of the counting practices of the Nahuatl provided in the task to the questions posed. He provided an appropriate response for 6 and 51 (line 2). The examples provided gave an inconclusive process for constructing 56. Jabo provided two possible solutions (line 4) and then (correctly) decided on the solution he thought was closest to the examples provided (line 6).

Student is required to revise his/her effort.
In a number of cases students' work required revision or additions. This performance index does not suggest that the student will undertake the required revisions independent of the teacher or other forms of assistance. An example to illustrate the index has been taken from a student's work on the Yoruba counting system. The student had correctly completed the other questions but had written 16 as 15+1 rather than "4 from 20" as suggested by the examples in the task. I intervened to encourage her to revisit this problem.

I: What does their 15 mean?

S: 15, they have got eedogun.

I: And what does that mean in English?

S: 5 from 20, so may be, I see there minus that

I: Yes, so what do you think they would do with 16?

S: To get to 16, they would say 20, no 4 from 20, minusing so there would be 20 minus four.

The student was able to revise her response (line 6) based on the probing questions I posed during the didactic exchange. Without further learning activities, but with appropriate intervention the student was able to correct her response.

Student requires additional learning experiences.
This performance index was applied when the nature of the students' errors suggested that additional learning experiences in this area might be most appropriate for their continued development. To illustrate this index I have used students' inconsistencies and the students' penchant for using a 10 cycle counting system (table one).

Inconsistencies: In the Nahuatl Counting task students wrote (in English) the numbers 50 to 60 as they thought the Nahuatl people would describe them. One student provided the inconsistent response given below.

50 = 10 + 40	51 = (2*20) + 10 + 1
52 = (10 * 5) + 10 + 2	53 = 25 + 25 =50 + 3

The student continued to apply an erratic collection of strategies to describe the numbers up to 60. The strategy used by the student was therefore classified as internally inconsistent. Considering that all counting systems demonstrate an internal consistency, this student required additional learning experiences. Similarly some students' strategies were internally consistent but were not consistent with the examples provided on the worksheet (e.g. 36 means 20+(15+1) and 51 means (2×20)+(10+1)). This is evidenced in the example below.

50 = 50	51 = 50 + 1
52 = 51 + 1	53 = 52 + 1
54 = 53 + 1	55 = 55 etc.

Reverting to a 10 cycle strategy: In the Yoruba Counting task students were asked to provide the English equivalent for the Yoruba numeral for 55 (5 from 3 times 20) and 100 (5 times 20). Examples of students' responses are given below:

55	= 10 * 5 + 5	**100**	= 10 from 10 * 11
	= 5 from 6*10		= 10 * 10
			= 10 *2 * 5

These students had reverted to a 10 cycle counting strategy possibly because it was more familiar. The given 5 and 20 cycle system was applied to smaller numbers but discarded for 55 and 100.

Student requires attention to previous outcomes.
This index is used as a description for students who demonstrated a lack of understanding of the counting system used in the language of instruction. As illustrated in table one, it is primarily evidenced by students who introduced an additive counting style rather than adopt the learned place value system.

For example, in the Yoruba Counting task students were asked to write the Yoruba word for 35. A number of students wrote "ogun eewaa aarun" which literally means 20, 10 and 5. The Yoruba people would have used words that mean "5 from 40."

Conclusions

In the reported educative assessment tasks students had an opportunity to engage various culture's counting strategies as an initial experience for developing an understanding of mathematics as a culturally situated activity. Table one provides a summary of the students' capacity to engage such tasks and is intended as a starting point for teachers as they attempt to assess students in the new curriculum. The effectiveness of educative assessment as a means to include the cultural nature of mathematics in the school mathematics curriculum continues to be investigated.

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A LONGITUDINAL STUDY OF CHILDREN'S THINKING ABOUT DECIMALS: A PRELIMINARY ANALYSIS

Kaye Stacey and Vicki Steinle
Department of Science and Mathematics Education
The University of Melbourne

Abstract

This paper reports preliminary results of a longitudinal study of children's understanding of decimal notation. A large sample of students completed a short test that enabled their understanding to be classified into four categories and changes over periods of up to two years to be tracked. When they attained expertise, students almost always retained it, even if it is simply following memorised rules. A small core of students retains "longer-is-larger" misconceptions. In contrast, students seem to move in and out of "shorter-is-larger" misconceptions. Improvements and hypotheses to be investigated in the future are noted.

Introduction and Background

It is now well documented that many students throughout schooling and indeed many adults have difficulty understanding the notation used for decimal fractions. The recent Third International Mathematics and Science Study showed that internationally about a half of 13 year old students could select the smallest decimal number from a multiple choice list of five decimals (data held at the Australian Council for Educational Research). Similar results have been known for many years in several countries. The aim of this paper is to present preliminary results from a longitudinal study that is tracing the development of students' thinking about decimal notation. This is of interest in its own right because being able to interpret decimals is important in a variety of everyday contexts (e.g. interpreting digital displays and calculator answers) as well as for mathematical tasks such as rounding and using significant figures. It is also a case study of how students' understanding and misunderstanding develops with progress through school and in the context of various types of instruction.

There are several ways of classifying the erroneous rules that students may apply when ordering decimals (Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989; Sackur-Grisvard & Leonard, 1985). The coarsest classification is that some students select "longer is larger" (e.g., deciding 0.125 is larger than 0.3) whilst others select "shorter is larger" (e.g., deciding 0.3 is larger than 0.496). Stacey and Steinle (1998), working with interview and written data, traced the various ideas behind

these erroneous rules, identified further misconceptions and developed a diagnostic test. This Decimal Comparison Test takes about five minutes and asks students to select the larger from 30 pairs of carefully chosen decimals. It enables ten patterns of thinking to be diagnosed. Some of these patterns of thinking are "longer-is-larger" misconceptions, some are "shorter-is-larger" misconceptions and others belong to neither of these. Steinle and Stacey (1998) present evidence that some misconceptions about decimal notation appear to be the result of instruction. In other cases, these misconceptions arise when ideas interfere with each other. Although future analyses will use the refined classifications, this paper reports student progress only in terms of four major categories:

- longer-is-larger misconceptions (resulting from any of five identified patterns of thinking and possibly others),
- shorter-is-larger misconceptions (resulting from three identified patterns of thinking and possibly others),
- apparent-experts (may possess excellent understanding or may apply correct rules not understood or may have an identified misconception (Steinle et al (1998))
- unclassified (since the criteria for classification are quite stringent, this large group includes students thinking about decimals in unknown ways and others who are inconsistent).

Cross-sectional data (see Figure 1, taken from Stacey and Steinle, 1998b) provides a picture of the incidence of various ways of thinking about decimal notation and how it varies with age. The Longer-is-larger category decreases from Grade 5 (32%) to Year 10 (5%), the trend suggesting that it is unlikely to be common in adult life. The Shorter-is-larger category plateaus at about 10%, which suggests that this general belief may continue into adulthood. The percentage of task experts also plateaus to about 60% in Year 10, which suggests that there are many adults who have difficulty understanding decimal notation. The task expert category of Figure 1 is somewhat smaller than the apparent-expert category used in this paper, because students with identified misconceptions have been removed from it and placed with unclassified students to form the category "Other". This adjustment is of the order of 5% and so Figure 1 can be taken as a reasonable guide to the number of students in the four categories used in this paper. This paper moves from the cross-sectional analysis to the beginning of a longitudinal analysis, which traces the movement of individuals in the overall data and reports on two questions:

- do students stay in the same category or move frequently from one to the other?
- what are the common paths through the misconceptions to attaining expertise?

The longitudinal sample and testing

This section presents preliminary results of the longitudinal study from 1995 to 1997. The sample was originally selected to contain a good mix of schools and to maximise the possibility of following students when they changed from primary (Years 0 to 6) to secondary school (Years 7 to 12) at approximate age 12 years. It consists of classes from:

- one state secondary school in a low socio-economic area and its three "feeder" primary schools,
- one church secondary school in an middle socio-economic area and its main feeder primary school,
- one private girls school in a high socio-economic area with both primary and secondary students,
- two large state primary schools situated in the same middle socio-economic area and the two high schools to which their students mainly progress and
- one church girls' secondary school in a high-middle socio-economic area.

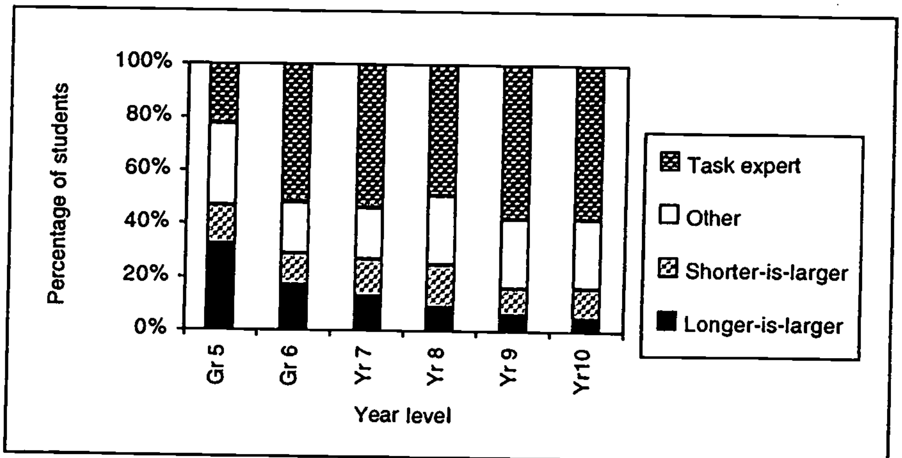

Figure 1: Distribution of classification by grade/year level

Students were tested with the Decimal Comparison Test at most once every six months, making a total of five testing times in the data under consideration, from the end of 1995 to the end of 1997. However, schools tested less often than this, for various reasons including different dates for joining the program. In this period, no individual student or class completed the test more than four times. The year level distribution of students is shown in Table 1. Note that many students are counted more than once, some up to four times each. In total, 5383 tests have been analysed, although there is no longitudinal data yet for many students. The large numbers in the lower year levels is due to selecting students whose progress could be followed until 1999.

Table 1. Year level distribution of students completing test 1995-1997

Year level	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10	Total
Number of students	336	965	874	1690	658	497	363	5383

The number of students who have completed the Decimal Comparison Test exactly one, two, three and four times is shown in Table 2. The students who have completed only one test play no further role in the analysis in this paper, as it aims to track change of individuals from one test to others. For this analysis, the first time an individual undertook the test will be called Test 1, the second time will be called Test

2 and so on. The tests are numbered for the individual, rather than by the date administered. Therefore for some students Test 1 was in 1995 while for others it was in 1996 or 1997. For some students, Tests 1 and 2 have been taken six months apart, whereas for others they may be one year or even 18 months apart if the student was absent on some testing days. This is an unsatisfactory feature of this preliminary analysis that will be addressed in subsequent work.

Table 2. Number of students by number of tests completed.

Number of tests	One test	Two tests	Three tests	Four tests
Number of students	1590	1198	307	119

Results

Changes of classification over consecutive tests.

Table 3 illustrates the changes in classification that occur over consecutive tests. The abbreviations A, L, S and U refer to apparent-expert, longer-is-larger, shorter-is-larger and unclassified groups, respectively. The cross-tabulation shows the movement of students who have tested in one category to other categories at the individual's next testing. The numbers are amalgamated from all tests. The data is therefore from Test 1 followed by Test 2, from Test 2 followed by Test 3 and from Test 3 followed by Test 4. As noted above, these changes are mostly over a period of about six months, but will also include changes over longer periods where students missed out on intermediate testing. This anomaly in the data will be eliminated when the final analysis is done, to give a better measure of change over six months.

The 426 students who have done the test more than twice contribute several times to the data. To illustrate, there were 165 instances where a student showed longer-is-larger thinking at one test and was an apparent-expert when next tested. Some of these individual students will have been tested again and contribute to the table again, in the group of 835 students who are apparent-experts on the prior classification.

Table 3. Changes in classification over consecutive tests (N = 2169)

Earlier classification	**Later classification**			
	A	**L**	**S**	**U**
A (N= 835)	732 (88%)	15 (2%)	29 (3%)	59 (7%)
L (N= 732)	165 (33%)	334 (46%)	94 (13%)	139 (19%)
S (N = 348)	125 (36%)	44 (13%)	120 (34%)	59 (17%)
U (N = 254)	98 (39%)	49 (19%)	50 (20%)	57 (22%)

Table 3 shows that from one test to the next, almost all of the apparent-experts stayed as apparent-experts and about one third of other students became apparent-experts. Nearly half of the longer-is-larger students (in fact two-thirds of those who did not become experts) re-tested as longer-is-larger. The shorter-is-larger students moved more than the longer-is-larger, but still about one third stayed in the same category.

Amongst those who did not become experts, over half remained as shorter-is-larger. As might be expected, the unclassified students spread most evenly across the other misconception categories. It will be important to repeat this analysis separating students by age group, as age is likely to be an important determinant of the speed and direction of change.

Changes of classification over at least one year.

Table 4 is similar to Table 3 except that it shows the changes in classification that occurred between testings typically at least twelve months apart. The numbers are amalgamated from Test 1 to Test 3 and Test 2 to Test 4. Therefore individuals who have been tested four times (there are 119 of these) contribute twice to the 545 comparisons in the table. The comparisons in this table come from students across a wide age range. About two thirds of the sample were in Year 4 or 5 at first testing. The oldest was a small group tested once per year in Year 8, 9 and 10. Some of the data is over one year, from students who were tested regularly, but where tests were missed it is over a period of up to 2 years.

Table 4 Changes in classification over at least one year (N = 545)

Earlier classification	**Later Classification**			
	A	**L**	**S**	**U**
A (N= 135)	125 (93%)	2 (1%)	1 (1%)	7 (5%)
L (N= 271)	104 (38%)	94 (35%)	23 (8%)	50 (18%)
S (N = 80)	44 (55%)	9 (11%)	19 (24%)	8 (10%)
U (N = 59)	25 (42%)	14 (24%)	8 (14%)	12 (20%)

Table 4 again shows that almost all of the apparent-experts re-tested as apparent-experts whilst about two-fifths of longer-is-larger students and unclassified students and over a half of the shorter-is-larger students became apparent-experts. This is consistent with shorter-is-larger thinking being somewhat more sophisticated than longer-is-larger thinking. Of those classified students who do not move to being experts, most stayed in their original category (56% of longer-is-larger and 52% of shorter-is-larger) but there was some movement into other misconception categories, not just towards expertise. The unclassified students spread most evenly across the other misconception categories.

We had previously expected that it would be more likely that longer-is-larger students would become shorter-is-larger than vice versa, because we hypothesised that students may move into the more sophisticated misconception on the way to expertise. However, since movement is equally frequent in both directions in both Tables 3 and 4, the data does not support this hypothesis. A further analysis using the finer grained classification system and broken down by age is required.

Changes of classification over two years.

Table 5 is similar to the previous two tables except that it shows the changes in classification that occurred from Test 1 to Test 4. In this sample, the only students who had been tested four times were from the two large primary schools in the middle socio-economic area. They were all tested initially in Year 4 and for the fourth time in the second half of Year 6 or in Grade 5 initially and for the fourth time in the second half of Year 7, when they had moved to a nearly secondary school. For this reason, the Table is labelled as indicating changes over two years.

Table 5. Changes in classification over two years (N = 119).

Earlier Classification	**Later Classification**			
	A	**L**	**S**	**U**
A (N= 16)	16 (100%)	0 (0%)	0 (0%)	0 (0%)
L (N= 74)	46 (62%)	10 (14%)	5 (7%)	13 (18%)
S (N = 11)	9 (82%)	2 (18%)	0 (0%)	0 (0%)
U (N = 18)	15 (83%)	0 (0%)	2 (11%)	1 (6%)

Table 5 again shows that all the students who were apparent-experts at first testing stayed as apparent-experts. These students are likely to have been among the most able students (having achieved early expertise in Year 4 or Year 5) so it is not surprising that they test again as experts at the fourth testing. Most of the shorter-is-larger and unclassified students became apparent-experts over the two years, which again supports the observation that these students have more sophisticated thinking than the longer-is-larger thinkers. In the latter group, only about two in three achieved apparent-expert status. At this level of analysis, there is no evidence to support the hypothesis that students who tested initially in one category, but do not move to expertise, stay over two years in the same category. However a closer look at the initially longer-is-larger students who move to the unclassified category is warranted. Because of the strict definitions employed for categorisation, it is possible that some of the 18% of formerly longer-is-larger students may still hold broadly longer-is-larger ideas yet not meet the criteria for that classification. This analysis would give a better insight into whether these students are essentially stuck in the one category or are on the way to expertise.

How representative is this group of students who have done the test four times? There are several reasons why they may be a better group than the rest of the sample. To be present on the four days of testing indicates that are likely to be regular school attenders with relatively stable schooling. The fact that their teachers have made the effort to test four times indicates commitment on their part. By Test 4 in Years 6 or 7, there are 86 (72%) apparent-experts and only 12 (10%) longer-is-larger thinkers. These proportions do indeed seem somewhat better than the proportions reported previously for the whole sample and summarised in Figure 1 above. This bias will require careful treatment in the next analysis.

A first attempt at following individuals

Table 6 is an initial view of the paths that students take through a series of tests. It displays data from the 119 students who completed the test four times (the same data set as Table 5). As noted above, these students were either in Year 4, moving through to Year 6 or in Year 5 moving through to Year 7. Each cell shows the number of students in the category at that particular test. There are, for example, 86 students (72% of the total of 119) testing as apparent-experts at Test 4. At Test 1, 16 students tested as apparent-experts. The table also records that all 16 of these were apparent-experts at Test 4. Similarly there were 34 apparent-experts at Test 2 and 32 of these individual students were apparent-experts at Test 4. Combining the information in Table 6 with the data in Tables 3, 4 and 5, we see that there is very little movement out of the apparent-expert category, suggesting that students retain knowledge of how to complete the test.

Table 6. Numbers of individuals in each category at each of the four testing times and, in brackets, the number of those students who retested in the same category at Test 4.

	Test 1	Test 2	Test 3	Test 4
Apparent-expert	16 (16)	34 (32)	48 (45)	86
Longer-is larger	74 (10)	48 (6)	33 (8)	12
Shorter-is-larger	11 (0)	23 (4)	19 (2)	7
Unclassified	18 (1)	14 (2)	19 (7)	14

Most of the students who have tested as longer-is-larger initially have also become experts by Test 4. The number in this category steadily reduces. However, Table 6 indicates that there is probably a small core of students who remain persistently in the longer-is-larger category. This will be further investigated with the finer grained analysis and is expected to be a larger effect in a less selective sample.

The shorter-is-larger category shows a different pattern. This category grew in Tests 2 and 3, but the low numbers who persisted in this category in Test 4 indicate that students move in and out of it over time. This is consistent with our experiences when we interviewed students from a class of Grade 5 and 6 students (not in this sample). These students had recently been studying fractions and so it seemed that the number of shorter-is-larger students who interpreted decimals as fractions (reciprocal thinkers) was inflated by the recent experience. The pattern of pathways for students who are unclassified seems similar to the shorter-is-larger pattern.

Discussion

The purpose of this paper was to report a preliminary analysis of data showing students' progress in their understanding of decimal notation. Several ways in which the analysis can be made more revealing have been highlighted: by following individuals, by following classes so that teaching effects can be observed, by

separating the analysis by age group and by using the finer classification system. This will be done when an extra year has been added to the data set, so that there are more long runs of data from individuals and more histories of intact classes.

The preliminary analysis has provided the following results to be confirmed later. There is quite marked stability of classification. Of course there is a general trend towards expertise, but those students who do not achieve expertise tend to remain in the same category. Even after a passage of at least a year (i.e. first to third test) about half of the students retaining a misconception are classified in the same way. This is a significant result given the stringency of the classification criteria and it confirms our informal data (from interviewing previously classified students) that the test, although taking only a few minutes, is highly reliable.

Students in different classifications behave differently. Apparent-experts nearly always stay in this category. This would be expected of students who "really understand" decimals. However, at least in the context of this test, the skill of decimal comparison is well retained even by those who use a rote-learned rule (e.g. compare digits from left to right or add zeros).

A small group of students seem to persist in the longer-is-larger category. On the other hand students seem to move in and out of the shorter-is-larger category and are more likely to move to expertise. Unclassified students are similarly likely to move towards expertise, contrary to a previous finding on a group of 50 students (Moloney and Stacey, 1997) they were more likely to move to expertise than the shorter-is-larger students. Following individual paths will help in unravelling students' thinking further and eventually providing better guidelines for teachers.

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EXPLORING STUDENTS' SOLUTION STRATEGIES IN SOLVING A SPATIAL VISUALIZATION PROBLEM INVOLVING NETS*

Despina A. Stylianou, *University of Pittsburgh, Pittsburgh PA*
Roza Leikin, *Technion, Israel Institute of Technology, Israel*
Edward A. Silver, *University of Pittsburgh, Pittsburgh PA*

Background

Many scholars have argued that visual and imagery based processes play an important role in mathematical learning, problem solving and reasoning (e.g., Clements & Battista, 1992; Dreyfus, 1992; Hershkowitz, 1989). A full understanding of the nature of students' mathematical competence and creativity requires examination of not only computational and logical problem-solving abilities, but also associated visual skills. Spatial transformation tasks have long provided a useful medium for investigating students' visual processes related to mathematical competence.

Translating between three-dimensional solids and their two-dimensional representations is one kind of processing that is particularly important to mathematics. Michelmore (1980) argues that "it is of great value to be able to visualize and represent three-dimensional configurations and to comprehend the geometrical relations among the various parts of a figure." In particular, problems related to the *development of solids* afford students opportunities to develop the visual skills mentioned by Michelmore. Problems of this type often require students to make translations between 3D figures and their 2D *foldouts* or *nets* by focusing on the relationship among the various parts of the solid both in 2D and in 3D. A net is a diagram of a hollow solid consisting of the plane shapes of the faces so arranged that the cut-out diagram could be folded to form the solid (Borowski & Borwein, 1991).

Even though problems regarding nets are included in some textbooks and on some assessments, only a few research studies have examined students' reasoning about nets. These studies (e.g., Bourgeois, 1986; Little, 1976; Mariotti, 1989; Potari & Spiliotopoulou, 1992) indicate either that students' thinking about nets is affected by the complexity of the geometric figures and/or that some nets of a given solid (e.g., a cube) may be more difficult than others. However, the reasons for these variations in difficulty of tasks related to nets are not clear. Mariotti (1989) hypothesizes that "constructing the correct net of a solid implies coordination of a comprehensive mental representation of the object with the analysis of the single components (faces, vertices and edges)" (p. 263). The findings of her study support this hypothesis as students appear to be more

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successful in recognizing those nets that require only relatively few transformations from the solid to the net. However, there is little information with respect to how students coordinate and analyze the components of a 3D solid when transforming it into a 2D net or vice versa.

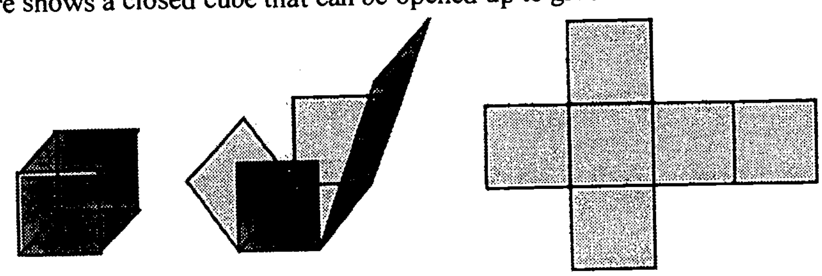
The purpose of our study was to build on previous findings and to deepen our understanding of students' work on problems which involve translations between 3D figures and their 2D net-representations. We focused on students' problem-solving strategies when *constructing* different types of nets. It is important to know the approaches that students take when faced with a task that requires them to construct (and not only recognize) different nets for a given solid and to examine the difficulties related to each type of net with respect to the strategies students use.

Method

The task shown in figure 1 was administered to eight eighth-grade students in the U.S. in a clinical interview setting[1]. It can be shown that there are only 10 possible nets for a cube (figure 2). Previous studies (Mariotti, 1989) have shown that the E-net is the most intuitive and familiar, and therefore, it was chosen to serve as the example for this study. Each student was asked to think aloud, and to explain each stage of his or her solution processes. All interviews were audiotaped and detailed notes were kept for each student's work.

Make a Net

The picture shows a closed cube that can be opened up to give a net:



One can open up a cube in many different ways and get different nets. In the space below draw ***different*** nets that could have come from a closed cube. If you find it necessary you may use the model of a cube.

Figure 1: The task used in the study

Each student was first given the task without any manipulatives, and was asked to solve and explain the problem. When students were no longer able to make progress in this way, they were offered cut-out manipulative nets to use. In the first case, we say that the students worked in mental mode (M-mode). In the second case, we say that students worked in concrete mode (C-mode). When working in the C-mode students were allowed to cut and tape a net or a cube in any way they preferred (a cube could be obtained by folding and taping a cut-out

[1] This was the first of six tasks related to nets that were given to students. Students had approximately one hour to solve all six tasks and the time each student allocated to each task varied.

net using the inverse procedure of the one demonstrated in Figure 1). Students' responses were analyzed for the students' solution strategies and the mode in which the students worked (M- or C-mode).

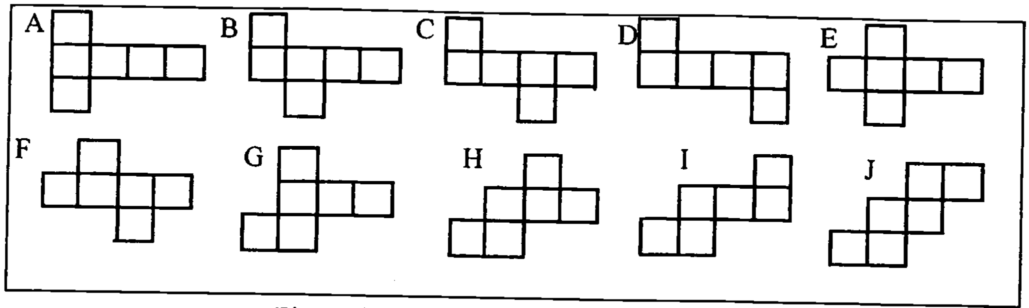


Figure 2: The ten possible nets for a cube

Results

All eight eighth-grade students produced a net while working in the M-mode, and most produced several more nets in M-mode. Seven students also obtained nets in C-mode. The students produced an average of about five nets each.

Two main approaches to the problem were observed: working in 3D with the cube, and working in 2D with the net. When *working in 3D* students indicated verbally or with hand movements that they were attempting to produce different nets by unfolding the cube in different ways. For instance, students indicated that they opened up the cube as is shown in figure 1, or by "rolling" it into a net. Students could work in 3D either mentally (mentally visualizing the unfolding of the cube and noting the resulting net) or with the use of concrete materials (literally cutting a paper cube in different ways and showing the resulting net). When *working in 2D* students focused on producing a net on the 2D plane and then translated that net into a cube. Once again, students could work in 2D either mentally (drawing a possible net and then mentally checking whether this can be folded up to form a cube) or with the use of concrete materials (by taping together paper squares to form a net and folding them up into a cube).

Furthermore, when choosing to approach the task either from a 2D or 3D perspective, students demonstrated that they could use different strategies which could be roughly classified under two categories: *trial-and-error strategies*, and strategies involving *systematic changes of either the net or the cube* (Table 1).

The most interesting strategies are those in which a systematic way of producing nets was used. As table 1 indicates, students used systematic ways to solve the problem in both 2D and 3D. The first systematic strategy we observed in 3D was to unfold the cube in different ways; this was used both by students who worked in the M-mode and by students who worked with concrete materials in the C-mode. The second systematic strategy in 3D was only used by students who worked in C-mode. In this case, students built a cube with a face missing and then systematically taped the cut-off face to different edges in order to obtain different nets. For example, when RA cut off one of the faces and formed "a cube with a missing face", say the front face, he systematically connected the cut-off face first to the bottom edge and then to the right edge. Each time the student would open up the cube to obtain a new net. Students who employed this strategy appeared to

have a good understanding of the concept of a net, how it can be formed, and how it can be changed.

Transformations in 3D	Transformations in 2D
a. Systematic changes of a cube - systematically fixing different face of a cube as a bottom and unfolding the cube to obtain the new net - removing one or more of the square faces of the cube creating a partial net cube (a cube with missing face(s)) and considering how the cut-off face(s) can be connecin a way that it would result to a new net	*a. Systematic changes of a net* - keeping four squares attached together in a row and moving the other two squares (one on the top the other underneath) around that row (nets A - F in figure 2) - fixing one square as a "bottom" and moving the other squares accordingly - fixing a pattern of squares and adding more around
b. Trial-and-Error - randomly unfolding the cube and checking whether this action resulted in a new type of net.	*b. Trial-and-Error* - randomly drawing or connecting together six squares and checking to see whether these would fold up to a cube.

Table 1: The different strategies used by the students

In 2D the most commonly used systematic way of changing the net was to move squares around a given pattern. Similarly to the students who realized that by connecting a face to different edges of an open cube resulted to different nets, students who engaged in this strategy realized that by connecting a square next to different parts of a partial net they may produce different valid nets.

Five students chose to draw/construct a "line-of-4" and to systematically move the two remaining squares along that "line-of-4" (figure 3). This strategy occurred both in the M- and the C-modes. Some students were able to justify their strategy and the validity of the nets they produced by explaining that the "line-of-4" connected squares when folded wraps around to create a "cube with missing sides" and the two squares can be attached anywhere along each of the sides to create the "missing sides" of the cube. In this case, students appeared to be confident about the effectiveness of their strategy, and often did not worry about checking the validity of the nets they produced. We must also note that despite the fact that five students seemed to be drawn to this strategy, none of them completed the systematic construction of all six nets (A-F nets) which can be constructed using this strategy.

Another common systematic strategy for producing nets in 2D was to perform rotations and reflections of a net. Students constructed nets that were obtained by rotating or reflecting a net that they constructed. Most of the students who drew or constructed nets that were only rotations or reflections of one net indicated that they understood that their nets were not different from each other and that they could actually be obtained by moving a concrete net around the table. For these students the drawing of rotations or reflections of any net was probably a response

to the requirement of the problem to produce as many nets as possible. Thus, they drew rotations or reflections of each new net they produced. Some students, however, did not appear to identify the equivalence between a net and its rotations.

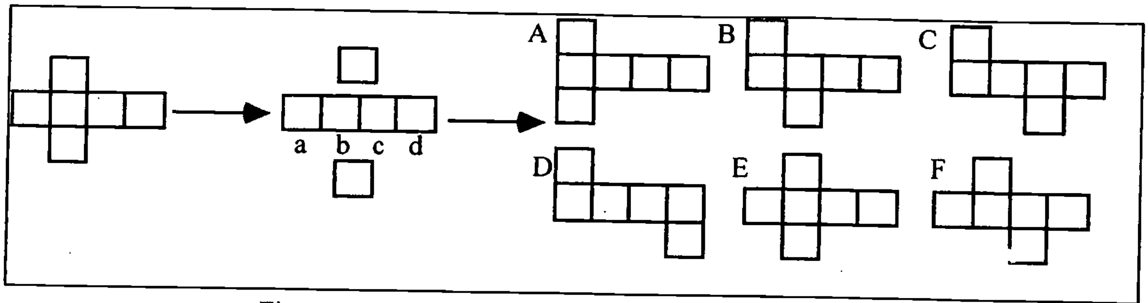


Figure 3: A systematic construction of nets A - F

A second goal of this study was to identify which are the nets that present students with the most difficulty, and also to identify the strategies that students use to obtain the "difficult" nets of the cube. Previous research (Little, 1976, Mariotti, 1989) indicated that students find the E-net to be the easiest. Mariotti (1989) in particular, suggests that the number of transformations on each element of the cube as it is "unfolded" represents an index of difficulty. The E-net is easy because it can be straightforwardly "opened up" (as shown in figure 1), while, for instance, to obtain the D-net one needs to "follow a "rolling" strategy where the composition of more transformations is involved" (Mariotti, 1989, p. 262).

Since we gave students an E-net as part of the task, it is not possible to confirm Mariotti's findings directly. But it is interesting to note that almost all of the students produced an E-net as one of their solutions. Table 2 shows all the nets that each of the students in our study produced in the order they produced them, and also describes the mode (mental or concrete) and the strategy they used to produce each net (for example, student ST produced 3 nets: A-, B-, and G-nets. The first two, A- and B-nets were produced in M-mode while she systematically moved two squares along a "line-of-4" squares in 2D. The G-net was produced when she mentally unfolded the cube in a way that was different than the one shown in figure 1). As the table indicates, half the students (SD, RA, CJ, and AH) responded to the task by first re-producing the E-net. Their verbal protocols suggest that they mentally unfolded the cube in the most straightforward way, thus, confirming the first part of Mariotti's findings.

In Mariotti's study, though, students found the second net (here indicated as the D-net) to be difficult. Mariotti explained this difficulty by stating that "the case of type [D] figures [are harder] because in the process of reconstruction each element is transformed many times successively. To solve the task it is necessary to follow, in one's own mind, the transformations of the single element, so that the number of transformations represents an index of difficulty. As a remark it is interesting to observe that the presence of a symmetry in the situation is not always noticed and used" (p.264). However, the D-net, was not as difficult for students in our study as Mariotti's results would have predicted. Along with the A-, B-, C-, E-, and F-nets, this net was also relatively easy for students to produce. In fact these are the nets that students were more likely to produce first (not considering the E-net that was

already given). Apparently, the unexpectedly common use of the systematic strategies in 2D, particularly the use of the "line-of-4" strategy, in which students kept four squares attached together in a row and moved the remaining two squares along those four, helped students easily obtain these nets, without having to resort to translations between the 3D cube and the 2D nets.

Name	Nets	Mode	Strategy	
ST	A, B, B	M	Moves 2 squares along line of 4	2D systematic
3 nets	G	M	Unfolds a cube systematically	3D systematic
SD	E, E, E, E	M	Unfolds cube randomly	3D trial and error
5 nets	C, B, D, A	C	trial and error	2D trial and error
RA	E,	M	Unfolds cube randomly	3D trial and error
7 nets	B, B, C, F, A	M	Moves 2 squares along line of 4	2D systematic
	G, H, C	C	Unfolds cube with missing face	3D systematic
AW	A, E	M	Unfolds cube randomly	3D systematic
4 nets	D, B	C	Moves 2 squares along line of 4	2D systematic
	Two incorrect nets	C	Trial and error	
VR	A, B, C, D, E, F, J	M	Moves 2 squares along line of 4	2D systematic
9 nets	Incorrect net	C	Trial and error	2D trial and error
	G, H	C	Corrects the previous error	2D systematic
AM	A, E, E, A, A,	M	Moves 2 squares along line of 4	2D systematic
7 nets	C, F, B	C	Moves 2 squares along line of 4	
	G, G, H, H	C	Trial and error	Trial and error
CJ	E	M	Unfolds cube randomly	3D trial and error
3 nets	H, G, G	C	Trial and error	2D trial and error
AH	E, E	M	Unfolds cube randomly	Trial and error
3 nets	H, G, Incorrect net	C	Trial and error	

Table 2: Distribution of nets students produced with respect to the types of the nets and the strategies used

Table 2 also indicates that the most commonly produced nets were the A and B (6), G (5), C and H (4), D and F (3), and J (1) nets. None of the students produced any I-nets. However, when we look at the strategy that students used to produce these nets, it is clear that when students worked systematically (i.e., the nets were not produced accidentally) then they were more likely to produce nets that maintained the "line-of-4" characteristic rather than other types of nets. The A-net, one of the most commonly produced nets, was also the net that was produced first by half of the students. It was produced by all the students who systematically changed the net either in 2D or in 3D. In fact, 5 of the 6 students who produced this net did it while making systematic changes to nets in the M-mode. Finally, it is interesting to note that, while nets A, B, C, D and F were most commonly produced when students worked systematically, nets G and H were produced by trial-and-error as frequently as they were produced systematically (see Table 3).

	A	B	C	D	F	G	H	I	J
Systematically	5	5	3	2	3	3	2	--	1
Trial & error	1	1	1	1	--	2	2	--	--
Total	6	6	4	3	3	5	4	--	1

Table 3: Distribution of nets with respect to production strategy

Discussion

One of our goals for this study was to examine the difficulties related to this task with respect to the strategies students use. Brown and Wheatley (1997) argue that students' ability to decompose and recombine images, that is the ability to break down a visual image into simpler parts and then recombine those parts into new images, is an important component of imagery in problem solving. Indeed, in the problem under consideration, it was necessary for students to notice that the given net is composed of squares that can be separated from one another and recomposed again into a different formation. However, our students did not simply decompose the net into squares. Rather, the most successful students worked using patterns of squares in a systematic manner. In fact, our students used primarily one specific pattern, the "line-of-4" pattern. Hershkowitz (1989) identified three types of visual reasoning when introducing and understanding mathematical concepts: (a) based on the whole figure, (b) based on non-critical attributes, and (c) based on critical attributes. The results of our study support the findings of Hershkowitz in the context of reasoning about correspondence between 2D and 3D representations of a cube.

Those students who attempted to find a systematic way of producing new nets focused on finding some critical characteristics of the net that remain invariant in all the nets and correspond to critical characteristics of the solid. It may be the case that students perceived the "line-of-4" as a core invariant characteristic of the cube-net; when folded-up this line of squares forms the core of the cube while the two other squares fill in two empty faces. Further, it is possible that students also perceived the "line-of-3" (perpendicular to the "line-of-4" in the given E-net - figure 1) as another critical characteristics of the net. This may explain the frequency in which students produced an A-net (the A-net maintains both characteristics) and produced it first.

Further we may attribute the common production of the G-net to the students' attempt to maintain the invariance of the "line-of-3" characteristic (the G-net is the only one among those that do not maintain the "line-of-4" that maintains the "line-of-3"). The hardest nets to obtain in the M-mode were the I- and J-nets, those that do not maintain either of the two salient, and presumed by the students to be constant characteristics of the E-net. Similarly, the drawing of symmetric equivalent nets can be attributed to an attempt to find and use more invariants in the net. When students focused on finding and fixing as many invariant characteristics as possible, students produced equivalent nets (notice, for example, how AM produced multiple copies of the A-net when working systematically – see Table 2).

Finally, another issue that deserves attention is the fact that none of the students who chose a systematic way of approaching the task completed the construction of all six nets (A-F nets) which can be obtained using this strategy. This could be due to the limited time that students had to solve the problem. Nonetheless, this issue brings up questions regarding students' ability to work systematically when faced with problems that are highly visual or spatial. Despite the fact that in six cases students attempted to use systematic ways of producing nets (five in 2D and three in 3D; two student worked systematically both in 2D and 3D), their attempts were often disorganized resulting in the omission of some nets and the reproduction of others. The difficulty that students faced in approaching this problem systematically is an area that deserves further investigation.

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COMPARING STUDENTS' RESPONSES TO CONTENT SPECIFIC OPEN-ENDED AND CLOSED MATHEMATICAL TASKS

Peter Sullivan, Elizabeth Warren, Paul White

Australian Catholic University

The results reported here are one component of a larger project that seeks to explore a range of aspects of the classroom potential of content specific open-ended tasks. We propose a way of discussing such tasks, and justify our focus both on open-endedness and the focus on content. In particular we compare the responses of students to comparable closed and open-ended tasks, and also explore the effect of using specific contexts for such tasks. We propose a method for exploring differences in facilities of open-ended and closed tasks and suggest that both types of tasks can contribute to effective classroom programs.

The research is based on three assumptions: tasks are the critical prompts for mathematical activity, tasks that are to some extent open are more likely to produce rich activity (Christiansen & Walther, 1986), and explicit attention to mathematics is a key characteristic (Cyril, 1998).

This is an investigation into the student responses to tasks which are, on one hand, open-ended to stimulate high quality thinking, and, on the other hand, content specific to emphasise the mathematics being learned. It is intended to make inferences from the results on the potential of such tasks both for classroom use and for assessment.

Open-ended Tasks

There is no agreement on classification of tasks. The following discussion seeks to situate the tasks used as the focus of this investigation in the context of other related tasks, and to clarify some terms and the way we use them.

We use the word *task* rather than the more common word *problem*. It is possible to have problems that are closed, and it is also possible to have open tasks that are not problems in that a solution path is known to the student, but which nevertheless provoke rich mathematical activity.

We take the *task* to be the statement presented to students that serves as the prompt for their work. Their *activity* is the thoughts and actions in which they engage in response to the prompt. The *goal* is the result the students seek as a product of their activity in response to the task statement. Each has the potential to be open or closed. *Closed* implies there is only one acceptable pathway, response, approach, or line of reasoning. *Open* refers to the existence of more than one (preferable many more than one) possible response.

While the direction of the *activity* can be fixed or closed we are only interested in tasks that stimulate open activity – that is, students will ordinarily follow different

approaches to the task goal. Tasks that stimulate closed activity may well contribute to a effective mathematics curriculum. Nevertheless, we see that at least some opportunities for working on tasks that stimulate open activity by students are important. We suspect that open activity fosters some of the more important aspects of learning mathematics, specifically investigating, creating, problematising, mathematising, communicating, and thinking, as distinct from merely recalling procedures.

Our focus then is on open goals, and we use the term *open-ended* to describe tasks that have such goals.

We recognise that tasks that have open statements contribute to a broad and rich curriculum and have potential to make a significant contribution to mathematical learning. Examples include investigations (Wiliam, 1998), the use of problem fields (Pehkonen, 1997), problem posing (Leung, 1997), and the open approach (Nohda & Emori, 1997). Nevertheless this is an investigation of tasks that have a specific focus on aspects of the mathematics curriculum. The content specific nature of the tasks implies that the statements need to be closed.

Content Specific Open-ended Mathematical Tasks

There are a number of advantages of having the content focus explicit. One advantage is that the students' attention is drawn to the mathematics and so their learning may be more directed, as well as making the role of the teacher clearer. A further rationale for the content focus of these tasks is that, with the current attention to testing and explicit statements of curriculum outcomes in many countries, teachers feel an imperative to be able to identify the mathematics content being addressed.

The term *content specific open-ended tasks* can be illustrated by means of some examples:

A number has been rounded off to 5.6. What might be the number?

The mean height of four people in this room is 155cm. You are one of those people. Who are the other three?

Find two objects with the same mass but different volume.

We believe that such tasks make a useful contribution to a mathematics curriculum in that they:

- provide similar advantages to less content specific open-ended tasks, and even many of the advantages of tasks with open statements in that students can investigate, generalise, seek patterns and connections, communicate, identify alternatives;

- address conventional content explicitly and so are more easily integrated into mathematics curricula;

- have a teaching focus sufficiently similar to what teachers usually do and so are easy to implement; and

- provide a bridge to tasks with open statements for the students.

In earlier research on the use of such tasks, Sullivan, Clarke & Wallbridge (1991) found that:

- pupils can respond to such tasks at a range of levels especially if prompted by the wording of the task;

- the quality of response within a single grade levels varied, and that student responses improved in higher grades;

- students at all levels gave better answers to questions which drew on content which had been learnt some years previously;

- students can learn from the activity; and

- students often gave only one response even if they could have given more.

Sullivan, Bourke and Scott (1997) conducted a detailed investigation of a classroom implementation using a program based solely on content specific open-ended tasks. Most tasks posed were open-ended, and there were few teacher explanations. The students engaged in personal constructive mathematical activity and there were no management or organisational difficulties created by the approach. Observation of individual students and interviews confirmed these impressions and indicated that teaching based on open-ended tasks is suitable for both students who are confident at mathematics and for those who lack confidence.

It seems that open-ended tasks offer significant possibilities for stimulating the active involvement of students in learning and doing mathematics in classroom situations.

As it happens, there is some opposition to the concept of such content specific open-ended tasks. Becker and Jacob (1998) quote a Prof Wu who was critical, inter alia, of the following curriculum statement:

> Students understand and use the relationship between concept of perimeter and area, and relate these to their respective formulas. (Becker & Jacobs, 1998, p 6)

Wu claimed that "There is no relationship whatever between perimeter and area, ... unless it is the isoperimetric inequality" (Becker & Jacobs, 1998, p 6). Wu had previously (Wu, 1994) been critical of problems like:

> *If the perimeter of a rectangle is 30m, what might be the area?*

He claimed that since students could not give complete answers, the question should not be asked. We disagree. It is precisely the exploration to the link between perimeter and area that is the strength of this task. Indeed, the failure to compare and contrast the related concepts seem to be at the heart of many of the difficulties which students find in responded to perimeter and area questions.

Ultimately the worth of a particular task or task type can be judged on the quality of the responses which the students give, and whether they engage in thinking mathematically for themselves. That is the focus of this investigation.

Comparing Responses of Students to Open-ended and Closed Tasks

The results reported here are from a larger project that seeks to explore responses of students to comparable open-ended and closed tasks. This aspect of the project sought to compare responses of students to tasks that address similar content, and to explore the effect of situating tasks in particular contexts. These aspects have potential to inform understanding of the classroom potential of such tasks and particularly to suggest ways in which such tasks can be productively integrated with the rest of the mathematics program. The key variable is the quality of responses students give to such tasks.

Responses were sought from approximately 1200 students from schools in three Australian states. Data were also collected in Indonesia but these are not reported here. In each state, more than two schools responded to the tests. Classes were selected to maximise comparability by gender mix, socio-economic status, urban location, school size, and experience of the teachers. The responses presented are from students in Year 8 classes in each of New South Wales, Queensland, and Victoria.

Seven instruments requiring written responses were developed. Each differed according to the mix of open-ended/closed and context/context free tasks. In addition to seeking to compare responses to similar open and closed tasks, the test forms sought information on the effect of the use of contexts in the phrasing of the tasks. Generally, the same students completed more than one test form. Where possible the tasks were designed to be comparable with items from large-scale tests used elsewhere.

The tasks were piloted in three states to minimise difficulties with the wording, and to ensure that the content was compatible with the syllabi. From the pilot results we included a specific prompt to cue more than one response to the open-ended tasks. This has a disadvantage in classroom situations in that it may limit the potential for students to generalise answers. In this case, it gave us a better indication on whether students could produce multiple responses. We also found that the mix of tasks seemed to have little effect.

The tasks on each of the instruments were scored individually. The closed tasks were scored as either correct or incorrect. Each of the open-ended tasks were scored using the following codes:

- *i* one or two correct responses
- *ii* some correct, some incorrect
- *iii* three or more correct responses

The tasks were basically grouped around three contexts: to gauge the effect of using *differing units* of measure; to ascertain children's understanding of the interrelationship between *perimeter and area*, and third to determine children's

understanding of the relationship between *embedded rectangles*. Only the results of the first of these are reported here.

In each case both open-ended and closed tasks were constructed, and each of these were posed in context and context free formats. The open-ended tasks used a prompt "give at least three answers" since during trialing we observed that many students were reluctant to give more than one response.

The following presents the results with tasks selected from various test forms to facilitate comparisons.

Responses to the *Differing units* Tasks

The tasks requiring the use of differing units were designed to be comparable to the following item taken from Department of Education (1991).

A rectangular rug has an area of 2 square metres. The rug is 40 centimetres wide.
How long would it be?
5 m *8 m* *20 m* *50 m*

The Department of Education (1991) study reported that 52% percent of year 9 students responded correctly.

Table 1 presents the four tasks derived from this item, and shows the percentage of correct responses for each task.

Table 1
Responses to different unit *tasks*

Task	Type	Response Code	% of correct responses	n
A rectangle has an area of $2m^2$. It is 40cm wide. How long is it?	Closed No context	Correct	7	226
A rectangle has an area of $3m^2$. What might be the length and width of the rectangle? (give at least three answers)	Open-ended No context	i ii iii	8 6 9	226
A rectangular rug has an area of $2m^2$. The rug is 40 cm wide. How long is the rug?	Closed Context	Correct	15	315
A rectangular rug has an area of $3m^2$.. What might the length and width of the rug?	Open-ended Context	i ii iii	11 10 23	315

The number of correct responses seems low, and certainly lower that the Department of Education (1991) question. Two possible explanations are that the one additional year of schooling makes a difference, and that the multichoice format made the earlier item easier.

More students responded correctly to the closed task when posed in context (although the facility is still very low) and when posed context free.

The responses to the open-ended tasks when posed in context are much higher than the context free ones. In context, 44% of students gave one or more correct responses to the context task, compared with 23% for the form which did not use the "rug" context. It seems that the rug context has assisted some students in this case.

In each case, the students gave more correct responses to the open-ended task than to the closed ones. It appears from the results that the open-ended task may have been easier than the closed task for these students. An examination of the types of responses for each task gives some insight into why the open-ended question seemed easier. The following presents responses given by of students who gave correct responses.

For the closed tasks:

some converted to 0.4 m then calculated correctly
a few converted to 20 000cm2
many converted 40 cm to 4 m
some converted both to "non standard units"

For the open-ended tasks:

all of those who gave one or more correct responses used 3m x 1m
many gave 1.5m x 2m as one of their responses
most other correct responses used decimals in some way
some gave 2m x 1.5m as one of their answers

Nearly 50% of students gave a response that was correct numerically but used incorrect units. This suggests that the need to use compatible units of measure may have been the main source of difficulty.

Many students were able to give three responses. Even though there were still fewer correct responses to the open-ended task than one might hope, there were more than for the closed task.

Based on these responses, and on discussion on the elements of the tasks, the components needed for success on each were delineated. Figure 1 summarises the components, with the elements common to both items recorded in the middle column of the table.

Steps to solve the closed tasks only	Steps common to both tasks	Steps to solve the open-ended tasks only
Match information to dimensions Convert m^2 to cm^2 OR cm to m Do division	Visualise the shape(rug) Read Comprehend Formula for Area (A=LxW) Reverse the formula (LxW=A) Encode Record	Select 2 numbers whose product is 3 Recognise multiple possibilities Generalise

Figure 1: Components for each task.

In this case, is possible that the closed task is more difficult because of the need to convert from one unit to the other and then to do just that. Difficulties with the units may have impeded students from exhibiting their understanding of the area concept. The open-ended task does not necessarily require unit conversion, and it seems that most students who responded correctly did so without converting. Hence, the open-ended task gave more information about students' understanding of area but gave little information about their facility with unit conversion. The use of an appropriate prompt may be needed to elicit this information.

In the classroom, the open-ended task may be useful for initial exploration of the area concept with either non whole number sides or mixed units, and the relationship between differing units of measure. The closed task seems to be potentially useful as an assessment item, assuming students have had the necessary unit conversion experiences. A brief analysis of texts at this level suggests that there are few exercises on area or perimeter that require unit conversion.

Conclusion

This part of the project sought to compare student responses to comparable closed and open-ended questions. One one set of data is presented here, but these concluding comments refer to all of the results.

One goal was to explore the effect of context on the students' responses. For the closed tasks, the context seemed to help for one set, but made no difference in tasks that included diagrams. It is possible that the diagrams performed a similar role to the context. For the open-ended tasks, the context helped for two sets, but seemed to make a third set more difficult. It is suggested that in this case, since it dealt with a complex concept, that the context made the task more difficult.

Another goal was to compare responses to closed and open-ended tasks. In two sets, students found the open-ended tasks more difficult suggesting that these required thinking above and beyond that required for the corresponding closed tasks. In the other case the open-ended task was easier.

An analysis of the responses and a breakdown of the elements of the tasks seemed to explain these differences. The open-ended tasks which were more difficult required students to link two concepts and to use such links to conjecture and generalise. Such open-ended tasks may serve the role of stimulating students' thinking to higher levels.

Both the context and the open-endedness seemed to effect the focus and student response to the tasks. It seems that each of the type of tasks would contribute productively to classroom programs and teachers could be encouraged to plan to use contexts and open-ended questions productively. In some cases the open-ended tasks may serve as a useful preliminary exploration of topics; in other cases, they may be better left until later.

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DYNAMIC SCAFFOLDING AND REFLECTIVE DISCOURSE: THE IMPACT OF TEACHING STYLE ON THE DEVELOPMENT OF MATHEMATICAL THINKING

Howard Tanner & Sonia Jones: University of Wales, Swansea

Abstract

The mathematical thinking skills project (Tanner & Jones, 1995) reported that classes which followed a course emphasising metacognitive skills were not only more successful than controls in assessments of those skills but also in assessments of mathematical development. However the size of effect was quite small, and ethnographic data revealed significant variations in teaching style from teacher to teacher. Further analysis identified four characteristic teaching styles. This paper discusses the effectiveness of the different styles with emphasis on the two most successful groups: the dynamic scaffolders and the reflective scaffolders.

Introduction:

The Mathematical Thinking Skills Project (Tanner & Jones 1995) aimed to develop and evaluate a thinking skills course to accelerate students' cognitive development in mathematics. The course was based on the earlier Practical Applications of Mathematics Project (Tanner & Jones, 1994) which had identified the metacognitive skills of planning, monitoring, and evaluating as necessary for practical problem solving.

We aimed to develop and evaluate an enrichment course for the first two years of secondary school (ages 11 to 13) focusing on the development of metacognitive skills.

Metacognition

Metacognition is a "fuzzy" and elusive term which is used to cover a range of ill defined interacting categories which share certain resemblances (Brown, 1987, p106). Metacognition has two aspects: (a) the awareness that individuals have of their own knowledge, their strengths and weaknesses, their beliefs about themselves as learners and the nature of mathematics; and (b) their ability to regulate their own actions in the application of that knowledge (Flavell, 1976; Brown, 1987).

The former aspect is passive in character and is characterised here as metacognitive knowledge or "knowing what you know". The second refers to the "active monitoring and consequent regulation and orchestration" of cognition (Flavell, 1976, p232) and is characterised here as metacognitive skill.

The earlier project (Tanner & Jones, 1994) identified classroom practices which might facilitate the development of metacognitive skills. These included the use of social structures to frame pupils' behaviour and constrain them to act as experts rather than novices, e.g. by slowing down impulsive behaviour and encouraging examination of several problem formulations; the development of a discourse in which differences in perspective were welcomed; the use of focussing questions in scientific argument; and the encouragement of reflective discourse for peer and self assessment.

One key practice was referred to as "start stop-go". Pupils were asked to read a problem, think in silence for a few minutes, and then discuss possible plans in small groups. The teacher then led a brainstorming session which focussed attention on key features. After the class had started work they were stopped at intervals and groups were asked to report on progress to encourage monitoring.

What were we trying to achieve?

These classroom practices were taken as the basis of a project to develop and evaluate a mathematical thinking skills course. The thinking skills targeted were metacognitive rather than cognitive. That is the course focused on the processes rather than the content of mathematics in the context of practical problem solving and modelling. It was hypothesised that *near transfer* would be found, meaning that students' metacognitive skills would be enhanced leading to improved performance in modelling problems. It was further hypothesised that the development of metacognitive skills would lead to improved mathematical thinking and that students would subsequently demonstrate *far transfer* through improved performance in the content areas of mathematics which had not been targeted by the course.

How do students learn to think mathematically?

Learning to think mathematically is more than just learning to use mathematical techniques, although developing a facility with the tools of the trade is clearly an element. From a constructivist viewpoint, the learner is considered as a sense-maker and an active negotiator of meaning and a distinction is often drawn between mathematical thinking and the knowledge base, strategies and techniques described as mathematics. Additionally, however, mathematical thinkers acquire a socially accepted way of seeing, representing and analysing their world, and an inclination to engage in the practices of mathematical communities (Schoenfeld, 1994, p60).

Cobb et al (1997, p269) claim that a "mathematical disposition" may be developed in an indirect manner through participation in "reflective classroom discourse". In reflective discourse, teachers should manage the interplay of social norms and patterns of interaction to create opportunities for pupils to reason for themselves and "engage in reflective thinking or reflective abstraction" (Wood, 1996, p102-103).

One of the issues which arises is the extent to which the teacher acts as a genuinely, neutral moderator of discussions amongst co-participants or as a director and guide of pupils' learning. There is an obvious power imbalance between teachers and pupils in classrooms and teachers' comments carry great weight. What is significant is the manner in which the power is expressed in action (Cobb et al, 1992, p486).

Two different forms of interaction can be described as scaffolding to support learning: *funnelling* and *focusing* (Bauersfeld, 1988; Wood, 1994). In *funnelling* it is the teacher who uses the thinking strategies and carries out the demanding tasks to lead the discourse to a predetermined solution. The social processes of the classroom hide the mathematical structure, which the pupil may only construct by choosing to reflect on

regularities in the actions performed. "Context- and problem-specific routines and skills" are likely to result (Bauersfeld, 1988, p37). Mathematical logic and meaning are replaced by the social logic and meaning of the interaction. In *focusing* the teacher's questions draw attention to critical features of the problem which might not yet be understood. The pupil then has to resolve perturbations which have thus been created (Wood, 1994, p160).

The discourse mode of teaching may lead to higher levels of understanding and thoughtfulness in mathematics. In "reflective discourse", teachers "proactively support students' mathematical development" by guiding and initiating shifts in the discourse so that "what was previously done in action can become an explicit topic of conversation" and thus "participation in this type of discourse constitutes conditions for the possibility of mathematical learning" (Cobb et al, 1997, p264 - 269). We intended the character of the discourse to lend social status to "the disposition to meaning construction activities" which is a "habit of thought" that can be learned (Resnick, 1988, p40).

The research design

The project mixed quantitative and qualitative methodologies. In a quasi-experiment 314 students aged between 11 and 13 from six schools followed a mathematical thinking skills course and were compared with matched control groups using pre-tests, post-tests, delayed tests and structured interviews. However, as the activities and teaching approaches were novel for most of the teachers involved, the project had many of the characteristics of action research.

The teachers of the experimental classes attended an induction day and then 4 action research network meetings spread throughout the five months of the intervention at which teaching experiences and progress were discussed. Thus statistical data collected from the written tests were supported by qualitative data collected during meetings, in interviews with teachers and pupils, and through participant observation by the researchers in experimental lessons.

Written test papers were designed to assess pupils' cognitive and metacognitive development based on a neo-Piagetian structure. The metacognitive skills of question posing, planning, evaluating and reflecting were assessed through a section in the written paper entitled "Planning and doing an experiment". Metacognitive self knowledge was also assessed by asking students to predict the number of questions they would get correct *before* and *after* each section (referred to here as *forecasting* and *postcasting*). Fuller details may be found in Tanner & Jones, 1995 or Tanner, 1997.

The overall results

Multivariate analysis of variance (MANOVA) was used to analyse the three levels of test (ie: pre, post and delayed) and two types of class (ie: control and intervention). A covariate approach (using pre-test scores as covariates) was used to add power to the analysis by adjusting for the small inequalities which existed between groups at the start of the quasi-experiment. For simplicity, only the multivariate results are given here (Table 1) (but see Tanner & Jones, 1995 or Tanner, 1997 for further details).

In assessments of the active metacognitive skills of planning, monitoring and evaluating both the control and the intervention classes improved over the period of the quasi-experiment. However the intervention classes improved more than the control classes in the post-test and this improvement was sustained in the delayed-test. The effect size was small (0.19), but significant at the 0.1% level for the written tests (Table 1).

As these active metacognitive skills had been taught in similar practical mathematical modelling contexts, such "near transfer" might be considered unsurprising. Its achievement was non-trivial, however, as the pupils were required by the assessments to form their own problems within open situations, plan, identify and control variables, choose simple strategies, monitor their work, collect and organise their data, find relationships, evaluate and reflect on their results.

In the assessments of Passive metacognitive knowledge or "knowing what you know" the results were not clear cut. Although the intervention classes improved in their forecasting more than the control classes in the post-tests and this improvement was sustained in delayed-testing, the effect of type of class was not significant at the 5% level. The postcasting of the intervention classes improved more than the control classes in the post-tests and this improvement was sustained in delayed-testing. This was significant beyond the 5% level but was limited to an extremely small effect size of 0.02 (Table 1).

Table 1: Multivariate tests of significance for the effect of type of class

Variable	Hotellings	F value	Hypoth DF	Error DF	Sig of F	Effect size
Metacog skill	.235	43.67	2	371	.000	.191
Forecast	.013	2.30	2	363	.101	.013
Postcast	.022	3.78	2	341	.024	.022
Cognitive dev	.021	3.89	2	369	.021	.021

The assessment of mathematical cognitive development showed a similar pattern to the active metacognitive skills but a smaller effect. The intervention classes improved more than the control classes in the post-test and the advantage was largely sustained at the delayed-test but the effect size (0.02) was extremely small (Table 1).

The content of the cognitive section of the written paper had not been taught directly by the course. Furthermore, as the intervention classes had experienced less teaching in their normal mathematics curriculum during the quasi-experiment, it might have been expected that the control classes would generally have outperformed the intervention classes. This small overall effect is claimed, therefore, to be an example of mathematical thinking skills paying for themselves through far transfer.

Although the overall results show a positive result for the intervention, albeit with a small

size of effect, when the results of individual classes were examined a far more complicated pattern emerged. It became clear that some classes had made very large gains over their controls whilst others had achieved little more or even less than their controls. Further analysis was conducted in the light of the qualitative data collected.

The four teaching styles

Analysis of the qualitative data collected through participant observation led to the classification of the teachers into four characteristic groups according to the teaching styles employed (see Tanner, 1997 for detailed descriptions).

The *taskers* focused on the demands of the task rather than the targeted metacognitive skills. The *rigid scaffolders* focused on planning, but rather than helping pupils to develop their own plans, aimed to develop the teachers' preferred approach. The scaffolding support provided by their questioning constrained pupils' thinking, funnelling them down a pre-determined path. These two groups of teachers were the least successful. The *taskers'* classes showed no advantage over their controls in any test. The *rigid scaffolders* showed an advantage in only the metacognitive delayed test with a very small effect size (0.09). This paper focuses on the other two groups of teachers.

The *dynamic scaffolders* made full use of the social structure of "Start-stop-go" to frame their pupils' behaviour and constrain them to act as experts rather than novices. Their scaffolding was dynamic in character and was based on participation in a discourse in which differences in perspective were welcomed and encouraged. The most significant participant in the discourse was the teacher, who validated conjectures and used focusing questions to control its general direction ensuring that an acceptable whole class plan was generated. The participation framework was equivalent to "legitimate peripheral participation" within an apprenticeship model of learning (Lave &Wenger, 1991), and the autonomy and responsibility of the pupils was limited by the teacher's desire to negotiate a plan to a pre-determined template. Although a whole class plan was developed, the negotiation and participation was sufficient to ensure that students referred to "our plan" in terms which indicated a sense of ownership. The discourse focused on both procedural knowledge and conceptual knowledge and during the planning and monitoring sessions, articulation and objectification of explanation was encouraged, making the explanation itself the object of the discourse. This was the only form of evaluation or reflection used by the *dynamic scaffolders*, however, and is characterised as "reflection in action" as opposed to "reflection on action" (Schön, 1990).

The *dynamic scaffolders* were very successful in accelerating the development of the active metacognitive skills of planning, monitoring and evaluating in the context of mathematical modelling, that is in near transfer, with a small to medium effect size (0.36) which was significant beyond the 0.1% level (Table 2). However they failed to achieve a significant advantage for their classes in either passive metacognitive self knowledge or "far transfer" into the content areas of mathematics. It is conjectured that, although active metacognitive skills may be necessary in the learning of new knowledge, they are not sufficient. Metacognitive self knowledge may also be necessary for far transfer.

Table 2: Multivariate tests of significance for the effect of type of class for *dynamic scaffolders*

Variable	Hotellings	F value	Hypoth. DF	Error DF	Sig of F	Effect size
Metacog skill	0.550	21.27	2	77	.000	0.36
Forecast	0.040	1.52	2	76	.226	0.04
Postcast	0.015	0.53	2	70	.590	0.02
Cognitive dev	0.020	0.80	2	78	.453	0.02

The reflective scaffolders also used the social structure of "Start-stop-go" to lead the discourse in their classrooms. They granted their pupils more autonomy, however, encouraging several approaches to the problems rather than constraining the discourse to produce a class plan. Pupils thus had to evaluate their own plans in comparison with the plans of other groups in the posing, planning and monitoring phases of the lessons. The participation framework had fewer of the characteristics of apprenticeship, with pupils taking a greater responsibility for an end product of their own design rather than limited responsibility for an element in the design of a "master". The characteristic feature of the *reflective scaffolders,* however, was their focus on evaluation and reflection. During interim and final reporting back sessions, scientific argument was encouraged to make the explanation an object of the discourse. Peer and self assessment was encouraged through group presentations of draft reports before redrafting for assessment. They deliberately generated a reflective discourse (Cobb et al, 1997) after activities to encourage self evaluation and reflection on process. Collective reflection does not equal reflected abstraction, but it is conjectured that during collective reflection, opportunities arise for pupils to reflect on and objectify their previous actions as they engage in reflective discourse (Cobb et al, 1997).

The *reflective scaffolders* were very successful in accelerating the development of active metacognitive skills, achieving near transfer in practical modelling situations with a medium size of effect (0.4) which was significant beyond the 0.1% level. They also succeeded in accelerating the development of passive metacognitive self knowledge in forecasting and postcasting. The effect sizes were very small (0.07 and 0.14), but significant beyond the 5% and 1% levels respectively (Table 3). They were the only group of classes to achieve this and it is conjectured that this was due to their emphasis on self evaluation and reflection. The *reflective scaffolders* also succeeded in accelerating development in the content domains of mathematics measured by the cognitive test, again the only group of classes to achieve this far transfer. The size of effect was small (0.21), but was statistically significant beyond the 0.1% level and approximated to a year's development (Table 3).

Table 3: Multivariate tests for the effect of type of class for *reflective scaffolders*

Variable	Hotellings	F value	Hypoth. DF	Error DF	Sig of F	Effect size
Metacog skill	0.652	43.34	2	133	.000	0.40
Forecast	0.073	4.73	2	130	.010	0.07
Postcast	0.161	9.82	2	122	.000	0.14
Cognitive dev	0.272	18.09	2	133	.000	0.21

Discussion

The "Dynamic scaffolders" were operating a model of cognitive apprenticeship which included authentic tasks, student autonomy and dynamic scaffolding but although they were very effective in teaching for near transfer, they failed to achieve far transfer. Individual construction was subordinated to the dynamics of the apprenticeship model, whereas the *reflective scaffolders* encouraged both cognitive apprenticeship and individual construction. The dynamics of "Start-stop-go" were internalised through participation in social processes. Learning this procedural knowledge was achieved through an apprenticeship model, organised and controlled by the teacher. Through participating in a scientific discourse led by an expert using dynamic scaffolding, pupils learned to internalise the processes of scientific argument and "argue with themselves". They also learned that mathematics "made sense" and that they could "make their own sense" of what occurred by making their own tentative conjectures and constructions and linking them with prior schemata.

Participation in reflective discourse encouraged reflective abstraction and the objectification of explanation. It is conjectured that the processes of mathematisation and problem solving, once objectified through individual construction, were thus knowledge rather than mere information and thus capable of being used elsewhere. Collective reflection provided both a social model and an opportunity for reflected abstraction. The reflective discourse focused on the processes of mathematisation, abstraction and generalisation in the service of understanding structure. Participation in reflective discourse may have encouraged the development of a mathematical disposition or point of view (Schoenfeld, 1994; Cobb et al, 1997) which might be expected to be transferable.

The active metacognitive skills of planning, monitoring and evaluating are generic rather than specific and these skills can survive near transfer to similar modelling contexts. It is conjectured that participation in reflective discourse can encourage objectification and the development of metacognitive self knowledge thus enhancing the transfer potential of such skills. It is further conjectured that the combination of active metacognitive skills and passive metacognitive knowledge supports both the application of old mathematics to new contexts and the learning of new mathematics.

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Learning to Question: A Major Goal of Mathematics Teacher Education

Dina Tirosh, Tel-Aviv University, Tel Aviv, ISRAEL

Abstract. *Mathematics education departments make efforts to revise their teacher education courses in response to recent documents that call for dramatic reforms in the teaching and learning of mathematics. This paper briefly describes the "Student Thinking About Rational Numbers" (STAR) course, it then analyzes three episodes drawn from the beginning, the middle and the end of the course, focusing on transitions in prospective teachers' understanding of mathematical procedures, shifts in their conceptions of what counts as an acceptable mathematical explanation, and developments in their awareness of the strengths and the limitations of practically -based and mathematically-based explanations. Finally, some implications for mathematics teacher education are drawn.*

Prelude

Janet, a beginning elementary fifth grade school teacher, described the standard multiplication-of-fraction algorithm in her class, and asked her students to work out several examples. Some minutes later one of the most talented students asked:
David: It is so easy to multiply fractions. I just multiply the tops and the bottoms. I don't understand why we can't add fractions in a similar way.
Janet demanded some clarifications:
Janet: I don't understand your question...
David: For instance, if I have to add ½ and ¼, I can add the 1 and the 1, and then the 2 and the 4, instead of doing all the complicated things with the common denominator.

Janet interpreted David's question to indicate that he still faced difficulties adding fractions. She immediately reacted:
Janet: If it is still hard for you, David, we should all do some more examples...
David: I know how to add fractions. My question is why we do it in such a way. It is strange that addition is complicated while multiplication is easy.

Janet was quite shocked. She certainly agreed that adding fractions was much more demanding than multiplying them. Yet she regarded these standard algorithms as self-evident and incontrovertible. After class, Janet discussed her "shocking experience" with two of her experienced colleagues, and they both confirmed that such a question is rarely asked in mathematics classes. Janet reckoned that David might ask the same question in the next meeting, and believed it was her responsibility to come up with an appropriate answer - but she was unable to come up with any. She did not know what to do.

Recent Reforms Encourage Students to Ask Questions

How will Janet proceed? What will she do? Let's leave her for a while, and discuss her colleagues' comment that such a question is uncommon. David's question, and many other "why-type" questions (e.g., Why ½ is bigger than ¼?, Why invert and multiply?, Why .2=.20?, Why is division by zero undefined?) are indeed rarely asked

in most classes. The traditional sequence of mathematics instruction, consisting of providing answers to the previous day's assignment, teacher's brief explanation of a new content, and working on a new assignment while the teacher moves around the class answering questions, fails to encourage students to understand why things are the way they are (e.g., National Council of Teachers of Mathematics, 1991).

A major recommendation that emerged as a result of the recent calls for dramatic reforms in the teaching and learning of mathematics is that teachers should welcome students' questions (e.g., Australian Education Council, 1991; NCTM, 1991). The professional standards for teaching mathematics, for instance, envisage mathematics classes in which students and teachers are actively engaged in exploring mathematical situations. According to these recommendations, teachers are expected to ask, and to stimulate students to pose questions such as "Why is it true?", "What would happen if..." What if not?" (NCTM, 1991). The professional standards and other reform documents recognized that this kind of teaching is significantly different from the traditional one, and that such changes require that teachers have long-term support and adequate resources. Those who participate in the preparation and the professional development of teachers are faced with many related issues, including: What knowledge will enable teachers to bring their instruction in line with the current recommendations for reform in instruction? What processes could foster change in teachers' conceptions of mathematics and mathematics instruction? This paper describes the "Student Thinking About Rational numbers" (STAR) method course, which was developed to address these issues.

The STAR Course

The goals of STAR clearly resonate with those set by the various reform documents: (1) encourage a view of mathematics as a human-made domain; (2) enhance prospective teachers' knowledge of mathematics; (3) promote prospective teachers' knowledge of students' conceptions; and (4) advocate a view of mathematics classrooms as learning communities in which individuals develop personally meaningful mathematical ideas as they participate in the interactive construction of mathematical meanings.

STAR is structured around a large data-base of activities aimed at developing prospective teachers' understanding of mathematical concepts, increasing their knowledge of children's conceptions, and enriching their repertoire of instructional strategies. These activities have been developed on the basis of the large body of recent knowledge regarding children's and prospective teachers' mathematical conceptions and ways of thinking about rational numbers. Most tasks included in the data-base are "why-type" questions, presented as if they were asked by elementary school students. Prospective teachers are generally encouraged to work in small groups, to suggest responses and to present them in class. This constitutes the basis for discussions on mathematical and pedagogical aspects of mathematics instruction.

A first version of this course was developed in 1994. Since then, the ways of thinking about and with rational numbers as they are revealed by prospective teachers during

the courses are sources of inspiration for constant modification of already developed activities and for the formation of new ones. Currently, the information related to an activity in the data-base is presented in three parts: a description of the activity; general description of related research findings; and suggestions for discussions that could originate from the activity. The data discussed in this article are from a class of 24 prospective elementary teachers who specialized in teaching mathematics in their second out of the four year teacher education program. Preprogram data suggest that before the course, the vast majority of the prospective teachers tended to perceive mathematics as consisting of rules and procedures that students should practice and memorize. I shall concentrate on three episodes that occurred at different times during the course and focus on transitions in the prospective teachers' views regarding mathematical concepts, and on the gradual development in their awareness of strengths and limitations of practically-based and mathematically-based explanations.

Activity 1: Can Fractions Be Added In An Easier Way?

It has often been reported that students tend to add fractions by "adding the tops and adding the bottoms" (e.g., Carpenter, Cobitt, Reys, & Wilson, 1976). The first task that the prospective teachers encountered related to this tendency:

Task 1. You are discussing operations with fractions in your class. In the course of this discussion, Ken says, "It is so easy to multiply fractions. I don't understand why we don't add fractions the same way: adding the tops and adding the bottoms"?

Participants were advised to discuss Ken's question in small groups and to present their responses in class. Here we focus on the first part of the class discussion, highlighting the negotiations between the participants on what counts as an explanation to this question (All names used for study participants are fictive):

Teacher: How would you respond to Ken's suggestion?

Sandra: I'll explain that adding fractions is different from multiplying fractions. I'll explain that when we add fractions, we do not add the tops and the bottoms, we have to find the common denominator.

Rachel: We decided to explain that it is impossible to add two fractions, unless they have the same denominator.

Mary: We would explain that in addition you must have the same family of fractions, but in multiplication it is not necessary.

Sandra, Rachel and Mary did not explain "why". Jane and Sharon commented on this:

Jane: I'm sure Ken will not appreciate these explanations. They do not tell why he can't add fractions in the way he suggested... I never thought about it... I add fractions the way I was told... It's the rule...

Sharon: I also never thought about the reasons for adding fractions the way we do. It's a strange question. It is also strange to think about adding fractions in a different way. I can only explain how to add fractions but not why, or why not.... Do students really ask such questions? That's scary!

There was general agreement with Sharon and Jane's remarks. The majority of the class acknowledged that they accepted the standard addition-of-fraction algorithm as a

rule and never questioned it. This was a turning point in the discussion. The participants started providing practically-based and mathematically-based arguments against adding the tops and adding the bottoms.

Karen: I will take a sheet of paper and demonstrate that when we take ½ of the paper and add to it ¼ of the same paper we get ¾ of the paper. But when we add the tops and the bottoms we get another sum: 2/6.

Anna: Your explanation shows that we get different answers when we use these different methods. It does not explain why the tops and bottoms method is not correct. I have another explanation [she wrote on the board]:

$2 \times \frac{1}{2} = 1$ and $2 \times \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$ but if we use the tops and bottoms method, we get $2 \times \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 2/4$, so we get a contradiction.

Betty: Karen also showed that if we use the tops and bottoms method, it leads to a contradiction. She showed that if we add ½ and ¼ in the tops and bottoms method, we get less than ½, and this is impossible. We add to½ a number that was not zero, and not negative, so the sum must have been bigger than ½.

Karen: You got me wrong. I meant something totally different. I used the sheet of paper because I'm sure children in elementary school need concrete examples. We can also make a cake in class, taking ½ kilogram of wheat and ¼ kilogram of sugar, weighting these quantities, showing that we get ¾ kilograms, and not 2/6.

The teacher then decided to draw the participants' attention to the profound differences between Karen's explanation and those of Anna and Betty, introducing the notions of practically-based and mathematically-based explanations.

Teacher: What is the difference between Karen's and Betty's explanations ?

Ruth: Karen uses concrete examples from daily situations. Betty did not do that.

Teacher: And what about Anna?

Anna: My explanation also does not use daily situations. I prefer my type of explanation, because I get confused with these illustrations.

Teacher: Can we come up with more examples of concrete, practically-based explanations? Can we also find other mathematically-based explanations?

In this episode we observed four types of prospective teachers' reactions to the "why-type" question that they were dealing with: explaining how (instead of why); rule-based arguments; practically-based arguments; and mathematically-based arguments. The participants demonstrated that the sum resulting from adding the tops and adding the bottoms is inconsistent both with their daily experience and with the definitions of other mathematical operations. Toward the end of this session, the notions of practically-based and mathematically-based explanations were discussed.

Activity 2: Can Fractions Be Divided In An Easier Way?

Division of fractions is often considered the least understood topic in elementary school, and children's success rate on these tasks is usually very low (e.g., Hart, 1981). Children tend to divide fractions by dividing the tops and dividing the bottoms (e.g., Tirosh, 1996). It is important to stress that when dividing fractions, unlike when

adding them, the standard "invert and multiply" algorithm and the "dividing the tops and the bottoms" method result in the same quotient.

The episode presented here took place twelve weeks after the course started. To put it in context, we should say that after working on addition of fractions, the class had dealt for several weeks with activities related to various models of mathematical operations. The twelfth lesson focused on intuitive models of division (see, for instance, Fischbein, Deri, Nello & Marino, 1985). The first task demanded the calculation of ½ : ¼ . One participant commented that it is strange that although multiplication and division are related, multiplication of fractions is so simple while division is so complicated. The instructor decided to take this opportunity and extend the discussion on definitions of mathematical operations. She requested the prospective teachers to consider this comment in groups and to report on the process they went through and their final conclusions.

All prospective teachers assumed that the quotient resulting from the "dividing the tops and dividing the bottoms" method would be different from that resulting from the application of the standard division-of -fraction algorithm. They tried "different types of numbers" and were confident that they would eventually find a "counter example". After about 10 minutes, Laurie commented:

Laurie: I think that in principle we can use the tops-and-bottoms method to divide fractions. We all tried many examples with different numbers, and always got the same results. So why do we have to invert and multiply? It is annoying to use the complicated way and not the easy one.

Laurie's comment suggested that she was quite confident that the tops-and-bottoms method is appropriate for dividing fractions. She based her decision on an accumulation of inductive evidence, involving comparison of results yielded from the method in question (dividing tops and bottoms) to results of a method that she regarded as correct (invert and multiply). She also expressed her view that the tops-and-bottoms method is much easier than the invert-and-multiply algorithm. Still, she appeared to feel obliged to obey a certain authority decreeing that division must be done in a certain way (invert and multiply).

Dana, however, presented a different opinion:

Dana: I think we can use this method in some cases, but in other cases we get stuck. I took ½ : ¼ . I divided the tops and got 1, but in the bottoms I got ½ . I couldn't go on and had to use the invert-and-multiply method.

The participants seemed to accept Dana's argument. Then, Victoria raised a "why-type" question which led to the following interchange:

Victoria: So we are stuck with the invert-and-multiply algorithm. I know how to do it, but I don't know why we do what we do.

Karen: We can use the measurement model. If we take ½ : ¼ we can think of how many ¼ make a ½ , and the answer is 2.

Anna: But, this does not explain why we first invert and then multiply.

Here, we see clear signs that Victoria differentiated between "describing how" and "explaining why". Her comment was made by the end of the lesson. For the next meeting, some participants brought to class a common teacher guide which suggested to use measurement problems and concrete materials to explain the invert-and-multiply algorithm. During this meeting, the prospective teachers worked in groups, attempting to understand the suggested explanation. After about 15 minutes, Anna commented:
Anna: This explanation holds only for some cases. It works only when the dividend is a multiple of the divisor.
Karen: I like the explanation in the teachers' guide. It illustrates the idea.
Anna: But it does not show why we can always invert and multiply.
Karen: It is always like that. When we teach that 1+2=3, we take one cube and two cubes to have three cubes. But, we can't do it with 1+ (-2).
Anna: We shouldn't show 1+2 with blocks. It will confuse the children later...
Betty: Right. Later they will think that 5-7 is 0, because there are no cubes left....
Karen: But what can we do? Children need concrete examples. They do not understand if we do not use something concrete...

A central issue in this snapshot is the legitimacy of using concrete materials that, due to their own properties (e.g., having physical dimensions) can potentially represent only some instances of a mathematical phenomenon. Karen conceived such concrete representations as essential tools that support the learning of mathematical concepts and procedures in elementary schools. Anna and Betty expressed their concerns about possible, long-term implications of the use of such models. Notably, accessibility to children and generality are two central issues in the research mentioned both for and against the use of practically-based explanations (e.g., Orton & Frobisher,1996). These argumentations were spontaneously raised by the participants as they reflected on suggested explanations of the invert-and-multiply method.

In this episode, the prospective teachers questioned the status of the invert and multiply method for dividing fractions which they had previously regarded as incontrovertible, and dealt with the possibility of using other methods to divide fractions. Gradual changes were observed in participants' conceptions of what could count as an explanation to "why-type" questions: "explaining how" and stating that "this is the rule that we have to follow" were no longer regarded as legitimate reactions, and practically-based explanations were evaluated on the basis of their generality and accessibility to children.

Activity 3: Can Zero Divided By Zero Be Zero?

Division by zero is one of the instances that violate the intuitive belief that every mathematical operation results in a numeric answer. Students and teachers tend to view this undefined mathematical operation as a "rule to remember" (e.g., Ball, 1990).

During the STAR course, a participant raised the issue of division by zero in the last session, which was planned to be devoted to general reflection on the course:

Teresa: I have learnt to ask why mathematics is the way it is. I still don't know why it is impossible to divide zero by zero.

Many of the participants wanted to react. Molly was the first:

Molly: I think 0:0 should be 1, because any number divided by itself is 1.

Teresa: But I can say that 0:0 should be 0 because 0 divided by any number is 0.

Anna: I know why 0:0 is not 1 [she wrote on the board]:

$\frac{0}{0}=1$; But: $\frac{0}{0}=\frac{0+0}{0}=\frac{0}{0}+\frac{0}{0}=1+1=2$. So $\frac{0}{0}=1$ and $\frac{0}{0}=2$, so 1=2 .

The class seemed deeply impressed. Then, Shelly reacted:

Shelly: So, it could be 2. Because if 0:0=2 then 2·0=0 and that's correct.

Betty: No. This will not work, we can do what Anna did before, showing that 1=2=3...So 0:0 could not be one. We should find why 0:0 is not zero--- it must contradict something...

The class worked for quite a while. Then Laurie commented:

Laurie: We should try harder. We should look in books and ask people. But, in my mind it is reasonable that 0:0=0. Is it possible that those who determined that 0:0 is impossible thought that it could be either 0 or 1, and did not notice what Anna did about 0:0=1?

This comment represents a dramatic change in Laurie's attitudes towards mathematics rules. By the end of the course, she was willing to doubt the validity of a well established mathematical definition and to consider its revision. Other participants seemed enthusiastic about this possibility too.

This episode suggests that by the end of the course, the participants tended to view mathematical statements as human-made, arguable conclusions, and not as absolute, immutable truths. Traditional explanations of mathematical rules were also challenged and reexamined, taking account of their possible, long-term negative effects.

Epilogue

Let's return now to the prelude, and see if participation in STAR could have helped Janet solve her problem which was to explain why fractions are not added by adding the tops and adding the bottoms. This specific issue was dealt with in STAR, and it is therefore reasonable to assume that had Janet been a participant in STAR, she would have approached the student's question by using one or more of the practically-based or mathematically-based explanations discussed in the course. Participation in STAR could have helped Janet and other prospective as well as practicing teachers answering "why-type" questions that were discussed during the course, but one would expect that the contribution of such a course be more profound.

The paper suggests that engaging prospective teachers in joint work on "why-type" tasks encourage them to reflect on their own understanding of mathematical concepts and structures and to acknowledge the discrepancies between "knowing that" and "understanding why". The paper illustrates that participants in the course gradually developed new sociomathematical norms regarding what counts as acceptable mathematical explanations: Rule-based justifications were no longer accepted as

legitimate explanations, and practically-based and mathematically-based explanations were pursued and examined in light of their accessibility to children, their generality and their possible long-term effects on students' mathematical conceptions and ways of thinking (Yackel & Cobb, 1996). Episode 3 demonstrates that by the end of the course, participants in STAR were willing to question well-established mathematical statements which at the beginning of the course were perceived as unquestionable. The description of these episodes should transmit not only the changes in the participants' conceptions of mathematical explanations, but also their deep involvement and enthusiasm on the part of the prospective teachers.

A word of caution should be added here. The first paragraph of this epilogue, argued that it is reasonable to assume that participation in STAR would have helped Janet (and others) in answering the "why-type" questions that were discussed during the course. Yet, so far we have not followed participants in STAR into their own classes, and thus at this stage of our study we lack the information needed to evaluate the program's effects on participants' own teaching. The next stage of our research will concentrate on studying the various effects of STAR on the actual teaching of the participating prospective teachers.

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Does the understanding of variable evolve through schooling?

María Trigueros
Departamento de Matemáticas
ITAM, México

Sonia Ursini
Departamento de Matemática Educativa
CINVESTAV-IPN, México

Abstract

Previous research has shown that even after several algebra courses starting university students still have serious difficulties in understanding the elementary uses of variable. Our concern now is to study how the understanding of variable evolves through schooling. For this purpose we conducted a study involving students aged 12- 18 years. Results obtained show that students' conceptions of variable do not substantially improve as more algebra courses are taken. We consider that students' difficulties are not of a cognitive or epistemological nature, but that they are a consequence of current didactical approaches.

Introduction

Many studies have been conducted to understand cognitive processes leading to the construction of the concept of variable. Special attention has been paid to students' difficulties at different school levels (Warren, 1995; Stacey and MacGregor, 1997, Boulton-Lewis et al., 1998). It has been found that even after several algebra courses starting university students still have serious difficulties in understanding elementary uses of variable (Ursini and Trigueros, 1997). Several questions arise from these findings. In particular we are interested in studying how the understanding of the concept of variable evolves trough schooling and in comparing the evolution of the different uses of variable to find out which, if any, is more strongly developed. In this article we present the findings of a study that addressed these questions.

Theoretical framework

Variable is a multifaceted concept. It includes as a whole different aspects. In order to reach a competent handling of elementary algebra students need to cope at least with what we consider the most relevant of them: unknown, general number and variables in functional relationship. An acceptable understanding of these aspects requires some basic capabilities as those stressed by Ursini and Trigueros (1997) in their decomposition of the concept of variable. This decomposition points to the importance of interpreting, manipulating and symbolising each one of those aspects when dealing with algebraic situations. They stress as well the value of being able to handle the variable as a mathematical object integrating its different aspects in one concept and shift between them in a flexible way.

Methodology

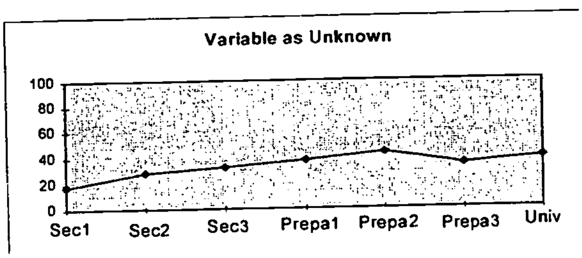
In order to study how the understanding of the concept of variable evolves through different school levels, the questionnaire designed by Ursini and Trigueros (1998) was used with 98 students aged 12-18 years. The distribution of students was as

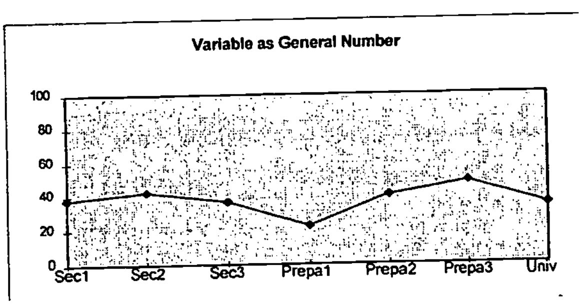
follows: 37 students in the three levels of Mexican middle education (12 – 15 years old); 30 students in the three levels of Mexican high school (16 –18); and 31 starting college students enrolled in social studies, business and economics majors.

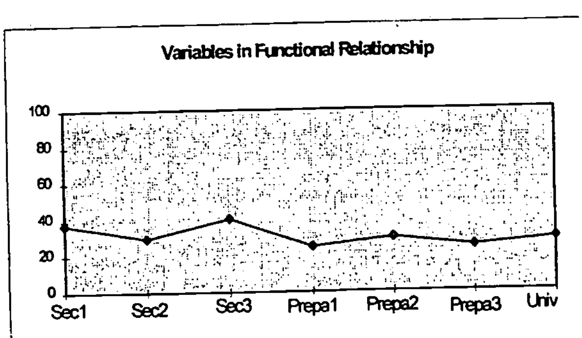
Students' responses were classified as correct, incorrect and not answered in order to make a global quantitative analysis per group and per student. The mean percentage of correct answers given by the students of each level was calculated in order to be able to compare them (Lozano, 1998). Based on the theoretical framework a qualitative analysis of students' responses was made, and some students were interviewed. This analysis provided a general overview concerning students' capability to work with different uses of variable and information concerning the evolution of this capability.

Results and Analysis

In the following graphs we present the mean percentages of correct answers given to the questionnaire by the students from different groups. In all the graphs the different groups are labelled as Sec1, Sec2, Sec3 for the first, second and third year of middle school; Prepa1, Prepa2, Prepa3 for the first, second and third year of high school; and Univ for the university group.







As can be observed in the graphs there is very little variation between the percentages of correct responses in the different groups. Moreover, the percentages scarcely exceed 40%. Overall these results show that during schooling students undergo a very slow evolution in their understanding of the different uses of variable. Even though a small raise in the curve for *variable as unknown* seems to indicate a slight evolution in students' understanding of this use of variable, a decrease in the curve for Prepa3 students shows that their understanding is not solid. This decrease might be due to the fact that at that school level students are not taking algebra courses but starting with Pre-Calculus or Financial Mathematics where variable as unknown is not emphasised any more. Regarding *variable as general number*, we observe that as students progress through algebra courses their understanding of this use of variable diminishes.

In the graph it appears as a decrease at Sec 3 and Prepa 1 levels. This result can be attributed to the emphasis given in those courses to procedures for solving equations. In contrast when students take other courses, like Analytic Geometry or Calculus (Prepa 2 and Prepa 3) the understanding of general number slightly improves. But it decreases again, as it can be seen, for college students. These fluctuations reveal, as in the case of variable as unknown, that their understanding of the use of variable as general number is fragile and not consistent enough.

Variables in functional relationship are not well understood by students at any school level. It can be observed in the corresponding graph that the percentages of correct responses are very low. It is surprising to realise that students who have not taken any algebra course do better in these questions. This fact raises the question of the role that courses play in the development of mathematical concepts. It seems that students are more able to use their imagination and other resources when they have not been exposed to formal algebra courses.

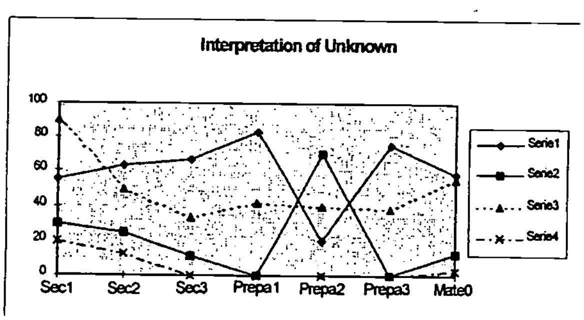
In order to gain a better understanding of students' capability to interpret, symbolise and manipulate each one of the uses of variable considered, their responses were analysed according to these dimensions. This analysis enabled us to develop a classification of responses for each of the uses of variable and have a deeper understanding of how students' conceptions change through their education.

Interpretation of variable

In the case of *variable as unknown* we found that students:

- interpret the symbolic variable in an equation as an entity representing values that can be determined (serie 1);
- interpret the symbolic variable in an equation as an entity that can take any value (serie 2);
- recognise in a simple problem, leading to a linear equation, the presence of an unknown that can be determined (serie 3);
- recognise in a problem leading to a quadratic equation, the presence of an unknown that can be determined (serie 4).

The following graph presents the evolution of students' interpretation of variable as unknown.



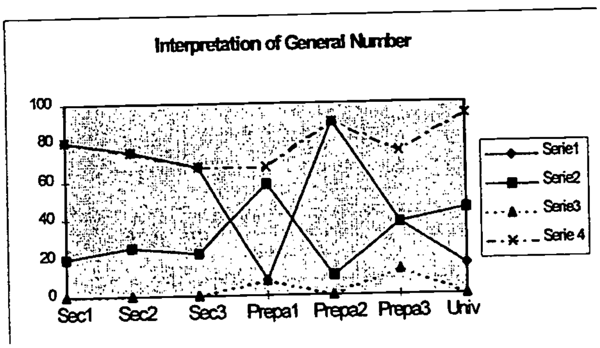
As can be seen in this graph interpretation of variable as unknown gradually improves while students take algebra courses (serie 1). This coincides with a gradual decrease of the interpretation of the unknown as representing any value (serie 2). However, when students start taking other mathematics courses in which

unknown is not emphasised and other uses of variable are present, they seem to experiment confusion. The interpretation of variable as representing an unknown value that can be determined decreases dramatically while its interpretation as representing any value increases. This provides evidence of the difficulty students have to deal with different uses of variable at the same time and the tendency to focus only on one of them at a time.

Even when working with very simple problems, students have strong difficulties in identifying and interpreting the unknown (serie 3 and 4). Again, it is amazing to realise that students perform much better before having been exposed to algebra courses (sec 1) and that their capability to interpret the unknown gets lost with schooling. A reason for this could be that algebra beginners, lacking in memorised algorithms and routines, face the problems without biases in a more intuitive fashion.

Regarding *variable as general number* we found that students:

- interpret the symbolic variable in an open expression or a tautology as representing any number (serie 1);
- interpret it as an entity representing a determined value (serie 2);
- interpret it as an undetermined object they can operate with (serie 3).
- can see a pattern (figure and/or number) (serie 4).

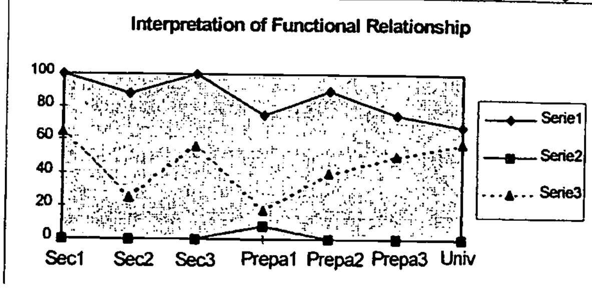


As can be observed in this graph when variable is used as general number students capability to interpret it correctly gradually decreases (serie 1), in spite of some fluctuations, while an incorrect interpretation of it as unknown increases (serie 2). We consider that this is a consequence of the emphasis given in algebra courses to variable as unknown. In accordance with the graph representing the interpretation of unknown, in Prepa 2 they recover temporally the capability to recognise a general number. However, it has to be consider that at this school level they tend to interpreted in that way any use of variable. In spite of many algebra courses students don't develop an understanding of the variable as an object they can operate with (serie 3). On the other hand in all levels they are acceptably good in seeing patterns (serie 4). This is considered (Mason et al., 1985) as a necessary step in the generalisation process, but they are not able to express it as it will be seen further.

For *variables in functional relationship* we found that students:

- recognise the correspondence between numerical quantities (serie 1);
- recognise the joint variation of related variables (serie 2);

* think of the functional relationship only in terms of discrete values (serie 3).

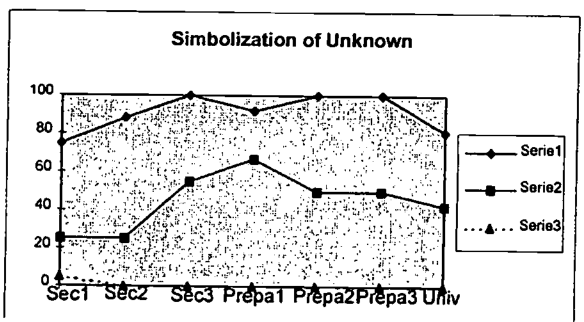


In general students can interpret related variables as representing quantities in correspondence whichever the representation used (table graph analytic expression) (serie 1). However, as it can be observed, this capability slightly diminishes during the school years. The interpretation of variation in terms of discrete values (serie 3), although fluctuating, persists in all school levels. Clearly absent in all the school levels is the capability to conceive the joint variation of related variables (serie 2). These results indicate that the school mathematics courses do not address in a satisfactory way the idea of joint variation, which is a fundamental notion when dealing with functions.

Symbolisation of variable

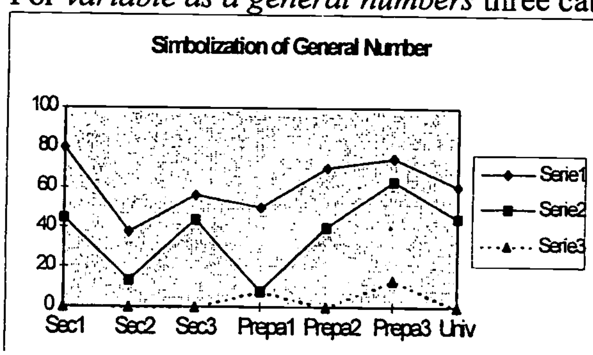
In the case of *variable as unknown* students' answers were organised in three groups:

* translate simple sentences to algebraic language (serie 1);
* pose an equation to solve linear problems (serie 2);
* pose an equation to solve quadratic problems (serie 3).



Almost independently of the algebra courses taken, students can translate simple sentences to algebraic language (serie 1). Students' capability to pose equations to solve linear problems (serie 2) gradually improves during the algebra courses (Sec 2, Sec 3 and Prepa 1) but it diminishes after these reaching prepa1) certain stability. At all school levels students are unable to pose equations to solve quadratic problems even when they are simple (serie 3). They have difficulties to identify the unknown of a problem and to link it with the data in a coherent way.

For *variable as a general numbers* three categories were found:

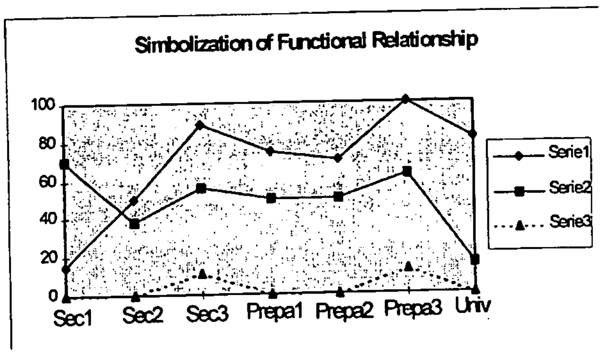


* translate simple sentences involving general number to algebraic language (serie 1);
* represent algebraically the general term of a numerical sequence (serie 2);
* represent algebraically the change involved when going from one step to the next in a numerical sequence (serie 3).

This graph shows a fluctuating capability to translate simple sentences to algebraic language (serie 1). Only in the case of very simple numerical sequences students can produce the general term, although there are strong fluctuations in the different groups (serie 2). The symbolisation of change involved in the generation of a sequence causes great difficulties at all school levels (serie 3).

In the case of *variables in a functional relationship* students were able to symbolise the relationship when:

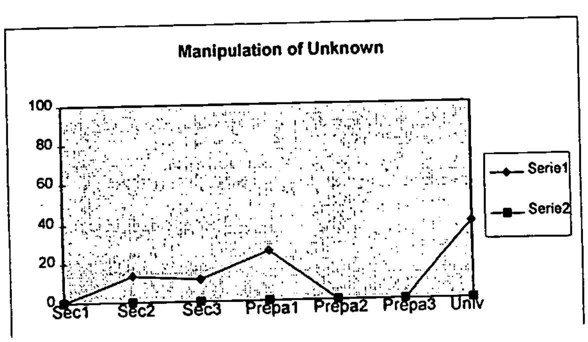
- a simple verbal expression was given (serie 1);
- a table was given (serie 2);
- when a verbal problem was presented (serie 3).



These results suggest that students learn to represent algebraically simple verbal expressions involving a functional relationship as they progress through algebra courses (serie 1). But this is not the case when they deal with verbal problems (serie 3). They have difficulties in recognising the joint variation of the variables involved and this seems to be an obstacle for symbolising a relationship. Again, it is amazing to find that younger students show a greater capability to symbolise information given in a table than university students (serie 2).

Manipulation of variable

The following graph shows students capability to manipulate *variable as unknown* when it appears in linear equations (serie 1) and quadratic equations (serie 2).



The lack of capability to manipulate the symbolic variable in any equation is striking. It is also evident that this capability almost disappears when students are not taking algebra courses, even though, as we have mentioned before, these courses strongly emphasise equation's solving.

In the case of *variable as general number* we classified students in two groups, considering those who could:

- simplify algebraic expressions (serie 1);
- expand algebraic expressions (serie 2).

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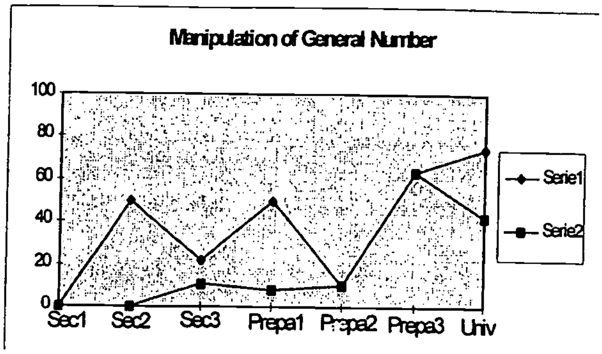
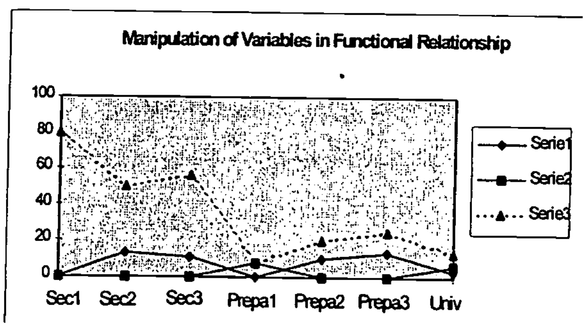


Students' capability to manipulate general numbers varies a lot in different levels. However, the general tendency shows an improvement in this capability. Moreover, it seems that it is easier for them to simplify expressions than to expand them.

For *variable in functional relationship* students' responses were grouped as follows:

- substitute a number to the independent variable in order to determine the value of the dependent variable (serie 1)
- manipulate an analytic representation of a function to determine variation intervals (serie 2)
- manipulate an analytic expression in order to produce its graph (serie 3)



It can be observed that all students have difficulties to manipulate variables in functional relationship. This is apparent both when it is necessary to determine values by substitution (serie 1) as when intervals have to be found (serie 2). These results are intimately related with the difficulty students have in dealing with related variation. A striking decrease in the percentages reflecting the capability to draw a graph was also observed (serie3). Once more students starting their study of algebra perform better than those in upper levels, in particular they are much better than starting university students.

We suggest that the difficulties students have in manipulating variables is tightly linked to their difficulties in interpreting and using variables. At different school levels they are taught manipulation but, without understanding its usefulness, they are forced to memorise techniques and to use them when they consider it appropriate. A competent use of manipulation requires the understanding of the meaning of the role played by the variable within a specific situation.

Concluding remarks

This results show that students' conception of variable does not substantially improve as more algebra courses are taken. They do not develop a comprehension of variable

as a multifaceted entity in spite of the presence of the different uses of variable in algebra courses. It is important to remark students' incompetence to discriminate between different uses of variable and how their interpretation of variable is strongly influenced by the specific topic being treated in class at a particular moment.

These results suggest that instead of promoting a deep understanding of variable and the development of intuitive algebraic ideas, current teaching practices seem to obstruct them. Also it is important to stress that errors committed by algebra beginners are not remedied by instruction and they prevail up to university levels. Students seem to develop only one of the aspects proposed by Bell (1996) as characterising algebraic thought, namely, resolution of arithmetic problems by step-by-step methods working from given data to unknowns or by global perception and use of multiple arithmetic relations.

We consider that these difficulties are not of a cognitive or epistemological nature, but that they are the result of current didactical approaches. We suggest that the different uses of variable, as well as their relationship, should be made more explicit to students. Variable is a fundamental concept in understanding mathematics and other sciences. A reconsideration of the way it is taught is needed.

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USING A HANDBOOK MODEL TO INTERPRET FINDINGS ABOUT CHILDREN'S COMPARISONS OF RANDOM GENERATORS

John M. Truran
University of Adelaide, Australia

Kathleen M. Truran
University of South Australia

The "Handbook Model" developed by the first author is used here to provide an integrated summary of research into children's comparison of random generators which is accessible to teachers but highlights the known complexities of individuals' responses. Linking psychology with pedagogy highlights the meaning of earlier, only partially successful, attempts to codify responses. Children's construction of probabilistic meaning in such situations must also take account of their inconsistent heuristics and approaches, the need for a generalisable non-numeric probability estimator, and the need to verify judgements in ways which allow for the uncertainties involved in any stochastic situation.

The Concept of a Handbook Model

J. Truran (1998) proposed using a "Handbook Model" for codifying research findings on specific topics and to contribute to the "model-building" stage of the development of "normal science" (Romberg, 1983) in mathematics education. The Model uses a prose summary to integrate findings within a hierarchy of contexts. It eschews a discursive style providing a general summary of reported findings in favour of the more segmented approach found in handbooks used within hard sciences. The hierarchy makes research findings readily accessible because each part of the text is tightly focussed, and the prose is an effective medium for expressing complexities. So the findings are more likely to be accessible to teachers with specific needs, and may more easily complement standard curriculum documents and textbooks. The Handbook Model provides one example of how mathematics education research might be expanded to encompass, not only psychological investigation, but also "teaching processes as an object of study as such, as well as the epistemology of mathematics from a teaching/learning perspective" (Balacheff, 1997).

Researching Children's Comparison of Random Generators (RGs)

Typically subjects are presented with two RGs and asked which they would prefer to use to obtain a desired outcome. Two seem to be sufficient for establishing the heuristics used—Spinolla (1997) used three and Ritson (1998) four, but without identifying new heuristics. Here we discuss only symmetric Two-Outcome Random Generators (TORGs), where each RG has finite numbers of favourable elements (called here "G", for "good") and unfavourable ones ("B", for "bad"). A typical example would be two urns: one containing 4 G and 3 B balls, and one containing 5 G and 4 B balls, summarised here as {4-3; 5-4}. The better choice (if there is one) is listed first and called "X", while the less good one is called "Y". Subjects know that "both are equally good" is an acceptable answer. Such cases are indicated by using a colon rather than a semi-colon, e.g., {2-3: 4-6}, and the smaller number of elements is listed first. We discuss here only comparisons between RGs of the same

form because children do not see mathematically identical RGs with different structures as being the same (K. Truran, 1994). For example, selecting a table-tennis ball from an urn containing six balls is seen as "different from [mathematically identical dice or spinners] because they roll everywhere and some could be in corners and you wouldn't know" (boy, Year 7, aged 12: 4). While not all of the methodological difficulties in this research have been addressed by all of the researchers discussed below, we consider that any methodological weaknesses are not sufficiently serious to invalidate the broad findings relevant to this paper.

Fischbein (1975, pp. 82–89) proposed that children might be making perceptual, not probabilistic, judgements. But J. Truran (1994) reported children's language which showed an awareness of the probabilistic nature of the task, and, for certain and impossible events (where proportional reasoning is not possible), that some children were moving towards formal mathematical language and the development of a probability scale. Furthermore, Cañizares, Batanero, Serrano & Ortíz (1997, p. 50) observed heuristics reported in the stochastics literature but not in the proportion literature, such as the "equiprobability bias"—two outcomes are seen as equiprobable regardless of the RG—(Lecoutre, 1992) and the "outcomes approach"—making a prediction about a single outcome rather than a statement about a probability function—(Konold, 1989). So comparison of probabilities may well require proportional or perceptual skills, but it also requires stochastic understanding.

Focus of This Paper

This paper examines, so far as space will allow, research into children's comparison of probabilities to develop conclusions which have sufficient generality to be useful for classroom teachers. Its purpose is not to construct a quantitative meta-analysis of the form described by Glass, McGaw & Smith (1981). Rather, it may be seen as a qualitative equivalent with a specific focus—using research to enhance pedagogy.

Summary of Research Findings

In Chapter 6 of Piaget & Inhelder (1951/1975) children's responses to such situations are classified within a developmental hierarchy based on their skill in co-ordinating the various elements of the situation (the numbers of each colour and their relevant totals), especially their ability to relate parts with wholes. Many similar experiments have followed from this, using three types of analysis.

Classifying Comparison Heuristics

The most common approach has been to determine the heuristics employed and to use some measure of complexity to arrange them in order. In general, the hierarchy has started by assessing children's ability to co-ordinate the parts and wholes of each RG, with higher levels based on whether their comparisons were additive (e.g., $a - b$ v. $c - d$) or multiplicative (e.g., a/b v. c/d). Freudenthal (1978, pp. 293–294) suggested that a/b v. c/d is cognitively easier than a/c v. b/d, especially when dimensions are involved, but this needs further investigation. In any case, some multi-

plicative approaches cause particular difficulties when denominators are 0. The principal purpose of most researchers has been to establish a hierarchy to be used as a measure of development, rather than to explain how the development occurs.

Way (1997) has summarised and codified six of these summaries, three from research into proportional reasoning, and three from research examining the comparison of probabilities. She defined ten categories, and made reasonable matches between these categories and those suggested by other workers. However, because the approaches differed markedly in the number of categories they defined, the match was inevitably far from perfect.

Matching Situations with a Cognitive Scale

An alternative approach has been to rank questions into a hierarchy, rather than responses, to claim that a subject who has correctly answered all questions up to a certain level has reached a certain stage of cognitive development. For proportion, this approach has been carefully developed by Noelting (1980) using a battery of comparisons covering "all possible variations of the particular situation" (p. 218) each of which was matched with a particular Piagetian level.

Noelting's approach has been taken up by Cañizares et al. (1997), but with a much more restricted range of questions. They found some conflict with Noelting's findings, possibly from the slightly different forms of their questions, and possibly because their stochastic setting brought subjective responses into play which would not have arisen in Noelting's deterministic problems.

Matching Heuristics with a Cognitive Scale

Finally, Watson & Moritz (1998) have used written reasons for responses to problems to assign cognitive achievement levels within the neo-Piagetian SOLO Taxonomy (Structure of Observed Learning Outcomes), based on the level of co-ordination of the relevant features of a problem. The size and longitudinal focus of the survey (with testing every two years) make its results of special value. Only one comparison problem was used ({6-4: 60:40}) and significant group improvements were found as children moved from Year 3 to Year 7, and Year 6 to Year 10. However, 37% of the children did not increase their assessed cognitive level of response in four years, which matches Ritson's (1998, p. 211) finding that some heuristics are remarkably persistent over time. Watson & Moritz also found that those who did develop did so in a remarkably diverse set of ways.

The Problem of Inconsistent Heuristics

These three different approaches have yielded no consistent general conclusions. Their striking common feature has been the diversity of responses offered. "Success in difficult items was due to strategies valid for this problem, but not for the general case," observed Cañizares et al. (1997, p. 54); "children as young as 5 years possess a repertoire of strategies to select from in reaction to the type of ratio pairs presented to them," said Way (1997, p. 574). Our personal experience suggests that

the heuristics are chosen to minimise computational demands, as is suggested by the following examples from a Year 8 boy (aged 13: 7):

{2-3; 1-2}
Box X
Can you say why?
With box Y, for each G one there are 2 Bs and in this one it's, on one G there are 2 Bs and in the other one there is only one.

{2-1; 3-3}
Box Y.
Why box Y?
You've got equal chances in box Y and you are outnumbered in box X, no sorry I'm going for G, you've got more chance in X because there's more of them.

Because researchers tend to use small, familiar numbers, this hypothesis has not been examined in detail. Nor has the effect of replacing small numbers by larger and/or less familiar numbers. Yet Collis (1975) has shown that number size and ability to achieve "closure" are relevant for understanding real numbers, so are likely to be relevant here as well.

This inconsistency of response has several implications. Firstly, it strongly suggests that data from small numbers of questions with familiar numbers will be poor indicators of the level of cognitive development obtained. It is probably not possible to rank either responses or questions into a hierarchy. Certainly, Piaget's research team knew that their questions did not represent "a ladder of difficulties in the development of the theory of probability" (Folder "calcul. probab.", Box "Hasard", Archives Jean Piaget, Geneva).

Secondly, we need to be aware that children are often not conscious that their thinking is inconsistent as the follow example from a Year 8 boy (aged 12: 11) suggests:

{4-3; 5-4}
Both the same.
Why?
Because there is one more G in X than what you have B and one more G in Y than what you have B.

{2-4: 3-6}
The same again because you've got 4 B and 3 G, you've got 4 Bs which is double the two Gs and in Y you have got 3 G and 6 Bs so it is double the G again.
Is that the same sort of method that you were using on the question before that?
Yes.

Thirdly, researchers need to appreciate and investigate further the significance of this inconsistency for the classroom teacher. Working with proportion, Hart (1984) showed that specific teaching could help to eliminate additive comparisons, but the improvements tended to be ephemeral, possibly because of weakness in working with fractions. Given that there are so many heuristics employed, and that they tend to be employed sub-consciously, this result suggests that deliberate correctional teaching is unlikely to be successful.

The Value of a Graphical Probability Measure

To our knowledge, only one piece of research has tried to find an holistic way of overcoming inconsistency of approach. Acredolo, O'Connor, Banks & Horobin (1989) asked children to place a mark on a probability number line to indicate the probability of success in a stochastic situation, thus allowing comparisons to be made indirectly by examining the relative positions of the marks. As a result children concentrated more on the relative values of both elements, and probably also on their totals as well, and performed much better than had been reported in research involving standard comparison of proportions, perhaps because they better appreciated that holistic estimates were required. This seems reasonable on theoretical grounds: the procedures were non-verbal, did not emphasise specific component numbers, dealt with only one proportion at a time, and encouraged closure at a point on the number line. But this valuable finding, which builds on the tendency which children already have to examine all of the relevant data, does not provide them with any reason for not reverting to their intuitive approaches.

Difficulties in Verifying Judgements

In stochastics, providing counter-examples to encourage accommodation (in the Piagetian sense) is difficult, especially in classrooms, Because large numbers of tests are needed, especially for naïve students. We propose two games—race-track and gang-plank—which our experience has suggested may be helpful, but this hypothesis has not been tested rigorously.

The games are inherently interesting, capable of elaboration, and encourage comparison of RGs. In a basic race-track game, two or more players operate their chosen RG in turn, and move forward one square if they obtain the desired outcome. In a basic gang-plank game one player stands in the middle of a gang-plank with an odd number of squares where the outer squares represent the ship's deck and falling off into the shark-ridden ocean respectively. A positive outcome allows one step towards the deck; otherwise, it is one step towards the sharks!

We know that individual experiences can prejudice judgements (the "availability" heuristic) as may be seen in this quotation from a Year 9 girl (aged 13:8) a few minutes after she had (a) played two gang-plank games with 2G 1 B, and had lost the first and won the second, and (b) made 9 successive draws from an urn with 2G 1B and obtained 4G and 5 B:

{1-2; 2-5}
> Box X.
> Why?
> The smaller number seems to come up more. In box Y there's more B ones, there's a fair few more B ones than G ones and in box X there is still a better chance of coming up with a blue one.

{3-3; 4-5}
> Box Y.
> Why?
> Coz the lesser number seems to come up more and in box X they're both equal.

Just as Hart (1984) found the calculator to be helpful in reducing the cognitive load for working with fractions, so it is likely that computer simulation may be helpful for testing decisions made when comparing probabilities in either the standard form or in that used by Acredolo et al. (1989). Pratt (1998, ch. 8) has shown that the opportunity provided by a computer-world for rapid, repeated experimentation is helpful for changing some children's interpretations, but for some children the micro-world may not be seen as an exact model of the real-world.

So at this stage, research can suggest that computer simulations of RGs in a game context may be effective in providing counter-examples which will discourage the inconsistent use of different heuristics, but these suggestion still need to be tested.

A Handbook Model Summary of the Research Findings

Here, as was also reported in J. Truran (1998), we have shown that children's responses to stochastic situations are inconsistent. A Handbook Model which is directed at teachers who have to think on their feet requires providing a unifying structure for interpreting such behaviour. Research and theory suggest that the stochastic nature of RGs should be emphasised by using an interesting context where judgements may be tested many times, and suggesting holistic ways for estimating probabilities to by-pass the known inconsistent intuitive approaches.

As with J. Truran (1998), to save space, and to highlight the main themes, references are omitted and terms defined in the main part of this paper are not re-defined. Most may be easily deduced from the discussion above.

Comparison of Two-Outcome Random Generators (TORGs)

Deciding which of two TORGs to use to achieve a desired outcome in, say, a race-track or gang-plank game is valuable for investigating chance processes and for encouraging and then testing careful choices. Making a good choice in a situation like {a-b; c-d} normally uses skills which are related to fractions, ratio and proportion, but stochastic situations introduce extra constraints.

Children's Varied and Inconsistent Numerical Strategies

The probability scale is defined only between 0 and 1 inclusive, so some comparisons, such as {0-2; 0-4} involve values at its extrema. Such situations are useful for helping children to see the value of a probability scale defined in this way.

Many heuristics for making choices may be observed in classrooms, often from the same child. These choices are not fickle: they represent serious attempts to find a computationally easy solution, and hence vary with the numbers presented. The most common incorrect ones compare the absolute number of either Gs or Bs, or make subtractive comparisons like $b - a$ with $d - c$. Logically correct solutions used by children involve proportional comparisons using any of three basic approaches, viz., comparing $\frac{c}{a}$ with $\frac{d}{b}$, or $\frac{b}{a}$ with $\frac{d}{c}$, or $\frac{a}{a+b}$ with $\frac{c}{c+d}$. The first two of these fail for cases at the extrema of the scale if the denominator is zero; only the third is always appropriate, and is, of course, the standard way of measuring probability. It is the least likely to be used by naïve children because it involves both addition and division.

While it is useful for a teacher to recognise that all these heuristics might be used, it is even more important to realise that children do not use any one consistently. Rather they adjust their heuristics to the numbers involved, usually to achieve simple arithmetic, so the mere provision of a wide variety of a different situations is unlikely to lead by itself to children's developing a totally consistent and generalisable approach.

Possible Stochastic Strategies

Children may well use heuristics which arise specifically from the stochastic situation. These may include making purely subjective decisions, considering that both outcomes are equally likely (by analogy with coins), or thinking that they are being asked to make a prediction of outcome (perhaps based on previous experiments) rather than a long-term judgement based on the structure of the RG.

Problems of Verifying Proposed Solutions

Correct responses are hard to verify, because they may still lead to failure in an individual game, especially if the race-track or gang-plank is relatively short. Computer models can be an effective, time-saving aid here, but some children may not accept that they are true models of "real life" situations.

Pedagogic Implications

Many children do not develop sound comparison methods over time without assistance. Teaching designed to eliminate specific poor heuristics is unlikely to be of lasting value. Using computer models in "estimate-test-revise estimate" situations will often prove helpful, especially if the estimates are done by marking them on a probability number scale. This encourages the students to co-ordinate simultaneously all the relevant aspects of the problem. Race-games and gang-plank games can be good motivators.

Conclusion and Implications for the Future

We have shown that when comparing proportions in stochastic situations naïve children use a variety of approaches with little consistency. The "standard" probability approach is not intuitive, as implied by many curriculum documents, but may be encouraged with skilful teaching. We have presented a structure for conveying the research findings to teachers which sees psychological findings as relevant to the environment in which children construct their meaning of probability. Finally, by focussing on what pedagogues need from researchers, we have highlighted important gaps in the research findings, such as determining what classroom approaches are most likely to engender sound and stable understanding. Without decrying the value of pure research, we observe that such questions are more likely to be seen as worth funding by those who hold the relevant purse-strings.

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ARE DICE INDEPENDENT? SOME RESPONSES FROM CHILDREN AND ADULTS

Kathleen M. Truran
University of South Australia

John M. Truran
University of Adelaide, Australia

This paper examines some understandings of the concept of independence of dice by comparing the results of an Italian study with that of four Australian studies. Roughly comparable groups produced quite different results, and similar reasons based on control were used to justify quite different responses. Some implications of these findings are discussed, especially with respect to the place of intuition in probabilistic thought and the lack of understanding of probability held by some by pre-service teachers.

BACKGROUND

Independence is an important and difficult probabilistic idea. "People seem to find it hard to consider an event as separate and detached from a series of similar events in which it occurs" (Cohen & Hansel, 1955, p. 178). "The belief that successive outcomes of a random process are not independent is supposedly one of the most common misconceptions about probability" (Konold, 1989a, p. 203). Yet independence is frequently given inadequate attention by researchers and teachers. Most research into children's understanding of independence has examined predictions of the next outcome in a sequence (e.g., Green, 1986). But Fischbein, Nello & Marino (1991) examined independence from a different perspective, and here we present and compare results from some replications of this study.

SELECTION OF A SUITABLE THEORETICAL MODEL

There are two principal models we might have used. Piaget & Inhelder (1951/1975) described understanding of probability using conceptual levels and claimed that a full understanding did not develop until about age 12. Fischbein (1975), on the other hand, claimed that effective probabilistic intuitions may be identified in children as early as pre-school age. But Green, working within a Piagetian framework, suggested that because of the deterministic aspect of mathematics, sometimes for probability 'performance declines with age' (1982, p. 774). Furthermore, speaking of young adults, Konold, Pollatsek, Well, Lohmeier, Lipson (1993, p. 193) have observed "that incorrect reasoning frequently occurs … and that a subject can switch from correct to incorrect reasoning, while reasoning about what an expert would consider to be the same situation". Clearly neither of the principal models is totally comprehensive. Indeed, as Shaughnessy (1992, p. 485) has observed:

> [t]here are very few models of conceptual development in probability and statistics. … The difficulty with building models in research on stochastics is that if a model were to try to incorporate the results of all the different types of studies by mathematics educators and psychologists, that model would run the risk of being so complicated that it may be of no practical use either to researchers or to teachers.

We have chosen to follow Fischbein's approach using intuition because such divergent responses are found at different ages, and because of the important finding by Fischbein et al. (1991) that some children do not regard the simultaneous tossing of three dice as mathematically equivalent to tossing them one after the other. For these children, when random generators (RGs) are operated together they are seen to lose any independence they might have.

Interestingly, the eighteenth century French polymath and encyclopaedist, Jean le Rond d'Alembert, also believed that the outcomes from successive and simultaneous tossings would be different (Todhunter, 1865, p. 279). If such a view was held by a trained mind at the time when the theory of probability was being developed, it is not surprising that it is held by many naïve children today.

FISCHBEIN'S STUDY

Fischbein et al. (1991) gave a questionnaire to c. 300 children attending schools in Pisa, Italy, aged from 9–14 years who were in Years 4 & 5 (Middle Primary) and 7, 8 & 9 (Junior Secondary). This was to obtain "a better understanding of the nature and origins of some probabilistic intuitive obstacles" (p. 523) prior to writing materials for teaching probability in Italian elementary and junior high school classes. Some of the students had experienced a formal teaching programme in probability, others had not (p. 529); only those students not involved in the teaching programme are included in our analysis. The questionnaire was given in the children's usual classroom setting, and comprised 14 questions, including:

> Are you more likely to get five on each of these three dice by rolling one dice three times or by rolling all three dice together, or are they both the same? Can you say why?

FOUR REPLICATIONS

We report four replications of Fischbein's question. An exact replication was not possible because the precise wording of the question was not reported, but the wording for a parallel question for two coins was:

> When tossing two coins which result is more likely: to get 'head' with one coin and 'tail with the other, or to get 'head' with each of the two coins; or is the probability the same for both results? (p. 532)

Given the problems of any translation, and the fact that children's reasons are cited in the paper, we believe our reconstruction is sufficiently accurate to permit the comparison of results. It is not clear whether Fischbein's students were interviewed or wrote their reasons on the questionnaires—the latter seems the more likely.

The first two replications were administered to Australian primary school students aged from 7–12 years, as part of a larger study investigating children's perceptions of the behaviour of different, but related RGs, and whether they see the outcomes of such RGs as independent. The smaller of the two replications was a pilot study, but both groups produced remarkably similar results, even though the children came from several different schools. None of the children had experienced prior

teaching of probability. Data was collected from group tests administered in a normal classroom, with each RG demonstrated at the time the question was asked, and then left in the children's view while they considered and made their responses. They wrote onto prepared sheets, and a random set of students was interviewed soon after to clarify their understandings.

The other two replications were given to sets of Australian Pre-service primary teaching students, with ages ranging from 19 to c. 50, in 1996 and 1998. These are discussed after the discussion on school aged groups..

RESULTS FROM SCHOOL AGED GROUPS

While ages and year groups in both school-aged studies are not identical, we believe that there is sufficient similarity to make credible comparisons. We are much more concerned with understanding the reasoning of those involved than with precise age comparisons. But where there are gross differences between groups of approximately equal age, we believe it is appropriate to try to find the reason. The results are shown in Table I, which is followed by some points of clarification.

Table I
Percentage distribution of "Best way to get 3 fives on 3 dice"—School Students

	Fischbein Year 4 & 5 n = 102	Truran Year 5 n = 43	Truran Year 5 n = 135	Fischbein Year 7, 8, 9 n = 139	Truran Year 7 n = 36	Truran Year 7 n = 98
No answer	12	0	0	16	0	0
Both the same	50	6	5	63	19	16
Successive	23	67	64	14	63	59
Simultaneous	13	27	29	4	13	20
Other (unspecified)	2	0	2	3	0	4
Combination	0	0	0	0	5	1
	100	100	100	100	100	100

The 'No answer' response in the Italian study was not found in our study, and may have been due to the way the test was administered. 'Successive' refers to rolling the three dice separately; 'Simultaneous' to all three dice being rolled together. The term 'Combination' was used to refer to the response of one student in our study who specified tossing two dice together and then one alone. It was claimed that a better result would occur that way, because three dice together were "too many".

DISCUSSION OF SPECIFIC RESPONSES

Two features of the results are of special interest. The first is that both sets of Australian students in our study made very different responses from the Italian ones; the second is that different children used the same reason—control—for choosing the different responses, 'Both the Same' and 'Successive'.

Both The Same

Most Italian students selected 'Both the Same' as the correct answer while very few Australian students did so. This is surprising: as part of our larger study students had been given situations where the option '... or are they both the same?' was

always offered. Where appropriate, this option was increasingly chosen with increasing age, so we had not expected responses that indicated such apparent lack of confidence in the independence of RGs. This suggests that the unexpected Australian responses indicate a real, wide-spread confusion about the ideas involved.

We cannot offer an explanation for this difference. We note that the results from Fischbein's work with students who had received probability teaching (not included in our table), were not markedly different from this naïve group. We shall meet a similar unexpected difference in the results from the tertiary students.

It is clear that the 'Both the Same' response increases a little with increasing age, but the difference and the sample sizes are rather too small, and the older Italian group is too diverse, to draw many conclusions from this.

Successive

Nearly $\frac{2}{3}$ of both the Australian year groups chose the "Successive" response, and it is also the most frequently chosen of the incorrect Italian responses. The reasons for this are not conclusive even though all of our students offered a reason for their choice. Some claimed that they had more control over successive tosses; a common response was, "If you throw the dice together they rub against each other, and the numbers change". They believed each die had an intended trajectory and outcome which 'rubbing together' would affect, and inhibit, their control of the die.

> AK (Year 7, F, 11:3) Well because when you shake them in your hand they sort of get all jumbled, and if you just concentrate on one at a time you have a pretty good chance. But all three, it's sort of a bit harder to concentrate on all three getting five, because it's not really that you would get five like that.
>
> M-AN (Year 5, F, 10:5) I think toss them one at a time because they won't bump into each other. Like if you get a five and another one bumps into it, you'd get another number. So I think one at a time.

Some Italian children expressed the same views.

> Because the dice do not knock against each other and therefore do not follow diverse paths (Year 7).

Simultaneous

Far fewer chose this strategy, but those who chose it, both Australian and Italian, believed they could better exert a constant influence on all three dice.

> AD (Year 7, M, 11:6) Because then you'd throw them the same height and the same way, so there's a better chance of getting all the same numbers.
>
> KS (Year 5, F, 9:8) Well probably if you had them all on one [uppermost before they were tossed] it be easier to get all fives. So if you're holding them on one and then you throw them all together they'd come down on fives.
>
> Because the same force is imparted (Year 5).
>
> One can launch in the same way (Year 7).

It is clear that the majority of children who did not choose 'Both the Same' did so because they believed they could exercise some control over the dice. However, these children expressed two quite different judgements about how best to effect this control. Fischbein et al. (1991, p. 529) found a slight tendency for more junior secondary students who had been taught probability to offer 'Simultaneous' than for those who had been taught.

It is not surprising that children have offered 'control' as a reason for their choices. This phenomenon has been reported by many researchers (e.g., J. Truran, 1985). One group has written

> The ability to calculate proportions as such does not necessarily signify an understanding of probability. A realisation of the impossibility neither of controlling or predicting the outcome of the immediate event is crucial (R. & R. Falk & Levin, 1980, p. 183).

The phenomenon was also found in response to other questions during the Australian study (K. Truran & Ritson, 1997). But, as well, although some of the Australian students had also expressed belief that there is a force, beyond their control, which determines the eventual outcome of an RG or that the RG itself 'knows' the result the child wants and behaves accordingly (K. Truran, 1995), they did not offer such explanations for this question. 'Animistic' responses have been extensively studied (Wollring, 1984), but, to our knowledge, the influence of question form on the persistence of such an approach has not been investigated.

What we find surprising in these results is that "control" has been used to justify two totally different responses. Perhaps the common feature, as we shall discuss in more detail later, is that children do not appreciate that randomness arises from the interaction of a *large* number of inter-dependent events.

Tertiary Group

Of the two tertiary groups few students had completed any Year 12 academic mathematics course. Both groups were studying a methodology subject including a single lecture and an associated workshop on probability. The 98 group also did a small probability experiment as part of an assignment. Responses to the question from the 96 group were written during a subsequent workshop, and were two or three lines long. Responses from the 98 group formed part of an examination where a response of about 15 lines was suggested. No students were interviewed individually. The 96 group had a weaker mathematics background than did the 98 one: increased employment prospects are attracting academically better students to teaching.

Table II summarises these results, which are mainly from young adults, and are strikingly different from each other. The responses of the 96 group are very similar to both Australian primary school groups, with nearly $\frac{2}{3}$ opting for 'Successive'.

While the vast majority of the 98 group gave the "correct" answer, the majority of the "incorrect" answers belonged to a new category—'Ambivalent'. Such responses

typically began with 'Both the Same', then went on to choose a different response, which seemed to be the preferred response. Two examples are:

> It doesn't matter it is merely chance that you will land all three on five. However, if one was to throw each die individually on the same angle from the same height and with the same force. The chances of rolling all three of five are increased. (98).

> It does not matter which method you use as there will always be a $\frac{1}{6}$ chance of rolling a 5. However it would be easier to control the experiment if you tossed one dice at a time (98).

Table II
Percentage distribution of "Best way to get 3 fives on 3 dice"—Tertiary Students

	1996 n = 87	1998 n = 140
No answer	0	0
Both the same	25	81
Successive	57	6
Simultaneous	18	3
Ambivalent	0	10
	100	100

Because the 98 students were advised to write about 15 lines we perhaps received more revealing answers than from those using questionnaires with limited response space, or from relatively focused interviews. Given that the independence of RGs had been discussed in the class lecture, it is possible that the responses represent a "textbook" response, followed by a more personal one. Ritson has observed that primary students 'coached' for an important exam shortly after revert to their 'real belief', and this may be what is happening here (K. Truran & Ritson, 1997, p. 239).

Some reasons for the 'incorrect' responses were the same as given by the children. But others are of some interest, given that these students will receive no further formal instruction in probability, and will be teaching in schools in two years time.

A few students provided no reasons for their 'incorrect' response, but relied on dogmatic assertion.

> The best and correct way is to toss all three at once (96).

> It is better to throw three separately. Throwing three together lowers the odds as the chance lessens as you add each die (98).

One gave an example of the 'Combination' argument, again without reasons.

> It wouldn't matter if I tossed all three dice at once or if I tossed them one at a time. ... It is much easier to obtain two fives if you are throwing two dice at a time as the chance to obtain a five is quicker and more possible (98)

Many students, even those who made the 'Both the Same' choice, saw this question as an actual experiment rather than a theoretical situation, in a way which is reminiscent of the 'Outcomes Approach' defined by Konold (1989b). Their concern

was with illustrating the idea to a class, which some saw as impractical because of the "long time it would take"

Finally, for some the question seemed other-worldly. Two asked if it were a trick question, and several stated, 'usually you would do them one at a time anyway'.

These findings corroborate that of Watson (1995, pp. 120-121) who wrote that "the inclusion of probability and statistics in recent years has meant that many high school teachers find that they are not adequately prepared by their own education to teach these topics". But the findings also have things to say about the nature of probabilistic intuition.

CONCLUSION

This question, whose very structure discourages the use of animistic responses, has been particularly valuable in bringing some misconceptions further into the open. Its moderately complex structure suggests that it may be one of the situations which Green described, as mentioned above, where performance declines with age.

While it is probably true that some probabilistic intuitions exist at a very early age, the results from this question show that they do not transfer well into a moderately complicated, but not unusual, situation. It is less clear that the students possess ways of judging whether a generator is producing random outcomes, even, for the adults, five years of secondary schooling (which will have included some work on probability) have not clarified these misconceptions. Since deciding whether two RGs are independent is essentially a subjective process (J. & K. Truran, 1997, pp. 90–92), and since we can see that this skill is not always intuitively acquired by many children and adults, it seems that it needs more explicit attention in schools. The suggestion from Konold quoted at the beginning of this paper that it is a lack of understanding of independence which is the principal problem becomes more precisely that it is a failure to know how to identify random generators. The principal problem seems to be a belief that the initial tossing is more powerful than the subsequent large number of forces which impinge on the RG.

Fischbein et al. claimed that:

> The general idea is then, that the outcomes *can be controlled* by the individual. The mathematical, probabilistic structure has not yet been detached from the concrete circumstances and considered in its abstract generality (p. 530).

We have earlier written that

> [s]uch strong beliefs in the physical behaviour of dice frequently over-ride any understanding of independence. Not only do RGs have no constancy of existence in the mind of many children [and adults], but when they are operated together, they are seen to lose any independence they might have had (J. & K. Truran, 1997, p 94).

We would now argue that the issue is deeper still. For both children and adults an understanding of how to decide if an generator is random is a critical pre-requisite

for developing a sound concept of independence. Such a skill is probably not intuitive, but learned.

One final comment. When we set this question we assumed (from our own childhood experiences) that subjects would normally toss dice using a cylindrical shaker. It has become clear that many use only their cupped hands. The extent to which this influences their responses is not known, but needs to be investigated. because the shaker ensures that many forces operate on the dice without direct human intervention.

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Testing the Cultural Conceptual Learning Teaching Model (CCLT): Linkage between children's informal knowledge and formal knowledge

Wen Huan Tsai
National Hsin-Chu Teachers College, Taiwan
Tsai@nhctc.edu.tw

Thomas R. Post
University of Minnesota
Postx001@tc.umn.edu

The purpose of this research was to test the effects of teaching students arithmetic based on the Cultural Conceptual Learning Teaching Model (CCLT) and to create a learning environment for children to link their informal and formal knowledge of mathematics by using lessons based on their cultural activities. 625second graders in sixteen classes selected from four schools were involved in an experimental teaching design. Children who learned arithmetic based on cultural-conceptual activities scored higher than children who learned arithmetic based on nation-wide textbook in computation problems, word problems, and simulated problems. Children involving the CCLT group solved everyday task problems more flexibly, more accurately, and more efficiently than children in the control group.

Introduction

Several studies (Lave, 1989; Carraher, carraher, & Schliemann, 1987) have suggested that situation in which arithmetic problems occur play an important role in eliciting different types of strategies in solving problems. In-school situations are likely to elicit formal procedures, and out-of-school situations are likely to elicit informal procedures. There are growing evidences that schooling does not contribute to performance outside of school and that knowledge acquired outside of school is not always used to support in-school learning (Ginsburg & Allardice, 1994). Although this gap is evident, there are not enough researches trying to design a curriculum to bridge this gap. This study was to develop a learning-teaching model, the Cultural Conceptual Learning Teaching Model (CCLT), and tried to link children's formal knowledge and informal knowledge based on this model.

Developing the Cultural Conceptual Learning Teaching Model (CCLT)

Many researchers have suggested that the strategies of solving daily problems created by children could be used as a starting point to learn school mathematics (Bishop & Abreu, 1991; Hiebert, 1988). Children gain the informal knowledge through cultural activities of daily life. According to Resnick's view (1987), when teaching mathematics, teachers need to consider the cultural aspects that are meaningful ways for students to make sense of the abstract symbols of school mathematics. Brown, Collins, & Duguid (1987) also emphasize the importance of the relationship among activity, concept and culture. They suggest that learning must involve all of the three. Based on above points of view, this study develops a learning teaching model called the Cultural Conceptual Learning Teaching Model

(CCLT) (see figure 1). This model tries to combine individual, activity, concept, and culture together.

Figure 1 shows that CCLT contains three learning environments: construction environment, connection environment, and reapplication environment and six learning stages: play stage, construction stage, connection stage, re-application stage, practice stage, and reflection stage. Play stage provides some cultural-conceptual activities for children to do role-plays. In this stage, children share, take, negotiate, and construct their immediate experiences to solve arithmetic problems with peers and old comers (teacher or expert children). In the construction stage, the teacher designs a guide sheet that has structural objectives that need to be accomplished by students. In connection stage, based on children's experiences or strategies, teacher tries to connect children's experiences to mathematical symbols and procedures. In re-application stage, teacher provides another similar cultural-conceptual activity for children to re-apply the learned mathematical concept. In practice stage, children are provided some opportunities to practice school mathematics in everyday situations. In reflection stage, children are trained to monitor their thinking and to be aware of where and how they can apply school mathematics in their everyday activities. Authors call the learning model as Cultural Conceptual Learning Teaching Model (CCLT).

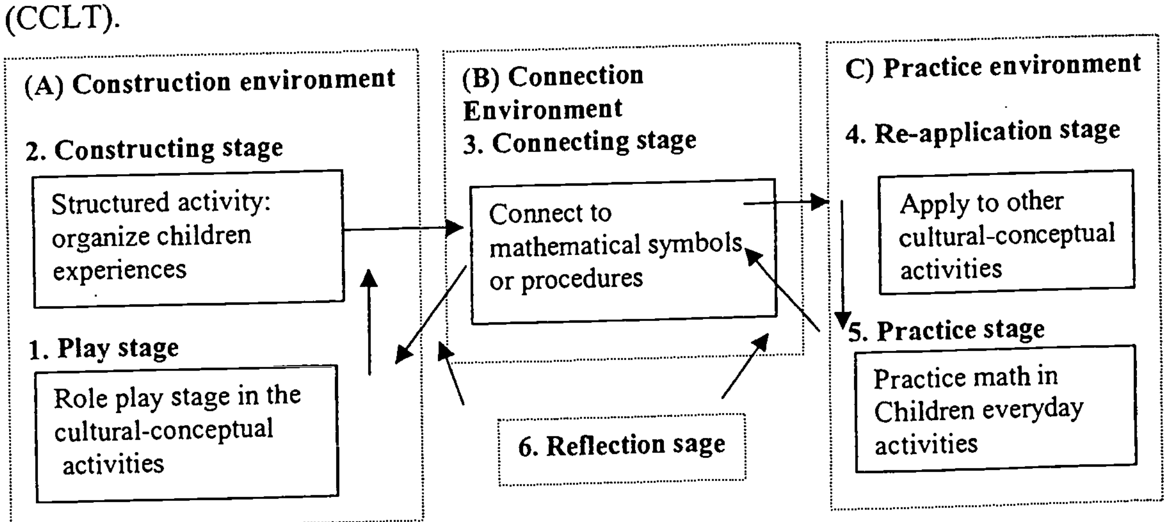

Figure 1: The Cultural Conceptual Learning Teaching Model (CCLT)

How to integrate children's cultural conceptual activities into classroom lessons and to improve children's learning of mathematics is the main purpose of this research. According to Piaget's view (1977), children's cognitive development depends on the cultural activities that are provided and the psychological development of children. Most children in elementary school are staying in the concrete-operation stage. Their thinking is closely tied to their immediate experiences. The environment of socioculturally organized activities in which children can learn is referred to as the zone of proximal development (Vygotsky). Therefore, there are two steps to develop the cultural-conceptual activities. In the first

step, the interview method is used to explore children's everyday cultural activities, social interactions, sign forms and cultural artifacts, and prior understanding, and to find out what kinds of children's cultural activities were related to the mathematical concepts of this study: three-digit place value, addition, and subtraction. In the second step, the cultural activities are organized according to those mathematical concepts. The conceptualized cultural activities were called the cultural conceptual activities.

According to Dienes' theory, mathematics learning contains four basic principles: dynamic principle, perceptual variability principle, mathematical variability principle, and constructivity principle (Post, 1992). This research modified the perceptual variability principle and the mathematical variability principle to design the cultural conceptual activities. Teaching activities are designed based on the dynamic principle and the constructivity principle. This study hypothesizes that children's learning arithmetic based on the cultural conceptual activities through the CCLT teaching model not only would improve their learning in school mathematics but also their abilities to solve everyday problem tasks.

Methodology

Sixteen-second grade classes in Hsin-Chu, a city of Taiwan, participated in this study. Half (n = 8) of the classes were assigned randomly to the treatment group and half to the control group. Teachers of the treatment groups participated in a one-week workshop, and met together on each Saturday within the period of study to design teaching activities. The researcher presented the interview data of children's everyday activities to the teachers of treatment group and designed the cultural-conceptual learning activities with their cooperation. There are five cultural conceptual activities in the CCLT model: Three activities of counting lucky money in an red envelope (New Year activity) and two activities of shopping and selling toys. The other teachers (n=8) were involved in a control group and participated in two 2-hour workshops focused on discussion of the arithmetic content of the national textbook.

Three different tests were used to get a broader understanding of the effects of teaching children arithmetic based on the CCLT model. They were Standardized Test (ST), Three Testing Conditions, and Interview Task Problems. The Standardized Test was developed by the Taiwan Provincial Elementary School Teacher Training and Research Institution. Three Testing Conditions contained Computation Problems, Word Problems, and Simulated Problems. The idea of testing conditions came from the research of Nunes, Schliemann, and Carraher (1994) but was redesigned and modified according to Ginsburg's (1982) classification of computation problems and Carpenter's (1985) classification of word problems. For testing the differences between the CCLT group and the control group in solving everyday arithmetic problems, students were required to accomplish the task problems. The task problems came from parts of Saxe's (1991) research problems and were modified by the researchers to fit the research questions of this study. There are two interview task problems: Identifying currency from a numeral (ICN) and identifying a numeral from

a pile of currency (INC). When students solved these task problems, their answers were recorded correct or incorrect and their strategies also were recorded. Each test, except the interview task problems, was taken by all. Four students were randomly chosen from four levels on the Standardized Test in each class as the target students to be interviewed. Three testing conditions and Interview task all contained three-digit and four-digit number. The three-digit problems were designed to test the teaching effect and the four-digit problems were designed to test the transfer of learning.

Results

Standardized Test

The Standardized Test was administrated before teaching and treated as the covariate for the other test. However, there is no statistically significant difference between the CCLT group ($M=28.14$; $SD=1.98$) and the control group ($M=29.66$; $SD=4.10$) on the Standardized Test ($F(1,12)=2.20$; $P=1.64 > .05$). Therefore, the Standardized Test can not be used as the covariate for comparing the CCLT group and the control group.

Children's Different Achievement in Different Teaching Contexts

This section describes the hypothesis being tested that the CCLT group would score higher on the simulated problems than the control group. On the other hand, the control group would score higher on the computation problems and word problems than the CCLT group. The hypothesis has been rejected. Table 1 summarizes the means and the standard deviations of three test conditions.

Table1: Means and Standard Deviations of Test Results on the Overall Test, Three Digit Test Conditions, and Four Digit Test Conditions.

Three test conditions:	Treatment		Location	
	CCLT	Control	Urban	Suburban
	M (SD)	M(SD)	M (SD)	M(SD)
Overall Test:				
Computation Problems (I=24)	10.38(.33)	9.73 (.81)	10.34 (.50)	9.76(.75)
Word Problems (I=24)	8.45(.62)	7.66 (.71)	8.26 (.90)	7.85 (.58)
Simulated Problems (I=24)	8.48(.60)	7.64 (.74)	8.27 (.74)	7.86 (.80)
3D Test:				
Computation Problems (I=12)	7.30(.21)	6.58 (.51)	6.98 (.30)	6.63(.51)
Word Problems (I=12)	5.58(.43)	5.0 (.48)	5.43 (.58)	5.16 (.47)
Simulated Problems	5.54(.43)	5.01 (.49)	5.33 (.51)	5.23 (.53)
4D Test:				
Computation Problems (I=12)	3.35(.17)	3.15 (.32)	3.36 (.25)	3.14(.25)
Word Problems (I=12)	2.87(.22)	2.56 (.27)	2.84 (.34)	2.69 (.13)
Simulated Problems (I=12)	2.94(.23)	2.62 (.31)	2.94 (.28)	2.63 (.27)

M = Means, SD = Standard Deviation, I= item. 3D =three digit numeral; 4D =four digit numeral.

Table 2 summarizes the analysis of variance of different test conditions taken by treatment groups and locations. On the overall test problems, the three- digit test problems, and the four-digit test problems, the CCLT group scores significantly

higher than the control group. Therefore, this study indicates that students who learn arithmetic based on their cultural activities improve their learning of both formal mathematics (computation problems and word problems) in school and informal mathematics (simulated problems) outside of school better than the students who learned arithmetic based on the traditional textbooks. Another impressive result is that the CCLT group does the transfer of learning more ably than the control group. This research does not address the issue of the main effect of three test conditions so the researchers did not do the follow-up t-tests. Although this research does not focus on this issue, it is interesting to note that children in urban school performed better on both school transfer problems and out-of-school transfer problems than children in suburban schools.

Table 2: Summarization of Analysis of Variance of Different Testing Conditions by Treatment Groups and Locations

Testing condition	F tests: Between T	L	T × L	Within C	T × C	L × C	T×L×C
Overall Test	9.81**	3.79	.95	86.11**	.14	.15	.76
3D Test	9.35**	2.05	1.17	108.36**	.13	.59	.20
4D Test	8.12*	7.35*	.40	31.67**	.38	.77	2.12

P < .05; ** p < .01; T = Treatment; L = Location; C = Test Condition; 3D =three digit numeral; 4D =four digit numeral.

Identifying the relation between numeral and currency values

Children learn counting and summing up money and the price of each item of merchandise from everyday activities. The interview task problems are to examine children's ability to deal with the relationship between numeral and currency value. This task consists of ten items. Each of the items deals with three-digit or four-digit number. Each item is written on one card. Children's strategies for solving the task that deals with the relationship between numeral and currency value are analyzed as follows.

(1) Children's strategies used in solving the ICN problems.

When presented a card, each child was asked to pick the currency up from piles of fake money to match the given number on the card. Children's responses are classified into two categories: correct and incorrect. Children's strategies used in solving the task were identified as formal method and informal method. For instance, when a child was given the number 567, he solved it by picking 5 hundred-dollar bills, 6 ten-dollar coins, and 7 one-dollar coins (written as $5 \times \$100 + 6 \times \$10 + 7 \times \$1$). This strategy is identified as a formal method. Other strategies were characterized as informal methods, such as one 500-dollar bill, one 50-dollar bill, one 10-dollar coin, and seven 1-dollar coins (written by $1 \times \$500 + 1 \times \$50 + 1 \times \$10 + 7 \times \$1$). Formal methods were presented in textbooks, but children are found to have a preference for using informal method to solve everyday problems.

(2) Children's strategies used in solving the INC problems.

Children's cultural activities frequently deal with counting money. A task includes ten piles of fake money. Each pile consists of coins and bills with various values. A child was presented a pile of fake money at one time and was asked to count the amount. In the INC problems, five are three-digit numeral and five are four-digit numeral. Students' correct answers and strategies used in solving the problems were analyzed. Four strategies used by children were identified: grouping, grouping with iterating, iterating, and others. *Grouping strategy* means that students regroup the currency by the same bill or coin denomination, and then regroup the currency by thousands, hundreds, tens, and ones and then report the sum according to one, tens, hundreds, and thousands. *Grouping with iterating strategy* means regrouping followed by counting the currency one by one from one pile to another pile. Without regrouping, children's strategy of counting the given money one by one is identified as an *iterating strategy*. When the strategies are not included in the three strategies, they are identified as *others*. For instance, students who don't know the way of counting, even if they do the regrouping.

(3) The results of solving two interview task problems

Table 3 summarizes the means, standard deviations of correct answers given and frequencies of the strategy used by students, and ANOVA analysis for between-group differences in identifying ICN problems and on identifying INC problems.

Table 3: Summary of Means, Standard Deviation, and ANOVA Results for Between-Group Differences On Identifying the Relationship between Numeral and Currency Values Test

Sub tests (number of items)	CCLT M (SD)	Control M (SD)	F tests
Identifying ICN problems			
3D Test (5)	4.97 (.25)	4.72 (.92)	1.675
formal method	1.16 (1.05)	2.91 (1.30)	34.945***
Informal method	3.81 (1.03)	1.78 (1.29)	48.527***
4 D Test (5)	4.49 (.18)	4.47 (1.22)	5.285*
formal method	2.22 (1.01)	3.66 (1.01)	34.929***
Informal method	2.75 (1.05)	1.19 (.93)	39.784***
Identifying INC problems			
3D Test (5)	4.43 (.79)	3.03 (.90)	38.673***
Grouping strategy	3.6 (1.22)	1.00 (.84)	62.171***
Grouping with iterating strategy	1.41 (1.10)	2.97 (.97)	36.317***
Iterating strategy	.47 (.51)	.94 (1.01)	5.471*
Others	.062 (.35)	.94 (.53)	.077
4D Test (5)	4.47 (.72)	3.22 (1.24)	24.433***
Grouping strategy	3.94 (1.34)	.94 (.95)	106.603***
Grouping with iterating strategy	.78 (1.18)	2.98 (1.1 8)	54.917***
Iterating strategy	.13 (.51)	.88 (1.13)	12.977**
Other strategy	.031 (.18)	.22 (.91)	.255

*P < .05; **P<. 01; ***P<. 001; 3D = three digit numeral; 4D = four digit numeral.

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Overall, children who learn arithmetic based on the cultural-conceptual activities perform significantly better than children who learn arithmetic based on the national textbook on four-digit problems of ICN, three-digit problems of INC, and four-digit problems of INC. No significant difference is found between groups in identifying three-digit problems of ICN. These results reveal that different teaching methods do not affect children in solving simple ICN problems but they do affect children in solving complicated larger number problems (four-digit problems). Different teaching methods also affect children's use of different strategies to solve the ICN and INC problems. Children in the CCLT group use informal methods to solve the ICN problems more often than children do in the control group. Conversely, children in the control group use formal methods more often than children do in the CCLT group. On the INC problems, children in the CCLT group use the grouping strategy more often than children do in the control group. Conversely, children in the control group use both the strategies of grouping with iterating and iterating to solve the INC problems more often than children did in CCLT group. These results reveal that different teaching methods lead children to using different strategies to solve their everyday problems.

Conclusion and Discussion

This study reveals that children who learn arithmetic through the CCLT learning model based on the cultural-conceptual activities score significantly higher than children who learn arithmetic based on the traditional textbook no matter what they solve computational problems, words problems, or simulated problems. Those results support Hiebert's suggestions (1988) that in school settings children's informal knowledge could serve as basis for the development of understanding of mathematical symbols and procedures.

Other evidence shows that there is no significant difference between two groups in solving the three-digit ICN problems, however,. the treatment group solves the four-digit ICN problems at a higher rate than the control group. In fact, most of the children in Taiwan are often given several hundred dollars by their parents to buy foods, drinks, or school supplies for themselves, therefore both groups can solve the tree-digit ICN problems easily, but few children are given thousand dollar bills to buy things. This result indicates that connecting children's everyday experiences with mathematical symbols and procedures can improve children's transfer of learning when they solve the four-digit ICN problems.

The CCLT group uses more efficient and flexible strategies to solve task problems as compare with the control group. One possible reason is that the CCLT program provided an arena for children to assimilate, accommodate, negotiate, and reconstruct their rich experiences between peers and teachers more frequently than the control group provided, and in this way children in the CCLT group increase their understanding more than students in the control group. In the cultural conceptual activities, children are full participants and act as "whole persons" to conduct their

business based on their prior understanding. This learning environment is very different from the traditional teaching based on the national textbook that follows the teacher's guide step by step. However, further studies need to consider how to increase the degree of students' participation in classroom and everyday activities. Students learning mathematics need to extend their learning to the entire social community, parent's involvement with children's mathematics learning, not just in school.

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PROSPECTIVE TEACHERS' ACCEPTENCE OF THE ONE-TO-ONE CORRESPONDENCE CRITERION FOR COMPARING INFINITE SETS[1]

PESSIA TSAMIR

Kibbutzim Teachers' College   Tel Aviv University

This paper examined the tendencies of 121 prospective teachers who had studied a Cantorian set theory course and 71 prospective teachers who had not, to declare that "1:1 correspondence", "inclusion" and "single infinity" are acceptable criteria for the comparison of infinite sets. It also investigated their tendency to accept illustrated comparisons conducted by 1:1 correspondence and the effects of these illustrations on the participants' acceptance of the three above mentioned criteria. The findings usually show significant differences between participants who had Cantorian set theory and those who had not. Prospective teachers who had studied frequently accepted 1:1 correspondence, both as a general method for comparing infinite sets and for specific comparison tasks. Yet, a substantial number of them declared that inclusion and single infinity are also suitable for comparing infinite sets. Moreover, many tended to reject the illustrated solution by 1:1 correspondence of an unusual comparison task,

The notion of actual infinity is a crucial one in mathematics and plays a major role in the theoretical basis of various mathematical systems. Still, its acceptance in the mathematical community was not smooth and simple. When Cantor introduced his theory of infinite sets late in the 19th century, a number of great mathematicians and philosophers, such as Poincaré and Gauss, fiercely objected to it. Moreover, even those who enthusiastically accepted the notion of actual infinity, like Cantor himself, Hahn, Russell, Hilbert, and Dedekind, highlighted its counter-intuitive nature: it goes against the grain of our everyday experience and of finite sets.

Indeed, research in mathematics education over the last two decades, has indicated that, when comparing infinite sets, students intuitively use a wide range of criteria, but generally neglect to use 1:1 correspondence for their comparisons (e.g., Borasi, 1985; Fischbein, Tirosh, & Melamed, 1981). Students' responses seem to be representation-dependent and influenced, for instance, by non-significant, visual aspects (e.g., Duval, 1983). Moreover, students reach contradictory conclusions by using various criteria, such as, "single infinity", (i.e., all infinite sets are equal), "inclusion" (i.e., a set that is included in another set has fewer elements than that set), and "incomparable" (i.e., infinite sets are incomparable). However, they usually are unaware of these contradictions (e.g., Tsamir & Tirosh, 1992).

Questions that arise are: how would an intervention that presents students with the Cantorian set theory, affect their tendency to accept 1:1 correspondence as the criterion for comparing infinite sets, and what kind of intervention should be used for this purpose?

[1] "Comparing infinite sets" here means comparison of the number of elements in infinite sets.

The common view of leading, 19^{th} century mathematicians who accepted the notion of actual infinity was that the problems rooted in its counter-intuitive nature could "easily" be overcome. Russell, for instance, claimed that:

> ... a little practice enables one to grasp the true principles of Cantor's doctrine, and to acquire true and better instincts as to the true and the false. The oddities then become no odder than the people at the antipodes, who used to be thought impossible because they would find it so inconvenient to stand on their heads.
>
> Russell, 1916/1988, Vol. 3, p. 1560

One may wonder how little "a little practice" exactly would be and how "true and better instincts" are actually acquired?

Mathematics educators reported a number of attempts to promote students' acceptance of 1:1 correspondence as the single criterion for comparing infinite sets. For instance, Tirosh (1991) described satisfactory results of a set theory course for middle school students. The course used, among other things, the cognitive conflict approach, which was based on research findings regarding students' intuitions about infinity. Two additional interventions centered on different representations of infinite sets. The first was aimed at raising cognitive conflict and promoting high school students' awareness of their inconsistent ideas when comparing infinite sets (Tsamir & Tirosh, 1994). The second presented a sequence of assignments advancing from an anchoring task to a target (counter-intuitive) task using the analogy approach (Yehoshua, 1994). In another intervention, Sierpinska (1989) presented suggestions to two girls (10 and 12 years old), for creating 1:1 correspondence between matching elements in infinite sets. One of the girls accepted the "pairing idea", while the other found it counter-intuitive. However, the subjects in all the above mentioned studies were younger than the students who study Cantorian set theory as part of their curriculum.

Nowadays, the concept of actual infinity is customarily presented at the college or university level to mathematics majors and to prospective mathematics teachers for secondary schools. It is usually introduced as an axiomatic system, by consistently and sequentially presenting axioms, basic notions, definitions and theorems within the Cantorian set theory, with reference to Zermelo and Fraenkel. Comparison tasks are discussed pointing to 1:1 correspondence or to the examination of the powers of the sets as the methods for determining equivalency within this theory. However, students' intuitive tendencies either to compare infinite sets by inclusion or to grasp infinity as a single infinity are usually neglected. Thus, it seems important to investigate prospective teachers' tendencies to accept these criteria for the comparison of infinite sets, before and after studying a set theory course.

Therefore, the questions posed in the present study were: (1) Do prospective teachers regard the criteria 1:1 correspondence, inclusion and single infinity as being suitable for the comparison of infinite sets? (2) Are illustrated ways of applying 1:1 correspondence accepted by prospective teachers for the

comparison of infinite sets? (3) How may the illustrated solutions affect the prospective teachers' acceptance of 1:1 correspondence, inclusion and single infinity for the comparison of infinite sets? and (4) Will there be significant differences between prospective teachers who already studied the topic and those who did not?

Methodology

Participants were 181 prospective secondary school mathematics teachers, sampled from Israeli state teachers colleges. Seventy-one of them had never studied Cantorian set theory [N-ST] and 110 had completed a year long Cantorian set theory course [C-ST] three months prior to this research.

The participants were given about 45 minutes to answer, in writing, a questionnaire, which consisted of the following parts:
Part I – an explanation illustrating the notions of 1:1 correspondence, single infinity and inclusion, asking the subjects to determine whether each of the criteria seemed suitable for comparing infinite sets.

Part II – five illustrations of the use of 1:1 correspondence to compare given pairs of infinite sets. Three of these were:

1. $A=\{1/2, 1, 1 1/2, 2, 2 1/2, 3, \ldots\}$ $B=\{1, 2, 3, 4, 5, 6, \ldots\}$
2. $B=\{1,2,3,4,5,6,\ldots\}$ $E=\{3,4,5,6,7,8,\ldots\}$
3. $I=\{4,8,12,16,20,\ldots\}$ $J=\{1,4,9,16,25,\ldots\}$

Which all have the same number of elements ($\aleph_0$). One pair of sets related to the points on two concentric circles having the same number of elements (c). The last pair consisted of non-equivalent sets. The comparison dealt with the number of points in a segment (power c) with the number of midpoints in that segment, created by an infinite process of halving the segment and the sub-segments ($n \Leftrightarrow 2^n-1$, power $\aleph_0$). In all cases, participants were asked to state whether they found the suggested solution acceptable and to justify their judgments.

Part III – was a repetition of part I.

The participants received each part of the questionnaire only after handing in the previous one. After the written assignment, ten participants in each group were also interviewed orally, in order to get a better insight into their ideas.

RESULTS

Part I: Declaring 1:1 Correspondence, Inclusion and Single Infinity as Acceptable Criteria for the Comparison of Infinite Sets

Table 1 shows that in stage I of the research, 1:1 correspondence was the criterion most frequently accepted by prospective teachers who had C-ST and inclusion was the criterion most frequently accepted by those who had N-ST. Single infinity was the least accepted criterion by both N-ST and C-ST participants. It should be noted that even among those who had C-ST, about 17% still viewed single infinity as suitable for comparing infinite sets and about 32% of them still viewed inclusion as suitable.

Participants' justifications for their use of the various criteria were of the following two types:

1. This criterion is the best – emphasizing that the suggested criterion must be used for comparing infinite sets. About half of those who accepted 1:1 correspondence presented such a justification. Interestingly, about 20% of the C-ST participants who regarded 1:1 correspondence as a suitable criterion, justified this claim in terms of 'power', by claiming that *"actually, in order to compare the number of elements one should examine the powers of the sets, but 1:1 correspondence is also quite OK"*.

Table 1

Frequencies (in %) of Accepting 1:1 Correspondence, Inclusion, and Single Infinity for Comparison of Infinite Sets in Parts I and III

Criterion accepted:	**1:1 Correspondence**		**Inclusion**		**Single infinity**	
Study of Set Theory	N-ST n=71	C-ST n=110	N-ST n=71	C-ST n=110	N-ST n=71	C-ST n=110
Part I	54.3	76.4	63.4	31.8	31.4	17.3
Part III	73.9	91.6	60.9	32.4	54.3	25.5

Moreover, about 60% of the N-ST and half of the C-ST participants who accepted inclusion expressed the idea that *"the use of inclusion is ideal because it allows you to reach definite conclusions"*. About 40% of the N-ST and half of the C-ST participants who claimed single infinity to be a suitable criterion, justified their claim by stating that *"infinity is always the same infinity, so obviously you should follow only this criterion"*.

2. Practical considerations – expressing a general attitude that any criterion may be applied when suitable. About 30% of the N-ST participants presented this explanation for each of the three criteria. Among the C-ST participants, about 20% of those who accepted 1:1 correspondence, 30% of those who accepted single infinity and 40% of those who accepted inclusion presented this line of reasoning. They claimed, for instance, that *"not in all cases is there inclusion, but when one set is included in the other, it is only natural to use this criterion"* or *"it is hard to find matching elements, but when possible, this method should be used to compare the sets"*.

The justifications given by the prospective teachers to explain their rejection of the various criteria were of the following four types:

1. Direct counter-arguments – excluding the use of a criterion due to its attributes or the nature of infinity. Most frequent claims were: (a) *"a criterion that always gives the same conclusion is useless"*. This was given by participants who rejected single infinity, about 40% of the C-ST and about 70% of the N-ST; and (b) *"even when I find 1:1 correspondence between matching elements, I can only control a finite number of "pairs", so I can never be sure*

that the matching rule keeps working infinitely". This was presented by about 60% of all participants who rejected 1:1 correspondence.

2. Indicating another criterion as more suitable for comparing infinite sets. This line of justification was used by 15% of the N-ST participants who rejected single infinity because *"one should use inclusion"*; and by about 20% of the N-ST participants who rejected inclusion because *"there is one, single infinity"*. Among the C-ST participants, about 25% of those who rejected inclusion and 40% of those who rejected single infinity argued that power is the only correct criterion. Surprisingly, the idea of power even served about 20% of the C-ST participants who rejected 1:1 correspondence to explain their rejection. They claimed, for instance, that *"we have learned that only the powers of sets indicate the numbers of elements they have. Obviously, in order to compare the number of elements in two sets, one needs to know these numbers"*.

3. Practical considerations – excluding the use of a certain criterion due to its practical limitations. About 30% of all participants justified the rejection of inclusion by claiming that *"the criterion is inadequate, as frequently there is no inclusion relationship between the sets"*. Similarly, about 30% of the N-ST participants who rejected 1:1 correspondence said, *"in many cases it is extremely difficult to find a way to match the elements"*.

4. Consistency considerations – expressing the need to preserve consistency. Only when rejecting inclusion did a few N-ST and C-ST participants claim that *"inclusion is not valid as it can contradict solutions arrived at by using 1:1 correspondence."*

Part II: Acceptance of Illustrations of Using 1:1 Correspondence when Comparing Infinite Sets

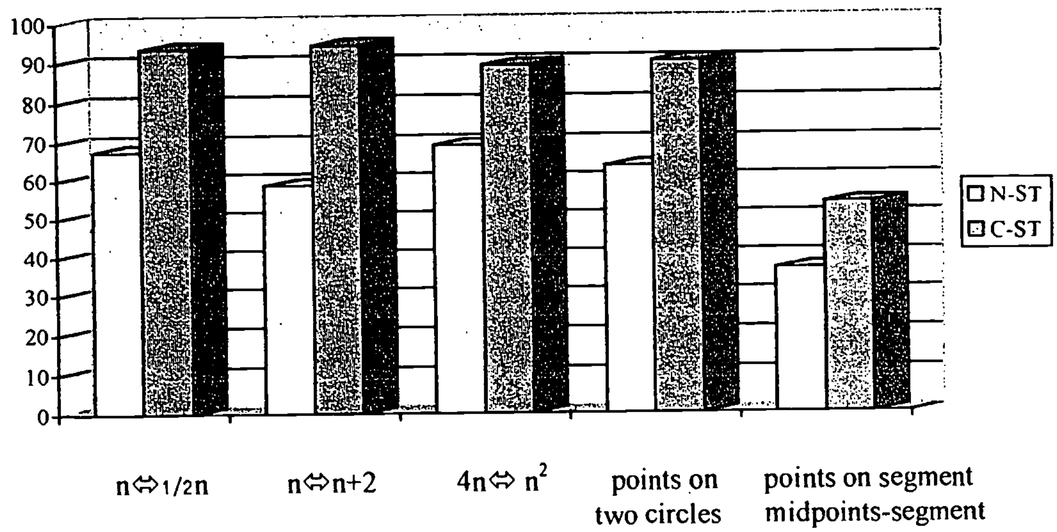
Graph I shows that for each of the four problems that presented equivalent sets, the tendency of C-ST participants to accept the illustrated 1:1 correspondence (about 90%) was significantly higher than that of the N-ST participants (about 60%). In problem 1 ($n \Leftrightarrow 1/2n$) $x^2=22.6$ DF=2 $p<0.01$, in problem 2 ($n \Leftrightarrow n+2$) $x^2=36.04$ DF=2 $p<0.01$, in problem 3 ($4n \Leftrightarrow n^2$) $x^2=13.2$ DF=2 $p<0.01$, and in problem 4 (points on two circles) $x^2=20.9$ DF=2 $p<0.01$. However, in the fifth problem, which dealt with an unusual presentation of non-equivalent sets, there were insignificant differences between the tendencies of N-ST participants and C-ST participants to accept the illustrated 1:1 correspondence. Only about 40% of the N-ST participants and about 50% of the C-ST accepted the suggested comparison of the number of points in a segment (power c) with the number of midpoints ($n \Leftrightarrow 2^n-1$, power $\aleph_0$).

Most prospective teachers provided no justifications for their acceptance or rejection of the given solutions. Nevertheless, among those who did give justifications, N-ST participants most frequently explained rejecting the given solutions to the equivalent sets by specifying another, more preferable, criterion.

Most prevalently they related to inclusion – *"One set is included in the other, thus they don't have the same number of elements"*, but they also mentioned single infinity – *"1:1 correspondence is unnecessary. It is known that infinities are always the same"*. However, C-ST participants who accepted the suggested 1:1 correspondence frequently confessed that the given solutions reminded them of rules they had studied. Those who rejected the suggested solution, however, usually accepted the equivalency, but said that power and not 1:1 correspondence was the criterion to be used.

Graph I

Frequencies (in %) of the acceptance of illustrated '1:1 correspondence' solutions



The one justification most commonly used to reject the fifth solution, used by all participants (both N-ST and C-ST) was that *"By endlessly choosing midpoints, one eventually deals with the whole segment. So, the collection of all midpoints matches the collection of all points."* Others used the single infinity idea, and a few prospective teachers argued, in all cases, that infinite sets are incomparable.

Part III: Re-Declaring 1:1 Correspondence, Inclusion and Single Infinity as Acceptable Criteria for the Comparison of Infinite Sets

Table 1 shows that when advancing from Part I to Part III there was an increase in the percentage of N-ST and C-ST participants who declared that 1:1 correspondence is suitable for comparing infinite sets. However, the growing acceptance of 1:1 correspondence was not accompanied by an increased tendency to reject the other criteria. The same rates of N-ST and C-ST participants as in part I viewed inclusion as acceptable, and surprisingly, more N-ST and more C-ST than before accepted the single infinity criterion.

Usually participants either gave no justification or wrote that they had already explained their judgements in Part I. Prospective teachers who justified their claims often followed two lines of reasoning: (a) *"I was convinced by the illustrations in Part II that 1:1 correspondence can be very useful even in cases where I couldn't provide the matching rule"* or *"the previous examples [in Part II] reminded me"* (used by prospective teachers who rejected '1:1 correspondence' in Part I and accepted it in Part III); and (b) *"All these questions confuse me and make me believe that infinity is extremely strange. There is probably a single infinity"* or *"Inclusion seems more reasonable than creating a correspondence between an infinite number of pairs"* (used by prospective teachers who accepted 1:1 correspondence in Part I but rejected it in Part III). A few participants consistently expressed in both Parts I and III the notion of single infinity to reject 1:1 correspondence, while some others argued in both parts that *"infinite sets are incomparable"*.

DISCUSSION

The findings of this research, in line with those of previous research with younger students, indicate that prospective teachers intuitively regard the criteria of 1:1 correspondence, inclusion and single infinity as suitable for the comparison of infinite sets (e.g., Tsamir & Tirosh, 1992). Participants who had not followed a set theory course tended to view inclusion as most applicable, but even a substantial number of C-ST participants declared that inclusion or single infinity were acceptable.

Moreover, practical considerations and the availability of the various criteria played an important role in prospective teachers' decisions whether to accept or reject a specific criterion. However, similar, practical considerations led to contradictory conclusions. On the one hand "to accept" (e.g., *"It is not always possible to use 'inclusion', but as this is a criterion which is easy to use, it should be used when possible"*); and on the other hand "to reject" (e.g., *"Sets are not always inclusive, so this is not a good criterion"*).

Interestingly, C-ST participants used the consideration of powers to reject 1:1 correspondence as a valid criterion, failing to grasp that 1:1 correspondence is the notion underlying powers. Moreover, even the C-ST participants examined each problem in isolation with no regard to the contradictions which may arise when accepting more than one of the above mentioned criteria as being valid. Consistency was only rarely mentioned as the means by which to decide whether to accept a suggested criterion, and as a means to determine validity in mathematics (see also Tsamir & Tirosh, 1994).

After having been presented in Part II with illustrations of the use of 1:1 correspondence as a criterion for comparing infinite sets, in Part III there was an increase among all participants in the acceptance of 1:1 correspondence. However, the rate of N-ST and even C-ST participants who accepted inclusion was unchanged, while the rate of the acceptance of 'single infinity' rose. It seems that while "familiarizing" the N-ST with the correct solution and

"reminding" the C-ST participants of it strengthens their tendency to accept this criterion, it does not weaken their tendency to reject other criteria.

The findings also indicate that, as could be expected, prospective teachers who had studied Cantorian set theory exhibited a significantly higher tendency to accept 1:1 correspondence than prospective teachers who had followed no course. These findings were detected in both types of questions — when asked to state whether 1:1 correspondence is a suitable criterion for the comparison of infinite sets, and when asked to judge illustrated solutions to specific comparison tasks. However, this tendency was no longer apparent when the participants were presented with an unusual comparison task – the fifth problem in Part II. This problem was unusual both in that it intuitively triggered the single infinity notion and in that it was a type of problem not dealt with in C-ST courses.

In light of these findings, it seems that "showing students the right way" does not decrease their tendencies to accept incorrect, intuitive ideas when comparing infinite sets. Thus, students' primary intuitions should be taken into consideration when planning instruction. Students should be made aware of their tendency to view all infinities as equal and of their tendency to use considerations of inclusion. Moreover, they should understand that using more than one of these criteria to compare infinite sets will lead to contradiction. Some ideas for practical applications of these conclusions will be offered in the oral presentation.

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-THE CONCEPT OF A VARIABLE; GAUGING STUDENTS' UNDERSTANDING

Elizabeth Warren, Australian Catholic University

This paper explores the use of written tests and semi-structured interviews in ascertaining students' understanding of the concept of a variable. A written algebra test was administered to 379 students and from the results students were selected for a semi-structured interview. The types of questions asked and the medium in which they were asked appeared to influence the responses given. It is conjectured that these issues must be considered when endeavouring to reach a richer understanding of the students' perception of the concept of a variable.

Algebra has long held a place of distinction in the mathematics curriculum (NCTM, 1989; National Statement for Australian Schools, 1991). Few have contested the importance of algebra as it is seen as 'the language through which most mathematics is communicated' (NCTM, 1989, p. 150). Critical to the algebraic domain is the variable construct. The literature identifies common misconceptions students experience when interpreting algebraic symbols.

Common misconceptions of the concept of a variable

Through his large-scale study of students' interpretations of literal terms, Kuchemann (1978, 1981) identified many misconceptions. These were:

Letter evaluated: Students assign numerical values to letters at the outset of a problem. For example, when asked to describe the expression $2+3x$ children often assign a value to x, such as 1, and compute the answer. Thus $2+3x=2+3x1=5$.

Letter not used: Here, students ignore the letters, or at best acknowledge their existence but without giving them meaning. For example, the algebraic expression $2x+8y+3x$ is equated to $13xy$. Such an answer is obtained by simply adding up all the numbers, then writing down each letter that occurs.

Letters used as objects: Here, students regard the letter as shorthand for an object or as an object in its own right. For example, $2a+3b$ represents adding 2 apples to 3 bananas. This is referred to in the literature as 'fruit salad' algebra (Booth, 1988).

Letters used as a specific unknown or constant: Students perceive the letter as a specific but unknown number. For example, the expression L+M+N would never equal L+P+N as N cannot equal P. Even though both N and P are acknowledged as variables they must always be different values from each other as they are represented by different letters of the alphabet.

Letter used as a generalised number: Here, students perceive the letter as representing, or at least as being able to take, several values rather than just one. For example, if students are asked to list all the possible values for the expression $x+y=10$, they will list more than one of the whole numbers which will satisfy the condition.

Kuchemann (1981) found that most students (aged 13-15 years) could not cope consistently with items that required the use of the letter as a specific variable.

A further three misconceptions identified in the literature were as follows:-

Changing the variable symbol as changing the referent. That is, different variables must take on different values. For example, the expression $3n$ is not the same concept as $3x$ as n and x could never be equal (Booth, 1984; Chalouh & Herscovics, 1983; Wagner & Parker, 1993).

Assigning numerical values to letters according to their rank in the alphabet, for example, a=1 and z=26, or if x=3 then y=4 and z=5. (Booth, 1984; Chalouh & Herscovics, 1983).

Assigning the letter as a subdivisional label, for example, 3a refers to the first part of the problem (Chalouh & Herscovics, 1983).

Misconceptions also occur when examining expressions. Some of these are:-

Closure: Some students exhibit a need to have a 'single' answer. For example, a+b becomes ab (Chalouh & Herscovics, 1988).

Equal sign: In arithmetic '=' tends to mean compute,. 'makes', or a place for the answer. Students fail to recognise the equality relation between the left and right hand side of the equation (MacGregor, 1991).

This research is a replication study. Kilpatrick (1993) claimed that mathematics education 'has suffered from a lack of replication studies [which] ... would help confirm and refute conclusions drawn from previous work.' This is particularly imperative in ascertaining students' understanding variable of the variable concept. Most of the misconceptions identified in the literature were delineated from the results of large studies involving written tests. The types of questions asked and the medium in which they were asked could influence the responses given. Through written responses students may not be able to exhibit their full understanding or misunderstanding of the concept of a variable. This paper goes beyond pure replication. It compares the results from written responses and a semi-structured interview and explores how each contributes to reaching a richer understanding of students' perceptions of the concept of a variable.

Methodology

This study comprised whole class testing to select students for in-depth interviews. The sample for the whole class testing comprised 379. The students' ages ranged between 12 years and 2 months and 15 years and 10 months. Half of the sample were in their first year of algebraic studies and half were in their second year. Both schools chosen for the study consisted of students from lower-middle socio-economic status, with a variety of ethnic backgrounds represented.

Whole class test: The whole class test was administered as a written test and completed within a one-hour period. The test consisted of three components, namely, generalising from visual patterns, generalising from tables of data, and understanding the concept of a variable. The results from 3 of the 8 questions selected for the variable component of the written test are discussed in this paper. These 3 questions were believed to probe students' view of the variable, their need for closure of algebraic expressions, their propensity to concatenate algebraic expressions, and why students allowed "= constant" to limit possible values for the variable. These questions were also used in the semi-structured interview.

Semi-structured interview: The results for each component were ranked and three students were selected at the 1^{st}, 25^{th}, 75^{th} and 100^{th} percentile for each of the written components. Each interview was videotaped and transcribed.

Instrument The selected questions were:

Question 1. *This question is about t+t and t+4.*

- *(a) Is t+t ever larger than t+4? If so when?*
- *(b) Is t+4 ever larger than t+t? If so when?*
- *(c) Are t+t and t+4 ever equal? If so when?* (Modified Harper, 1979)

This item was modeled on an item used by Harper (1979) and Quinlan (1992). The item used in their studies was as follows:

This is a question about t+t and t+4

(a) Which is larger, t+t or t+4? WHY? *(b) When is t+t larger?*

(c) When is t+4 larger? *(d) When are they equal?*

In the interview stage of the pilot study, concern was expressed about the inconsistency in the wording of the Harper item. For example, *"When is t+t larger?"* seemed to alert to the fact that perhaps *t+t* could be larger. This proved confusing for students who had chosen *t*+4 for part (a). Hence the question was reworded so that such inconsistencies no longer existed.

Question 2. *For a school excursion, 3 buses take f students each and 4 cars take g student each.*

(a) Give the total number of students taken by these buses and cars.

(b) One car leaves early with g students. How many students remain? (Quinlan, 1992b)

This question was created to obtain information on students' ability to interpret the meaning of letters when they referred to quantities as objects rather than mere numbers (as in question 1) and to carry out operations on numerical variables without knowing their values. This question was not reworded for this study.

Quest 3. (a) *If c+d=10, tick ALL the meanings that c could have:*

3 10 12 7.4 the number of apples in box

an object like a cabbage an object like an orange

(b) If c+d=10, what happens to d as c gets bigger?

(c)If c+d=10, and c is always less than d, what values may c have

The origin of this item is threefold. It is based on the CSMS project question (Hart, 1981) *"What can you say about c if c+d=10 and c is less than d,"* (Kuchemann 1980, p. 67) and on Harper's (1979) equation task, *"If x+y=10 when is x less than y."* Kuchemann (1980) regarded this question as one which required students to view letters as generalised numbers. Quinlan (1992, p. 106) added parts (a) and (b) to the original question. Part (a), measured the types of possible meanings students are prepared to accept for alphabetic symbols in algebra. Part (b) tested students' understanding of the covarying relationship between the two variables (i.e., as the value of *c* varies so does the value of *d*). This question was adapted for the semi-structured interview. In the semi-structured interview students were asked *"If p+m=12, put a ring around all the possible values p can have. 4, 12, 15, 0, 3.9, -2, an object like a pear, the number of children in*

the class." Students were then asked to articulate their reasons for including or excluding certain values.

Results

Written component For each part of question 1, students were required to give two responses. Students were asked to reply either in the affirmative or negative to the question and then elaborate on their response with an appropriate reason. Table 1 summarises the results for this question.

Table 1 *Percentage of Students who correctly responded to Each Part of Question 1*

Question 1	Correct response	Valid reason
(a) Is t+t ever larger than t+4? If so when?	46.4	28.8
(b) Is t+4 ever larger than t+t? If so when?	45.9	29.0
(c) Are t+t and t+4 ever equal? If so when?	43.0	36.1

Less than half the students answered this question. The results indicate that giving a valid reason for the equality situation was easier than the other two situations. The rewording of the Harper Quinlan question appeared to result in a greater percentage of students reaching correct solutions. While Quinlan (1992) found that 50% of his sample (517 students) responded correctly, 241 of these had completed at least 2 more years of schooling with 108 students being in their final year. Thus only 27% of students with comparable formal algebraic studies correctly responded to the Harper Quinlan question.

Responses to both 2(a) and 2(b) seemed to fall into four distinct levels Table 2 presents a description of the levels together with the percentage of students whose responses were considered to be at that level.

Table 2 *Levels of responses to Question 2 & % of correct responses*

Level	Description	2 (a)	2(b)
0	Did not attempt or assigned a number	29.0	31.4
1	Incorrect algebraic response	14.5	21.4
2	Correct response but the need for closure	7.7	10.8
3	Correct response	48.5	36.4

Students found 2(a) easier to respond to than 2(b). This question was not reworded for this research and results were similar to those obtained by Quinlan (1992).

Table 3 summarises the percentage of students who chose the various options in 3(a)

Table 3 *Percentage of Responses to each Choice for Question 3(a)*

	Choice						
	3	10	12	7.4	number in box	object cabbage	object orange
Accepted	76.5	51.5	11.9	66.5	40.4	22.7	21.4

The majority of students accepted 3 and 7.4 as possible values for the variable *c* but were less confident in allowing *c* to be 10. When *c* is 10 the value of *d* is zero. Only twelve percent accepted *c* as 12. When *c* is 12 the value of *d* is negative. These findings seem to indicate that the nature of the product, "equal ten," of the equation plays a significant role in limiting the permissible values for the associated variables. This could reflect reluctance by the students to assign zero or negative values to the variable *d*. This trend was difficult to delineate in the Kuchemann (1980) format of the question. This issue was investigated in the semi-structured interview. Only 40.4 percent of the students accepted that *c* could be the number of apples in a box, whereas up to 22.7 percent allowed *c* to stand for an object such as an orange or cabbage. This seems to indicate that the students had difficulty in accepting the variable as a generalised number and in differentiating between this interpretation and the variable as an object.

Levels were developed for classifying responses to 3(b) and 3(c). For 3(b), the levels indicated whether the students could see the covarying relationship between *c* and *d* in the equation $c+d=10$. 68.3% of the students reasoned correctly that *d* would get smaller. For 3(c), the levels indicated the possible values students would accept for *c*. Table 4 summarises the results for this question.

Table 4 *Levels of Responses for Question 3(c)*

Level	Description	% response
0	omitted response or incorrect list of numbers e.g., 1,2,3,4,5,..10	29.4
1	one or two numbers as response e.g., 3,4	4.5
2	1,2,3,4 – included only integer responses	42.9
3	included 0, negatives and/or fractions	20.6
4	< 5	2.1
5	< 5 and included 0, negatives & fractions	0.05

As can be seen from the above results very few students responded correctly to question 3(c). Over half of the students tested gave at least four values for *c*. Kuchemann (1981, p. 108) found that nearly 40% of his sample of 14 year old students (n=1000) gave only one value for *c* and that 30% gave four or more. For this study, approximately 43% of the students gave four values for *c*. As indicated by Quinlan (1992), it appears that the inclusion of the first two components helps students frame appropriate answers to (c).

In summary, these 3 questions from the written test indicated that the rewording of questions can influence the responses given. Also, when examining the relationship between two variables in an expression of the form $x+y=c$, where c is a constant, the value of the constant seems to play an important role in the permissible values students will accept for the variables themselves. Students exhibited a reluctance to choose values for either *x* or *y* that were greater than c. Even students who were considered by their teacher to be very capable of understanding algebraic concepts, included the letter standing for an object in a number of their responses. A very small percentage of the

students perceived the variable as a generalised number. Most students tended to see the variable as being represented by up to 4 specific integer values. These trends were investigated further in the semi-structured interview.

Semi-structured interview Twelve students were chosen to be interviewed on their understanding of the concept of a variable. The following section presents a summary of students' perceptions.

Variable as an object: The results for question 3 from the written algebra test indicated that 22.7% of students allowed c to stand for 'an object like a cabbage' and 21.4% allowed c to stand for 'an object like a orange' (see Table 3). From an analysis of the written responses, some of the more able students seemed to hold this misconception. The inclusion of "object like a pear" in the third interview question probed why students accepted the variable as standing for an object. Not one of the twelve students interviewed accepted p as "shorthand for pear." When questioned, those students who ringed 'an object like a pear' seemed to assign a number to p. For example, Jason at the 1st percentile said, "Yes it could be about the number of pears and number of apples." Michelle, also at the 1st percentile said, "Could be 14 pears." One student at the 100th percentile could clearly articulate why 'an object like a pear' was not a valid option. Mary said, "Object like a pear and 12 have no relationship," indicating that she viewed 'pear' as an object and not as a number. The two other students at the 100th percentile said, "p could be an object like a pear only if m was 11 pears and then you would have 12 pears." The interview seemed to indicate that, contrary to the belief held when analysing the written component, simply ringing 'object like a pear' did not necessarily mean students saw p in expressions like $p+m$ standing for pear, but rather as 'a pear', that is, 'one pear'. This seems to indicate that relying solely on written responses could lead to erroneous conclusions with respect to students' understanding of variable concept.

Specific unknowns: From the students' responses there seemed to be a growth of acceptable values for the variable. Students at the 1st and 25th percentile made decisions about the two expressions $t+t$ and $t+4$ using one specific value for the variable. Adam said, "$t+t$ is bigger when $t=5$, $t+4$ is bigger when $t=2$ and they are the same when t equals 4." By contrast, one student at the 75th percentile reached a conclusion by using a series of specific unknowns for t (e.g. "$t+t$ is bigger when $t=5, 6, 7, 8$….).

Generalised number: For the expressions $t+t$ and $t+4$, two students at the 75th percentile and three students at the 100th percentile immediately articulated the general case, that is, "$t+t$ is larger when t is greater than 4, $t+4$ is larger when t is less than 4 and they are equal when $t=4$." It was conjectured that they had an understanding of the variable as a generalised number.

Closure: For question 2, 3 students at the 1st and 25th percentile wrote $c+f=$, indicating a need for closure. All the students at the 75th and 100th percentile did not need to close the expression, writing $c+f$.

Product component of an equation: For the expressions $p+m=12$ it seemed that the '=12' component of the question was not the only factor that limited the number of acceptable

values for p. The addition sign before m also seemed to play an important role. The six students at the 1st and 25th percentile failed to accept 15 for the value of p. A common response was "15 plus anything is over 12," indicating that the value of m must be positive as it was preceded by '+'. Of particular interest was the fact that they all accepted '-2' as a value for p. One student stated that, "-2 was alright as it is below 12."The other six students at the 75th and 100th percentile accepted 15 as a valid value for p, stating that m would need to be negative.

Concatenation of algebraic expressions: Three students exhibited a need to concatenate the variables. All of these students were either at the 1st or 25th percentile. For question 1, Chris at the 1st and James at the 25th percentile indicated that they would have '*cf* trucks'. When comparing $t+t$ and $t+4$, Chris also suggested that, "$t+t$ is $2t$ and $t+4$ is $4t$ so $t+4$ would be bigger."

In summary, the results of the interviews indicated:-Students at the 1st and 25th percentile were characterised by their need to evaluate, close and concatenate algebraic expressions, and by perceiving the variable as representing a specific unknown. Two students at the 75th percentile and the three students at the 100th percentile perceived the variable as a generalised number and did not concatenate or close any of the algebraic expressions.

Discussion and conclusion

Both the results of the written test and the interview support Kucheman's stages of understanding. Not only did the less able students, students at the 1st and 25th percentile, seemed to perceive the variable as representing a specific unknown but they also exhibited a need to close algebraic expressions. By contrast, the more able students did not need to close algebraic expressions and they perceived the variable as a generalised number.

The results also seem to indicate that care must be taken when conjecturing about students understanding from written responses. First, the framing of the question itself could limit the types of responses given. The use of appropriate prompts could help elicit students' full understanding of the concept of a variable. For example, as indicated by Question 3 in the written test, the inclusion of (a) and (b) helped students frame a response to (c), and for question 1 the rewording resulted in a greater percentage of students with similar algebraic experience reach correct solutions.

Second, the medium in which questions are posed needs also to be considered when reaching conclusions from students' responses. From an analysis of the results of the written algebra test it seemed that many students perceived the variable as a letter standing for an object. The results of the interview seemed to indicate that this assumption was too simplistic. None of the students interviewed believed that p was "shorthand for pear." All the students who selected p as standing for an object like a pear, had valid reasons for its inclusion and these reasons relied on p standing for a number of pears and not pear. While these results point tot he limitation of asking questions only in written format, further probing is needed to ascertain how the context of the problem can influence this common misconception (Clement, 1982). Also, for the

expressions $p+m=12$ the product component (=12) is not the only factor that limited students' thinking. It seemed that the placement of '+' before the variable also played a key role. Students at the 1st and 25th percentile believed that '$+m$' meant that the value of m must always be positive. This seems to indicate that some students are not capable of delineating between '+' as the operation of addition and '+' indicating a positive value. Both of these misunderstandings are difficult to identify in a written test format.

The results of this study seem to confirm generalities found in previous studies. First, the wording of questions can influence the type of response given. A slight change in how the question is framed can lead to a very different array of responses. Second, adding prompts, lead ins and explain components to questions appears to help students frame appropriate responses. The role of these components needs to be further explored. Third, the value of interviewing and listening to students explanations to how they solve problems can not be underestimated when gaining insights into students' understanding of algebraic concepts. Fourth, when teaching arithmetic and algebra there is a need to delineate between '+' as an operation and '+' as denoting a positive number. Recent research has already acknowledged this need for '-' but not for '+'. The results of this study also indicate that rewording of questions can significantly influence the responses given by students. Rewording of questions in previous areas of mathematical research needs to be carried out in order to both confirm and refute conclusions drawn.

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TYPES OF RESEARCH IN MATHEMATICS EDUCATION

Dylan Wiliam
King's College London

Different approaches to research are described in terms of the different emphases accorded to the hermeneutic notions of text, context and reader. Knowledge-building in mathematics education is defined as a dual process of establishing warrants for particular beliefs, and eliminating plausible rival hypotheses, where 'plausibility' is established either by explicit reference to a theoretical frame, or implicitly within a discourse. These perspectives are then integrated by a classification based on whether the primary source of evidence is reason, observation, representation, dialectic, or ethical values. It is then argued that educational research, as well as building knowledge by the process identified above, requires subjecting the consequences of the research to the ethical judgements of the community.

Introduction

The community of PME is heterogeneous, and while there appears to be reasonable agreement about the purposes of research in the Psychology of Mathematics Education, there is much less agreement as to how that research should be conducted and disseminated and what is to count as evidence or knowledge (Lester, 1998). Supplementing the traditional paradigms of experimental psychology, recent work has borrowed heavily from ethnographic traditions of research, and more recently still, there has been an increasing interest in 'action research'—that is research carried out by teachers in their own classrooms for their own benefit. With this proliferation of paradigms, there is a danger that debates about the quality of research are clouded by differences in researchers' views about what counts as evidence rather than about the quality of the research. The purpose of this paper is to present a framework for thinking about research in mathematics education, with a view to clarifying the debate about the quality of that research.

Evidence and inference

The relationship between different approaches to research in mathematics education can be clarified by the use of some ideas from hermeneutics. Traditionally, it had been assumed that an utterance, picture, piece of writing etc (collectively referred to as *text*) has an absolute meaning. In hermeneutics, it is acknowledged that the same text has different meanings when presented in different contexts, and when presented to different readers. For example, when a student says that the work that she has been asked to do is "boring", in one context, and to a particular teacher, this might be an informed comment that the work was too repetitive, not sufficiently challenging, and unlikely to effect any meaningful learning. In another context, or to another person, "boring" might mean almost the opposite — work that is too challenging, or even threatening. The text (in this case "It's boring") will be interpreted differently in different contexts, and by different readers (eg teachers). These three key ideas — text, context and reader — are said to form the *hermeneutic circle*.

In educational research the 'text' is usually just 'data'. Sometimes the fact that the data has to be elicited is obvious, as when we sit down with someone and ask them some questions and tape-record their responses. At other times this elicitation process is less obvious. Observing and making notes on a teacher's actions does not feel like 'eliciting' evidence. It feels much more like the evidence presenting itself. However, it is important to realise that the things I choose to make notes about, and even the things that I observe (as opposed to those I see), depend on my personal theories about what is important. In other words, all data is, in some sense, elicited. This is true even in the physical sciences, where the physicist Werner von Heisenberg remarked that "What we learn about is not nature itself, but nature exposed to our methods of questioning" (quoted in Johnson, 1996 p. 147).

For some forms of evidence, the process of elicitation is the same as the process of recording the evidence. If I ask a school for copies of its policy documents in a particular area, all the evidence I elicit comes to me in permanent form. However, much of the evidence that is elicited is *ephemeral*, and only some of it gets recorded. I might be interviewing someone who is uncomfortable with the idea of speaking into a tape recorder, and so I have to rely on note-taking. Even if I do tape-record an interview, this will not record changes in the interviewee's posture which might suggest a different interpretation of what is being said from that which might be made without the visual evidence. The important point here is that it is very rare for all the evidence that is elicited to be recorded.

During the process of elicitation and recording, and afterwards, the evidence is *interpreted*. Research based on approaches derived from the physical sciences emphasises *text* at the expense of *context* and *reader*. The same educational experiment is assumed to yield substantially the same results were it to be repeated elsewhere (for example in another school), and that different people reading the results would be in substantial agreement about the meaning of the results. Other approaches will give more or less weight to the role played by *context* and *reader*. For example, an ethnography will place much greater weight on the *context* in which the evidence is generated than would be the case for more positivistic approaches to educational research, but would build in safeguards that different readers would share, as far as possible, the same interpretations. In contrast, a teacher researching in her own classroom might pay relatively little attention to the need for the meanings of her findings to be shared by others. For her, the meaning of the evidence in her own classroom might well be paramount.

In what sense, then, can the results of educational research be regarded as 'knowledge'? The traditional definition of knowledge is that it is simply 'justified true belief' (Griffiths, 1967). In other words, we can be said to know something if we believe it, if it is true, and if we have a justification for our belief. There are at least two difficulties with applying this definition in educational research.

The first is that even within a subject as precisely defined as mathematics or science, it is now acknowledged that there are severe difficulties in establishing what, exactly, constitutes a justification or a 'warrant' for belief (Kitcher, 1984). The

second is that these problems are compounded in the social sciences because the chain of inference might have to be probabilistic, rather than deterministic. In this case, our inference may be justified, but not true!

An alternative view of knowledge, based on Goldman's (1976) proposals for the basis of perceptual knowledge, offers a partial solution to the problem. The central feature is that knowing something is, in essence, the ability to eliminate other rival possibilities. For example, if a person (let us call her Chris) sees a book in a school, then we are likely to say that Chris knows it is a book. However, if we know (but Chris does not) that students at this school are expert in making replica books that, to all external appearances, look like books but are solid and cannot be opened, then with a justified-true-belief view of knowledge, we would say that Chris does not *know* it is a book, even if it happens to be one.

Goldman's solution to this dilemma is that Chris knows that the object she is looking at is a book if she can distinguish it from a relevant possible state of affairs in which it is not a book. In most cases, the possibility that the book-like object in front of Chris might not be a book is not a relevant state of affairs, and so we would say that Chris does know it is a book.

However, in our particular case there *is* a relevant alternative state of affairs—the book might be a dummy or it might be genuine. Since Chris cannot distinguish between these two possibilities, we would say that Chris does not know.

Within educational research, therefore, we can view the task of producing 'knowledge' as having two requirements. The first is establishing that the inferences that are made from the evidence are warranted. This is something at which most researchers are relatively good. The second requirement, honoured more in its breach than its observance, is establishing that the chosen interpretation is more warranted than plausible rival interpretations.

Such a process can never be complete—there are no 'off-the-peg' methods; only a never-ending process of marshalling evidence that the chosen interpretation is *a) supported by the available evidence, and b) more warranted than plausible rival interpretations*.

This solution to the problem of 'knowledge' in education is only partial, because it leaves open what counts as a *plausible* rival hypothesis. In practice, even in the physical sciences, this is decided by the consensus of a community of researchers. Sometimes what is and is not plausible is made absolutely explicit, in the form of a theoretical stance. In other words, a researcher might say "because I am working from this theoretical basis, I interpret these results in the following way, and I do not consider that alternative interpretation to be plausible". More often, communities of researchers operate within a shared discourse that rules out some alternative hypotheses, although these tend to be implicit and are often unrecognised.

To sum up, evidence is elicited, recorded in some form and interpreted (not necessarily in that order!). The interpretations are validated by the elimination of plausible rival interpretations of the evidence, and the definition of what counts as 'plausible' is determined by the discourse within which the validation takes place.

Meanings and consequences

The above description has dealt with the production of 'educational knowledge', which, although it acknowledges the role of context in interpreting text, still places substantial emphasis on the production of shared meanings within a community of researchers.

During the 1980s, this concern with sharing of meanings across readers was questioned in what is sometimes called *action research*. In action research, what is important is the potential of the research to transform practice in the individual school or even for the individual teacher. Even if the research has different meanings (or even if it is meaningless) for those in different contexts, this is not a problem, as long as it has meaning for the teacher. There is no doubt that action research has huge transformative potential for the individuals involved in the research, but many have argued that it cannot be classed as research per se, because the research makes no effort to produce meanings that are shared beyond the immediate context and readers.

My concern here is not, however, with whether action research is valid research or not, but to show how it can fit into a theoretical framework and to examine how it differs from other approaches to research.

In all research, there is a tension between the *meanings* and *consequences* of research. For example, it would not be unusual for a researcher to discover something about a teacher's practice (perhaps through interviews with students) that appeared to be preventing the students from learning effectively. The question is, then, should the researcher communicate this to the teacher? In the traditional research paradigm, the answer would be a resounding 'no'. Feedback by the researcher to the teacher might change the teacher's subsequent behaviour, thus rendering the results of the research much more difficult to interpret. At the other extreme, many advocates of action research would say that such important evidence should be fed back to the teacher, and if this changes what is being investigated, then so be it. Put crudely, in traditional *text*- and *context*-focused research, unfortunate (or non-existent!) *consequences* are frequently justified and legitimated by the need for shared *meanings*. In action research, weakness in the extent to which meanings of the research findings are shared are justified and legitimated by the consequences of the research.

A full consideration of the nature of educational research must, therefore, take account of the consequences as well as the meanings of the research. The role of consequences in the validation of research is made much more explicit in the classification of inquiry systems developed by Churchman, to which we now turn.

Inquiry systems

Different methods of inquiry were investigated by Churchman who regarded all kinds of inquiry as being classifiable into 5 broad categories, each of which he labelled with the name of a philosopher (Leibniz, Locke, Kant, Hegel, Singer) he felt best exemplified the stance involved in adopting the system, and in particular, what is to be regarded as *evidence*.

More detailed accounts of the systems can be found in the work of Churchman (1971) and his colleagues (Mitroff & Sagasti, 1973; Mitroff & Kilmann, 1978), and Messick (1989). However, it is perhaps easier to understand the framework when it is applied to a 'real' research question in mathematics education — should students be allowed unrestricted access to calculators when learning mathematics?

One approach to this problem is to use only rhetorical tools to attempt to establish the truth of the proposition. For example a report by the London Mathematical Society, the Institute of Mathematics and Its Applications and the Royal Statistical Society (1995) argues that "To gain a genuine understanding of any process it is necessary first to achieve a robust technical fluency with the relevant content" (p9).

This would be an example of what Churchman calls a *Leibnizian* inquiry system, in which certain fundamental assumptions are made, from which deductions are made by the use of formal reasoning rather than by using empirical data. In a Leibnizian system, reason and rationality are held to be the most important sources of evidence. Although there are occasions in educational research when such methods are appropriate, it is usually far more appropriate to use some sort of evidence from the situation under study (usually called empirical data) in the inquiry.

The most common use of data in inquiry in both the physical and social sciences is via what Churchman calls a *Lockean* inquiry system. In such an inquiry, evidence is derived principally from the observations of the physical world. Empirical data is collected, and then an attempt is made to build a theory that accounts for the data. This corresponds to what is sometimes called a 'naive inductivist' paradigm in the physical sciences, and is most appropriate for well-structured problems.

In the context of our investigation into the use of calculators, we might design an experiment in which students were tested on their mathematical attainment, randomly assigned to one of two groups: one given unrestricted access to calculators and one given no access to calculators, and then re-tested after some period of teaching. From the resulting data, we would then attempt to build a coherent account of what was going on (see, for example, Hembree and Dessart, 1986).

The major difficulty with a Lockean approach is that, because observations are regarded as evidence, it is necessary for all observers to agree on what they have observed. Because what we observe is based on the theories we have, different people will observe different things, even in the same classroom.

For less well-structured problems, or where different people are likely to disagree what precisely is the problem, a *Kantian* inquiry system is more appropriate. This involves the deliberate framing of multiple alternative perspectives, on both theory and data (thus subsuming Leibnizian and Lockean systems). One way of doing this is by building different theories on the basis of the same set of data. Alternatively, we could build two theories related to the problem, and then for each theory, generate appropriate data (different kinds of data might be collected for the two theories).

For the issue of access to calculators, this could involve development of two alternative theories. For example, we might examine the relative effectiveness of calculator use and non-calculator use in terms of achievement, or in terms of

attitudes towards mathematics and confidence. It is not immediately apparent where these two theories overlap and where they conflict, but by attempting to reconcile the alternative conceptualisations, new theories can develop.

This idea of reconciling two (or more) rival theories is more fully developed in a *Hegelian* inquiry system, where antithetical and mutually inconsistent theories are developed. Not content with building plausible theories, the Hegelian inquirer takes the most plausible theory, and then investigates what would have to be different about the world for the *exact opposite* of the most plausible theory itself to be plausible. The tension produced by confrontation between conflicting theories forces the assumptions of each theory to be questioned, thus possibly creating a synthesis of the rival theories at a higher level of abstraction.

An Hegelian approach to our inquiry into the use of calculators would involve researchers who have adopted an 'achievement' perspective on the use of calculators to think through what would have to be different about the world for the exact opposite of their theory to be true. Those who adopt the 'attitude' perspective would do the same, which might then result in sufficient clarification of the issues to make a synthesis of the two perspectives, at a higher level of abstraction, possible.

The differences between Lockean, Kantian and Hegelian inquiry systems were summed up as follows by Churchman:

> The Lockean inquirer displays the 'fundamental' data that all experts agree are accurate and relevant, and then builds a consistent story out of these. The Kantian inquirer displays the same story from different points of view, emphasising thereby that what is put into the story by the internal mode of representation is not given from the outside. But the Hegelian inquirer, using the same data, tells two stories, one supporting the most prominent policy on one side, the other supporting the most promising story on the other side (Churchman, 1971 p. 177).

However, the most important feature of Churchman's typology is that we can inquire about inquiry systems, questioning the values and ethical assumptions that these inquiry systems embody. This inquiry of inquiry systems is itself, of course, an inquiry system, termed *Singerian* by Churchman after the philosopher E A Singer (see Singer, 1957). Such an approach entails a constant questioning of the assumptions of inquiry systems. Tenets, no matter how fundamental they appear to be, are themselves to be challenged in order to cast a new light on the situation under investigation. This leads directly and naturally onto examination of the values and ethical considerations inherent in theory building.

In a Singerian inquiry, there is no solid foundation. Instead, everything is 'permanently tentative'; instead of asking what 'is', we ask what are the implications and consequences of different assumptions about what 'is taken to be':

> The 'is taken to be' is a self-imposed imperative of the community. Taken in the context of the whole Singerian theory of inquiry and progress, the imperative has the status of an ethical judgment. That is, the community judges that to accept its instruction is to bring about a suitable tactic or strategy [...]. The acceptance may lead to social actions outside of inquiry, or to new kinds of inquiry, or whatever. Part of the community's judgement is concerned with the *appropriateness of these actions from an ethical point of view.* Hence the linguistic puzzle which bothered some empiricists—how the inquiring system can pass linguistically from "is" statements to "ought" statements— is no puzzle at all in the Singerian inquirer: the inquiring system speaks exclusively in the "ought," the "is" being only a convenient *façon de parler* when one wants to

block out the uncertainty in the discourse. (Churchman, 1971 p. 202; my emphasis in fourth sentence).

The important point about adopting a Singerian perspective is that with such an inquiry system, one can never absolve oneself from the consequences of one's research. Educational research is a process of *modelling* educational processes, and the models are never right or wrong, merely more or less appropriate for a particular purpose.

A Singerian approach to calculator use would then look at all possible perspectives, but also at the ethical and value positions underlying such perspectives. Even if restricting access to calculators results in higher average achievement, who are the winners and losers, and what are the resulting costs to society? Such difficult questions can be avoided within Leibnizian, Lockean, Kantian and Hegelian inquiry systems, but must be confronted within a Singerian inquiry system.

Educational research can therefore be characterised as a never-ending process of assembling evidence that:

1) particular inferences are warranted on the basis of the available evidence;
2) such inferences are more warranted than plausible rival inferences;
3) the consequences of such inferences are ethically defensible.

Furthermore the basis for warrants, the other plausible interpretations, and the ethical bases for defending the consequences, are themselves constantly open to scrutiny and question.

Conclusion

In this paper, different approaches to educational inquiry have been characterised in terms of the hermeneutic notions of *text*, *context* and *reader*. Traditional 'positivistic' forms of research seek to produce texts (eg data, research findings etc.) whose meanings are shared by different readers, and across a variety of contexts. Other approaches (particularly those sometimes labelled 'qualitative') acknowledge the context-dependent nature of the research findings, but nevertheless seek to produce texts whose meanings are widely shared. However, research results that have widely shared meanings appear to be more difficult for teachers to 'make sense of' and to make use of in improving their practice.

The approach sometimes called 'action research' addresses this by not even trying to generalise meanings across readers—what matters is the meaning of the research findings for the teacher in her own classroom. This lack of generalizable meaning for action research is justified by its potential to transform the practice of the individual teacher. There appears, therefore, to be a trade-off between meanings and consequences. Put crudely, in action research, the lack of shared meanings are justified by the consequences, while in other kinds of research, the lack of consequences are justified by their more widely-shared meanings.

The tension between meanings and consequences was then further explored in terms of Churchman's five-fold classification of inquiry systems, based on what is taken to be the primary source of evidence:

Inquiry system	Source of evidence
Leibnizian	Reasoning
Lockean	Observation
Kantian	Representation
Hegelian	Dialectic
Singerian	Ethical values

Adopting a Singerian perspective, it was argued that educational research involved marshalling evidence that:

1) the interpretations made of the data were warranted;
2) the interpretations were more warranted than plausible rival interpretations;
3) the consequences of such interpretations were ethically defensible.

From the point of view of the individual researcher, the important message is that nothing that is written about the process of research relieves the individual researcher of the responsibility for the research she undertakes, and what happens as a result of that research.

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A MATHEMATICS ANALOGUE OF CHOMSKY'S LANGUAGE ACQUISITION DEVICE?

Carl Winsløw
Department of Mathematics
Royal Danish School of Educational Studies

Abstract. The aim of this paper is to suggest a perspective on mathematical knowledge and its learning which is radically different from both Platonism (an old favorite of mathematicians) and constructivism (currently popular among educators,); in particular, to stress that the denial of one does not imply the other. The starting point is what Chomsky (1986) calls Plato's problem: *explain how we know so much, given that the evidence available to us is so sparse.* Some major theories related to this problem in the contexts of linguistics and cognitive psychology are discussed. In particular, I propose a mathematics analogue of Chomsky's language acquisition device, and discuss the relations between the two that emerge from the description of mathematics as a linguistic 'register' (Pimm, 1991; Winsløw, 1998).

1. Background: LAD, LASS and so on.

Plato's problem arises for Chomsky (1957, 1988) in the following way: it can be theoretically argued that natural language is too complex to be acquired from finite input alone (thus, the constructivist idea of an initial *tabula rasa* must be abandoned). The radical Chomskyan proposal to solve Plato's problem for language acquisition is the hypothesis of the *language acquisition device* (LAD): an innate[1] mental structure, common to all human beings, which takes, as input, primary linguistic data – a finite number of more or less 'correct' utterances in the language to be acquired – and produces a 'grammar' of this language, i.e. a system of rules which allows the learner to speak the language creatively – to 'perform' linguistically. The original LAD hypothesis (Chomsky, 1965) is very strong concerning the detailed 'mechanics' of the device; as a reconstruction, I have represented the whole mechanism in process-diagrammatic form in Fig. 1. For a detailed explanation, the reader should consult (Chomsky, 1965, §1.6-8); the basic idea is that, given the finite input, LAD chooses among an infinity of in-built 'possible grammars' one which is in optimal consistency with the input.

The scientific status of LAD is, so far, unclear; at best, it is an unverified hypothesis about the innate language competence of the human mind, but as it is even quite unclear how it could be scientifically rejected, a Popperian critique might claim that *no substantially stronger claim to scientific status can be made for it than for Homer's collected stories from the Olympus* (Popper, 1963, §1). Of course, the point

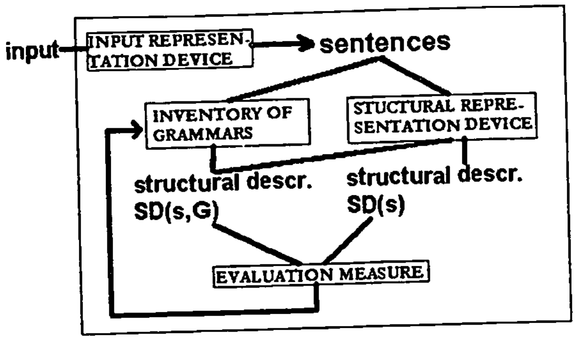

Figure 1: LAD

of bringing up the idea of a LAD here is not to reject it in this way (let alone that Popper's falsification criteria are widely acknowledged to be impracticable in any science). The aim is, first, to point out the LAD model as an outstanding example of an unverified hypothesis which has inspired many fruitful research efforts, in this case not only in linguistics, but also in educational psychology, as evidenced e.g. by (Bruner, 1983) and (Karmiloff-Smith, 1992). Secondly, and more importantly, alike hypotheses form the indispensable lifeblood of research regarding the mechanisms of cognition, where measurable evidence in a direct physical sense is still very sparse, in spite of recent progress (to which we return below). A parallel may be found e.g. in nineteenth century chemistry, which used notions such as valence and molecule as *convenient myths devised to help organize experience* (Chomsky, 1986, p. 7).

Later works on language acquisition have provided various auxiliary hypotheses and modulations, such as pointing out the necessity of taking into account the interaction between input and output in a language community setting, e.g. the language acquisition support system (LASS) discussed in (Bruner, 1983), or proposing domain-specific cognitive 'modules' as discussed (and questioned) in (Karmiloff-Smith, 1992). Recently, the notion of *metaphor* has been used as a central explanatory model for language related cognitive mechanisms, including natural language, e.g. in (Lakoff, 1987); the idea is also exemplified in the context of mathematics (Lakoff-Núñez, 1997).

Until fairly recently, the study of cognition had to be based solely on evidence of performance, with no possibility to gain direct evidence of cognitive structures[2]. The interest of theories (systems of hypotheses about cognitive structure) derived from analysis of performance is increased by current developments in cognitive neuroscience which suggest that measurable evidence for such theories may become available; see (Dehaene, 1997) for a leisurely introduction to this exciting area as it applies to mathematics. As in any experimental science, techniques to gain

evidence have to be complemented by theories suggesting what to look for; only explicit (and, at least initially, simple) theories will do here.

2. Linguistic aspects of mathematical knowledge.

The striking similarity between discourse in natural language and mathematical discourse (in the sense of (Winsløw, 1998)) has at least the following aspects, which form together a strong indication that Plato's problem arises for mathematical knowledge much in the same way as for language[3]:

- the complexity of the rules governing discourse,
- the extent to which consensus may be reached (within a community of discourse) about the rules,
- the implicit nature of the vast majority of those rules.

We already mentioned (in sec. 1) one sense of the first point in the case of natural language (even when considering only syntax), and Chomsky's argument is actually to demonstrate the possibility of embedding a non-finite state mathematical structure into English[4]. The third point is in fact an immediate corollary of the first, and poses Plato's problem exactly as for natural language. The second aspect is not theoretical but empirical in nature, but is theoretically essential to counterbalance the other two; without it, neither natural languages nor mathematics would represent useful means of communication. The paranthesis about consistency referring to a community is essential, though; it marks the difference between *languages* and *registers*, which we proceed to explain (see also Pimm, 1991). A *language* is a system of structured knowledge applicable to express human thoughts, while a *register* of language is a systematic way of using language in some specifiable setting[5]. Thus, the second aspect is about registers (within natural languages, and in mathematical discourse).

To penetrate deeper into the connection between natural language use and mathematical discourse, we shall study more closely how they interact. As suggested in (Winsløw, 1998) and further elaborated below, mathematical registers integrate natural language use with the use of certain symbolic languages:

1. Parts of the word and sentence inventory of natural language are *semantically defined or redefined* for use in the mathematical register.
2. Symbol strings may replace word strings in phrases of natural language, whose syntax remains otherwise unchanged. Replacement does, in this context, not mean that an ordinary meaningful phrase in verbal language is transformed into a phrase of the mathematical register by simply replacing some syntactic elements (e.g., a noun) by a symbol string, but rather that the phrase in the mathematical register can be *formally derived* from a syntactically correct (but possibly meaningless) phrase in verbal language by such a replacement[6]. Very specific rules can be given about how replacement takes place within a given mathematical register.
3. Finally, the symbolic inventory (producing symbol strings) has its own syntax, as explained in (Winsløw, 1998, Sec. 3.2). The basic concepts (called *universal*

syntactic features) here are those of *object*, *relation* and *operator*, and for syntactic relations among symbol strings the derived concept of *transformation*. The syntax of symbol strings (at a deeper level than the universal syntactic features, for instance, the construction of objects), and not least their *semantics*, can be given in detail only for very specific subregisters, as it is highly context dependent (cf. e.g. Woodrow, 1982).

The most important point here is the crucial *relation* between natural language and mathematics – the fact that mathematical knowledge, understood as knowledge of the mathematical register, is to a large extent *depending on* knowledge of natural language. The impact of childrens language background on their performance in mathematics learning contexts has indeed been the subject of numerous studies[7], and it has been shown beyond reasonable doubt that this influence is both strong and many-sided, even when discounting external factors that correlate with natural language capacities[8]. We need hypotheses – convenient myths – regarding the intrinsic source of those correlations. As was argued in this section, such a model of mathematics acquisition will have to relate to (or even contain) a model of language acquisition. One may consider the LAD as a simple yet significant model of the last sort, and we use it as a working hypothesis for natural language acquisition in the following.

3. Contours of a minimal MAD.

Our task now is to determine a minimal set of competencies for acquiring mathematical knowledge which are not themselves acquired, but which are part of the human cognition apparatus. It is obvious that minimality is most desirable, as we are not willing to settle with the easy but mystifying Platonic solution (claiming *all* mathematical knowledge to be 'built in'). On the other hand, as argued in sec. 2, if the overall aim is a model *accounting* for all mathematical knowledge – not just the chess-type parts – then this minimal set is not empty. In fact, language acquisition competencies are necessary but not sufficient. The existence and importance of such a minimal solution is a main point of this paper. Notice that *uniqueness* is neither implied nor claimed to be implied.

The discussion in Sec. 2 does suggest some elements of the cognitive faculties which the human capacity to learn and foster mathematical knowledge seems to necessitate. The most basic one is the ability to perceive symbolic language and to distinguish it from natural language. This it not in itself a faculty which is solely related to mathematics in a strong sense (like mathematical registers as considered here); the human use of symbolic inscription and signification is likely to be as old and broad in scope as natural languages. We may think of this as a supplementary feature belonging to the 'input representation device': the capacity to perceive and distinguish symbol type input in different surface forms (audile and visual). This faculty is related to 'mental imagery' as studied in cognitive

psychology (Paivio, 1971) and some of its effects are (in a more direct way) studied in iconography. Where the properly mathematical enters the scene is in the ability to represent and manipulate structures involving states and processes of symbolic entities. This faculty clearly pertains to structural description of represented symbol input, that is, it belongs to what is called the structural description device in Chomsky's LAD. It accounts for the universal syntactic features of mathematical knowledge: some symbols are perceived as distinguished *objects* which may be found in the state of *relation* to other objects, while other symbols represent processes by which objects are changed (*operators*); and finally that such processes induce new relations when related objects are changed by the same operator (the state structure is changed by what we call *transformation*). However, the concrete transformational rule relate different structural descriptions and hence it belongs to the domain of 'grammar', just as in the case of natural language. Assuming further the capacity of evaluating different transformational grammars of symbol language against structural descriptions of represented input, we have situated all the elements of a MAD (construed in analogy and even in addition to the LAD) required to account for the universal syntactic features of mathematical symbol language.

To accomplish our task set out at the beginning, we must reconsider the phenomena of (re)naming and replacement (point 1 and 2 in sec. 2) by which the mathematical register is brought together as a whole.

Naming, which effectuates semantic (re)definition, carries greater weight in mathematics than the mere assignment of lexical signifiers (Sfard, 1991). I believe it is important to distinguish two different naming procedures here, as suggested by the following examples.

(a) 'A function G from A to B is a subset of $A \times B$ such that for every $a \in A$ there is precisely one $b \in B$ such that $(a,b) \in G$.'

(b) 'Let G be a function.' (Equivalent: 'Let a function be given; call it G.')

The first kind assigns a name to a mathematical structure, and this relation name-structure becomes part of the 'lexicon' much as in the case of natural language. If we are, at a later point, told that a function is a non-commutative group, this leads to conflict. By contrast, the second kind of naming feeds a short-memory name-symbol lexicon which can easily be changed the same way by later namings. The last naming of a symbol prevails and must often be available in the lexicon *before* applying the syntactic parts of the structural representation device to the whole sentence (as in 'Let G be a function with $G' \geq 0$...'). Any revision of the structure-name lexicon must be based on the structured description of the whole sentence, as in the feeding of the natural language lexicon. To sum up: the structural representation device contains two semantic components handling two kinds of naming, one which feeds a symbol-name lexicon directly, and one which marks (parts of) the sentence as a naming of structure, which (via the evaluation measure) causes a relation name-structure to be

activated in the corresponding lexicon. Both lexicons are part of the grammar used in (and affected by) the structural description of a sentence.

The simplest way to accomodate the phenomenon of *replacement* at this final stage, seems to be twofold: an 'inverse replacement' component of the structural representation device, adding to the structural description a base phrase from natural language from which the sentence can be derived by lexical insertions; and an addition to the grammar, consisting of rules for replacement by symbol strings (just as rules of replacements are built into the grammar of natural language, cf. (Harris, 1965, §2)).

The tentative model of a MAD developed above is summarised schematically in Fig. 2, which contains (and is built upon) Chomsky's LAD as interpreted in Fig. 1.

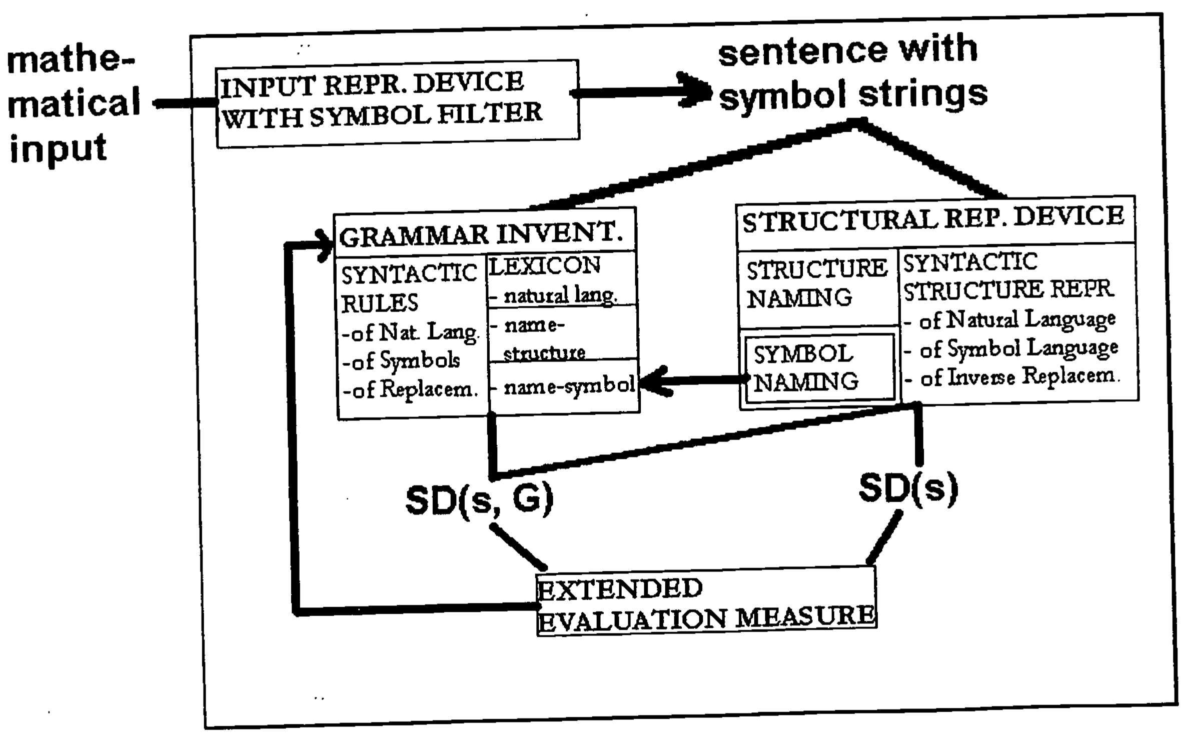

Figure 2: MAD

This picture only aims to reflect the 'acquisitionist' aspects of learning, but it should be clear that output (and hence participation in mathematical discourse) is not ignored because of this. On the contrary, the value of the evaluation measure (specific lexicons and syntactic rules) at any time reflects the linguistic type knowledge based on which the learner performs mathematically (or, as LAD is built in, using natural language alone). What cannot be accounted for by such a model is the actual

production of output. However, it is clear that in addition to linguistic type knowledge, a central item is *memory and associative use of it.*

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Notes.

[1] Notice that this does *not* imply acceptance of Plato's idea world, but may be explained (as Chomsky does) as a product of evolution.
[2] As an early exception, Skemp (1987) noticed with enthousiasm what he saw as a neuropsychological confirmation of his analytically based theory of visual-geometrical thinking as opposed to verbal-algebraic thinking; Skemp's theory was published in 1971, and the relevant work in neuropsychology (due to Shannon) in 1980.
[3] This is in a way also the original setting, as described in Plato's dialogue *Menon.*
[4] In (Chomsky, 1957, n.3), it is also explicitly noted that *any formalized system of mathematics or logic will fail to constitute a finite state language.*
[5] The definition of concept of *register* goes back to (Halliday, McIntosh, Strevens, 1964; p. 87), although formulated in different wording and with a narrower definition of language than given here.
[6] In more technical terms: the *phrase structure* is the same, but in addition to usual lexical insertion, some elements of a phrase may be filled by symbol strings. One may think of the latter as 'replacing' a natural language string by the symbol string, although no such replacement actually takes place.
[7] A classical collection of such studies may be found in (Cocking and Mestre, 1988).
[8] That is, one has identified even purely *intrinsic* effects, in the sense of (Saxe, 1988).

DIVISION WITH FRACTIONS IS NOT DIVISION BUT MULTIPLICATION: on the Development from Fractions to Rational Numbers in terms of the Generalization Model Designed by Dörfler

Takeshi YAMAGUCHI
Fukuoka University of Education, Japan

Hideki IWASAKI
Hiroshima University, Japan

(Abstract) Division with fractions is the "capstone" of elementary school arithmetic and the "cornerstone" of algebra. It is a teaching material of calculation depending finally upon quantities and considering division as the special case of multiplication. Conceptual understanding of both the expression and algorithm on it is traditionally planed based on the proportional relation about two concrete quantities in the elementary school of Japan. Only simple procedure "invert and multiply", however, is confirmed as a result of teaching and learning for lack of "constructive abstraction" in the sense of Dörfler's generalization.

Most students cannot reach at the idea of division by way of schema of proportion, and moreover schema of share and measure known by students cannot induce and deduce division with fractions. We, therefore, propose here that schema of comparison is quite applicable for making the expression about it, and algebraic treatment learned by students is much relevant for understanding the algorithm of the expression. As a matter of fact, both of them were abstracted constructively in the classroom lesson of sixth grade(twelve years old). The educational significance mentioned at the opening paragraph was realized in this practice, and this teaching and learning process was more successful than the present treatment based on the proportional reasoning.

1. Conceptual Understanding of Division: separation of division with natural numbers and that with fractions

Conceptual understanding of division with fractions is traditionally and carefully taught on the base of proportional relation between two concrete quantities in the sixth grade of Japan. Proportional reasoning plays an important role there, and moreover in the transition from elementary mathematics to advanced mathematics as Lesh, Post, and Behr say as follows (1988, pp.93-94):

> *We view proportional reasoning as a pivotal concept. On the one hand, it is the capstone of children's elementary school arithmetic; on the other hand, it is the cornerstone of all that is to follow.*

What is firmly established in the sixth graders, however, is only the procedure "invert and multiply" concerning a divisor. Most of them cannot make a reason why division becomes multiplication suddenly in the case of fraction as well as they cannot often make a division expression to the word problem including fractions. Such an outstanding difference between the concept and procedure is due to the following fact: the schema of share and measure concerning

divisions with natural numbers learned by students cannot be assimilated and accommodated to that of proportion on division with fractions. Constructive abstraction about this, therefore, is almost impossible on the presupposition of partitive and quotitive divisions because their characteristics are quite different from that of division with fractions except dividend(See Table 1). They are definitely division as it is, and cannot induce or deduce multiplication with the inverse of a divisor.

The idea of Lesh, Post, and Behr is theoretically right but is not practically right in the case of division with fractions as the national achievement survey showed in Japan. Many teachers also know this fact from their experience. The expression and algorithm of division with fractions, therefore, should not be induced and deduced over the proportional relation between two concrete quantities. The expression of it can easily get the meaning in the schema of comparison which is as fundamental as that of share and measure. The algorithm of it can be deduced from the calculation rule which students have already learned. This is our proposal and point.

Table 1. Characteristics of Partitive and Quatitive Division

	Partitive Division	Quatitive Division
Image	Sharing a quantity into equal parts	Measuring a quantity with some quantity
Meaning	To ask how much the unit is	To ask how many units there are.
Dividend	Rational number	Rational number
Divisor	Natural number	Rational number
Quotient	Rational number	Natural number
Remainder	No existence	Principally existence

2. The Result of the National Survey concerning Division with Fractions

The achievement survey, which was implemented by Ministry of Education in Japan, 1994 and had 16,000 samples in each grade fifth and sixth, shows that 27.2 % of students correctly answered the word problem mentioned at Table 2 to make an expression of division with fractions. On the other hand, 89.7 % of same students calculated the division expression with fractions $2/7 \div 3/4$ correctly. Furthermore in the same survey, 65.9 % of students in fifth grade correctly answered the word problem mentioned to make an expression of division with decimals which has the same mathematical structure as the case of fractions. There is much difference between division with fractions and that with decimals in making an expression.

Both are typical problems for introducing division with decimals and fractions in each grade fifth and sixth of the elementary school. To make a division expression as well as to understand its algorithm in both cases are taught and learned on the base of proportional relation between two concrete quantities in the classroom lesson. Making a division expression first, the case of decimals and fractions are extrapolated into the case of natural numbers referring concrete quantities. Next its algorithm is explained using usually two number lines which

Table 2 Achievement tasks of division with fractions and decimals

(Problem for fifth graders) There is a steel pipe. Its weight is 4.2 kilogram to the length of 3.5 meter. How much weight is it for one meter steel pipe? Write an expression to get an answer.
(Problem for sixth graders) There is a water tank. Water is poured 5/6 liter per 2/3 minute into the tank. How much water is poured for one minute? Write an expression to get an answer.

represent proportional relation figuratively. Proportional reasoning, therefore, plays a fundamental roll to understand not only the division expressions but also their algorithm.

But there is big difference in the rate of correct answer between two problems although they have the same mathematical structure. According to the Dörfler's generalization, "constructive abstraction" and "intensional generalization" are well conducted in classroom lesson about division with decimals consequently speaking. On the other hand, division with fractions has a serious problem in the process of intensional generalization for understanding the algorithm even if it succeed in constructive abstraction for making the expression.

Division with decimals is division and not multiplication. It is a kind of extension of division with natural numbers. But division with fractions is not division but multiplication. This algebraic reconstruction cannot be overwhelmed only by proportional reasoning. We think that whether there is this transformation or not reflects the results of the national survey consequently, and this is the problem of "constructive abstraction" and "intensional generalization".

3. Analysis of Teaching and Learning of Division with Fractions from Dörfler's Generalization Model

(1)Critical Analysis of Division with Fractions Based on Schema of Proportion from Dörfler's Generalization Model

In the teaching and learning process of division with fractions, division is supposed to be integrated into the special case of multiplication with the extension of meaning of division. Therefore we can regard conceptual understanding of it as a kind of generalization process. In this sense, Dörfler's generalization model (1991) as shown in Fig.1 gives us some suggestions which clarify the reason of serious detachment between making the expression and understanding the algorithm about division with fractions.

In Dörfler's generalization model, the pre-stage of "symbols as objects" is called "constructive abstraction", the objective and method of activity are reflected upon first and the properties of elements of activities and the relation of elements are extracted as a result. Furthermore, the applicability of these properties and relation is considered in the "extensional generalization". After this stage, symbolical description of the invariants is detached from the original context and the symbol itself has become object of thinking. This is to say "symbols as

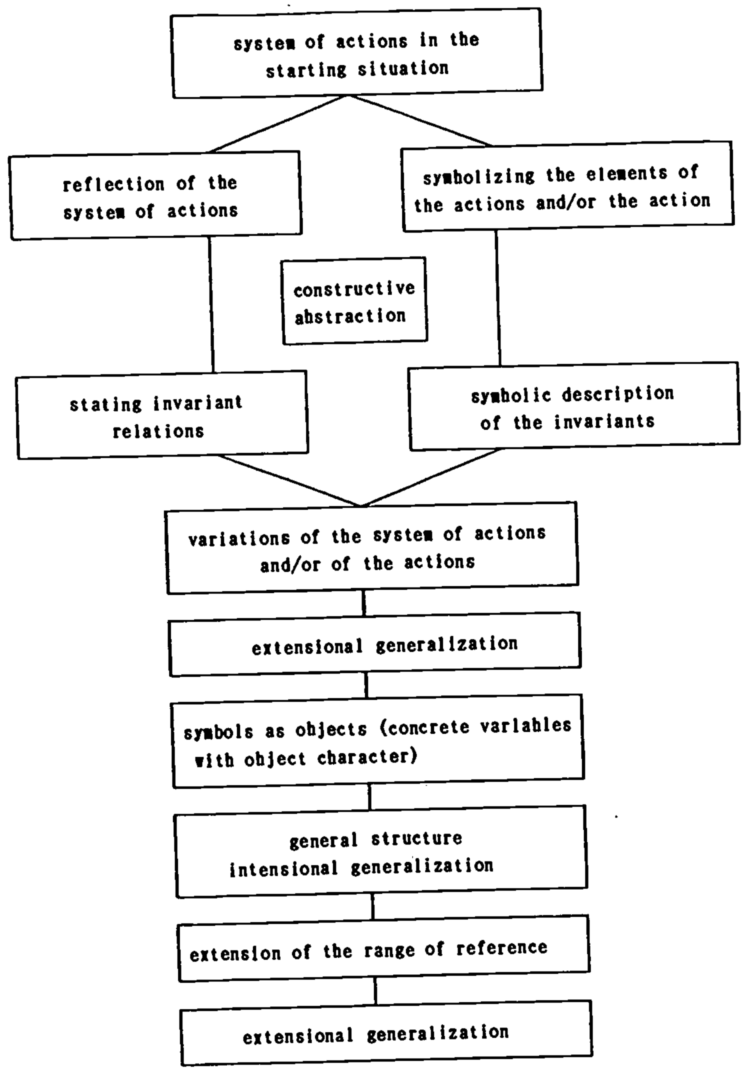

Fig.1 Dörfler's generalization model (1991,p.74)

objects", which connotes to proceed to "intensional generalization". It is a symbolic operation to multiply divisor and dividend or denominator and numerator of fraction by the same number on the assumption of "symbols as objects".

Conceptual understanding of calculation is composed of an expression and its algorithm. The expression is induced from a word problem using a number line and extrapolation particularly in the case of division with both decimals and fractions. After that their algorithms is deduced in terms of proportional reasoning.

Making an expression corresponds to constructive abstraction and understanding its algorithm to intensional generalization in the Dörfler's generalization. The algorithm of division with decimals is intensionally generalized into that of division with natural numbers through proportional relation between two quantities. On the other hand, the algorithm of division with fractions is intensionally generalized into proportional relations, and transformed into multiplication. Most students suppose that they invent new algorithm "invert and multiply" from division. They, therefore, repeat constructive abstraction for understanding the algorithm although they arrived at "symbols as objects" through it.

The similar processes in the different meaning interfere mutually and restrain each other. The process of making an expression consequently fades away from cognition and the procedure "invert and multiply" fades in it and is firmly established as a result. We think this is the reason why there is a big difference in the rate of correct answer of the national survey. In other word of the Dörfler's generalization, the division with decimals could have "constructive abstraction" but the division with fractions could not have "constructive abstraction". Then we should

choose another schema and devise the teaching way other than the schema of proportion about division with fractions.

(2)Division with Fractions Based on Schema of Comparison as the Alternative Framework

In order to overcome this issue, we propose new instructional framework based on "schema of comparison". Schema of comparison has two mathematical aspects which are *difference* and *ratio* between two quantities. One typical word problems in the context of schema of ratio is "*There are the tape A and tape B. The length of tape A is 3/5 meter and that of B is 2/7 meter. How many times is tape A as long as tape B ?*". This kind of situation more easily induces the division expression than that of the proportional relation.

After making the expression, however, it is a significant issue that schema of comparison can not work in itself as the driving force which forms conceptual understanding of the procedure "invert and multiply". So students are requested to remind of some rules about fractions and division learned by them, and to figure out the algorithm as follows:

(A)

$$\frac{3}{5} \div \frac{2}{7}$$
$$= (\frac{3}{5} \times 35) \div (\frac{2}{7} \times 35)$$
$$= (3 \times 7) \div (2 \times 5) = \frac{3 \times 7}{2 \times 5}$$
$$= \frac{3}{5} \times \frac{7}{2}$$

(B)

$$\frac{3}{5} \div \frac{2}{7} = \frac{3 \times 14}{5 \times 14} \div \frac{2}{7}$$
$$= \frac{3 \times (14 \div 2)}{5 \times (14 \div 2)} = \frac{3 \times 7}{5 \times 2}$$
$$= \frac{3}{5} \times \frac{7}{2}$$

These ideas are not related to quantities directly but mathematical operation of fraction and calculation. Furthermore these activities would contribute to build up new conceptual network among properties of fraction, rules about calculation, and cognitive schema of division depending on partitive or quatitive division, which is the essence of the "intensional generalization" in the sense of Dörfler's generalization.

4. Design and Evaluation of Teaching Practice Based on Schema of Comparison

(1)The Result of Pre-test of Division with Fractions

Prior to the teaching practice in this study, we conducted the pre-test of division with fractions. Subjects in this pre-test were sixth graders at Ukiha Elementary School in Fukuoka, Japan. Problems in this pre-test are shown as below. Problem 1 is about division with fractions on the base of schema of proportion and Problem 2 is about that of comparison. We should note that sixth graders had already learned division with fraction before we gave them this test.

---

1. We can paint 2/5 m^2 of the board per 3/4 *dl*. How much area of the board can we paint for 1 *dl* ?
2. The length of Kaori's tape is 2/3 meter and that of Shiho's tape is 5/7 meter. How many times is Shiho's tape as long as Kaori's tape ? (Note: Kaori and Shiho are names of Japanese girls.)

Results of this pre-test are shown in Table 3. Only four of sixth graders (14%) made correct expression in Problem 1, although they had already learned this subject matter. In addition to this, table 4 shows the number of students who made the expression of division in Problem 1 and 2 respectively. According to this table, we can see the tendency that the word problem based on schema of comparison is more closely related to division expression than that of proportion. As Problem 2 is briefly requested to find the value of ratio between two quantities, the schema of comparison is accessible to that of measure for quatitive division even if both divisor and dividend are fractions. This is one of reasons why this difference between division based on schema of proportion and that of comparison is caused. According to these two results, at least, we can suppose that making an expression based on schema of comparison is easier for sixth graders than that based on schema of proportion.

Table 3. Results of Pre-test (N = 28)

	Problem1 (Schema of Proportion)		Problem2 (Schema of Comparison)	
	The Number of Students	Method	The Number of Students	Method
Correct	4	2/5 ÷ 3/4 (4)	9	5/7 ÷ 2/3 (9)
Wrong	2 0	3/4 × 2/5 (8) 3/4 ÷ 2/5 (7) 4/4 × 2/5 (2) Others (3)	1 7	2/3 ÷ 5/7 (13) 5/7 × 2/3 (3) 5/7 − 2/3 = 3/4 (1)
No Response	4		2	

Table 4. The Number of students who made the expression of division

Problem	Schema	The number of students		
1	Proportion	11	Correct	4
			Wrong	7
2	Comparison	22	Correct	9
			Wrong	13

(2)Design and Evaluation of Teaching Practice Based on Schema of Comparison

Based on the results in the pre-test, we designed teaching practice of two lessons based on schema of comparison for sixth graders. The objective of this teaching practice was to verify empirically that students could make the expression of division with fractions based on schema of comparison and deduce the procedure "invert and multiply" logically, referring and using properties of fractions and rules about the calculation which students had already learned. The class was, for necessity, divided into small groups of 6 to 7 members so that they felt it easy to make discussion. We report here the brief sketch of sixth graders' activities below.

At the beginning of this lesson, the following problem situation was presented by a teacher: "*Hana has the tape whose length is 3/4 meter. Yuki has 2/5 meter. Compare the length of Hana's tape with that of Yuki's tape*". At the earlier stage of this lesson, to this task, some students showed three types of method as follows:

(a) $\frac{3}{4}-\frac{2}{5}=\frac{3\times5}{4\times5}-\frac{2\times4}{5\times4}=\frac{15}{20}-\frac{8}{20}=\frac{7}{20}$

Hana's tape was longer than Yuki's one by 7/20 meter.

(b) $\frac{3}{4}\div\frac{2}{5}=\frac{3}{4}\times\frac{5}{2}=\frac{15}{8}$

The length of Hana's tape was 15/8 times as long as Yuki's tape.

(c) $\frac{3}{4}:\frac{2}{5}=\frac{15}{20}:\frac{8}{20}=15:8$

(The length of Hana's tape) : (The length of Yuki's tape) = 15 : 8

Most students could easily make an expression of division with fractions such as (b) or understand meaning of this expression by others' explanation.

After this, teachers asked students to explain the reason why we could calculate division with fractions by the algorithm of "invert and multiply" since they had already learned it. Students, however, felt much difficulty as we anticipated it because the proportion schema could not be driven for the explanation and the comparison schema could not either generate the algorithm of multiplication with the inverse of divisor in itself. Then a teacher need to inform of some mathematical knowledge known by students. The teacher helped them to reconfirm the property of fractions such as "*The value of fraction is constant when both denominator and numerator are multiplied by the same number.*" and to make sure of rules about division such as "*The quotient of division is constant when both divisor and dividend are multiplied by the same number.*" (for example, "0.6 ÷ 0.2 = 6 ÷ 2"). There happened to be a student who explained the way of calculation by the idea of transformation of division with fractions into that with decimals as follows.

$$\frac{3}{4}\div\frac{2}{5}=(3\div4)\div(2\div5)=0.75\div0.4=75\div40=\frac{75}{40}$$

Though this idea itself did not produce the algorithm "invert and multiply" at all, students could apply another meaning of fractions such as "a/b = a ÷ b".

Then a teacher gave students a word of advice to multiply both divisor and dividend by any other numbers than 100. Following this advice, one students wrote the following explanation on the blackboard and explained in detail.

$$\frac{3}{4}\div\frac{2}{5}=(\frac{3}{4}\times20)\div(\frac{2}{5}\times20)=(3\times5)\div(2\times4)=\frac{3\times5}{2\times4}=\frac{3\times5}{4\times2}=\frac{3}{4}\times\frac{5}{2}$$

Some opinions about this explanation were exchanged among students in the class as follows:

S1: Why is the multiplier twenty in this expression ?

S2: It is the reason why twenty is the least common multiple of four and five.

S3: Why is the expression " $(\frac{3}{4}\times20)\div(\frac{2}{5}\times20)$ " transformed into the expression "(3 × 5) ÷ (2 × 4)" ?

S4: The reduction of four and twenty or that of five and twenty produce this transformation of the expression.

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S3: I don't understand your explanation.

S4: We can reduce twenty to five in the fraction $\frac{3 \times 20}{4}$ like this (reducing twenty to five) and get the expression "3 × 5".

S5: I think that the fraction " $\frac{3 \times 5}{2 \times 4}$ " will be transformed into the expression " $\frac{3}{2} \times \frac{5}{4}$ " by this manner. Is it OK ?

S6: Because we can change "2 × 4" into "4 × 2", we can get the algorithm " $\frac{3}{4} \times \frac{5}{2}$ ".

Students in the class: All right. I can understand the meaning of the procedure "invert and multiply".

As this protocol shows, students in this class deduced the algorithm of division with fractions logically for themselves, referring properties of fractions and rules about division, and negotiated the meaning of it through the social interaction among students in the class. These exchanges of ideas represents the process of building up new network by means of combining prior cognitive schema of division with properties of fractions and rules about division.

5. By Way of Conclusion

We consider division with fractions as the transitional teaching material from arithmetic to algebra. According to Lesh et al., proportional reasoning is supposed to play a role of bridge between them. And division with fractions is carefully taught in sixth grade in Japan in terms of proportional reasoning. But the conceptual understanding of it is not taken place but only the procedure "invert and multiply" is firmly remained as shown in the national survey. Therefore it is our research concern of this paper to overcome this difficulty on the instruction of division with fractions by setting up new instructional framework based on schema of comparison as the alternative instructional framework.

As a result, in the teaching practice in this paper, we have confirmed that sixth graders easily made an expression of division based on schema of comparison, and that most of them can deduce the procedure "invert and multiply" logically and understand conceptual meaning of this algorithm sufficiently, referring properties of fractions and rules about division.

In the present practice of division with fractions, proportion schema does not work substantially as the bridge. On the other hand, comparison schema facilitates to cope algebraically with the division expression because the expression is well established by comparison schema but can not be calculated in terms of it. In this sense, division with fractions realizes its teaching objective mentioned at the head of this section. This is our proposal and main finding in this paper.

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TEACHING MATHEMATICAL MODELING WITH A COMPUTER ALGEBRA SYSTEM

Nurit Zehavi and Giora Mann

Weizmann Institute of Science, Israel   Levinsky College of Education, Israel

Abstract

Here we present an approach to studying mathematical models using a Computer Algebra System (CAS). Pupils (ages 13-14, n = 141) were introduced to the mathematical model of the classical problem "How long did Diophantus live" They examined the model by solving problems, manipulating the parameters, and inventing similar story problems; and found that the model is more than an equation and that the implicit restrictions are an integral part of the model. The interaction of pupils with the computer disclosed some of the cognitive processes involved in the modeling activities and caused teachers to reflect on their methods of teaching.

Introduction

Probably all generations of pupils, who studied algebra, were introduced to the famous problem from the Greek Anthology, "How long did Diophantus live". It is known that Diophantus (about 250 AD) solved arithmetical problems, not equations, but he was interested in *exact* rational solutions. Therefore, his title as the father of algebra is a matter of historiographical study (Boyer, 1968, pp. 196-216). Father of algebra or not, his place in algebraic textbooks is secure.

Algebraic word problems are notoriously difficult to solve for most pupils. The challenge for educators is to somehow overcome the difficulties. We describe here a few attempts that are relevant to our approach. Stacy and MacGregor (1995) used different verbal descriptions of the same problem to encourage pupils to form mental representations. Many pupils, as they worked, extended their mental models to encompass additional features of the mathematical structure. Hoz et al. (1997) studied in depth the role of structural and semantic factors in the solution of speed problems. They recommend that teachers use similar/isomorphic problems in order to clarify the inference of the relations embedded in the structure of the problems. Inspired by cognitive science, educators attempted to use computers to enhance word problem solving. One such system, *Animate* (Nathan, 1992) creates animations of the story problem derived from the algebraic equations constructed by the pupils. Experiments with the system indicated that the animated simulations of the problem provided

experiential feedback for error detection and correction (Nathan, 1998). In this paper we approach mathematical modeling using a Computer Algebra System.

The appearance of Computer Algebra Systems (CAS) on personal computers in the mid-eighties opened the way to new teaching strategies and curricular developments. Research on the use of CAS in mathematics education is relatively new. Most of the work in this area concentrated on the student-knowledge-technology triangle, realizing that computer algebra software does not answer questions; rather it reacts to an action and produces something that needs to be interpreted. Therefore it is essential to teach pupils how to make the most effective use of the output of the software (Hunter et al., 1995). It is commonly believed that a CAS, by freeing students from syntactic manipulation, allows them to concentrate on semantic or conceptual aspects of algebraic reasoning. Pozzi (1994) studied this in a qualitative case study with students of ages 16-17. He has pointed out that students, who do not fully comprehend a CAS output, will develop informal and possibly erroneous ideas of what the computer is doing. Pozzi has concluded that using a CAS may necessitate a closer conceptual understanding of the algebraic manipulations. With a CAS executing the procedures efficiently, we can now reflect on relationships that connect the formal notation and the procedures for performing mathematical tasks (Zehavi, 1996, 1997). The implication of studies on learning with a symbolic manipulator, is that the teacher plays a far more complex role while teaching in this new environment (Heid, 1996). This was the motivation for Drouhard (1997) to generalize the *didactic triangle* (student-knowledge-teacher) by defining the *double didactic pyramid* with two new vertices, the CAS and the group. He challenges researchers to pursue studies in mathematics education that investigate the construction of knowledge in CAS environment.

The MathComp project

The MathComp (mathematics on computers) project was initiated in 1996 in the Science Teaching Department at the Weizmann Institute of Science with the aim of integrating CAS into teaching, to improve the learning of mathematics with the following goals.

Student goals:

1. Creating a network of relations between concepts and the procedures leading to them. This became possible because CAS frees the student from the tedious work involved in carrying out the procedures;
2. Enriching the mathematical language used by students while they "do" mathematics;
3. Developing independent and critical thinking especially while using CAS;
4. Using technology to stimulate and motivate students to do mathematics.

Teacher goals:

1. Using technology to increase teachers' awareness of the cognitive aspects of learning and to design tools to deal with these aspects;
2. Challenging rethinking of curricular and didactical aspects of mathematics learning and teaching;
3. Developing independence and creativity when incorporating CAS into the professional lives of mathematics teachers.

These goals seem adequate to any technology-based educational project. Indeed, what really matters are the ways to achieve these goals. We decided to develop *mathematical learning units with computer algebra* to be worked out in a computer lab. The units accompany the syllabus for junior high school mathematics. Each of the units that have been developed includes a variety of tasks aimed at improving skills, understanding concepts, and investigating problems. These tasks create a network of mathematical connections, which in turn, give deeper meaning to the subject at hand and open windows for new mathematical experiences.

The basic assumption in the design of the MathComp units is that for the time being, pupils do algebra with paper and pencil in class, and use CAS in the lab to broaden learning opportunities and to promote mathematical understanding. This assumption is affected by practical circumstances, as well as, by the current state of CAS in education. The topics that are dealt with in the units were selected by considering the potential use of *Derive* (by Soft Warehouse) to achieve new didactic opportunities, new learning strategies, and new curricular relations. It is our hope that the units will act as a stimulus for teachers developing their own tasks. Here we bring a unit for Grade 8 "Equations and problems". We explain the rationale of this unit and the formative development process of the tasks. An exploratory study in four classes that worked on the unit using the CAS software *Derive* is described, and the conclusions of the study are discussed.

Equations and problems: A learning unit with *Derive*

The unit "Equations and problems" deals with building a model (equation) for a 'story' problem. It is clear that CAS cannot translate a story into an equation - so how can a CAS support modeling activities? Our idea was to choose story problems that do not fall into the common schemas (e.g., velocity problems, mixture problems) for which teachers and their pupils 'pretend' to have algorithms for solving the problems. The old puzzle about the *School of Pythagoras* (Problem 1) seemed suitable for our needs. This problem also includes fractions, which are both intimidating and difficult to calculate for pupils, but not for CAS. The idea was that pupils will interact with the software by using the *solve* command and other commands for feedback and for debugging their models. After solving the problem, we do not proceed immediately to another problem; rather, we challenge the pupils to invent similar word problems of

their own. The main goal of such an activity is to shed light on the equation itself as an object for exploring, and not as just a tool for finding an answer to the puzzle. The design of similar problems forces the designer to realize that there are implicit restrictions in the model, in addition to the equation itself.

Problem 1: Pythagoras' school

Pythagoras, who lived in the sixth century BC, ran a school. He was once asked how many students are in his school. After thinking for a while, he said:

1/2 of the students do mathematics.

1/4 of the students deal with science.

1/7 of them are silently exercising their mind.

And in addition to all the above we have three girls.

How many students were in Pythagoras' school?

We have observed that pupils are motivated to play "teacher" and to express themselves by inventing problems close to their heart. Naturally, some girls expressed their disagreement with the "three girls" item. However, in the first experiments we realized that they write whatever comes to their heads, sometimes without writing the algebraic model. Those who set equations and applied the solve command often got 'unreasonable' solutions (e.g., a negative solution for the equation $x/2 + x/3 + x/4 + 5 = x$). We therefore need to prepare them to reflect on their stories/models. Hence we added tutorials for making explicit the implicit restrictions of the model. For example: (a) "Pythagoras' secretary reported that 1/2 of the students do math, 1/3 do science, 1/6 think in silence, and there are also 3 girls". The model that describes this story simplifies to $x + 3 = x$, which evidently has no solution; (b) How do we change the problem so that the number of students in the school will be 280 instead of 28? The second example (which, in fact was suggested by a pupil) leads to the relation between the number of girls in the school and the total number of students.

The unit includes a built-in assessment tool based on the famous story about Diophantus (Problem 2). The model is similar to the previous one, only a little more complex. Both models simplify to an equation of the form $25/28 \bullet x$ + "number" = x. Again, pupils are encouraged to write their own story. (Note that we modified the authentic story, in which a son was born and later died, because we realized that the context of students' stories was affected by the tragic fact.)

Problem 2: Diophantus' life

A puzzle, similar to the following, was written on the grave of Diophantus:

Diophantus spent 1/6 of his life as a child, 1/12 as a young man, and more 1/7 of his life as bachelor.

Five years after he got married he left his hometown.

He returned to his hometown 4 years before his death.

Diophantus stayed away from his hometown ½ of his life.

How long did Diophantus live?

Exploratory study

An exploratory study was carried out in four Grade 8 classes (n = 141) of average and above ability. The students worked on three *Derive*-based units in which they became familiar with the following computer procedures: substituting numbers in expressions, solving (in)equalities, and performing operations on both sides of an (in)equality (using the *F4* key). Then they were asked to solve Problem 1 and to explain how they used *Derive*. We show three lines from a *Derive* file created during one pupil's work:

#7 $x - \frac{x}{2} + \frac{x}{4} + \frac{x}{7} = 3$ *User*

#8 $\frac{25x}{28} = 3$ *Simp(#7)*

#9 $x = \frac{84}{25}$ *Solve(#8)*

In the following lines the student modified the equation until he got the correct one. All four teachers were impressed by the fact that after about 10 minutes, all the pupils managed to get the correct answer, "28 students", with only a little help from the teacher and their classmates. This was not so when students in other classes worked on the problem with paper and pencil.

In Table 1 we summarize students' work on Problem 1 based on what students wrote in their worksheets and on *Derive* files that were created during their work. In row 1 we see two equations that represent the two basic types of equations that students came up with. The first is a straightforward translation of the story, and the second is

the result of some data processing that pupils did in their heads. (It is not surprising that the pupils who constructed the second model could later find more easily than the others that we get the solution, x = 280, if we change the number of girls from 3 to 30). In the last row we have the percentages of pupils who concluded their answers, without being explicitly asked, with the calculation of the number of students engaged in various activities of the story.

Table 1: Distribution of solutions to Problem 1

Solution to Problem 1: School of Pythagoras		Class A n = 38	Class B n = 33	Class C n = 36	Class D n = 34
EQ	$\frac{x}{2}+\frac{x}{4}+\frac{x}{7}+3=x$	84%	100%	75%	60%
	$x-x(\frac{1}{2}+\frac{1}{4}+\frac{1}{7})=3$	16%	0%	25%	40%
PROCEDURES	soLve, (Simplify)	35%	45%	30%	73%
	Simplify, soLve	60%	15%	40%	18%
	F4 - Operations on both sides , Simplify, (soLve)	5%	40%	30%	9%
Calculation		5%	3%	33%	20%

We discussed with the four teachers the distribution in each class and the differences between the classes. Each teacher described his didactical strategies, techniques, and goals in teaching algebra in the class. Together we came up with the conclusion that pupils' work on the problem with *Derive* reflect their previous experience. The teachers could make specific inferences to their teaching.

Another source of insight to the teachers was the story problems that students wrote that followed Problem 2. The number of complete story problems (i.e. story, equation and reasonable solution) in each class varied from 33% to 61%. To illustrate a typical story and model formulated we show in Table 2 a *Derive* file related to the story of Uncle Moses who challenged his friends, on his birthday, with the puzzle: *How old am I?* The file illustrates, somewhat transparently, the monitoring, evaluating, and planning processes that were part of pupils' reasoning. We can see that he used the *solve* command for monitoring the model, and the *simplify* command for evaluating the expressions. It seems that in his planning he referred to the fact that the sum of the fractions should be less than 1(see line #7), but the role of the "number" in

determining the solution was not yet clear to him (see the trial and error in lines #10 - #12). Based on the findings of the study, we are currently revising the unit. We are considering including of a parametric form of the model, which can easily be manipulated by the software, to enhance the understanding of the structure of model.

Table 2: Modeling the story of Uncle Moses' birthday

1	2
#1: "Uncle Moses' birthday"	#9: $\frac{19\cdot x}{20} + 7 = x$
#2: $\frac{1}{2}\cdot x + 3 + \frac{1}{3}\cdot x + 4 + \frac{1}{4}\cdot x = x$	#10: $\frac{1}{2}\cdot x + 3 + \frac{1}{4}\cdot x + 4 + \frac{1}{7}\cdot x = x$
#3: "soLve"	#11: $x = \frac{196}{3}$
#4: $x = -84$	#12: $\frac{25\cdot x}{28} + 7 = x$
#5: "Simplify"	#13: $\frac{1}{2}\cdot x + 3 + \frac{1}{4}\cdot x + 3 + \frac{1}{7}\cdot x = x$
#6: $\frac{13\cdot x}{12} + 7 = x$	#14: $x = 56$
#7: $\frac{1}{2}\cdot x + 3 + \frac{1}{4}\cdot x + 4 + \frac{1}{5}\cdot x = x$	
#8: $x = 140$	

Conclusion

In designing learning units in a CAS environment we have at least three concerns: (a) What is the balance between student interaction with the system and student reflection? (b) Does the complexity of the system simplify the parts that cause the student the most trouble? (c) How do conceptual and procedural knowledge interact when the CAS is performing the procedures?

In this study we have found that the use of technology does not replace reflection on the part of the student, but rather enhances it. When a pupil is using CAS to solve a word problem and immediately gets a solution, he is free to check if the solution does not make sense. In such a case he knows that the source of the trouble is the model/equation and not the algebraic manipulations. He may reflect on the way he constructed the model and try to resolve the problem. This is a big advantage over the conventional way of teaching, where pupils do not have the immediate control over

their work, and therefore they have to wait until the teacher reacts to their solution before they start to reflect, if at all.

The weight we placed on the model emphasizes the fact that usually the complete model of a word problem includes, in addition to the equation(s), some implicit restrictions. Because the pupils were asked to play the role of the teacher by inventing their own story problems, they realized the importance of those implicit restrictions. Thus, the design of the unit leads in a natural way to parametric equations, which we plan to investigate further. Our experience so far indicates that the use of CAS in this unit and in others creates networks of relationships between conceptual and procedural knowledge that facilitate mathematical understanding.

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BOYS, MATHEMATICS AND CLASSROOM INTERACTIONS: THE CONSTRUCTION OF MASCULINITY IN WORKING-CLASS MATHEMATICS CLASSROOMS.

Robyn Zevenbergen
Griffith University

This paper explores the interactions between teacher and students in attempting to understand the ways these contribute to the construction of masculinity with the primary school setting. Research suggests that not all boys or girls are winners in mathematics, but rather, it is important to consider the intersection between gender and other variables such as ethnicity, rurality, class and poverty. This paper examines the intersection of gender and social class – namely boys and working class. It analyses a transcript taken from a series of videotaped lessons during the last year of primary school (12-13 year old).

A considerable amount of research has investigated aspects of mathematics and mathematics education which have resulted, or have the potential to result in, the marginalisation of girls in and through mathematics. Such research has been very powerful in changing the status and outcomes for girls and women in the study of mathematics. However, as has become increasingly evident across Australia and internationally, there is a recognition of the polarisation of the performance of boys. In part, such foci are a result of a backlash against many of the political and economic reforms sweeping the world. Traditional male roles have come under challenge within the economic and workplace reforms of the post-Keynesian reforms. Such changes have seriously challenged the role and place of men in society, and just as women's movements had taken up the challenge in the 60s for girls, men's movements are now taking up the challenge for boys' education.

Boys simultaneously have a clear at a disadvantage and advantage in the study of mathematics. In the 1999 Year 12 results for Queensland, boys dominated both the top and lowest 10 percentile in results. Similar results have been noted in other state-wide testing regimes. However, as Teese, Davies, Charlton, and Polesel (1995) have noted that it is not gender alone that compounds success, but also socio-economic privilege, rurality and culture. In Australian schools, in both literacy and numeracy, Aboriginal girls outperform Aboriginal boys, but as a group, they score considerably lower than other Australian students (Gilbert & Gilbert, 1998; Zevenbergen, 1996); and those living in remote areas do not perform as well as those in urban areas (Gilbert & Gilbert, 1998; Lamb, 1997). Similar patterns occur for working-class students who do not do as well as their middle-class peers, but in study of mathematics, middle-class girls are likely to score as well as their middle-class male peers, whereas working-class girls are less likely to score as well as their working-class male peers (Teese et al, 1995).

In examining the teaching of mathematics from feminist standpoints, studies of questioning in mathematics classrooms have shown similar results where boys are asked significantly more questions than girls, (Leder, 1990) boys are asked more challenging questions whereas girls are more likely to be asked closed questions; girls tend to be more focused on writing and presentation whereas boys are concerned and rewarded for experimentation (Forgasz & Leder, 1995, Walkerdine, 1989). Whereas this literature arose from the needs to recognise the girls in mathematics, it also is useful in documenting potential areas for boys' education. Just as the gender identity literature has alerted educationalists and the wider society to the restricted versions of masculinity open to boys, so too does the literature on girls and mathematics. The options open to boys, while may be more empowering, they are only empowering for a restricted number of boys.

Studies of classroom interactions have demonstrated a regularity in the interactions which become *taken-for-granted* to constitute what becomes part of a mathematics culture. The culture of the mathematics classroom has been well documented and had often been described by teachers and students within restricted frameworks. Teachers' widely-held beliefs that they are responsible for teaching students specified procedures in order to solve mathematical problems (Goodland, 1983; Stodolosky, 1988) facilitates the development of traditional forms of classroom cultures which are so familiarly recalled by students. Most often it is described as teacher directed, students undertaking routine exercises and pencil-and-paper testing procedures (Leder & Forgasz, 1992). With a greater emphasis on inquiry modes of pedagogy, mathematics classrooms which have adopted these approaches have been placed under scrutiny. Bauersfeld, Krummheuer and Voigt, (1989) along with Wood (1994) note that the changes in the epistemological approach adopted by the teacher produces changes in the patterns of interactions within the reformed classroom.

"Triadic dialogue" (Lemke, 1992) has been documented as one of the most frequently occurring interactions within the classroom and is not unique to any particular curriculum area. This form of interaction consists of three key parts: the teacher initiates a question to which the answer is usually known by the teachers; the student responds; and the teacher then evaluates the response (Sinclair & Coulthard, 1976; Mehan, 1982). Such interactions serve the purpose of controlling the interactions as well as the content of the lessons. Lemke (1990) argues that the rules for interacting are not explicitly taught and hence students come to learn them through participation in the interactions. However, Lemke also notes that the patterns of interaction are not consistent across the three phases of a lesson. Triadic Dialogue is common in the introductory phase of a lesson where the teacher attempts to keep tight control of the content and students. Hence a significant amount of power resides with the teacher. Similar observations are made of the concluding phase of the lesson. However, during the work phase of the lesson, the patterns of power are somewhat more equal and students can express their lack of understanding. This phase of the lesson adopts patterns of interactions that are somewhat more equal

between teacher and students. The role of teachers' questions are critical in controlling the interactions with classrooms. As Lemke (1990) has shown through Triadic Dialogue, questions are used to control the flow of the lesson, the content to be covered, the behaviour of students and to provide progressive evaluation of student learning and lesson implementation.

Wood (1994) contends that many of these interaction types found in mathematics classrooms do not require the students to be working mathematically to participate. In her work with inquiry classrooms, she contends that the questions asked are substantially different in form where the answer is often unknown; and seeking further information and/or encouraging reflective thinking from the students (p.152). Extending the notion of interaction to include extended interactions, two key forms of interactions have been identified as potentially contributing to mathematical understanding. Funnel interactions (Bauersfeld, 1988; Voigt, 1985) are those where the students "narrowing of a joint activity to produce a predetermined solution procedure by teacher...[by] providing students with leading questions" (Wood, 1994, p. 155). Focussing patterns of interactions occur when the teacher uses carefully worded questions to guide the students to the more critical components of the task. In so doing the teacher poses problems to the students so as to ensure that the interaction is focused on the student and that she/he assumes responsibility for the resolution of the problem.

These studies have been useful in identifying micro and macro patterns in classroom interactions, but they do not address gendered aspects of the interactions. Such studies are valuable in identifying key aspects of the hidden components of the classroom culture which may be instructional in understanding the ways in which masculinity is exercised in mathematics classrooms.

Studies of classroom talk of who talks, for how long, with what forms of questions and about what, show that boys dominate the linguistic space usually within masculinist and often aggressive ways. However, as Gilbert and Gilbert (1998) are quick to recognise, it is not all boys who are advantaged by such interactions. Just as narrow readings of femininity constrained girls education, so too can the narrow versions of masculinity constrain boys education. This project critically examines the interactions within a mathematics classroom in order to understand the ways in which masculinity is being constructed in and through mathematic for working-class boys.

The Study

The study was conducted in a working-class school in a major regional centre of Australia. The school was selected on the basis of its nationally computed disadvantage index. The class "self-selected" with the teacher volunteering to participate when the school was approached to participate in the study. The class was in the last year of primary school (12-13 year olds), mixed gender with approximately equal representation of boys and girls. There were approximately 30 students in the class. The teacher adopted an interactive approach to teaching mathematics in which debate and discussion were actively encouraged. The project involved the

videotaping of weekly mathematics lessons for one school term (11 weeks). All videotapes were transcribed and analysed. For the purposes of this paper, the analysis was conducted on the introductory phases of the mathematics lessons and the ways in which interactions were gendered.

The lesson to be discussed in this paper is one which is focused on co-ordinates. The lesson begins with the teacher sketching a very crude map of Australia on the board over which a 3 x 2 grid is drawn with the y-axis labelled A and B, the x-axis as 1-3. The teacher uses this grid work to revise co-ordinates before setting the students to the task in which they have a series of co-ordinates which they must plot and then produce a sketch. Students are able to discuss and help each other with the task during the work phase. The conclusion of the lesson is very brief with teacher and students discussing their drawings, any areas of difficulty – including the incorrect co-ordinate provided in the activity sheet.

The following is a short segment from the introductory phase

1. T: This is a very rough map, making it as clear as mud. OK. Got that. 123AB. Can you look up here please?
2. B: Do we have to do it?
3. T: Just look at this. Just look at it. OK. Right, this mud map is a mud map of Australia. Here is our little map of Australia. Tasmania down here.
4. B: Where's New Zealand?
5. [T draws in New Zealand]
6. B: It's not a very good one!
7. T: The other island over here …. Near enough, OK.
8. B: Yeah rub off New Zealand
9. T: If we were looking up a particular place on this map, you would use the grid reference and the grid reference is the numbers and letters. Can anybody tell me the grid reference for let's see….. Tasmania? What is the grid reference for Tasmania?
10. B: B2
11. T: Put your hand up please. Rebecca?
12. G: B2
13. T: Yes. So …B2. And what if you wanted to go from Tasmania to say… Perth. What is the grid reference for where you would find Perth? Tate?
14. B: B1
15. T: Right, Thank you. Now it would look this. So you would say from B2 to B1. Now it's pretty unclear exactly [referring to map], I mean it could be from here to here because we're not exactly certain. Who can tell me what it would be if you want to draw or show someone from Tasmania to Darwin?
16. B: [calling out] I don't know where Darwin is.
17. T: Guess if you don't know exactly…..Yes, what do you think it will be?
18. B: A1.

19. B: Yeah
20. T No, we want to go from …to
21. B: B2 to A1
22. T: Right B2 to A1….thank you.

The introduction continues on as the teacher and students plot a few more cities and draw the connecting lines between two points. The interactions continue in a very similar fashion, with no girl being asked a question until the work phase where one seeks clarification of the task.

All lessons analysed produced similar results, confirming earlier research on the girls and mathematics. Boys were asked substantially more questions that girls, frequently in excess of 80% of the questions. In some cases, very few, if any questions were asked to the girls in the introductory phase of the lessons. In 3 lessons, no girls were asked any questions in the introductory phases. These outcomes convey very subtle message to boys and girls about the nature of mathematics and mathematics teaching. From a feminist standpoint, this could be seen to be very disempowering for girls while being very empowering for boys. However, such a position does not consider how *some* boys (and girls) are being marginalised from the study of mathematics. In considering the social-class variable, these interactions are very disempowering for girls and arguably could be a significant factor in the marginalisation of working-class girls in and through mathematics. As noted, this group is one of the most-disadvantaged groups in Australian education. However, as it will be shown, the interactions can equally be disempowering for the working-class boys.

In considering the Triadic Dialogue identified by Lemke (1990), it is clear that such a strategy is useful in controlling the classroom and content of the lesson. At the end of line 13, where the teacher asks Tate where is Perth, to which the boy answers correctly, the teacher then evaluates the response, acknowledges that it is correct and then is able to move on to the next step of the lesson content. In contrast, in interactions 9 to 13, the teacher asks the question, but admonishes the student for calling out and then asks another student (a girl) to respond. From this interaction, it can be seen that when a student violates the taken-for-granted rules of the classroom (in this instance, it is improper to call out) the response is ignored and the student's response is rendered invisible and another response is called for. Using this simple Triadic Dialogue, the teacher is able to keep the lesson flowing effectively both in terms of the mathematical content where the students are being asked to identify co-ordinates for different locations (eg B1) and the tracks between two co-ordinates (eg from B2 to A1). Simultaneously, the interactional pattern is useful in controlling the social norms of the classroom – rules such as putting hands up to give answers, sitting in seats, not writing during instructions and so forth are often made explicit. However, as Lemke (1990), the patterns of interaction are never made explicit, they must be learned through participation.

Although not apparent in this extract from the transcript, this lesson and others contained interactions that supported notions of funnel interactions and focussing interactions. Such extracts support the notion that such interactions facilitate a stronger mathematical understanding to be developed. In contrast, the interactions in this extract support only minimal mathematical working. They tend to be low level and more at the level of revision and/or consolidation.

What is worthy of discussion is the ways in which gender, vis-à-vis masculinity, is being constructed for this group of boys. As noted in the introductory sections, working-class boys do not perform as well in mathematics as middle-class boys and middle-class girls (Teese et al, 1995). In line 2, the first challenge from the floor is made when a boy asks whether he has to do this work. The unspoken rules of classroom behaviour is that there is little or no leeway in participation. There may be coersive means by which the students may resist participation, but these are most frequently left unspoken. After seven years of formal schooling most students know this rule. In this interaction, the boy openly challenges the teacher as to whether participation is compulsory. The teacher's response indicates the type of participation that he is expecting.

In the fourth line a student (from New Zealand) challenges the teacher and asks where New Zealand is to which the teacher responds by drawing New Zealand on the map. Another male student interjects that New Zealand should be removed from the board. This friendly banter was common in the classroom but it was almost always instigated by a boy. The verbal challenges and rebuffs made by the boys can be seen to be masculine ways of interacting and reinforced in the public domain of the introductory phase. The challenges being drawn out through the interaction are among the boys and are not overt challenges to the teacher.

In line 16, a boy calls out, but unlike like 10, is not punished. Given that the top western quarter of Australia lies in the quadrant A1, then the location of Darwin is not really an issue. In this case, the student knew the location of Darwin, but was making overtures about the poor drawing of Australia. The boy has challenged the teacher's control of the lesson through this transgression, but has been able to ward off any retribution from the teacher. The transgression has been successful, and the boy's reputation was left in tact.

The disruptive behaviour in classrooms is often for the purposes of attention seeking. Frequently such behaviour was generated by the boys to gain approval from their peer group. Interviews with students at the end of the project confirmed that the most popular students were those who had strong profiles in the classroom. The profiles were not often associated with academic performance, but rather the social aspects of classroom antics. For example, one student thought that Daniel was the "best kid in the class because he could muck around a bit, you know, give the teacher a bit of a hard time. He made us laugh." Such comments indicate that status within the peer group for boys is often achieved through transgressing the social norms of the classroom.

Disruptive behaviour can be a seen as exciting in that it pushes the boundaries, exerts a sense of independence and challenges authority. In some cases, the pushing of boundaries may be valid when the lessons are boring, slow or repititive, but in other cases it may not be so legitimate. By displaying disruptive behaviour, the boys challenge the teacher's authority and in so doing, can be seen to be engaging in risk-taking behaviour. In some cases, the boys were successful in transgressing the social norms of classroom behaviour, in other instances they were not. When unsuccessful, they risk humiliation among their peers. Overall, disruptive behaviour is a key component of boys' school culture. In the playground, the events can be relived and celebrated with laughter and backslapping, cementing the bonds of boyhood and camaraderie. Boys' behaviour is closely linked to their public persona and their need to establish themselves in the public arena in a way which they feel is acceptable. For the boys in this classroom, this often was seen to take on quite masculinised ways of working and resisting the school mathematics culture. In so doing, they were effectively and progressively excluding themselves from mathematical content and knowledge.

In another study (Zevenbergen, 1995), it was found that such disruptive behaviour was not as evident in middle-class mathematics classrooms. Boys and girls complied with the Triadic Dialogue so that both social and mathematical norms were unchallenged. In these classrooms, students were exposed to far greater mathematical content and language than their middle-class peers, suggesting an area for the potential differences between class and gender.

Conclusions

Boys' education has alerted researchers to the unique aspects of boys' schooling. The study of interactions in mathematics classrooms provides scope for understanding how the classroom context contributes to the personal development of how students come to understand and construct themselves as learners of mathematics. How students construct and are constructed by classroom dynamics is an area worthy of in-depth investigation. As has been shown here, the interactions within a working-class classroom offer restricted outcomes for learning mathematics. Potentially the interactions ,restricted in the content, are constrained by the transgressions posed by students. Such transgressions appear to be male dominated both quantitatively and qualitatively. In many cases, the boys are positioning themselves within practices which reflect their emerging senses of masculinity. The challenges posed by the boys tend to disrupt the flow and content of the mathematics lessons. To this end, both the working-class boys and girls are potentially exposed to a restricted offerings of mathematical understanding. As a consequence of such actions, it is suggested that emerging sense of the masculine self, may contribute to the marginalisation of working-class boys and girls in the study of mathematics

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