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ABSTRACT

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Reform in Differential Equations: A Case Study of Students' Understandings and Difficulties

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Abstract

This study investigated six students' understandings of and difficulties with qualitative and numerical methods for analyzing differential equations. From an individual cognitive perspective, the following obstacles were found to influence the development of students' understandings: the function dilemma, the tendency to overgeneralize, interference from informal or intuitive notions, and the complexity involved with graphical interpretations. From a sociocultural perspective, students' understandings were constrained by the use of technology that was disconnected from the learning process, instruction that did not seek out students' explanations, and classroom interactions that implicitly established procedure-based mathematical justifications.

Reform in Differential Equations: A Case Study of

Students' Understandings and Difficulties

The typical engineering or physical science student begins his or her university studies in mathematics with a year of calculus, followed by differential equations in the second year. In the past decade there has been a nationwide effort to revitalize the calculus curriculum and its instruction, but less well-publicized and far less well-researched are the changes occurring in the content, pedagogy, and learning of differential equations. Current reform efforts in differential equations are decreasing the traditional emphasis on specialized analytic techniques for finding exact solutions to differential equations and increasing the use of computing technology to incorporate qualitative and numerical methods of analysis.

The vast majority of differential equations cannot be solved with the numerous analytic techniques that have been the mainstay of the traditional introductory course in differential equations. Numerical and qualitative methods, however, are applicable to all differential equations. Numerical methods are most easily employed with the aide of technology and, in many cases, they provide accurate approximate solutions to differential equations. With qualitative methods, one obtains overall information about solutions by viewing the differential equation geometrically and by analyzing the differential equation itself. For the purpose of this study, the terms qualitative methods and graphical methods are used interchangeably. West (1994) provides many illustrative examples of how numerical and qualitative methods are used to analyze differential equations and several recent textbooks (e.g., Blanchard, Devaney, & Hall, 1996; Borrelli & Coleman, 1996; Coombes, Hunt, Lipsman, Osborn, & Stuck, 1995; Kostelich & Armbruster, 1996; Lomen & Lovelock, 1996; West, Strogatz, McDill, & Cantwell, 1996) reflect these new directions.

Research on student understanding of differential equations, however, has lagged behind these reform efforts. This is problematic as curricula developed without the link to research into student thinking is likely to be less effective than curricula that take student cognition into consideration. In one of the few published studies on student understanding in differential equations, Artigue (1992) investigated first year French university students' understandings of and difficulties with qualitative methods for analyzing first order differential equations. In Mexico, Solís (1996) describes the framework and theoretical background for a future study that will examine how students might use visual and analytic modes of thinking about linear differential equations. Except for these two studies, the vast majority of published reports in the United States have been expository or illustrative in nature. The purpose of the study reported here was to examine students' understandings of and difficulties with qualitative and numerical methods for analyzing differential equations and to identify the factors that shape these understandings and difficulties in one such approach to reform. For new areas of interest such as differential equations, understanding the difficulties that students may have is an important first step for research into student learning. Such knowledge, when linked and synthesized with future studies, will provide a sound basis for future curricular and instructional design.

Theoretical Framework

The theoretical orientation employed in this study is based on the emergent perspective (Cobb & Bauersfeld, 1995), which strives to coordinate the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). From this viewpoint, mathematical learning is viewed "as both a process of active individual construction and a process of enculturation into the mathematical practices of the wider society" (Cobb, 1996, p. 35). These two perspectives treat the role of representations (such as graphs, expressions, etc.) somewhat differently. From the individual cognitive perspective, these representations are characterized "as means by which students express and communicate their thinking." From the sociocultural perspective, these representations are treated "as carriers of either established meanings or of a practice's intellectual heritage" (p. 35).

The study also draws on the work of Herscovics (1989) and his description of the various cognitive obstacles associated with learning algebra. Herscovics described three kinds of obstacles (or factors): epistemological obstacles, obstacles associated with the learner's process of accommodation, and obstacles induced by instruction. The later two he termed "cognitive obstacles" and he credited the French philosopher Bachelard with introducing the term "epistemological obstacle."

Epistemological obstacles naturally arise with revolutionary shifts in the dominant scientific assumptions, methods, and conceptualizations of a discipline. For example, Newton's groundbreaking work in mathematical modeling through differential equations and Poincaré's geometric approach for viewing solutions to differential equations can be considered revolutionary shifts in thinking (Kuhn, 1970). Proposed new paradigms for knowledge development such as these naturally encountered objections and confusion. Herscovics noted that,

> just as the development of science is strewn with *epistemological obstacles*, the acquisition of new conceptual schemata by the learner is strewn with *cognitive obstacles*. And just as epistemological obstacles are considered normal and inherent to the development of science, so should cognitive obstacles be considered normal and inherent to the learner's construction of knowledge. (p. 61)

Herscovics used Piaget's theory of equilibration to frame and relate his discussion of the cognitive obstacles associated with the process of knowledge construction and those induced by instruction. Although this is a good start, different theoretical perspectives are needed when considering these two types of cognitive obstacles. From the individual cognitive perspective, a constructivist theory of learning is used as a lens to examine the difficulties encountered by individual learners. However, when viewing the obstacles induced by instruction, the sociocultural perspective is more appropriate. As Cobb (1996) notes, the particular perspective "which comes to the fore at any point in an empirical analysis can then be seen to be relative to the problems and issues at hand" (p. 35).

In order to clarify these two cognitive obstacles described by Herscovics and the different theoretical perspective employed in thinking about each, the terms "individual cognitive obstacles" and "sociocultural cognitive obstacles" are used. The difficulties that students have can then be viewed with either of these lenses. It is most likely the case that a combination of these obstacles comes into play at any one time and no one lens is entirely adequate. For the purposes of discussion, however, individual and sociocultural cognitive obstacles are discussed separately.

Method

The Setting

Six students (Gordon, Sean, Robert, Debra, Tom, & Jeff) from one section of an introductory course in differential equations for scientists and engineers at a large mid-Atlantic university volunteered to participate in the study. The six students were part of an intact class of 18. The class met for three 50-minute lectures per week in a room with no computer resources. The course used the fifth edition of the textbook *Elementary Differential Equations* by Boyce & DiPrima and a reform-oriented computer supplemental text *Differential Equations with Mathematica* by Coombes, Hunt, Lipsman, Osborn & Stuck. Students in the class were assigned four problem sets from the Mathematica supplement as a means for developing and deepening their understandings of qualitative and numerical methods for analyzing differential equations. These assignments counted for 20% of their course grade. The target class under investigation was not an experimental section, but rather the typical configuration for this course.

Data Sources

From an emergent perspective, it is necessary to collect data that will allow for both individual and sociocultural analyses. This requires individual student interviews and classroom observations. Data therefore included transcripts from four semi-structured individual interviews with the six students; fieldnotes from every class session; instructor and other mathematics department faculty interviews; copies of students' quizzes, exams, computer assignments; and an end-of-the-semester questionnaire administered to all students in the target class as well as to students in six similar sections of the course. I also audio-taped every class session. Audio-tapes were used to supplement my classroom fieldnotes.

Each of the four 60 to 90 minute interviews consisted of three to five problems, all of which were reviewed by two mathematicians for validity and appropriateness. Students were asked to "think aloud" as they worked through the problems. Since the purpose of the interview was to explore students' understandings and difficulties, a semi-structured interview format was used (Schoenfeld, 1985). Thus, although the problems were set in advance, I was free to expand on these questions and to probe students' thinking in order to get as complete a picture of their understandings as possible. Students were frequently asked, "Why did you do ...?" or "How would you explain to another student why ...?" Such intervention is more likely to elicit interesting behavior and lead to understanding students' thinking (Schoenfeld, 1985). The problems were adapted from other reform-oriented textbooks, exam questions from previous years, prior research, the supplementary text, or written by myself. Each of the four interviews took place as soon as possible after students had completed each of the four Mathematica assignments. This was an appropriate time to conduct the interviews since the Mathematica assignments were intended to develop and deepen their understandings of numerical and qualitative methods. None of the problems required the use of Mathematica (version 2.2) since figuring out the correct syntax often takes considerable time, but most of the problems made use of previously generated output from Mathematica. In this way, I was able to gain insight regarding students' understandings without the frustrations students often encounter with syntax.

Data Analysis

The overall method of data analysis was analytic induction. This strategy involves "scanning the data for categories, developing working typologies and hypotheses on an examination of initial cases, and then modifying and refining them on the basis of subsequent cases" (LeCompte and Preissle, 1993, p. 254).

Analysis of the data began with data collection. I attended and audio-taped every class session, keeping field notes on questions raised by students, development of concepts and methods, references to technology, use of graphical representations, and instructional format. As soon as possible after each class was over, I used my field notes and classroom audio-tapes to write a one to two page summary of these details for each class session.

Analysis of the interview transcripts began with a written summary of responses to each question. I then looked for convergent, inconsistent, and contradictory evidence (Mathison, 1988) in students' quizzes, exams, computer assignments, and in my classroom observation notes. Descriptions of students' understandings and difficulties were then formed. The multiple data sources were then read and reread for consistency and accuracy of propositions. These descriptions were then related to and linked with existing descriptions of students' understandings and difficulties in other areas of mathematics.

Results

Two of the goals of the supplemental text were to "guide students into a more interpretive mode of thinking" and to enhance students' ability to qualitatively and numerically analyze differential equations (Coombes et al., 1995, p. 5). The instructor's numerous comments in lecture about the positive role of graphical and numerical methods of analysis reveals that he views these as worthy goals. The findings suggest, however, that these goals were less than adequately achieved. Consequently, there was a gap between the intended curriculum and the achieved curriculum. For example, students had difficulty making important conceptual, symbolic, and contextual connections to the various graphical representations used to visualize solutions to differential equations. Some qualitative and numerical methods of analysis were learned in isolation from other aspects of the problem and the concept of stability and its connection to the reliability of numerical methods was not well understood. These difficulties, among others, are analyzed according to what I see as the factors or learning obstacles that help shape these difficulties. The discussion of results focuses on what I found to be the major difficulties encountered by these six students. Thus, instead of detailing student responses to each interview problem, I highlight particular responses to those interview problems that are paradigmatic of specific difficulties and worthy of further investigation. Considering the dearth of literature regarding the teaching and learning of differential equations to date, this holistic, interpretive analysis is appropriate at this point.

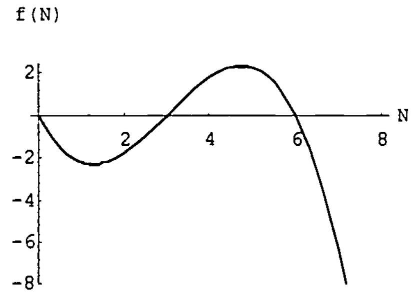
Individual Cognitive Obstacles

The following categories of individual cognitive obstacles were developed as a means to frame and understand students' difficulties in this course: the function dilemma, the tendency to overgeneralize, interference from informal or intuitive notions, and the complexity involved with graphical interpretations. These categories were culled from the literature on students' understandings and difficulties with concepts and methods in algebra, functions, and calculus.

The function dilemma.

Students' difficulties understanding the concept of function is well-documented in the literature. There are several surveys of this research (e.g., Eisenberg, 1991; Leinhardt, Zaslavsky, & Stein, 1990; Thompson, 1994) and one-time publications dedicated to this topic (e.g., Harel & Dubinsky, 1992; Romberg, Fennema, & Carpenter, 1993). It is not surprising then, that students have a difficult time with the notion that solutions to differential equations are functions. This difficulty is compounded by the fact that students' past experience with the word "solve" usually required them to find an unknown *number* (or numbers). In differential equations, however, the "unknown" that one is interested in is no longer a number, but a function. The following problem which was part of the first exam and discussed during the first interview illustrates this dilemma.

Suppose a population is modeled by the equation $\frac{dN}{dt} = f(N)$ where $f(N) = -4N(1-\frac{N}{3})(1-\frac{N}{6})$. The graph of $f(N)$ is shown below.



Graph of $f(N)$ vs. N

(a) What are the equilibrium solutions?
(b) Which of the equilibrium solutions are asymptotically stable? Which are unstable?
(c) For the following values of the initial population $N(0)$, what is the limiting population?
(i) $N(0) = 2$, (ii) $N(0) = 3$, (iii) $N(0) = 4$, (iv) $N(0) = 7$

All six students provided appropriate responses for parts (a) and (b). For part (a), students determined the equilibrium solutions (N = 0, 3 & 6) by either examining the factored form of the differential equation or by reading the values off the graph.

For part (b), students typically created a new sketch by first drawing three horizontal lines and labeling them 0, 3, and 6. Then, by either examining the graph provided or by plugging in values into the differential equation, they determined the sign of dN/dt. If dN/dt was positive, they drew an increasing curve and if dN/dt was negative, they drew a decreasing curve. The following sketch produced by Gordon is representative of this approach for part (b).

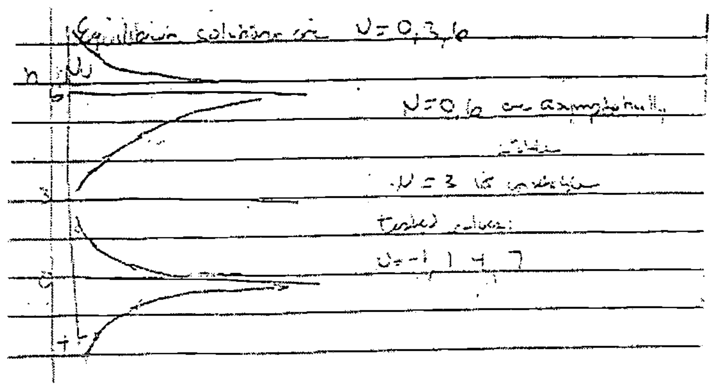


Figure 1. Gordon's sketch for part (b)

Based on this new sketch, students correctly determined that 0 and 6 were stable equilibrium solutions and that 3 was an unstable equilibrium solution.

However, for part (c), four of the six students, Sean, Robert, Gordon, and Jeff, provided responses that were incompatible with the expected response. The incorrect responses on part (c) consisted of attempts at solving the differential equation, evaluating d*N*/d*t* at the given initial condition, and concluding that the various populations tend to either 0, ∞, or 6. Why would students be able to do parts (a) and (b), but fail to make appropriate connections between this analysis and the long term behavior for various initial populations? During the interview, the answer to this question was quite clear. Students did not view the sketch they created (like the one created by Gordon in Figure 1) as a representative plot of the *functions* that solve the differential equation. In the words of Gordon, the sketch he made was "just a test for stability." Sean and Jeff had similar perspectives. In fact, this additional sketch was so disconnected from the problem at hand that Sean stated one could actually place the horizontal lines labeled 0, 3, and 6 *in any order* and still come to the correct conclusions regarding stability.

From an individual cognitive perspective, the sketch in Figure 1 is an expression of Gordon's thinking. When Gordon and the other three students said that the equilibrium solutions were 0, 3, and 6, these were only *numbers* in their mind and not constant *functions*. As a result, the sketch in Figure 1 (and others like it) was "just a test for stability." It was not a graphical expression of the functions which satisfy (or solve) the differential equation. This resulted in an inability to make appropriate connections between the equilibrium solutions to autonomous differential equations and the long term behavior of solutions that start off near the equilibrium solutions. Students could find the equilibrium solutions and even classify them as stable or unstable, but they did not appear to think of equilibrium solutions to differential equations as collections of functions, but rather as a collection of numbers where the derivative is zero. In a traditional course, the typical complaint is that students often learn a series of analytic techniques without understanding important connections to other aspects of the problem. The results here suggest that these students learned to carry out a graphical method of analysis without understanding important connections to other aspects of the problem.

The tendency to overgeneralize.

Students' tendency to overgeneralize algebraic operations like the distributive property, operations like differentiation and integration, and the limiting process are well documented in the literature. The interviews with students in differential equations revealed several specific instances of this tendency to inappropriately generalize as well.

In a direction field-differential equation matching activity in the first interview, students were provided with 4 direction fields and asked to match each direction field with one of 8 given differential equations. When solving this problem, students tended to overgeneralize the notion of an equilibrium solution. If a differential equation is autonomous (i.e. of the form $dy/dt = f(y)$)

then equilibrium solutions exist whenever the derivative is zero. That is, the constant function $y(t) = c$ (for the appropriate value(s) of c) is a solution to the differential equation. Students tended to think that such equilibrium solutions existed whenever the derivative was zero, even for differential equations that were not autonomous. For example, when examining the differential equation $dy/dt = y - t$, students tended to say that $y = t$ was an equilibrium solution and they looked for a direction field with vectors pointing along the line $y = t$. Three of the six students exhibited similar inappropriate generalizations at some point during their attempt to solve the direction field-differential equation matching task (See Rasmussen (1997) for a complete discussion). The reason for this inappropriate generalization is likely linked to students' notions about what constitutes a solution to a differential equation. More specifically, this overgeneralization is a natural extension of the notion that an equilibrium solution to a differential equation is a number where the derivative is zero (rather than as a constant function that satisfies the differential equation).

Another instance of an inappropriate generalization, this time with numerical methods, surfaced in the second interview. Artigue (1992) conjectured that students' graphical image of the Euler method was similar to that of circles approximated by interior or exterior regular polygons. Inspired by this conjecture, students were asked to determine which of the following two sketches best illustrates the Euler method approximation to the initial value problem $dy/dt = y$, $y(0) = 1$ with a step size of 0.5. The dashed line is the approximate solution and the solid line is the exact solution.

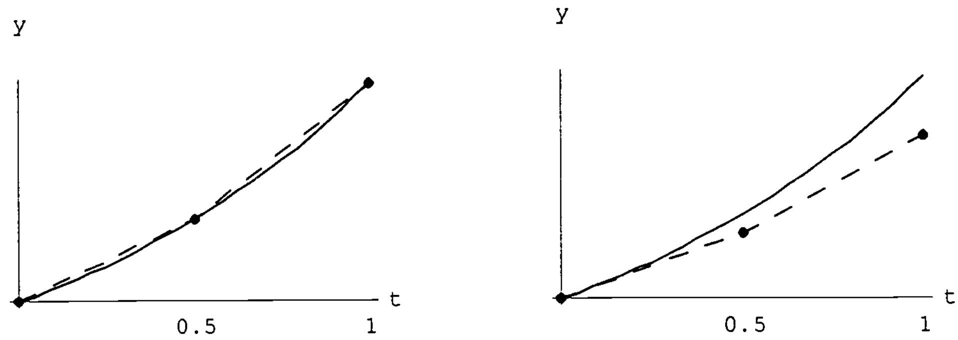
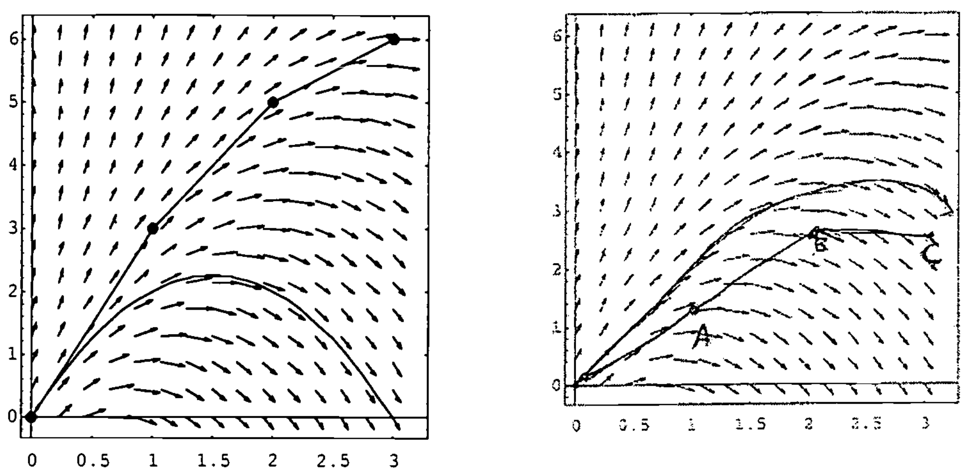


Figure 2. Euler method approximations

In line with Artigue's conjecture, one student incorrectly chose the first sketch as the one that best illustrates the Euler method approximation. Another student said that the first sketch does not depict the Euler method approximation, but that it does illustrate Mathematica's built-in numerical solver. This suggests that Artigue's conjecture is not an artifact of the specific numerical approximation, but rather an artifact of students' image of numerical approximations in general. For example, this image of numerical approximations might be an inappropriate extension of approximation methods like the trapezoidal rule for approximating definite integrals.

Another inappropriate generalization that arose from this problem was the notion that approximate solutions in general always "look like" the exact solution. For the Euler method in particular, Debra and Tom explained that their image of the Euler method was that it uses the slope of the exact solution at each step in the approximation. For example, when $t = 0.5$ in the second graph of Figure 2, Tom explained that the slope of the second line segment was the same as the slope of the exact solution at the point $(0.5, y(0.5))$. This faulty image of the Euler method may be intertwined with the inappropriate notion that approximate solutions always look like the exact solution. If so, this overgeneralization of numerical methods in general and the Euler method in particular helps explain student responses on the following problem from the second interview.

Students were provided with the direction field to the differential equation $dy/dt = (2y - 5x + 3)/(x + 1)$ and asked to sketch the exact solution and the Euler method approximation for the initial value condition $y(0) = 0$ with a step size of 1 unit. Figure 3a depicts the expected sketch while Figure 3b shows a typical sketch for four of the six students.



a. b.

Figure 3. Expected solution and typical student solution

As shown in Figure 3b, the approximate solution curves students sketched tended to follow the same overall pattern of the exact solution with little regard to the direction field. The fact that their sketches and accompanying explanations essentially disregarded the direction field may suggest that they have limited understanding of direction fields and/or that they learned to carry out the Euler method algorithm without fully understanding what the symbols referred to. This later interpretation may have roots in instruction that emphasized procedural competency over conceptual understanding. On the other hand, the Euler method sketches may in fact be graphical instances of overgeneralization previously referred to. More specifically, students conceivably have an overgeneralized notion that approximate solutions always just follow the exact solutions. They tend to generalize that approximate solutions "look like" exact solutions. It is also possible that the word "approximate" has informal meanings that encourage this graphical overgeneralization. To the mathematician, the word approximate does not necessarily mean always close to, but perhaps for students this is a strong connotation. The issue of language and the informal or intuitive notions associated with particular ideas is considered next.

Interference from informal or intuitive notions.

In several of the interview problems, the findings suggested that students did not fully understand the notion of stability and its implications for numerical methods. I suggest that many of these difficulties can be traced back to an informal and somewhat inappropriate notion of stability. In particular, students tended to associate stability with regularity and predictability: much like that of a stable home environment, perhaps. For example, Gordon incorrectly thought that the direction field for $dy/dt = (2y - 5x + 3)/(x + 1)$ typified stability because "all the solutions are going in the same direction. They are all increasing and then at some point starting to decrease....you'd say they are stable." In contrast, instability was associated with wildly different and unpredictable behavior. This inappropriate (but reasonable from an informal viewpoint) conception of stability has serious ramifications when considering issues of reliability for numerical methods.

In some ways, students' notions of stability are similar to the well-documented misconception that graphs of functions are necessarily smooth and without abrupt changes or other irregular behavior. It seems that students tend to think that solutions to differential equations (which are functions, after all) behave in ways that are not too strange. This is most definitely not the case for nonlinear differential equations. For example, in the first problem of the second interview, students were provided with the direction field for the nonlinear differential equation dy/dx = $y(x - y)$ as shown in Figure 4 and asked to describe how the limiting behavior depends on the initial point in the (x, y) plane.

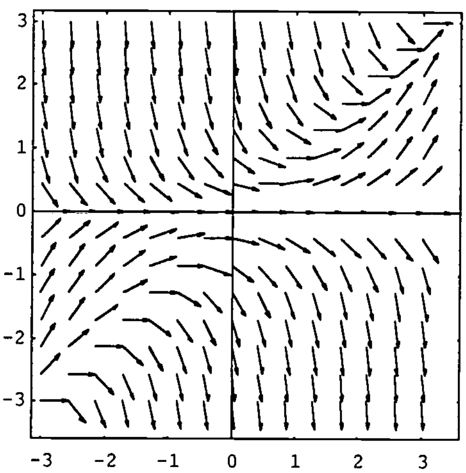


Figure 4. Direction field for $\frac{dy}{dx} = y(x - y)$

Three of the six students, Debra, Sean, and Jeff, thought that solutions starting off in the upper left-hand region of the direction field would tend to 0 as x approaches infinity. After Debra described the behavior of such a solution, I asked what she thought the slopes on the direction field would be when x was 10 and y was positive, but near zero. She responded,

Debra: I think it would be....it would still be, let's see. It looks like it would be kind of positive [plugs in a small number for y and 10 for x in the differential equation]. It would be positive, but it would be really small, that's what it looks like to me. Some positive small number. I don't know if that means it would be curving up some, but y would be getting closer and closer to 0, so I guess it would be....I don't know. It would be really really small, but it would be positive. I don't think it would make a significant increase.

Thus, despite her own calculation that the slope of the tangent vectors would be positive, she decided in the end that it wouldn't "make a significant increase." That is, the solution would remain asymptotic to the x-axis.

Similarly to Debra, Jeff was prompted to compare his predictions with the differential equation. But unlike Debra, Jeff changed his mind about the apparent asymptotic behavior after determining the vectors would point away from the positive x-axis. However, Jeff remained skeptical because he thought the solution $y = 0$ was stable for *all* values of x. "I don't understand why it would want to go away from that line [referring to the equilibrium solution $y = 0$]." This skepticism was rooted in his understanding of stability, as illustrated in the following excerpt.

Jeff: If something is stable, that means that it likes the way it is. So if I were stable, I'd like to stay that way. If I were unstable, I'd try to get stable as much as I could. So that means if you're stable, you won't want to go away from that line, but if you want to become stable, then you have to approach that line.

Thus, personally and graphically, Jeff sees stability as a desirable trait. It appears that Jeff's personal understanding of stability was a strong motivating factor in his original conjecture that solutions would become asymptotic to the x-axis.

The conjecture that some solutions would become asymptotic to the x-axis is perhaps not unreasonable. In fact, it is consistent with our everyday experiences. Consider, for example, the simple pendulum with no external force. Under normal circumstances, the simple pendulum swings back and forth, eventually coming to rest in the expected position. Paraphrasing Jeff, the simple pendulum "wants to become stable." But, as Hubbard and West (1991) point out, nonlinear differential equations are "strange and mysterious" (Hubbard & West, 1991, p.1). Analyzing such equations requires highly interconnected graphical, symbolic, and theoretical understandings. These students are, however, in good company. Gleick (1987) made the following statement about the famous mathematician Smale.

> Smale made a bad conjecture. In the most rigorous of mathematical terms, he proposed that practically all dynamical systems tended to settle, most of the time, into behavior that was not too strange. As he soon learned, things were not so simple. (p. 45)

Thus, although it is intuitively appealing, it is not always the case that solutions that hint at asymptotic behavior continue that behavior.

Complexity involved with graphical interpretations.

A common obstacle when dealing with graphs is the tendency to interpret a graph as a literal picture of the situation being modeled, and that seems to be a pitfall for students in differential equations as well. For example, in the third and fourth interviews, students tended to interpret the angular position versus time graphs as literal pictures of the pendulum behavior, despite having considerable exposure to such graphs both in lecture and in their Mathematica assignments. Another difficulty with graphs encountered by students was the tendency to spontaneously parameterize non-parametric graphs. This is likely a result of inexperience with coordinating parametric and non-parametric graphs together with a tendency to interpret the graph as a literal picture.

Another difficulty students encountered when trying to make sense of the various graphical representations (direction fields, exact and approximate solution curves, phase portraits, and vector fields) was the failure to make important connections between the graph and key conceptual or theoretical notions, such as the existence and uniqueness of solutions, dependence of solutions on initial conditions, and the relationship between stability and reliability of numerical methods. All students in the study demonstrated an inability to make these connections and the tendency to ignore these connections occurred in every interview. As

Eisenberg (1992) points out, this can, in part, be attributed to the fact that graphs are typically complex, concentrated collections of information and hence there is an increased cognitive demand. The amount of information contained in direction fields, vector fields, and phase portraits that needs to be unpacked is considerably greater than that the graph of a function of one variable. It is perhaps then, little wonder that students experience such difficulty interpreting them.

Sociocultural Cognitive Obstacles

In the preceding section, individual cognitive obstacles (or factors) that help shape students' difficulties with qualitative and numerical methods were discussed. These obstacles provided some insight into the development of students' understandings and helped frame specific difficulties. The picture is incomplete, however, without an analysis and discussion regarding the role of technology in the learning process and the culture of the classroom learning environment.

Role of technology.

Based on the interview data and the survey administered to students in seven different sections of the course, most students felt that the use of Mathematica was disconnected from the rest of the course and, although they liked the aspects of visualization, they felt it did not contribute much to their overall understanding about differential equations. Sean expressed a common sentiment when he referred to the Mathematica projects as a "one time thing." Sean, like many other students, just wanted to get the syntax down and move on and do his learning elsewhere. This disconnect between the Mathematica assignments and the rest of class has instructional roots. Mathematica outputs were not discussed in class and, although the instructor made references to Mathematica, it wasn't used in any direct way in lecture to assist learning.

Mathematica was peripheral to the course and, in most cases, peripheral to students' understandings. In this regard, we have to see the use of technology as problematic.

It is a bit naive, however, to simply cite the instructor for not integrating Mathematica more thoroughly into the learning process. Besides the typical barriers that instructors face when attempting to incorporate technology into the curriculum (e.g., time, availability, experience, beliefs and values, etc.) there are influences outside the classroom that led to the current state of integration, or lack thereof. Based on conversations with faculty members, the departmental support for reform has been lukewarm. More specifically, although the department made it policy that Mathematica (or Maple) is to be used in all sections of the course, there was not the extra support and encouragement given to assist faculty in figuring out how to best integrate technology into their instruction. No one of administrative influence stepped forward and said, for example, "Yes, this is the direction we want to head and the faculty should spend their time reading the supplement and seriously working it into their course." The department wanted it to happen, no doubt, but no one in a position of administrative authority "sold" it to the faculty in a way that encouraged serious integration into their teaching. Partly as a result of this lukewarm embrace, the supplement was instituted in a way that required little or no faculty investment. One may argue that this path of least resistance was a wise choice, for otherwise it may not have been implemented at all. However, this path does not appear to have had the intended effect on students' understandings. In a sense, the preceding discussion can be framed in terms of external cultural influences on the microculture of the classroom, which, in turn, had ramifications on individual students' mathematical understandings.

Just as it is too simplistic to look solely towards the instructor for the manner in which technology is or is not integrated into the classroom, it is naive to think that all modes of integrating technology in the classroom will significantly affect students' understandings. How would, or rather could, the integration of technology affect students' learning? To begin to address this, I now turn to the kinds of mathematical discussions that went on in the target classroom *without* technology. The assumption is that understanding current instructional practices without technology will help inform future instructional practices *with* technology.

Classroom environment: Social and sociomathematical norms.

From the sociocultural perspective, the development of students' understandings of mathematics is influenced by classroom interactions and the discourse in which they participate. The interactions in the target classroom were typically one directional--the instructor talked and the students (presumably) listened. Schoenfeld (1988) describes how this type of instruction typically leads to the belief that only the very bright are capable of creating mathematics or really understanding mathematics. As a corollary, students must then accept what is passed down from above with the expectation that they can make sense of it for themselves (p. 151). Students in this study evidenced this belief that mathematics must just be "accepted" on several occasions. Moreover, the previous discussion of individual cognitive obstacles strongly suggests that students were not making important connections in the mathematics being "passed along."

It is also informative to consider examples of classroom interactions that involved exchanges between the instructor and students. The vast majority of such exchanges can be characterized as "question asked-question answered." The instructor was certainly respectful of students' questions, but they were not used as a springboard for further discussion or mathematical development. My point is not to denigrate the instructor's intentions, but rather to shed some light on the mathematical activity and discussion that is a factor in the development of students' understandings.

For example, when covering the linearization of the system of differential equations that model the behavior of the pendulum, the instructor drew a picture of the situation, noted the variables involved, wrote down the nonlinear differential equation, calculated the Jacobian, and concluded that the linearization is $x' = y$ and $y' = -x$. At this point Robert asked, "How did you get the $x' = y$ and $y' = -x$?" The instructor's response was,

I: Because that's the rule. Remember that the linearized system is $x' = Ax + By$ and $y' = Cx + Dy$ and the partial of G with respect to y is 0 and the x' is already linear so you don't have to touch it.

Unfortunately, we never get to find out what Robert's real question is. Is it a conceptual question? Is he just having trouble following the procedure? We don't know. The results from the fourth interview do suggest, however, that students don't understand the notion of linearization much beyond being able to do it. We also know that the students accepted the instructor's response as sufficient justification, which sheds some light on what it means to understand mathematics in this course. Besides capturing this procedure-based approach, the preceding exchange illustrates the implicit acceptance of those involved regarding what counts as an acceptable mathematical explanation and justification. The response, "Because that's the rule," to Robert's question was accepted as a mathematically sufficient justification. Such implicitly established norms are referred to as "sociomathematical norms," since they are distinctly different from general social norms (Yackel & Cobb, 1996). Other sociomathematical norms are implicitly negotiated understandings of what counts as mathematically sophisticated, efficient, different, or elegant.

It is interesting to note that in the interviews we had to try and establish a different sociomathematical norm. I purposefully say "we" because establishing what counts as an explanation or justification is understood to be socially negotiated. We also had to negotiate the general social norm that explanations of solutions and ways of thinking were expected. Based on students' comments and the fact they came to explain their thinking in detail without prompting, my sense is that we did indeed establish different social and sociomathematical norms. To illustrate how these norms are continually negotiated, consider the following comment made by Debra in response to my request for further clarification on the idea behind the characteristic equation.

Debra: We might have derived it in class, but we don't remember why we do that or what it's supposed to mean.

Debra's comment indicates that she recognizes that the social and sociomathematical norms regarding justification and explanation in the classroom are different from what I am asking her to do. In a sense, we are negotiating what will be accepted in the interview as an explanation, and she is making it clear to me that these expectations are different from those implicitly established in the classroom.

As a final example of the way in which mathematical justification and explanation was established, I consider one of two atypical classroom interactions. It was atypical in the sense that it involved a somewhat extended exchange between the instructor and the students. It was typical, however, in that mathematical sense-making, connections, justifications, and explanations were not part of the discussion. In the following example, the instructor introduced the method of undetermined coefficients. The instructor put the differential equation $y''-4y'+3y=e^{2x}$ on the board and asked if anybody could guess a solution.

S1: Guess e^x?
I: That gives us zero. That won't work. Try another guess.
S2: Try e^{2x}.
I: Let's try. Let's figure out what $L[e^{2x}]$ is. That seems Ok.

The instructor writes $L[e^{2x}] = 4e^{2x}-8e^{2x} + 3e^{2x} = -e^{2x}$ on the board and says,

I: Doesn't work, does it?
S3: How about e^{-x}?
I: Don't give up so fast on e^{2x}. Do you really want to give up so fast? So $L[e^{2x}]$ gives us $-e^{2x}$. That didn't work, did it? Do you really want to give up on this so fast?
S2: $-e^{2x}$?
I: Yeah, then we get .....[the answer he was looking for]

This same type of exchange was repeated with four other different differential equations. In the end, the instructor told the students to look at the "top of page 159 where there is a table that shows you what to try." I think the instructor's intentions here are good. It is my interpretation that he wants students to develop a "sense" for the method through illustration and pattern recognition. However, little about the interaction seems to encourage the kind of reflection needed to develop such a sense. In no instance did the instructor pursue *why* students thought that a particular function was a solution. For example, the suggestion of e^x was quickly brushed aside and not pursued. The suggestion of e^{2x} was pursued because that was very close to the anticipated response. I would argue, in fact, that students pick up on what the instructor wants, which was not their motivations, ideas, or strategies, but the expected answer. This, in turn, limits and constrains students' learning opportunities.

In summary, several factors influencing the development of students' understandings and difficulties have been highlighted, including the non-use of technology in the classroom, external classroom decisions, and classroom interactions that led to procedure-based justifications and explanations. The latter of these factors is seen as particularly problematic. Classroom interactions were used primarily as a means to "get to" the answer. A culture of inquiry into the heart of mathematical connections and reasoning was not pursued and this climate constrains the development of students' mathematical understandings.

Discussion of Results

One of the most surprising and unsettling results was the manner in which students learned a graphical technique for analyzing first order autonomous differential equations as an isolated skill, disconnected from the long-term behavior of nearby solutions. Individual cognitive obstacles related to this finding were discussed in the preceding section. I would also like to suggest that this finding has related instructional and curricular roots. Instructionally, getting an answer was far more highly valued than the processes involved in obtaining that answer. Students also practiced problems where they examined the graph of the differential equation (dy/dt vs. y) solely in order to classify the equilibrium points. What seems to have happened is that students proceduralized this graphical technique without having a sound idea about what solutions to differential equations are. Instructional activities and ensuing discussions about those activities would do well to keep this in mind. Although not discussed in this paper, a similar situation occurred for systems of differential equations. Students could classify the equilibrium points quite well, but had little sense of how it all fit together.

The instruction and classroom interactions described in this study revealed that procedures and final answers were more highly valued than students' strategies and their ways of thinking about the subject. This is fairly typical of traditional instruction and is not meant to imply "poor" teaching on the part of the instructor. This traditional mode of instruction, however, is seen as a contributing factor to the difficulties and disconnected ways of thinking exhibited by the students in this study. I suggest that instruction shift toward a more inquiry-oriented approach where students' interpretations, strategies, and ways of conceptualizing representations and concepts become objects of discussion.

Consistent with this call for more sense-making in the classroom, one of the students in the study, Jeff, commented that, "I wish he [the instructor] would just hand out a bunch of graphs and let us think about them." This comment highlights the fact that the various graphical representations typically used in numerical and qualitative analyses are complex, concentrated collections of information and that students do not always "see" what experts see in them. Jeff's suggestion that the instructor "hand out a bunch of graphs" and let us think about them is certainly one possibility to help students develop mature mathematical understandings of fairly sophisticated representations. This approach may be characterized as "top down" since students would be attempting to make sense of mathematically complete representations. In other words, conventional representations could be used as the starting point for classroom discussions and investigations that focus on helping students see what the experts see.

Another approach, which I think holds more promise, would be to take a more activity-oriented route, as discussed by Meira (1995). Gravemeijer, Cobb, Bowers, and Whitenack (in preparation) explain how this approach takes

> account of both the reflexive relation between symbolizing and sense making, and the dynamic character of this relation. It is while actually engaging in the activity of symbolizing that symbolizations emerge and develop meaning within the social setting of the classroom. (p. 11)

In terms of differential equations, these symbolizations would include direction fields, phase portraits, etc. This is a decided shift from traditional modes of instruction where one of the instructor's goal is to explain what these representations (or symbolizations) mean and the mathematical connections embodied therein.

Educational research in the teaching and learning of undergraduate mathematics is a relatively recent phenomenon and the topic of differential equations is even newer. As with the start of many new areas of research, a firm grounding regarding students' conceptions is an essential component for future curricula and instructional approaches. The information gained on students' understandings is one of the strengths of the research reported here. An in-depth examination of students' understandings and difficulties with qualitative and numerical methods of analysis revealed many areas where students lacked the ability to make fundamental representational and conceptual connections.

This study did more than simply document students' difficulties, however. The emergent theoretical perspective was used as a means to understand and interpret these difficulties. Many of the difficulties encountered by students in this study should be viewed as natural and inherent in the learning process. As Herscovics (1989) stated, "the acquisition of new conceptual schemata...is strewn with cognitive obstacles" (p. 61). From the individual cognitive perspective, students' difficulties were framed in terms of obstacles that learners encounter in most areas of mathematics--beginning with the transition from arithmetic to algebra. These overarching obstacles include the function dilemma, the tendency to overgeneralize, informal or intuitive notions, and the complexity involved with graphical interpretations. Understanding the nature of students' difficulties is the first step to thoughtful instruction and curricula designed to help students through the difficult process of constructing new knowledge.

The second lens that was used to interpret students' difficulties was informed by the sociocultural perspective. The main focus of my analysis was on the classroom interactions and the implicit manner in which "understanding" was established. As Schoenfeld (1992) points out, "if we are to understand how people develop their mathematical perspectives, we must look at the issue in terms of the mathematical communities in which students live and the practices that underlie those communities" (p. 363). An examination of the discourse in the target class revealed that students' thinking processes and strategies were not expected to be explained and that correct answers and procedure-based justifications were acceptable mathematical explanations.

Although the preceding picture is discouraging, there is good reason to believe that the situation can be significantly improved. Instructional changes ushered in by the calculus reform movement are finding their way into all levels of undergraduate mathematics education and they need to find their way into the teaching of differential equations. Curricular efforts in differential equations and other undergraduate mathematics courses need to be take seriously what we know about the learning process and organize curricula and instruction to take this into account. Lastly, technological advances continue to offer opportunities to rethink and reorganize what and how we teach.

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