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ABSTRACT

It is not complex to teach the algorithm for multiplying fractions so that children can multiply numerators and denominators to arrive at correct answers. However, it is a challenge to teach so that students build understanding of multiplication of fractions, extending what they have already learned about fractions and about multiplication of whole numbers. This paper describes three days of instruction with fifth grade students in March during the school year. Rather than offer polished lessons, this paper offers a glimpse into actual classroom instruction to provide a platform for discussing mathematical instruction of a traditional skill in a way that focuses on children's thinking and reasoning. (Author)

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Abstract

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It's not complex to teach the algorithm for multiplying fractions so that children can multiply numerators and denominators to arrive at correct answers. However, it's a challenge to teach so that students build understanding of multiplication of fractions, extending what they already learned about fractions and about multiplication of whole numbers. This paper describes three days of instruction with fifth graders in March of the school year. Rather than offer polished lessons, the paper offers a glimpse into actual classroom instruction to provide a platform for discussing mathematical instruction of a traditional skill in a way that focuses on children's thinking and reasoning.

Building Understanding of Multiplication of Fractions

Over my years of teaching, I've worked to hone my skills at establishing classroom learning environments that allow and encourage students to think and reason mathematically. This is a complex challenge, I think, and an important one for teachers, calling for a balance among strengthening our mathematical understanding, finding ways to engage students in mathematical thinking, and learning to listen to and learn from students. Also, it's helpful for teachers to have models of teaching that have the goal of promoting mathematical thinking and understanding.

In this paper, I offer an example of an attempt at such instruction by describing three days of instruction with fifth graders in March of the school year. The class was one that I had taught intermittently during the year and also during the previous year when they were fourth graders. One reason I chose to describe these particular three days was because the topic was multiplying fractions, a topic with which I wasn't very confident in my own teaching and which I know has concerned many teachers. My plan was to build on what the students knew about multiplication of whole numbers and what they had already learned about fractions in order to help them make sense of multiplication of fractions.

Another reason I chose these lessons is because they are more works in progress than they are polished stones and, therefore, offer much opportunity for discussion. After these three days, it was clear that the children needed more time to think about and discuss multiplication of fractions. Also, I needed more time to think about the pedagogical choices I made, what I might better have done, and what I might do from here. While it's not comfortable to offer up examples of teaching with rough spots, it seems that doing so has the potential for looking at important issues of promoting children's mathematical learning. I hope that the descriptions of the three days will raise questions that can help you clarify your own thinking. Following the descriptions, I offer some of the reflecting I did as a result of teaching these lessons.

Day 1

I've long relied on the teaching strategy of teaching something new by relating it to what students already know. To begin a conversation about multiplying fractions, I asked the students what they know already knew about multiplication. They weren't responsive, not sure what I had in mind. Actually, I had three things in mind to discuss with them about multiplication of whole numbers that I would use as a basis to talk about multiplication of fractions. One was that when we multiply whole numbers, we expect the answer to be larger than the factors (which isn't always the case with fractions). Another is that we can look at multiplication as repeated addition. And the third is that we can represent multiplication by using an area model. I knew that the students had had previous experiences with each of these three ideas.

Since the students weren't forthcoming with responses to my question, I began by offering something. "Here's an idea about multiplying whole numbers," I said. "When you multiply, things get bigger, and we expect the product to be larger than the factors we're multiplying."

Tanya had a contradiction to my idea. "Not when you multiply by 1," she said. "Then it stays the same."

"What stays the same?" I asked.

"Like when you multiply 6 times 1, the product is 6," she said.

"Not with zero," Maria said. "Then the answer is zero."

"So let me revise the idea," I said. "When you multiply whole numbers, except when one or both of the factors is zero or 1, the product is larger than the factors." The students seemed satisfied with this idea. I'm not sure it was necessary to push this clarification, but I wanted to be sure later that the students were aware that when multiplying fractions, the product could be smaller than one or both of the factors.

I then wrote on the board: "2 x 3" and as I started to write the answer, several of the students offered 6. I completed the sentence:

$2 \times 3 = 6$

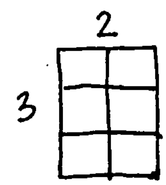
I then said, "Suppose you didn't know that 6 was the answer. Or suppose that you were trying to explain to someone who hadn't yet learned about multiplication how to figure out the answer. How could you explain how to get 6?" Many hands shot up. I called on Leanne.

"You could go 3 plus 3," she said. I wrote "3 + 3" on the board.

Saul had another idea. "You could do two plus two plus two," he said. I wrote "2 + 2 + 2" on the board.

"These both show how multiplication relates to addition," I said. "Multiplication is a fast way of adding. Another way to show 2 time 3 is to draw a rectangular array. Does anyone remember how to do this?" Again, hands shot up. The students were familiar with representing multiplication as rectangles. If they hadn't been, I would have taken time to develop this idea as I think it's key for them to think about multiplying fractions. I called on Maria.

She said, "You draw a rectangle that has two squares on the top, then you draw two squares underneath and then two more so there are three squares going down." I drew as she described. [Illus of 2-by-3 array]



I then said, "I've learned from teaching math for a long time that some people like thinking about numbers, some like thinking about shapes, and some like both." Some students commented about which they liked better. "It's okay to have a preference for one over the other, but it's important that you can think about both. If you know how multiplication relates to addition and you also know how multiplication relates to rectangles, I can easily teach you about multiplying fractions."

The students had worked on a problem a few days ago: How many eggs are there in two-thirds of a dozen? They had figured out that the answer was eight eggs. I hadn't attended the class on the day when they solved the problem, but since they already knew the answer, I was interested in having them think about why their answer made sense in the two ways we thought about multiplication -- repeated addition and making rectangles.

I wrote on the board "2/3 x 12" and asked, "Why is this problem about eggs a multiplication problem?"

Leanne said, "Well, it's really 2/3 of 12."

"Does '2/3 of 12' mean it's a multiplication problem?" I asked. The students were stumped.

"Maybe it's really division," Tim said. Others were also confused.

"Suppose I asked you how many eggs there were in three dozen?" I asked. "How would you do that?" Jimmy said you could add 12 three times. Candace said that was the same as doing 12 times 3. Others agreed. I wrote "3 x 12" on the board.

"Three times 12 means three groups of 12," I explained. "Candace, you said 12 times 3, which really is 12 groups of 3. You'd get the same answer as for 3 times 12, but when we're talking about dozens of eggs, we're really thinking about groups of 12. That's why I wrote 3 times 12 instead of 12 x 3, but both are fine."

"It's 36," Alan said. Others agreed.

"What if the problem were to figure out how many eggs in two dozen?" I asked. "What math problem could I write?" I called on Katie.

"It's two times 12," she said. "It's 24." I recorded this on the board.

"What about a problem for how many eggs are in one dozen?" I asked. There was some silence for a moment and then Douglas said, "Oh, I know, it's 1 times 12." I recorded this on the board, then wrote "1/2 x 12" underneath.

3 x 12 = 36

2 x 12 = 24

1 x 12 = 12

1/2 x 12 =

"What do you think "1/2 times 12" means and what do you think the answer is?" I asked.

Hands flew up to answer. They all knew that there were six eggs in half a dozen. To reinforce out why these were all multiplication problems, I read each of the using "groups of" instead of "times." I pointed to each problem in turn and said, "Three groups of 12 equals 36, two groups of 12 equals 24, one group of 12 equals 12, one-half of a group of 12 equals 6."

I added one more comment. "Sometimes instead of saying 'one-half group of 12' we just say 'one-half of 12' to mean the same thing. In the same way, a problem like '2/3 of 12' is a multiplication problem."

Then I added to the list the problem they had worked on:

3 x 12 = 36

2 x 12 = 24

1 x 12 = 12

1/2 x 12 = 6

2/3 x 12 = 8

I asked, "If you look at the pattern of the answers, who can explain why the answer '8' makes sense for 2/3 times 12?"

Douglas explained, "Because 2/3 is in between 1/2 and 1, so the answer should be between 6 and 12."

"Okay, now back to looking at why 8 makes sense in a few other ways. First by adding. How could I wrote 2/3 times 12 as an addition problem?" The class was stumped again. I waited a moment and still no one volunteered.

"Watch as I do it," I said. The children caught as I wrote on the board:

2/3 + 2/3 + 2/3 + 2/3 + 2/3 + 2/3 + 2/3 + 2/3 + 2/3 + 2/3 + 2/3 + 2/3 =

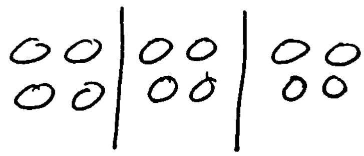
"If we add 2/3 plus 2/3 12 times, how many thirds do we get?" I asked. I waited to give the students a chance to think.

"It's 24 thirds," Michael said.

"That's the same as 8," Jimmy said. "Three goes into 24 eight times."

"So that's a way to look at the multiplication problem as an addition problem," I said.

"I know another way to solve it," Sally said. She came up to the board and drew a dozen eggs, two rows with six in each as they appear in a typical egg carton. "If you divide it into thirds," she explained, "there are four eggs in one third. So there has to be eight eggs in two-thirds of a dozen." [Illus]



After Sally sat down, I recorded numerically what Sally had drawn:

1/3 x 12 = 4

2/3 x 12 = 8

"Now we have two different ways to see how to solve 2/3 times 12," I said. "Let me show you how we can solve it with a rectangle." I explained as I did this. "Let see, I need a rectangle that is 2/3 by 12. I'll draw the 12 side first." I drew a long line vertically, divided it into 12 units, and erased the extra on the bottom. "Now I need a 2/3 side. Well, I'll start with just one square across so I have a 1-by-12 rectangle." I completed the rectangle. "But I only need the rectangle to be 2/3 across the top. Let's see, I'll divide the rectangle into thirds the long way. Then I can shade in the 2/3-by-12 part. Now all we have to do is see how many square units the shaded rectangle is worth." [Illus]

Jason had an idea. "It's 24 little pieces," he said.

"And how much is each little piece worth?" I asked.

He hesitated for a moment and then said, "They're a third."

"So the rectangle is worth 24 thirds?" I asked. He nodded. I pointed to the answer we had gotten when adding 2/3 twelve times.

"It's the same!" Candace said.

Michael had a different way to explain. "It takes three thirds to make a whole," he began. "So you could color in three of the little pieces to make one." I shaded in three thirds to make an L-shaped piece.

"And you think that what I shaded is worth 1 whole?" I asked.

"Yeah," Michael said. "The piece on the bottom that sticks out could fit on the top."

Alan offered another explanation. "It's 2/3 and another 1/3," he said.

"There are eight of them," Michael continued.

"Eight of what?" I said.

"Those L's," he said. I colored them all in to verify that the 2/3-by-12 rectangle was worth 8 units. [Illus]

Alan then raised his hand. "I know a shortcut way to do it," he said. "You make 12 into 12/1, then with 2/3 x 12/1, you multiply and get 24 on the top and 3 on the bottom and that's 24 thirds and that's 8." Alan had learned the standard algorithm. When children offer a method that is different from what I'm trying to teach, I'm careful not to dismiss what they know, but to accept what they offer and try and incorporate it into the lesson. In this instance, other students helped.

"I don't get why you can write 12 over 1," Tim said.

"It just means 12," Alan said. "It's 12 wholes instead of something like five thirds or something else." This seemed to satisfy Tim.

"Why did you times across the top?" Leanne wanted to know.

"That's how I learned to do it," Alan answered.

Although I intended to keep the focus away from the standard algorithm, this was a good opportunity to address how shortcuts can be useful and why this works. "I think I can explain," I said. "Look at the rectangle. When we divided the short side into thirds to show 2/3 instead of a whole, we wound up dividing the whole rectangle into thirds. And the part of the rectangle that we were considering had two columns of thirds, one column of 12 thirds and another column of 12 thirds. That's the 2 times 12 part to give us 24 thirds, those 'little pieces' that Jason was referring to." I didn't expect this to be clear to everyone. The notion I was trying to convey is that when someone has a shortcut, the shortcut has a reason for working. When students discover, for example, that when you multiply a number by 10, you can just add a zero to the number, the shortcut becomes a quick and useful way to perform the computation. But it can be explained as a pattern that applies to all situations and, therefore, is a useful shortcut. This is what algorithms are -- efficient shortcuts to perform computations. It's important, however, that shortcuts rest on a base of understanding. At this time in the children's learning, I was intent on keeping the focus on their understanding, not on showing a shortcut.

I shifted the conversation to another problem. I wrote on the board:

3 1/2 x 5

"About what is the answer?" I began. "It's always good to start with an estimate so that we have something to refer to when we get an answer to be sure our answer is reasonable."

"I think 17," Jimmy said.

"How come?" I asked.

"Well, I know that 3 times 5 is 15, so it will be a little more than that because 1/2 of 5 is 2 1/2. So I think it's 17 1/2."

"Is that your estimate?" I asked.

"I think it's the exact answer," Jimmy said.

"Who could figure the answer by adding?" I asked.

Candace said, "You add 3 1/2 five times." I wrote on the board:

3 1/2 + 3 1/2 + 3 1/2 + 3 1/2 + 3 1/2

Candace went on to add. "I know that 3 1/2 plus 3 1/2 is 7, and 7 plus 7 is 14, and 14 and 3 1/2 more is 17 1/2." I recorded:

3 1/2 + 3 1/2 + 3 1/2 + 3 1/2 + 3 1/2

7 7

14 + 3 1/2

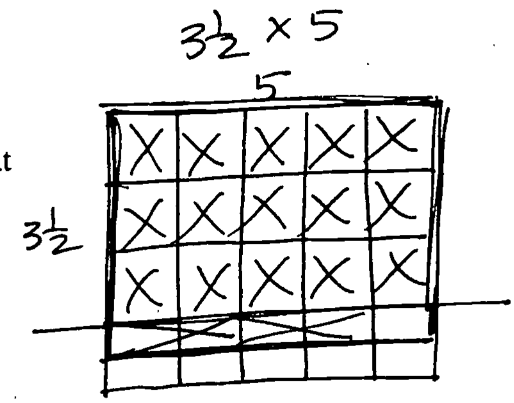
17 1/2

Sally raised her hand. "I have another way," she said. "I know that five 3s make 15 and five 1/2s make 2 1/2 and 15 and 2 1/2 makes 17 1/2." I recorded on the board:

Five 3s = 15

Five 1/2s = 2 1/2

15 + 2 1/2 = 17 1/2



"Now let's see if we can use a rectangle to figure out that the answer is 17 1/2," I said. Again I modeled, explaining as I drew. "I'll start with a rectangle that is 4 by 5. Then I'll draw inside a rectangle that is 3 1/2 by 5. Now I'll count up the squares inside. Let's see, there are 15 whole squares. Well, that makes sense; it's the same as the 5 times 3 that Sally did. Now I'll piece together the halves. There are five halves, so that's two whole squares and one half more. So there are 17 1/2 squares all together."

I drew a line dividing the rectangle into two parts, 3-by-5 on the top and 1/2 by 5 on the bottom and said, "This shows what Sally did. Here's the 3 times 5 part on the top and the 1/2 times 5 part on the bottom."

I also pointed out how the rectangle showed what Candace had suggested. "Each vertical line has 3 1/2 squares in it," I said, "and there are five of them. That's the same as Candace adding 3 1/2 five times."

"My way still works," Alan said. "I change 3 1/2 into 7/2 and 5 into 5/1 and do 7/2 x 5/1 and that gives me 35/2 and that's 17 1/2." There were several comments of confusion. "I don't get it." "Why does that work?" "What are you doing?"

I pointed to the rectangle. "Let's divide each of the whole squares in half. We already have the seven halves on the bottom. And if we divided the 15 whole squares into halves, we'd have 30 of them. Thirty plus the 5 more makes 35 halves."

"Let's try one more together," I said, "and then you can try some on your own. Remember, that first you should make an estimate. Before we try another problem, let's do some practice estimating." I wrote problems on the board, telling them that to show that a number was an estimate they could write a "wavy circle" around it. I modeled this.

"What's an estimate for 6 1/3 x 2 1/2?" I asked, writing the problem on the board.

Tanya answered 15 and explained that it had to be more than 12, so she thought 15 was close. I wrote and showed what I meant by a "wavy circle":

6 1/3 x 2 1/2 = 15

Then we estimated and discussed several more problems:

15 x 2/3 = 10

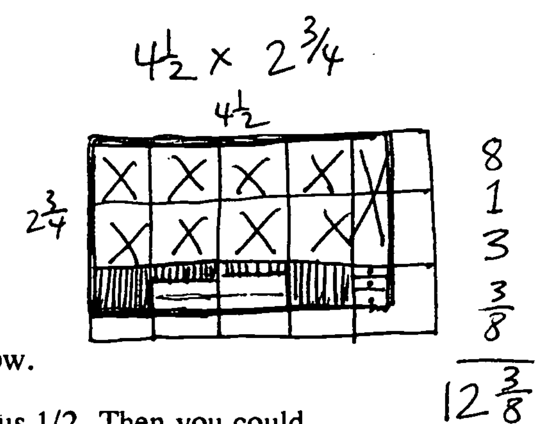
3 1/2 x 2/3 = 2 3 1/8

4 x 3/4 = 3

7 1/2 x 2 2/3 = 18

4 1/2 x 2 3/4 = 10

"Let's figure the last one to see what the exact answer is," I said. "Would anyone like to try drawing a rectangle?" There were several volunteers and I called on Saul. He came up and carefully drew a 3-by-5 rectangle. He divided the bottom row of squares into fourths and marked off 2 3/4 and then he divided the right-hand column into halves and marked off 4 1/2. He drew Xs to mark whole squares and figured out the answer. The class cheered for him.



"How could you do it by adding?" Candace wanted to know.

"I know," Alan said. "You could make the 4 1/2 into 4 plus 1/2. Then you could add 2 3/4 four times. Then you have to do half of 2 3/4." I wrote on the board:

4 1/2 x 2 3/4

(4 + 1/2) x 2 3/4

(2 3/4 + 2 3/4 + 2 3/4 + 2 3/4) + 1/2 of 2 3/4

"I don't get how that works," Tim complained.

"It works, I think, but it's too complicated," Sally said. "It's too hard that way," Jason said. Meanwhile Alan was lost in thought, as he often can get in math class, continuing to figure out what he had started. The others were mostly just lost.

I commented, "Not every method makes sense for every problem. What Alan is doing will produce the correct answer, but it may not be a way that makes sense for you to use. What's important is that you have a way to make sense of a problem and be able to represent it mathematically and explain it to someone else."

It was the end of class, so I ended here and planned to return with some problems for them to try on their own. Only when I see what students can do by themselves can I assess what they understand.

Day 2

I began class by writing "3 x 2 1/2" on the board. "About what is the answer?" I asked, looking for the children to give an estimate that would provide a way to check the reasonableness of their accurate answer. Some of the students were able to figure that 7 1/2 was the exact answer. Others ventured that the answer would be close to 6 or 7.

"Who is willing to come up and show how you would solve the problem?" I asked. About a third of the class volunteered. I called on Tanya. When she came up to the board, she initially lost her confidence, something that I've seen happen with students once they face the vastness of the board and realize that they are in front of the class. After a bit, she started to talk about her idea and, with my encouragement, recorded her thinking.

She began, however, with a false start. "You have 1/6," she said. She wrote 1/6 on the board and then remained silent.

"Tell us where the 1/6 comes from," I probed.

"Well, first I did 3 times 1/2," Tanya said. She wrote "3 x 1/2" on the board and then said, "Oh, no, it's not 1/6." She erased the 1/6 and replaced it with 1 1/2. Then she regained her confidence and completed the problem. "And 3 times 2 is 6, so you add 1 1/2 and 6 to get 7 1/2." She wrote:

3 x 1/2 = 1 1/2

3 x 2 = 6

1 1/2 + 6 = 7

Other students were interested in sharing their ideas. I called on Jason and he came up to the front of the room.

"Well, I just did it in my head," he said. "I don't know how to write it." Jason is able to figure mentally with a good deal of facility but often has trouble knowing how to represent his thinking on paper.

"Tell us what you did in your head," I said, "and I'll help you figure out what to record on the board."

In one breath Jason said, "First I thought three 2s and that's 2, 4, 6 and then I did 3 times 1/2 and then I added." I realized that Jason had done essentially what Tanya had, but he thought about it in a slightly different way. I pushed him to write down how he figured out the three 2s. And when I pushed him to explain how he knew 3 times 1/2, he reported that he counted by 1/2s and was able to record that. He wrote:

2, 4, 6

1/2, 1, 1 1/2

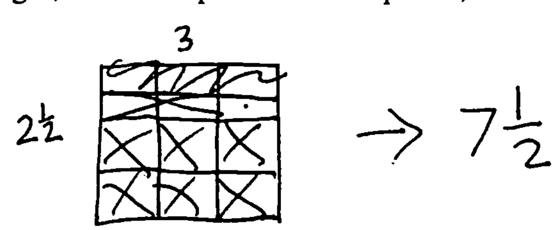
6 + 1 1/2 = 7 1/2

I talked with the class about the importance of learning to represent their thinking mathematically. I gave a brief commercial for the Standards 2000 draft, telling them that representation was one of the important goals of the new math standards for the country. I then returned to having other students come up to explain their thinking.

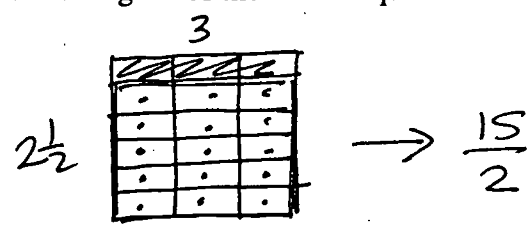
Alan did the problem using the rule he has learned about multiplying across the numerators and denominators. He wrote:

3/1 x 5/2 = 15/2 = 7 1/2

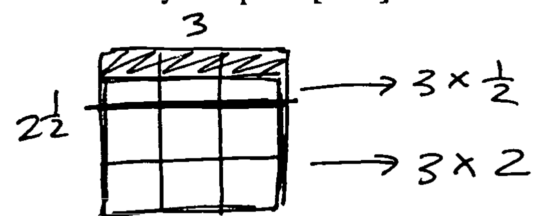
Jimmy came up to try and show how to do the problem using a rectangular array. He drew a 3-by-2 1/2 rectangle, counted up the whole squares, combined the halves, and got the answer of 7 1/2. [Illus]



I pointed out to the class that the rectangle shows us why the 15/2 Alan got makes sense. "If you think about cutting all of the whole squares into halves," I told them, "there are 15 halves." [Illus]



I also showed how they could see Tanya and Jason's thinking in the rectangle by looking at the 3-by-2 part and the 3-by-1/2 part. [Illus]

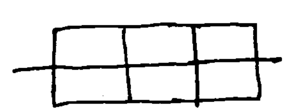


Michael had a question. "I don't get where Tanya got 1 1/2," he said. He had been pondering this and wasn't able to make sense of it.

"Do you want to explain, Tanya?" I asked.

Tanya nodded and said, "It's three 1/2s, so you go 1/2 plus 1/2 plus 1/2." I recorded this on the board and Michael seemed to be able to see that 1/2 plus 1/2 was 1 and 1/2 more made 1 1/2.

Sally had another way to explain. "If you drew a rectangle with three squares," she said, "and then cut it in half, you'd have 1 1/2." I drew three squares adjacent to each other and cut them in half. Michael seemed to understand this as well. [Illus]



Douglas had another way to solve 3 x 2 1/2. He came up and explained, "I doubled both factors and that's 6 times 5 which is 30. And then I divided by 4." From the whole number work we had done in December, Douglas remembered that doubling both factors resulted in a product that was four times larger. He wrote on the board:

6 x 5 = 30

30

÷ 4

I stopped him to correct his representation for division. I gave him two options for recording:

30 ÷ 4

30/4

Saul didn't understand why Douglas divided by 4. "Why don't you divide by 2?" he asked.

"Because I doubled both factors," Douglas said. Saul was still confused.

"Let's look at an easy problem with whole numbers and see what happens when you double both factors," I said. I wrote on the board:

2 x 3 = 6

4 x 6 = 24

"Oh, I see," Saul said. "You double twice."

"Let's see what happens if we double just one factor," I said. I wrote on the board:

2 x 3 = 6

2 x 6 = 12

Leanne said, "If you just doubled the 2 1/2, then the problem would be 3 times 5 and you only have to divide by 2." I wrote on the board:

3 x 2 1/2

3 x 5 = 15

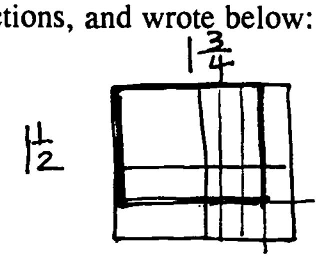
15/2 = 7 1/2

"It's easier that way," Jimmy commented.

I then gave the students five multiplication of fraction problems to try on their own. I asked them to do each in two different ways, using rectangles for one way. The problems I assigned were:

1. 7 x 1 1/2
2. 1/2 x 2/3
3. 1 1/2 x 1 3/4
4. 8 x 2/3
5. 3 1/3 x 2 1/2

I think I made a pedagogical goof by including the third problem -- 1 1/2 x 1 3/4. It was too hard for most of the students because eighths got involved. While some students persevered to make sense of it, many were confused and frustrated. A few who were able to solve the problem did so by drawing an array. Maria, for example, drew a rectangle, divided it into four sections, and wrote below: [Illus]



1 + 1/2 + 3/4 + 3/8

1 + 4/8 + 6/8 + 3/8 = 2 5/8

Other students, however, preferred to tackle the problem numerically by using the distributive property. Tanya, for example, wrote:

1 1/2 x 1 3/4

1 1/2 x 3/4 = 1 1/8

1 1/2 x 1 = 1 1/2

1 1/2 + 1 1/8 = 2 5/8

She had gotten stuck, however, when she tried to figure out 1 1/2 x 3/4. She used the distributive property there as well, figuring 1 x 3/4 and 1/2 x 3/4. She knew that 1 x 3/4 was 3/4, but she wasn't sure about how to figure out 1/2 x 3/4. She came to me at that time for help.

"Do you know what 1/2 x 1/4 is?" I asked her.

Tanya thought for only a moment and said, "Half of a fourth is an eighth."

"So, how much do you think is half of three-eighths?" I then asked. Tanya saw immediately that it had to be 3/8. (I tried this reasoning with several other students and it seemed to make sense to them also.) Tanya then added to what she had written:

1/2 x 1/4 = 1/8

1/8 x 3 = 3/8

1 x 3/4 = 3/4

3/4 + 3/8 = 9/8 = 1 1/8

I think this is extraordinary thinking for a fifth grader, at least for the fifth graders in this class. Only a few others were able to reason as Tanya did,

That night I pored over the students' papers to see what I could learn about their reasoning. Practically all of the students were able to solve the first problem -- 7 x 1 1/2 -- both numerically (using the distributive property by multiplying 7 times 1 and 7 times 1/2, then adding) and with a rectangular array (drawing a 7-by-1 1/2 array and counting up whole squares). [Work attached -- #1, #2]

Two papers showed different and unique methods. Sally used squares, but she didn't think of a rectangle of dimensions 7 and 1 1/2. Instead she used the squares as wholes and reasoned about parts of the whole. [Work attached -- #3]

Leanne first tried to solve the problem by doubling both factors and dividing by 4, as Douglas has done. She multiplied 14 times 3 to get 42, then divided 42 by 4 to get 10 with a remainder of 2. This answer didn't look right to her. She knew she needed a fraction and didn't know how to deal with the remainder. Then Leanne took another approach. She decided to solve the problem using repeated addition and wrote:

1 1/2 + 1 1/2 + 1 1/2 + 1 1/2 + 1 1/2 + 1 1/2 + 1 1/2 = 10 1/2

Then she drew a rectangular array and again arrived at 10 1/2. [Work attached -- #4]

Work on the other problems was a mish-mash of correct thinking and misconceptions. A common error the students made when trying to solve 1 1/2 x 1 3/4 was multiplying 1 x 1, then multiplying 1/2 x 3/4 (on which they generally made an error), and finally adding together the two partial products.

I decided that the next day I'd give them other problems to try, choosing ones with which they could have some success. I planned to ask them to choose at least two problems to solve from five that I'd offer. I planned to list the five problems in what I thought was their order of difficulty and encourage them to take on a challenge if they were willing. Also, I'd again ask that they solve each problem in at least two ways, one by reasoning numerically and one by using rectangles.

Day 3

To begin class, I talked with the students about the previous day's work. I began with a short speech about my pedagogical belief. I said, "I've learned as a teacher to expect two things from students when they are learning something new." I wrote on the board:

Confusion

Partial understanding

I had them give me their ideas about what these meant. Leanne asked what I called it after they weren't confused any more. Alan say that it would be comprehension then. I agreed and said that that's when "partial understanding" changed to "robust understanding."

Then I told the students that I realized that some of the problems I assigned yesterday were difficult. I asked them how they felt about working on those problems. Most of the students thought that some of the problems were okay but agreed that some were too hard. I told them I was interested in helping them learn how to make sense of all problems that involved multiplying fractions.

"I have some more problems for you to try," I said. "This time I've listed them from easiest to hardest. I'd like you to do at least two of them, and you can choose which ones to try. If you get stuck, I'll come and help."

"Can we work together?" Douglas wanted to know.

"It's fine to talk with one another because that's a good way to learn," I said. "But you each need to do your work on your own paper. Also, you need to be able to explain what you write on your paper, so be sure you understand the help you receive." I reminded the students that they were to make estimates first. Then I wrote the problems on the board:

1. 3 x 3 1/3
2. 5 x 3 1/3
3. 3 x 2 1/4
4. 1 2/3 x 2 1/2
5. 2 1/2 x 5 1/4

It was interesting to see who tackled the harder problems first and who started at the top of the list. Some of the stronger students began with the easy ones and some of the weaker students took on a challenge. I guess that my views of their mathematical strength doesn't necessarily match their views.

Multiplying a fraction by a whole number seems accessible to most of the students, though some struggled to figure out 1/3 x 5 as a partial product for the second problem. Multiplying two mixed numbers is still difficult for about two-thirds of the class. The problem persisted on this day's work of students multiplying the two whole numbers together and then the two fractions. Michael, for example, after feeling successful with the first problem, tried the fourth -- 2 1/2 x 5 1/4. He estimated 10 and then wrote:

2 x 5 = 10

1/2 x 1/4 = 1/8

Michael also drew a rectangle with correct dimensions but seemed simply to assume that it contained 10 1/8 squares. [Work attached -- #5] I didn't notice what Michael had done during class because he didn't ask for help and plenty of other calls for help were keeping me busy. Examining students' work afterwards is essential, I think, for getting insights into what they do and don't understand.

To figure 2 1/2 x 5 1/4, Jason did the same as Michael, multiplying 2 x 5 and 1/2 x 1/4 and then adding to get 10 1/8. He had drawn a rectangle but its dimensions didn't match the problem or figure into his thinking. This last problem was the first Jason tried and he asked me to check what he did. I knew that Jason was comfortable multiplying two-digit whole numbers and I decided to see if I could help him connect the thinking he used in those sorts of problems to this problem. I asked if he could multiply 14 times 23. He said yes and wrote "14 x 23" horizontally on his paper. Then he rewrote it vertically, figured the four partial products, and added. [Work attached -- #6]

"What if you wrote the fraction problem vertically?" I asked Jason. He did, then immediately went to work to figure the four partial products, and added them to get the correct answer of 13 1/8. He changed his original answer.

"Why did you get a different answer this time?" I asked.

"I left some parts out the other way," he answered. I wasn't sure that Jason really understood why four partial products make sense for multiplication. After we talked, Jason went back and did the first two problems, getting the answers correct but not showing how he reasoned. Also, the one other rectangle he drew again had incorrect dimensions. He then spent time making a drawing of a face at the bottom of his paper and writing "give me more harder fractions." [Work attached -- #6]

Tanya's paper shows her correct solutions for four of the problems, including 2 1/2 x 5 1/4. Tanya, as Jason did, figured four partial products, but she was able to show me where they all appeared on the rectangular array she also drew. As on the previous day's assignment, her work is clear and correct. [Work attached -- #7]

Douglas is very capable reasoning numerically but finds using rectangles completely inaccessible. He got the correct answers for the four problems he did and his reasoning for multiplying 1 2/3 x 2 1/2 shows how he correctly used the distributive property and figured with fractions. [Work attached -- #8 (2 pages)] But his drawings of rectangles make no sense. The pedagogical question I have to resolve is how hard to push Douglas (and students like him) to make sense of the geometric model. I'm satisfied that he can reason numerically, but I feel that he needs to strengthen his ability to reason spatially.

On today's assignment, Sally drew rectangles correctly. She didn't bother drawing one, however, for the third problem. "I'm sure I'm right," she told me when I questioned her, "and I wanted to try one of the harder ones." Sally copied the fourth problem incorrectly from the board, but her solution to the problem she did is correct. It was a challenge for me to figure out how she reasoned, but I finally tracked her recording and saw that her thinking was solid. [Work attached -- #9]

Candace's work is also typical of some of the other students. She did fine with problems that called for multiplying a whole number by a fraction but was confused when multiplying two mixed numbers. For 1 2/3 x 2 1/2, she multiplied 1 x 2 to get 2 and then went to multiply 2/3 x 1/2. To do this, she converted the fractions to sixths and added -- 4/6 + 3/6 = 7/6. She wrote 2 7/6 for her answer. Just a few minutes before the end of class, Candace showed me her paper. [Work attached -- #10, 2 pages] I commented, as I looked at the first three problems, about how what she did made sense to me. I asked her a few questions to hear her reasoning: "How did you know that 3 times 1/4 was 3/4?" "Where did the 1/3 + 1/3 + 1/3 + 1/3 + /13 come from?" She answered those easily. When I asked her what made her think about converting 2/3 and 1/2 to sixths in the last problem, Candace answered, "So I could add them." "I don't understand why you were adding 2/3 and 1/2," I responded. My question caught her off guard and this, with the reality that it was one minute before P.E., seemed to lead Candace to a shutdown. She couldn't explain and started to get upset. "Let me think more about what you did," I said. "Everything else on your paper makes good sense to me, and I'll get back to you if I have trouble figuring this out." Candace nodded with relief and went back to her seat to be dismissed for P.E.

Reflections

It's clear that the students need more time to think about, discuss, and solve multiplication problems like these. Studying the students' papers gave me some hints about what to do next. First of all, I thought that the whole class would benefit from discussing the problem that

Candace was grappling with -- 1/2 x 2/3. I realized that I hadn't given them much experience with multiplying two fractions. That was because I had decided that an easier beginning to the topic of multiplying fractions was to start with multiplying whole numbers times fractions because that led to a fall-back strategy of multiplication as repeated addition. This still made sense to me so I thought that perhaps another time I should keep problems to just those until all of the students could solve them in several ways -- by repeated addition, by using the distributive property, by constructing a rectangle and figuring its area.

After that, I thought, problems where one factor is 1/2 and the other is a whole number, fraction, or mixed number might be good to tackle. I thought this because finding half of something is accessible to the students, such as when Tanya and others knew that half of 1/2 was 1/8 when I asked that to help them think about 1/2 x 3/4. When Candace was facing 2/3 x 1/2, I suspected that she was trying to do something that she knew how to do with the numbers. If I had asked Candace how much was half of 2/3, however, I felt sure that she'd know that it was 1/3. She never thought, however, about 2/3 x 1/2 being the same as 1/2 x 2/3 which called for taking half of 2/3.

Then again, I thought, all problems with 1/2 as a factor aren't so tidy. The problem of 1/2 x 3/4 was the part that made the problem on the first day's assignment so tough. It would be interesting to focus on problems like these where Candace's skill of converting could come in handy. Thinking about half of 6/8 is easier than thinking about half of 3/4.

Following this line of thinking about teaching multiplication of fractions, I might then give students different types and more complex problems to think about, helping them see that while some methods work well for some problems (repeated addition, for example, when one factor is a whole number), they have to choose a method that makes sense for the numbers at hand. This is the same goal I have when teaching multiplication of whole numbers. But the part that bothered me about this pedagogical direction was that it seemed to contradict the view I've long held that sequencing material hints at sequencing children's thought processes. It isn't a pedagogical approach that I've liked to use.

Also, in these lessons I was addressing all problems abstractly, using only a geometric area model for them, a model which wasn't helping some of them at all. Thinking about eggs was the only introduction of a real-life context into these three days of instruction. Perhaps I should think more about making contexts an integral part of learning about multiplying fractions.

A question then occurred to me: What is my expectation for the students regarding multiplication of fractions? I don't want to resort to teaching them the standard algorithm of multiplying numerators and denominators. While Alan knows the rule and it serves him to be efficient, unless he can also reason why and also has other ways to figure, the rule has the limits that we all know about and have encountered with students for years. And I think that until students have a firm understanding of how to use an area model for multiplication, I don't have a way to help them see why the standard algorithm that Alan knows makes sense. I don't want to teach students procedures that they can't understand.

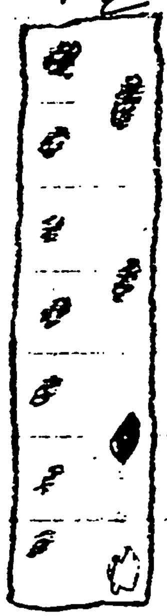
The good news is that even with the confusion and partial understanding that existed, the class discussions, my individual conversations, and some of the students' written work revealed that the students were capable of understanding in ways I hadn't imagined. At the same time, I have to deal with the realities of helping all students learn and of dealing with my worry that I'm causing as much confusion as I'm fostering learning.

There was a great deal from these three days for me to think about.

(1) $7 \times 1\frac{1}{2} = 10\frac{1}{2}$

$1\frac{1}{2}$

7



$7 \times 1\frac{1}{2}$

7 $3\frac{1}{2}$

$$\begin{array}{r} 7 \\ + \; 3\frac{1}{2} \\ \hline 10\frac{1}{2} \end{array}$$

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$7 \times 1\frac{1}{2}$ #2

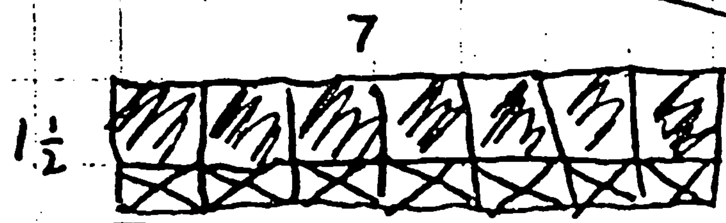
Key ▨ = 1
X = $\frac{1}{2}$

① $7 \cdot 1\frac{1}{2} =$ ~~10~~ $10\frac{1}{2}$

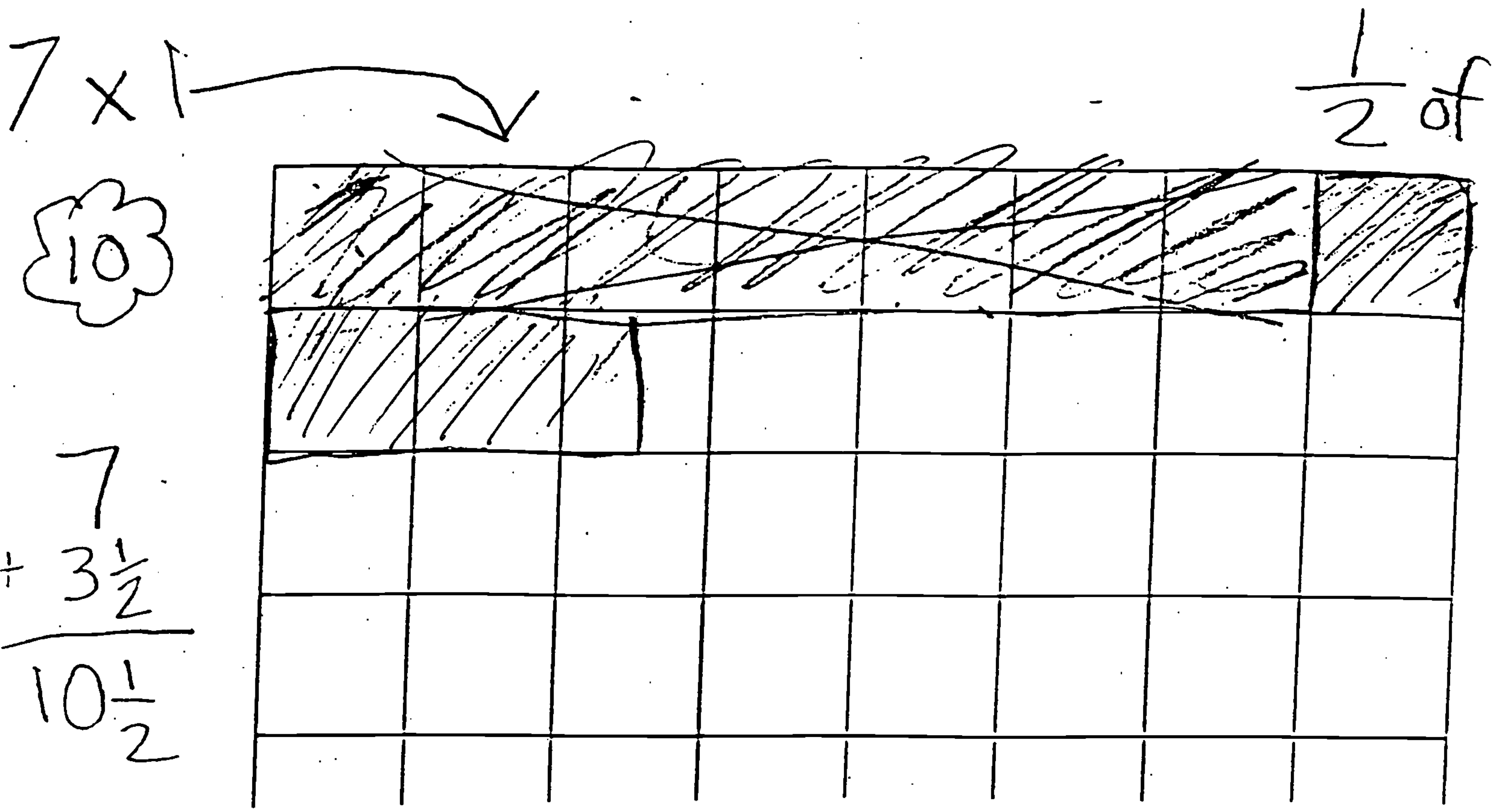
$7 \cdot 1 = 7$

$7 \cdot \frac{1}{2} = 3\frac{1}{2}$

$7 + 3\frac{1}{2} = 10\frac{1}{2}$



Sabine $7 \times 1\frac{1}{2}$ #3



Lily $7 \times 1\frac{1}{2}$ #4

(12)

Lily

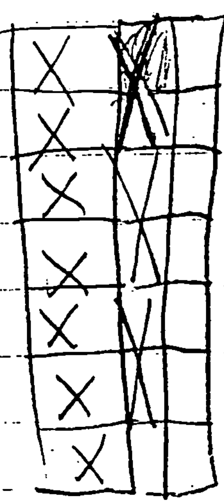
$7 \times 1\frac{1}{2}$

$14 \times 3 = 42$

$$\begin{array}{r} 10 \\ 4\overline{)42} \\ \underline{40} \\ 2 \end{array}$$

$$\begin{array}{r} \overset{1}{1}4 \\ \times\ 3 \\ \hline 42 \end{array}$$

$1\frac{1}{2} + 1\frac{1}{2} + 1\frac{1}{2} + 1\frac{1}{2} + 1\frac{1}{2} + 1\frac{1}{2} + 1\frac{1}{2} = 10\frac{1}{2}$



$10\frac{1}{2}$

Matthew

① $3 \times 3\frac{1}{3} = 10$

($9\frac{1}{2}$)

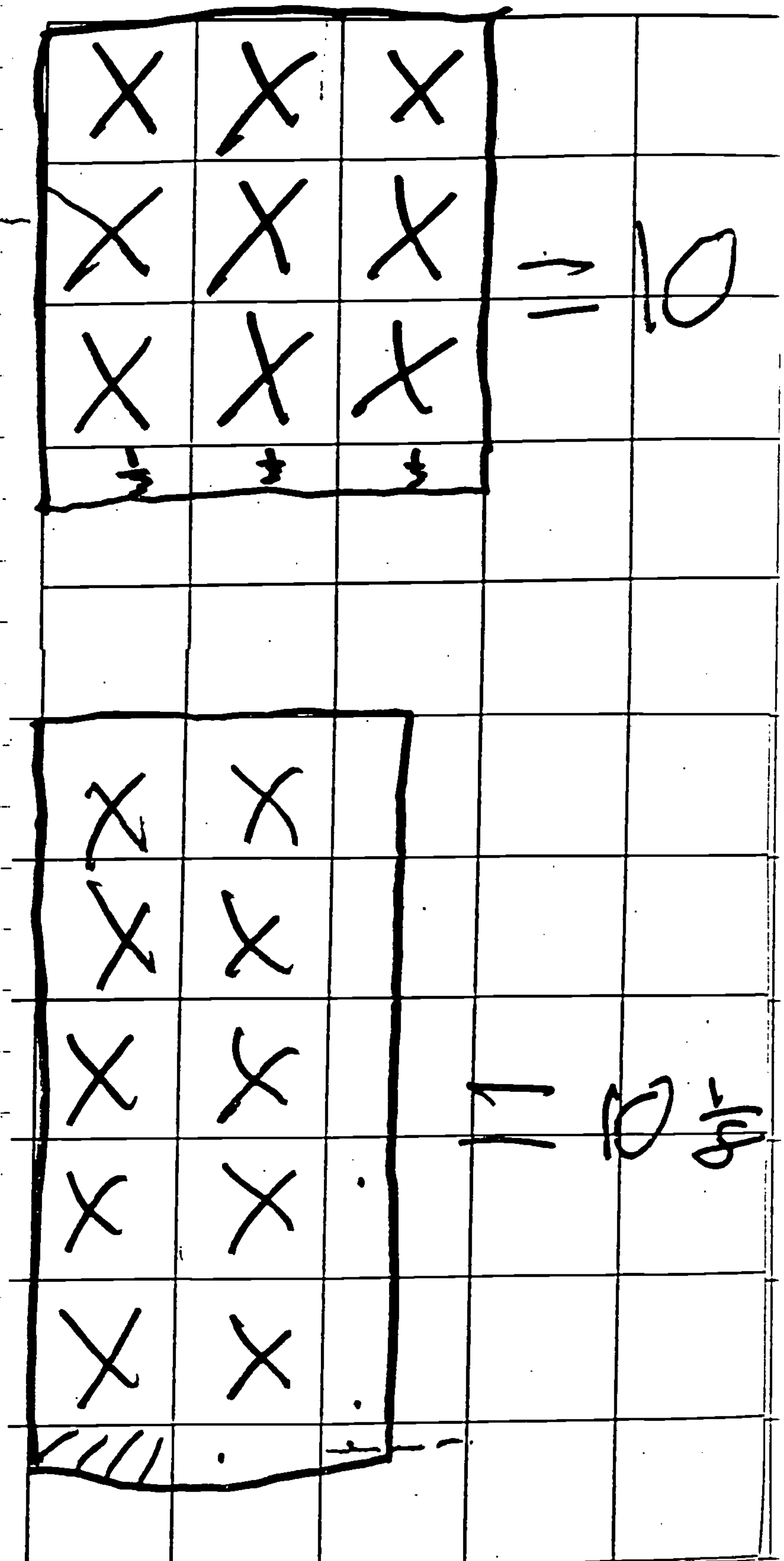
$3 \times \frac{1}{3} = 1$

⑤ $2\frac{1}{2} \times 5\frac{1}{4} = 10\frac{1}{8}$

(10)

$2 \times 5 = 10$

$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$



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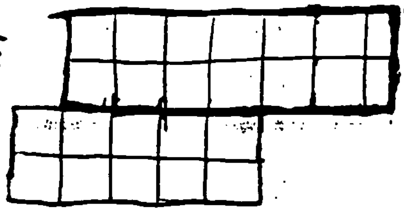
Jesse #6

(5) = $13\frac{1}{8}$

(1) = 10



(2) = $16\frac{2}{3}$

(4)
$$\begin{array}{r} 2\frac{1}{2} \\ \times\ 1\frac{2}{3} \\ \hline \end{array}$$

14 × 23

$$\begin{array}{r} 23 \\ \times\ 14 \\ \hline 12 \\ 80 \\ 30 \\ 200 \\ \hline 412 \end{array}$$

$$\begin{array}{r} 2\frac{1}{2} \\ \times\ 5\frac{1}{4} \\ \hline \frac{1}{8} \\ \frac{1}{2} \\ 2\frac{1}{2} \\ 10 \\ \hline 10 \\ +\ 2 \\ \hline 12 \\ +\ 1 \\ \hline 13 \\ +\ \frac{1}{8} \\ \hline 13\frac{1}{8} \end{array}$$

give me more harder Fractions

#

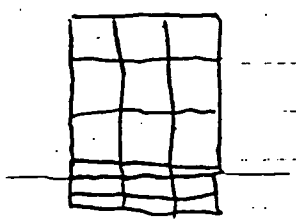
Taylor

① $3 \times 3\frac{1}{3} = 10$

$3 \times \frac{1}{3} = 1$

$3 \times 3 = 9$

$9 + 1 = 10$

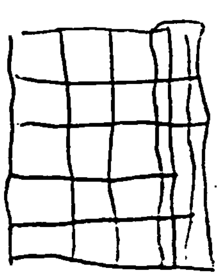


② $5 \times 3\frac{1}{3} = 16\frac{2}{3}$

$5 \times 3 = 15$

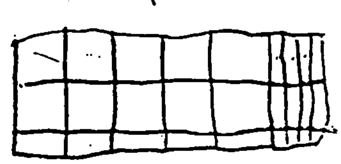
$5 \times \frac{1}{3} = 1\frac{2}{3}$

$15 + 1\frac{2}{3} = 16\frac{2}{3}$



⑤ $2\frac{1}{2} \times 5\frac{1}{4} = 13\frac{1}{8}$

$2\frac{1}{2} \times 5\frac{1}{4}$



$2\frac{1}{2} \times 5\frac{1}{4}$

$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

$\frac{1}{4} \times 2 = \frac{1}{2}$

$5 \times \frac{1}{2} = 2\frac{1}{2}$

$5 \times 2 = 10$

$13\frac{1}{8}$

③ $3 \times 2\frac{1}{4} = 6\frac{3}{4}$

$2\frac{1}{4} \times 3$

$3 \times \frac{1}{4} = \frac{3}{4}$

$3 \times 2 = 6$

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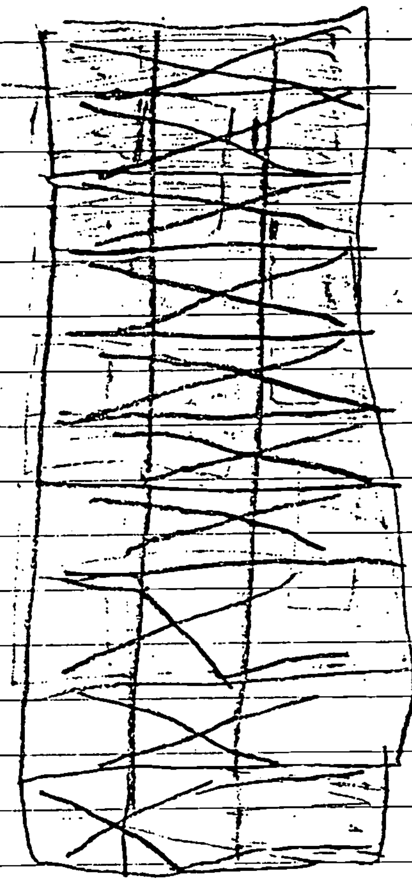
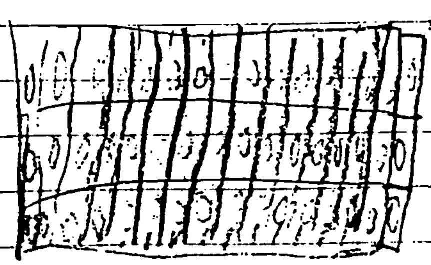
3/10
Brendan

1 3x3 $\frac{1}{3}$ (6) = 10



= $\frac{10}{1}$

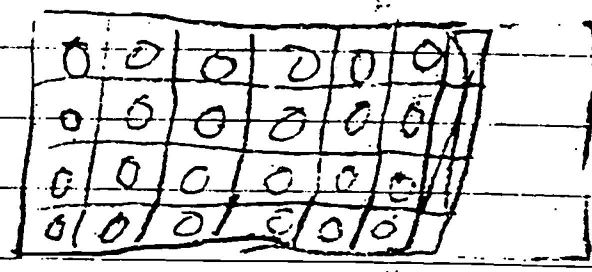
2. 5x3 $\frac{1}{3}$ (15) = 16



= $\frac{16}{1}$

Brendan (cont) #8 k

(3) $3 \times 2\frac{1}{4}$ (6) $= 6\frac{3}{4}$



(4) $1\frac{2}{3} \times 2\frac{1}{2}$ (3) $= 4\frac{1}{6}$

$1\frac{2}{3} \times 2 = 1\frac{1}{3} + 2 = 3\frac{1}{3}$

$1\frac{2}{3} \times \frac{1}{2} = \frac{5}{6}$

$\frac{1}{2} \times \frac{3}{3} = \frac{3}{6}$ $\frac{1}{3} \times \frac{2}{2} = \frac{2}{6}$

3-10-9
Sabin

1. $3 \times 3\frac{1}{3}$ (10) 10

$3 \times 3 = 9$
$3 \times \frac{1}{3} = +1$
10

1	2	3	$\frac{1}{3}$ ← $\frac{1}{3}$
4	5	6	$\frac{1}{3}$
7	8	9	$\frac{1}{3}$

$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

2. $5 \times 3\frac{1}{3}$ (16) $\frac{2}{3}$

$5 \times 3 = 15$
$5 \times \frac{1}{3} = 1\frac{2}{3}$
$16\frac{2}{3}$

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$ ← $\frac{1}{3}$

$16\frac{2}{3}$

3. $3 \times 2\frac{1}{4}$ ($6\frac{3}{4}$)

$3 \times 2 = 6$
$3 \times \frac{1}{4} = \frac{3}{4}$
$6\frac{3}{4}$

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4. $1\frac{1}{3} \times 2\frac{1}{2} =$ ($3\frac{1}{3}$)

$1 \times 2 = 2$
$\frac{1}{3} \times 2 = \frac{2}{3}$
$2\frac{2}{3}$
$1 \times \frac{1}{2} = \frac{1}{2}$
$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

1	2	$\frac{1}{2}$ — $\frac{1}{2}$

$\frac{4}{6} + \frac{2}{3}$ $\frac{1}{6}$ $\frac{5}{6}$ $2\frac{2}{6}$

$\frac{1}{2} = \frac{3}{6} + 2$

$2\frac{2}{3}$
$\frac{3}{6} \leftarrow \frac{1}{2}$
$\frac{1}{6}$
$3\frac{1}{3}$

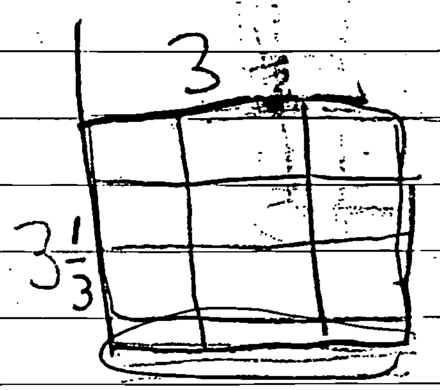
#10 a

Camille

$3 \times 3\frac{1}{3} = 10$

$3 \times 3 = 9$

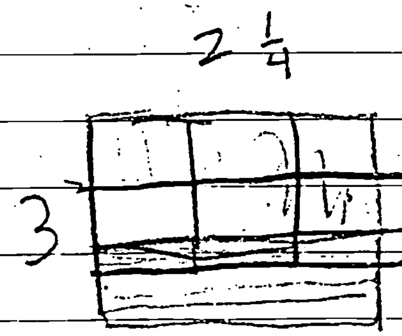
$\frac{1}{3} \times 3 = \frac{1}{10}$



$3 \times 2\frac{1}{4} = 6\frac{3}{4}$

$3 \times 2 = 6$

$3 \times \frac{1}{4} = \frac{3}{4}$



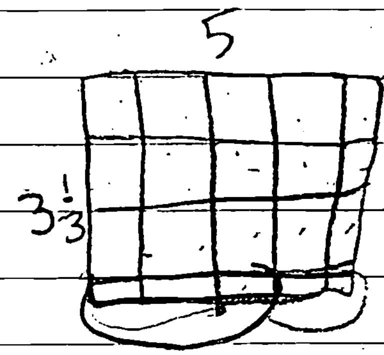
$= 6\frac{3}{4}$

Camille #10b
(cont)

$5 \times 3\frac{1}{3}$

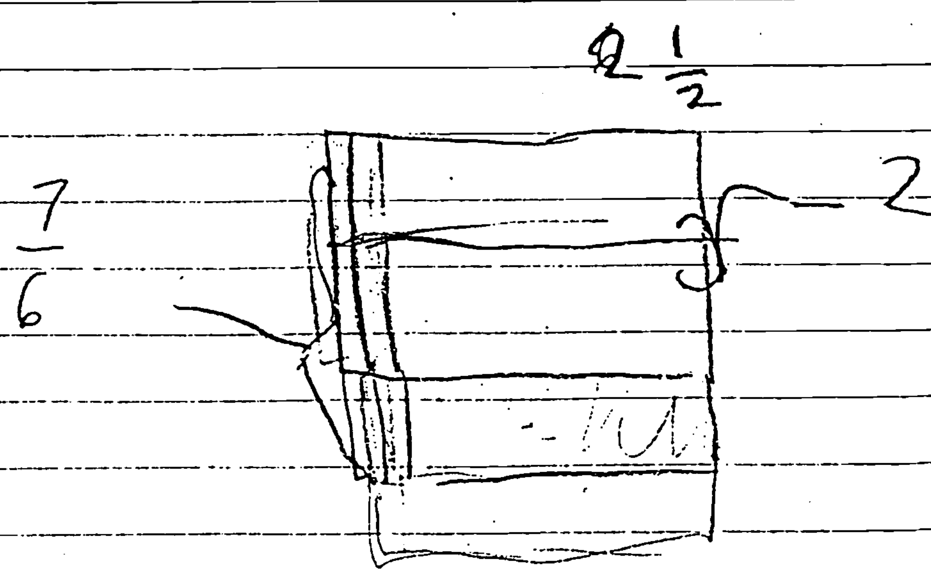
$5 \times 3 = 15$ $16\frac{2}{3}$

$\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ $\frac{1}{3} + \frac{1}{3}$



(4) $1\frac{2}{3} \times 2\frac{1}{2} = 2\frac{7}{6}$ $\frac{4}{6} + \frac{3}{6}$ $\frac{7}{6}$

$1 \times 2 = 2$



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