DOCUMENT RESUME

ED 420 494 SE 061 475

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TITLE Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (19th, Bloomington-Normal, IL, October 18-21, 1997). Volume 1.
INSTITUTION ERIC Clearinghouse for Science, Mathematics, and Environmental Education, Columbus, OH.; International Group for the Psychology of Mathematics Education. North American Chapter.
SPONS AGENCY Office of Educational Research and Improvement (ED), Washington, DC.
PUB DATE 1997-10-00
NOTE 353p.; For Volume 2, see SE 061 476.
CONTRACT RR93002013
AVAILABLE FROM ERIC/CSMEE, The Ohio State University, 1929 Kenny Road, Columbus, OH 43210-1080.
PUB TYPE Collected Works - Proceedings (021) -- ERIC Publications (071)
EDRS PRICE MF01/PC15 Plus Postage.
DESCRIPTORS Algebra; Cultural Influences; *Educational Assessment; Elementary Secondary Education; Functions (Mathematics); Geometric Concepts; Geometry; Graphs; *Mathematics Curriculum; Mathematics Education; *Mathematics Instruction; *Problem Solving; *Statistics; Teaching Methods; *Thinking Skills

ABSTRACT

This conference proceedings volume for PME-NA-XIX contains a total of 87 reports: one plenary session report; 39 research reports; 20 short oral reports; 25 poster session reports; and two discussion group reports. Only the plenary and research reports are full reports; the others are generally one-page abstracts. The full reports include: (1) "Participation as Fundamental in Learning Mathematics" (James G. Greeno); (2) "An Undergraduate Student's Understanding and Use of Mathematical Definitions in Real Analysis" (Barbara S. Edwards); (3) "Mis-Generalization in Calculus: Searching for the Origins" (David E. Meel); (4) "Students' Cognitive Approaches to the Concept of Rate" (Rodolfo Oliveros and Manuel Santos-Trigo); (5) "Effects of Different Instructional Approaches on Calculus Students' Understanding of the Relationship between Slope, Rate of Change, and the First Derivative" (Donald T. Porzio); (6) "The Process of Periodicity" (Gilli Shama and Nitsa Movshovitz-Hadar); (7) "The Relationship between Written and Verbal Performances: A Study of First Year Calculus Students' Understanding of the Derivative" (Kathleen G. Snook); (8) "Mathematical Patterns in the Middle Grades: Symbolic Representations and Solution Strategies" (Joyce Wolfer Bishop); (9) "The Relationship of Undergraduates' Beliefs about Learning Algebra and Their Choice of Reasoning Strategies for Solving Algebra Problems" (Albert D. Otto, Cheryl A. Lubinski, and Carol T. Benson); (10) "Teachers' Beliefs and Student Failure in Algebra" (Daniel K. Siebert); (11) "Mandated Assessment Instruments: How Do Teachers Value Them?" (Karen Bell and Thomas J. Cooney); (12) "Using Assessment Practices as a Tool for Changing Teaching Methodology" (Daniel J. Brahier);

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Proceedings of the Nineteenth Annual Meeting

North American Chapter
of the International Group
for the

Psychology of Mathematics Education

Volume 1: Plenary Papers, Discussion Groups, Research Papers, Short Oral Reports, and Poster Presentations

October 18-21, 1997
Illinois State University
Bloomington/Normal, Illinois U.S.A.

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PME-NA XIX

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Editors:
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Published by
The ERIC Clearinghouse for Science, Mathematics, and Environmental Education
Columbus, OH

Cite as:

Dossey, J. A., Swafford, J. O., Parmantie, M., and Dossey, A. E. (Eds.). (1997). *Proceedings of the Nineteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Document development:

David L. Haury, ERIC/CSMEE Executive Editor
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Accession number: SE 060 729

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This publication was funded by the Office of Educational Research and Improvement, U.S. Department of Education under contract no. RR93002013. Opinions expressed in this publication do not necessarily reflect the position or policies of OERI or the Department of Education.

History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and of the North American Chapter (PME-NA) are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Preface

This program began with a meeting of interested volunteers in October 1996 at Panama City, Florida during the 18th PME-NA meeting. The results of the ideas discussed and suggestions made were taken to a meeting of the local program committee at Illinois State University where the theme of the psychological underpinnings of mathematics education was selected. This theme became the focus of two of the plenary sessions. Perspectives from cognitive psychology about the foundations of learning mathematics are presented in a paper by James Greeno. Children's intuitions about numbers are discussed from the perspective of developmental psychology in a paper by Robbie Case. A special memorial lecture in honor of Alba Thompson given by Suzanne Wilson is also planned, but the paper dealing with reform and issues surrounding it was not available at the time these proceedings went to press. Alba was both an active member of PME and a former faculty member at Illinois State University. Hence, the local program committee decided to include this memorial to her in the program of the PME meeting held in her former city of residence.

Included in the *Proceedings* are 68 research reports, 9 discussion groups, 40 oral reports, and 41 poster presentation entries. The research reports and the one-page synopses of discussion groups, oral reports, and poster presentations are organized by topics following the pattern begun with the *Proceedings* of the 1994 PME-NA meeting. Additionally, an alphabetical index by author is provided in both volumes. Initially 238 proposals were received with 218 for research reports, 11 oral reports, and 9 discussion groups. Proposals for all categories were blind reviewed by three reviewers with expertise in the topic of submission. Cases of disagreement among reviewers were refereed by a subcommittee of the Program Committee at Illinois State University. The process resulted in the acceptance without reassignment of about 33% of the research report proposals with an overall acceptance rate across all categories of about 58%.

Submissions for the *Proceedings* were made on disk and read by the editors. The format of the papers was adjusted to make them

uniform but substantive editing was not undertaken. Papers are grouped by topic area for the table of contents and cross referenced alphabetically in the index to both volumes by the first author.

The editors wish to express thanks to all those who submitted proposals, the reviewers, the 1997 Program Committee, and the PME-NA Steering Committee for making the program an excellent contribution to the growing body of research and discussions about psychology and mathematics education. The Program Chairs would like to extend our special appreciation to the mathematics education faculty at Illinois State University for their support and generous contributions to the preparations for the conference.

Jane O. Swafford
John A. Dossey
October 1997

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PARTICIPATION AS FUNDAMENTAL IN LEARNING MATHEMATICS

James G. Greeno and The Middle-School Mathematics through Applications Group
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This paper considers participation in practices of mathematical inquiry, understanding, and reasoning as fundamental in the processes of learning mathematics, as an alternative to considering mathematical skills or conceptual understanding as the fundamental factor. Rather than considering participation as an instrumental condition for acquiring skills and understanding, this view considers skills and understanding as means by which people participate in practices in which mathematical concepts and methods are understood and used. Emphasizing educational aims involving participation also supports a focus on students' development of personal identities as learners, knowers, and users of mathematics. The Middle-School Mathematics through Applications Project is discussed as an example.

What is fundamental in a learning process is a theoretical question, and since we have different theories of learning, we have different answers to the question. In the current state of psychological theory, there are three main candidates for the position of "fundamental." Roughly, the candidates are skill, understanding, and participation. I will sketch the ideas of skill and understanding as a background—these ideas about the fundamentals of learning are quite familiar. I will discuss the idea of participation as fundamental to mathematics learning in more detail. This idea is less developed in our discussions of mathematics education. I propose, however, that the idea of participation has some advantages. I believe that with this idea, we can develop a more coherent theoretical account of learning, and a conceptual basis for more coherent practices of education, than we have been able to with theories in which skill or understanding plays the fundamental role.

I take the question: "What is fundamental?" to be about what we think is most important as a basis for other aspects of learning and for further learning. To ask what is fundamental, in this view, is to ask what needs to be attended to most in considering what kinds of activities should be arranged for students to learn in next. If something is fundamental, it deserves special attention as a kind of prerequisite. The current state of students' progress in respect to a fundamental aspect of knowing will shape the kind of further progress that they can make as they engage in further learning activities. If these activities are designed on the assumption that students have some fundamental capabilities, and they do not, then the intended outcomes of those activities are not likely to be accomplished.

Background: Skill and Understanding

For many decades, the main debate about the nature of learning in mathematics has been a contest between behaviorist and cognitive assumptions. In the middle decades of this century the debate was stated in terms of a theory of stimulus-response connections, often associated with Thorndike (1922), versus a theory of gestalt understanding, often attributed to Katona (1940) and Wertheimer (1945/1959). In the 1960s, the behaviorist theory was developed with much sophistication, principally by Gagné (1965). These ideas have been used as a basis for much educational design and technology, especially in mathematics, in the form of behavioral objectives, that have been the organizing principle for designing much of the curriculum and testing. The cognitive side of the debate has been developed in the theory of information processing by a great many people, including Newell and Simon (1972), as well as much of the voluminous literature on children's mathematical thinking and conceptual growth (e.g., Carpenter, Fennema, & Romberg, 1993; Gelman & Gallistel, 1978; Harel & Confrey, 1994; Steffe, Cobb, & von Glasersfeld, 1988).

In the behaviorist view, skills are fundamental. The most important concern for a student's early learning is the laying down of a firm foundation of skill to build upon. And at any stage of a student's learning, we need to know what skills the student has acquired, because that determines which new skills the student is ready to learn. Assessment of a student's learning consists of her or his performance on a test that is constructed by sampling the skills that constitute the curriculum.

In the cognitive view, the fundamental issue is how students are able to organize their information and activity in relation to general concepts. Understanding depends on general schemata that provide coherence when information is encountered and strategies that support successful problem solving. Questions about what new learning students are ready for emphasize what conceptual structures they have available so that they can comprehend new material in a coherent, sensible way. General methods of problem solving are emphasized, so that students will be able to use the procedures that they learn when they are appropriate.

We are all familiar with the dilemmas that are inherent in the values of skill and understanding. Although there is agreement that skill and understanding both have importance, people differ in their judgments of the relative importance of these aspects of students' knowledge. Part of the dilemma arises from two principles that are both valid, but they lead to a practical conflict. (1) When someone learns a skill, it is better if it makes sense, rather than being a mechanical matter of going through some arbitrary motions. (2) On the other hand, when someone learns the meanings of some concepts, it is better if the learner knows what the concepts are about. This sets up a chicken-and-egg

problem for learning mathematics. If we teach skills to students who don't yet understand the underlying concepts, we risk their being learned as arbitrary procedures. But if we teach concepts that support understanding of mathematical operations to students who don't yet know how to perform the operations, we risk the concepts being learned as meaningless abstractions.

In the past 20 years or so, considerable progress has been made in understanding how to cope with this dilemma. Part of the improved clarity has come from distinguishing between two aspects of understanding procedures: general problem-solving strategies and theoretical concepts. The theory of problem solving in cognitive science helped us to understand how activity is organized when students solve problems successfully and to appreciate the importance of having student learn general patterns of reasoning that support successful problem solving. Important general intuitions about problem-solving strategies that were discussed by Polya (e.g., 1967) and were developed as explicit theories in cognitive science (Greeno & Simon (1989) provided a review). In these theoretical analyses, strategic aspects of problem solving are represented explicitly as processes for adopting plans and setting subgoals to support organized problem-solving activity effectively. An emphasis on general problem-solving abilities was reflected in NCTM's *Agenda for Action* (1980), and the cognitive theory of problem solving has been used in development of some instructional systems that can be very helpful for students who are learning to solve standard problems in the mathematics curriculum (Anderson, Boyle, & Reiser, 1985).

Progress has also been made in understanding the psychology of theoretical concepts. A major advance in developmental psychology has provided a much stronger understanding of children's intuitive understanding of concepts in many domains, including mathematics. We now recognize that children have significant understanding of concepts of number, operations, and functions that organize their reasoning and comprehension, although much of this understanding is implicit (Greeno, 1992), and curricula that connect students' mathematical learning to the intuitions that they bring to learning are being developed apace (e.g., Curcio & Bezuk, 1994).

Although these developments provide important concepts and methods for coping with the skills-understanding dilemma, I do not believe that they provide much progress toward a resolution of the dilemma, either in theory or in educational practice. A different framing is possible, however, by shifting the focus of our theoretical and practical attention to mathematical practices, analyzed as activity systems, rather than individual cognitive and behaving agents.

Participation Viewed as Fundamental in Learning

My main proposal in this paper is to consider participation in practices as being fundamental in students' knowing, learning, and understanding mathematics. For our theoretical agenda, this would mean that we would use con-

cepts about participation as the principal basis for our explanations of the phenomena of students' mathematical activity. For our practical work, this would mean that we would design materials and activities for learning in mathematics with primary attention to the participation structures that they support.

This proposal is consistent with a recent movement in the philosophy of mathematics and science, which is shifting the focus of epistemological attention from the contents of mathematical and scientific documents to the processes of developing bodies of practice and, through those practices, developing the conceptual, material, and methodological structures that are the substance of mathematical and scientific disciplines. Philosophers who are developing this view include Kitcher (1983; 1993), Longino (1990), and Tymoczko (1985), partly in response to studies in the sociology of science (e.g., Latour & Woolgar, 1979/1986; Pickering, 1992), which have focused on the social organization of practices in scientific communities.

In this view, someone who knows mathematics is able to participate successfully in the mathematical practices that prevail in one or more of the communities where mathematical knowledge is developed, used, or simply valued. Learning mathematics by an individual is a process of becoming more effective, responsible, and authoritative in the ways that an individual participates in mathematical practices (Lave & Wenger, 1991; Stein, Silver, & Smith, in press). Learning also occurs at the level of communities, such as classrooms, that develop more effective practices that accomplish the functions that the community values and affords participation by its members in ways that are functional for the community.

Through their participation in a community over time, and through their participation in the various communities to which they belong, individuals develop identities as learners and knowers of mathematics (Wenger, in press). These identities are constructed through the interactions that individuals have with other learners and with people who represent communities of mathematicians and people who use mathematics in their work and everyday lives. An individual's identity as a mathematics learner is reflected in her or his self-concept as well as by the concepts of that person by other members of the communities that he or she participates in.

Design Principles Focused on Participation and Identity

Recent discussions and development of curricula are placing greater emphasis on aspects of students' participation in mathematical practices, focusing on their contributions to a classroom community of learners and their identities as responsible and authoritative mathematical learners and knowers. As the authors of the NCTM *Standards* put it, "First, 'knowing' mathematics is 'doing' mathematics. A person gathers, discovers, or creates knowledge in the course of some activity having a purpose" (NCTM, 1989, p. 7).

It is important to remember that participation in practices is not limited to school settings in which people are trying to involve students in their learning in new ways. Every way of organizing a classroom provides social arrangements in which for students participate. The question is what those opportunities are, and what the students learn how to do by participating in the classroom activities that are available. On the other hand, the proposal to treat participation as fundamental involves recognizing that students are always participating in an activity in some way. For example, students need not be cooperating in the activity that a teacher has set up in order to be participating. A student's participation may consist of ignoring what is going on in the classroom or causing a disruption.

Skill-Oriented Didactic Participation Structures

In a traditional didactic classroom that works well, many students learn to participate by listening to and watching the teacher demonstrate and explain mathematical procedures, by demonstrating their understanding and learning by answering the teachers' questions and working problems at the board, and by practicing (in the narrower sense) the procedures that they have been shown so they can perform successfully in tests. Knowing mathematics, in these classrooms, is mainly a matter of having acquired skills, and students participate mainly by displaying the abilities that they have acquired, by displaying their inabilities, or by displaying their indifference to the expectations that everyone should learn to perform the procedures.

A great deal is known about the variety of ways in which students learn to participate in traditional skill-oriented mathematics classes. Successful students learn to attend carefully and make enough sense of the procedures they learn to be able to remember them and use them to learn procedures that come later. This involves suspension of sense-making, to a considerable extent. Like the audience in a theater, students need to accept that the activities of mathematics generally do not relate to their experiences in the nonschool world very directly. Students who insist on being told why they need to learn mathematics, other than for the sake of succeeding in this school activity, are not the most serious participants in the activity, and there is a good chance that they will come to understand mathematics as a collection of arbitrary rules, rather than as an intellectually meaningful activity.

At the same time, students who insist on making sense of mathematics in its own terms can have a challenging and rewarding intellectual experience, even in skill-oriented classrooms. Making sense of mathematics is not trivial, especially when it is taught almost entirely as a structure of procedural skills. The occasional student who persists in understanding how mathematical procedures work is a very active learner, who builds an understanding that contains a great deal of conceptual understanding, even though that understanding may be largely implicit.

A major feature of the traditional didactic participation structure is that it distinguishes successful from unsuccessful students clearly and one-dimensionally. Students learn where they are on this one-dimensional scale, and only a few of them are understood, by themselves and the rest of the community, as being successful. Much of the evaluation depends on displaying one's ability, so students who are disinclined to display their knowledge are not likely to become recognized as mathematically successful. The affordances for failure in these classrooms are quite strong. Because so much of the meaning of mathematics learning is to identify which students are more successful than the others, it is crucially important that some students are identified as being the ones that are unsuccessful. That Americans define the window of success narrowly, so that most students are identified as being unsuccessful, is an important additional feature.

The patterns of participation that are available in these classrooms are likely to result in quite different identities. A few students come to understand themselves, and be understood by others, as mathematically gifted—they have both motivation and talent in the subject. Other students develop identities of not being interested in mathematics, sometimes because of its lack of connection with their nonschool experience. And many students develop identities of being mathematically untalented.

Understanding-Oriented Didactic Participation Structures

Although an emphasis on learning correct procedures has been common in much mathematics teaching, there has always been a concern that it would be better if students could understand what they are taught to do. This concern is the basis of a considerable amount of curriculum design and teaching that is focused on the meanings of mathematical concepts and principles. A common form of this pedagogy takes the form of explaining the mathematical ideas to students or directing them through activities that exemplify the mathematical principles, helping them to arrive at understanding of the mathematics with the activities as examples. Although there are important differences between these methods and skill-oriented didactic teaching, they involve similar participation structures. The teacher directs the students in well-ordered activity sequences that students follow, and much of the discussion is focused on whether the activities are carried out correctly.

An important difference between methods that focus on students' understanding and methods that focus primarily on correct procedures is that the content of the activity is focused on mathematical meanings and the goal of the activity is students' understanding. This can make a very large difference in what students learn about the nature of mathematics. Mathematics is presented as an intellectual discipline, in which the material is supposed to make sense. Part of the teachers' job is to convey mathematical concepts and principles to

the students, and part of the students' job is to grasp these concepts and principles. On the other hand, a shift in content does not imply a fundamental shift in the participation structures of the classroom. Students' participation can still be mainly receptive, in the sense that their activities are mainly following directions and displaying their success by answering questions correctly.

The affordances of understanding-oriented didactic instruction for students' development of identities are quite similar to those of skill-oriented didactic instruction, although the stakes are higher. The alternatives of an identity of being mathematically motivated and talented, or uninterested, or weak, are available in the practice, as they are with more procedurally-oriented classrooms. The difference is that with more emphasis on concepts and principles, the "motivated and talented" students are identified as being good at mathematical understanding, the "uninterested" students lack interest in mathematical ideas (perhaps because of their abstractness), and the "untalented" students aren't very good at understanding mathematics.

Participation Structures of Collaborative Learning

The reforms that have become the agenda of mathematics education include fundamental changes in the participation structures that are recommended for mathematics classrooms. Efforts to adopt these changes are increasingly widespread, and many members of PME are playing central roles in these efforts. Well-known examples of these projects include Lampert's (1988) and Ball's (1993) teaching, the Algebra Project (Moses et al., 1989), the QUASAR Project (Silver & Stein, 1996), Cobb et al.'s (1991) problem-centered project, the Cognition and Technology Group at Vanderbilt (1994), and there are many others. One focus of all these efforts is to establish classroom practices in which students participate actively in their mathematics learning in ways that are not afforded by traditional didactic instruction. There are important implications for assessment, some of which have been discussed by Greeno, Pearson, and Schoenfeld (1997).

The participation structures in these programs provide quite different affordances for the development of students' identities as mathematical learners and knowers from those of didactic learning environments. Activities are designed to enable students to contribute ideas and questions in discussions, and to contribute to the class's judgments of the validity of their own and other students' questions, proposed answers, arguments, and explanations. In some cases, the main topic is mathematical meaning, and discussions are organized so that students' intuitive understandings of number and quantity support their abilities to contribute. In other cases, the topic is not primarily mathematical, but is designed so that mathematical concepts and methods can be used advantageously for understanding and reasoning about the topic, and students' understandings of the primary topic provide them with further support for making

meaningful contributions. These activities are more open to multiple kinds of contributions, and students can contribute to the success of their classroom community's activities in different ways. When these various kinds of contributions are acknowledged and appreciated, students are supported in developing identities as effective contributors to their community.

As an example, I will briefly describe the version that we have developed at IRL and Stanford in the Middle-school Mathematics through Applications Project (MMAP) (Goldman, Moschkovich, & Knudsen, 1995; Greeno & MMAP, in press). The project is a collaboration of curriculum developers, teachers, and researchers, which is one of several curriculum projects that the National Science Foundation funded several years ago in response to the NCTM *Standards*. We have developed software and print curriculum materials that present mathematics mainly as a resource for a variety of design activities, which are supported by computer programs that function as computer-aided design systems that are appropriate for middle-school students. We have developed four computer-based learning environments that support students' work on projects in architectural design, scientific modeling of populations, construction and analysis of lexicographic codes, and cartographic planning. Each of these computer systems is used in several curricula that can be used in part or all of the middle-school mathematics program. For example, a curriculum for the architecture environment has students design living and working space for a research team that will spend two years in Antarctica. Their designs need to include spaces that meet the needs of the research team and to comply with constraints of available space and costs for constructing and heating the building. A curriculum for the biology modeling environment has students develop hypothetical policy recommendations for the State of Alaska about controlling populations of wolves that live on public lands, considering that they are predators for caribou. Their policy recommendations are to be supported by mathematical models.

The materials and curricula that we have developed are designed to support active participation by students in formulating and evaluating mathematical problems and questions, as well as solutions, answers, conjectures, conclusions, examples, explanations, and arguments. Students work, usually in groups of from three to five members, on a project such as designing a building or constructing a model of changes in the sizes of populations. While the students are designing a building or constructing a model, mathematical problems emerge. The questions that motivate mathematical reasoning are grounded in the students' project activity. Mathematical concepts and methods are functional in their activity, providing resources for their understanding and reasoning about questions that they are engaged in.

We do not consider the solution of problems embedded in the students' projects as sufficient in their mathematical learning. In addition to those activi-

ties, the curriculum materials include support for explicit teaching of mathematical concepts, principles, and methods. These include relatively brief teaching units, called mathematical activities, that take up a few lessons, and longer units, called mathematical extensions or investigations, that can take several lessons. These units of explicit mathematical activity are related to the students' project activities, which motivate and illustrate the explicit generalizations of ideas and methods that the students more implicitly in their projects.

Theoretically, the mathematics learning that occurs with these materials and curricula fit well with activity structures that were recommended by Dewey and, more recently, by Lave and Wenger. Dewey (1910/1978) characterized thinking as cognitive activity that is prompted by an incoherence of understanding or an impediment in the flow of activity. He emphasized the importance of reflection on the meanings of concepts in relation to the activities that can be understood in terms of the concepts (Dewey, 1938). The intention of our curriculum is to create occasions in which the main activities of design projects give rise to questions and problems that can be resolved using mathematical concepts and methods, and the mathematical activities, extensions, and investigations provide reflection on the meanings of concepts that are involved in those solutions. Lave (1988) also emphasized that mathematical problems emerge as snags in ongoing practical activity, and argued for activities in school that involve mathematical reasoning more meaningfully (Lave, Smith, & Butler, 1988). Lave and Wenger (1991) discussed trajectories of participation in communities that correspond to learning in which individuals become more effective in contributing to the functions of communities, and Wenger (in press) has discussed ways in which trajectories of participation across time and in different communities correspond to the development of individuals' identities. Our goals for this functional approach include greater agency in the students' participation, and through this participation, the development of their identities as successful learners, knowers, and users of mathematics.

We have a significant commitment to conducting research about the processes of learning and teaching that occur with the materials and teaching practices that are developed in MMAP. I will briefly mention three analyses that have been conducted to illustrate our research approach.

Cole (1995) reported case studies in which she analyzed participation structures in students' design group activities and presentations that they made to their class using the Antarctica curriculum. The group that she analyzed made notable progress in two ways. First, the meaning of their project changed, from being focused mainly on constructing correct answers for tasks that the teacher assigned to more integrated concerns with the features of their building design, including a greater sense of authority as they formulated issues and argued for features of the building. The members of the group adopted different functional participatory roles as their work progressed, such as taking responsibility for

evaluating the appropriateness of room sizes and for constructing unique features of their design.

Bushéy (1997) studied discourse in students' problem solving, focusing on ways that problems with mathematical content emerged in their activity. For example, in designing a building, students encountered problems involving the sizes of rooms represented by sizes of spaces in their representations, which required significant proportional reasoning to resolve. This mathematical reasoning played an integral role in their understanding of representations that they constructed in their design projects, and thereby had significant functional meaning in their activity. This contrasted with the same students' work on standard problem sets involving conversion of units of length and area, in which they carried out computations that lacked referential meaning.

Knudsen (1994) analyzed participation structures focusing on the contributions of teachers in organizing and leading the classes' learning activities. She has developed a framework that distinguishes three dimensions of teachers' activity that she calls *facets of teaching, planning and improvising,* and *concerns.* Facets of teaching include identifiable kinds of functional classroom activity such as guiding a group of students' design process, discussing relations between classroom activities and students' experiences out of school, facilitating the interactive work of student groups, and assessing and adjusting materials and methods to facilitate the progress of different groups of students. Planning and improvising are modes of teachers' work that reflect both the need for and the limitations of preparation for teaching when the activities of learning are distributed and open-ended. Concerns are categories of awareness that function in teachers' work as broad organizing principles. These include fostering students' mathematical understanding, equity of students' access to and participation in mathematical activities, maintaining a balance between progress on general educational goals and progress in the detailed requirements of classroom activities, and reflection on and revision of their own teaching.

Conclusion: Participation as an Organizing Aim in Education

I have proposed *participation* as a unifying concept that reflects several aspects of the goals of mathematics education reform. This proposal can be considered in two ways, and I will close by mentioning the distinction between them.

One way to consider this proposal can be called *participatory instrumentalism.* This is a hypothesis that participation is an important condition for acquiring mathematical knowledge. There is much evidence that this is the case, and it is a hypothesis that fits well with constructivist views. After all, if learning is a process of constructing mathematical knowledge, students' behavior is the crucial factor in whether they will be successful in this constructive work.

The other view, which is more radical, can be called *participatory fundamentalism*. According to this view, mathematical knowledge *is* participation. That is, what we mean by knowing mathematics is sustained participation in mathematical practices, contributing to the functioning of groups and developing an identity as an engaged and competent learner, knower, and user of mathematics. In this view, rather than thinking of participation as an important condition for learning mathematics, we can understand learning of mathematics as learning to participate effectively in mathematical practices.

This fundamentalist view of participation does not exclude skills and conceptual understanding as important aspects of mathematical knowing. This view does, however, reframe the roles of skill and understanding, treating them as aspects of participation. Individuals can participate more effectively in mathematical practices if they are fluent with computational procedures and can contribute successfully to mathematical understanding in the communities in which a person is engaged. In this view, students' mathematical skill and understanding are instrumental to their participation, which contrasts with the cognitive and behaviorist views, in which participation is instrumental to students' acquiring mathematical understanding or skill.

In this view, the classic debate between skill and understanding does not disappear, but it is framed differently. Instead of arguing whether skill or understanding is more important (unqualified), we need to specify the kinds of mathematical practices that we consider important, and carry out a serious analysis of the kinds of skill and understanding that will support effective participation in those practices.

If this view of participatory fundamentalism is adopted, the most important questions are (a) what are the mathematical practices that are most important for students to learn to participate in, and (b) what are the learning activities in which students can best become able to participate effectively in those practices. The mathematical practice that dominates much of current mathematics education is test taking, and it is important to ask whether we consider that particular mathematical practice as valuable as other kinds of practice that could be emphasized more. We can take the NCTM *Standards* and other related documents as assertions that practices other than test taking should be given more emphasis.

We need to recognize that the practices of mathematical reasoning and communication that are emphasized in mathematical reforms have a much weaker basis in research and theory than the practices of taking tests. This motivates an important research agenda for the field. The discussion of relative importance of different kinds of practices as educational aims will be more valid if we can provide more specific and coherent descriptions and explanations of the systems and processes of learning that are focused at all of the levels of analysis

that are involved in the several alternatives. At this point, we have analyses of individual behavior and cognition that are developed significantly. Although there is much still to be accomplished with research on individual mathematical behavior and cognition, the analyses that we have of processes at the level of participation in mathematical practices are much less well developed. Research that develops analyses of participation in mathematical practices seems, then, to warrant a high priority for us.

Acknowledgment

This research is supported by grants from the National Science Foundation.

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ADVANCED MATHEMATICAL THINKING

AN UNDERGRADUATE STUDENT'S UNDERSTANDING AND USE OF MATHEMATICAL DEFINITIONS IN REAL ANALYSIS

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The purpose of this paper is to focus on one undergraduate mathematics major's understanding and use of the formal mathematical definitions of real analysis. The researcher's analysis of four task-based interviews with Stephanie, a junior mathematics major, focused on Stephanie's understanding of the role of mathematical definitions and the logical and conceptual complexities of certain definitions. Many of Stephanie's understandings and beliefs conflicted with those accepted by the mathematics community which resulted in difficulties for her when she attempted to understand formal mathematical definitions. An understanding of Stephanie's understanding can contribute to the improvement of the undergraduate learning experience in mathematics.

Background

The purpose of this paper is to present some of the results of a study which characterized the way undergraduate mathematics majors understood, or attempted to understand and use, mathematical definitions with which they had varying levels of familiarity. This paper, which is part of a larger study, addresses one student's understanding and work with definitions in light of such factors as the student's understanding of the role of definition in mathematics, and her understanding of certain concepts from real analysis which underlie the formal mathematical definitions.

Over the past few years there has been an increased interest in studying advanced mathematical thinking (Tall, 1992). Although no one claims to be able to specify indisputably the characteristics which uniquely describe advanced mathematics, Tall has written, "Advanced mathematical thinking . . . is characterized by two important components: precise mathematical definitions (including the statement of axioms in axiomatic theories) and logical deductions of theorems based upon them" (p. 495). For many students the increased attention to definitions, theorems and proof occurs in their post-calculus courses, and the transition from calculus to these courses can be difficult.

There is an abundance of anecdotal evidence and some research which indicates that students struggle with proof-writing in their first courses in such areas as real analysis and abstract algebra. Hart (1986) studied the connection between students' proof-writing abilities in abstract algebra and the depth of the students' conceptual understanding, and recommended that the emphasis in advanced mathematics should not be on *how* to do proof, but should be on "stabilizing unstable concept systems" (p. 137). Moore (1994) found that the

proof-writing difficulties for students in introductory real analysis were closely related to their understanding of the concepts involved as well as their ability to state and use the formal mathematical definitions (p. 252).

In the literature definitions have been described as being either "logical[1]" or lexical (Landau, 1989). A "logical" definition "attempts to analyze things in the real world" (p. 120) such as a definition for the word "love" might try to describe love in a way that would be generally acceptable to most people. A lexical definition, on the other hand, delineates the characteristics or features of a given concept in such a way that the concept is completely determined by that definition. In a sense, the concept is created by the definition. Mathematical definitions are lexical, and therefore understood by the mathematics community to be precise and unambiguous. At the same time there is a flexibility available with (lexical) mathematical definitions in that one may freely create a new definition which will fit one's needs if there is no existing definition that will do so.

Although mathematicians assume definitions have basic importance in mathematics, very little research has been done which investigates how students deal with them. The notion of *concept image-concept definition* (Vinner, 1991), emphasizes not the student's struggles with the formal mathematical definition, but the learner's concept image in which the definition may or may not play a part. Issues such as whether one has flexibility in creating, interpreting or using definitions appear in the literature (Landau, 1989; Tall, 1994; Wilson, 1990) but research on post-calculus undergraduate student's use of definitions is rare.

Theoretical Framework

This study is grounded in certain epistemological and pedagogical beliefs of the researcher. The mathematical understanding that a student creates is dependent upon two things - the individual's viewpoint and his or her previous knowledge. Since it is "human nature" for us to avoid states of confusion and conflict, if a new experience causes cognitive conflict with our current understanding our goal is always to return to a state of equilibrium. Through further actions and reflective abstraction we deal with cognitive conflicts in a way that either leads to new or modified understandings or rejects the conflicting experience. In this way "learning" occurs. However, it is precisely one's view of things that determines if a new experience is in conflict with previous knowl-

[1] The word "logical" in "logical" definition is not consistent with the use of the word logical in mathematics. For this reason in this paper, whenever "logical" definitions are mentioned, the word "logical" will be in quotes.

edge. This is individual and thus it is quite possible that a cognitive conflict for one person will not be viewed as a conflict by another.

One's accumulated mathematical knowledge can be described in terms of a *concept image - concept definition* framework (Tall, 1992; Vinner, 1991). Using the concept of limit as an example, the *concept image* can be described as the collection of all understandings about a certain idea, in this case the limit, held by an individual. These may include primitive notions of limit as well as the understandings that have developed formally in mathematics courses. The *concept definition* is the body of words used to designate that concept. It may not be known to the learner, may be "separate" from the concept image or may be included, possibly incorrectly or incompletely in the learner's concept image. The *evoked concept image* is that part of an individual's concept image which is brought forth in any given situation where the individual finds the need to use a given concept. It is the researcher's belief that only an individual's *evoked* concept image is "knowable" to the researcher, and, more importantly, that it may vary from situation to situation and over time.

Part of the enculturation of college mathematics students into the field of mathematics involves their acceptance and understanding of the role of mathematical definitions - that the words of the formal definition embody the entire meaning of the concept or entity being defined. If students do not understand the role of the definition in this way they may allow their previous and emerging concept images to dictate the meaning of a definition rather than the words of that definition. Thus, a student's understanding of the role of mathematical definitions is itself part of his or her concept image.

Method

Stephanie was one of eight undergraduate students, junior- and senior-level mathematics majors, who volunteered to participate in the study. The students were members of an introductory real analysis course. The researcher interviewed each of the students four times and audio-taped and video-taped and verbatim transcripts were produced which included a record of non-verbal actions as well. During the interviews the researcher presented to the students formal definitions for such notions as an infinite decimal, pointwise continuity, connected set and an absolutely continuous function. The researcher asked the students to read the definition and then questioned the students as they reasoned about the meaning of the definition. Students were asked to explain their understanding of any mathematical terminology they used as well as to explain any conjectures they made.

Data analysis involved several careful readings of Stephanie's interview transcripts. The researcher began writing the "story" for Stephanie early in the analysis process, revising daily as the emerging themes from the transcripts became refined and clarified.

Results

Stephanie's understanding of the role of mathematical definitions seemed to be primarily a "logical" one. She talked about mathematical definitions as being discovered rather than created and as being open to the interpretation of others to establish their authenticity. During the first interview she spoke of addition as a "concept that existed, but we defined it." In the fourth interview Stephanie said that when one presents a new definition "you have to like have your colleagues review it, make sure it's legitimate and there's no errors." Since a lexical definition creates the concept it defines it is inconsistent to question its legitimacy. Further, a student with a "logical" understanding of mathematical definitions who sees a conflict between her understanding of a definition and her understanding of the concept defined by that definition is likely to allow the conceptual understanding to dominate. Stephanie seemed to have a procedural understanding of many of the concepts discussed in the interviews. In the first interview she said she could "find" a derivative but when asked what it meant she said, "I have no idea." In the interviews when Stephanie talked about the limit she often described it as a process of approaching some number although she also said the limit was a number. Stephanie reason procedurally when she worked on the definitions for an infinite decimal and the continuity definitions. In the interview on the absolutely continuous function definition, Stephanie chose $f(x)=x^2$ as an example and carried out a procedure involving the calculation of the lengths of intervals and the summing of function value differences to determine if the function was absolutely continuous.

In the first interview Stephanie was asked to discuss the sequence S_n={.9,.99,.999,..._n is a natural number}, its related infinite decimal, .999..., and a definition for an infinite decimal that the professor in the Concepts in Real Analysis course had presented in class. The definition was given as follows.

> **Definition.** Let $c_1, c_2, ..., c_n, ...$ be an infinite sequence of integers with 0_c_i_9. The number sup {$.c_1c_2...c_n$_n=1,2,3,...} is denoted by $.c_1c_2...c_n...$ and is called an infinite decimal.

Stephanie was asked to discuss the potential equivalence of .999... and 1 and then was shown the definition for an infinite decimal which had been presented in class. Essentially the definition stated that every infinite decimal has two representations which, in the case for all c_i equal to nine, were .999... and 1. Before seeing the definition, Stephanie was adamant that .999... did not equal 1 even though the researcher tried to convince her otherwise. She said that .333... was equal to 1/3, but that was because one could divide 3 into 1 and get .333... According to Stephanie, "If you divide 1 into 1 you don't get .999...!" There was a procedure for .333... but there was not a procedure for .999.

If Stephanie had interpreted the formal definition as intended it most probably would have created cognitive conflict for her, but she did not. As she discussed it, Stephanie seemed to have an adequate understanding of the term "supremum," since she explained that the supremum of a set was its least upper bound and that there could be no member of the set between it and any other upper bound of the set. Further, she acknowledged that 1 was an upper bound. However, for Stephanie .999... was the least upper bound even though one could always make a larger number by "adding" another nine to the end of the string of nines in the decimal. She said, "That's what the repeating is, it just goes on forever and ever." Stephanie, perhaps unconsciously, interpreted the definition to fit her prior conceptual understanding.

It is important to emphasize that Stephanie seemed to have the conceptual understanding necessary to interpret the words in the definition for infinite decimal. Her rejection of that definition did not seem to be on the basis of an inability to reason about its written meaning. It seemed that Stephanie had found a meaning for infinite decimal that made sense to her, based upon her understanding of infinite processes perhaps, and her "logical" interpretation of mathematical definitions made it reasonable for her to interpret the meaning of the formal definition based upon her previous conceptual understandings.

Discussion

As mathematics students like Stephanie move from courses that are more procedural into theoretical, proof-intensive courses it becomes more important that they know how to use and understand mathematical definitions. Stephanie's previous mathematical experience may have de-emphasized familiarization with formal definitions and emphasized an establishment of more robust understandings of the concepts themselves. Often definitions are memorized and then the formal words are shoved aside in favor of knowing, as one participant said, "what it really means." However, a robust understanding of the role of mathematical definitions is needed too if students are to be able to use them effectively in more theoretical settings.

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MIS-GENERALIZATION IN CALCULUS: SEARCHING FOR THE ORIGINS

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This paper discusses how a mis-generalization leads to an erroneous belief held by some calculus students studying from traditional texts. Mis-generalization can be the result of a variety of conditions and this paper will characterize them. The focus of this paper will be on a previously unidentified mis-generalization, held by some students, that a corollary to Rolle's Theorem is applicable across the general class of functions. In addition, an examination of student reliance on polynomial-based functions when providing calculus examples will be reported.

Introduction and Theoretical Framework

Much of the previous research on student mis-generalization of mathematical concepts or procedures has focused on whole numbers, fractions, decimals and algebra (Brown & VanLehn, 1982; Hiebert & Wearne, 1986; LeFevre, 1984; Mack, 1990; Matz, 1980, 1982; Resnick & Omanson, 1987; Resnick et al., 1989; Sleeman, 1981, 1984). The studies concluded that students have exceptional capabilities for extrapolating salient properties from sets of examples; although mis-generalizations have been attributed to problems in identifying necessary and sufficient characteristics for distinguishing examples and nonexamples of concepts. In particular, overgeneralization, the classification of a nonexample as an example of a concept, results from not distinguishing the concept's fundamental attributes necessary for making discriminations; whereas undergeneralization, the classification of an example of a concept as a nonexample, ensues from the identification of attributes inconsistent with the concept as a whole (Dempsey, 1990).

Hiebert and Carpenter (1992) identified an origin of mis-generalization when they noted that "many errors in mathematics may result from students' attempts to build connections within overly restricted domains" (p. 89). The restricted domains, holding an insufficient cache of examples and nonexamples, obstruct proper generalization of a concept definition. In particular, perceived characteristics may not hold universally thereby becoming obstacles to understanding. The restricted domain can impact how students define a concept and decide whether a given mathematical object is an example or nonexample of a concept. According to Vinner and Dreyfus (1989), a student's image of a concept is resultant from experience with the provided examples and nonexamples of a concept. In particular, all the various images descriptive of a concept presented to a student, whether they be examples or nonexamples, have the potential to be used in the student's formulation of a personal concept defini-

tion. However, a student's personal concept definition may not correlate well with the concept's mathematical definition (Vinner & Dreyfus, 1989). The reason for this is that students have been found to formulate their personal definitions and decisions based on a *concept image*, that is, the set of all mental pictures associated in the student's mind with the concept's name, together with all the properties characterizing them. This creates the possibility of a situation where "the set of mathematical objects considered by the student to be examples of the concept is not necessarily the same as the set of mathematical objects determined by the definition" (Vinner & Dreyfus, 1989, p. 356). Thus, the set of examples and nonexamples presented to a student can impact both the accuracy and development of a personal concept definition as well as influence decision making based on the concept definition.

At first, this might seem to infer that if a large set of examples and nonexamples are presented to students, then students would recognize the important features and develop a personal concept definition consistent with a concept's mathematical definition. However, the use of a large, overly restricted set of examples and nonexamples to characterize a concept's salient qualities does not always achieve the desired goal. A robust set of examples and nonexamples, in terms of scope, quality, and quantity, must be presented. The diversity of the set should aid students in determining what further conditions should be placed on their student-developed definitions and result in definitions more consistent with a concept's mathematically correct definition.

According to Tall (1989), the use of simplified examples as the motivating basis for a concept has some problematic aspects. Polynomials form the primary example space for most treatments of calculus concepts in standard calculus texts even though they have attributes which do not hold for the general class of functions. As a result, the use of polynomial functions as the central example space potentially can lead to a variety of incorrect and inadequate beliefs which can form the basis for obstacles to the development of understanding calculus and analysis concepts. In particular, Tall (1990) found beliefs that "a function must be given by a formula (and only one formula is allowed); that every function is differentiable, except possibly at a few isolated points; that the graph of a function looks fairly smooth with reasonably shaped maxima and minima; that graphs always have tangents; that a tangent touches the curve at one point only and does not cross the graph, etc." (p. 56). This paper extends Tall's listing by identifying how instruction using polynomial-based functions as the primary example space leads to an additional mis-generalization.

Methodology of this Study

During the Fall of 1995, students in a large calculus class ($n = 85$) at a level 1, private research university participated in a series of examinations focused

on student reliance on polynomial-based functions and the impact of such reliance. The study triangulated on students' dominant conceptualization of exemplar functions for calculus through three instruments. The first instrument focused student responses to the following statement:

> True or False. If $f(x)$ has n real roots, i.e., for $c_1, c_2, c_3, \ldots, c_n$, $f(c_i) = 0$ and $c_i \in \Re$, then $f'(x)$ has at most n real roots. Explain why or why not and provide an example.

The second and third instruments requested the provision of examples of functions with certain attributes and the identification of the type of function generally chosen by the student to ascertain if a mathematical statement about a calculus concept is valid or not.

Of the 85 students, 45 responded to all three instruments. As a result, the sample was restricted to these 45 students and their responses to the assessment items form the basis of this study's conclusions. Students' responses to the instruments were categorized and linked to distinguish a variety of typical responses. The analysis focused on qualitative differences between responses and students' inclination toward polynomial-based functions and if this predilection held under different conditions. Such information provided a indicator of the strength of reliance on polynomial-based functions and their use in examining calculus concepts.

Results

The study found that the presentation of calculus concepts, using a constricted example space composed of predominantly polynomial-based functions, harbored the possibility of students mis-generalizing characteristics associated with polynomials to the general class of functions. In particular, 37 of the 45 students were found to incorrectly associate the derivative of a function that had n real roots to have (a) less than n roots ($\hat{n}_1 = 7$), (b) at most n roots ($\hat{n}_2 = 12$), or (c) at most $n-1$ roots ($\hat{n}_3 = 18$). Students' typical explanations revealed that the conclusions were based upon polynomial-based examples. For instance, students made comments typical to these:

1. "False, the derivative of a function will always have *fewer real roots than the original function*. For example, 3rd degree polynomial functions have at most 3 real roots, while 2nd degree polynomial functions have 2 real roots";
2. "True – If the function has n real roots then the derivative of the function *cannot have more than n roots* because when you find the derivative of a function, the degree of the function is lowered and thus the number of real roots is going to be less than or equal to n. An example is the function $f(x) = x^3 + 2x^2 + 3x - 6$ has 3 possible roots and the

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derivative of the function is $f'(x) = 3x^2 + 4x + 3$ which has the possibility of at least 2 roots"; and

3. "False – $f'(x)$ is the derivative of f and the derivative's power is (functions power -1). The number of roots of a function is equal to the exponent of the function. $\therefore$ f has n roots. $\therefore$ f' has $n-1$ roots. Which means $f'(x)$ can have *at most* $n-1$ *real roots* and not n real roots".

In contrast, eight of the 45 students concluded that it was impossible to determine the number of real roots by indicating that it was possible to obtain more roots from the derivative than were found with the original function. Generally, these conclusions were based upon either generic examples (see figure 1), i.e., those not tied explicitly to a particular functional statement, or to specific examples which identified original functions with no roots and then showed their derivatives to have at least one root, i.e., $f(x) = 5$ and $f'(x) = 0$.

These results point to the conclusion that in deciding the veracity of the provided statement, students generally appealed to polynomial-based functions as a means of reconciling the statement and their statements echoed the corollary to Rolle's Theorem that "a polynomial of degree n has at most n real roots". The students who established the first item's falsehood and provided a correct and reasonable explanation reached beyond polynomial-based examples to consider a more global sense of functions and functional representations.

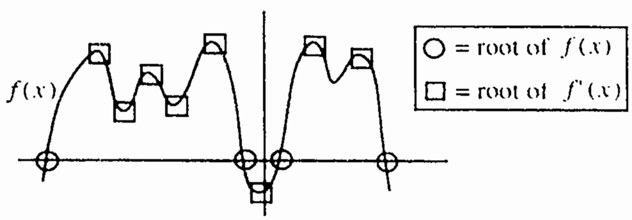

Figure 1. Generic representation of a function with more roots for $f'(x)$ than $f(x)$.

This ability of the students who obtained a correct and reasonable answer to the first instrument's item to reach beyond polynomial-based functions was again evidenced in the analysis of the students' responses to the second and third instrument's items asking students to provide an example of a function which: (a) is increasing over its domain, (b) is both increasing over its domain and concave up over its domain, (c) has at least 5 roots (i.e., 5 or more) over its domain, (d) is always increasing over its domain but changes concavity at some x-value in its domain, (e) is integrable over its domain but not differentiable over its domain, (f) $\int_{-a}^{a} f(x)dx = 0$, (g) is decreasing over its entire domain, (h) is decreasing and concave down over its entire domain, and (i) is always

positive, always increasing, and changes concavity at only one point over its entire domain. A two-group Wilcoxon test, examining the percentage of polynomial-based responses to the above questions with respect to the first instrument's responses, found a significant difference ($T_w(8,37) = 101$, $\alpha < .05$) indicating a comparably reduced appeal to polynomial-based functions by students providing a correct and reasonable answer to the first instrument.

Analysis of student responses to the question "When you are asked to provide an example in calculus, what type of function (would you) think of initially? why?" revealed that 32 of the 45 students reported that polynomial-based functions would be the type of function used to develop examples in calculus. Four additional students identified a blend of polynomial and transcendental functions as aiding their decisions. Only nine students indicated an initial example function coming from a domain not necessarily having a polynomial-based component. It was of interest to note that of the eight students correctly establishing the falsehood of the statement of item in the first instrument, six of them stated that they would initially investigate either a polynomial-based function or a function with a polynomial component. The general reasons cited for the initial use of polynomial-based functions included:

1. "[I] used them the most so much over the last 4 - 5 years."
2. "[A] student is usually asked to take the derivative of a polynomial function on exams."
3. "[I]t's what is easiest to work with and what I am most comfortable with."
4. "It's the most fundamental function in mathematics."

Obviously, students recognize what has been modeled by instructors, texts, and examinations.

Conclusions and Implications

This paper has identified an additional mis-generalization evidenced by multiple students studying calculus from a standardized calculus text. In particular, it was found that students were consistently presented with a restricted domain of examples and non-examples derived from polynomial-based functions. This consistent presentation resulted in many students improperly extrapolating a corollary to Rolle's Theorem, i.e., "a polynomial of degree n has at most n real roots" to be applicable to the entire class of functions. The student responses revealed that their conclusions were predominantly drawn from the mere examination of polynomial-based example functions and the reasons provided for using such functions for decision making in calculus focused on the preponderance of such functions in class examples, texts, and assessments. Those students who were able to avoid making such a mis-gener-

alization revealed that they were able to appeal to either generic examples of functions or non-polynomial-based functional representations.

This study has implications for student learning and pedagogy. The recognition of traits from examples and nonexamples is an integral aspect of developing understanding of a concept. However, conceptual errors are almost inevitable as students develop their own understandings of mathematical concepts and classroom instruction cannot provide students with a sufficiently comprehensive experience to eliminate this possibility (Resnick et al., 1989; Davis & Vinner, 1986). Instruction needs to use diverse representations, robust with quality examples and nonexamples, to minimize the occurrence and virility of these conceptual errors. This should improve the presentation of mathematical concepts and help students avoid some of the obstacles associated with them.

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STUDENTS' COGNITIVE APPROACHES TO THE CONCEPT OF RATE

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What type of learning activities help students develop mathematical resources and strategies that can be applied successfully in their study of mathematics. This is an important question that involves the design of instructional problems in which students have the opportunity to use their mathematical resources and to apply several strategies during their interaction with the problems. That is, problems are used as a vehicle to problematize the content and they are taken as a platforms to reflect on other related mathematical ideas. This study documents the work shown by high school students during the study of the concept of rate. It focuses on the analysis of the type of question that students discuss as a means to understand this concept.

Introduction

Problem solving has become an important instructional component in the learning of mathematics. Research studies have provided useful information about the main ingredients that influence the way students solve problems; however, there is little research on the impact that problem solving instruction actually produces in regular mathematical classroom. A problem solving approach may include instructional activities in which students are encouraged to participate actively in discussions that help them solve diverse problems. Thus, problems are identified as the main vehicle to understand the mathematical content, to propose conjectures and analyze the information given in the problem, and to discuss and pursue diverse methods of solution. Therefore, it is important to document what students do while the instructional activities implemented in the classroom ask students to problematize the subject. Specifically, this study documents what students show when they study the concept of rate under the perspective of a mathematical problem solving approach.

Background to the Study

The study took place in a first calculus class in grade 12. During the development of the course, the students were encouraged to work on a set of problems. These problems were designed by the instructor and the researcher in accordance with the main topic to be studied throughout the course. The central theme that is used to analyze the students' mathematical behaviors is the concept of rate. This concept has been identified as difficult to understand by the students and its understanding plays an important role in the study of the concept of derivative (Thompson, 1995). To analyze the work shown by the students as a result of the activities implemented during the course, it was decided

to focus the analysis on two complementary activities that appeared consistently during the class development:

(a) The students' participation in the small group discussions.

(b) The students' participation during the whole class discussions.

Since there is interest to document the students' participation while working cooperatively, it becomes important to mention that the learning activities implemented in the classroom included aspects such as:

(a) The students were aware of their responsibility as member of a small community. Here they knew that it was important to express their ideas and to listen to the others. To understand one mathematical idea or problem meant that each member of the group had grasped it and explained it to the others.

(b) The students' evaluation process throughout the course took into account the group work involved.

(c) The students spent a significant amount of time working as a group in the classroom, in the computer lab, and in homework assignments.

Conceptual Framework

It is important to mention that we based the analysis of the students' work on the idea that the process of learning any mathematical concept is an ongoing process in which it is possible to identify important changes in the students' understanding of that concept. Thurston (1995) pointed out that "people have very different ways of understanding particular pieces of mathematics" (p. 30). He cites an example in which the concept of derivative could be seen from different angles:

(i) as a ratio of small change (infinitesimal), (ii) as a symbolic manipulation of sign (the derivative of x^n is nx^{n-1}, etc.), (iii) as a formal definition which involves the concept to limit, (iv) as the geometric representation (the slope of a line), (v) as the instantaneous speed of f(t), when t is time, and (vi) as an approximation in which the derivative of a function is the best linear approximation to the function.

As Thurston mentioned, the above list does not include all the possible ways of thinking about derivative; however, an explicit discussion about what each conception entails, and what connections are important among them could help students to make more robust their understanding of the concept of derivative. Therefore, the analysis of the students' work focused on to what extent the students were able to show conceptual ideas related to the concept of rate while dealing with a variety of mathematical tasks.

It is important to mention that the use of the small and whole group discussion during the session was based on the ideas suggested by Reynolds, et

al. (1995). Analysis of the following questions helped organize the work shown by the students and are used as framework to present the main results:

(1) In which type of mathematical aspects did the students exhibit certain types of difficulties in their conceptualization of the concept of rate?

(2) What type of discussion (questions, explanation, and arguments) helped students to spot their difficulties and to improve them?

(3) When did students show significant progress in their understanding of the concept of rate? What evidence showed that they were able to use and interpret data that include tables, graphs and algebraic representations, etc.?

Results

An important result that emerged from the data is that working in small groups helped students identify a variety of conceptions that they held about the term rate and how to operate with the rate concept. For example, one idea that seemed to be problematic for the students while working on the first problem was that they used the procedure to calculate the arithmetic mean to obtain the average flow. The written report handed in by one small group included the following ideas.

It was clear that the students needed to discuss and contrast differences between the meaning of rate of flow and the way of finding the average flow and the idea of arithmetic mean. Indeed, the presence of these ideas during the small group discussions was an aspect that motivated the students to consult textbooks and their notes in order to attach mathematical arguments to their work. It seems that the students were aware of the importance to present mathematical arguments to support their ideas.

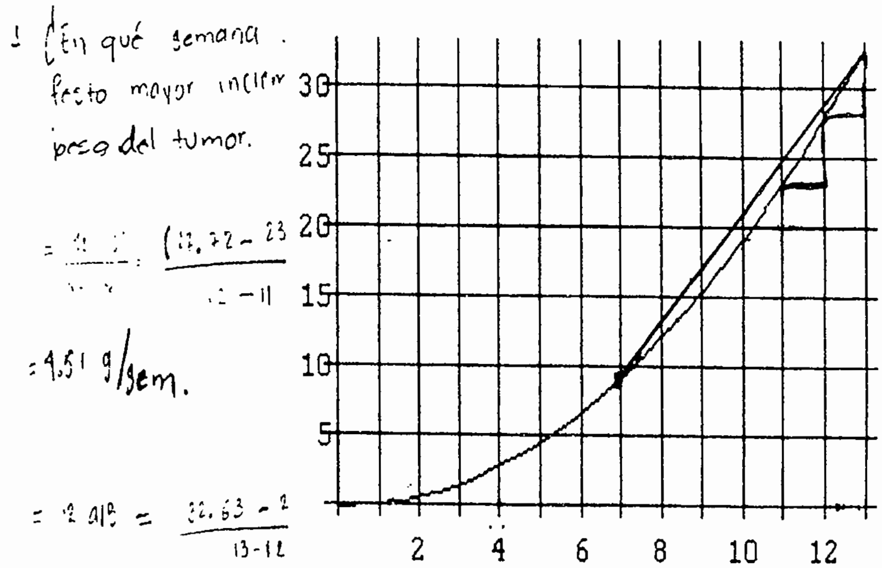
Another aspect that appeared consistently in the students' approaches was to see the instantaneous flow as direct quotient between two quantities. That is, they took the data given in the table as final records and not as a points of reference to approximate the instantaneous flow.

A problem that was used during the interview with pairs of students involved a situation which described the growth of a tumor. Data here were given in three different forms of representation: a table, a graph, and an algebraic representation. With this information, the students were asked to pose and respond to three questions. Hence, the students needed to understand clearly the information of the situation in order to deal with the data. For example, the statement of the problem was clear to indicate that the three forms of presenting the data represented the same information of the problem. However, there were some students who thought that the information came from three different situations. For example, Ann and Maria thought that they had to formulate one question for each representation of the data.

Formulating the questions resulted in being another difficulty shown by the students. It seems that getting a sense of the data is a process that involves working for a while with the richness (or lack of it) included in it before trying to put the information into a relationship. For example, some students realized that the three representations of the data offered different advantages or disadvantages to analyze the phenomenon. While the table representation shows discrete information, the graph representation shows a continuous phenomenon. The algebraic representation has the advantage of allowing the student to find more information accurately and easily than the other two representation. That is, they could find, for example, what happens to the tumor at 7.99 weeks. In fact, these differences become explicit in the students' work after they had worked on one of the first questions they had proposed.

The list of questions that students proposed included: What is the average weight of the tumor at the 8th week? How much did the tumor increase its weight between the fourth and eighth weeks? In which week did the tumor reaches its highest growth? How much did the tumor increase in the 9th week? How could you explain the behavior of the tumor in relation to the graph? What is the rate of growth of the tumor in the 8th week? Although one may think that the activity of asking the students to formulate questions gives them various options and these may or may not be related to the concept of rate, it is important to mention that the instructional problems discussed during the class and other assignments that the students worked involved directly the concept of rate. Therefore, in this task, it was important to evaluate the extent in which students were able to identify the concept of rate as means to problematize the given information.

A pair of students worked on finding the rate of growth of the tumor at the 8th week. During their initial interaction, they discussed aspects related to the differences between the terms average growth and rate of growth. In fact, some students used the term average to indicate rate of growth initially and they became aware of their differences after they got involved in some calculations. For example, Martha and Amir decided to find the average weight of the tumor but they introduce the idea of slope of a line to calculate it. It seems that these students focused their attention to the graphical representation to identify points which were close to 8. However, they realized that this representation did not give them the exact value of the weight of the tumor at for example 7.9 or 8.1 weeks. It was here when these students were aware of the potential of the algebraic representation. Thus, they found the value of $p(t)$ with $t = 7.9$ and 8.1. These students also observed that the tumor does not grow evenly and they posed the question: In which week the tumor reaches its highest growth? To answer it, they decided to check the behavior of the graph and compared the run and rise for each interval. They concluded (incorrectly) that during the 9th week the tumor reaches the highest growth. It is important to mention that they never tried to check their responses.



Conclusions

To what extent students exhibited consistency in their use of problem solving strategies to deal with situations that involved the concept of rate of change? This is a fundamental question that needs to be discussed in terms of the students' work. There is indication that students were aware of the presence of this concept in situations that included contexts such as moving objects (speed), growing of a tumor, or filling a water tank. This was an important instructional goal. It was observed that students, in general, used the method of calculating slopes of secants to get approximations to the slope of the tangent in a particular point (growing of the tumor in the 8th week, for example). However, it was not clear if the students actually thought of the idea of limit as an important concept to relate the calculations of slopes of secants to the concept of rate. That is, the students only reported the calculations of two or three slopes without mentioning the limit process. Dealing with data that included different representations helped students discuss the meaning of instantaneous rate within the context of the situation. Finally, it is important to mention that although the instructional activities encouraged the students work as a part of a community, this aspect does not appear consistently in the students' interviews. It seems that the students need more time to assimilate and exploit this type of practice.

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EFFECTS OF DIFFERENT INSTRUCTIONAL APPROACHES ON CALCULUS STUDENTS' UNDERSTANDING OF THE RELATIONSHIP BETWEEN SLOPE, RATE OF CHANGE, AND THE FIRST DERIVATIVE

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The impact of the instructional approaches used in three different differential calculus courses on students' understanding of the relationship between slope, rate of change, and the first derivative was investigated. A total of 100 students were taken from traditional calculus courses, a course that made use of graphics calculators, and the electronic course, Calculus & *Mathematica*. Results from interviews with 12 students from each course indicated a greater lack of understanding of the relationship between the first derivative of a function at a point and its slope or rate of change at that point amongst the traditional and graphics calculator students than amongst the Calculus & *Mathematica* students. When analyzed using a theoretical framework derived from the work of Hiebert and Carpenter (1992) and Dubinsky (1991), the results suggest differences between the students from the different courses may be related to the amount of time students spent working on problems designed to help them make connections between different representations of the concept of slope.

Over the last decade, various studies have provided evidence that conceptual understanding can be improved by using technology to increase emphasis on concept development and decrease emphasis on computational skills (e.g., Boers & Jones, 1993; Cooley, 1996; Lauten, Graham, & Ferrini-Mundy, 1994; Park, 1993). These studies typically compare test scores of students in traditional and "technology-rich" calculus courses. These tests usually consist of procedural and conceptual problems taken from the traditional course's curriculum. Since the goals of "technology-rich" calculus courses typically differ from those of traditional courses, problems taken from a "traditional" curriculum may not provide a fair or valid comparison of students from the different calculus course. One possible way to overcome this difficulty is to look not at test scores but at how students solve problems. This was the basis for the present study which compared students from three calculus courses on their abilities to use numerical, graphical, and symbolic representations when solving calculus problems. The portion of this research addressed in this paper focuses on the impact of the instructional approaches in these courses on students' understanding of the relationship between slope, rate of change, and the first derivative.

Subjects and Environment

Participants were undergraduate students from intact classes from three calculus courses at a large midwestern university. Each course was the first in a four-quarter sequence designed for mathematics, science, and engineering students. One was a traditional differential calculus course like those typically taught at most colleges and universities. In this course, new material was presented by the instructor during lecture and problems from homework assignments and examinations were discussed by a teaching assistant during recitation. Students in this course were allowed to use graphics calculators in class and on homework assignments, but could not use them on quizzes or examinations. Graphics calculators were never used as part of course instruction.

The second course was similar in content to the first course but was designed so instruction and assignments stressed use of symbolic representations and graphical representations generated via graphics calculators. Symbolic and graphical representations were used on a regular basis when new material was presented by the instructor during lectures and when problems were solved by the teaching assistant during recitation. Students were required to have a graphics calculator and use it during class, on homework assignments, and during portions of their examinations. It should be noted that the university required 75% of the problems on each examination in the course to be equivalent to those used in the "traditional" course. This meant that the problems were to be solved using *only* symbolic methods (F. Demana, personal communication, April 18, 1994).

The third course was Calculus & *Mathematica* (Davis, Porta, & Uhl, 1994), an electronic calculus course designed around the computer algebra system *Mathematica*. During class, students worked individually or in assigned cooperative groups on the different *Mathematica* notebooks that comprised the course. These notebooks were live electronic documents consisting of a mixture of static text and active programs that could be activated by students to view examples or alter by students to create their own solutions to problems. Course assignments emphasized the use of symbolic, numerical, and graphical representations and contained numerous problems designed to help students make connections between different representations of concepts. Except for an occasional instructor-led class discussions on problems that were causing significant difficulties for the students, no lecturing was done in this course.

Theoretical Framework

A framework derived from the theories of Hiebert and Carpenter (1992) and Dubinsky (1991) was developed to help analyze differences in students' understanding of different concepts. Hiebert and Carpenter propose a theoretical framework for defining understanding. The premises behind this frame-

work are that knowledge is represented internally, internal representations can be connected, and when internal representations are connected, they produce networks of knowledge. They theorize that a mathematical idea or procedure is understood if its internal representation is part of a network of knowledge and that the degree of understanding is determined by the number and strength of the connections in the internal network containing that representation. Under this framework, differences in students' understanding of the relationship between slope, rate of change, and the first derivative can be analyzed in terms of the internal networks of knowledge students were likely to form based upon the instruction received in the different calculus courses.

Dubinsky (1991) applies Piaget's notion of reflective abstraction to advanced mathematical thinking to form a theory of mathematical knowledge and its construction. Dubinsky posits that reflective abstraction occurs as part of a student's construction of new knowledge during the process of solving problems. If a problem is solved successfully, then the student assimilates the problem and solution into one or more schema. If the problem is not solved successfully, then the student may or may *not* make accommodations in existing schema to handle the unsolved problem. Dubinsky's (1991) theory is relevant to this research since, along with differences in the type of technology used, there were differences in the types of problems emphasized by the instructors and solved by the students in each course. It was applied to situations where differences in students' understanding could not necessarily be explained through analysis of differences in the networks of represented knowledge likely to be formed by students from the different courses.

Data Collection

Data collected included twice-weekly class observations of each course, a posttest, and 36 student interviews. Class observations were made to document use of numerical, graphical, and symbolic representations during instruction and on assignments. The posttest was used to assess students' abilities to use different representations when solving calculus problems. The following posttest problem was designed, in part, to assess students' abilities to use different representations to solve problems dealing with the slope, or rate of change, of a function.

> The population of a herd of deer is given by the function $P(t) = 4000 - 500(\cos 2\pi t)$ where t is measured in years and $t = 0$ corresponds to January 1.
>
> a. When in the year is the population at its maximum? What is that maximum?

b. When in the year is the population at its minimum? What is that minimum?

c. When in the year is the population increasing the fastest? decreasing the fastest?

d. Approximately how fast is the population changing on the first of July?

The problems used on the posttest were designed by the researcher so that they might be solvable by *any* calculus students, no matter what calculus course they completed. Content validity was established by a panel of mathematicians and mathematics educators from across the United States.

The posttest was given during the final week of classes to 100 of the students (40 traditional, 24 graphics calculator, and 36 Calculus & *Mathematica*). From this group, 12 volunteers from each course were chosen to participate in a one-on-one interview with the researcher. The interviews took place 4-8 weeks after the completion of the course, lasted between 25 and 45 minutes, and were audiotaped. The interviews were used to clarify how and determine why students used different representations when solving the posttest problems. They also provided an opportunity to have students solve these problems using representations different from those used on the posttest, and explain their reasoning for using these representations, while being observed and prompted by the researcher. Student responses during the interview, particularly those related to uses of different representations to solve the problems, formed the basis for the researcher.

Findings

Analysis of the data suggests that the traditional and graphics calculator students had more difficulty than Calculus & *Mathematica* students recognizing the relationship between the first derivative of a function at a point and its slope or rate of change at that point. During the interviews, these students could describe how to use the first derivative to determine the local extrema of the function but did not make the connection between the slope of the graph at the local extrema and setting the first derivative equal to zero, as the following interview excerpts suggest.

Subject G12: I would probably find the derivative. And then just find . . . that equal to zero.

Interviewer: *Why zero? . . . Why does it tell me a maximum occurs?*

Subject G12: Uh - cause that's just the way derivatives work.

Interviewer: *What information is the derivative giving me? . . . Is there any way that maybe the derivative is somehow connect with slope? Does that ring a bell?*

Subject G12: No.

Interviewer: *Why is the first derivative 0 at the local maximums and local minimums?*

Subject T11: If you kinda follow the graph along, at those points, it doesn't have a slope.

Interviewer: *So the first derivative gives me slope?*

Subject T11: I think. If you look at the maximum, then the graph completely changes and goes in a different direction. Then, at a certain point on there, it doesn't have - I mean it has a slope but the slope is 0. That's another way to look saying that, you know, that's where the slope is 0.

Interviewer: *Is the first derivative - is it giving me slope of the function? Is that what it does?*

Subject T11: I don't want to exactly say that it gives you what it is cause I'm not really sure what you plug in to give you the exact - I mean it tells you where it's zero.

Many graphing calculator students could not remember how to use the first derivative to locate local extrema. Nine of the 12 students interviewed were unable to describe how to locate extrema algebraically without prompting. Two traditional and two graphics calculator students could not describe this technique even with prompting. Traditional and graphics calculator students also had difficulty recognizing that how fast the population changed on one day corresponded to the value of the first derivative for that day, as the following interview excerpts suggest.

Interviewer: *What am I asking for when I ask how fast the population's changing? . . . Does any function give me that information? ... Does the derivative give me that information?*

Subject G8: I don't think so.

Interviewer: *What does the derivative give me? . . . What information does it give me?*

Subject G8: Gives you the max and min.

Interviewer: *Does it give me anything else? For example, say I put .25 in [P(t)]. I'll end up getting about 1000π. What does that number represent?*

Subject G8: The number of deer.

Interviewer: *Is that what the first derivative tells me is rate of change? How fast it's changing?*

Subject T7: Um - I don't know. I totally forget. Um -

Interviewer: What information does the first derivative give me? What does it tell me?

Subject T7: Max, min, intervals where the function's increasing or decreasing, and - I don't know.

Interviewer: When I put in one-half, the derivative gave me zero. What does that zero tell me?

Subject T7: That obviously gives you the . . . max. Um. I don't know.

Few Calculus & *Mathematica* students had the same difficulties as the other students describing solutions to the various problems. The following interview excerpts, which represent typical response by these students, illustrate how they seemingly had a better grasp of the relationship between slope, rate of change, and the first derivative than the other students.

Subject C8: The population is increasing the fastest for when P(t) is greatest ... decreasing the fastest for when P(t) is smallest, because P(t) represents the instantaneous growth rate of the function. . . . I know that the [first] derivative would have to be at its lowest and highest points in there. I could take the second derivative . . . and find where those points are zero, . . . finding where the maximum and minimum values for [the first derivative] are.

Subject C3: [G]et what halfway would be and plug that into the equation [to solve part d].

Interviewer: Of the first derivative or of the population?

Subject C3: The population.

Interviewer: Will that give me how fast the population's changing?

Subject C3: Actually, no. You wanted to put it into the derivative. You have to find the change, and not the actual population. . . . So, you'd put it into the derivative.

An analysis of the instructional approaches of each course in terms of the study's theoretical framework helps provide possible explanations for the differences in students' understanding of the relationship between slope, rate of change, and the first derivative.

Analysis of Findings

In the traditional course, instruction heavily emphasized use of symbolic representations to present concepts and solve problems. Students were not provided with opportunities to develop different mental representations of slope

beyond those based upon symbolic representations. As such, students in this course were more likely to form a disjoint or weakly-connected internal networks of knowledge related to the concept of slope which would help explain their lack of understanding of the relationship between slope, rate of change, and the first derivative.

In the graphics calculator course, instruction emphasized use of graphical and symbolic representations to present concepts and solve problems. As such, students in this course were more likely to form well-connected internal networks of knowledge related to the concept of slope since they had opportunities to develop different mental representations of slope. However, class observations indicated that students solved few problems designed to help them make connections between different representations of concepts. Thus, they had little opportunity for the type of reflective abstraction during problem solving necessary for the construction of knowledge relating these representations. Thus, graphics calculator students may only have formed weakly-connected internal networks of knowledge related to the concept of slope, which would help explain their lack of understanding of the relationship between slope, rate of change, and the first derivative.

In Calculus & *Mathematica*, instruction emphasized use of multiple representations to present concepts and solve problems. As such, students in this course, like those in the graphics calculator course, were more likely to form well-connected internal networks of knowledge related to the concept of slope since they had opportunities to develop different mental representations of slope. Calculus & *Mathematica* students also solved problems designed to help them make connections between different representations of the same concept. Thus, unlike the graphics calculator students, they had the opportunity for the type of reflective abstraction during problem solving necessary for the construction of knowledge relating these representations. As such, Calculus & *Mathematica* students were more likely to form strongly-connected internal networks of knowledge related to the concept of slope, which would explain their greater understanding of the relationship between slope, rate of change, and the first derivative they exhibited during the interviews.

Discussion

Findings from this study point out the need for further research on the effects of instruction emphasizing use of multiple representation in the presentation of concepts. Difficulties experienced by graphics calculator students suggest viewing multiple representations of a concept does not necessarily help students develop better understanding of the concept. Apparently, having students solve problems designed to help them make connections between different representations of a concept, as Calculus & *Mathematica* students did, rather than having connections pointed out to them, as class observations indicated

was the case in the graphics calculator course, is also important. The need for further investigation becomes even more apparent when one considers technology that can create multiple, dynamically linked representations of different concepts *for* students will soon be readily available for use in the classroom.

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THE PROCESS OF PERIODICITY

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This paper presents results from a study of the ways students in grades 3 to 12 perceive periodicity. The study was conducted in two consecutive phases: a qualitative phase followed by a quantitative one. The study revealed common students behaviors. Two of them are a tendency to assign a direction (from left to right) to a periodic process, and to not distinguish a periodic process from its non-periodic product. These and other findings constitute supportive evidence to the claim that students grasp periodicity of phenomena and of functions as a process, rather than an object.

In 1992, as we started to investigate the ways students perceive the notion of periodicity[1] very little (if any) previous research in this area had been published. Consequently, we set up our study to establish a field-grounded theory. Naturally, issues related to validity and reliability of the instruments, as well as generalizability of the results, were among our concerns. Hence, the research was designed in two consecutive stages.

1. A qualitative stage included:
   (a) Semi-structured interviews with 7 teachers.
   (b) Open observations at mathematics lessons in grades 6, 9, 11, 12, and of non-mathematics lessons concerning periodicity in grades 3, 11.
   (c) Semi-structured interviews with 28 students in grades 3, 6, 9, and 11.
2. A quantitative stage followed the qualitative one and included a survey of 895 eleventh grade, math majors, in sampled high-schools, in Israel. For this stage, a paper-and-pencil questionnaire of 121 items was prepared, based upon the results of the first stage. It included two sub-sets of close items: (a) 39 items examined period identification; (b) 76 items examined conceptions of periodicity. Structural validity of each sub-set was verified by Factor Analysis and by Multi Dimensional Scaling. The questionnaire included also a third sub-set of 6 open questions. These were used to validate externally the close items.

Presented below is part of the results from this study (for a full account see Shama 1995).

[1] A non-constant real function f is periodic in its domain, if there exist T(0 such that for every d in the domain $d \pm T$ is also in the domain and $f(d)=f(d \pm T)$. A non-constant series {an} is periodic if there exist $T>0$ such that for every n, $a_n=a_{n+T}$.

Understanding Periodicity

This section presents qualitative evidence supporting the claim that *most of the students construct their concept of periodicity upon time dependent processes.*

At the beginning of each interview, the interviewer asked the students to give examples of periodic phenomena. Each interviewee gave several examples to periodic phenomena that are time dependent. From the 28 interviewees, 24 gave as a first example one that is time dependent. Ninety-three percent of interviewees' spontaneous examples were time dependent, e.g., "the seasons of the year", "the monthly appearance of the moon", "the day-night change", "animals breathing" and a 9th grader said with no hesitation, "the clock is the most periodic thing there is, because those hours always repeat themselves, it goes all the time."

The interviewer also asked the students to define a periodic phenomenon and a period. Of the 28 interviewees, 16 used terms of time and process in their definitions, e.g., a student from grade 3 defined, "A periodic phenomenon is that all the time, all the time it does the same action ... We are doing the same movement all the time, this is periodicity in my opinion," a student from grade 6 defined, "It repeats itself all the time, it doesn't change much. In each time unit it is repeated," a student from grade 9 defined, "It is an action that happens, and in the very moment it ends it is starting again — it happens again and again," and a student from grade 11 defined "It is a process that repeats itself after a certain period of time."

In the second stage of the study, students were asked in three open items to define a periodic phenomenon, to define a periodic function and to define a period. Of the subjects, 52% used terms of time and process in their answers.

The concept of periodicity as a process seemed to be reinforced by teachers. As observed, the following periodic movements were addressed in non-mathematics lessons: motion of celestial objects in grade 3; harmonic motion in grade 11. In observed mathematics lessons periodic processes were approached. A process of repeating division was used to describe periodic decimals in grade 6 and in grade 9. A process of a cyclic movement around the unit circle, and a process of movement over a graph, both were used to describe the periodicity of trigonometric functions in grade 11 and in grade 12. This time dependent processes were found to be the foci of attention in dealing with periodicity in grades 3 to 12.

A Periodic Process Yields a Periodic Object

The four parts of Figure 1 have a common characteristic. The process of drawing each of them is periodic, to repeat on an algorithm, but the result does not represent periodic function nor periodic series. This section presents evi-

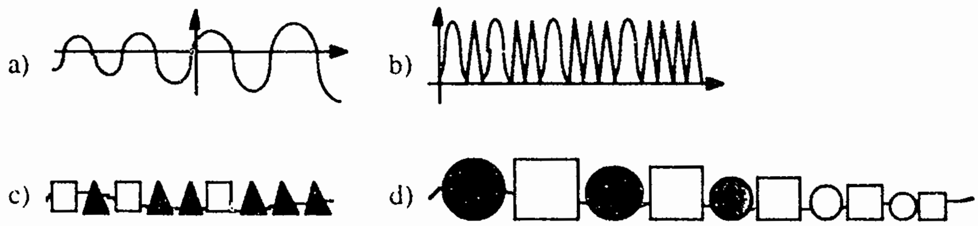

Figure 1. Non-periodic graphs and series perceived as periodic.

dence supporting the claim that *students tend to assume that a periodic process yields a periodic object, such as graph, number or series.*

In the first stage of the study, interviewees gave the drawings in Figure 1 as examples to periodic phenomenon. Interviewees brought also other non-periodic examples, which erroneously seem to them as periodic, for example, "0.74374307430074300074300000", "the order of the day at school is periodic — we are doing the same thing, but it is increasingly difficult from year to year", "running faster and faster."

In the second stage of the study, students were asked whether a function defined for all real numbers, which is graphically presented in a limited domain in Figure 1a or in Figure 1b, is periodic. They were also asked whether an infinite chain, which is drawn in Figure 1c or in Figure 1d, is periodic. Eleven percent of the subjects identified at least three of the four examples on Figure 1 as periodic[2]

Assigning a Direction (from left to right) to a Periodic Process

This section presents evidence supporting the claim that *students assign a direction (usually from left to right) to a graph of a periodic function and to a visual representation of a series.*

Both stages of the study provided evidence that students prefer to locate graphs of periodic functions to the right of the y-axis. In the first stage of the study, 14 interviewees (from grade 9 and 11) were asked to bring an example of a periodic function. Five of them drew all their examples to the right of the y-axis (as in Figure 2a). In the second stage of the study, answering the same question, 77% of the students drew a graph almost completely to the right of the y-axis.

[2]The items in the same sub set of the questionnaire were classified through the validation tests. The items in Figure 1 were distinguished as one group, as well as those in Figure 3

A B

Figure 2. Directions in graphs of periodic functions.

In both stages of the study, students were asked to continue a finite graphical representation of a periodic function or a periodic series. In the second phase of the study, 37% of the subjects extended the two given representations to the right (as in Figure 2 b).

In the second stage of the study, the students were asked to determine whether the graphs, that appear in Figure 3, represent periodic functions or not. Fourteen percent of the subjects identified both graphs in Figure 3 as graphs of a periodic function. Nine percent identified one of those graphs as representing a periodic function.

Following the first stage of the study, two habits of the students were revealed. We have found that students prefer to locate graphs of periodic functions to the right of the y-axis. We also found that students tend to continue a graph of a periodic function, and a representation of a periodic series, to the right. These findings led the hypothesis that there are students who think that periodicity has a direction. Therefore, in the second stage, we included questions that will not be answered correctly by such students (see Figure 3). Errors in students' answers to these questions were indeed found as demonstrated above.

A

Here is a limited domain graphical representation of a function that is defined for all real. Write "Yes" if you think the graph represents a periodic function, and "No" if you think the graph represents a function that is not periodic.

B

Here is a limited domain graphical representation of a function that is defined for all real. Write "Yes" if you think the graph represents a periodic function, and "No" if you think the graph represents a function that is not periodic.

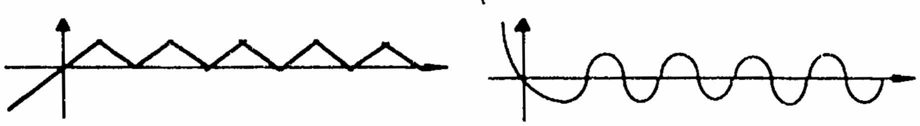

Figure 3. Items from a pen-and-pencil questionnaire*

A
B
C

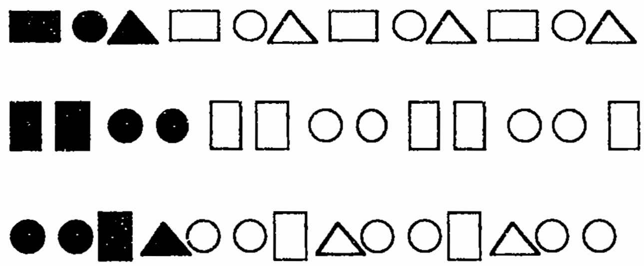

Figure 4. Basic period of a periodic sequence in the left end (black) and in the right end (white).

Left-End Period Preference

This section presents evidence supporting the claim that *students prefer a basic period located on the left end of the graphical representation.*

In the first stage of the study, interviewees were asked to identify a period in several examples. Seventy-five percent of their answers pointed to a period at the left end of the representations. Among other assignments, all interviewees were asked to identify a period in three series of geometrical shapes as in Figure 4. Twenty-three of 28 interviewees chose a left-end period, as marked in Figure 4 in black. The other 5 interviewees chose a right-end period, as marked in Figure 4 in white.

In the first stage of the study, in the observed lessons both teachers and students preferred a period located at the left side, or a period of a function next to the y-axis. In trigonometry lessons, angles between 0 and T were chosen as representatives of a set of solutions of a trigonometric equation, which has a period of length T.

In the second stage of the study, subjects were asked in 39 items to decide whether a marked part of a periodic function or periodic phenomenon is its period or not. Only in five of those items a left-end fundamental period was marked. These five items obtained the highest percentage of correct answers. Ninety-three percent of the subjects answered correctly most of these five items.

Discussion

According to Sfard (1991) there are two aspects of the conception of mathematical notion to be considered: an operational aspect and a structural one. Operational conception of a notion is based on processes, algorithms and actions. Structural conception of a notion means: "...treating mathematical no-

tions as if they referred do some abstract objects..." (p.4). Regarding the notion of function, two stages that precede understanding as a process were identified by Breidenbach, Dubinsky, Hawks & Nichols (1992). They found that in many cases the concept of function never develops beyond these first two stages. In the few cases it does, it rarely goes beyond the perception of a function as a process (Harel & Dubinsky, 1991).

We have found that both the concept image and the concept definition of the notion of periodicity, held by students, relate to its operational aspect. Most of the students constrict their concept of periodicity upon time dependent processes. Many of the students include, in their definitions of periodicity, the notion of time or the notion of process.

The understanding of periodicity as a process may explain students' tendency to assume that a periodic process yields a periodic object, such as graph, number, or series. This tendency may cause some errors, as shown.

Students tend to assign a direction (usually from left to right) to a graph of a periodic function and to a visual representation of a series. Students also prefer a basic period located on the left end of the graphical representation. A connection between these two findings can be explained as follows. A process, with an attached direction, is also characterized by a staring point. Students select the "first" period of this process.

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THE RELATIONSHIP BETWEEN WRITTEN AND VERBAL PERFORMANCES: A STUDY OF FIRST YEAR CALCULUS STUDENTS' UNDERSTANDING OF THE DERIVATIVE

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A ten problem Derivative Test was designed and used as a written instrument and in talk-aloud problem solving interviews to explore the relationship between the levels of understanding of the derivative indicated by subjects' written and verbal performances. Each problem on the Derivative Test was categorized by type of derivative interpretation: Geometric, Physical, Algorithmic, or Relational, and by manner of presentation: Traditional or Nontraditional. Subjects' written performances were analyzed quantitatively, while their verbal performances were analyzed qualitatively using mapping techniques and the Combined Model of Understanding, an assessment instrument developed by the author. A synthesis of these analyses indicated that relationships between subjects' performances depended upon the type and presentation of derivative problems. The study also suggests the usefulness of the Combined Model of Understanding in analyzing qualitative data.

The ideas of calculus long identified as essential for modeling and explaining phenomena in disciplines such as physics and engineering, are now recognized as playing a similar role in many other disciplines. In calculus, changing systems are modeled by differential equations which describe relationships between variables in terms of rates of change or derivatives. To model and solve interdisciplinary applied problems involving change or motion, such as optimization, rectilinear and curvilinear motion, logistics growth, structural analysis, mechanical vibrations, and economic models, one must understand the various interpretations of the derivative and recognize its appropriate use. Many students who pass a calculus course, however, do so without a clear understanding of some of the fundamental concepts of calculus. Existing research also indicates significant gaps in students' conceptual understanding of the derivative (Ferrini-Mundy & Graham, 1994; Orton, 1983; Selden, Selden & Mason, 1994).

Student understanding is commonly assessed based on what is displayed in writing and often these displays are interpreted by instructors without regard for the students' thought processes (Davis, 1992; von Glaserfeld, 1987). Researchers report that students are rewarded for correctly performing rituals and algorithms and yet these reproductions of their textbooks' or instructors' examples do not accurately reflect their levels of understanding (Dreyfus, 1991; Selden, Selden & Mason, 1994; Tall, 1991). This study examined subjects'

understanding of the concept of the derivative by investigating the relationship between the levels of understanding indicated by subjects' written and verbal performances when solving derivative problems. While subjects' written performances were quantitatively assessed, determination of subjects' understanding as indicated by their verbal performances required complex qualitative analysis.

Research in the areas of understanding (Davis, 1992; Pirie & Kieren, 1994; Sierpinska, 1990; von Glaserfeld, 1987), procedural and conceptual knowledge (Hiebert & Lefevre, 1986; Tall, 1991), and concept development (Dreyfus, 1991; Dubinsky, 1991; Sfard; 1991) framed this study. In order to investigate subjects' levels of understanding of the concept of the derivative, this research was synthesized to first develop a Combined Model of Understanding which provided general descriptions of acts which indicate understanding at various levels. Descriptions in the Combined Model of Understanding were then adapted to describe levels of understanding of the concept of the derivative in order to analyze the level of understanding of the derivative indicated by subjects' verbally revealed conceptions.

Methodology

To investigate subjects' understanding of the derivative, the Derivative Test and two interviews were employed. The Derivative Test was designed to collect information on subjects' levels of understanding of the derivative indicated by their written and verbal performances. The interviews were designed to collect verbal Derivative Test performance data on subjects' understanding by using a talk-aloud problem solving methodology.

The ten problem Derivative Test was designed to assess subjects' understanding of the geometric and physical interpretations of the derivative, as well as their algorithmic proficiency with differentiation rules. In addition, subjects' abilities to describe the derivative and form relationships between various representations of the derivative were assessed. Problems on the Derivative Test were presented in two different formats referred to as Traditional and Nontraditional.

The Derivative Test was administered as a written instrument in a classroom setting to 225 first year college calculus students at the beginning of the Calculus II course. Tests were quantitatively scored using a predetermined scoring scale. Results of the written Derivative Test were used to identify levels of subjects' understanding of the derivative. Eleven subjects then participated in two individual interviews where they re-solved the ten Derivative Test problems in a "talk-aloud" manner. Analyses of data from five of the interviews were used in the study. Cognitive mappings and Pirie and Kieren (1994) mappings were developed for each problem for each subject to capture the interview data in a condensed and visual format. These mappings were then

analyzed within the framework of the Combined Model of Understanding to determine subjects' levels of understanding of the derivative indicated by their verbal performances.

A synthesis of the quantitative and qualitative Derivative Test analyses was used to determine if there existed a relationship between the levels of understanding of the derivative indicated by subjects' written performances and those indicated by their verbal performances.

Results and Conclusions

Written Performances

Quantitative data from the written Derivative Tests were used to determine descriptive statistics for all problems and subproblems, and for the Geometric, Physical, Algorithmic, Relational, Traditional and Nontraditional problem groups. Analyses of these quantitative data identified various classification levels of subjects' understanding of the derivative. Based on their Derivative Test scores, subjects' written performance understanding levels were classified as either HIGH or LOW in Traditional and Algorithmic problems, referred to as the T-A category, and as either HIGH or LOW in Nontraditional and Relational problems, referred to as the N-R category. Each subjects' overall written performance on the Derivative Test was then classified as HIGH-HIGH, HIGH-LOW, LOW-HIGH, or LOW-LOW indicating their level of understanding in the T-A and N-R categories, respectively. The following table contains the results of the written performances.

Table 1. Classification of 225 Subjects Based on Written Derivative Test Scores

Classification	HIGH-HIGH	HIGH-LOW	LOW-HIGH	LOW-LOW
Frequency	13	21	34	157
Percent	5.8	9.3	15.1	69.8

Verbal Performances

Interview data in the form of cognitive mappings and Pirie and Kieren mappings for each problem for each subject were analyzed in terms of the Combined Model of Understanding to describe the levels of understanding indicated by subjects' verbal performances. The cognitive mappings provided the pieces of information and connections between those pieces of information subjects used in their solution processes. The Pirie and Kieren mappings represented the progression of subjects' solution processes through the levels of Pirie

and Kieren's (1994) theory of understanding. The Combined Model of Understanding included descriptions of three primary levels of understanding: Level 1 (Low), Level 3 (Intermediate), and Level 5 (High). Secondary levels, Levels 2 (Low-Intermediate) and 4 (High-Intermediate) indicated that a subjects' level of understanding fell between two primary levels. The levels of understanding indicated by subjects' verbal performances were determined for the T-A and N-R problem categories resulting in two number designations. The five interview subjects were classified as 4-4, 4-3, 4-2, 3-3, 2-2. A classification of 4-3, for example, indicates a High-Intermediate level (Level 4) of understanding of derivative problems in the Traditional and Algorithmic groups and an Intermediate level (Level 3) of understanding of derivative problems in the Nontraditional and Relational groups.

Relationship of Performance

The quantitative and qualitative data from each of the five subjects' written and verbal performances were synthesized to examine the relationship between the indicated levels of understanding of the derivative. There was not a consistent relationship between all subjects' written and verbal performances. One subject whose written performance levels of understanding were consistently HIGH across problem groups also revealed a high-intermediate to high level (Levels 4 and 5) of understanding across problem groups in his interview. Other subjects whose written performances were inconsistent across problem groups revealed inconsistent levels of understanding in their verbal performances. Similar levels of understanding were observed in both written and verbal performances in the Traditional and Algorithmic groups, while a consistent relationship was found between subjects' actual written Derivative Test scores in the Physical group and their verbal performance levels. Analysis revealed no relationships in the Nontraditional, Relational, and Geometric groups.

The small sample size made it difficult to confirm possible explanations for the presence or absence of relationships between subjects' written and verbal performance levels. Two subjects, whose performance levels in the T-A category were not related, exhibited difficulty in recalling formulas and algorithms. Their written Derivative Test solutions showed incorrect algorithms and errors on Traditional and Algorithmic problems, but when verbalizing their solutions during interviews, they reconstructed many of the required procedures from conceptual knowledge and formulated correct solution processes. When instruction emphasizes procedures and definitions, it seems subjects' performances on Traditional and Algorithmic problems should be similar. In this case, however, because they could not recall procedures and definitions during the written test, it appears their written solutions masked some of their understanding.

Two subjects' performances were not related in the N-R category. One

subject displayed a HIGH written performance level, but during the interview his apparent conceptual insights were mitigated by inconsistencies across problem groups and difficulties in applying specific algebra and calculus procedures. The second subject whose written performance level was LOW revealed understanding at Level 3 during her interview. She verbally expressed insights at an even higher level, but often dismissed them in favor of inappropriately applying rules and procedures to some Nontraditional and Relational problems. Her actions may have stemmed from an emphasis on these procedures during instruction.

Relationships found in the Traditional and Algorithmic groups may be explained by the familiar nature of these problems. Subjects solved these problems by following prescribed processes or standard algorithms. The relationship within the Physical group may be explained by the fact that after initially introducing the derivative using the geometric image of the slope of the secant line, algorithmic rules were introduced and classroom discussions focused on familiar applications, such as velocity, which involve the physical interpretation of rate of change.

One implication of this study is that results of assessments which use only problems from either the Traditional-Algorithmic (T-A) or Nontraditional-Relational (N-R) categories, or from only one of the Geometric, Physical, Relational, or Algorithmic groups, may not fully capture a student's level of understanding of the derivative. Although the use of talk-aloud problem solving offers students more opportunity to reveal their depth of understanding, this technique is difficult to apply as an assessment tool in the classroom. It is possible that if the subjects in this study were exposed to many different types of problems both in class and on exams, and if they had opportunities to discuss and verbalize solutions to those problems, then their written performances might more closely match their verbal performances.

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FOSTERING STUDENTS' CONCTRUCTION OF KNOWLEDGE DURING A SEMESTER COURSE IN ABSTRACT ALGEBRA

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Student difficulties in abstract algebra are a source of concern in mathematics departments across the country (Dubinsky et al., 1994). However, if learning truly is a constructive process (von Glasersfeld, 1987), then a constructivist perspective may hold promise for the learning and teaching of advanced mathematical structures such as abstract algebra.

The authors co-developed and co-taught an undergraduate course in abstract algebra grounded in a theory of "action-process-object-schema" (Dubinsky et al., 1994). A series of concrete contexts were used to foster students' ability to construct knowledge of abstract algebra. Within these contexts, students transformed their physical actions into mental actions and their mental actions into first processes, and then objects. Students often worked in small cooperative groups and were invited to engage in a "Socratic dialogue" during whole-class discussions. Moreover, new notation was held to a minimum and the introduction of any notational device was delayed until students had an opportunity to construct a mental image of the referent.

At the close of the semester, every student in the class completed a brief survey. Using a six-point Likert scale, students were asked to assess the degree to which these instructional strategies had proven helpful. Mean student responses to the survey items ranged from a low of 4.9 to a high of 5.4, with 5 representing "agree" and 6 representing "strongly agree."

During the semester, most students responded appropriately to the instructional strategies that were used and many commented favorably about them. Students communicated their appreciation for an interactional style with which they could engage, rather than a lecture format. These comments were perhaps best summarized by a student who wrote, "I understand better as a result."

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THE ROLE OF GENERAL LOGIC ACTIONS DEVELOPMENT IN TEACHING MATHEMATICS

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The foundations of the Stage by Stage Development of Mental Actions Theory (further DMA theory) were developed in the 1950s-60s by Russian psychologists A. Leontiev (Leontiev, 1972), P. Galperin (Galperin & Talizyna, 1979), and T. Talizyna (Talizyna, 1975). We developed DMA theory in application to computer based instruction (Bouniaev, 1996). The process of instruction is considered to be a process of developing mental actions with objects of the studied field.

The objective of the presentation is to analyze the structure of different actions in undergraduate mathematical courses to be developed in students' minds. We'll show that these actions as a rule have a general logic component and a specific component. Examples of general logic actions are: classification, attributing to the concept, etc. .c. implication, etc.; . . . . To illustrate our theoretical concepts we consider examples from calculus, particularly the example of finding limit of rational function. *.c. examples from calculus, particularly the example of finding limit of rational function.* Analysis of specific actions that must be developed in the calculus course shows that it is this general logic component that causes students mostly difficulties in performing the entire action. *.c. Mostly it is this general logic component that causes students difficulties in performing the entire action.*

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LEARNING RATE OF CHANGE

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Objectives

Rate of change is important in mathematics but it is a difficult concept for high school and college students to learn (Nemirovsky & Rubin, 1992; Monk, 1992). This study investigates knowledge of precalculus, calculus, and postcalculus students and how it may support their learning of rate of change.

Theoretical Framework

Knowledge as strategies has been used by others to study learning. Siegler and Jenkins (1989) examined strategies 4- and 5-year-olds used for addition and how these strategies helped them learn other strategies. Smith (1990) investigated strategies 11-, 14-, and 17-year-olds used to work rational number tasks and formed one view of learning rational number. This study considers strategies high school and college students use with rate of change and how these contribute to learning.

Methods and Data

In individual taped interviews, 12 precalculus, 15 calculus, and 10 postcalculus students worked three tasks in which two people start at opposite corners of a room, walk toward each other, pass, and proceed to opposite corners: (1) slow down, pass, and then speed up; (2) maintain a steady pace the whole way; (3) speed up, pass, and then slow down. Students were asked to construct graphs and tables showing the distance between the two people at each point in time and to explain how these showed the two people slowing down, speeding up, and maintaining steady pace.

Results

When making graphs, students drew different shaped arcs or plotted points and referred to distance traveled each second or visual slope t explain how their graphs showed slowing down, speeding up, or constant pace. When making tables, most students read points from their graphs and referred to distance traveled each second to explain how their tables showed varying speed. For these students, distance traveled each second (changes over intervals) was a primary way to think about rate of change. Students also connected shape of graph and visual slope to varying rates of change.

Implications

To help students learn more about rate of change, teachers and curriculum writers should capitalize on students' tendencies to use changes over intervals in making sense of varying change in situations involving graphs and tables of values. They should also make a more direct effort to connect shape of graph and visual slope to rate of change since these features of graphs are also used by students when thinking about rate of change. Furthermore, they should think about how changes over intervals and shape and visual slope of graph and other knowledge students have of rate of change in situations involving graphs and tables of values could be used to help students learn about rate of change in situations involving functions as represented by equations.

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TEACHERS' CONNECTIONS AMONG VARIOUS REPRESENTATIONS OF SLOPE

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This study examined the connections that teachers make among various representations of slope: algebraic, geometric, physical, functional, trigonometric, and ratio (Stump, 1996). It investigated secondary mathematics teachers' concept images and concept definitions (Tall & Vinner, 1981) of slope, mathematical understanding of slope, and pedagogical content knowledge (Shulman, 1986) of slope. It also compared the knowledge of preservice and inservice teachers. Data from 18 preservice and 21 inservice teachers were collected from paper-and-pencil surveys. Preservice teachers were undergraduate students enrolled in a secondary mathematics methods course. Inservice teachers were from the mathematics departments of four high schools. Random subsets of eight preservice teachers and eight inservice teachers were selected for interviews.

The geometric representation of slope was included in all teachers' concept images and concept definitions. Functional and physical representations were cited more often than algebraic, trigonometric, or ratio representations.

Both preservice and inservice teachers had trouble recognizing the parameters of a linear equation that involved only literal symbols. Some had difficulty answering questions involving rate of change, and several failed to recognize the trigonometric representation of slope. Inservice teachers had greater understanding of the trigonometric representation of slope.

Physical representations were most often included in teachers' descriptions of classroom instruction. Geometric and functional representations were the second and third most frequently mentioned. Algebraic and ratio representations were named less frequently.

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SYNTHESIS BETWEEN VISUAL AND ANALYTIC ACTS IN LINEAR DIFFERENTIAL EQUATIONS THROUGH A GRAPHICAL ARGUMENT

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This work deals with mathematics at the undergraduate level, specifically with differential equations. We have some evidence that today the instruction, in this topic, is more centered in the algebraic and analytic modes. In our work we found a graphical argument that allows us to relate "graphical arithmetic" with the composition of functions (Cordero & Solis 1997). We work with the fundamental structure: $y(x) = A\,[f(x) + b] + B$ which is obtained from a linear iteration that starts with $y(x) = ax + b$, where the analysis of graphic behaviors is fundamental. The function is now conceived as an instruction that organizes behaviors. This idea, carried to the domain of differential equations, is the argument that allows us to relate the *graph* of the solution with the algebraic expression of the equation, but, when it is worked out via the fundamental structure above allows us to simulate graphical behaviors that connect with the analytic structure of the equation. There are now a new kind of problems, whose main goal is to simulate and predict the graphical behaviors of the solution when the coefficients of the equation are varied.

To apply to students, we will design activities, in which we will consider five situations where the aforementioned arguments are discussed: 1) variation of parameters, 2) simulation of first order linear differential equations, 3) "tendencial behavior" of the solution, 4) analytic arguments and 5) generalization of "tendencial behavior".

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ON THE RELATIONSHIP BETWEEN CONCEPTUAL AND ALGORITHMIC ASPECTS IN INTEGRAL CALCULUS: AN EXAMPLE IN KINEMATICS

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This study is based on the approach called "Academic Mathematical Discourse" (AMD). The problem that we are going to deal with consists of a disequilibrium between the conceptual and the algorithmic aspects in the teaching of integral calculus. That is, the students are taught procedures to calculate integrals using integration methods only through drill exercises, and in a way separate from the conceptual part. It is only when they see the "applications", they study some notions related to integration. Because of the characteristics of the problem we seek the help of the theory of the conceptual fields. (Vergnaud, 1990). In this presentation we are only going to deal with one of the objectives of our research: to obtain information about what "to comprehend an algorithm" may mean. In the domain of AMD, one necessary condition to arrive at the relationship between the conceptual and algorithmic aspects is to propose a specific problem that requires solution by integration (Muñoz, 1996a,b). Through a problem in kinetics we are going to discuss the following points: 1) Problems of the following type: If a body falls freely from a certain height starting at rest, this has a constant acceleration. To calculate the next position of the body at any instant of time t and at a particular instant t_n, if we ignore the air resistance. 2) With respect to what "to comprehend an algorithm" might mean, we make the following premise: the student does not acquire pre-established procedures by simple conditioning through repetitive exercises, but what he acquires is rules that may and should be applied to new problems. He acquires them only if he comprehends them, that is, if he realizes the relationship that they maintain with the relational structure of the problems to which they are applied to (Vergnaud, 1991). Among the rules of the extreme point, trapezoid, mid-point, Simpson and Taylor series, we identify the relationship that they maintain with the relational structure of the specific problems in kinematics. In this case the implicated relationships are between the quantities of distance and time. Furthermore, the structure is formed according to how each procedure of the rule allows the passage from one quantity to another, or if the operations contain only one type of quantity. It is necessary to clarify that we do this identification by having a socially established canonical procedure, and a kinematics problem whose solution requires integration. However, how to make sure that the student would realize the relationship of the rule to the structure of the problem, is a research problem that we will deal with later. 3) When

forming the relational structure, we discuss the notions of prediction, accumulation, constantification of the variable, that have been studied in some form by Cantoral (1990) and Cordero (1994). Finally, we comment that this is an ongoing research and it is directed toward the identification of the genesis of the aforementioned notions and rules when the student is confronted with a problem situation that requires to be solved by integration.

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SOME REPRESENTATIONS AND THEIR INTERACTIONS WITH MENTAL CONSTRUCTIONS IN THE CONCEPTS OF CALCULUS

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To describe the understanding of the concepts of calculus consists not only in thinking about a structure that organizes the concepts of calculus and looking for an epistemological framework that describes the mental constructions that a student makes. It is also necessary to consider that there exists a Calculus that is taught in schools; that is, it is necessary to consider the didactic transpositions in Calculus.

Therefore, it is important to describe the understanding of the concepts in the context of different epistemological frameworks for the Calculus and the phenomena in the representations between different frameworks. This description of understanding acquires the very characteristics of a functional, and not a structural, nature. It is true that the mental constructions characterize the levels necessary to reach a certain understanding of the concepts, but the functional nature must pay attention to the restrictions of knowledge in relation with the mental constructions. For example, from a perspective of the didactic transposition it is not sufficient to consider the concept of function in calculus as a structure of relations between two sets. It is necessary to look at its development in the concept of different epistemological frameworks and in the phenomena of representations. If f(x) is the expression or description of the continuous variation of a certain quantity, or if f(x) is a formula, or if f(x) is an organizer of graphical behaviors, what are the different representations, their forms and levels? What are the different planes of representation and possible homomorphisms between them? What are the locally coherent operating procedures that are derived from the representations?

The nature of these relations necessarily leads to carrying out visual and analytic acts, that can be explained through processes and objects constructed by the relationships which are established between the internal and external representations. Therefore, our hypothetical statement is that a graphical situation favors more constructing processes, objects, encapsulations, and deencapsulations moving in a mathematical topological context. That is, the description of the form of the graph is sometimes local and sometimes global: pieces of the graph and the graph as a whole. Furthermore, depending on the representations that come into the play in the descriptions, operational procedures are derived to establish the relationships.

ON UNDERSTANDING OF RELATIONS BETWEEN FUNCTION AND ITS DERIVATIVE

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This study tries to explain how students understand the relationship between function and derivative. Relationships will be studied through situations that deal with contexts that move back and forth between graphic and analytic aspects.

The nature of these relationships carries visual and analytic acts. These can be explained through processes and objects, but these processes and objects are constructed by external and internal representations. So our hypothetical point is that in a graphic situation the student must necessarily construct processes, objects, encapsulations and de-encapsulations. These constructions take into account a mathematical content in a topological sense, that is, the description of the graph form is sometimes local and, at other times, global: the piece of the graph and the graph as a whole. However, the procedures depend of the representations that play a role in the descriptions.

The effects of the interchange of the mathematical content, representations and procedures are elements that let us to describe the "structure of development" of understanding. In this sense, we are designing situations to move back and forth among different contexts: *graphic context -> analytic context, analytic context -> analytic context, analytic context -> graphic context and graphic context -> graphic context.* For example, the graphic tendency behavior between the derivative and primitive functions could carry to situations that to move back and forth several contexts. The graphic of **f'** and its position in relation to the coordinates provides information on the graph of the function primitive **f**. But on the other hand, knowing what happened with the behavior of **f'** and **f** and adding a constant to **f'**, i. e **f'+ a** it is logical to ask about the new graphic behavior of **f** . This type of situation results in a quantitative analysis.

In our study, these situations will be sequences of questions provoking disequilibrium in student conceptualizations.

The theoretical framework on which the analysis of this study is based in relation to the mental construction: actions, processes, objects, and shames. However, because the nature of the study depends on a specific situation, we will only use conceptual tools of this theory that help or explain the situation.

UNDERSTANDING OF CHAIN RULE IN A GRAPHIC CONTEXT

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We are looking for explanations of the understanding when the concepts are in a different kind of context. In this sense, our main questions are focusing on the representations phenomenon and the procedures that depend of the context. As a special case, we are paying attention to the chain rule concept in a graph context. And we have found an argument in relation to the graph of the function. Currently, we are trying this argument within an epistemological framework. In this framework it describes how the graph context helps to construct special graph operations that we have called "tendency behavior of the graph." In this sense, "to recognize patterns of functions and their graphs" plays a special role in the construction of the chain rule concept.

We are designing situations with the use of graph calculators where the variation conception is necessary for the construction of the special graph operations.

Therefore, in this work we are looking for elements in order to appreciate cognitive aspects that are in the construction of chain rule by observing graphical procedures verifying heuristic value and revising the suggested mental structures in the light of empirical experience.

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METHODOLOGY TO STUDY TEACHER CHANGE IN A REFORM CALCULUS CURRICULUM

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In order to examine the extent to which one secondary school teacher's beliefs and practice may have changed during the first year of implementation of a reform calculus curriculum, the following research methodology was designed. The data was used to construct, using grounded theory, a case study about the implementation of a reform calculus curriculum at the high school level. The methodology was as follows. At the beginning of the year, the teacher was interviewed regarding teacher beliefs, knowledge, and lesson planning. This interview was re-administered halfway through and at the end of the first year of implementation: In another interview, the teacher reconstructed lessons from the previous text so that instruction from both curricula could be compared. This interview focused on lessons in key calculus topics such as limits, continuity, conceptual and procedural development of the derivative and integral. In addition, a quantitative beliefs instrument was administered at the beginning and end of the first year of implementation. This survey focused on teacher beliefs regarding the discipline of mathematics and its teaching and learning.

Classroom observation data were also collected. The focus was a period of daily observations for eight weeks. The remainder of the school year, a systematic sample of observations was made. All lessons pertaining to the key calculus topics mentioned previously were also observed. Each observation was followed by an interview focusing on the teacher's perception of the lesson. Interviews focusing on the teacher's goals for the chapter and the extent to which these goals were met were also conducted at the beginning and end of each text chapter. In addition, the following data were collected from both the previous text and the reformed text: (a) teacher lesson planning notes, (b) evaluation instruments and classroom handouts, and (c) two student class notebooks to help document calculus lessons not directly observed.

THE EFFECT OF CONTEXT ON STUDENTS' PROCEDURAL THINKING

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In his recent analysis of students' ability to construct Venn diagram representations of multi-operation set expressions, Hodgson (1996) reports that although the task elicits an "unexpected" number of errors, the distribution of errors across operations is not uniform. In particular, expressions containing both complementation and union operators elicited a majority of the observed errors. Moreover, the predictable (rather than random) nature of these errors suggests that error-prone students actually construct and implement procedures to complete the task, a belief that production system models and follow-up interviews later confirmed. This research, which represents a continuation of the preceding study, compares students' efforts to construct Venn diagram representations with their ability to determine the elements that correspond to set expressions. Specifically, this research examines an interesting phenomenon in which students can successfully identify the elements corresponding to a set expression (one involving the complementation and union operators), yet are unable to construct the Venn diagram representation of the very same expression. In-depth analysis of students' solutions reveals consistency in their approach to each task, suggesting similarities in the procedures used. However, the fact that errors occur in one context and not in another suggests that context plays an important role in the initial proceduralization of each task. In particular, students' proceduralization of the underlying mathematical definitions - the union of two sets and set complementation - appears to be context dependent. In addition to the presentation of students' work and the author's interpretation of the data, this poster explores the implications of these findings for education (What do the results imply about the teaching and use of mathematical definitions in the classroom?) and invites the feedback and participation of viewers.

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Algebra and Algebraic Thinking

MATHEMATICAL PATTERNS IN THE MIDDLE GRADES: SYMBOLIC REPRESENTATIONS AND SOLUTION STRATEGIES

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This study investigated the strategies used by twenty-three seventh- and eighth-grade students to answer questions about four sequential perimeter and area problems. The study explored the strategies used to find perimeter and area; generalize the pattern relationships and express them symbolically; assess the validity of alternative expressions; and solve equation-evoking situations, that is, questions that could be answered by solving an equation. The data suggest at least four clusters of reasoning about sequential perimeter patterns: (1) Students model and count to find the perimeter without identifying a relationship between the number of a figure and its perimeter or area, (2) Students attempt to describe the critical relationships with a single operation, (3) Students perceive the relationship in terms of the perimeter or area of consecutive figures, and (4) Students recognize the critical relationships and express them symbolically.

Purpose of the Study

The far-reaching changes which have swept through American society during the past century have created a need for a redefinition of mathematical competence (National Council of Teachers of Mathematics, 1989). The changing workplace now requires workers with "mathematical power" based on the abilities to explore, conjecture, and reason logically (Committee on the Mathematical Education of Teachers of Mathematics, 1991). Calculational skills receive less emphasis while effective reasoning about quantities and quantitative relationships has become more important (Thompson & Thompson, 1995). Because algebra provides concepts and language that facilitate reasoning about relationships within problematic situations, the shifting emphases in the workplace have contributed to a renewed interest in the teaching and learning of algebra. In the *Curriculum and Evaluation Standards for School Mathematics (CESSM)* (1989), the National Council of Teachers of Mathematics (NCTM) recommends that experiences with patterns and relationships provide an introduction to algebraic concepts in grades K-4 and be extended to focus on analyses, representations, and generalizations of function relationships in grades 8.

If the study of patterns is a valid means of preparing students for algebra, then more information is needed describing how children think about patterns. The purpose of this study was to explore seventh and eighth grade students' thinking about patterns by conducting task-based interviews in which the students were asked to solve problems about sequential perimeter and area prob-

lems modeled with pattern blocks and tiles, generalize the relationships related to the patterns, represent the relationships symbolically, and identify other valid symbolic expressions of the pattern. The students also encountered equation-evoking situations which a student with knowledge of formal algebra might address by solving an equation. The research questions were as follows: What are the strategies that middle school students use to solve pattern problems involving perimeter and area? What are the relationships among the following: (1) the strategies middle school students use to reason about pattern problems, (2) the symbolic representations the students develop, (3) their interpretations of symbolic representations, and (4) their strategies for solving equation-evoking situations?

Theoretical Frameworks

The concept of quantitative reasoning as described by Thompson (1993, 1994) and Thompson and Thompson (1995) provided the theoretical framework for the design of this study. True algebraic power requires an awareness of underlying relationships and should be built on a foundation of quantitative reasoning (Thompson & Thompson, 1995). This reasoning is developed in situations where students reason about quantities and quantitative relationships, and use arithmetic notation to represent their reasoning (Thompson, 1993). The design of the study was also guided by Carey's (1991) study which found that the domain of numbers in problems influenced the problem-solving strategies used by children and that the use of alternative number sentences provided a revealing context for studying children's thinking.

Two models provided structure for describing children's thinking about the pattern problems in the study. The dual nature of Sfard's (1991) framework for describing the acquisition of mathematical concepts is particularly appropriate for the studying the development of algebraic thinking: the operational aspect relates to the modeling and skip-counting strategies used by some students in this study, and the structural aspect corresponds to the abstract strategies such as solving equations also used by some students.

The three stages of concept development described by Sfard (1991) provide a framework for characterizing the most frequently used strategies for evaluating symbolic expressions. Some students who were asked to decide whether a given expression described a particular mathematical pattern accomplished the task by substituting the numbers for every known figure into the expression, an operational approach suggestive of interiorization. Other students related alternative expressions to different characteristics of the physical models, a strategy suggestive of the condensation phase. Another group of students compared the given expressions to other student-invented or symbol-card expressions and stated that the pairs of expressions were either equivalent or not equivalent, in effect treating the expressions as objects separate from the

patterns which generated them. This strategy is suggestive of reification in respect to the identification of expressions which describe pattern situations. The findings of this study support Sfard and Linchevski's (1994) observation that reification appears to require the introduction of symbolic notation, possibly because compact symbolic expressions facilitate structural thinking.

The other model resulted from research into children's solution strategies on simple addition and subtraction problems (Carpenter & Moser, 1984). Like the first-, second-, and third-grade students solving addition and subtraction problems in Carpenter and Moser's study, students in this study modeled the problem situations, skip-counted, and sometimes used a more abstract strategy. For first- and second-graders, recalling number facts directly is an abstract strategy; among the seventh- and eighth- grade students in this study, application of an equation or an expression was the most abstract strategy used. Carpenter and Moser demonstrated that children may use a combination of strategies that over time gradually shifts from less abstract to more abstract strategies; specifically from direct modeling to counting, and eventually to derived facts or recall. Although the present study was not longitudinal, some students adopted increasingly abstract strategies as they worked through four pattern problems.

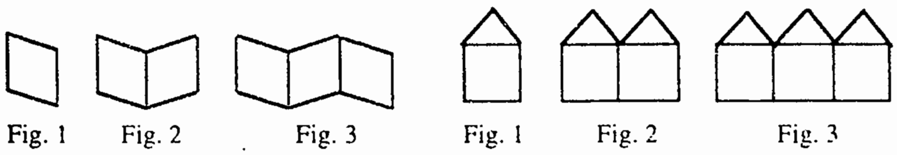
Methods of Inquiry

This study followed the naturalistic paradigm of qualitative research (Miles & Huberman, 1994; Stake, 1995) in the respect that the data were collected in a school setting and the intent of the research was to develop an integrated understanding of students' reasoning about pattern problems (Miles & Huberman, 1994). An emergent research design (Stake, 1995) was used because there exists no model for describing children's thinking about patterns and because it allowed for adjustments to accommodate information which developed in the course of the study.

Each student in the study participated in a two-part interview during which he or she engaged in four sequential pattern problems: three perimeter problems modeled with pattern blocks and one area problem modeled with tiles. For each problem the students were shown the first four figures of the sequence. Then they were asked to predict the perimeter or area for several figures from later in the sequence. (For pattern problems, see Figure 1.)

Students were also asked how they would tell someone to find the perimeter or area of the figure no matter what the number of the figure was, in effect a request for a generalization of the pattern. Then students were asked to write what they had said with mathematical symbols and numbers. Students were also shown a series of symbol cards and asked whether or not the expressions on them described the pattern in the problem. Finally, students encountered an equation-evoking situation for each problem in which they were given a perimeter or an area and asked to find the number of the figure. At every step students

Problem 1 [Perimeter = $2n + 2$] **Problem 2** [Perimeter = $3n + 2$]



Problem 3 [Perimeter = $2n + 3$] **Problem 4** [Area = $2n - 1$]

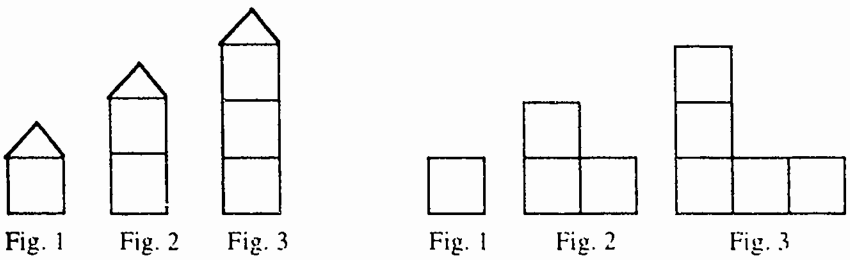


Figure 1. Sequential Perimeter and Area Problems

were asked to explain their reasoning. All written work completed by the students was collected and saved.

Data Sources

The interviews were audio- and video-taped and later transcribed. The transcribed notes and the students' written work were coded according to strategies used by the students, accuracy of outcomes, and implications for student understanding. The four sections for each problem were analyzed separately to determine the range of strategies and identify clusters of similar strategies, then data from the four sections of the problems were analyzed to identify relationships among the strategies for the four sections.

Results

The data suggest at least four clusters of thinking about the type of mathematical pattern used in this study. It appears that students' ability to recognize the critical relationships of a pattern and describe them symbolically strongly influences the strategies used to solve problems about the patterns. The ability to relate the elements of an expression to the physical characteristics of the

model also appears to support quantitative reasoning about the problem situation.

Model and Count. In the most concrete cluster, students modeled the figures and counted to find perimeter or area and to solve equation-evoking situations. When asked to generalize a method for finding perimeter or area, students in this cluster gave verbal directions for counting the number of block-sides in the figure. Their most effective strategy for identifying alternative expressions was substituting a number into an expression and comparing the results to a known perimeter or area. They demonstrated little or no awareness of the relationships between the number of the figure and its perimeter or area.

Single-Operation Relationship. In the second cluster, students demonstrated an awareness that there exists a relationships between the number of the figure and its perimeter or area, but they did not fully understand the relationship, expressing it in terms of only a single operation. These students might write an equation such as $3 \times n = p$ for a $2n + 3$ situation. Some students in this cluster did not recognize that the relationship incorporated two operations, while other students recognized that some aspects of the figure remained constant while others varied, but they appeared not to know how to express symbolically such a relationship; in both cases students incorporated a single operation into their generalizations. These students attempted to find perimeter or area and solve equation-evoking situations by applying a single operation in one case and reversing it in the other. Since their understanding of the pattern was incorrect, they were unsuccessful.

Consecutive Figures. Students in the third cluster perceived the patterns in terms of the relationships between the perimeter or area of consecutive figures. Students who recognized that the perimeter or area increases by one or two from one figure to the next often skip-counted to find perimeter or area and solve equation-evoking situations. Skip-count strategies were effective within a domain of small numbers but did not generalize well to all situations. In a situation where consecutive perimeters increase by 2, students in this cluster might write $x + 2$ where x represents the perimeter of the previous figure. Several students who used skip-count strategies to find perimeter and area later recognized the relationship between the number of the figure and the perimeter or area and adopted strategies consistent with the fourth cluster.

Appropriate Symbolic Expression. Students in the fourth cluster recognized the relationships between the number of the figure and its perimeter or area and expressed the relationships symbolically. Students in this cluster were more likely to explain their calculations in terms of the characteristics of the figures, and all but one demonstrated that they could manipulate the terms of the expression without reference to the figures on which they were based. Students in this cluster demonstrated that they could reason quantitatively in a manner which would support formal mathematics.

Conclusions

The results of this study suggest that problems of this type provide an appropriate opportunity to develop students' ability to reason about mathematical relationships and express them symbolically. One possible reason that this type of problem appears to promote the development of quantitative reasoning is that the relationships are accessible through a variety of number patterns and interpretations of the figures. Well-orchestrated classroom discussions could provide students opportunities to describe the pattern relationships in familiar language and then connect their language and understanding to symbolic representations of the relationships. A level of classroom discourse which elicits and explores a variety of expressions for each pattern situation is needed to provide adequate opportunities for students to develop an appreciation for the meaning of an expression. Special attention must be given to identifying and expressing the relationships among components in the pattern because those students who did not develop valid symbolic representations of the relationships were unable to use equations to answer questions about the patterns, and that powerful concept is critical for algebraic reasoning.

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THE RELATIONSHIP OF UNDERGRADUATES' BELIEFS ABOUT LEARNING ALGEBRA AND THEIR CHOICE OF REASONING STRATEGIES FOR SOLVING ALGEBRA PROBLEMS

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This study involved 25 undergraduate mathematics specialists who are planning to teach at the K-9 level. We examined both their beliefs about learning and teaching mathematics and their reasoning strategies. Findings suggest that students whose reported beliefs reflect an internal locus of mathematical authority and an importance placed on understanding in a mathematics learning community are more successful in developing their reasoning strategies.

Although it is widely recognized that there are several facets to school algebra (Bednarz, Kieran, & Lee, 1996; Kaput, 1995), it is also accepted that students leave their school experiences with a narrow perspective of what constitutes doing mathematics, particularly algebra (Kieran, 1992; Sfard & Linchvski, 1994). Beliefs about the learning and teaching of mathematics play an integral role in developing this perspective. Lampert (1990) and Schoenfeld (1992) found that students' beliefs about mathematics develop as a result of their classroom experiences, reflecting the perspective that mathematics is a fixed static discipline and that doing mathematics means memorizing, manipulating symbols, and identifying and applying the right equations. Further, Schoenfeld (1994) emphasized the importance of student belief about the locus of mathematical authority within the classroom culture. This authority, he noted, ultimately needs to reside "deeply in individuals and collectively in the mathematical community"; he believes that learning environments must develop "a community of mathematical judgment" that use standards to determine the veracity of their mathematical discourse (p. 62).

Although there have been studies of how beliefs impact the mathematical performance of teachers, the goal of our study is to investigate the relationship of college students' beliefs about the learning of algebra to their selection and quality of reasoning strategies for solving algebra problems. In this paper, we present information about what students perceive to be the locus of authority for the learning of algebra and discuss in what ways their beliefs may affect the reasoning they use to solve algebra problems.

Background

The mathematics course in which data were collected focused on developing reasoning and problem solving skills in algebra. Instruction assumed that

algebraic understanding is actively constructed by each student and includes the student being able to reason and make sense of algebraic situations and to communicate to others their sense-making and reasoning. The overall instructional pedagogy was to use problems as the main vehicle by which to develop this algebraic reasoning and sense-making and to help students become their own arbiters of the validity of their mathematical reasoning. Instruction focused on the use of problem situations for which expressions and equations could be assigned meanings from the problem. The ideas of "quantity-based algebra" (Thompson & Thompson, 1993) and "conceptual orientation" (Thompson, Philipp, Thompson, & Boyd, 1994) were important instructional components. The processes of exploring, investigating, conjecturing, explaining, and justifying were used consistently with students to help them develop their reasoning and problem solving skills in order to "deflect inappropriate teacher authority" (Schoenfeld 1994, p. 62). Social interaction among students and questioning from the instructor provided a framework for students to develop confidence in their mathematical reasoning. The importance of memorization of techniques and procedures was de-emphasized by allowing students to use all their notes on quizzes and exams.

Rationale

During the fall semester of 1995, two of the authors were team teaching two sections of this course. Reflecting on what was occurring in our classes, one of us noted that, on quizzes, students tended to mimic the reasoning process of other students or of the instructor which had occurred during the closure discussion of in-class problems. These students focused only on what calculations had been done, rather than relying on their own sense-making of the discourse. The other instructor, having assessed students' pre- and post-class beliefs about the learning and teaching of algebra, noticed that students who matured in their problem solving strategies more often believed learning was their responsibility and not based on careful teacher explanations.

Thus, our observations and assessments, combined with the robustness of resistance by many of our students to establish their own reasoning and understanding, suggested that we look at the belief systems of these students with respect to the strategies they selected as they proceeded to "do algebra." The decision to look at the role of authority when assessing mathematical correctness was an additional area to explore, especially when one student responded to the question "What is represented by this algebraic expression ?" by stating she didn't know, "This is how I was taught to do it in high school" suggesting that the authority of correctness rested in her high school teacher and not in her own reasoning and understanding.

Methodology

The subjects were 25 pre-service K–8 teachers, specializing in mathematics, at a midwestern university. Each had taken a standard one-semester mathematics course for K–8 teachers and were in the first course in a sequence of specially designed courses for mathematics specialists. Their self-reported backgrounds in high school mathematics are approximately: 25% with three years (two years of algebra and one year of geometry); 15% with the traditional four years, and 60% with 4 years including at least one additional college course in mathematics.

The data consisted of responses collected in the spring of 1996 from 1) a Survey on Students' Beliefs about Learning Algebra (SBLA) (Otto & Lubinski, unpublished), 2) the first quiz, 3) the final exam, and 4) final grades which in our classes reflect students' quantitative reasoning and communication of this reasoning both verbally and in writing. Pre- and post-responses to three of the five prompts on the SBLA were analyzed. These prompts were chosen because they provided the most information on locus of authority. The prompts are: Describe the role of an algebra teacher; Describe the role of an algebra student; and Describe a memorable algebra experience you have had. Three researchers categorized responses as reflecting an internal or external locus of authority or mixed. Discussions among raters continued until 92% agreement was reached.

Classroom Culture

The first problem assigned for the class to solve was the following:

> A bus travels up a 1 mile hill at an average speed of 30 mph. At what average speed would it have to travel down the hill (1 mile) to average 60 mph for the entire trip?

The intent of this problem was to establish a degree of cognitive dissonance to establish the importance of making sense of the use of symbolic expressions. (All students used, in one form or another, the representation for a simple average and got the incorrect answer of 90 mph.) Ensuing discussions of this problem were conceptually oriented.

On the sixth day of class, the following problem was introduced: "Is there a temperature that has the same numerical value in both Fahrenheit and Celsius?" The students knew the boiling and freezing points on each scale. No other information was provided. Discussion of this problem took parts of several class periods. Initially, the problem was solved computationally by subtracting multiples of 9 and 5, respectively, until a common value was found on both scales. Some students remembered conversion equations or developed them using point-slope methods. Then, they either graphed both equations to

find the common point or set "F" equal to "C" and solved using Algebra I procedures. Finally, the focus of the discussion turned to the development of the equations used to convert from one temperature scale to the other, redirecting the mathematical discourse from a computational orientation to a conceptual orientation. The emphasis was on making sense of the algebraic parts of an equation for conversion, such as $C = (5/9)(F - 32)$. Students were asked to explain what quantity was being described or represented by $F - 32$, then by $5/9$, and finally what quantity was being represented by multiplying the quantity $F - 32$ by $5/9$. A similar discussion ensued for the conversion equation $F = (9/5)C + 32$. Focusing these discussions on what was being represented, not on what calculations were being done, encouraged students to make sense of the algebraic expressions. This was done so that students could gain confidence in their own ability to judge the correctness of these formulas instead of relying upon previous memorization. The role of instructor throughout these discussions was one of probing and questioning students' responses with questions such as, "What number is being represented by that expression?"

Results

Students' Beliefs about Learning Algebra and Their Final Grades

Students' initial beliefs indicated general dependence on others for their learning; they stated that algebra teachers were responsible for providing examples, explaining material clearly, and teaching multiple ways to arrive at an answer. Being patient and having knowledge of the material were recurring themes. A typical response was, "To be able to explain algebra at different levels according to the learning ability of the student. The key to teaching algebra would be patience, because mainly students will want to block algebra out and some will just not comprehend it at first." Reported beliefs about students were that they were to listen, watch the teacher, ask questions, and practice (problems similar to examples presented in class). One student wrote, "Should listen and watch the teacher when they are learning new areas for the first time. Ask questions if they are lost and don't understand something." A recurring theme was to be open minded. Initially, memorable algebra experiences referred to getting good grades, having a teacher that explained the material well, developing understanding (procedural), and having fun.

At the end of the semester, more responses than at the beginning reflected an internal locus of authority. For example, one student wrote, "The role of an algebra teacher is to NOT give answers, but to let the students figure out things on their own. . . The algebra students need to contribute to problems and class discussions to not only help themselves learn, but to help others learn as well. This is difficult to do if the work is not done before class." Nine students

indicated that this class was most memorable. One student said, "I believe that it was this semester working on our quizzes in our group. It made me proud that we worked very hard on explaining, to our best, our answer. I was very excited when I was able to find an answer for the story problems. . . I can actually spend hours on end trying to find an answer and be able to explain how I derived it." Responses such as these reflect an internal locus of authority.

We looked at change on a continuum from external to internal locus of authority and classified students as either external, internal, or mixed on each of their three responses to the SBLA prompts. Then, we noted whether they focused on understanding in their end-of-the-semester response involving their most memorable experience. Initially, all students' responses were classified as external for the prompts about the role of the algebra teacher and the role of the algebra students. At the end of the semester, all students made some change towards the internal end of the continuum: ten were still external; seven were mixed; and seven were internal (one student's data were unavailable).

In regard to a memorable experience, students were initially classified as 15 external, four mixed, and six internal. At the end of the semester, six students were classified external, six mixed, 11 internal, and two uncodeable. Of the 11 students classified as internal: three received As, four Bs, and four Cs; nine mentioned the most memorable experience as being this class; and nine focused on the importance of understanding in their end of semester responses. It is important to note that the majority of students receiving Cs and Ds did not mention this class as a memorable experience nor the importance of understanding; their responses tended to emphasize a focus on procedures and three of them mentioned experiencing math anxiety. Of the 25 students, there were three As, eight Bs, nine Cs, and five Ds.

All students changed from external toward internal locus of authority on the continuum in regard to the prompts on teacher's role, student's role, and memorable experiences. We classified these changes as either minimal, some, or great. Of the three students who received As: one had a minimal change and two had some; and all noted the importance of understanding and the impact of this class. Of the eight B students: changes were minimal for three, some for three, and great for two; four noted the importance of understanding; and three realized the impact of this class on their thinking about mathematics learning and teaching. Of the seven Cs: 1 had minimal change (mentioning both the importance of understanding and this class), five had some, and two had great change. Of the four Ds: three had minimal change and one had missing data.

First Quiz and Final Exam

The following question appeared on the first quiz:

In order to work with new chemical compounds in a laboratory, it is necessary to develop two new scales, named "Up" and "Down" for

measuring temperature. In one room the Up scale is showing a temperature of 52 degrees while the Down scale is showing a temperature of 61 degrees. In a second room the Up scale is showing a temperature of 132 degrees while the Down scale is showing a temperature of 201 degrees. Develop a formula for converting from the Down scale to the Up scale.

Slightly more than half (13 out of 25) of the students obtained a valid conversion equation. Two of the 13 students used information about slope and equations of lines, but provided only calculations with minimal explanations; they earned As in the course, having shown some change in beliefs, recognition of the importance of both understanding and this class. No student referenced the need for a constant rate of change to employ the equations. One student mimicked what was done in class by using the original data, but did not provide a coherent explanation for what was being represented by the algebraic expressions; this student received a final grade of C, showed some change in beliefs, but never reflected an internal locus of authority in responses, and mentioned neither understanding nor the impact of this class.

Ten of the 13 students appeared to have chosen a strategy from the problem situation resembling the conversion of Fahrenheit to Celsius by subtracting multiples of seven and four until the Down scale had a 0, or 0 on the Up scale for those doing the reverse direction. Their choice of this strategy is appropriate, but indicates an unwillingness to use the reasoning that developed from class discussions. These students did not recognize the need to still provide a meaningful explanation for their algebraic expressions. Several students provided no or minimal explanations, appearing to just track the numbers by directly mapping to numbers used in class solutions. Some provided explanations that caused one to doubt their understanding. Some explanations emphasized what operation was being used rather than what the result of the operation represented, reflecting a computational orientation. No student provided a complete explanation. Six of these students received As or Bs in the course; one earned a D.

The work of students who did not get the right equation ranged from virtually nothing to explanations that lacked meaning. A couple of attempts to use quantitative reasoning on the original data met with little success. Most of these students received Cs or Ds in the course. Two of the three that received a B made great changes in their beliefs and one realized the importance of understanding.

On the final examination the following problem appeared:

Scientists are using two different temperature scales to conduct experiments on a newly discovered chemical compound. For simplicity the

two temperature scales are called the A and B scales. On the A scale, water boils at 173 degrees and on the B scale water boils at 152 degrees. It is also known that an 8 degree change on the A scale corresponds to an 11 degree change on the B scale. Develop a formula for converting from the B scale to the A scale. (You may not use formulas from Algebra 1 and 2. If in doubt, check with me.)

Of the 25 students who took the final exam, only seven presented any solution that had substance. The other 18 students did little more than find the rate of change between the two scales, an expected strategy since they had access to their notes. Of particular interest was the attempt by several of these students to incorporate 21 into their explanation without stating what the 21 represented (the difference between the two boiling points). Little attempt was made by these students to employ any reasoning about the quantities involved. One student perhaps best described her situation when she wrote, "I tried so hard to understan and make sense of this problem, but I just could not do it." The majority of these 18 students received Cs or Ds.

Of the seven students who provided a correct equation, only one tried to assign meaning to the equation using the original data, but her explanation focused on calculations rather than on what the expressions represented. This student had done poorly on quiz one, made great change in her beliefs, did not write about the class being memorable, but did note the importance of understanding. Six students adopted the quiz strategy of adjusting corresponding points on the two scales until the B scale reached 0 and then mimicked the strategy of converting from Celsius to Fahrenheit. As with the first student, their explanations focused on calculations rather than on the meaning of algebraic terms; however, their explanations were clear and relatively complete. With the exception of one, all of these students received an A or B and all used conversion equations with weak explanations on quiz one.

Discussion

As we review our data, we realize the students overall are not reflecting the conceptual orientation we strive to achieve within the classroom culture; however, we recognize the progress they have made towards developing an understanding of mathematics that they had never before realized possible. We concluded that those students whose beliefs changed to reflect a more internal locus of authority, who placed an importance on understanding, and who acknowledged the impact of this class to their learning developed their ability to explain their reasoning strategies (as reflected in their grades). These connections are more clearly recognized as we looked at where these students were at the beginning (first quiz) and end (final exam) of the semester in relation to their reasoning strategies.

As students of mathematics, many of these students would be considered "calculus ready." We may ask how their computational learning style will impact their performance in higher level mathematics. They might bring the type of manipulative focus described by White and Mitchelmore (1996) to these classrooms; further, the issue of developing a conceptual orientation in regard to their reasoning strategies might be compounded if their locus of authority is external.

As teachers, we expect them to teach in a manner consistent with reform recommendations. Thus, they must be able to make sense of the mathematics themselves and believe that it is possible for their future students to decide on the correctness of their own reasoning. Our results suggest that this process will not be easy. It will be equally as important for them to change their beliefs as to experience doing mathematics.

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TEACHERS' BELIEFS AND STUDENT FAILURE IN ALGEBRA

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Previous studies imply that algebra teachers' beliefs may affect instructional practices and student learning in algebra. The two summer school algebra teachers in this study believed that their success at helping students learn algebra was largely due to their ability to relate with their students and to foster a relaxed yet motivating classroom environment. Teacher practices for dealing with failing students were consistent with their beliefs about what caused the failure, and sometimes had undesirable consequences for students. Beliefs about algebra teaching and learning were consistent with instructional practices. Both teachers focused on teaching procedures and calculations, and believed that students' mistakes were best corrected by warning students of commonly made errors and showing them the correct steps to use while doing the assigned exercises. Their heavy emphasis on procedures may have led their students to develop an instrumental understanding of algebra.

Research pertaining to the teaching and learning of algebra has often focused on teaching practices and student thinking to explain why many students have great difficulty learning algebra (see Kieran, 1992). One explanatory variable that has frequently gone overlooked in this body of literature is teachers' beliefs. There are many ways that teachers' beliefs might affect student learning, perhaps the most well-documented one being the relationship between teachers' beliefs and instructional practices (Thompson, 1992). Thompson (1984) found that teachers' beliefs and conceptions concerning what constitutes mathematics and mathematical activity were fairly consistent with their instruction. This suggests that teachers' beliefs about algebra and algebra teaching and learning may affect their instructional practices, which in turn can impact student learning.

Teachers' beliefs about their students may also affect instruction and student learning. In his observations of a first year high school mathematics teacher, Gregg (1995) noted two beliefs Ms. Weston, the teacher in his study, held that might have contributed to student difficulties in learning mathematics. She believed that certain students in her classes had limited abilities for learning mathematics, and that other students were willfully refusing to learn. This led her to give up on some of her students, which significantly altered the attention and instruction they received. These two beliefs may play a prevalent role in algebra learning and teaching, because many students who struggle with algebra might be perceived by their algebra teachers as fitting into at least one of these two categories.

In this paper, I will describe a study I conducted to examine how algebra teachers' beliefs about their students, algebra, and algebra learning and teaching were related to their instructional practices. By doing so, I hope to suggest ways that teachers' beliefs might influence student learning in algebra.

Method

Two high school mathematics teachers from an inner city high school in Southern California participated in this study. Both teachers were teaching algebra units from integrated mathematics courses during summer school. I employed ethnographic methods of data collection, which included a week of classroom observations and three or four interviews with each teacher, all of which were conducted during the third and fourth weeks of summer school. Data collection during classroom observations consisted mostly of field notes, although two-thirds of the observations were audiotaped. I occasionally referred to these tapes to supplement my notes. The interviews were audiotaped and transcribed in full. During these interviews, I not only probed teachers directly about their beliefs concerning their students and algebra teaching and learning, but also questioned them about instructional practices that I had seen in their classes. I hoped that by framing questions within the context of their teaching, I would obtain a more accurate report of their beliefs, since general questions or hypothetical situations might lead the teachers to voice beliefs about an "ideal" or "non-real" situation.

My beliefs about the nature of knowledge and the learning and teaching of algebra guided my observations, data gathering, and analysis. First, my acceptance of radical constructivism led me to recognize both the teachers and myself as active constructors of knowledge. This in turn implied that the teachers and I most likely had different perceptions of what happened in their classrooms. As a result, I faced the challenge of attempting to view classroom situations from their perspectives, while at the same time realizing that at best I would only achieve a "good fit" between my explanations and the data I collected, not an exact account of their perceptions. Second, the realization that students also constructed knowledge led me to assume that teachers understanding their students' thinking was a critical part of teaching algebra. Consequently, I looked for ways that teachers accessed and responded to student thinking in their classes. Third, I valued mathematical understanding that included not only the ability to perform calculations, but also the knowledge of when and why to perform them. This type of understanding is often referred to as relational understanding (Skemp, 1978). Discourse in the classroom that focuses on conceptual understanding rather than calculations is an important means for helping students develop relational understanding (Thompson, Philipp, Thompson, & Boyd, 1994). Fourth, Kieran (1992) identified several concepts and mathematical structures, such as variables and functions, that students usu-

ally study formally for the first time in algebra. This led me to search for evidence that teachers were sensitive to students' struggles with these new concepts and for teachers' beliefs concerning what teaching practices were best for helping students acquire them.

Results and Discussion

At the time of the study, Peter and Jason, the names by which I shall refer to the teachers in my study, had been teaching high school mathematics for 3 and 25 years, respectively. Peter's major and Jason's minor in college were mathematics. Both teachers confided to me that they were covering the same units, teaching the same lessons, using the same tests, and assigning the same homework problems as they had during the regular school year. This was done in part to reduce their workload of having to teach three or four different lessons each day. At least two-thirds of their summer school students were retaking their courses, having failed the first time. Nonetheless, both teachers reported that the percentage of students who were passing their classes halfway through summer school was much higher than the district pass rate during the regular school year.

Beliefs About Social Relationships In The Classroom

Both Peter and Jason believed that a large part of their success in teaching summer school could be attributed to their ability to relate to their students. They felt this was an important factor in motivating students to learn. Peter was particularly concerned about being perceived as a "traditional" mathematics teacher:

P. I'm probably different. . . . Like I wouldn't want to be a traditional math teacher.

I. Where do you see yourself as being different?

P. Uh, I don't know. . . . I'm just a lot different, I think, than the math teachers that I had. . . . Maybe [I'm] more understanding. . . .

I. What do you mean by understanding?

P. Uh, understanding that . . . it's OK to make a mistake. . . . where maybe other people would say, "No, that's not how you do that!"

Peter's use of the term understanding seems to refer to his sensitivity to students' feelings, an interpretation I find consistent with my observations in his classroom. Peter related very well to his students. He had a lot in common with them, and often capitalized on shared interests, such as using a song lyric from a popular song to catch the attention of his class. I enjoyed being in Peter's class because of the relaxed and respectful atmosphere there. Students took turns speaking, asked permission to leave their seats, said "bless you" when someone sneezed, spoke in quiet voices, worked diligently during independent

practice, and seemed genuinely pleased to be in class. Peter acknowledged that he pushed students to "be nice" and complete their work. It was a common occurrence in class to hear him say things like "Ladies and gentlemen, please be" (i.e., please be ladies and gentlemen), or "Maria, I'm missing your exploration write-up."

Jason also acknowledged the importance of being sensitive to students' feelings and interests. However, he attempted to relate to students in a different way than Peter:

> J. I try to make it a personal thing. . . . I try to think back on my childhood. Why was I afraid to screw up someplace? Because I knew that . . . my parents would be disappointed in me, and they would let me know they were disappointed in me, and that would hurt me. And I know my kids feel the same way about me. Now these are not my own children, but I can use a lot of the same theory on them.

Student-teacher relationships in Jason's classroom were also respectful and relaxed. Jason seemed very approachable, and actively encouraged questions from his students. He allowed students to talk with one another at almost anytime during class, including lectures. He permitted these discussions so that students could engage in "peer tutoring," which he said he hoped would compensate for the lack of organized group work in his class. Consequently, there were often many student conversations taking place in Jason's class at any given moment. Jason tried to keep track of those who were not working, and would "cajole" or "prod" them to get back to work.

Beliefs About Student Failure

Peter had two failing students in his class of 30. When I asked him why they were failing, he responded as follows:

> P: [They] just didn't do any work. I don't think the work is too fast. It's not that it's so challenging that they can't do it. Well, it's challenging that they have to keep on working. Some people aren't used to working.

Peter seemed to feel that failing students in his class did not have appropriate work habits. However, it did not appear that he felt they were willfully choosing to fail his course, like the teacher Gregg (1995) observed. When I asked Peter what he did to help these students, Peter responded that he gave them opportunities to make up missing assignments. I also noticed that he would occasionally walk over to these students during independent practice, wait by their sides until they took out a sheet of paper, and then talk them step-by-step through the first one or two problems, telling them what to write down as they went along. Peter's belief that student failure was caused by poor work habits led him to treat failure as a motivational problem.

Five of Jason's 33 students were failing his class. Jason felt that one of these five students was not getting adequate rest, and that this was the cause of her poor performance. Jason described the remaining four as follows:

> J. There's another couple of kids that I haven't really gotten to know that well yet. But there are a couple that...probably don't know their times tables. They probably don't know their addition facts...They've probably struggled all their lives in math....We could talk about right brain left brain. We could talk about the creative types. But there are a couple that...seem to struggle.

Jason's comments seem to indicate that he believed that some of his students had a limited capacity for learning mathematics. Jason said that when he first began teaching, he felt he could teach anyone anything. Early experiences in the classroom convinced him that some students could not learn, that "a dense fog" surrounded their brains. Even so, Jason still believed that every student should be able to pass his class. Jason indicated in an interview that the way he helped failing students was by reviewing material and providing one-on-one help. During my observations, however, I saw only one instance where Jason provided a failing student with individual assistance. Reviews were also infrequent, usually taking place only before an exam. My observations yielded no indication that Jason had altered his instruction in any way so as to specifically address the needs of the failing students. Whether intentional or not, such action was consistent with his belief that these students could not learn mathematics regardless of how hard he tried to teach them.

Beliefs about Teaching and Learning

Both Peter's and Jason's instructional activities were traditional in nature. Peter frequently lectured, providing students with step-by-step procedures they could use to do the exercises in the text. While Peter occasionally used the discovery learning activities in the book to introduce new topics, these were done as whole class activities, where Peter performed the task at the front of the class and students copied at their desks. Peter felt this was the best way to engage students in these hands on activities: "And it was real neat, too. . . . As we did it together, they could check their work, 'cause their work should have been the same thing that we had on the board." Similarly, Jason also based his instruction on lectures that focused on learning procedures so that students could do the exercises in the problem sections. In both classes, very little time was spent on explaining why the procedures worked or why they were used, nor did my discussions with the teachers indicate they valued this type of activity.

Peter and Jason's emphasis on procedures and computations was consistent with their beliefs about what difficulties their students had with algebra and what they should do to help their students. Both teachers seemed to be well aware of the assignments or sections their students would find difficult; they

would frequently warn their students about common mistakes or tell me about the procedures their students struggled with. However, at no time during the interviews did either teacher exhibit an awareness of the relationships or structures in algebra that might have required their students to develop new conceptions or modify existing ones. Instead, their talk focused on the wrong steps students performed. This emphasis on procedures was compatible with their belief that student mistakes could be corrected by showing them the proper procedure or forewarning them of common errors. It also very likely served as a barrier to their students' development of a relational understanding of mathematics.

Conclusion

Both teachers in this study were very successful in establishing a classroom environment where students felt relaxed and which motivated students to do their work. Given the background of the students and the school setting, this in itself may be viewed as a major accomplishment, and perhaps one of the biggest reasons for high student pass rates in their classes. The teachers' methods (or lack thereof) for helping failing students seemed to be consistent with their beliefs about what was causing their students to fail. In particular, Jason's belief that some of his students were failing because they lacked the ability to learn mathematics was accompanied by no apparent attempt to assist these students. Their emphasis on procedures and computations most likely led their students to develop a "rules without reason" (Skemp, 1978, p. 9) understanding of algebra, a type of understanding analogous to knowing how to navigate the streets of a city by only a few well-known routes. Students who have this understanding of algebra may have a difficult time navigating the mathematical cityscape in their next mathematics classroom, especially if it offers variations of or innovations on the procedures they learned over the summer.

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ALGEBRA I STUDENTS' IDEAS ABOUT COVARIATION

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Covariation, the relationship between changes in one quantity and simultaneous changes in another, is viewed as a pivotal concept as students develop their ideas of algebra (Harel & Confrey, 1994). For example, when considering an increasing quantity, students use ideas of covariation to determine if another quantity is increasing, decreasing, or remaining constant. The purpose of this study was to explore students' understanding of covariation of linear functions through the use of graphical, tabular, and verbal representations. Specifically, we were interested in the following questions. What ideas do Algebra I students have concerning the concept of covariation? How stable are these ideas?

Algebra I students from grades 8 and 9 completed a 30-minute written assessment comprised of nonstandard, open-ended items involving translations among graphs, tables, and words. The participants' responses were analyzed in a qualitative manner based on multiple sorts of the data. Four themes of covariation emerged from the data analysis - dependency, linear patterns of covariation, generalizability of patterns, and multiple patterns of covariation. (Algebraically, multiple patterns of covariation are represented by piecewise-defined functions.) Only one-third of the participants indicated stable understanding of dependency, almost two-thirds of the participants indicated stable understanding of linear patterns of covariation, and about one half of the participants indicated stable understanding of generalizability of patterns. A major finding of this study was that the majority (80%) of participants demonstrated stable understanding of multiple patterns of covariation, even though students in precalculus and calculus often have difficulty with piecewise-defined functions. These data suggest that perhaps a curricular approach which enabled students to build on their previous knowledge of graphic, tabular, and verbal representations of multiple patterns of covariation would enhance students' understanding of piece-wise defined functions in precalculus and calculus courses.

COLLEGE ALGEBRA: STUDENT SELF-PORTRAITS

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This pilot study examined the mathematical beliefs and conceptions of College Algebra students. Subjects came from College Algebra classes at two Southern universities. The study addressed the following research questions: (1) What are students' beliefs regarding the purpose, setting, and content of a College Algebra course? (2) How do students' general beliefs about mathematics influence their performance in a College Algebra course? (3) What mathematical conceptions do students develop during a College Algebra course?

A mathematics beliefs and attitudes survey developed by Yackel (1984) was administered to the College Algebra students. In addition, 20 students participated in a series of individual teaching interviews, which occurred bi-weekly and lasted about 40 minutes each. The interviews included a session during which the student solved a set of learning tasks; in addition, the student was allowed to introduce his or her own problems and questions. Since both researchers also served as instructors, the interviews provided them with on-going feedback for teaching the classes. Data sources included videotapes of the interviews, the researchers' field notes, the students' written work, and transcripts of the tapes. Case studies were developed from the data analysis.

The results indicate that College Algebra students hold rigid beliefs about mathematics and the role it plays in their lives. While many students were familiar with the content of College Algebra, their belief that they already knew the material was not borne out in subsequent interviews and tests, nor were they convinced of its utility in either their everyday lives or with regard to their academic majors. While it was expected that our interviews with students would reveal their "mis-conceptions", other interesting results were found among academically successful students. While these students achieved levels of competence adequate to pass the course, their interviews indicated that they held, at best, fragmented understandings about algebra concepts.

Our future work will focus on developing action-based models of student belief systems, that will help explain the students' deep-rooted "misconceptions". We will also consider how the use of technology impacts the beliefs that students bring to a College Algebra class. We believe our work will yield novel approaches for working effectively with future College Algebra students.

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THE MEANING OF ALGEBRAIC VARIABLES: AN EMPIRICAL STUDY WITH STUDENTS AGED 16-18

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This work is part of a wide theoretical-observational study performed with students from approximately 17 years of age, who studied the second year of high school in Mexico City. It is inscribed in the inquiry about the construction of meaning of the lineal equations with two unknown values, inaugurated by Herscovics (1980), and tries to establish bonds among the operational uses of the algebraic variables with superior order uses: those we have acknowledged as analytical uses of algebraic variables, and which take place in the tasks of translating from geometric field to the algebraic one.

We observe the type of thought that is unfolded when the student is asked to obtain the equation of a given straight line that does not cross the (0,0), inasmuch as the Cartesian plane has been introduced and a geometrical meaning of the slope of straight line given.

Depth observations were performed with three cases in clinical interview. The analysis of what occurred in the interviews was carried out based on the actions performed by the students during the join resolution between the interviewer and each of the selected students, each of whom worked on the following problem: "Find the equation of the circumference that crosses point (6,2), and that is tangent to the straight line $2x + y = 16$ in the point (8,0)". Any of the possible analytical solutions will demand that the student face repeatedly the production of different lineal equations. Such production constituted the empirical motive of the research presented here.

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CONTENT ANALYSIS OF ALGEBRAIC WORD PROBLEMS FOR GENDER-BASED REASONING PATTERNS

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The question of how to enhance student conceptual understanding of mathematics drove this research. For years, Perry's (1970) stages of male intellectual development served as the baseline for mathematics curriculum. The Belenky, Clinchy, Goldberger, and Tarule (1986) research noted that female cognitive development differed in the number of stages and steps from Perry's model. Based on these findings, this research looked into whether female students formulated their mathematical concepts into reasoning patterns that differed from the established male standard.

This field studies design employed content analysis to explore reasoning characteristics for gender-based patterns. The researcher took five female, five male, and two common reasoning characteristics from Perry's (1970) and Belenky's et al. (1986) early stages of intellectual development. Expert mathematics educators identified which gender was the primary user of each characteristic. The twelve reasoning characteristics were used for the content analysis coding.

The subjects were forty high school students: twenty male and twenty female students between the ages of 15 to 18, taking a third year of required mathematics. The subjects lived in a middle class, white, suburban neighborhood in southwestern Ohio. During the one hour interview, each subject used the prescribed "think aloud" protocol while working on two algebraic word problems.

The data analysis revealed three observations. First, expert mathematics educators placed the twelve reasoning characteristics into gender groups that differed from the single gender studies. Secondly, the subjects used only three reasoning characteristics at a significantly different rate based on gender. Third, female subjects spoke almost twenty-four percent more phrases than the male subjects. Reasoning characteristics used by male and female students did not differ greatly, but teachers' perceptions of the reasoning characteristics preferred by each gender did.

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GENERALIZATION OF PATTERNS AND RELATIONSHIPS BY PROSPECTIVE ELEMENTARY TEACHERS

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Prospective elementary and middle school teachers' ability to generalize algebraic patterns and functional relationships were investigated using open response written items and interviews. The mathematics content domain analyzed were arithmetic sequence, direct variation, linear relationships, geometric sequences, and inverse variation. Subjects were asked to describe and represent contextual problems with an equation and to solve problems involving specific cases. Results show that the majority could generalize linear relationships and arithmetic sequences. However, these prospective teachers experienced greater difficulty when asked to represent problems involving inverse variations and geometric sequences. Sfard's model of conceptual development was used a framework for examining students' understanding.

A HYPERTEXT RESOURCE FOR THE TEACHING AND LEARNING OF SCHOOL ALGEBRA

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Examinations of the current reform movement's attempts to change the nature of mathematics teaching suggest that the proposed, "new" ways of teaching require a high level of teacher subject matter knowledge (e.g., Ball, 1996). This project explores the potential of hypertext documents as a resource for teachers seeking to develop their subject matter knowledge and change their instruction. Situated in the domain of school Algebra, the materials provide teachers with oppportunities to understand how reformers are reconceptualizing Algebra as an intellectual field of study which can be explored through a variety of coherent paths, rather than a series of discrete, hierarchical skills to be mastered. This reconceptalization of Algebra views the x's and y's of Algebra as a representation of relationships between quantities and emphasizes the use of other representations as well, like tables, Cartesian graphs, diagrams, gestures, and everyday language (e.g., see chapters in Romberg et al., 1993 or Bednarz et al., 1996).

Central to the structure of the hypertext environment are tasks designed to support student exploration and whole group discussion. The materials also include links to essays on the mathematical content of Algebra, teacher-written narratives about the use of these tasks, and rich annotations of the problems. Annotations include typical student strategies, purposes of the tasks, reasons for particular wordings, and more. The tasks can be browsed in a course/topic mode which presents paths that particular teachers have chosen with specific classes and a mode which views the curriculum as an examination of four sorts of operations on representations of functions.

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Assessment

MANDATED ASSESSMENT INSTRUMENTS: HOW DO TEACHERS VALUE THEM?

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A large Southeastern school district designed a reform-based curriculum and created mandated assessment instruments to reinforce and drive the curriculum. Although the primary goal of introducing the assessment instruments was to inform teaching, teachers' focus on the numerical score affected their perceived value of the instruments. Using the instruments compelled teachers to adhere to the curriculum more closely but also diminished some teachers' sense of autonomy. Teachers' concerns focused on technical aspects of the instruments such as the amount of time required to administer and score them. However, teachers who valued the pedagogical usefulness of the instruments reported fewer technical problems.

Assessment for a Standards-Based Curriculum

In the reform atmosphere permeating discussion of mathematics education there has been an evolution in the thinking of many mathematics educators and researchers away from a reductionist view of mathematics to a cognitive approach (Putnam, Lampert, & Peterson, 1990), wherein children are seen as active constructors of their own mathematical knowledge. Consistent with this approach to learning, the NCTM *Standards* (NCTM, 1989) recommends that teaching focus on the development of conceptual understanding, reasoning, and problem solving. This constructivist theory (Cobb & Bauersfeld, 1995) framed the study.

Teachers' assessment of student understanding should be aligned with the philosophy implicit in a curriculum (NCTM, 1989, 1995.) Although substantial research on assessment is currently being conducted, implementing alternative assessment techniques in the classroom poses problems not encountered in research settings (Marshall & Thompson, 1994). When teachers begin using alternative assessment methods, they report loss of predictability and increased demands on their time (Cooney, Bell, Fisher-Cauble, & Sanchez, 1996). More importantly, if teachers' beliefs and attitudes affect the way in which they implement curricula and interpret assessment results (Thompson, 1992), then teachers who view mathematics as rule-driven and who believe children learn through passive absorption of discrete bits of information are likely to encounter difficulty adjusting to a *Standards*-based curriculum.

In an effort to reform the teaching of elementary school mathematics, a large Southeastern school district created a curriculum, *Continuous Achievement*, to implement the goals of the NCTM *Standards* (NCTM, 1989) and to

meet the needs of a diverse student population. The curriculum promotes development of conceptual understanding, reasoning and problem solving skills and is designed to actively involve students in "doing" mathematics by requiring them to interact with physical objects and other students. *Continuous Achievement* was developed by a team of curriculum specialists, called Instructional Resource Teachers (IRTs) and teachers from within the county under the direction of the County Coordinator for Mathematics. The knowledge and skills expected to be learned at each grade are divided into three levels and students are grouped by level of instruction so that each student can progress at his or her own pace.

After the curriculum was implemented throughout the district, assessment instruments, called *Cumulative Assessment*, were created to promote the use of the curriculum and to inform instruction. Teachers are required to administer these instruments before the students can advance to another level.

The assessment instruments are closely tied to the curriculum, both conceptually and contextually. In the assessment process a student may be required to model an addition problem using Dienes blocks or to draw and label a bar graph from data. Many questions are open-ended and grading is partly based on teacher observation. The scoring rubrics provide general guides but permit the teacher to use his or her own discretion in determining a "reasonable" answer. The IRTs informally assist teachers with curriculum and assessment decisions.

Purpose Of The Study

Continuous Achievement and *Cumulative Assessment* represented a dramatic shift away from more traditional curriculum designs and tests. A change of this magnitude can have a profound impact on teachers, both in their perceptions of their role as teachers and their understanding of the mathematics of their students. We focused on the impact of *Cumulative Assessment* on teaching in the elementary grades and investigated the following questions:

1. In what ways do the teachers' views of the role of the mandated assessment instruments reflect or diverge from those of the Curriculum Coordinator and the IRTs?
2. In what ways do the teachers use the mandated assessment instruments to inform instruction?

The research was not designed to provide a quantitative analysis of teachers' reactions to the *Cumulative Assessment* or to measure the impact of the assessment instruments on teaching. Rather, our objectives were to begin to understand whether teachers' views and goals coincided with those of the designers of the instruments and whether the instruments did, indeed, inform instruction.

Methodology

In the spring of 1995, semistructured, 45 minute audiotaped interviews were conducted with the County Coordinator for Mathematics, four Instructional Resource Teachers and ten elementary teachers. The IRTs were from different socio-economic areas and had been identified as people who were concerned with assessment and who would be interested in the study. In turn, the IRTs were asked to schedule interviews with teachers who were evenly distributed over grade level, had a wide range of teaching experience and who would be likely to provide a range of attitudes and impressions about the assessment instruments. The opinions of the teachers were not known to the interviewer in advance.

The interviews focused on the teachers' methods and attitudes toward administering and scoring the assessment instruments, the teachers' perceived value of the assessment instruments, and the changes in instruction that had been expected to occur or did occur as a result of using the instruments. The interviews were analyzed using analytic induction, which involves scanning the data for themes and relationships, developing hypotheses, and modifying them on the basis of the data (LeCompte, Preissle, & Tesch, 1993).

Results

Question 1

According to the Curriculum Coordinator, one of the goals of the assessment instruments was to promote the goals of *Continuous Achievement* and to encourage the teachers to use the curriculum. She indicated that some teachers had continued to use the old textbooks in a lockstep manner despite the fact the county had adopted their own *Continuous Achievement* program. Miriam[1] (IRT) pointed out that the assessment instruments compelled the teachers to adhere to the curriculum more closely because it is "hard to test something you're not teaching." Patricia (teacher) said that some teachers actually learned what was in the curriculum when they used the assessment instruments. Anne (teacher) admitted that the assessment instruments helped her follow the curriculum because she had to make sure she "covers all the objectives." However, Elizabeth (teacher) felt that she was "not trusted" as a teacher to use her own assessment methods with her students. Deborah (teacher) stated that "the reason we're giving these tests is because Big Brother is looking over our shoulder and not thinking that we're doing the right thing," although she also admitted that "the truth is, there are plenty of teachers who aren't doing what they are supposed to be doing in the classroom."

[1] All names are pseudonyms.

The Curriculum Coordinator reported that although teachers were required to report the scores generated by the assessment instruments, the primary goal of assessment was to inform teaching. However, three teachers mentioned their concerns about the numerical scores. For example, Angela (teacher) questioned the accuracy of a score that requires so much subjective judgment and the validity of a score that is generated from cooperative effort. She thought that the amount of guiding that she does when she administers the instruments compromised the validity of the scores. She was adamant that the instruments provided valuable feedback about her students but because the students worked cooperatively, she felt that the scores were not a valid measure of their mathematical ability. To her, tests should be a means of "finding out what kids can do independently."

The developers of *Cumulative Assessment* felt that the assessment process itself provided opportunities for student learning. However, at least two teachers indicated that they thought that assessment and instruction should be mutually exclusive activities. That is, if they are not mutually exclusive activities then teachers would not feel that time spent on assessment was bought at the price of instructional time. For example, Elizabeth (teacher) did not seem to think that the time students spent on Cumulative *Assessment* was as valuable as the regular class time because she mentioned the instructional time that was "lost" due to the time spent on assessment. Tina (teacher) was aware of instruction that occurred in the assessment process, but thought that too much instruction invalidated the assessment scores. She admitted that when she administered the instruments she "kind of reteaches it" but said that tests should be a measure of whether children have "mastered" the material.

Question 2

Cumulative Assessment was designed to inform instruction. Wendy (teacher) said that she learns which children can work independently and became aware of how much she was helping them in nontesting situations. Learning observation techniques necessary to administer the assessment instruments have helped her to become a better observer in her teaching. She also said that she has changed her teaching of problem solving strategies because of the questions that are on the assessment instruments.

Although Angela was uncomfortable with the scores she recorded she said that the instruments are a "wonderful tool" for informing instruction because they test students' problem solving ability and require "real high-level thinking skills."

All but one of the teachers felt that the assessment instruments were similar to their informal assessment activities and they expressed mixed views about the value of the instruments, depending on whether they felt the instruments were merely redundant or a verification of their observations. Virginia (teacher)

said that although she "probably knows beforehand" how her students will do, the instruments help her decide what to emphasize in her teaching. Elizabeth (teacher) said that she receives no new information from *Cumulative Assessment.* She said, "I've been assessing all along, doing thumbs up, thumbs down," and Marie said that while there was some value in the assessment instruments they didn't warrant the time spent on them. Referring to an item that requires students to count money, Marie said, "it takes a lot of time for each child to count out coins when I've seen them count out coins fifty times and I know whether or not they can do it." Two teachers said they appreciated having the written documentation the instruments provided for communication with parents. Another two teachers admitted that if they were to spend more time analyzing and reflecting on the results of the instruments they would derive more benefits from them.

Teachers' immediate concerns were centered more on the instruments themselves. Every teacher in the study mentioned the length of time it took to administer the instruments. There were also comments about the mechanics of test administration such as having to collect the materials and the difficulty of assessing students in groups rather than individually. There were also concerns about scoring consistency.

Upon reviewing the teachers' comments it became apparent that teachers who felt that the assessment instruments had strong pedagogical value also reported having fewer problems with the technical aspects, such as distributing manipulatives and grading. Similarly, teachers who felt that the instruments did not inform instruction were most critical of the technical aspects. To test this hypothesis we assigned numbers to the individual teachers based on comments they made about the pedagogical value of *Cumulative Assessment*. Teachers who believed that the instruments provided beneficial information about their students received a "five." Those who felt that the instruments were not useful received a "one." Of course, most teachers' views were not only mixed but varied in emotional intensity, so the scoring was necessarily subjective and was partially a function of our general impressions of the teachers' beliefs and attitudes, as evidenced by the interview data.

We also rated the teachers by the numbers and kinds of comments they made about the technical aspects involved in administering the instruments. Again, those who said they encountered minimal problems received a "five" while those who felt that the technical problems bordered on being insurmountable received a "one." Table 1 summarizes our ratings. Note that four teachers' rating on the two categories were identical, three teachers ratings differed by one point, and the other two teachers differed by two points. One teacher did not discuss pedagogical issues.

Although causality and direction of impact are open to interpretation, we suspect that pedagogical value and the problematics of technical issues are in-

Table 1 Ratings of Teachers' Views of *Cumulative Assessment*

	Ease of Administration				
Pedagogical Value	**1**	**2**	**3**	**4**	**5**
5					Patricia
4				Virginia	Wendy
3	Anne				Angela
2		Elizabeth	Marie		
1	Tina	Deborah			

tertwined, that is, the teachers' perceived pedagogical usefulness of *Cumulative Assessment* subjectively magnifies or diminishes the problems they encounter in administering the tests. In turn, technical difficulties lower the relative pedagogical value the teachers place on the assessment process. Teachers reported that they assessed on an ongoing basis and so being required to generate a score with an unwieldy test may be have been viewed as a needless use of valuable teaching time. However, the majority of the teachers felt that they received useful information from *Cumulative Assessment*. Most teachers found value in at least parts of the instruments and said that they would continue to use them even if they were not mandated.

Conclusion

Philosophically, some teachers may agree that assessing and teaching are overlapping and even inseparable activities. On the other hand, they may think that when a numerical score is generated from their assessment information, the information should illustrate what a student can do alone and unaided. However, as teachers gain experience by observing students and by discussing student responses with colleagues they might feel more comfortable when they assign numbers as a measure of student reasoning abilities.

Little research has been done on the relationship between assessment and grading in the elementary classroom. None of the teachers in the study used the results of the assessment instruments as part of their grading system even though they were required to record the scores. Senk, Beckmann & Thompson (1997) studied the assessment practices of secondary teachers and suggested that determining aggregate measures of assessment is a difficult task for teachers and should include more than numerical information. One of the teachers in the *Cumulative Assessment* study, Angela, who had been uncomfortable as-

signing numerical scores to the students based on the instruments, suggested making notations on the students' records. Other teachers had expressed concerns about the consistency of scores, suggesting that written records might be a viable and preferable alternative to numerical representations of student ability.

There is considerable rhetoric about the role of assessment, particularly alternative assessment, in our schools today. In this study teachers expressed mixed views about the assessment instruments and about alternative assessment in general. The length of time required to administer the instruments seemed to be the teachers' greatest concern. The instruments are relatively new, so practice administering and scoring will alleviate some of the stress, and changes in the instruments and reporting methods have already helped.

This study illustrates that teachers' views are not always consistent with the views of those who design such instruments. As teachers gain experience using the assessment instruments they may learn new ways of assessing knowledge and carry this knowledge into their daily teaching activities.

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USING ASSESSMENT PRACTICES AS A TOOL FOR CHANGING TEACHING METHODOLOGY

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An inservice project, funded by the Eisenhower Program for Mathematics and Science, was conducted with approximately 40 teachers in the Midwest. After a full-year of staff development, teachers showed significant gains in their knowledge and use of several authentic assessment strategies. In addition, the project reversed several of their fundamental beliefs about the nature of assessment and mathematics and empowered them to emphasize the mathematical process skills of problem solving, reasoning, and communication in their classrooms. As teachers reflected upon alternative assessment strategies, they also tended to rethink curricular and instructional issues.

The *Curriculum and Evaluation Standards for School Mathematics* (1989) of the National Council of Teachers of Mathematics (NCTM) made it clear that a shift in the curriculum from isolated procedure memorization to one of inquiry and problem solving implies a significant change in the ways teachers assess student progress. Clearly, if the objectives are increasingly process-oriented and involve the use of manipulatives and an emphasis on writing and communication, the assessment procedures also need to change. Curriculum, teaching, and assessment are almost inseparable, and a change in one necessitates a change in the others.

The *Assessment Standards for School Mathematics* (1995) emphasized that assessment is much more than assigning grades. Instead, it is a process by which the teacher gathers information about what the student can do and thinks about mathematics. As such, assessment plays a number of roles in the classroom, not the least of which is to provide day-to-day, formative data about how students are attaching meaning to mathematical concepts to enable to teacher to better serve the needs of his/her students. Documents by the Mathematical Sciences Education Board, *Measuring Up: Prototypes for Mathematics Assessment* (1993) and *Measuring What Counts: A Conceptual Guide for Mathematics Assessment* (1993) argued for the development of meaningful assessment tasks to "provide concrete illustrations of the important goals to which students and teachers can aspire" (NRC, 1993, p. 3). Therefore, the implementation of alternative assessment techniques simultaneously promotes the use of problem solving and inquiry in the classroom. Resnick and Resnick (1992) wrote that a change in assessment practices can have a significant impact on classroom teacher behaviors. Furthermore, the use of authentic assessment strategies, such as rubrics, projects, and portfolios, significantly contribute to

academic achievement (Kerr, 1996) and student motivation to do mathematics (Chinni, 1996).

While the changing of assessment practices appears to be a cornerstone in mathematics education reform, there is a significant lack of research regarding "typical" classroom assessment practices in general. Most of the current research relates to secondary mathematics teachers and their strategies. For example, one study (Senk, Beckmann, & Thompson, 1997) showed that secondary teachers still rely heavily upon tests, quizzes, and homework to assess student progress and that most test items tend to be low-level and are not open-ended, while a previous study by Garet and Mills (1995) suggested that assessment practices are still dominated by short-answer and multiple-choice tests and that there has been little change in the use of these techniques over time.

In 1996, a year-long inservice program was conducted with approximately 40 classroom teachers, grades K-12, in one county in the Midwest. The project, titled "Assessment Project for Erie County Teachers" (ASPECT), was designed to assist teachers in the implementation of the NCTM assessment Standards. Instructors for the project were university mathematics educators, teamed with classroom teachers who had previously undergone leadership training. The project was funded by a grant from the Dwight D. Eisenhower Mathematics and Science Program.

Teachers were involved in a series of inservice sessions through the Spring of 1996, which included testing a variety of teaching and assessment strategies in their classes and sharing results. During the summer, they were involved in nearly 40 hours of intensive work to develop assessment plans for the following school year. Finally, in the Fall of 1996, teachers attended a series of follow-up sessions, designed for them to share ideas and to fine-tune their plans. As an outgrowth of the ASPECT project, a number of research questions were pursued by the project director and the instructional team: (1) How did the participants' knowledge about and use of authentic forms of assessment change as a result of their participation in the project? (2) Besides instructing teachers on "how to" use alternative forms of assessment, what other effects did the project have on the participants? (3) What were the key factors that convinced teachers that they needed to rethink and/or change their assessment and teaching practices?

Method

Participants in ASPECT were pre-tested in March of 1996, prior to the project. They were asked for their opinions about assessment, their knowledge about various assessment techniques, and the degree to which they were using those strategies. During the Spring sessions, participants wrote journal entries about the progress they had made in rethinking their classroom practices. In the Summer, participants were interviewed in small groups and asked a variety

of questions regarding the effectiveness of the program and the degree to which they had changed their strategies. Finally, in the Fall, participants were surveyed and asked many of the same questions they had been asked in the Spring. While formal classroom observations were not part of the evaluation, the project director visited several project participants on-site to discuss their progress.

Since most of the survey questions were answered through the use of a Likert Scale, and participants coded their pre-test and post-test sheets, the data was used to determine whether knowledge about and use of alternative assessment strategies had shown a statistical change. Furthermore, open-ended responses to survey questions and transcribed interview data were used to qualitatively examine the effectiveness of the program and the changes in beliefs and practices of ASPECT participants. Triangulation of quantitative survey data, qualitative survey data, journal entries, interview data, and informal school visits were used to provide a consistent "big picture."

Results

Question #1: How did the participants' knowledge about and use of authentic forms of assessment change as a result of their participation in the project?

On the pre-project and post-project surveys, participants were asked to rate their knowledge of assessment strategies on a Likert scale from 1 (No Knowledge) to 5 (Expert). Likewise, they rated the extent to which they used the same strategies from 1 (Never) to 5 (Once Per Day). A paired-sample, two-tail t-test was used to analyze the changes in knowledge level and use of these strategies. The quantitative analysis yielded the results shown in Table 1.

Overall, participants expressed the most significant knowledge gains in areas related to communication in mathematics—the use of interviews, writing and journaling in the classroom and scoring those products on rubrics, and the use of portfolios to assess progress. Project participants were most likely to have adopted the use of writing in the classroom (including journals), portfolios, and rubrics to assess student products. A relative decline in the use of observations in the classroom appears to be due to the fact that the project developed formal observation skills, and participants may have responded to the question about observation as an informal process when they filled-out the pre-project surveys.

Question #2: Besides instructing teachers on "how to" use alternative forms of assessment, what other effects did the project have on the participants?

Survey and interview data were collected to detect changes in attitudes and beliefs of the participants toward assessment and some pedagogical issues. Five teacher beliefs were identified as having made significant shifts during the pro-

Table 1 *Survey Data On Changes in Knowledge Level and Use of Various Assessment Strategies*

Assessment Strategy	t-value (Knowledge)	t-value (Use)
Portfolios	5.774 *	4.255 *
Journaling	4.625 *	4.960 *
Investigations	2.130 *	0.683
Open-Ended Questions	2.927 *	2.929 *
Interviews	5.964 *	3.138 *
Observations	2.414 *	-1.455
Rubric Scoring	6.026 *	5.688 *
Writing	6.583 *	3.861 *
Performance Tasks	0.000	-0.903
Standardized Tests	-0.236	0.776
Multiple Choice Tests	-1.000	0.845
Competency Tests	0.495	0.776

* Significant 2-tailed t-value, ... < 0.05.

gram. For example, the most significant change of thinking regarded the statement that "The main purpose of assessment is to assign grades." Initially, participants agreed with the statement, but by the end of the project, they had reversed their belief to strongly disagree. Table 2 includes a list of the five changed beliefs.

Another belief that showed a marked reversal in thinking, although not statistically significant, was, "Students are generally incapable of assessing their own progress and need the assistance of a teacher." The results indicate that not only did the project change teacher behaviors, but it also began a change process in their beliefs, which may be the most important product of the program. For example, if one believes that the mathematical thinking process is important, he/she will emphasize more than just the "final answer." Not only will this teacher tend to use alternative assessment strategies, but he/she may be more likely to select richer problems and activities.

Question #3: What were the key factors that convinced teachers that they needed to rethink and/or change their assessment and teaching practices?

Cuban (1993) wrote that teachers will only change their beliefs and practices when they perceive the reform as being a benefit to themselves and their students. Clearly, the project demonstrated these benefits. On the final survey, participants were asked to cite specific experiences that may have caused them

Table 2 *Teacher Beliefs Showing a Significant Reversal as a Result of ASPECT*

1. The main purpose of assessment is to assign grades.
2. When assessing a checklist item, it is important to evaluate all children in the class on the same day.
3. When analyzing a child's work sample, the emphasis should be on the final answer.
4. The use of technology in assessment in mathematics classrooms is inappropriate.
5. The only truly objective form of assessment is a multiple choice test.

to rethink their practices. Three major themes emerged in the final surveys and interviews: (1) the importance of field-testing, (2) the connection between curriculum, teaching, and assessment, and (3) the value of discussing progress with peers.

Many of the participants talked about how field-testing assessment ideas had convinced them of the value of those strategies. For example, one participant wrote that "[using journals in my class this year] gives me an opportunity to evaluate my own teaching strengths and weaknesses." Another said that student portfolios "allowed me to conduct conferences easily as I showed parents what we had been doing in math." Several teachers discussed how conversations about assessment got them to rethink what they saw as important in curriculum and teaching. For example, one participant stated, "In our discussions in class I decided that I needed to improve communication skills and I wanted my students to explain how they solved the problems, not just write down the answer." Another said that "discussing the NCTM [Standards] really started me thinking and helped me to change." One participant commented that "the background information about the NCTM standards . . . showed me how I needed to change the way I teach as well as how I assess." Finally, participants frequently described the power of meeting with colleagues to discuss successes and challenges, consistent with the writing of Chittenden and Wallace (1992). They found it beneficial to meet with grade-level counterparts while planning and implementing assessment plans. One participant talked about the importance of sharing progress with peers and that interacting with other teachers is "something we don't get to do often enough."

Discussion and Conclusions

While the focus of ASPECT was on assessment, the process inevitably involved an analysis of the entire teaching and learning process. While partici-

pants in the project grew in their knowledge and use of alternative methods of assessment, they also demonstrated emphasis changes in content and pedagogy. Teachers felt empowered to encourage problem solving, reasoning, and communication in the classroom as they developed means by which to assess process skills.

Fullan (1982) pointed-out that some classroom teachers simply do not have the desire to change at all. Participants in ASPECT were clearly self-selected because they were interested in improving their performance as teachers. But when the participants were faced with the complex interplay of the teaching and learning process, many stated in interviews that they had "gotten more than I had ever expected from the program." Two major questions about this inservice process remain to be answered: (1) Will there be significant gains in student achievement in these teachers' classes over the next few years as a result of their participation in ASPECT? After all, the bottom line for most teacher enhancement programs is the effect on the students. (2) Is the change "permanent" for the participants, or will they snap back to more traditional teaching and assessment strategies over time? In California, for example, many teachers who were extensively inserviced on the mathematics framework still did not change their teaching styles (Cohen & Ball, 1990). Goodlad (1984) stated that, despite our best efforts, teachers still tend to teach as they were taught. Perhaps the key to continued success is for teachers to network with others who are attempting similar changes, as evidenced by recent research on assessment practices by Meisenheimer (1996).

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A PROPOSED METHOD FOR ASSESSING TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE

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Case studies of two kindergarten teachers were conducted in order to understand: 1) the knowledge they acquired of the mathematical thinking of their children; 2) the ways in which they acquired that knowledge; and 3) the uses they made of that knowledge. A Prediction/Solution Comparison was developed to assess the teachers' knowledge of the mathematical thinking of their children. Each teacher selected problems, predicted whether each child in her class would correctly solve each problem and the strategy the child would use to do so, and administered the problems to the children. The teachers' predictions were compared to the children's solutions. In addition to providing information about the teachers' knowledge of the mathematical thinking of children in their classes, the procedure provided information about the teachers' knowledge of research-based information on children's thinking. It is suggested that discussing the mathematical thinking of a teacher's children can provide insight into the teacher's pedagogical content knowledge.

Inherent in much of the recent reform literature in mathematics education is the position that learning occurs when connections are formed between new information and already existing knowledge structures or when new information leads to cognitive conflict and, therefore, to reorganization of existing structures in order to resolve the conflict (Hiebert & Carpenter, 1992). An implication of this view of learning is that individuals construct their own knowledge as they add to and reorganize their knowledge structures. Related to the belief that individuals construct knowledge is a view of teachers, not as transmitters of knowledge, but as facilitators of children's construction of knowledge within the social context of the classroom. Also related to this belief, is the idea that teachers, as well as children, construct knowledge in classroom contexts.

It is widely accepted that what teachers know influences their instruction. However teachers' knowledge, like all knowledge, is complex and is made up of many components that are connected in cognitive structures in memory. Because of the many strong connections, the components of teachers' knowledge cannot easily be separated (Fennema & Franke, 1992). Researchers, however, have generally focused on specific components of teachers' knowledge as if they could be separated from others. Many studies have focused on teachers' mathematical content knowledge. Such studies have often failed to find a significant relationship between teachers' knowledge of mathematics, as measured by the number of college mathematics courses the teacher had taken or performance on mathematics tests, and the learning of the students in their classes (Eisenberg, 1977). The argument has been made that the failure of this research

to show a relationship was due to the methods used to assess the teachers' knowledge (Carpenter, Fennema, Peterson, & Carey, 1988).

Since Shulman wrote about pedagogical content knowledge in 1986, a number of studies have indicated that there is a relationship between teachers' pedagogical content knowledge, including their knowledge of students' understanding of subject matter content, and their instruction (Carpenter, et al., 1988; Grossman, 1990; Wilson, Shulman, & Richert, 1987). A theme of this work is that teachers' pedagogical content knowledge develops as teachers plan for instruction and teach. A conclusion that can be drawn from this view of knowledge development is that, if we are to understand teachers' knowledge, we must examine it in the context of their teaching. In this paper a procedure that was used to examine teachers' knowledge of specific content, research-based information on children's mathematical thinking will be described.

Two kindergarten teachers teaching mathematics were studied. The foci of the study were: 1) the knowledge the teachers acquired of the mathematical thinking of individual children in their classes; 2) the ways in which they acquired that knowledge; and 3) the uses they made of that knowledge in subsequent instruction. The teachers' beliefs about learning and teaching and knowledge of research-based information about young children's mathematical thinking was also investigated to understand how it contributed to their knowledge of the thinking of their children.

Methodology

The subjects of the study were Ruth Anderson and Sarah Wilson (names are pseudonyms), two all-day kindergarten teachers who had participated in workshops conducted by the Cognitively Guided Research Project (CGI) and who were selected because they were reported by workshop leaders to be teachers who knew about their children's mathematical thinking. At CGI workshops the teachers had opportunities to learn about research-based information on children's thinking about word problems. That information is organized into a framework with two main components. First, there are several types of word problems that young children are able to solve; these problems are categorized based on the action or relationship in the problem as well as the location of the unknown. For example, "Melissa has five Beanie Babies. How many more Beanie Babies does she need to get to have 11 Beanie Babies?" involves a joining action. The unknown is how much Melissa's number of Beanie Babies needs to change to get from 5 to 11. Thus, the problem is classified as a Join (Change Unknown) problem.

Second, there are different strategies that children use to solve those problems. These strategies fall into three main categories: Direct Modeling, Counting, and Facts. Direct Modeling involves using counters to act out the action or relationship in the problem. For the problem above, this would entail counting

out five counters, counting out more counters until there were 11 altogether, and then counting the second set of counters to get the answer six. Counting strategies also involve following the action of the problem; however, they entail counting up or back to arrive at a solution. For the problem above, a child would say "five" and then count on from five to 11, extending one finger until 11 was reached. The answer would be the number of fingers extended. There are several subcategories of Direct Modeling and Counting strategies; the subcategory that is used depends on the structure of the problem. Fact strategies are of two types. For the above problem, a child using a Derived Fact strategy might say, "The answer is six, because five and five is 10, so five and six must be 11." A Recalled Fact strategy entails knowing the fact called for in the problem; for the problem above, this would mean knowing that five plus six equals 11.

As teachers explored the problem types and children's solution strategies at the workshops, they considered the relationships between a particular type of problem and the strategies that could be used to solve it. In addition, they had opportunities to discuss ways in which they could use the framework to learn about the thinking of their children and consider how they could use what they learned about their children to inform instruction. A more complete description of the workshops can be found in Fennema, Carpenter, Franke, Levi, Jacobs, and Empson (1996).

Participant/observational fieldwork methods were used in this study. Extensive observations of the teachers' instruction were conducted over a four month period. During that time the teachers were also interviewed several times. At the end of the observation period, a procedure that will be called a Prediction/Solution Comparison was carried out; it is this procedure that is the focus of this paper. The purpose of the Prediction/Solution Comparison was to determine what the teachers knew about the mathematical thinking of individual children. Each teacher was asked to choose mathematics content through which she could demonstrate her knowledge of her children's mathematical thinking. Both teachers chose to use word problems. Each teacher wrote a list of word problems and predicted whether each of her children would correctly solve each problem and the solution strategy each child would use. She then read the problems to small groups of children, allowed the children time to solve the problems individually, and had each child explain his or her solution strategy. We both took notes on the children's accuracy and reported strategies and then discussed and compared our notes shortly afterward. The discussions were audio-taped. Copies of the problems the teachers used and their written predictions were collected.

Analysis of the data from the Prediction/Solution Comparison consisted of comparing each teacher's predictions with how her children reported solving the problems. As transcripts of the audio-taped discussions were analyzed and

the teachers' descriptions of how the children solved problems were compared to their predictions, discrepancies between the teachers' statements about solution strategies and the research-based information became apparent. In order to learn more about the discrepancies, I interviewed each teacher again. In that interview, I referred to the problems from the Prediction/Solution Comparison and asked the teachers to further explain their predictions. For example, if a teacher had said that a particular child would Directly Model a problem, I asked the teacher to show me what she had meant the child would do. These interviews were also audio-taped. The statements the teachers made during these interviews were compared to the research-based information on children's thinking that had been the focus of the workshops they attended.

Results

A great deal was learned about the teachers' beliefs about learning and teaching mathematics, knowledge of research-based information about young children's mathematical thinking, and instruction from the observations and interviews. The Prediction/Solution Comparison provided data about each teachers' knowledge of the thinking of individual children in her class about word problems. Those results are discussed elsewhere (Warfield, 1997).

The Prediction/Solution Comparison was also used as a means of learning about the teachers' understandings of research-based information on children's mathematical thinking. The originators of the Cognitively Guided Instruction project believe that individuals construct their own knowledge; they recognize that teachers will select from the information shared by workshop leaders and will adapt the information they select as they use it in their own classrooms. The teachers who participated in this study had quite different understandings of the research-based information.

Ms. Wilson knew the problem types introduced in the CGI Workshops. For the Prediction/Solution Comparison she prepared a list of problems with each correctly labeled as to type. Ms. Wilson also understood the distinctions among the strategies that children use to solve problems and the connections between a problem of a specific type and the strategies that can be used to solve it. However, even when she did know the name and definition of a strategy, she did not, in all cases, use the name in the way she knew it was defined. She knew, for example, that direct modeling involved modeling the action or relationship in a problem, and she often talked about children directly modeling problems. When she did this, she generally qualified her statement. For example, one of the problems she posed in the Prediction/Solution Comparison was "Mickey Mouse had 12 pieces of cheese. He gave some to Minnie Mouse. Then Mickey had 8 pieces left. How many pieces of cheese did he give to Minnie?" She predicted that several children would directly model that problem. To do so according to the definition of direct modeling, would entail count-

ing out 12 counters, taking counters away until 8 were left, and then counting the counters that had been removed. Ms. Wilson said, however, that there were different ways the problem could be directly modeled. A child could get 12 counters and take away 8, could start with 8 counters and add more until 12 was reached, or could get 12 counters and 8 counters and match to see how many more were in 12 than 8. She told me that she knew she was not using the term correctly but said it made more sense to her to use direct modeling to refer to strategies in which the child used counters to make all of the objects in the problem and then to describe what the child did with those counters. Ms. Wilson usually described what a child did to solve a problem, rather than using the terminology from the workshops. Although for the most part Ms. Wilson had learned the information shared at the workshops, she had adapted it so as to make it more useful to her.

Ms. Anderson also knew the problem types; she labeled the problems she wrote for the Prediction/Solution Comparison as to type and was able to discuss the distinctions between types. However, when she was asked to demonstrate strategies, it became clear that her understanding was incomplete. During the Prediction/Solution Comparison, she posed the problem "Manuel had some candy bars. His mom gave him six more candy bars. Now he has 10. How many candy bars did his mom give him?" She said the problem could be solved by: Direct Modeling, a variety of Counting strategies, Derived Facts, and Recalled Facts. She was asked to demonstrate each of these. Direct Modeling, she said, would involve making a set of six cubes, adding more cubes while counting "7, 8, 9, 10," and then counting the cubes that were added. She went on to say, "But that's not a true Direct Model . . . because what they may do is they may get the 6 out and then they get the 10 out and they do." She demonstrated matching the two sets. In neither of these descriptions did Ms. Anderson consider the essential component of Direct Modeling, that of following the action in the problem. Several Counting strategies were possible according to Ms. Anderson. A child could say "6" and count on to 10 using fingers to keep track. The answer would be the number of extended fingers. A child could extend 10 fingers, count backward while folding down one finger per count until six fingers were remaining and keep track of the number of fingers that had been folded down. Or a child could start with the 6 and count "7, 8, 9, 10." Since there were four counts, the answer would be four. Again, Ms. Anderson did not consider the order in which things happen in the problem. Ms. Anderson said that she did not know the difference between a Derived Fact and a Recalled Fact. Ms. Anderson had not considered it necessary to attend in great detail to the information about strategies that was shared at the workshops.

Conclusions

Although information about the teachers' knowledge of the research-based information on children's mathematical thinking was collected during the observations and earlier interviews, most of what was learned about this knowledge came as a result of the Prediction/Solution Comparison and the following interviews. Focusing the discussions about the specific content from the workshops on the teachers' predictions and understandings of their students' actual solutions allowed me to investigate their pedagogical content knowledge as they used it in interactions with their children, that is, in the context of teaching. It seems possible that this strategy for coming to understand teachers' knowledge could also be useful in assessing teachers' knowledge of other content. For example, it might be possible to learn about teachers' understanding of other pedagogical content or, possibly, of specific mathematical content by asking them to select problems, predict their students' strategies for solving the problems, and then compare the students' problem solving with the teachers' predictions. The ensuing discussions would, it is hypothesized, provide information about the teachers' understanding of the content in ways related to teaching rather than in isolation as did the early studies.

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ASSESSING STUDENT WORK: THE TEACHER KNOWLEDGE DEMANDS OF OPEN-ENDED TASKS

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Many open-ended tasks in mathematics are designed to assess students' conceptual understanding. Using these tasks for valid assessment may require that teachers possess different levels or forms of content knowledge than more traditional forms of assessment. To investigate this conjecture about different forms of content knowledge, a study was designed in which undergraduate elementary education majors were given an open-ended task on fractions developed for fourth-grade students, and then were asked to score student work and justify the scores. Findings from the study revealed that one-fourth of the participants successfully completed all components of the activity, while another one-fourth were unable to respond adequately to the open-ended task itself. Giving valid justifications for the scores appeared to be dependent on having a conceptual understanding of the mathematics in the task and on possessing certain pedagogical content knowledge related to that task.

Introduction

Classroom teachers spend an average of 25 percent of their time engaged in assessment activities (Stiggins & Conklin, 1992). Such activities include developing or choosing methods of assessment, gathering the data, evaluating and interpreting the results, and using the results to guide instruction. In mathematics education there is a significant effort underway to reform the assessment practices of classroom teachers (National Council of Teachers of Mathematics, 1995). In particular, teachers are being encouraged to use a wide range of assessment tools, such as open-ended tasks and portfolios. Many of these tools require that teachers move beyond simply marking student work as "right" or "wrong." Instead, teachers are being asked to develop and use scoring rubrics and other methods designed to analyze more complex responses. It is our hypothesis that the use of alternative forms of assessment designed to measure conceptual understanding may demand different levels or forms of content knowledge from teachers who use them.

[1]The second author's participation in the preparation of this paper was supported in part by a grant from the National Science Foundation to the National Council of Teachers of Mathematics (grant no. RED-9453189). Any opinions expressed herein do not necessarily reflect those of the National Science Foundation or the National Council of Teachers of Mathematics.

In this paper, we present results from a study designed to investigate the nature of the mathematical knowledge essential to use one form of assessment—the open-ended task. Our focus is on the extent to which preservice elementary teachers can make mathematically valid inferences about students' written responses to such tasks. Specifically, we are interested in these research questions:

- To what extent can preservice elementary teachers respond successfully to an open- ended fraction task designed for students in grade 4?
- Can they describe the important mathematical concepts that underlie the task?
- How successfully can they score students' responses, using a pre-existing scoring rubric?
- How valid are the inferences they make about students' mathematical understanding?
- What is the relationship between their own understanding of the mathematics in the task and their ability to make valid inferences about students' understanding?

Background

It is widely accepted that content knowledge is a necessary, though not sufficient, ingredient when teaching mathematics for understanding (Wearne & Hiebert, 1988), and the lack of such knowledge among preservice elementary teachers is well-documented (e.g., Ball, 1990; Rech, Hartzell, & Stephens, 1993; Eisenhart et al., 1993). What is missing in the research base on teacher knowledge is the forms of content knowledge necessary for assessing and evaluating students' written work.

Pedagogical content knowledge is a construct that has been used to describe the knowledge necessary to convert content into a representation that enables learning to occur (Shulman, 1986). Though it has been used only in the broad context of teaching, we propose that there is a component of pedagogical content knowledge necessary for the specific use of open-ended tasks and the interpretation of student responses. In particular, a teacher who uses such a task effectively should be able to identify the mathematical content that the task assesses. Without this knowledge, the teacher's ability to make inferences about what a student knows may be seriously compromised. Another important component of pedagogical content knowledge related to assessment is the anticipation of misconceptions that students may have for that content.

To assess students' work on an open-ended task, it would seem reasonable to assume that both content knowledge and pedagogical content knowledge would be essential. Making inferences about students' mathematical under-

standing (or misunderstanding) requires that a teacher not only be able to recognize both correct and incorrect responses, but also be able to give a mathematically valid justification for those judgments. We suggest that this kind of knowledge is "assessment knowledge," and in this study we begin to describe some of its components.

Methods

Participants for the study were 58 undergraduate elementary education majors who were enrolled in a mathematics methods course at a major university in the eastern United States. The participants completed a multi-part instrument created for this study. The instrument was developed around an extended constructed-response question, which was a released task from the 1992 National Assessment of Educational Progress (NAEP) in mathematics, and ancillary materials such as the official scoring guide for the task and sample student responses. The task, called "Pizza Comparison" in an official NAEP publication (Dossey, Mullis, & Jones, 1993), appears in Figure 1.

There are three reasons that we chose the Pizza Comparison task for this study. First, it assesses understanding of an important concept for elementary school mathematics: relative size with respect to the common fraction 1/2. A complete and correct response to this task requires that a child understand 1/2 as a "meta-relation," where the ratio of one out of two parts of a continuous whole remains the same, even when the area varies (Hunting, Davis, & Bigelow, 1991). That is, relative size must be taken into account when evaluating fractions. Second, while the task elicits a conceptual understanding of 1/2, it also illuminates misconceptions such as the idea that 1/2 always equals 1/2 and that more pieces of pizza constitutes more pizza, regardless of the size of the pieces. Third, the understandings and misunderstandings elicited by the task have the

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Pizza Comparison (NAEP Grade 4 task)

José ate ½ of a pizza.

Ella ate ½ of another pizza.

José said that he ate more pizza than Ella, but Ella said they both ate the same amount.

Use words and pictures to show that José could be right.

Figure 1. NAEP task used in this study

potential to permit analysis of the relationship between content knowledge and assessment knowledge.

Participants in the study were asked to complete these activities individually: 1) respond to the Pizza Comparison task, giving their best example of an exemplary response; 2) describe in writing the important mathematical concepts embedded in the task and identify potential student misconceptions; 3) use the official NAEP rubric to score nine sample student responses, and 4) write a rationale for the score given to each sample response. Hereafter, these four aspects are called the "participant papers."

Data Analysis and Results

The authors evaluated a subset of 12 participant papers jointly and the remaining papers independently. The criteria for the evaluation of the papers were designed to match some of the research questions: 1) Was the participant's response to the Pizza Comparison task complete and correct? 2) Was the participant able to explain the task's important mathematical concepts? 3) Could the participant evaluate the nine student responses according to the official NAEP scoring guide? and 4) Were the participant's reasons for scoring mathematically valid? Decisions on each of these aspects were based on an evaluation guide. On the set of 12 common papers, there was 100% agreement on all four aspects for each paper. Of the 58 papers total, 1 paper had responses that were deemed too vague to be evaluated; this paper was omitted from further consideration.

The first analysis of the data was accomplished by looking at each of the four aspects separately. Of the 57 participants, slightly more than three-fourths were able to complete the Pizza Comparison task successfully or could identify the important mathematics in the task. Scoring responses and giving mathematically valid reasons for the scores were more difficult for the participants: only 53 percent of the participants scored responses accurately, and only 40 percent gave mathematically valid rationales.

The second analysis involved a holistic analysis of each paper across all four aspects. Given that there were four aspects for each participant, and given that the final classification for each aspect was either Yes (Y) or No (N), there were 16 possible cases (e.g., YYYY, YYYN, ... , NNNN). The 57 participants' papers fell into 11 of the possible 16 cases, as shown in Table 1. For example, papers that were classified as Case 5 belonged to those participants who successfully completed the task, were able to describe the mathematics, and could score student work according to the NAEP rubric, but who were unable to provide mathematically valid reasons for their scores.

Looking at Case 1, about one-fourth of the participants completed the Pizza Comparison task successfully, identified the important mathematical concept in the task, accurately scored the sample student papers, and gave mathemati-

Table 1 Distribution of Participants' Responses According to Cases

Case	Completes Task Successfully	Identifies Important Mathematics	Scores Responses Accurately	Gives Appropriate Rationales	Number of Papers in Case (%)
1	Y	Y	Y	Y	13 (23%)
2	N	Y	Y	Y	2 (4%)
3	Y	N	Y	Y	2 (4%)
4	Y	Y	N	Y	4 (7%)
5	Y	Y	Y	N	7 (12%)
6	N	N	Y	Y	2 (4%)
7	Y	Y	N	N	17 (30%)
8	N	Y	Y	N	1 (2%)
9	N	N	Y	N	3 (5%)
10	Y	N	N	N	1 (2%)
11	N	N	N	N	5 (9%)

cally valid reasons for the scores. On a more somber note, five participants (9 percent) did not successfully complete any of the four criteria.

Some cases are interesting to examine in more detail. For example, in Case 6 two participants who at first did not provide a complete and correct answer and who could not identify the important mathematics, seemed to "learn by scoring" and were successful in providing accurate scores and in providing mathematically valid rationales. Case 9 is also interesting because here the three participants were successful in scoring according to the NAEP rubric, despite their inability to do the task themselves, articulate the important mathematical concepts, and give valid rationales for the scores. Lastly, the 17 participants in Case 7 might be labeled as ones who do not possess a high degree of "assessment knowledge;" that is, these participants could answer the task correctly and identify the important mathematics, but they were not successful in transferring their own content knowledge into the assessment knowledge needed to evaluate student responses (at least according to the NAEP rubric) and to provide mathematically valid reasons for their scores.

Discussion

The data suggest that there may be three levels or forms of knowledge essential for validity assessing student work on open-ended tasks. The first level is content knowledge, by which we mean a conceptual understanding of the mathematics is the task. Such understanding seems to be a prerequisite for assessing student work on the task. As shown in Case 2, only two of the partici-

pants were unsuccessful on the Pizza Comparison task and yet were able to describe the mathematics adequately and also give valid reasons for their scores. All others who failed to answer the task completely and correctly were also unable either to describe the mathematics or to give valid rationales, or both.

The second form of knowledge that seems to be essential for valid assessment is pedagogical content knowledge, here evidenced by describing the mathematics in the task and predicting children's misconceptions. The four participants (see Case 3 and Case 6) who were unable to describe adequately the mathematics in the task, yet able to give valid rationales, seemed to learn from studying the NAEP scoring rubric which contained the major mathematical concepts in the task, and then to use this knowledge in their scoring. Nine other participants (see Cases 9, 10, and 11) who could not describe the mathematics were unable to give valid rationales.

We are proposing that assessment knowledge, the third level or form of teacher knowledge, is evidenced by the ability to make mathematically valid inferences about students' work, and that such knowledge is dependent on having a conceptual understanding of the mathematics in the task as well as possessing certain pedagogical content knowledge related to that task. In our study, only about one-fourth of the participants (see Case 1) showed evidence of having some degrees of assessment knowledge. Further analysis of the data and additional studies will be undertaken to determine whether these three forms of knowledge are verified, and whether they are, in fact, hierarchical.

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ASSESSMENT AS DIALOGUE IN COLLEGE DEVELOPMENTAL MATHEMATICS: GAINS IN STUDENT AND INSTRUCTOR UNDERSTANDING

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This study describes advantages and issues of an assessment strategy devised to align with the curriculum and pedagogy of a problem-based developmental mathematics course. The curriculum materials and pedagogy employed in the project courses were consistent with the constructivist perspective of learning as essentially reliant on student thinking and diminished by "teacher telling" . Adopting a perspective of assessment as dialogue led the researcher-instructors to allow students to resubmit assignments based on instructor feedback, which took the form of questions and suggestions for further thinking.

Instructor feedback on written work was guided by several complimentary perspectives, including:

- the fostering of a learning goal orientation for the student, as contrasted by goals of completion;
- the need for raising cognitive dissonance for students to work through by raising questions or providing relevant (counter)examples, rather than "correcting" student work;
- the advantage of using student misconceptions as "springboards for inquiry" rather than as failures to be corrected and avoided (Borasi, 1994).

Hard copies of student work with each submittal and including instructor feedback were kept by the researcher-instructors for qualitative analysis. These will be used to illustrate our findings in this short oral presentation.

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ASSESSING MATHEMATICAL KNOWLEDGE WITH CONCEPT MAPS AND INTERPRETIVE ESSAYS

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This study explored the use of student-constructed concept maps in conjunction with written interpretive essays as an alternative method of assessment in a Calculus I class. The objectives were to describe and document: 1) the use of concept maps and written essays to assess the connectedness of students' knowledge; 2) the correlation between students' scores on the concept maps and written essays, course exams, and final grade; and 3) the degree to which learning was enhanced by the use of concept maps and written essays.

Students were introduced to the construction of concept maps at the beginning of the quarter. Given a list of terms related to the concept of functions, they were asked to construct a concept map based on these terms and write an interpretive essay describing the relationships they perceived between the terms and elaborating on the relationships indicated on their concept map. A second concept map and accompanying essay using terms from differential and integral calculus were assigned at the end of the quarter as a summative activity. Each map and essay was scored using a holistic scoring criteria developed by the researcher. In the final component of the study, individual student performance on the concept maps and written interpretive essays, homework and quizzes, tests, final exam and course grade were compared.

Findings suggest that concept maps, used in conjunction with written interpretive essays, are a valuable addition to traditional assessment in mathematics classes. This dual approach provided substantial insight into the degree of connectedness of students' knowledge with respect to the given topics. The combination of these two instruments, each relying on a different avenue of expression, allowed students to communicate their knowledge in a more complete manner and readily identified many students misconceptions. In addition, the numerical scores assigned to the concept maps and interpretive essays were highly correlated with scores obtained on more traditional measures and final course grades.

DIFFERENT STRATEGIES USED TO SOLVE CONSTRUCTED RESPONSE SAT MATH ITEMS

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This research study examines the strategies students adopted to solve constructed-response(CR) SAT-Mathematics word problems. CR and multiple-choice (MC) items were administered to students representing a range of mathematical abilities. Format-related differences in difficulty were observed at the item-level. Analysis of students' problem-solving processes was a major factor in explaining difficulty differences. Traub (1993) has noted that some items are more difficult in the constructed-response format than in the multiple-choice format. ETS researchers Bennett, Rock and Wang (1991) were unable to document differences on computer science items when format appeared to make very different cognitive demands. The factors that cause such differences in results between item formats remain unclear. To better understand these differences we are taking a look at the methods that students use to solve CR items.

A sample of students taking the 1995 administration of the SAT and living in the midwest were obtained as part of a collaborative UIUC/ETS research study (Harnisch, 1996). Results from the analysis of the student work on the CR items and the analysis of video-taped protocols were summarized. Students' strategies used on different item formats were identified based on analysis of the tapes and student work. Strategy categories were developed for solving CR items (writing and solving algebraic equations, estimation, reasoning from given material, & plug-in). Item format differences revealed that the CR items removed the element of guessing and hints based on available distractors. CR items also placed a high value on the importance of calculation skills and having an accurate value.

The results from this study point to the complexity of the items and the interactions of the learner within the context. Improving measurement is not likely to be found in simply removing the options from the MC items to create CR items. To have improved understanding of students' mathematical reasoning requires that we consider multiple indicators of student learning, including those provided by standardized tests.

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Curriculum Reform

MATHEMATICAL ACTIVITIES IN INSURANCE AGENTS' WORK

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This paper describes the mathematical activities in the work of insurance agents. These practices do not involve only arithmetic computation but also estimating, using heuristics, explaining complex relationships between quantities, and describing diagrams.

Research in cognition and learning has pointed out the need for closing the gap between learning mathematics in and out of school (Carraher, Carraher, & Schliemann, 1985; D'Ambrosio, 1991; Lave, 1988; Saxe, 1991). This perspective redefines what mathematics is and extends mathematical activity to include more than using rote algorithms. Following this perspective, current curriculum guidelines and standards for mathematics (NCTM, 1989; California Mathematics Framework, 1989) call for engaging students in "real world" mathematics rather than mathematics in isolation of its applications. Detailed accounts of "real world" mathematical practices can serve in closing this gap in a principled manner by framing classroom activities on the basis of what different practitioners *actually do* at work, rather than what we *imagine they do* or what practitioners *say they do*.

The paper will summarize the mathematical activities observed and documented in the work of insurance agents and provide examples of the mathematical activities in life insurance sales conversations. The account of the mathematical aspects of insurance agents' work presents a view of workplace mathematical competency as involving communication and the use of representational resources. Many aspects of the work of agents and staff involve mathematical activities. Even a task as apparently routine as answering a question about a change in an auto premium requires the ability to understand and explain complicated mathematical relations. We present an account of workplace mathematical activities as not only mental or individual skills in computation, but also as a social skill: an ability to communicate about numbers, and to explain relationships between quantities in the context of selling or servicing insurance policies. For example, while calculating a premium involves mainly computation, the social aspect of this mathematical activity is the ability to explain these results clearly to a client.

Theoretical framework and methods

This study builds on previous research on the mathematical practices in everyday commercial situations such as candy selling (Carraher, Carraher, & Schliemann, 1985), dairy (Scribner, 1984), construction foremen (Carraher,

1986), and carpet laying (Masingila, 1994). The framework used to explore and analyze the mathematical practices of insurance agents draws on activity theory (Scribner, 1984) and conversation analysis (Goodwin & Heritage, 1990).

This study was part of larger ethnographic project focusing on insurance agents workplace practices. While conducting this project a team of ethnographers observed a total of 18 agents for approximately 3 to 10 days per agent. The ethnographers followed the agents daily activities such as client visits and phone calls, inspections of residences to determine or verify the value of a property, and staff training. The ethnographic approach to documenting workplace practices involved direct observation, extensive field notes, interviews of agents and staff, as well as audio (and in a few cases video) tapes of conversations between agents (or staff people) and clients.

Field notes, audio tapes, and videotapes were analyzed by a team with multidisciplinary backgrounds (anthropology, linguistics, mathematics education, and cognitive science) for instances of mathematical activity. From this analysis four situations emerged that were rich in mathematical activity and that became the focus of more detailed analysis: life insurance sales conversations, property re-inspections to determine square footage and value, explanations of how to choose a deductible amount for auto or home insurance, and answers to billing questions.

An Overview of Mathematical Activities in Insurance Work

The work of insurance agents and staff requires not only competency in arithmetic, but also the ability to explain to clients complex relations such as: comparisons of whole life versus term policies, the price structure of life policies, comparison of life insurance to other financial instruments, the tax consequences of insurance choices, changes in billing, and how to choose a deductible.

The mathematical practices documented in the work of insurance agents and staff include:

- estimating figures
- using rules of thumb for calculating quantities
- producing ballpark figures for quantities
- explaining the relationship between quantities in a policy (for example an increasing premium and constant benefit)
- using, constructing and explaining tables, graphs and diagrams that represent quantities (for example the floor plan of a house, a printout detailing changes in a life insurance policy over the years, or a diagram depicting the relationship between the two main quantities in a whole life policy, "guaranteed cash value" and "protection")

- using tools (for example, a roll-a-tape for measuring the perimeter of a building, an "Insurance to Value" calculator or a slope meter).

Mathematical Activities in Life Insurance Sales Conversations

Below we provide a description and analysis of the agents' mathematical activities in life insurance sales along two themes, communication and the use of representational resources. These two themes arose from our observations of the work of agents and the central work situations that involve numbers and quantities. We describe how agents' mathematical activities involve multiple ways of explaining policies and multiple ways of explaining and illustrating the relationships between quantities.

Communicating Explanations of Life Insurance Policies

Explanations of life insurance policies are central to life sales conversations. When explaining life policies agents are showing their competence as well as justifying the purchase of a policy. The issues involved in life insurance sales are multidimensional: comparisons of term, whole and universal life policies; comparisons to other financial investment options; tax implications; risk levels; return levels; rates of cost and rates of change of cost. Yet, while the underlying issues are complex, it is the task of the agent to explain them to the client as simply as possible. Part of the agent's expertise is in providing explanations and materials to manage this complexity.

One way agents manage this complexity is by selecting the factors involved in a policy to present, depending on the focus of the explanation. While explaining a policy, an agent includes or omits certain factors, such as the effect of taxes or fluctuations in the stock market. Sometimes the agent does not mention taxes, while other times taxes become a central part of the conversation; changes in the stock market are mentioned only when discussing projections about dividends; and inflation is usually not a part of the explanations. Table 1 presents in more detail the mathematical activities in life insurance sales conversations:

A central mathematical activity involved in these conversations is comparing quantities. Agents were observed: describing how a change in one quantity affects another quantity, describing the rate of change of one quantity over time, comparing two rates of change. Another important mathematical activity observed during life sales conversations was estimating quantity and size. During these conversations agents also often make estimates, suggest ballpark figures, and describe benchmark figures. In the excerpt below, the agent uses the rule of thumb "death benefit equals five times your present income" to estimate a reasonable death benefit:

Table 1 Mathematical Activities in Life Insurance Sales Conversations:

Agents' Talk	Mathematical Activity
Explaining one type of policy (Term, Whole or Universal)	• Explaining the rate of change of one quantity (Examples: how term premium increases quickly; how premium or protection for a whole life policy changes over time) • Explaining the relationship between two quantities (Example: in a whole life policy as cash value goes up, protection increment decreases) • Explaining what dividends do (Example: in a whole life policy dividends can pay the premium or accumulate as cash value or equity) • Explaining two kinds of mutually exclusive quantities (Example: cash value or death benefit) • Making estimates using rules of thumb and ballpark figures (Example: how much death benefit a client needs)
Comparing two policies (Whole and Term, Term and Universal, Whole and Universal)	• Explaining the relationship between two quantities (Example: the premium for a term policy and the premium for a whole policy) • Comparing two quantities that change (Example: over time the premium for a term policy increases and the premium for a whole policy remains constant)
Comparing a policy to another financial instrument (Whole to an IRA, pension, or annuity; a combination of Term and an IRA account to Whole).	• Describing the change in one quantity • Comparing two quantities • Comparing the rate of change of two quantities

Agent: *You should have at least five times your income. You know, so, if you make fifty grand, you should have about two hundred and fifty. That at least gets your wife, through about eight years.*

Another rule of thumb might be that for a thirty year mortgage policy, in fifteen years the policy will start accruing more cash value the client is paying in premium.

Using Diagrams

While some agents give clients prepared printed materials describing policies, other agents also make and use sketches. These sketched diagrams are not given to the client but rather used as tools in the conversation. The diagrams serve as "story boards" for stages of the conversation. Instead of presenting the client with a finished product, *a diagram unrolls as the conversation unfolds.* This can have several advantages over using preprinted materials:

- The agent takes the client along in the conversation by adding or changing pieces of the diagram.
- The agent can refer to a piece of a diagram as if it were the policy, making the policy a concrete object and aiding in gesturing.
- Some of the technical terms involved in describing a policy are written out and located in the sketch available to the client.

Some agents develop and use their favorite sketches. Figure 1 presents an example of a sketch used in a life sales conversation to explain whole life policies.

The diagrams in this conversation served as common references for agent and client. The different stages of the sketch were conversation pieces, something for the agent to refer to as he explained and compared policies. For example, the agent referred to some aspects of the policy as being the picture:

Agent: *They call this protection, this side right here, protection.* (Writing the word in to the upper triangle of the rectangle.)

The diagrams also helped to structure the conversation. As the agent drew each component of the diagram he presented a new piece of the explanation. The diagrams also served to summarize the relationship between quantities such as cash value and protection:

Agent: *The equity will increase and exactly the opposite in inverse relationship, the protection increment decreases.* (Drawing the diagonal)

Even though this diagram exists as printed material, developing the sketch within a conversation is different than showing the client a finished picture.

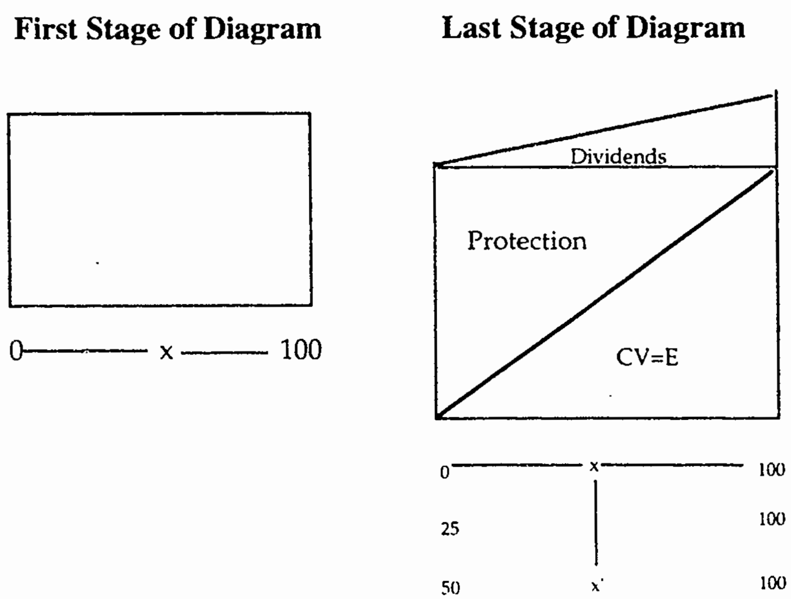

Figure 1 Sample sketch used in a life sales conversation to explain whole life policies.

Drawn sketches support explanations in life sales conversations in ways that are different than a completed diagram. A sketched diagram that is created during a conversation develops meaning within that conversation; the meaning of the different parts of the picture is developed with the conversation. As the conversation goes along, these sketches become the dynamic part of the conversation.

Conclusions

By uncovering the mathematical aspects of insurance work this study adds to the collection of accounts of mathematical cognition in everyday settings. This study also shows that the mathematical activities involved in insurance work, like mathematical practices in other commercial situations, involve communication and the use of social and material resources. This study is relevant to mathematics education because it shows that the mathematical practices in insurance agents' work do not focus on computation. Mathematics instruction aiming to be relevant to student's future experiences in workplaces needs to

address the importance of communicating and using representational resources as central aspects of workplace mathematical competency.

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The author would like to thank the team of researchers at IRL who collaborated on this research project: Chris Darrouzet, Larry Gallagher, Jim Greeno, Joe Harding, Nancy Lawrence, Charlotte Linde, and Charlene Poirier. I am also grateful to all the members of the insurance company who participated in this project.

A SEMIOTIC FRAMEWORK FOR LINKING CULTURAL PRACTICE AND CLASSROOM MATHEMATICS

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With the increasing recognition that *connections* are an important component in the pedagogy of school mathematics (National Council of Teachers of Mathematics, 1989), there is a need for a theoretical framework which addresses the ways in which real experiences and cultural practices of students may be connected with mathematics classroom pedagogy. In this paper, the objective is to construct such a theoretical framework, drawing on literature from semiotics and ethnomathematics. Examples and some evidence which suggests the efficacy of this framework in connecting school mathematics and mathematical ideas constructed from cultural practice, are drawn from the literature and from data collected in a research project in a multicultural high school mathematics class.

Abstraction and generalization are fundamental components of academic mathematics, defined as "the science of detachable relational insights" (Thomas, 1996). At the same time, mathematics is a cultural product (Bishop, 1988), and there is a growing literature suggesting that the potential for constructing mathematical ideas is present in everyday practices in all cultures. In multicultural classrooms, the cultural heritages of students may be viewed as a rich resource for learning and for fostering a classroom climate which promotes equity (Nieto, 1996). It is necessary, then, to reconcile the *specificity* of cultural practice with the *generality* of academic mathematics, the *concreteness* of many out-of-school activities with the *abstraction* of this mathematics, if everyday practice is to be useful in mathematical classroom pedagogy. It is argued in this paper that a semiotic framework (Whitson, 1994) provides connections between these two aspects of the construction of mathematical ideas. Symbolism and structure are key elements in the connecting of the domains of everyday practice and academic mathematics, and a science which addresses signs, their connections and meanings (i.e., semiotics), is eminently suitable for the development of a connecting framework.

Modes of Inquiry

Firstly, two examples from ethnomathematics literature will be used to show the capacity of a semiotic framework to connect cultural practices and formal academic mathematics. The first example is an extension of Marcia Ascher's (1991) analysis of the kinship relations of the Warlpiri of Australia as a dihedral group of order 8. The second example is Paulus Gerdes' (1986) math-

ematical treatment of Angolan Tchokwe sand drawings, which will be discussed in the presentation although lack of space precludes its inclusion here.

Secondly, evidence for the need for such a theoretical framework will be drawn from data collected in an ethnomathematics research project with high school students from diverse cultural backgrounds.

Theory Development and Evidence

In the Warlpiri system of kinship relations (Ascher, 1991), the population is divided into eight sections, simply numbered 1-8 by Ascher. Persons in section 1 may only marry spouses in section 5, those in 2 may marry those in 6, 3 in 7, and 4 in 8. The section of children from a marriage is determined by the section of the mother, according to the rule

$1 \rightarrow 4 \rightarrow 2 \rightarrow 3 \rightarrow 1$

$5 \rightarrow 7 \rightarrow 6 \rightarrow 8 \rightarrow 5$

This means that children of both sexes from a mother in section 1 will be in section 4; those from a mother in section 4 will be in 2, and so on. In this way the population is divided into two matrimoities or cycles, consisting of sections {1, 4, 2, 3} and {5, 7, 6, 8}. There are four patricycles, i.e., {1, 7}, {2, 8}, {3, 6}, and {4, 5}. If a boy is in section 1, his father is in section 7, and his grandfather is again in section 1, and so on. This system, which seems complex, and specific to the Warlpiri cultural practice, has a structure which is isomorphic to five of the eight symmetries of the square, if each *side* of the square is linked to a specific section of the matrimoity in order, clockwise. (This notation differs from Ascher's, in which the *vertices* symbolized the sections, although it gives a characterization analogous to the torus suggested by Ascher & Ascher in Powell & Frankenstein, 1997.) The symmetries used are as follows: starting from a particular individual who is in, say, section 1, four counterclockwise rotations are used for the relation "is the mother of", and a flip about a horizontal axis for the relation "is the spouse of". This symbolism takes all the relationships into account. Ascher showed further that if a table is constructed linking each of the eight sections through their relationships, a dihedral group of order eight is formed. Note that the "mathematical ideas" implicit in the structure belong to the Warlpiri and are intrinsic to their cultural practice. A "dihedral group of order eight", on the other hand, belongs to Ascher's mathematics, as she is the first to admit: "A Warlpiri, of course, does not go through this analysis. . . . A variety of diagrams were used to describe the Warlpiri kin system. The system is theirs but the diagrams were ours" (Ascher, 1991, p. 77). In this paper, a further construct, the "chaining of signifiers" is introduced from semiotics. This chain also belongs to a culture other than that of the Warlpiri; but this in no way diminishes the recognition of the value of the Warlpiri structure or its complexity. It merely serves a different purpose: in fact it could be characterized as

"wonderful" that constructs for different purposes in two very different cultures may be connected in this way. The feeling evoked in me is awe at the unity of humanity.

The increasingly abstract systems of symbolism in this example illustrate a chaining of signifiers (Walkerdine, 1988) which may be derived from the semiotic system of Jacques Lacan (Whitson, 1994; Presmeg, 1997). Unlike Charles S. Peirce who constructed a triadic theory of semiotics in the USA, Lacan's system was developed from the diadic theory of Swiss linguist Ferdinand Saussure, which addresses the relationships between signifiers and signifieds. The following figure illustrates a chaining of signifiers in the Warlpiri example.

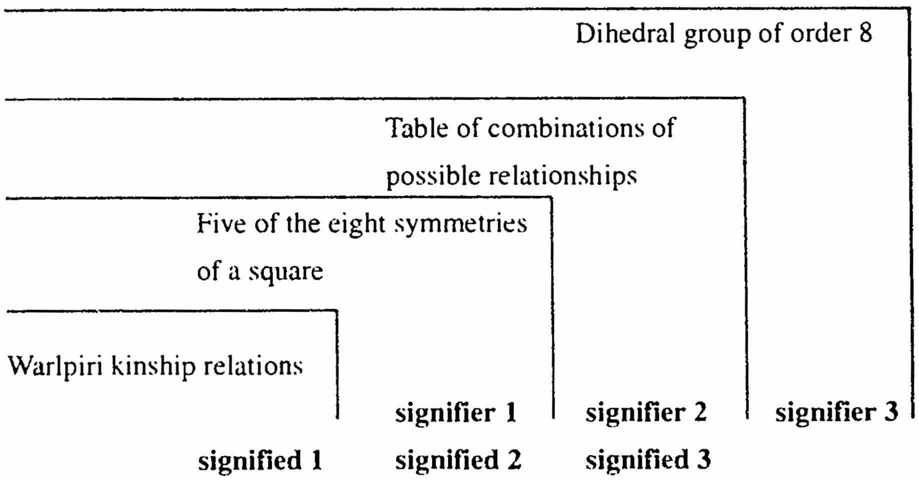

Figure 1 Chaining of signifiers in a progression of generalizations from the Warlpiri kinship system to a dihedral group of order 8.

Now one might question, 'Where does mathematics start in this chaining of signifiers?' The answer to this question hinges on a culture's definition of mathematics. The Warlpiri, if questioned, would in all likelihood not consider their kinship system to be mathematics, even though some definitions of *ethnomathematics* would include their practice as it is (Powell & Frankenstein, 1997). My position is that the Warlpiri are not "doing mathematics" merely by practicing their kinship system; but when they or others recognize the structure of their system *as a structure*, explain it to others for example by encoding it in a diagram, or in some other semiotic form, then there is mathematics. The definition of *ethnomathematics* which I use is simply "the mathematics of cultural

practice" (Presmeg, 1996), which includes what Ascher (1991) calls "mathematical ideas" used by the Warlpiri, as well as those of so-called academic mathematics, which is arguably a culture of its own. Discussion of definitions of ethnomathematics by writers such as Ascher, Pompeu, Borba, and others, could constitute a paper in its own right. Some of these definitions may be found in Powell and Frankenstein (1997). The definition of ethnomathematics given by D'Ambrosio (1985, also published as Chapter 1 in Powell & Frankenstein, 1997) is "the mathematics which is practiced among identifiable cultural groups, such as national-tribal societies, labor groups, children of a certain age bracket, professional classes, and so on" (p. 45). D'Ambrosio based his definition on a "ceaseless cycle" involving an individual in a model with three components, *reality, individual, action,* going back to *reality,* and so on. The intellectual action of the individual is an essential element, in a process which he called *reification*, used by sociobiologists as "the mental activity in which hazily perceived and relatively intangible phenomena, such as complex arrays of objects or activities, are given a factitiously concrete form, simplified and labeled with words or other symbols" (D'Ambrosio, 1985, p. 46). This characterization suggests the role of signification and symbolism, which can provide connections between cultural practice and academic mathematics in a semiotic framework, consonant with the theoretical position formulated in this paper.

In the use of the sides and symmetries of a physical square, say, made of cardboard (signifier) to illustrate the structure of the Warlpiri system (signified), the symbolism may not yet be of a level of generality to satisfy some definitions of academic mathematics. In the next link of the chain, the concrete square gives way to more abstract symbolism in a table. Finally, a generalized structure called 'a dihedral group of order 8' becomes the signifier for this specific table, which is now no longer the signifier, but the signified, in an academic mathematical structure. In this way, semiotic processes may be used to illustrate cultural connections as symbol systems are constructed in a bridge between cultures. In this way, *symbolism* provides possible connections between mathematical ideas "frozen" in practices (Gerdes, 1986), and academic mathematics. Different symbolism would facilitate the construction of different mathematical concepts.

A High School Research Project

The need for a theoretical model such as the one developed and illustrated in this paper was strikingly highlighted in a research project to investigate possible ways of introducing ethnomathematics in a high school mathematics classroom. The purpose of the project was to work with a group of students and their teacher to develop viable ways of using the cultural and ethnic backgrounds of the students as a resource for the learning of mathematics. The seven students involved in the project were from African American, Caucasian, Asian, and

Hispanic cultural backgrounds. In video or audio recorded interviews, they described rich activities based on four "h"s: their hobbies, hopes (career aspirations), homes and cultural heritages. These activities were an integral part of their lives. Other issues which were discussed were the nature of mathematics, the work done by their parents (and whether mathematics was involved in this), their achievement in and feelings towards school mathematics, and perceived links between mathematics and other subjects in the school curriculum. These students, and their mathematics teacher, with their current beliefs about the nature of mathematics, could not readily develop mathematical ideas from these practices. However, the research project (more fully reported in Presmeg, 1996) did give strong evidence for the richness of the experiences and activities in the lives and cultural heritages of the students. According to Nieto (1996), and strongly suggested in Bishop (1988), it should be possible for teachers to use such experiences and activities to facilitate students' construction of mathematical ideas, with a consequent affirming of cultural diversity. The present paper begins to illustrate how symbol systems are a connecting bridge in this endeavor. A semiotic framework thus has the potential to provide a basis for culturally relevant pedagogy in multicultural mathematics classrooms.

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EDUCATING NON-COLLEGE BOUND STUDENTS: WHAT WE CAN LEARN FROM MANUFACTURING WORK

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This paper reports a survey of the mathematical demands of "blue-collar" work in various workplaces involved in automobile manufacturing. It is oriented by current concerns that U.S. high school graduates are often ill-prepared for high-skill, high-wage work in a variety of industries. The study has examined what sort of mathematical knowledge and skills high school graduates need to perform competently in that industry. Two main findings have emerged: (1) the demands of high-volume assembly work appear well within the current content of the K–12 curriculum, while (2) more challenging and rewarding work, like CNC machine tool operation, require extensive spatial and geometric competence that the U.S. curriculum does not strongly support. International comparisons of mathematics curriculum and teaching (the TIMMS study) show that other leading manufacturing countries devote more attention to space and geometry in the middle grades than the United States.

In this short paper I take an instrumental perspective on mathematics education: that students learn mathematics to prepare for meaningful and rewarding work. I believe there are other equally important goals for mathematics education, but I adopt an instrumental view here because the role of schools in preparing students for work has been a central theme in recent educational policy discussions in the United States. Increased economic competition, dramatic changes in workplace technology, and poor educational achievement have focused critical attention on the performance of schools. Many American 18 year olds appear poorly prepared to meet the demands of modern industrial work, especially in mathematics, science, and technology (Murnane & Levy, 1996; U. S. Department of Labor, 1991). In contrast to other industrialized countries, particularly those whose industries compete well against U.S. firms, American curricula in mathematics and science appear shallow and unfocused (Schmidt, McKnight, Raizen, 1996). Current curricular reforms address that problem, but the pace and breadth of their impact on classroom practice is uncertain.

To examine the mathematical demands of work in one important industrial sector, the "Mathematics in Michigan's Industrial Workplaces" project has surveyed mathematics use in workplaces related to automobile manufacturing.[1]

[1] Thus far, twelve sites have participated. Twenty-four site visits have been conducted in 11 sites; 20 site visits have been conducted in the one site under detailed study. More sites are being added to expand the size, range, and adequacy of the workplace sample.

We have examined jobs and work practices legitimately open to students with high school diplomas, avoiding engineering, design, and management work that typically requires college or more advanced education. Empirical surveys of work practices can usefully complement policy analyses that report work practices more selectively (U.S. Department of Labor, 1991) and case studies of particular workplaces (Hall & Stevens, 1996). We began with observational methods: general tours of production floors followed by more focused observations and interviews of specific work areas (e.g., quality laboratories) and jobs. Where we found that work practices included mathematically interesting problems and processes, we returned to study them in successive cycles of analysis and observation and interview. Two broad questions orient the project:

- What mathematical competencies are required by production work in this industry?
- How has technological change effected these requirements?

This paper discusses results on the first question; analyses of the second question are available elsewhere (Douglas & Smith, 1997; Smith, 1997).

Sites were selected from different levels in the automobile production hierarchy: Final assembly, Tier I producers who supply parts and parts systems to final assembly , Tier II producers who supply to Tier I, Tier III producers, and after-market producers (e.g. parts and "add-ons"). Products ranged from finished vehicles; to wheels, air conditioner compressors, and anti-lock brake valves; to molds for making headrests and dies for forming sheet metal body panels. Work forces ranged from thousands to 30 employees. The sophistication in the production technology varied dramatically from 1940's manual machine tools to state-of-the-art, fully automated production facilities that required little manipulation by workers.

Two central assumptions have guided these investigations. We have accepted a broad view of mathematics that is not bounded by the content of the K–12 curriculum. It includes the classical domains of number and space as well as logical reasoning, analysis of causal systems, and spatial visualization. In particular, we have found it important to attend to non-metric spatial and geometric reasoning. We have also worked from the assumption (not yet disconfirmed) that some mathematical reasoning is visible on the surface of work activities. Examples include use of counting, measuring with various tools, adding data to 2-D graphs, and computing averages. Mathematical operations were both observed directly during tours and inferred from observed practices. In many other cases, substantial understanding of the production process was useful, if not necessary to identify the mathematical content in the work practices.

Other studies of mathematical work in non-school settings have inspired and guided the project. Sylvia Scribner's (1984) study of dairy workers pro-

vided a dramatic example of how numerical tasks can become strongly spatial. Her later work with colleagues provided background on the nature of CNC machining (e.g., Martin & Scribner, 1990), though their analytic focus was different. Analyses of mathematical work in streets (Saxe, 1991) and workplaces (Nunes, Schliemann, & Carraher, 1993) have shown that school taught procedures often do not structure people's mathematical thinking outside of school. Generally, we share with the ethnomathematics program a concern for mathematics in use in the daily lives and practices of different groups and cultures (d'Ambrosio, 1985).

Current Results

*High volume assembly production is **not** highly mathematics-intensive.* In ten final assembly, Tier I, and Tier II sites where thousands of the same part or system were produced, the main task was moving material through production lines. In these workplaces, mathematics use was generally limited to reading and recording numbers (e.g., from gauges to record sheets), counting, measuring with hand-held, manual and digital tools, computing averages, plotting numerical data on time graphs (e.g., in statistical process control [SPC] procedures), and converting among fractions, decimals, and percents. Beyond quality analysis via SPC, the specific mathematical procedures observed varied with the product type and kind of production technology on the floor. The sophistication of the technology was not related in any simple way to mathematical demand; examples of both "upskilling" and "downskilling" were observed (Douglas & Smith, 1997; Smith, 1997).

Some production-related work in these sites—in quality labs, "quality-circle" groups, and special expertise jobs—was more mathematically interesting and intensive. This work was open to high school graduates, but only a small number of workers were involved, most with substantial experience. In quality labs, maintained by all high-volume producers, workers test parts against their design specifications with manual or semi-automated Coordinate Measuring Machines (CMMs). A solid working knowledge of 2-D geometry is required to utilize measuring tools effectively, e.g., knowing that the center of a circle can be located if the locations of three points on the circle are known.

In quality circle analyses observed at one site, small groups of workers isolated segments of their production process, identified inefficiencies, and proposed potential improvements. Systems analysis and representation ("fishbone" analyses) and logical and inferential reasoning were required skills. Quality work at one other high-volume site required line workers to identify and correct variation in their parts. Analysis involved "reading in" a 2-D trace of the part, imposing a 2-D coordinate system on that trace, and checking the values of crucial angles and distances. With that numerical data, workers made manual adjustments to their machine tools. That more mathematical work was in the

hands of line workers at this site was due to this firm's decision to retain and rebuild "in-house" older machine tools, rather then investing in more sophisticated technology whose maintenance requires specific workers with special expertise.

In contrast, at the two sites that produced small numbers (usually ≤ 5) of highly engineered and machined products (e.g., molds and dies for high-volume assembly), production work was much more demanding mathematically. Workers operated computer-mediated (CNC) machine tools which required strong spatial visualization skills, knowledge of basic Cartesian geometry in 2- and 3-space, associated integer arithmetic, plane trigonometry, and ratio conversions between English and metric dimensions, in addition to mastery of computer operating systems, the CNC programming language, and manual machining (Martin & Scribner, 1990). Two interesting classes of non-school problems were observed: (1) "set-up," where machinists manually locate and lock down parts relative to their machines' fixed three axes of motion,[2] and (2) "mapping," where they trace 2-D and 3-D paths in advance of programming tool movements and then determine the Cartesian coordinates of each move and turn. As with the quality work described above, the most salient mathematical requirements of this work were spatial and geometric, though workers and managers alike associate "mathematics" only with numerical computation.

Though this observational work has been productive, there are clear limits to the approach. Initial general tours can easily miss the mathematical aspects of some practices, either regularly occurring or resulting from irregular problems or "breakdowns." Tours are often shaped by the guide s view of what is mathematics, which is often limited to numerical and symbolic computation. It is also difficult to gain sufficient understanding of tasks and workers' reasoning only through observations and informal questioning. Workers are not well practiced in explaining the character of their work to visitors.

To gain more extensive and detailed understandings of the mathematics of work , we are currently studying one mathematically-intensive workplace where dies for pressing car and truck exterior body panels are fabricated. Here we have moved to a wider combination of methods—extensive observation, interviews, and videotaped segments of work. I have "apprenticed" myself to a skilled machinist and learned the nature and flow of work, the language of machining, and machine tool itself. From this apprenticeship of observation and conversation, the structure of the work and common mathematical skills and problems have emerged. The next step involves documenting the machining of one die "job" from entry to exit from the shop. This record will permit more careful analyses of mathematical tasks and solutions than observation alone.

[2] Numbers of axes of motion can exceed three on modern CNC machine tools.

Discussion

The mathematical demands of the high-volume assembly work that we observed are consistent with published analyses of workplace skills. For example, Murnane & Levy (1996) include the ability to "do math" at a 9th grade level or better in their list of "new basic skills." But these "new" skills are not beyond the reach of current U.S. K–12 curricula, when taught seriously to all students. More high-wage, high-skill work like CNC machining, as well as quality work, turns on a stronger base of spatial and geometric abilities than current U.S. curriculum appear to support (Schmidt, McKnight, & Raizen, 1996). By contrast, Germany and Japan devote much more attention to space and geometry in the middle grades. If we are serious that schooling should prepare young people for the cognitive demands of work, we should pay closer attention to which skills are fundamental to a broad array of high-wage jobs. Representation, problem solving, and reasoning in 2- and 3-space is one such candidate competence.

Beyond matching curriculum with basic, valued workplace skills, more studies of mathematical reasoning in workplace settings are needed. Despite some celebrated exceptions, we know very little about reasoning and problem solving in these contexts, relative to the substantial range of work and embedded mathematics. Researchers must be prepared to find the mathematics in the work practices and not rely on workers and managers judgments of what is "math."

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CHANGES OF PRESENTATION IN MIDDLE SCHOOL MATHEMATICS TEXTBOOKS: THE CASE OF INTEGER OPERATIONS

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Previous efforts to identify contributing factors for cross-national differences in students' mathematics achievement (e.g., McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers, & Cooney, 1987) have led to the contention that the school mathematics textbook is a key factor, particularly in the cases of Japan and the U.S., (Mayer, Sims, & Tajika, 1995). The results from Mayer et al.'s study have indicated weaknesses within US mathematics textbooks in their presentations relative to conceptual understanding and problem solving. Since the textbooks analyzed in Mayer et al.'s study were published around the end of 1980s, it is important to know how current texts may have changed their presentations.

Lessons on addition and subtraction of signed whole numbers in six US textbooks were selected for analysis. The pertinent lessons were then coded independently by raters using criteria and procedures described in Mayer et al.'s study. The results indicated that the more recent US mathematics textbooks have devoted more space to worked-out examples, explanations, relevant illustrations, and less space for exercises and irrelevant illustrations than had the US texts in Mayer et al.'s study. These changes may be positive indicators of reform efforts to develop students' conceptual understanding and problem solving. However, the US textbooks still failed to present fully coordinated, multiple representations as part of their explanations of the integer operations, and had scant evidence of using inductive approaches in developing general rules for integer addition and subtraction.

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Data, Probability & Statistics

PROBABILITY INSTRUCTION INFORMED BY CHILDREN'S THINKING

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Responding to world wide recommendations that recognize the importance of having younger students develop a greater understanding of probability, this study designed and evaluated a third-grade instructional program in probability. The instructional program was informed by a cognitive framework that describes students' probabilistic thinking and also adopted a socio-constructivist orientation. Two classes participated in the instructional program, one in the fall (early) and the other in the spring (delayed). Following instruction, both groups displayed significant growth in probabilistic thinking that was not simply due to maturation. There was also evidence, based on four target students, that children's readiness to list the outcomes of the sample space, their ability to connect sample space and probability, and their predisposition to use valid number representations in describing probabilities, were key factors in fostering learning.

The importance of having younger children develop an understanding of probability is now widely advocated (e.g., National Council of Teachers of Mathematics, 1989). This emphasis on probability in the school curriculum has established the need for further, on-going research into the teaching and learning of probability (Shaughnessy, 1992).

While more is known about how students learn mathematics than how to apply knowledge about learning to mathematics instruction (Romberg & Carpenter, 1986), there is an increasing body of literature (Fennema, Franke, Carpenter, & Carey, 1993; Mack, 1990) that advocates the use of research-based knowledge of student's thinking to inform instruction. With respect to probability, there has been considerable research into children's thinking (Fischbein, Nello, & Marino, 1991; Piaget & Inhelder, 1975; Shaughnessy, 1992), but none has evaluated instructional programs that are guided by research-based knowledge of students' probabilistic thinking.

This study addresses the development and evaluation of such an instructional program. In particular, it seeks to: (a) use a framework that describes and predicts children's thinking in probability to construct a third-grade instructional program; and (b) evaluate the effect of two different sequences of the instructional program on children's thinking in probability.

Theoretical Considerations

The instructional program developed and evaluated in this study is based on two theoretical positions. The first, a cognitive framework that describes

children's probabilistic thinking (Jones, Langrall, Thornton, & Mogill, 1997), was used to provide the research base for informing instruction. The second is a pedagogical orientation that espouses learning within a socio-constructivist environment (Cobb, Yackel, & Wood, 1993).

Framework for Children's Probabilistic Thinking and Pedagogical Orientation

The cognitive framework assumes that probability concepts are multifaceted and develop slowly over time. In order to capture the manifold nature of probabilistic thinking, this framework (Jones et al., 1997) incorporates four key constructs: sample space, probability of an event, probability comparisons, and conditional probability, as they relate to one- and two-stage random experiments. The framework provides a coherent picture of children's probabilistic thinking by building on earlier probability research in sample space (Borovcnik & Bentz, 1991), probability of an event (Acredelo, O'Connor, Banks, & Horobin, 1989; Fischbein, Nello, & Marino, 1991), probability comparisons (Falk, 1987), and conditional probability (Borovcnik & Bentz, 1991).

For each of the key constructs, four levels of thinking were established and validated over a two-year period (Jones et al., 1997). Level 1 is associated with subjective thinking, Level 2 is transitional between subjective and naive quantitative thinking, Level 3 involves the use of informal quantitative thinking, and Level 4 incorporates numerical reasoning. These levels of thinking appear to be consistent with neo-Piagetian theories that postulate the existence of levels of thinking that recycle during developmental stages (Biggs & Collis, 1991). This framework provided the research base for informing the instructional program and creating the assessment instruments.

A further tenet of this study was that the potential for effective mathematical learning is optimized when the instructional environment is consistent with a constructivist view of learning. Such a view, grounded in the work of Piaget (1970), and extended by other researchers (Cobb et al., 1993), holds that mathematics learning is a process in which children internally reorganize their thinking to resolve situations that are problematic for them. Moreover, we have adopted the view that mathematical learning is an interactive as well as a constructive process (Cobb et al., 1993).

Methodology

Subjects

The sample for this study consisted of 37 grade 3 children from two intact classes at a University laboratory school. Children from these two classes participated in an instructional program in probability —one class in the fall semester (Early Instruction Group, n=18), the other class during the spring se-

mester (Delayed Instruction Group, n = 19). In addition, two children from each of the two classrooms were randomly selected as target students.

Procedure

Each semester's instructional program consisted of sixteen, 40-minute probability sessions over a period of eight weeks. Following the session opener, a whole-class exploration, twelve teacher education student mentors worked with pairs of children to solve probability problems. These mentors also acted as participant observers, collecting and organizing data on the children's thinking. All children in the study were assessed using a researcher-designed interview protocol at the beginning (September), middle (December), and end (April) of the school year.

Instructional Program

The instructional program consisted of probability problem tasks (Jones & Thornton, 1992) generated from the key constructs of the framework. The tasks were designed so that they would be accessible to children at different levels of the framework. In accord with the pedagogical orientation of the program, mentors were encouraged through weekly seminars to (a) use the framework to assess and build on children's understanding; (b) pose problems and questions rather than model solutions; (c) guide children to construct their own solutions; (d) maximize opportunities for pairs of children to engage in collaborative problem solving; and (e) challenge children to negotiate one or more solutions or approaches to a problem.

Data Collection, Instrumentation, and Analysis

Interview and observational data were gathered from three sources: (a) assessments conducted at the beginning, middle, and end of the school year; (b) mentor evaluations of the four target students from each instructional session; and (c) researcher narratives of observations on each of the four target students and their mentors.

The interview assessment based on the Probabilistic Thinking Framework comprised 20 tasks: five for sample space, four for probability of an event, seven for probability comparisons, and four for conditional probability. Two different procedures were utilized to code the interview assessments. The first procedure, used only with the four target students, involved double coding (Miles & Huberman, 1994) to establish probabilistic thinking levels on each of the four constructs over the three assessment points. The second procedure, which also involved double coding, was used to generate performance scores (maximum = 20) for *all* children in the study over the three assessment points. Each item was scored 1 or 0 according to a two-point rubric: 1 – children completely solved and justified their solution to the problem; or 0 – children were unable to solve the problem or justify their solution. In order to test differences between

instructional groups, a repeated measures ANOVA was also carried out using performance data as the dependent variable.

Data on the four target students were collected on Mentor Summary Evaluations (MSEs) of the children's probabilistic thinking during each session. Researcher Narrative Summaries (RNS), based on field notes and video observations, were also generated to describe each target student's thinking. Multiple codes were assigned to the 16 MSEs and 16 RNSs based on the framework, and a "within-case displays" approach (Miles & Huberman, 1994) was used to document changes in each target student's probabilistic thinking.

Results

The Effect of the Intervention Program: Analysis of Target Students

Target student analysis yielded four learning patterns: (1) growth in the development of systematic strategies for listing sample space outcomes was generally protracted for children who began the intervention at level 1; (2) growth in probability thinking is inhibited when children do not make *connections* between sample space and probability; (3) growth in probabilistic thinking was more pronounced for children who analyzed probability situations using both part-part and part-whole relationships; and (4) growth toward numerical probabilistic thinking was enhanced when children were able to use conventional or valid invented notation for recording probabilities.

With respect to the first two learning patterns, two of the target students, Jana and Kerry, were typical of children whose thinking in sample space restricted their growth in probabilistic thinking. In spite of instruction, both children were unwilling to identify all of the outcomes in a sample space, or unable to build a systematic strategy for listing outcomes. Even when Kerry's sample space thinking matured toward the end of the intervention, she was not able to make connections between the sample space and the probabilities of events within the sample space.

Corey and Deidra, whose thinking profiles were similar to those of Jana and Kerry prior to instruction, illustrate the third and fourth learning patterns. In contrast to Jana and Kerry, these students showed strong and consistent growth across most of the constructs. While Corey and Deidra's rapid growth in sample space thinking was an important factor, a more crucial factor was their predisposition to use part-part *and* part-whole relationships to analyze probabilities from a quantitative or numerical perspective. In using these relationships to describe probabilities, Corey used invented notation like "2 out of 6," whereas Deidra consistently used fractions or percentages.

The Effect of the Intervention Program: Analysis of the Two Instructional Groups

The probability performance of *all* children in both instructional groups was analyzed at each of the three assessment points. (Figure 1) Relevant means and standard deviations for the three assessment points for each group were, respectively: Early Instruction—$\underline{M}$ = 12.44, $\underline{SD}$ = 2.56; $\underline{M}$ = 15.39, $\underline{SD}$ = 2.70; $\underline{M}$ = 15.00, $\underline{SD}$ = 2.47; Delayed Instruction—$\underline{M}$ = 13.11, $\underline{SD}$ = 2.77; $\underline{M}$ = 13.95, $\underline{SD}$ = 2.32; $\underline{M}$ = 16.63, $\underline{SD}$ = 1.54. A repeated measures analysis of variance revealed significant differences for the three assessment points ($\underline{F}(2,70) = 12.88$, $\underline{p} < .001$) and a groups by assessment points interaction ($\underline{F}(2,70) = 6.38$, $\underline{p} < .01$). Further analysis using the Tukey-HSD test, showed that the interaction was disordinal, being produced by significant but reversed differences in the means of the two instructional groups at the middle and end assessment points.

Discussion

Although there has been a call for the development and evaluation of instructional programs that are informed by research-based knowledge of children's thinking, virtually all of the studies responding to this call have focused on whole number operations and fractions (Fennema, et al., 1993, Mack, 1990). Significantly this study investigated an instructional program that was informed by a framework describing children's probabilistic thinking and was based on a socio-constructivist orientation to learning (Cobb et al., 1993).

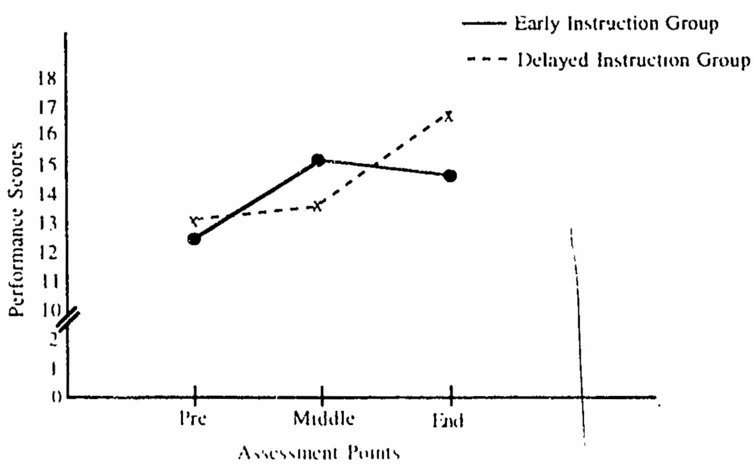

Figure 1 Performance at Pre, Middle, and End Assessments

A repeated measures ANOVA, used to evaluate the instructional program, demonstrated that both the Early and Delayed Instruction groups showed significant growth in performance following instruction. Moreover, because the Delayed Instruction group essentially acted as a control group between the first and middle assessment points, the significant difference in favor of the Early Instruction group at the middle assessment point provides further evidence that learning was not solely due to maturation. This quantitative analysis endorses the work of Fennema, Franke, Carpenter, and Carey (1993) by demonstrating that the probabilistic thinking framework can be used to design and implement an effective instructional program.

Notwithstanding the overall effectiveness of the instructional program, the size of the standard deviations for both groups at almost all stages of the study predicate substantial variation in the probabilistic thinking of these students. Some insights into these variations were revealed from the target student analysis. The differential effects appeared to be linked to four discernible learning patterns: children's initial level of thinking in sample space; their willingness to connect sample space and probability; their predisposition to use both part-part and part-whole relationships; and their ability to use conventional or valid invented notation to record probabilities.

Although this study was limited by the fact that instruction was carried out by mentors rather than by a classroom teacher, a number of implications can be drawn for probability instruction. Given the effectiveness of the framework in informing instruction, probability programs could benefit from the development of resources which incorporate a cadre of probability problems based on the constructs and thinking levels of the framework. Further, based on the evidence of mentors working with children in this study, teachers should find the framework helpful in monitoring children's probabilistic thinking, in identifying misconceptions, and in fostering learning.

Future research is needed to evaluate the viability of the framework for monitoring and fostering probability learning in regular classroom situations. Such research would assess the ease with which classroom teachers can use the probabilistic thinking framework to inform instruction. There is also a need for further research to investigate the long term effects of delaying systematic instruction in probability. Given the impressive growth in probabilistic thinking exhibited by children who had a predisposition to use numbers and invented notations, there may be merit in delaying systematic instruction in probability until children have greater "number power."

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STUDENT UNDERSTANDING OF STATISTICS: DEVELOPING THE CONCEPT OF DISTRIBUTION

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In the past decade, various countries have produced national mathematics education reform documents that recommend that students learn to produce, explore and interpret distributions of data meaningfully. The purpose of this study is to expand on what has been learned about student notions of the average as a representation of a distribution. For this paper we report interview data with Jim, a middle school student. We followed the investigation of Mokros and Russell (1995) and their framework for understanding how children develop an understanding of the concept of mean. Jim did not seem to fall clearly into any one of the strategy types identified by Mokros and Russell. His problem solving strategies for finding the mean seem to vary, but at the same time he maintains a consistent interest in making sense of the results of his computations. Jim did not appear to make sense of the mean as a statistical measure of a distribution.

Statistics is receiving increased attention in school mathematics nationally and internationally. In the past decade, countries such as Spain, Australia, Great Britain and the United States have produced national mathematics education reform documents in which statistics, or data handing, is a main component (Shaughnessy, Garfield & Greer, 1996). These documents recommend that students learn to produce, explore and interpret data in a meaningful way. Recently developed statistics curriculum materials, such as textbooks and computer software, appear to have been created in the spirit of these recommendations (e.g., Used Numbers, Connected Mathematics, Advanced Placement Statistics, Tabletop software and Datascope software). However, because data analysis has traditionally received so little attention in the school mathematics curriculum, the teaching and learning of statistics has received little attention, and thus little research is available to support the recommendations.

Shaughnessy et al. (1996) summarize the guidelines in the various national documents that recommend that students learn to collect and analyze data. Although it is not stated explicitly, it appears that a central goal of the recommendations for school statistics is that students develop an understanding of the concept of distribution. That is, they should learn to create visual and graphical representations of data so that they can "see" how the data are distributed; they should learn to compute and apply statistics, such as measures of central

tendency ("center") and variability ("spread"), as summaries of empirical distributions; and ultimately, they should learn to make inferences, decisions and predictions based on distributions. This consensus suggests that a fundamental aim is that students come to understand the concept of distribution by representing and summarizing distributions, and that they learn to think about how data are distributed in order to make judgments about data sets.

Although little research on student conceptions in statistics is available, there is a small body of research on student understanding of the average concept. Underlying this concept of center, however, is the more basic concept of distribution, which enables meaningful interpretation of measures of center (and other statistics) for a set of data. This is one reason that visual and graphical displays of data are central to data analysis—such displays enable us to see how the data are distributed. Research on student understanding of the concept of distribution is in the early stages, but research on the average as a representative value of a data set is available.

The purpose of this study is to expand on what has been learned about student notions of the average as a representative measure. We wanted to further clarify what is known about student understanding of the mean concept and to link our results to classroom practice.

Background

Researchers have studied student understanding of the average concept, which includes mean, median and mode, from a variety of perspectives and approaches. They have looked at weighted means (e.g., Pollatsek, Lima, & Well), mathematical properties of the mean (e.g., Strauss & Bichler), balancing and leveling models of the mean (e.g., Cai & Moyer, 1995; Pollatsek et al., 1981), and representativeness of the average (Mokros & Russell, 1995; Strauss & Bichler, 1988). Although these studies have identified important information about student difficulties with the average concept, they have revealed little about how students make sense of an average in the context of the distribution that it represents or summarizes.

Mokros and Russell (1995) provide some insight into how students make sense of an average in the context of real data. They studied how children construct and interpret representativeness for a real data set, and how children understand the mean and connect it with their informal understanding.

Mokros and Russell believe that "a well-developed notion of representativeness should include an understanding of the mean and how it works" (p.22). Their point of view draws on student understanding of other mathematical properties of the mean (e.g., the mean is a number not necessarily in the data set) and of the mean as a representation of a real data set. Mokros and Russell also believe that an investigation of student understanding of average should in-

clude an examination of how a child describes and constructs sets of data. Prior research studies have not included such an examination.

Mokros and Russell asked children in grades 4, 6 and 8 to solve different types of problems involving data, including two in which they had to construct sets of data. One of the data construction tasks is the Potato Chip Problem: In the first problem, the task was to put price stickers on pictures of nine bags of potato chips so the "typical or usual or average" price of the chips would be $1.38. The students were asked to make these price stickers without using $1.38 (the average value itself) in the data set. (p. 23) In the remaining tasks, they asked the children to interpret data and to solve at least one problem involving weighted means. They used the words average, typical and usual interchangeably to find out which word had meaning for the child, and continued with that word. The children were given graphs, pictures and other materials appropriate for each the problem (e.g., price stickers for the Potato Chip Problem). Mokros and Russell then identified and classified student solution strategies and usually found a preferred approach for each student. They arrived at five predominant approaches: modal, algorithmic, reasonable, midpoint and balance point. The modal and algorithmic approaches were characterized as "approaches in which average is not viewed as representative." The other three approaches considered the average to be based on the idea of representativeness.

Jim's Notion of Average

For this paper we report interview data with Jim, who attends a private church affiliated middle school. He is currently taking a pre-algebra mathematics class using a textbook that appears to reflect the vision of the NCTM Standards in probability and statistics for the middle grades (National Council of Teachers of Mathematics, 1989). Jim had completed a chapter on basic concepts from statistics, and was familiar with the arithmetic mean, graphing, and numerous other topics, including stem-and-leaf plots and box-and-whisker plots. He was also familiar with calculators, and had one available during the interview.

In our study we followed the investigation of Mokros and Russell (1995) and their framework for understanding how children develop an understanding of the concept of mean. The interview began by asking Jim to read the Potato Chip Problem. Jim's first reaction after reading the problem was one of confusion; he thought it was a "trick question" because he was not supposed to use $1.38 as one of the prices. After several probes Jim finally understood the task, and used a process of trial and error to solve the problem.

He confidently wrote down nine prices with the lowest $1.27 and the highest $1.50. He did not write the nine numbers in any particular order; he pro-

ceeded quickly and confidently, and did not appear to have any specific method for constructing the prices, except that they were all realistic, and all relatively close to the target price ($1.38). He added the numbers on a calculator, divided by 9, and asked if he could round up to $1.37. He appeared surprised that he came so close on his first try. Jim revised his list by reducing one of the prices of the bags by $0.01. Then he decided to add his new list without using the calculator, even though the only change he had made was to reduce one of the prices by a penny. When he completed his total "by hand" he got the wrong answer, but with a little prompting, he recognized that the correct total would have to be one cent less than his previous answer. At this point, his strategy solidified, and Jim continued to add to his total, getting closer to the desired sum, 9 x $1.38.

The interviewer reminded Jim that he had not yet solved the problem, since he had not assigned prices to the potato chips. When it did become apparent to him that he needed to price each bag of chips, he returned to his original list of nine prices and changed one price from $1.31 to $1.39 because he needed to add eight cents to his original total.

Jim did not seem to fall clearly into any one of the strategy types identified by Mokros and Russell (1995). He used the "average as an algorithm" strategy but clearly went beyond that level. When he listed his nine reasonable prices for the bags of chips, Jim showed evidence of the "average as reasonable" strategy. He did not appear to use the "average as a midpoint" and b[illegible]nce his prices. He almost demonstrated using average as a "point of balance," however, because he appeared to understand the quantitative relationship between the total and the average; he was able to go back to his original list and accomplish his goal by changing the price on one bag by eight cents.

Among the additional tasks used in this interview was the Elevator Problem, adapted from Pollatsek et al. (1981). This task dealt with weighted means: could 6 men averaging 180 pounds and 4 women averaging 125 pounds all ride on an elevator with a weight limit of 1500 pounds. Jim was also asked to find the average weight of the 10 people on the elevator. This task was a challenge for Jim, although with some questioning from the interviewer he was able to solve the Elevator Problem.

After Jim finished solving these two problems (plus two additional problems not discussed here) he was asked if he had done similar problems before. He replied that he had done statistics in his mathematics class, but "not with elevators," just with "lots of apples, lots of farmers, and lots of fences." He thought the problems were "pretty easy," but he said, "the one that really made me think was the first one." Jim indicated that after thinking carefully about the first one, he had a new understanding of the relationship of "the total amount like in exercise one [the Potato Chip Problem]" and the mean. He was able to use this new understanding in the other tasks that the interviewer presented to him.

Although Jim's understanding of the mean is not well developed, he has a number of characteristics that make him an interesting subject. Jim knows the algorithm for finding the mean. His problem solving strategies for finding the mean seem to vary, but at the same time, he maintains a consistent interest in making sense of the results of his computations. He shows some tendencies toward thinking of the average as a point of balance, but it appears to be his keen number sense that allows him to solve problems like those in this study. Jim thus thinks about the mathematical properties of the mean, and probably does not think about the mean in a statistical way, as a representation of a distribution.

Discussion

This report focuses on the interview with Jim, but in other interviews (not reported here), students in high school and college were unable to solve the Potato Chip Problem (e.g., Mellissinos, 1997) and the Elevator Problem (Pollatsek et al., 1981). Like Jim, many students who have shown difficulty with these problems knew the algorithm for finding the mean. One reason for the student difficulties may be that students have not learned to think about the mean as a representative measure of a distribution.

Mokros and Russell explain their notion of representativeness as follows: "As soon as there is the need to describe a set of data in a more succinct way, the notion of representativeness arises: What is typical of these data? How can we capture their range and distribution?" (p. 20). Their notion of representativeness involves capturing a distribution of data, but they use the terms "distribution" and "data set" interchangeably, so it is not clear exactly how they interpreted and established representativeness. These terms may be distinguished as follows: A data set is a collection of measurements of one or more characteristics (of objects or people). A distribution is an attribute of a data set that communicates how measurements in a data set are distributed across its range of values. This distinction is more than a matter of semantics. In a substantial way, Mokros and Russell based their study of children's concepts of the average as representative on the concept of distribution. Yet it is not clear how Mokros and Russell think about the concept of distribution themselves. Without a clear idea about distribution, it is difficult to make inferences about student notions of representativeness.

Without a clear idea about distribution, it is also difficult to make sense of data and data descriptions, such as statistics (e.g., mean) and graphs (e.g., histograms). Computer technology offers potential for students to come to understand the notion of distribution because it allows students to visually explore data. They can easily compute statistics and create graphs, rather than spend time calculating statistics and drawing graphs with pencil and paper. The use of computer technology, however, does not ensure that students will learn to

make sense of data. Student exploration of data must be guided by appropriate curriculum materials and instruction. If we better understand student notions about distribution, curriculum developers and teachers may be better able to facilitate student understanding of statistics.

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PRE-SERVICE SECONDARY SCHOOL MATHEMATICS TEACHERS' UNDERSTANDING OF DATA ANALYSIS AND THE FITTED CURVE

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In 1989 the NCTM Curriculum and Evaluation Standards for School Mathematics included a goal for secondary school students which stated, "The mathematics curriculum should include the continued study of data analysis and statistics so that all students can . . . use curve fitting to predict from data." Many studies have found that students develop increased conceptual understanding when taught mathematics using technology (Lauten, Graham, & Ferrini-Mundy, 1994; Marshall, 1996). But studies show that many students still have difficulty with graphical interpretation in general and with the relationships between graphical and symbolic representations in particular (Leinhardt, Zaslavsky, & Stein, 1990).

One of the interesting phenomena that was observed during this study was the difference in the treatment of numeric data presented in the problem and subject generated data. Whereas all of the subjects used the calculator as a tool to generate symbolic representations of the data presented in numeric form, not one subject attempted to produce such a solution to the tasks that were presented in problem format despite the fact that all of them attempted to solve the tasks by generating numeric data almost identical in form and format to the numbers presented in the other problems. All of the subjects attempted to solve the tasks numerically without attempting to graph or fit curves to any of the data that they produced. Initial analysis of the data contained in this study suggests that pre-service mathematics teachers have an incomplete understanding of data analysis and curve fitting.

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THE COMPLEXITY OF TEACHERS' KNOWLEDGE: AN EXPERIENCED TEACHER'S STRUGGLES WHEN CONSTRUCTING PICTORIAL REPRESENTATIONS ABOUT CONDITIONAL PROBABILITY

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The purpose of this paper is twofold: (a) to examine one teacher's knowledge of pictorial representations about conditional probability (CP), and (b) to examine the difficulties that the teacher faced when constructing some of the representations of this topic. The research reported here is part of a larger project (Contreras, 1997). I asked the participant to create a pictorial representation for three concepts related to CP using interviews and questionnaires. The interviews were audiotaped and transcribed. The three concepts examined were: (a) The definition of CP as P(A I B) = Area (A«B)/Area (B), (b) the definition of CP as $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, and (c) the conditional probability formula, P(A « B) = P(B)P(A I B). The teacher was able to construct a correct pictorial representation for each of the three concepts. Regarding difficulties, the participant did not struggle to provide a correct pictorial representation for (a) nor (c). I asked the participant to construct a pictorial representation for (c) twice because the first time he did not state explicitly why the pictorial representation he first constructed illustrated P(A « B) = P(B)P(A I B). During the second time some understanding of why the pictorial representation illustrated $P(A \cap B) = P(B)P(A \mid B)$ was explicit. The case (b) was more problematic because the participant was not able to construct a pictorial representation during three occasions. It was when the participant was working to provide a word problem for (b) that he had an insight on its pictorial representation. He then constructed the pictorial representation requested. The poster will display a sample of the pictorial representations and an analysis of the participant's thinking behind the representations to illustrate the complexity of his knowledge.

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FUNCTIONS AND GRAPHS

A SNAPSHOT OF DEVELOPMENTAL ALGEBRA STUDENTS' CONCEPT IMAGES OF FUNCTION

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This paper discusses the feasibility of using the function concept as a core idea in a technology-rich, college developmental mathematics course. Written surveys, before and after a "pilot" beginning algebra course, and task-based interviews are used to build a profile of developmental algebra students' concept image of function. Some quantitative results along with contrasting partial profiles of two students are discussed. The data suggest that the target students can develop, minimally, a process-level understanding of function, though the profiles indicate how complex and uneven the resulting understanding often is. The research suggests that using function as a core concept in developmental mathematics has potential, but particular attention must be paid to specific problem areas, such as graphs and the connections between various representations.

Introduction

U.S. college mathematics faculty encounter a sizable percentage of students who begin their college career in a non-credit mathematics course such as arithmetic, geometry, beginning algebra, or intermediate algebra. It is likely that many developmental algebra students have been severely debilitated by their previous exposure to mathematics. Succeeding with this population may require providing the students with a completely different educational experience, such as building a beginning algebra course around the function concept.

The theoretical framework for this research was initially described by DeMarois and Tall (1996) who suggest a structure for analyzing mathematical concepts along both breadth and depth dimensions. Schwingendorf et al. (1992) contrast the vertical development of function in which the process aspect is encapsulated as a function concept and the horizontal development relating different representations. They refer to these as depth and breadth respectively and investigate the way in which the student's concept image (Tall & Vinner, 1981) of function can be described in terms of these two dimensions.

DeMarois and Tall (1996) use the term "layer" to refer to various levels of the depth dimension in the development via cognitive process to mental object. The depth dimension has been discussed extensively in the literature, including Dubinsky's APOS construction (Cottrill et al., 1996; Breidenbach et al., 1992; Dubinsky & Harel, 1992), Sfard's (1992) process acting on familiar objects which is first interiorized, then condensed, and finally reified as an "object-like entity," and Gray and Tall's procept theory (1994). Following Davis (1983), Gray and Tall distinguish between a process that may be carried out by a vari-

ety of different algorithms and a procedure that is a "specific algorithm for implementing a process" (1994, p. 117). A procedure is therefore cognitively more primitive than a process. In the framework for this research, pre-procedure, procedure, process, concept, and procept are considered layers of increasing depth in the understanding of a concept. The breadth dimension is conceived as consisting of various representations, including geometric, numeric and symbolic. DeMarois and Tall use the word "facet" to build up a description of the breadth dimension. The facets of a mathematical entity refer to various ways of thinking about it and communicating to others, including verbal (spoken), written, kinesthetic (enactive), colloquial (informal or idiomatic), notational conventions, numeric, symbolic, and geometric (visual) aspects.

This research focuses on the understanding of function that students acquire as a result of completing a technology-rich, "reform" beginning algebra curriculum (DeMarois, McGowen & Whitkanack, 1996) in which "function" is the key concept and graphing calculators are required. Can adult students who arrive at college having had debilitating prior experiences with algebra develop a rich concept image of "function" through appropriate instructional treatment? Analysis of written responses to pre- and post-course surveys suggests statistically significant positive shifts in students' ability to answer questions involving function machines, equations in two variables, two-column tables, and graphs. Analysis of task-based interviews suggests that student understanding of various facets is sometimes deep and mature and other times shallow and misinformed.

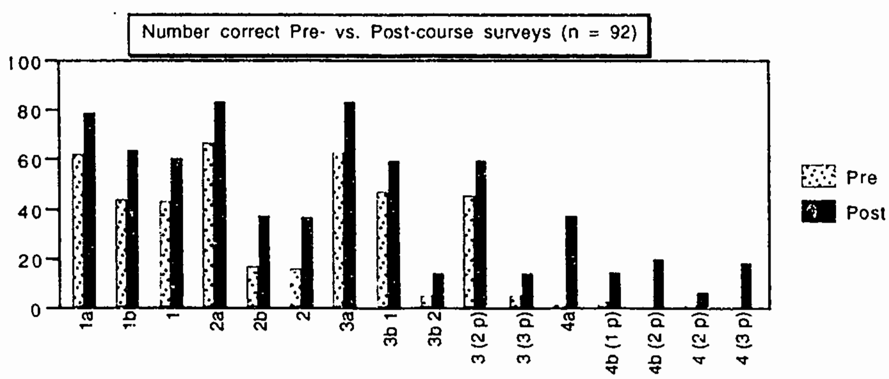
Method

The study was conducted on students enrolled in "pilot" sections of beginning algebra at 4 different community colleges. Ninety-two students completed written function surveys at the beginning (first day) and at the end (last day) of the course during the Fall Semester, 1996. Subsequently, three students at each site participated in task-based interviews which were conducted one to two weeks after the end of the course. The interviews were video- and audio-taped. All questions on the pre-course survey were repeated on the post-course survey and on the interviews. Additional questions on the post-course surveys were also asked during the interviews.

The common questions on the pre- and post-course surveys were analyzed quantitatively by measuring the significance of the changes in responses from beginning to end of the instructional treatment. The data collected during the interviews along with the written surveys were analyzed qualitatively to create before and after snapshots of the depth of student understanding of function.

Results

Due to space confines, partial quantitative and qualitative results on colloquial, symbolic, numeric, and geometric facets will be presented. Students were asked the same question about each of these facets on all 3 instruments: Question 1 involves a linear function expressed as a function machine; Question 2 involves a linear function expressed as an equation in two variables; Question 3 involves a quadratic function expressed as an input/output table; Question 4 involves a quadratic function expressed as a rectangular coordinate graph. Students were asked to find the output given the input (part a) and the input given the output (part b) thus assessing students' ability to apply a procedure (procedure layer) and reverse a procedure (process layer) for the colloquial (function machine), symbolic (equation in two variables), numeric (table), and geometric (graph) facets. A comparison of students' scores on pre- and post-course surveys appears below.



One point was awarded for a correct answer to part a and 2 points for a correct answer to part b. The additional weighting for part b reflects the added difficulty inherent in the "reversal" of the function. Since 3b and 4b had 2 answers, students were given 1 point for each correct answer. Each question is broken into parts a, b, and total score. Questions 3 and 4 are further subdivided depending on whether the student gave one or both correct answers to part b. On all four questions, a t-test for related cases indicates a significant positive change in total score from pre- to post-course survey (Question 1: $t = 4.07$; Question 2: $t = 4.61$; Question 3: $t = 5.16$; Question 4: $t = 7.56$. For all questions, $df = 91$ and $p < 0.0005$).

CH and GK are students in the same class who participated in the interview. CH is a female between 21 and 25 years of age who had 1.5 years of algebra prior to coming to college. CH earned an A in the beginning algebra course. Throughout her interview, she appears happy and confident, constantly smiling and anxious to attack each question. GK is a male between 30 and 35 years of age who had 1 year of algebra prior to coming to college. GK barely earned a C in the course. His face during the interview expresses utter agony. As a student of math he says he is very poor. He offers: "I don't seem to grasp it. It goes in one ear and out the other. It is very frustrating. I have one C in school and that was in this class."

CH's and GK's scores on Questions 1–4 (1 point for part a, 2 points for part b possible) appear in Table 1 along with their responses to the same questions on the interview.

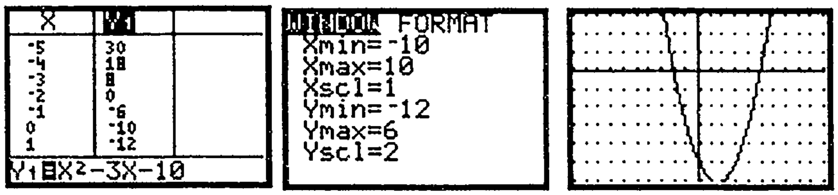
The data suggest that CH was at the process layer at the beginning of the course on the colloquial (function machine) facet and possibly the numeric (table) facet. By the end of the course, she had moved at least to the process layer on all 4 facets and demonstrated retention of her knowledge from the post-course survey to the interview. GK began the course at the procedure layer on the colloquial facet only. By the end of the course, he appears to be at the process layer on the symbolic facet though he couldn't demonstrate this during the interview. GK moved to the procedure layer on the numeric facet, remained at the procedure layer on the colloquial facet and at the pre-procedure layer on the geometric facet. Furthermore, he was inconsistent on several responses between the post-course survey and the interview.

Table 1 Points Awarded Pre-Course , Post-Course, Interview CH and GK

	Question 1			Question 2			Question 3			Question 4		
	Pre	Post	Int	Pre	Post	Int	Pre	Post	Int	Pre	Post	Int
CH a	1	1	1	1	1	1	1	1	1	0	1	1
CH b	2	2	2	0	2	2	1	2	2	0	2	2
GK a	1	1	1	0	1	1	0	1	1	0	1	0
GK b	0	0	2	0	2	0	0	0	1	0	0	0

A specific interview question investigated the boundaries that exist between three facets. An equation, a table, and a graph are displayed below for the same function.

$y(x) = x^2 - 3x - 10$



Students were asked a series of questions involving finding outputs given inputs and vice versa. Students were first asked to find the output if the input is –5. Both CH and GK responded with 30 immediately and indicated they used the table. Notice GK's comment regarding his preference.

GK 30. I used the table. It's a lot easier than messing around with the graph or using trace or zoom in. I like the tables.

Next they are asked to find the output if the input is 4. Note that this input does not appear in the table. CH responded correctly using the graph. GK's response is most interesting.

GK It's not showing on my table. If I had the calculator I'd scroll up or down then I'd hafta ...–6.

Intvw Tell me what you are doing.

GK Well the input is 4 so first I'd square it and then 3 times 4 is 12 subtract it and subtract 10 to get –6.

Intvw Okay–continue.

GK I plugged 4 in for *x*. In the equation.

Noteworthy is the fact that he avoided the graph, but was able to evaluate the function at 4 quite easily. The interview then shifted to asking students for outputs given inputs.

Intvw What are the input(s) if the output is 0?

CH (Looks in table) –2. (Switches to graph) And 5. First I use the table and then I looked at graph, saw the parabola, and saw there was another answer.

GK Input is a –2. Table.

Intvw Are there any others?

GK Not that I can see from this table.

Intvw Okay. Any possibilities from the other forms?

GK Probably but I just don't know.

While CH displayed a capability of using two facets simultaneously, GK was unable to move to the graph never locating the second input. GK remained firmly tied to his favorite facet: the table. CH's ability when given the output of 0 to use the table and graph simultaneously without prompting suggests she crosses this boundary easily. GK is much less flexible. He was unable to use the graph at all and was only able to use the equation procedurally.

Discussion

Comparison of pre- and post-course surveys indicates that students were able to demonstrate improved capabilities in interpreting a function defined:

1. by a function machine both from input to output and vice versa.
2. symbolically both from input to output and vice versa.
3. by a two-column table both from input to output and vice versa.
4. by a graph both from input to output and vice versa.

The data suggest that function machine may serve as a good entry point to function noting the high percentage of students capable of using that facet at the beginning of the course. On the other hand, graphs prove particularly difficult for students as indicated by the low success rate at the end of the course on this facet.

The interviews indicate that the growth of the concept image of function in students is complex and uneven. The cognitive links between facets is sometimes nonexistent, sometimes tenuous, and sometimes unidirectional. We see that, by the end of the course, CH was equally comfortable with a function machine, an equation, a table and a graph, but GK was really only comfortable with the table and function machine though he did indicate some ability with the equation. He virtually froze when any question arose relating to graphs. Keep in mind that this problem with graphs persists in a technology-rich environment in which students were required to buy and use graphing calculators throughout the course.

The student population, including CH and GK, for this research is a high-risk group who have had little prior success with mathematics. Using "function" as a focal point of their beginning algebra course, the authors hope to provide students with a vehicle to build meaning into their work with algebra. While some common misunderstandings about function appear in the data, the in-depth analysis suggests that function is not beyond the conceptual grasp of students at this level. Initial indications point to particular problems with graphs indicating some change in the curriculum to address these areas. Crossing boundaries from one facet to another proves difficult with students exhibiting inconsistent definitions when looking across facets.

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PRESERVICE TEACHERS' COGNITIVE APPROACHES TO VARIABLES AND FUNCTIONS[1]

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This study identifies different cognitive approaches to the concepts of functions and variables used by preservice teachers. In particular, the difficulties which arise with the concepts of variables and functions are described. Twenty-five undergraduate elementary and secondary school preservice teachers completed the Chelsea Diagnostic Mathematics Test for Algebra and a researcher-developed instrument focusing on the function concept. After the written instruments were completed, one-to-one individual interviews were conducted. Preservice teachers were assigned four levels of understanding both variables and functions. Almost all of the preservice teachers successfully worked with functions using sequential reasoning, but many struggled to produce equations for the general relationship. The findings indicate that a high-level of understanding of functions is always accompanied by a high-level understanding of variables but not necessarily vice versa.

Statement and Significance of the Problem

Perceiving a need to help those who are preparing to teach mathematics at both the elementary and the secondary school levels, this study attempts to describe undergraduate preservice teachers' levels of understanding and the difficulties which arise with the concepts of functions and variables. Functions and variables are central to the study of many areas of mathematics including algebra and calculus. Furthermore, the *Standards* (NCTM, 1989, 1991) emphasize that preservice teachers must possess content knowledge of concepts such as functions and variables with sufficient depth and breadth in order to later help their students learn and understand these concepts.

Theoretical Framework

This study is built on a research framework which includes the work of many researchers on constructivism (e.g., Piaget et al., 1977; von Glasersfeld, 1991), the epistemology of the function concept (e.g., Confrey, 1991; Even, 1990; Inhelder & Piaget, 1958; Piaget et al., 1977; Wilson, 1992), different approaches to variables (e.g., Collis, 1975; Kuchemann, 1978, 1981), and knowledge acquisition under different modes of representation (e.g., Moschkovich et al. 1993; Piaget et al., 1977).

[1]The research for this article is based upon the author's recently completed doctoral dissertation. This dissertation was completed under the direction of Dr. Helen Khoury, in the Department of Mathematical Sciences, at Northern Illinois University.

This study uses the four levels described by Piaget and his colleagues (Inhelder & Piaget, 1958; Piaget et al., 1977) as a way to classify the preservice teachers' understanding of the function concept. These four levels include: a failure to link the two variables in any systematic manner, an ability to complete ordered pairs on a trial and error basis but without an ability to coordinate, a sequential linking of the ordered pairs with a qualitative relationship, and a generalized and quantitative relationship.

This study also addresses the different approaches to the concept of a variable as described by Collis (1975) and applied by Kuchemann (1978, 1981). These approaches to letters (variables) include: letter evaluated, letter ignored, letter as object, letter as specific unknown, letter as generalized number, and letter as variable.

Research Methodology

Students

A total of 19 elementary school preservice teachers and six secondary school preservice teachers participated in this study. All 25 preservice teachers were enrolled at a small private liberal arts college in the Midwest of the United States. The six secondary preservice teachers, three males and three females, had each completed three semesters of calculus and at least two advanced courses. These students were enrolled in a course focusing on methods of teaching mathematics in the secondary school. The 19 elementary preservice teachers, one male and eighteen females, varied in their mathematical backgrounds, ranging from one year of high school algebra to some experience with calculus. These students were enrolled in the first course in a two semester sequence of content mathematics courses for elementary education majors.

Data Collection and Analysis

After initial data was collected concerning these preservice teachers' understanding of the variable concept using the Chelsea Diagnostic Mathematics Test for Algebra (Brown et al., 1984), a second written instrument was completed which focused on the preservice teachers' levels of understanding of the function concept. The different functional situations on the written instrument are set in a variety of different representational modes, including tabular, symbolic, graphical, and real world setting.

After the written instruments were completed, individual interviews were held between the researcher and each preservice teacher. These interviews each lasted approximately fifty minutes and focused on both the previously completed written work and several additional problems concerning functional relationships. The interviews were videotaped and later transcribed by the researcher. The written instruments and videotaped interviews constitute the data

used to document these preservice teachers' levels of understanding variables and functions.

Results and Discussion

All six of the secondary preservice teachers passed all four performance levels and seemed to demonstrate an understanding of variables which could be classified as both procedural and relational (Hiebert & Lefevre, 1986; Skemp, 1978). The results with the 19 elementary preservice teachers were more consistent with the findings of Kuchemann (1981), which focused on secondary school students. Forty-two percent ($n = 8$) of the elementary preservice teachers were unable to pass the highest performance level. These students relied almost exclusively on procedural understandings of the variable concept, typically applying the letter evaluated and letter as object approaches. Even among the 11 elementary preservice teachers who passed all four performance levels, few of these students demonstrated a relational understanding of the variable concept, as they struggled to apply the letter as generalized number or letter as variable approaches.

Several different cognitive approaches to functions were demonstrated by these preservice teachers. When problems were given in a tabular mode, a few of the elementary preservice teachers seemed unable to coordinate or covary the input and the output variables. In these instances, a pattern observed in the output variable was extended without regard to changes in the input variable. Their written responses did not demonstrate any knowledge of a generalized relationship between the input and output variables.

Other preservice teachers seemed able to covary the input and output variables but these covariations were restricted to a sequential approach. Thus, when the input variable skipped several integers, these students inserted the missing input values, either implicitly or explicitly, and then continued the pattern sequentially until the final output was found. These preservice teachers were unable to provide a generalized equation or other description of the underlying function. In several cases, the written explanations and interview transcripts demonstrate that these preservice teachers recognized that a generalized description did exist but that they could not find one. Among the six secondary preservice teachers, two applied this approach for an exponential function placed in a real-world setting and one applied this approach to both linear and exponential functions placed in a tabular setting. These findings seem to indicate that while these preservice teachers had prior experience with functions, they might have benefited from a greater emphasis on the search for and description of generalized patterns and relationships.

Finally, some of the preservice teachers were able to apply both sequential and generalized approaches to functions. Among the 19 elementary preservice

teachers, the number demonstrating both approaches varied from only 16% (n = 3) on both linear and exponential functions set in the tabular mode to 74% (n = 15) on a linear function set in the real-world mode. By contrast, the number of secondary preservice teachers demonstrating both approaches ranged from 67% (n = 4) on an exponential function set in the real-world mode to 100% (n = 6) on both linear and quadratic functions set in the real-world mode. Preservice teachers demonstrating these cognitive approaches appear not only to have both the necessary prior experience with functions but also to have constructed a concept of function which can be generalized, especially when functions are set in the real-world mode.

The results of this study seem to confirm Inhelder and Piaget's (1958) conjecture that there are different levels of understanding for the function concept. Since this study was not longitudinal, it could not establish whether or not the levels were hierarchical stages through which each student must pass. Rather, this study represents a snapshot of the levels at which these preservice teachers were operating at the time. The study found that the sequential approach to functions, or Level 3, is most common among elementary preservice teachers while the generalized approach to functions, or Level 4, is most common among secondary preservice teachers.

In addition to confirming the existence of these Piagetian levels of understanding the function concept, this study extends this area of research by reporting a relationship between the preservice teachers' level of understanding variables and their levels of understanding functions. In particular, the results suggest that a high level understanding of the function concept is associated with an equally high or even higher level of understanding of the variable concept. The study also found cases where a Level 4 understanding of variables was not associated with a Level 4 understanding of functions but the opposite phenomenon was not observed among the 25 preservice teachers in the study.

Specifically, out of the 25 preservice teachers, 68% (n = 17) demonstrated a Level 4 understanding of the variable concept. Within this group of 17, only 6 demonstrated a generalized approach to functions in every problem setting, 9 showed a generalized approach in some of the problem settings, and 2 used strictly sequential approaches for functions. Thus, for this group, a high level understanding of the variable concept seems necessary but not always sufficient for a high level understanding of the function concept. Among the remaining 32% (n = 8) of the preservice teachers, all elementary, their understanding of the variable concept ranged from Level 1 to Level 3, but none were able to apply a general approach for functions on a consistent basis.

Implications

This study has several important implications. First, several individual conclusions highlight several different ways that preservice teachers approach

variables. The secondary preservice teachers were able to approach variables in a generalized manner, whereas most of the elementary preservice teachers used a more procedural or instrumental approach to variables. The differences in mathematical backgrounds provide a possible reason for these differences. Even among the elementary preservice teachers, those who demonstrated a Level 4 understanding of variables usually had taken three or four years of high schooi mathematics while some of those who did not pass Level 4 did not take a second year of high school algebra. Thus, since a generalized understanding of variables is essential to teachers at the middle school level, this study might be used to argue in favor of requiring the equivalent of four years of high school mathematics for all teachers whose certification includes the middle school level.

Second, a similar and equally important impiication would apply to the result on preservice teachers' levels of understanding functions. Most of the elementary preservice teachers used a sequential approach to functions and experienced difficulty with exponential functions, while most of the secondary preservice teachers used a generalized approach and demonstrated success with linear, quadratic, and exponential functions. Once again, mathematical background appears to be one key factor, although cognitive development may also play a role. Since patterns in general and functions and relations in particular are being introduced in the middle school, the breadth and depth of the mathematical requirements for those teachers whose certification includes the middle school level may need to be reexamined in light of this study.

Third, this study identifies a link between the concepts of variables and functions. The finding that a majority of the elementary preservice teachers passed all four performance levels for variables while only two elementary preservice teachers demonstrated a Level 4 understanding of functions might mean that these students need more learning experiences which require the student to coordinate two variables.

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INVESTIGATING STUDENTS' KNOWLEDGE AND DEVELOPMENT OF FUNCTIONS IN A TECHNOLOGY-ENHANCED PRECALCULUS CLASS: A CONCEPTUAL FRAMEWORK

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Teaching and learning mathematical functions is an area that is being greatly influenced by the use of graphing technology (graphing software and graphing calculators) which allows users to have prompt access to multiple representations of functions including tables, equations and graphs. An increasing body of research on learning functions with graphing calculators suggests that its use can help students to integrate different representations of functions, which in turn should help to provide foundations for a more formal conception of functions. This session describes the conceptual framework that guided the data collection and data analysis to investigate students' knowledge and development of functions in a high-school precalculus course enhanced with graphing calculators over a period of nine months. The framework incorporated historical and psychological contributions (processes and objects) to the development of functions; concept image and concept definition; and multiple representations. Concept images took into account the different ways of thinking about a mathematical entity (geometric, symbolic and familiar examples among others) and students' experience (pedagogy and use of technology) with mathematical functions in this technology-enhanced class. At the end of the study, we revised the framework to incorporate findings from this research. In particular, three models (graph, equation and unique correspondence) of students' thinking about functions (Martínez-Cruz, 1995) are included in the revision. The presentation will conclude with implications of the modified framework for further research on learning functions in a technology environment.

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UNDERSTANDING OF THE ASYMPTOTIC BEHAVIOR OF A FUNCTION

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The study tries to explain the nature of the understanding of students of the concepts that are in relationship to the asymptotic behavior of the function. The explanation will be based upon the framework of the sequential mental constructions that the student makes: actions, processes, objects and schemes.

Specifically, we are analyzing what the student can construct when she or he attends a situation of asymptotic behavior. In this sense we are asking about the following aspects:

Which are the main relationships between visual and analytic acts that the student needs in a situation where, the asymptotic has vertical direction or horizontal direction, the representation of the variable zero and variation zero play an important role, comparison between graphs are stabled taking in to account the behavior of the graphs, there are different kinds of asymptotic behavior in relation with a accumulation point, convergence, and stay or stationary state?

The analysis of these aspects must provide elements that explain the disequilibrium points of the students. So, the analysis will be based on the framework of the menial constructions derived from the concept of reflection abstraction. In this sense, the mathematical knowledge and their acquisition is described in terms of schemes of the concepts. A collection of processes and objects could be organized in a structured way in order to form a scheme. The same schemes could be tried like objects and included in the organization of schemes at a higher level.

We will apply a mechanism that starts from a theoretical perspective formulating a epistemological framework of the concepts. It will check the hypothesis based the epistemological framework, and we will try to describe the links between mental constructions and understanding and how these are actualized by the student. The description of the links will be shown like a "structure of development".

At present, we are designing situations and activities using graphics calculators.

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TEACHING FUNCTIONS IN THE ELEMENTARY YEARS

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The concept of a function, traditionally introduced in grade nine, is extremely difficult for students of all ages to master. Research suggests two areas of particular difficulty: 1) understanding simple quantitative relations (e.g., Dreyfus & Eisenberg, 1984), and 2) translating among equivalent representations of a function (e.g., Markovits, Eylon, & Bruckheimer, 1986). In the present study a mini-curriculum was designed to improve grade six students' (N = 20) intuitive understanding in both these areas, thus laying the groundwork for more formal study in later years.

(1) *Quantitative relations.* Quantitative relations were introduced using the example of a Walkathon, in which money earned (y) depended on the distance walked (x). Students worked on and invented problems related to the Walkathon scenario, in which different rules were used for relating two variables.

(2) *Translating among representations.* Students were introduced to a standard computer graphing program, namely LOTUS 1-2-3. This program was configured to help students move among different representations, including algebraic, numeric, graphical, and natural language. Exercises involved the invention of new functions, and the exploration of the consequences of parametric variation using existing functions.

Pre and post test measures showed a strong and significant improvement in both of the above areas. Children's improved intuitions were also evident in interviews that were conducted and in explanations that were requested for test answers. The findings suggest implications for including an introductory functions unit in elementary mathematics curricula.

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STUDENTS' UNDERSTANDING OF FUNCTIONS

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The questions under investigation were those of determining: 1) what advanced high school students understand regarding the concept of function; and 2) to what extent their skills knowledge of functions is an indicator of their understanding of the underlying concepts. The subjects were 35 high school students enrolled in a precalculus course at a high school known for its rigorous academic programs. Two tests were administered to all subjects: one, a skills test; the other a conceptually based test. Five students were then interviewed individually. The skills test was used as a measure of students' skills knowledge. The concept test and interviews provided the data for measuring the students' conceptual understanding of function.

Scores on the Concept Test (M=44%) were much lower than the scores on the skills test (M=64%). Results replicate, for the most part, those found in previous research studies with undergraduates. The students tended to think of functions as graphs that pass the vertical line test; and that functions had to be continuous, involve numbers only, and have a single domain. Unlike the studies with undergraduates, the subjects also tended to reject some functions in graphical form if they had not seen the functions so noted by a teacher or textbook, even if the function passed the vertical line test. Some students appeared to be more dependent on their teacher and textbook for assurance of functionality than on personal experience. Linking families of functions with their different representations proved much more difficult for students than linking functions given explicitly. Across-time problems were also difficult for students to solve. In addition, most had difficulty viewing a function as an object itself; rather, they saw the graphical or analytical form of a function as the function itself, not as a representation of it.

Some explanations of the findings may be that: the students' teachers did not know how to teach for conceptual understanding; tests tended to drive the curriculum and these were, for the most part, skills oriented; or that the time allocated to studying functions was predominantly spent learning procedures and techniques rather than the concepts underlying the procedures.

ASSESSMENT OF STUDENTS' ABILITY TO REPRESENT A SCIENTIFIC EXPERIMENT BY AN ALGEBRAIC MODEL

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The Maryland collaborative for teacher preparation (MCTP) is a statewide program for undergraduate students who will become specialist teachers of mathematics and science in elementary or middle school. A fundamental feature of the MCTP are specially designed courses in mathematics and science that aim to teach mathematical modeling, through constructivist approaches. According to this aim, a tool for assessing students' abilities to develop a model was created. The tool was used, after a pilot-test, in two mathematics' classes and in two physics' classes for pre-service elementary teachers, offered at the University of Maryland.

The assessment is divided into two parts. In the first part, students working by themselves are presented with a cup, half-filled with water, that is hung from a spring. This scientific context is new to the students. Each student is asked to individually design a procedure for carrying out an experiment in order to find an equation that represents the relationship between the spring's length and the amount of water in the cup. They are not allowed to carry out the experiment. In the second part, pairs of students worked together in groups similar to those employed during the semester, and asked to choose the procedure and actually carry out the experiment. In both parts the students are directed to provide the details of their design, record all obtained or imaginary information and write the obtained or possible equation.

The assessment revealed many difficulties of students in all the phases of the modeling process, including planning and describing the experiment, through using representations to record the information, to constructing the equation. Most of these difficulties were in the individual work.

GEOMETRY AND GEOMETRIC THINKING

THE DEVELOPMENT OF STUDENTS' NOTIONS OF PROOF IN HIGH SCHOOL CLASSES USING DYNAMIC GEOMETRY SOFTWARE

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This article reports preliminary findings of a study that seeks to describe the development of students' notions of proof in high school geometry classes using the Geometer's Sketchpad software. Data obtained from task-based interviews of a total of ten students from five geometry classes was used to document the development of students' notions of proof throughout the school year. The students' proof schemes put forth by Harel & Sowder (in press) were used to categorize students' solutions of the geometry tasks. A first analysis of interview transcripts of two students shows that the proof schemes that the students exhibited early in the year were of the external type, symbolic or authoritarian. By the end of the academic year the students rarely resorted to authoritarian proof schemes but used more often empirical inductive proof schemes.

Establishing the validity of ideas is critical to mathematics, both for mathematicians and students. In everyday life people establish truth or consider something a proof if it is a convincing argument. Even mathematicians use intuitive and empirical methods when searching for the validity of ideas. However, mathematics findings are recorded in a deductive format and most traditional mathematics instruction and textbooks follow this format. This type of instruction has led students and teachers to believe that mathematicians make use only of formal proof, that is, logical, deductive reasoning based on axioms (Battista & Clements, 1995).

Historically, learning to write proofs has been an important objective of geometry courses for college-bound students, however, the relative emphasis that formal proof should have in high school geometry is debated among mathematics educators and mathematicians today. There is evidence showing that secondary mathematics students have difficulties understanding the concept. For example, formal deduction among students who have studied traditional secondary school geometry is nearly absent (Burger & Shaughnessy, 1986), and only about 30 percent of the students in full-year geometry courses that teach proof reach appropriate mastery of proof (Senk, 1985; Usiskin, 1982). It also is common to find that students' proof activities do not help them understand geometry concepts, or that the geometric knowledge involved in the proofs is compartmentalized and is not accessible to students for use in other proofs or problem-solving situations (Schoenfeld, 1987; Usiskin, 1987). Research on

students' learning of proof in traditional geometry courses also shows that most attempts to improve students' proof skills by teaching formal proof in novel ways have been unsuccessful (Battista & Clements, 1995). Furthermore, an orientation towards extreme formalism in proof does not reflect current mathematical practice or current philosophies of mathematics (Hanna, 1991).

In part because of these shortcomings of traditional approaches to geometry, curriculum reform documents have proposed new goals for the geometry class (NCTM, 1989). It is proposed that the meaningful justification of ideas should be a major goal of the geometry curriculum. The curriculum should require students to explain and justify their ideas. It should encourage students to refine their thinking, gradually leading them to understand the limitations of visual and empirical justifications so that they discover and begin to use some of the critical components of formal proof. Ideally, students should develop the ability to see a proof as a logical necessity.

Recently, the availability of dynamic geometry software has opened the possibility to approaches that may help move students toward meaningful justification of their ideas as an alternative to axiomatic approaches to learning geometry (Battista & Clements, 1995; Clements & Battista, 1994). Computer software such as The Geometer's Sketchpad (Jackiw, 1991) allows students to create simple geometric figures, explore the relationships in the geometric figures, make conjectures about their properties, and test their conjectures. For example, let us say that students are studying some properties of geometric shapes. Students could start explorations using manipulative materials such as geoboards to work with different shapes constructed using rubber bands. After some exploration students could make conjectures as to the mathematical relationships involved. Students could then test such conjectures and extend their explorations in a computer based environment using the software with its built-in measuring, table, and calculator tools. Students would then be encouraged to build validating arguments for their findings that could be scrutinized by others. When some of their findings are challenged by their classmates students could move away from considering visual appearance, or measurements of particular cases, as evidence for proof. Students could then start using some components of formal proof as a way to justify ideas meaningfully.

Although research indicates that a classroom environment like the one described above may be conducive to students understanding of proof (Fuys, Geddes, & Tischler, 1988; Piaget, 1928), the development of students' notions of proof in such environments has not been studied. The present study seeks to describe the development of students' notions of proof in technologically enhanced geometry classes. The study will document the experience of students enrolled in geometry courses that use the Geometer's Sketchpad at a local high school. The findings of this research will help inform the efforts of educators and policy makers who are trying to implement the NCTM *Standards* (NCTM,

1989). This study will also contribute to the body of research on students' understanding of mathematics, and it will help determine what are some features of tasks of dynamic geometry software that foster the use of meaningful justification in geometry class.

Procedures and Theoretical Framework

During the academic year of 1996-97 five geometry classes at a local high school were taught with the support of the Geometer's Sketchpad Software. The teacher was teaching this class for the second year using this technology. Data obtained from task-based interviews of a total of ten students from all classes was used to document the development of students' notions of proof throughout the school year. The goal of the questions and interview tasks was to elicit the strategies and methods used by the students to establish the validity of propositions in this computer-based environment, so that the development of their notions of proof during the school year could be documented (See Table 1 for a brief description of the tasks used during the interviews). Purposeful and maximum variation sampling were used to select students to be interviewed. Five students who by the end of the second month of the academic year had been identified by the teacher as "strong in proof," and five students who according to the teacher were "not as strong in proof" were selected to follow their development throughout the school year. This categorization was based on the students' work in homeworks, quizzes, and one test, however, the teacher emphasized that it was very relative as all of the students in these geometry classes were honors students. The students were interviewed by members of the research team three different times during the school year (during the third, fifth and eighth months of the academic year).

The students' proof schemes put forth by Harel & Sowder (in press) were used to categorize students' solutions of the geometry tasks. The notion of *proof scheme* put forth by Harel & Sowder is a psychological one. They use it to refer to what convinces a person, and to what the person offers to convince others. Their classes are the result of extensive work with college mathematics majors and include three major categories: external conviction proof schemes, empirical proof schemes, and analytical proof schemes. Students' justifications exhibit an external conviction proof scheme when they depend on an authority such as a teacher or a book (the authoritarian proof scheme), on strictly the appearance of the argument (the ritual proof scheme), or on symbol manipulations, with the symbols and/or the manipulations having no meaningful basis in the context (the symbolic proof scheme). Empirical proof schemes, on the other hand, rely on evidence from examples or direct measurements of quantities (the inductive proof scheme), or perception (the perceptual proof scheme). Finally, analytical proof schemes encompass mathematical proof although the emphasis is on the student's thinking rather than on what he or she writes.

Table 1 *Abridged Description Of The Geometry Tasks Used In Interviews*

Task 1, Session 1, Measure of the Exterior Angle of a Triangle.
Consider the triangle ABC where the side BC is extended to show the exterior angle ACD. Can you show that the measure of the exterior angle ACD is equal to the sum of the measures of the interior angles CAB and ABC?

Task 2, Session 1, Intersection Point of Selected Lines of a Triangle.
Do the medians (heights, angle bisectors) always meet in one point?

Task 1, Session 2, Sum of Two Consecutive Angles of a Parallelogram
Can you prove that the sum of any pair of consecutive angles in a parallelogram is 180 degrees?

Task 2, Session 2, Intersection Point of Selected Lines of a Triangle.
Do the medians (heights, angle bisectors) always meet in one point?

Task 1, Session 3, Measure of Angle Inscribed in a Semicircle.
AB is the diameter of a circle. C is a point on the circle. Consider the triangle CAB. What can you say about the angle ACB? What can you say about the triangle ACB?

Task 2, Session 3, Locus of the Midpoints of All Chords Drawn From a Fixed Point on a Circle.
Let A be a fixed point on a circle with center O. Consider the midpoints of all the chords that can be drawn from point A. What can you say about the locus of these midpoints?

Harel and Sowder emphasize that their taxonomy is not a hierarchical one, and that a given person may exhibit various proof schemes during one short time span. For the study reported here, we assigned numerical labels to the main proof schemes in order to be able to easily refer to them during the analysis of the interviews. Table 2 lists the labels used for our data analysis (these labels are also used in the vertical axis of the chart in Figure 1).

In order to document the development of the students' notions of proof, their solutions to the tasks in each interview were classified as one of Harel and Sowder's proof schemes. The interview transcripts were analyzed to examine what convinced the student about the truth of a statement and what the student would use to convince others. The student response was then assigned a label 1-7, and the proof schemes used by the student throughout the academic year were analyzed to search for patterns. The preliminary findings reported here

Table 2 *Labels Assigned To Harel and Sowder's Proof Schemes*

Proof Scheme	Label
External Ritual	1
External Authoritarian	2
External Symbolic	3
Empirical Perceptual	4
Empirical Inductive	5
Analytical Transformational	6
Analytical Axiomatic	7

are based on the analysis of the interview transcripts of two students; we will refer to them as Kate and Alex (an analysis based on data from all students, and an unabridged version of this paper will be available during the presentation). They were identified by the teacher as "strong" and "not so strong" in proof, respectively.

Findings

Figure 1 shows the proof schemes exhibited by Kate and Alex during six of the geometry tasks. Early in the academic year the students seemed to prefer to approach geometry tasks working with paper and pencil and exhibited some authoritarian proof schemes. For example, when Kate approached the task about the measure of the exterior angle of a triangle, and the task about the measure of two consecutive angles of a parallelogram, her first approach was to work on paper and to use equations. Her first approaches were labeled as external symbolic proof schemes because once she tried the same tasks using Sketchpad it became apparent that she had not associated much meaning to her initial justifications. When working on the computer she said "Now that I think about it. . . it makes more sense to me than looking at all the numbers (referring to the equations she used earlier) and going 'oh, OK.' This just seems a lot simpler."

One advantage of dynamic software is the possibility to make constructions that can be dragged around and still maintain the relationships among their parts. By the second interview, half of the academic year, most students were familiar with the notion of construction in this environment, and the use of dragging of constructions to verify that some properties hold. This was seen as progress in their notions of proof, as they were using the dynamic geometry software to produce constructions that represent infinitely many examples. This

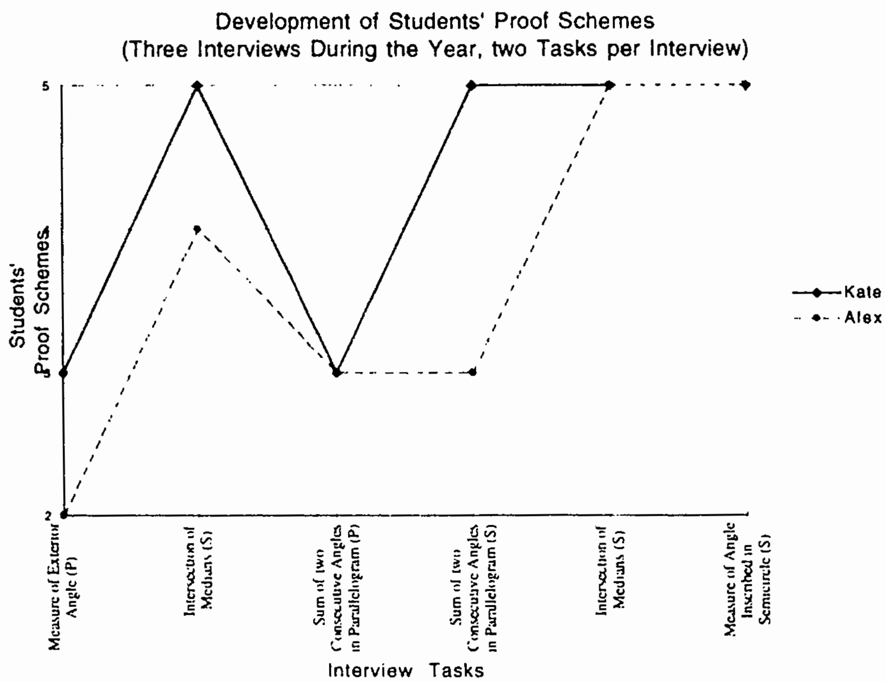

Figure 1 *Proof schemes used by two students when solvingsix geometry tasks.*

type of solution was labeled as an empirical inductive proof scheme because the students ascertained themselves and sought to persuade others by quantitatively evaluating their conjecture in one or more specific cases. A downside of this use of the software is the reluctance of some students to prove statements for which they have created a dynamic construction. The dynamic construction seems so powerful and convincing that it is difficult to engage the students in geometrical thinking in theoretical terms. shows how by the third interview both students were consistently exhibiting empirical inductive proof schemes (label 5). It was also interesting to find that the student who had been identified by the teacher as "strong in proof" seemed to abandon authoritarian proof schemes earlier than the other student.

To summarize, we found evidence that suggests that appropriate use of dynamic software helps move students toward meaningful justification of their ideas. The students we interviewed abandoned authoritarian proof schemes and used empirical inductive schemes often. Furthermore, when asked to think of situations where they would have to convince others of the validity of their explanations (their teacher, a mathematician), students usually displayed a good understanding of mathematical proof and realized that their empirical justifica-

tions would not suffice, although they not always exhibited an analytic proof scheme.

Acknowledgments

The author would like to thank Becky McGraw for allowing us to use her geometry classes for this research. The author greatly appreciates the assistance of Gudmundur Birgisson, Jean-Marc Cenet, Norm Krumpe, and Mike Lutz conducting and transcribing interviews.

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UNDERSTANDING ANGLE IDEAS BY CONNECTING IN-SCHOOL AND OUT-OF-SCHOOL MATHEMATICS PRACTICE

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In investigating a group of middle school students' out-of-school activities, we found that miniature golf was a familiar context for these students. We also found this context to be potentially rich for engaging students in thinking mathematically. Based on this data, we used this context for a four-week teaching experiment on geometry and measurement ideas with three sixth grade classes in Spring 1997. One aspect of our study is comparing students' understanding of angle ideas for students who were in the teaching experiment with students who experienced a more traditional geometry and measurement unit. We found that all the students progressed in their understanding of angle ideas and that the students in the teaching experiment classes were better at preserving the angle measure when copying an angle.

Introduction

The "Connecting In-school and Out-of-school Mathematics Practice"[1] project is beginning its third year. During this project we are (a) investigating how middle school students use mathematics concepts and processes in a variety of out-of-school situations, and (b) using ideas from the students' out-of-school activities to investigate whether students are connecting their in-school and out-of-school mathematics learning and practice.

During the first year we collected data about six sixth grade students' mathematics practice out of school through (a) activity sampling with electronic pagers and logs, (b) field observations of each student in out-of-school activities, (c) interviews with students about logs and observations, (d) logs kept by students and parents, and (e) interviews with students and parents about logs and their activity.

In our second year, we worked with a sixth grade mathematics teacher and her students to see how the students think about things mathematically and whether they make connections between doing mathematics in school and out of school. Through analysis of our first-year data, we identified a context that was familiar to all our respondents and with which we thought most students

[1] This project is funded by a National Science Foundation-sponsored grant (RED-9550147) awarded to the first author under the Faculty Early CAREER Development Program. The opinions expressed in this paper are the authors' and not necessarily those of the National Science Foundation.

would have some experience. This context—miniature golf—was used with the students in the classroom to investigate geometry and measurement ideas (as well as other mathematical ideas, such as ratio, that arose). For four weeks in March and April 1997, the students in three sixth grade classes participated in a teaching experiment where they investigated mathematical ideas, through the context of miniature golf. The first author taught the four week unit with the sixth grade teacher, while the second author observed and took field notes.

Perspectives and Guiding Frameworks

The work of Piaget, and Clements and Battista has been most influential in our thinking about students' understanding of angle ideas. Piaget, Inhelder and Szeminska (1960) identified four stages in the development of understanding that angular measure involves the principle of one-many correspondence, in particular that two coordinated measures are needed to construct an angle. The task they used had students reproduce an angle. In stage I (up to 4-5 years) and substage IIA (up to about 6 years), children were found to copy an angle by visual estimation with no attempt at measurement. In substage IIB, some measurements of constructed segments were taken, but no attempt was made to coordinate them. At level IIIA (7-8 years), all measurements of constructed segments were taken, plus some effort was made to coordinate them, by attempting to preserve the "slope" of one leg. At level IIIB, measures were taken of non-constructed objects (e.g., the distance between two unconnected points) and coordinated to construct the angle. By stage IV (10-11 years), students were constructing totally new measures, such as the perpendicular distance from a point to a line.

Piaget and Inhelder's (1959) study of how children come to understand the equality of angles of incidence and reflection identified three stages. In stage I (up to 7-8 years), they found that children are mainly concerned with practical success or failure, and often even the role of rebounds, let alone angle ideas are overlooked. Piaget and his colleagues identified this as the pre-concrete operations stage, where actions are never internalized as operations. Behavior is goal-directed and reason for success is not investigated. In stage IIA, concrete operations are identified. For example, actions are internalized and integrated with others to form general reversible systems with an awareness of techniques and coordination of behavior. By stage IIB (9-13 years), increasing awareness is seen in the relationship between the inclination of the cue and the path of rebound. But despite isolating all elements needed to formulate the law of equality of angles of incidence and reflection, they do not look for the reasons behind what they have discovered. Further, they do not think in terms of segmenting the total angle. By stage IIIA (by 14 years), the subjects formulate conditional one-way hypotheses and discover the equality. But the hypotheses are still related to the concrete correspondence. In stage IIIB (14-16 years),

they seek necessary reasons for what happens and this is the mark of formal thinking. As a result, their hypotheses are bi-directional. Further, they are able to consider all possible combinations in each case, another hallmark of formal thinking.

Clements and Battista have used Logo programming with students in elementary school, particularly third- and fourth-grade students, to investigate students' understanding of geometry ideas. In one study (Clements & Battista, 1990), they investigated whether Logo programming experience facilitated children's: (a) development of geometric concepts such as angle, angle size, and related arithmetic ideas, and (b) transition from the visual to the descriptive/analytic level of geometric thinking (van Hiele, 1986). They interviewed 12 fourth graders three times, at the beginning, middle, and end of 40 sessions of Logo graphics programming experience. The six Logo children, but not the comparison children, progressed from their original intuitive notions to more mathematically sophisticated and elaborate ideas of angle, angle size, and rotation. In addition, more Logo children explicitly mentioned geometric properties of shapes, indicating that they were beginning to think of the shapes in terms of their properties instead of as visual gestalts. Thus, there was support for the hypothesis that Logo experiences, especially those enriched with appropriate activities and discussions, can help children become cognizant of their mathematical intuitions and move to higher levels of geometric thinking.

In a more recent study (Clements, Battista, Sarama, & Swaminathan, 1996), Clements, Battista and their colleagues investigated the development of turn and turn measurement concepts within a computer-based instructional unit. They collected data within two contexts, a pilot test of the unit with four third graders and a field test in two third grade classrooms. The researchers conducted paper-and-pencil assessments, interviews, and interpretive case studies. Turns were less salient for children than "forward" and "back" motions. Students evinced a progressive construction of imagery and concepts related to turns. They gained experience with physical rotations, especially rotations of their own bodies. In parallel, they gained limited knowledge of assigning numbers to certain turns, initially by establishing benchmarks. A synthesis of these two domains—turn-as-body-motion and turn-as-number—constituted a critical juncture in learning about turns for some students. Some common misconceptions, such as conceptualizing angle measure as a linear distance between two rays, were not in evidence. This supports the efficacy and usefulness of instructional activities such as those employed.

Methods and Data Sources

The design of our study for this phase of the project has two aspects: (a) we are comparing the understanding of angles ideas for students who are investigating geometry and measurement ideas through the miniature golf context

with students who are having a more traditional geometry and measurement unit during this time period, and (b) we are analyzing students' understanding of angle ideas and connection making for students in the classes using the miniature golf context. This paper will deal only with the comparison aspect of the research.

For the comparison part of the study, we interviewed two to three students from each of six classes in pre-interviews and post-interviews, which were approximately ten weeks apart. In each of the "regular" classes (meaning not accelerated), we interviewed one student who has been labeled as needing help from a resource teacher, and two other students (one female and one male). Two teachers' (T1, T2) classes (two regular classes, one accelerated class) did their geometry and measurement unit for the year during the time period between the pre- and post-interviews and, in both cases, it was a fairly traditional unit, drawing heavily upon the textbook. The third teacher's (T3) classes (two regular classes, one accelerated class) did their geometry and measurement unit through the miniature golf context during this time period.

The chart below provides a concise look at the design for the comparison of students. The teacher for each class recommended students for us to ask to volunteer for the interview based on their ability to articulate and reflect, as well as gender.

Teacher	Content	Type of Class	Data Collection
T1	trad. geom	regular	3 students (1F, 1M; 1 Resource); pre- and post-interviews
T2	trad. geom	regular	3 students (1F, 1M; 1 Resource); pre- and post-interviews
T2	trad. geom	accelerated	2 students (1F, 1M); pre- and post-interviews
T3	golf geom	regular	3 students (1F, 1M; 1 Resource); pre- and post-interviews
T3	golf geom	regular	3 students (1F, 1M; 1 Resource); pre- and post-interviews
T3	golf geom	accelerated	2 students (1F, 1M); pre- and post-interviews

The data were analyzed using inductive data analysis procedures.

Findings

At the school where we conducted this study, all the sixth grade teachers (T1, T2, T3) introduced angle ideas through the use of a pattern block and hinged mirror activity (Burns & Humphreys, 1990). This activity involved students in finding the angle measures of different pattern blocks by using the hinged mirror to see how many angles of a particular block were needed to form 360°. For example, students found that six angles from the triangle were needed, and thus each of the angles was 60°. Near the end of this several-day activity, students were introduced to the protractor and learned how to measure and draw angles with a protractor. In the teaching experiment classes, this activity took place approximately three months before the miniature golf-based

geometry and measurement unit. Throughout the geometry and measurement units in the three classes, students were given opportunities to measure and draw angles using a protractor.

In the post-interviews, we found that all of the students had progressed in their understanding of angle ideas. For example, in the pre-interviews, many

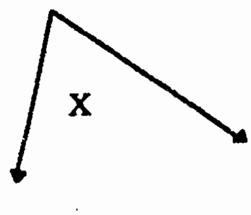

Figure 1

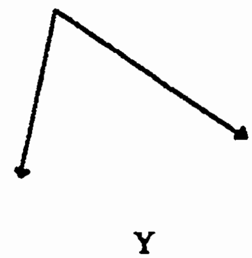

Figure 2

students thought that extending the rays of an angle made the angle bigger. A number of students also indicated that while a point placed in the region between the rays was inside the angle (Figure 1 below), a point placed in the region that would be between the rays if the rays were extended was outside the angle (Figure 2 below).

However, in the post-interviews, all the students agreed that extending the rays of the angle did not change the angle itself. All of the students also identified a point placed as in Figure 2 as being inside the angle.

Another task during the pre-interviews, and adaptation during the post-interviews, was the task that Piaget and colleagues used of reproducing an angle. In the pre-interview students were shown the drawing in Figure 3 below, given a sheet of clean paper, a ruler, string, a compass, and an eraser and asked to copy the figure as exactly as they could. The students were allowed to look at the figure as much as they wanted, and take any measurements they wanted, in between drawing it, but they were not allowed to look at it while drawing.

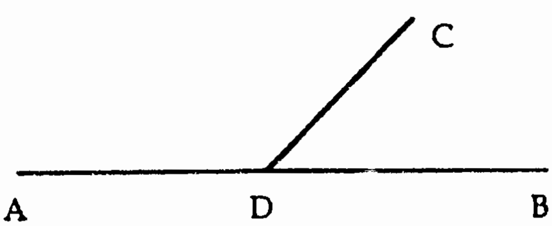

Figure 3

In the pre-interviews, many students (from both the teaching experiment classes and the other classes) had difficulty coordinating all the measurements. Common mistakes were (a) not locating D on segment AB, (b) not locating D correctly, and (c) not preserving the measure of angle CDB.

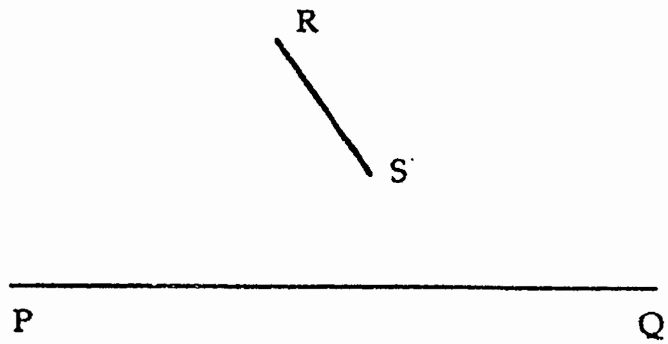

Figure 4

In the post-interviews, we gave the students the drawing in Figure 4 below and asked them to copy the figure as exactly as they could. During this task the students could look at the figure, take measurements, and draw on it, if they wanted, as they tried to copy the figure. They were not allowed to trace the figure. The students were given a sheet of clean paper, a ruler, a protractor, and an eraser.

In the post-interviews, we found that all of the students who had investigated angle ideas through the miniature golf context preserved the measure of the angle formed by segment PQ and the extension of RS, whereas a number of the students who were not in the teaching experiment classes did not preserve this measure. We speculate that this may be due to the fact that students in the teaching experiment classes had experiences during the unit where they had to coordinate a number of measurements. These students took measurements and made sketches of actual miniature golf holes, which often included curved edges and locating obstacles.

We will have a longer paper available at the research reporting session that will provide more analysis and details about this study.

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DEFINING AN EXTERIOR ANGLE OF CERTAIN CONCAVE QUADRILATERALS: THE ROLE OF "SUPPOSED OTHERS" IN MAKING A MATHEMATICAL DEFINITION

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A teaching experiment was conducted with two pairs of tenth grade students to assess their metacognitive behaviors in the process of making a mathematical definition and to explore the extent to which these students could be taught to be more reflective of their own mathematical activities. In the teaching experiment, a special attention was given to the role of teacher as facilitator of students' monitoring of their own activities by the role of "supposed others". In this paper a transcribed protocol was analyzed of a session in which one pair of students were working on a task that asked them to find the sums of exterior angles of kite and "boomerang" and that raised a new problem of examining the definition of exterior angle to an obtuse angle. Students' monitoring and verifying of their own activities were often observed in the form of responding to questions and critiques supposed by themselves. Students could distance themselves from the action of making a definition by attending to critiques from "supposed others".

Introduction

In Japanese curriculum, the meaning and role of definitions in mathematical reasoning are introduced to the students in eighth grade. Although teachers are expected to explain the meaning of definition by using a familiar concept like "isosceles triangle," it is one thing for students to learn a definition of concept and it is quite another to appreciate the role of definition in mathematics and how it is made. One of the students' difficulties related to definition is in appreciating its important role in mathematical argument or proof. Students have few opportunities for creating a definition of mathematical concepts and they are often unaware of both the constructive and tentative nature of definitions in mathematical reasoning and their important role in communicating with others.

The purpose of this paper is to elucidate the role of "supposed others" in the process of making a mathematical definition. Specifically, the study had two objectives: (1) to assess the metacognitive behaviors of tenth grade students working in pairs making definitions of unfamiliar concepts in plane geometry, and (2) to explore the extent to which these students could be taught to be more reflective of their own mathematical activities in terms of the role of "supposed others".

The study is based on the analysis of a course described in "The Nature of Proof" (Fawcett, 1938) that aimed to foster students' "critical thinking," in which

definitions and propositions were socially constructed by students and the teacher. Fawcett's course had a flavor of "an experiment in metacognition," as described by Crosswhite (1987), and suggested the importance of "critical thinking" fostered through the students' experiences of critiques from others. More recently, Borasi (1994, 1996) conducted a teaching experiment in which students experienced the need for monitoring and justifying their mathematical work when they were engaged in "error activities" including defining the familiar notion like a circle, though teacher's role is different from those in Fawcett's course. These studies suggest the importance of both the social construction of a mathematical definition and students' metacognitive activities in appreciating its nature and role.

From a general perspective, on the other hand, thinking can be seen as conversation with "generalized others" (Mead, 1934). Also Mason et al. (1982) discussed a significance of developing an "internal enemy" for thinking mathematically, and Hirabayashi and Shigematsu (1986) referred to the "inner teacher," an internalization of teachers' utterances into the students' own thinking through classroom experiences.

Drawing upon these sources, the study explored the role of metacognition in terms of critiques and suggestions by "supposed others" (Shimizu, 1993) in the social process of making a mathematical definition. By "supposed others," the author means a mental model one has of others who often ask questions and make critiques.

Methods

Subjects

A teaching experiment including sixteen instructional experiences over four months was conducted with two pairs of tenth grade students (8 experiences with each pair.). The students were selected based on their responses to a preliminary questionnaire survey with 35 tenth grade students for identifying the their backgrounds, which had been conducted one month before the teaching experiment. The items of questionnaire included broader questions to ask the meaning of the term definition and theorem respectively (e.g., "Describe briefly the meaning of definition.") , as well as more specific questions about kite (e.g., "Give the most concise name for the figure," asking the name of kite). Four students selected for the teaching experiment were those who had met with the concept of kite before but could not differentiate the meaning of definition from those of theorem.

Procedure

The teaching experiment was designed and taught by the author. A series of tasks was given to the students that asked them in pairs to find some properties and definitions of various quadrilaterals including kite and "boomerang".

Students' activities by themselves typically lasted about thirty to forty minutes, followed by interview/instructional sessions of twenty to thirty minutes duration. The teacher's roles in the interview/instructional sessions, which were similar to those of "teacher as facilitator" (Lester et al., 1989), included: (1) making the students' solution process explicit by having them reflect on it, (2) asking them to justify their solutions, and (3) facilitating their deeper understanding of the properties and definitions of quadrilaterals through a discussion with them. In addition, the teacher made conscious efforts to ask "why questions" at certain points in each session.

Task Used in the Session Analyzed

In the session which is analyzed in this paper, one pair of students (S & U) were working on the task of finding each sum of interior and exterior angles of both kite and "boomerang".

The Task

Find the sums of interior and exterior angles respectively of kite (Figure 1) and boomerang (Figure 2). An exterior angle (to the <BAC) is shown in Figure 3.

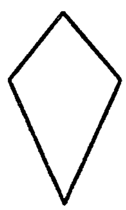

Figure 1

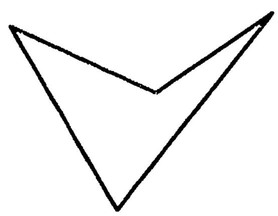

Figure 2

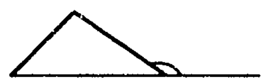

Figure 3

The task was supposed to raise new problems for the students to think about during their problem solving activity. Namely, the task was given to the students with an intention of asking them to consider what definition might be appropriate for the "extended" case, since the ordinary definition of an exterior angle (Figure 3) does not seem to fit in with an exterior angle to the obtuse angle in "boomerang" (Figure 2).

All the activities in the teaching experiment was videotaped. Transcribed protocols were made and submitted to the analysis. The protocols were analyzed by focusing on "problem transformations" (Shimizu, 1992), namely, the major turning points during students' solution process in terms of problem (re)formulating by them, to identify the major metacognitive behaviors of the students. The questionnaires before the teaching experiment and students' notes during and after the sessions were also submitted to the analysis.

Results

Students' activities by themselves lasted about thirty eight minutes, followed by the interview/instructional session of twenty one minutes duration. Because of the space limitation, students' activity on examining the definition of an exterior angle will be briefly summarized here.

After having concluded that the sum of interior angles of both kite and boomerang should be 2<R, they found that the sum of exterior angles of kite is 2<R by applying the relationship between an exterior angle and its interior opposite angles in a triangle. Then they started to discuss the case of boomerang.

Dividing the obtuse angle into two parts, namely, "a" and "d" in Figure 4, student U proposed that the exterior angle to the obtuse angle might be "2<R-(a+d)". Responding to this idea, student S pointed out that this idea was inconsistent with the definition of an exterior angle so far, by showing another exterior angle which seemed to be "too big" (Figure 5).

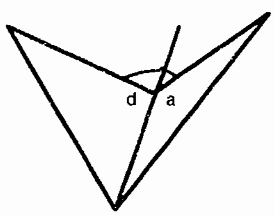

Figure 4

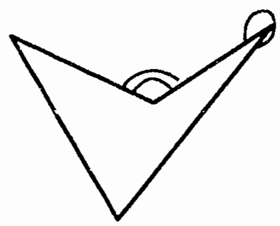

Figure 5

S (28:04) : *why, however,, . . . if the exterior angle . . . might be here, we should do the same thing to other angles.*

U (28:20) : exterior angles to other angles?

S (28:21) : Yahh, . . . cause, it doesn't fit in with the case of other angles so far. Then, 360 minus this angle, about 30, makes 330. It's why this guy is too big, isn't it ?

They finally found that the sum of exterior angles of boomerang was 540, when they applied the definition that student U had proposed, and that by using the definition they would lose the "consistency" with the original one.

In the interview/instructional session, the teacher suggested the way of "going around" to confirm the sum of exterior angles of kite is 2<R (Figure 6) and then asked students to examine the case of boomerang.

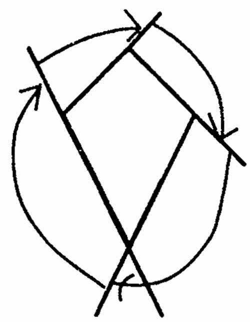

Figure 6

During the conversation between the students, student U proposed the idea of "signed angles" to incorporate two cases. At this point of the session, they were very flexible both in proposing and accepting new ideas concerning to an extended definition of exterior angle.

In their problem solving process, the monitoring and verifying of their own activities were often observed in the form of talking to "supposed teacher," though these behaviors were not sufficient for finding a new definition of an exterior angle. When they got the sum of exterior angles of boomerang as 540, for example, students S expressed her idea as *"but we might be in trouble if we were asked why?"* Students' utterances like this seem to be made by supposing possible questions an critiques by the teacher.

In contrast to the beginning of the teaching experiment, they showed flexibilities both in proposing new ideas like "signed angles" and in accepting "negative angles". A comparison of students' writings before and after the teaching experiment suggests that students' view on the nature and role of mathematical definition had been changed. Student U, for example, who had described the meaning of a definition as "what is given to explain a thing" in the questionnaire, mentioned to it in the interview session as *"we can make a definition by ourselves, depending on the situation."* Student S who had described a definition as *"Since many people have gotten the same answer, it is viewed a*s a rule" before the experiment, mentioned that she *"likes a definition as brief as I can"* in the interview session.

Discussion

Some characteristics of students' thinking were observed in the session. First, students often mentioned that they did not understand the definition of an exterior angle exactly. Namely, they made comments at certain points in their problem solving that indicated they were aware of their own understanding. Second, they often asked questions by themselves about what they seemed to

have already known. These characteristics were clearly connected to the statements and questions in which they mentioned to possible questions and critiques by the teacher. In this sense, by attending to critiques from "supposed others," it was found that students could distance themselves from the action of making a definition and were led to verify the appropriateness of their activities.

When one is asked to define an exterior angle to an obtuse angle, there are several possibilities to consider. To make a new definition by "extending" the original one itself is one way and to hold the same definition in a broader context with keeping certain consistencies is another. In the latter case, which is often the case in mathematics and which was chosen by S & U, the consistency between two definitions becomes of significance when they wanted to hold the sum of exterior angle as 2<R. For students S & U to maintain the consistency, the need for monitoring and verifying their work seemed to increase.

As was mentioned earlier, it is one thing for students to know a definition of a mathematical concept and it is quite another to appreciate the role of a definition in mathematics. The students in this study could distance themselves from the action of making a definition by attending to critiques from "supposed others" and through such activities their view on the nature of definition seemed to be changed. This observation suggests the importance of "supposed others" in helping students to appreciate the nature and role of definitions in mathematics. The constructive and tentative nature of definitions in mathematical reasoning should be emphasized through the experiences in which definitions are socially constructed with critiques from others.

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IDENTIFYING THE NATURE OF MATHEMATICAL INTUITIONS

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Secure understanding of formal mathematical concepts can be best accomplished when a learner assimilates the concept into a concrete, intuitive schema that the learner already possesses (Ginsburg, 1989). For example, the proposition that a plane ($ax + by + cz + d = 0$) in three dimensional space can be uniquely determined by fixing three points on the plane may be understood in reference to the fact that only three fingers (3 fixed points) are enough to carry a tray (a plane). Such intuitive models, based on our informal, physical experience, have enormous power in helping learners to understand abstract mathematical concepts (Kamii, 1978). However, the nature of the mathematical intuitions that learners already possess has rarely been studied systematically. In classrooms, math teachers have no clear basis for deciding what type of intuitive models may or may not work for the students.

The question is: what is the nature of learners' mathematical intuitions and how do they develop? My hypothesis is that many significant mathematical intuitions emerge in conjunction with physical, geometric schemas which develop in everyday life as learners engage in spontaneous activities such as play and the use of everyday objects. For instance, I have made systematic observations of a preschooler playing with wooden blocks. The child spontaneously considered how to build a flat roof (a plane) on top of a house he was constructing. After some trial and error, he completed the roof by placing long blocks in straight lines across several pairs of supporting columns at 2 fixed points in a parallel manner. This reflective scheme, originating in the child's spontaneous thinking concerning both physical and geometric properties of the objects, could serve as the foundation for the understanding of abstract mathematical concepts and relations. Identifying and analyzing such *latent mathematical schemes*, therefore, would help math educators make use of the power of learners' mathematical intuitions to the fullest extent.

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WHEN DOES THE POINT EXIST IN THE PLANE? SOME HIGH SCHOOL STUDENTS' CONCEPTION

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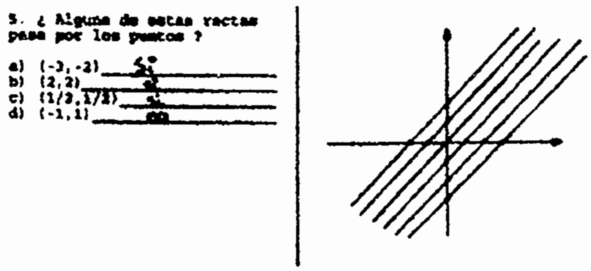
In research with high school students about the spatial location of points in the Cartesian plane, two faces of student performance appeared, first one when given the coordinates and one must draw the point on the graph and the second one when given the point on the graph and one must give the coordinates. These are different in the cognitive demands necesary to respond.

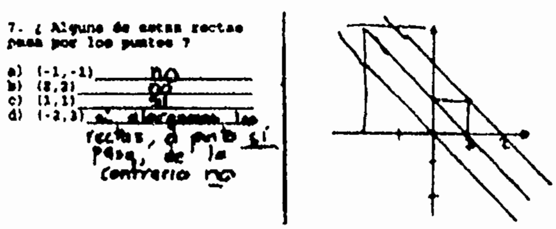
In this work with 221 sixteen-year-old high school students, we found misunderstanding when we asked these exercises:



Right answers in location of (-3,-2), 56.1%.
Right answers in location of (-2,-4), 85.3%.

In an informal review one student said "I can't say nothing because there is not a mark on -3 ". For the students the point does not exist because having not been drawn, it has no physical existence.

But this exercise was proposed to 87 fifteen-year old students we ask them to construct their own scales to solve some exercises and the scores changed to 85% . In appearence, the free construction of scales allows the assimilation of the symbolic existence of the point in the absence of marks on the graph. However, the same students had misconceptions in the solving of the next exercise :



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The note says, "If we can enlarge the lines the point pass on; else, it does not."

The 55.1% of right answers against against the average of 75.3% on the other three exercises, 17.24% of the students failed only in the location of (-2,3), despite their average of 89% on the rest of the items.

Again, these students believe that the point exists only if the mark on the paper exists. This is not a symbolic existence although they can graph points under instruction. Their conception says if you can see the point, then the point exists. If you can't see the point, it does not exist.

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Instruction and Planning for Instruction

PROBLEM-CENTERED LEARNING AND EARLY CHILDHOOD MATHEMATICS

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This paper describes research on classroom praxis based on a problem-centered approach incorporating a constructivist epistemology and principles of Adlerian Positive Discipline in a Grade Two classroom. The theoretical basis of the project is described and implications for classroom instruction highlighted. The constitutive contextual reasons which make mathematical engagement a meaningful experience are discussed in relation to a classroom which adheres to a child-centered approach to teaching and learning.

Considerable research is currently being generated by the significance of the relationship between the classroom culture and the individual child's development. Garofalo (1989) suggests, "The nature of the classroom environment in which mathematics is done strongly influences how students view the subject of mathematics, the way they believe mathematics should be done, and what they consider appropriate responses to mathematics questions" (p. 451). In short, the type of learning is influenced by the type of learning situation. Franke & Carey (1997) propose that in order to improve the teaching and learning of mathematics, it is critical to understand what it means for children to 'engage' in mathematics. Such a proposition implores a deeper look at the constitutive components of 'engagement' in mathematics education. Whereas "sociocultural theorists contend that the individual dimensions of experience are subsidiary to the social and cultural dimensions", and emergent theorists argue that "individuals jointly create interactional routines and patterns as they adapt to each other's activity" (Cobb, Jaworski, & Presmeg, 1996, p. 15), both theoretical stances focus upon the context of the learner, as a member of a classroom community, and in doing so establish grounds for considering 'engagement' (in the sense of mathematical-sense-making) as a contextual classroom artifact, dependent and resultant upon the nature of the classroom environment/learning situation. Hiebert et al. (1996) portrayed learning mathematics as being more effective when children engage in a culture in which they know they have the freedom and responsibility to develop their own solution methods. Geoghegan (1996) suggests that it is the synergistic coalescence of children's intentionality, creative endeavor and emotional security that act as unifying complementarities in establishing effective engagement in the learning of mathematics.

The present study was conducted in a classroom which focused upon learning wherein the construction of knowledge harmonized in a reflexive relationship between individual sense making and group (social) interaction. For students to begin to make sense of their learning they must not only feel some degree of willingness (intentionality) to resolve perturbations but also a confidence to go about searching for resolution in order to establish a new sense of constructed meaning. Thus students' making-of-sense is educed from their active forays into and explorations of new conceptualizations but, and most importantly, underpinned by a confidence in already assimilated (qua 'sensible') starting points. The teacher's job then is to consolidate and expand upon these points of reference by providing students with an environment and experiences that will encourage new meaning to emerge. The building of confidence and meaning go hand in hand and it is in the milieu of social dynamics that the stage is set for such forays to be enacted.

Method

The present study took place in a Grade 2 classroom with an experienced teacher who volunteered to participate after conversations on constructivism and imaging with one of the researchers and after involvement in a Positive Discipline program based on the philosophy of Adler and Dreikurs. A mathematics program grounded in a problem-centered learning approach (Wheatley, 1991) and constructivist epistemology was implemented. As part of a total classroom regimen aimed at creating a more self/other-respecting and autonomously-focused learning environment, a Positive Discipline program (Nelsen, Lott, & Glenn, 1993) was also implemented. Daily mathematics lessons lasted one hour, the initial ten to fifteen minutes being teacher-directed, whole-class imaging activities involving number and shape in which children were provided with a particular spatial and/or number challenge (Wheatley & Reynolds, 1991) and then asked to interpret and discuss "what they saw." During the remaining time children collaborated in pairs to solve assigned mathematics tasks and then presented their solutions in a whole class sharing time. Any additional time was spent in self-selected mathematics games and/or puzzles. The social dialogic interaction provided opportunities for children to cross-reference ideas, justify and rationalize different points of view and consider, through shared negotiation and reflection, personal and other's mathematical meaning.

Underpinning the classroom ethos of what it means to share and negotiate as problem solvers was the Positive Discipline program adhering to ethical and constructivist views of learning. Positive Discipline principles sought not to have the learner's progress externally controlled by the teacher; the teacher's creed was to follow children's thinking instead of leading it. Children's autonomy, respect for others, sense of interdependence, and willingness to take risks as creative problem-solvers were nurtured in an harmonious caring and

compassionate environment. Children were responsible for determining what was appropriate classroom (social) behavior in all situations, during all lessons, and were encouraged to discuss their concerns, insights, and resolutions throughout the day. This was achieved through class meetings and various open discussions which included the mathematics lessons.

Field notes and children's work samples were collected all year. For three days a week written observations and audiotape recordings were made during the hour-long mathematics lessons and then, immediately following, the teacher and the researchers spent one hour analyzing outcomes. More extended times during the semester were also used for further analysis and planning. Video recordings were made of interviews with six children selected to be a representative cross-section of classroom abilities.

Discussion

Some teachers as mathematics educators characterize learning in terms of individual children's development whilst others characterize learning as a process of acculturation. Some advocate that learning mathematics cannot be discussed separately from teaching mathematics— more so, that 'Learning' and 'Teaching' cannot be discussed separately (Lerman, 1996; Geoghegan, 1996). Classroom settings, social dynamics, goals, needs, emotions, and aspirations are all integral to cognitive development. From this world view it became evident that facilitating mathematical development in the classroom of the present study was constitutive of many significant contextual dimensions. The setting in which the mathematical experiences were enacted and concepts formulated was not a 'math lesson' determined in the traditional sense by such components as lesson plan, manipulatives, textbook, content, etc. but more akin to a hermeneutic dialectic ecology characterized by a child-centered philosophy and established upon sociocultural (including sociomathematical) norms. The children created the essential guide-lines for their daily experiences and, as proactive stakeholders in the formulation of their own experiences, became more engaged in the learning process. In making sense of each other's point of view *the children* were facilitating the teaching and learning process simultaneously with the teacher. The teacher's authority as the end point of knowledge determination increasingly subsided.

Early in the study the children were seemingly respectful of each other (quietly and patiently waiting for their own turn to present their ideas) but appeared to be, by and large, disengaged from their peer's attempts to verbalize conceptualizations during the sharing sessions. Maybe as a result of years of enculturation aimed at producing 'passive young children who are polite' many of the children did not respond beyond being quiet. Their initial manifestations of "listening with respect" were characterized by maximum patience and minimum attentiveness. The following example highlights the level at which the

children were "prepared" to function. In pairs, they had been asked to record their solution to a subtraction problem on an overhead transparency in order to present it to the rest of the class. The children did this with zeal, even decorating the border of the transparency with artistic embellishments such as flowers, hearts and faces, etc., in some cases almost obliterating the mathematical symbols. During the presentations, when the teacher asked the class to respond to or comment upon the solutions presented by different pairs of students, typical responses included: "That's a nice heart in the corner." "That's pretty." and "I like your drawing." Minimum engagement adhered to the mathematical thinking that had been involved despite the fact that several alternate solutions had been presented.

In order to develop collaborative participation daily morning whole-class meetings were held. These meetings sought to have opinions voiced, differences resolved, and classroom norms clarified—this was where the community was built. Through continuous interactive discourse generated by class meetings and daily forays into a wide range of problem-solving experiences children developed considerable capacities to remain attentively focused as they shared, challenged, negotiated, refuted and substantiated points of view, including their own unique mathematical thinking. Their respect for each other appeared to flourish as they began to appreciate the diversity of thinking featured in their class. Children changed markedly from being inactive to proactive participants in the teaching/learning process. The culture of the classroom, in nurturing empathy towards each person's point(s) of view, continually reinforced the value of sense-making through willing and confident participation—the hermeneutic nature of the dialectic educed connections and interpretations through comparison and contrast of divergent views. All experiences in the classroom e.g., author's chair, show-and-tell, spelling, social studies, science, etc. bore evidence of the students' engagement in peer interaction as they sought to make meaning of their experience; a stark contrast to their earlier modus operandi as passive listeners. They became increasingly focused on solving and resolving problems as active collaborators, and more and more compelled to justify meaning (qua shared knowledge) as they sought to explicate their perturbations. The following example from whole class sharing time in the mathematics period (April 21) exemplifies this change in focus: Elvis had written the following on the chalk board:

$$19 + 5 = 24$$

$$25 - 5 = 19$$

Taylor interjected: "You said nineteen plus five is twenty-four; then you said that twenty-five minus five is nineteen. That doesn't make sense." Notice that Taylor did not simply contradict Elvis by indicating that 25 – 5 = 20 (which Taylor knew to be the case). Instead he attempted to challenge Elvis' thinking by using Elvis' own argument.

Though findings from this study suggest that the children have developed mathematically, the question remains: "What constitutes mathematical development?" The children in this class have demonstrated the following characteristics of mathematical thinking: (a) they readily and openly debate and justify their mathematical reasoning with confidence and enthusiasm; (b) they are excited about their mathematical experiences; (c) they work collaboratively to negotiate differences of opinion about a broad range of mathematical ideas; (d) they have devised their own successful ways to operate with numbers rather than the teacher's; (e) they have expressed a variety of ways of conceiving patterns in number and spatial relationships including notions of fractions, negative numbers, symmetry, equality, commutativity and multiplicativity; (f) and they work imaginatively and willingly towards solving problems. "The tone of this class is one of respect and freedom—no repression or coercion—it revolves around trust" (project field notes). Children constructed mathematical concepts whilst constructing different ideas of what it means to learn, and these two developing dimensions of education coalesced to establish what it meant to do mathematics.

Conclusion

Friere (1985) has argued that decontextualized knowledge, thought without action, mystifies learning and knowledge and leads to oppression rather than empowerment. When children have the opportunity to openly and confidently present their thinking in a public forum without fear of recrimination or rebuttal, even though they know their ideas are open to cross-examination from their peers, they demonstrate their creatively diverse ways of thinking. Such an opportunity positions them as co-learners and co-teachers. Young children are constantly constructing ideas of what it means to engage in mathematics. As collaborators in the solving of problems, disposed towards a willingness to share, reflect, and trust in a collective attempt to respect personal and unique mathematical thinking, young children engage in the interactive constitution of mathematical meaning. Should children in every classroom demonstrate such a proactive disposition—then what? Bauersfeld (1996) suggests "consequently, we shall have to engage much more in interactions than in arranging for a set of tasks to be solved by the single child in competitive isolation . . . as teachers we will have to act much more carefully in all classroom interactions taking into account that our children actually learn along more fundamental paths, and actually learn deeper lessons than those that we think we are teaching them" (p. 6).

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SIMILARITIES AND DIFFERENCES OF EXPERIENCED AND NOVICE K–6 TEACHERS AFTER AN INTERVENTION: THE USE OF STUDENTS' THINKING IN THE TEACHING OF MATHEMATICS*

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This study examined relationships among experienced and novice teachers' pedagogical content beliefs, pedagogical content knowledge, and instructional decisions after a one-year intervention. Results indicate that the instructional decisions made by experienced and novice teachers who believe that students' thinking is important were similar as were the instructional decisions of experienced and novice teachers who believe less that students' thinking is important.

Research aligned with current mathematics reform recommendations suggests that to teach mathematics effectively means to teach with a reflective consideration of students' thinking (Carpenter & Fennema, 1988; Cohen, 1990; NCTM, 1989; Shulman, 1987); thus, the process of learning to teach effectively becomes an ongoing pedagogical-problem-solving endeavor. During this endeavor, teachers develop beliefs about what it means to teach and to learn that have a myriad of influences which include, but are not limited to, their own knowledge base and experiences with learning (Borko & Putnam, 1996).

Researchers believe that if teachers are to teach in new ways, focusing on understanding, then their own understanding and conceptions of mathematics are of utmost importance (Ball & Mosenthal, 1990). Even though knowledge of content may be necessary in order to teach effectively, it is not sufficient. In a reform-based learning environment, many studies have focused on the influence of both beliefs and pedagogical content knowledge which Shulman (1986) describes as "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9).

Novice teachers believe mathematics is a set of rules and procedures (Ball 1990) and their methodology typically models a "show and tell" approach, reflecting how they were taught. Pedagogical reasoning (Shulman, 1987) is relatively undeveloped in these teachers; that is, their ability is limited in respect to identifying and selecting strategies to represent critical ideas of a lesson and to adjusting instructional strategies to the thinking of their students. Several studies document their lack of understanding of the mathematics they teach (Ball, 1990). Moreover, preservice teachers are highly influenced by their cooperating teachers, and there is "a tendency for [them] to develop attitudes and be-

haviors that are dominant in the existing culture of the schools" (Brown & Borko, 1992, p. 223). Evidence does exist that novice teachers can learn to use pedagogical content knowledge about students' thinking during teaching (Philip, Armstrong, & Bezuk, 1993), but more studies are needed that can help us to understand how extensive and lasting this influence is.

In order to better understand the learning to teach process, our research focuses on novice teachers. We try to understand the learning to teach process in terms of what we know and have learned about experienced teachers. We try to shed some light on the concerns of the Mathematical Sciences Education Board (MSEB) (1996) when they noted that, "There are several well-known inservice efforts which are based on learning through mathematics teaching practice. What is the potential of such efforts for preservice teachers? What kinds of adaptations are necessary? How can preservice teachers best learn about how children learn mathematics?" (p. 8).

Intervention

As part of a five-year NSF grant that was developed to enhance the learning to teach process, we conducted an intervention to address issues raised in the literature. In order that preservice teachers could practice teach in learning environments consistent with the research-based university methods class we developed, we spent one year working with clusters of experienced teachers in a few schools. For the intervention, articles from research on children's learning in a variety of topic areas in mathematics provided the basis for the readings we used. Tasks were carefully developed to assist teachers to focus on their own understandings and conceptualizations of mathematics. To challenge teachers' beliefs about how students learn mathematics, videotapes of students solving problems were shown and discussed; teachers also interviewed students. Journal writing occurred daily. We continually focused on how to develop students' understanding and the ways to represent mathematics to make it comprehensible to students. One year after the experienced teachers had begun the project, preservice teachers joined them for a two-week summer session. During the following school year, preservice teachers were paired with the experienced teachers for both a clinical experience (six hours per week for one semester) and a student teaching experience (15 weeks for the next semester). This format allowed us to carefully choose teaching placements to ensure that the knowledge, beliefs, and attitudes present in a cooperating teacher's classroom were more consistent with the goals of our teacher education program reflecting recommendations from research (Borko & Putnam, 1996).

A mathematics methods course provided the preservice teachers with the research-based information and tasks that the experienced teachers had during their first year with the project. This course focused on developing pedagogical reasoning and content knowledge. There were many experiences in which

these preservice teachers were engaged that reflected the experiences we would like them to create in their own classrooms.

Biweekly meetings on content and pedagogy were conducted with all teachers during each school year in which they participated, two years for the experienced teachers and one year for the preservice teachers. These sessions were designed to help teachers acquire new knowledge and beliefs (Borko & Putnam, 1996), to develop pedagogical reasoning (Shulman, 1987), and to encourage a support structure of colleagues in order to integrate teachers' new pedagogical content knowledge and beliefs into their teaching practices (Borko & Putnam, 1996).

Theoretical Framework and Methodology

We embraced the perspective that pedagogical content knowledge is a critical component in developing an understanding about teaching and that it is important to be able to describe the degree to which knowledge of students' thinking plays a role in the instructional decision-making process of teaching. Thus, we compared the beliefs, knowledge base, and instructional decisions of experienced and novice teachers in order to determine how a similar one-year intervention affected each group.

Both experienced (N = 27) and novice (N = 21) teachers self-selected to participate in the intervention. All teachers completed a questionnaire on their beliefs about instruction, students' learning, and mathematics content relating to whole number operations. They completed a written survey that posed pedagogical problems related to classroom teaching and learning experiences and were asked to indicate their solutions for each. Videotapes of teachers' instructional sessions and follow-up interviews were made in order to provide more insight into their pedagogical-decision-making process.

Results

Data were analyzed after one year of intervention and thus are from two different years because the experienced teachers had begun one year prior to the preservice teachers. This comparison was made at comparable times in the intervention because an analysis of the data from the belief questionnaire indicated no significant difference between experienced and novice teachers' beliefs after the intervention. That is, the reported beliefs about teaching and learning of each group ranged similarly from a greater consideration of students' thinking (*GST*) to a lesser consideration of students' thinking (*LST*).

Comparisons between experienced and novice teachers were made with respect to knowledge and teaching practices because we believed this information would provide insights into more effective teacher education, particularly at the preservice level. Comparisons were made: 1) between all experienced and all novice teachers, 2) between 3 experienced and 3 novice teachers at the

upper end of the continuum in respect to their reported beliefs, 3) between 3 experienced and 3 novice teachers at the lower end of the continuum in respect to their reported beliefs, and 4) between the six teachers at the upper end and the six teachers at the lower end. This allowed us to determine if pedagogical content knowledge, pedagogical reasoning, and content knowledge were the same or different between experienced and novice teachers who had comparable reported beliefs after similar one-year interventions. Comparisons were done in three topic areas: whole number operations and place value; early algebraic reasoning; and geometry and measurement.

Whole Number Operations and Place Value

Teachers were given a situation in which students incorrectly computed a two digit subtraction algorithm involving regrouping with a zero in the minuend. The questions posed to the teachers were: "What would you do?" and "Why is this an appropriate action to take?" Forty percent of the novice teachers stated that their first step would be to ask the students to explain their thinking. Only one of the experienced teachers stated she would first ask students to explain their thinking. A majority of the experienced teachers' decisions involved asking students to resolve the computational error using a manipulative and almost half of these teachers specified which manipulative the students were to use, such as base ten blocks. Only a few of the novice teachers said they would suggest a manipulative. When manipulatives were suggested, more of the experienced teachers specified how the students were to use the manipulatives than did the novice teachers.

When comparing the six teachers at the upper end of the belief continuum to the six at the lower end, all considered having the students explain their reasoning. However, only the GST experienced teachers stated that this computation was not appropriate to give to students since their representations did not reflect an understanding, indicating an assessment decision. GST novice teachers did not assess the situation to the degree that the GST experienced teaches did. All six GST teachers indicated that this was an appropriate problem for a kindergarten or first grade class compared to the LST teachers who thought this more appropriate at the second grade level. All six GST teachers mentioned providing manipulatives for the students to explain their answers, but giving students a choice of manipulative to use was mentioned more often by the GST teachers than the LST teachers.

The LST novice teachers focused on the students correcting their mistakes and practicing procedures: "Students who did answer correctly can explain why we need to regroup . . . then a couple times a week give such a problem as warm up . . . 'this' will reinforce the idea of borrowing with continued extra practice"; while the LST experienced teachers focused on connecting the use of manipulatives to the algorithm. Five of the GST teachers clearly used students' thinking: one novice teacher wrote, "We would discuss what the numbers re-

ally represent and if their answer would make sense in relation to the numbers. . .I might even refer them back to using their invented strategies for solving the problem and explaining how they got that answer . . . because my main focus is that they understand the problem by using whatever process they choose." GST teachers also considered more options than the LST teachers who focused on procedures.

When examining teaching practices from videotapes and stimulated recall interviews of two experienced (one GST and one LST) and two novice teachers (one GST and one LST), we noted similar patterns. GST teachers have imbedded within their lessons problem-solving tasks based on students' interests and on a reflective assessment of their students' mathematical thinking and needs while the LST teachers focused on tasks that are disconnected from students' thinking. LST teachers' lessons have several examples of students explaining their computational processes with connections among strategies not being made. The GST teachers focused on having students describe their strategies and noted that different strategies exist. During interviews that followed the videotaping of their lessons, both GST teachers indicated knowledge about individual students' thinking, unlike LST teachers.

Since much of our intervention time was spent on the whole number operations and place value topic, we wanted to determine if pedagogical reasoning and beliefs about students' thinking would carry over to other topics; thus, we gathered data in the areas of early algebraic reasoning and geometry and measurement.

Early Algebraic Reasoning

A question was designed to investigate how teachers thought students would respond to a situation that involved counting the number of equal-length sticks needed to build a ladder. When asked how students would respond to determining the number of sticks needed to make a ladder with four and five steps, at least two-thirds of each group indicated that they expected students to suggest more than one possible strategy. Two strategies that were nearly always mentioned were a direct modeling or counting strategy and a recognition of a number pattern generated by adding additional steps to the ladder. When asked about the grade level for introducing this problem, over half of the novice teachers said third grade or higher whereas almost all of the experienced teachers said second grade or lower.

When we focused on GST teachers and LST teachers, we found no differences between groups of experienced teachers and novice teachers at either end of the continuum. The only difference we noted was that novice GST teachers stated they would use algebra problems at a lower grade level than novice LST teachers; thus novice GST teachers' responses appeared more like experienced teachers as a group.

Geometry and Measurement

We compared experienced teachers' to novice teachers' responses on a question that dealt with a situation involving a misconception about the relationship between area and perimeter. The experienced and novice teachers all approached the pedagogical problem in very similar ways. The majority of both groups focused on having the students explore the problem more fully. Most of the novice teachers indicated that they were interested in students communicating an understanding of the problem. Although the experienced teachers suggested many of the same strategies as the novice teachers (drawing, exploring, and discussing), their comments more often centered on having the children "see" what was happening as the dimensions of a shape varied. These teachers suggested having the students use a manipulative such as popsicle sticks or paper clips, thus limiting the shape to a polygon.

When analysis focused on the groups of teachers at the upper end and lower end of the belief continuum of students' thinking, we noticed that the novice teachers (five out of six) clearly stated the misconception while only one (the lowest on the continuum) experienced teacher did. One of the novice teachers wrote: "They are focusing on the length around [the shape] not the amount of space inside . . . I would ask them to accurately draw the different measurements and compare the amount of space. . . on the inside. . .that gives [the students] a chance to develop their understanding of area and perimeter and their relationships." One novice teacher suggested giving the students string to explore the relationship between perimeter and area. These responses indicated a willingness to accept all shapes, unlike the experienced teachers who appeared to limit the shapes to rectangles.

We also analyzed one question in regard to curricular knowledge. Results suggest that both experienced and novice teachers at the upper end of the belief continuum would more often introduce a variety of geometry concepts at kindergarten, than those teachers from either group at the lower end.

In a measurement situation, all teachers considered asking students to explain their thinking and reasoning to each other in order to resolve a difference of opinion as to the perimeter of a rectangle. Half of the novice teachers noted that it was important that students be aware of what units were being used; whereas none of the experienced teachers did. Experienced teachers at the upper end of the continuum went beyond just having the students explain; they also wanted students to discover their own errors, suggesting that these teachers were more likely to go in-depth with their consideration of students' thinking than the novice teachers at the upper end. The novice teachers believed it was important for students to communicate measuring strategies in order to verify their answers, but none of the experienced teachers did. As one novice teacher wrote, "I would have the two students show each together how they measure things to make sure they both measure the same way. . . .They would

also want to see what form [units] of measurement they are each using. . . . It would make them double check themselves."

Conclusions and Discussion

In regard to whole number operations, we concluded that novice teachers' decisions compare more closely to experienced teachers' decisions whose beliefs are similar to theirs, than to the novice teachers whose beliefs reflect a different consideration of students' thinking. GST teachers look more alike in their practices, as do LST teachers. Number of years teaching did not appear to affect decision making as much as beliefs about the role of students' thinking.

In general, our findings suggest that the experienced and novice teachers whose reported beliefs are similar share similarities in their pedagogical decision making and teaching practices. Our findings raise questions in regard to the experiences necessary for novice mathematics teachers. To what extent is the learning to teach with experience process expedited when beliefs on using students' thinking are emphasized in conjunction with an in-depth understanding of a variety of mathematics content? Preliminary analyses of teaching practices raise the question of how the two groups use manipulatives. Novice teachers appeared more influenced by research findings on manipulatives, not specifying as to what or how manipulatives should be incorporated into the learning process as much as the experienced teachers. To what extent does this play out in practice? Novice LST teachers tend not to incorporate manipulatives as much as experienced LST teachers who more often selected the manipulative students were to use.

Most teachers had little knowledge of students' thinking in both early algebraic reasoning and in geometry and measurement. Pedagogical decisions in these topics were vague or general. This has implications for methods courses. If an in-depth understanding of students' thinking is developed in one area and the need to focus on students' thinking does not carry to topics not addressed in university courses, how do instructors balance the need to develop in-depth understanding with the apparent need to address a variety of topic areas? This raises questions in regard to the carry over of pedagogical reasoning among topic areas in mathematics.

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FRACTION DIVISION INTERPRETATIONS

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Through a process of analyzing different algorithms, problem situations, and instructional models, we have identified five fraction division interpretations. The interpretations represent different problem situations that elicit distinctive solution strategies.

The most common interpretation used in instruction is measurement division. Division of fractions by whole numbers can be interpreted as partitive divisions. Related to partitive division interpretations is the determination of a unit rate (Ott, Snook, & Gibson, 1991). Fraction division can also be interpreted as the inverse operation of an operator multiplication. Examination of a concrete model (Carlisle, 1980) for fraction division yielded our final interpretation. This interpretation is the inverse of Cartesian products. Probability provides problem situations for this interpretation.

In summary, rational number divisions can be measurement divisions, partitive divisions, the determination of a unit rate, the inverse of an operator multiplication, or the inverse of Cartesian products. A model for fraction division based on these distinct interpretations will provide a framework for investigating the teaching and learning of rational number division.

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TEACHERS' USE OF MATHEMATICS CURRICULUM GUIDES

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This paper reports findings from a study examining how 21 elementary through high school teachers described their use of curriculum guides in their mathematics teaching. Given their prominence in mathematics instruction, curriculum materials are seen as a potential source of guidance for teachers in their efforts to change their teaching. Exactly what this guidance should look like is not clear because we have limited knowledge of how teachers interact with and use curriculum materials. The aim of this study was to contribute to our understanding of teachers' uses of curriculum resources in order to consider ways that curriculum materials might support change in mathematics teaching. Examination of the teachers' written responses to questions regarding their mathematics instruction revealed patterns in their ideas about curriculum materials.

While most of the teachers were inclined to draw problems or tasks for students from their texts, few looked to it to guide their enactment of these tasks. In fact, many teachers claimed that they rarely read the teaching suggestions. Most teachers claimed that informal assessments of students had the greatest influence on their decisions during teaching. Nevertheless, none of the teachers indicated that their text supported or contributed to these assessments of students.

Even though all but two teachers claimed to consistently use a textbook in one way or another, most were ambivalent about the role the text played in their teaching. They believed that "following a textbook" too closely was a sign of bad teaching. At the same time, they appeared to appreciate having a textbook as a recourse or curriculum guide. Some even mentioned feeling guilty for relying on their texts as much as they did.

While not necessarily representative of all teachers, these findings raise questions about how revised curriculum materials might support teachers in changing their mathematics teaching. In particular, the teachers' concerns about over reliance on curriculum guides together with their tendencies to rely on their own assessments of students in their decision making suggest that materials need to support teachers' decision making processes, rather than supersede them.

COLLABORATIVE INQUIRY FOR INSTRUCTIONAL CHANGE

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In the past decade of mathematics reform, professional development programs to retrain teachers have illuminated a need for school-based opportunities for reflective assessment of new instructional methods (Jones et al., 1994). As teachers examine methods within the context of their classroom and deliberate among old and new views in light of their own students' learning, they begin to restructure their pedagogical knowledge and beliefs (Crawford, in press).

This project describes a ten month professional development program to strengthen the mathematics instruction in middle schools within three school districts. Twenty-two teachers from nine middle schools participated in a seven day instructional phase in June designed to develop pedagogy with the use of manipulatives, graphics calculators, and investigative teaching. This was followed by three days of instruction in August focusing primarily on technology to ignite teachers for the coming year.

A strong implementation phase was designed to assist teachers with analysis and reflection of their new methods impact on student learning. During the instructional phase, each team of two teachers formulated a research question to investigate during the next year. They then designed an action research project in which they would collect data on their students' learning. Teachers were also given a reflective journal to write each week about a class in which they implemented a new strategy and the effects on student learning. For further project evaluation, teachers submitted one video tape and an audio tape of classes.

Teachers attended half-day follow-up sessions in October and January which involved teacher reflection, discussion and feedback. In March for the final follow-up, a Celebration of Learning was held with each teacher team presenting a portfolio display of the results of their implementation and action research projects.

Key elements of the project included: 1) flexibility in implementation as each team self-selected their action research project based upon their own individual readiness level; 2) support for different levels of implementation and change based upon each teacher team; 3) encouragement of teacher autonomy with teachers as self-directed learners; and 4) continuous reflection to foster construction of new beliefs and practices. Project evaluation data will be shared.

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A REVIEW OF RECIPROCAL TEACHING AND ITS APPLICATION IN SOLVING MATH WORD PROBLEMS

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This paper presents a review of the reciprocal teaching method, designed by Palinscar and Brown, which involves the instruction of strategies designed to foster reading comprehension in adequate decoders, but poor comprehenders. The four basic strategies that they developed were: summarizing, questioning, clarifying, and predicting. The unique feature of reciprocal teaching is the fact that soon the students are leading the group. Robust positive findings have been reported by Palinscar and Brown, and by others, who have adopted this teaching strategy.

A student's sense of academic learning is composed of both cognitive and motivational components. Intervention attempts work best when they are designed to improve the student's feeling of competence in academic matters, as well as in the specific cognitive activities. Metacognitive training has been shown to be especially worthwhile in students with learning problems. When students are trained in the processes of planning, checking, and monitoring their work, they should also be informed of what they are actually being asked to do, and why they are doing these activities. The aspects of reciprocal teaching and their reflections on cognitive and metacognitive activities, as well as on competency and motivation, are reviewed.

In most mathematics teaching, students are taught by a method of direct instruction. In an interactive learning environment all this changes: they are told why they are performing certain procedures, and are encouraged to understand the metacognitive issues at hand. By using the reciprocal teaching method of mathematics instruction, the teacher provides modeling, scaffolding, and coaching. Campione, Brown, and Connell devised a reciprocal teaching method for mathematics which incorporates the use of three boards: the planning board, the representation board, and the doing board. Results showed that students improved on the targeted word problems, as well as on several other problems which suggested that transfer had worked for them.

Another schema is being suggested by this writer for solving mathematics word problems. It consists of: 1) Statement of the word problem; 2) Question: What am I looking for? 3) Clarification: What information has been given to help assess the question? 4) Prediction: What interval should contain the answer? 5) Solution: How to solve the problem; and 6) Evaluation: Going back to check if this answer *really* fits the problem. Initial studies with this model showed an increase in student comprehension, less fear of word problems, and an increased interest in doing mathematics.

COOPERATIVE LEARNING IN MATHEMATICS: THE EFFECT OF PRIOR EXPERIENCE

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This study examines some of the factors that influence the effectiveness of cooperative learning of mathematics in small groups. Four preservice elementary teachers taking a sequence of two mathematics courses designed specifically for such students participated in the study. The course covered foundational mathematics through problem solving in cooperative groups. The participants were observed as they worked together in a small group during the first half of the second course. These observations were both in- and out-of-class as the group worked together on different aspects of the course. Most in-class observations were audio and video taped for more detailed analysis of group processes. At the end of the course, the participants were individually interviewed. They were asked to reflect a) on their prior experience in learning mathematics, of working with others in academic and non-academic activities, and of learning from and with others in such settings and b) on their year-long experience of learning mathematics within different cooperative groups. They were also asked the extent to which they would use cooperative learning methods whey they taught mathematics as well as how they envisioned cooperative groups as influencing their work and continued development as teachers.

Analysis of both observation and interview data suggest that while prior experience has a strong and on-going influence on the kind of cooperative learning that occurred within the group, the nature of the task often determined whether there was cooperative learning of mathematics or simply cooperation to successfully accomplish the task. The distinction is important. Further, the interviews indicate the preservice teachers' changing conceptions of the nature of mathematics and of who can do mathematics, as a result of their year's experience of learning within cooperative groups, and their growing awareness of the complex factors that determine the effectiveness of this method of teaching and learning.

INTERNATIONAL STUDIES

LEARNING AND TEACHING GRADE 5 MATHEMATICS IN NEW YORK CITY, USA, AND ST. PETERSBURG, RUSSIA: A DESCRIPTIVE STUDY

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During the 1994-95 academic year, one grade 5 mathematics class in New York City and one in St. Petersburg were selected to examine the intended and implemented curriculum, the level of mathematical preparation of teachers, students' progress, and students' and parents' attitudes. Only the curriculum, teacher preparation, and students' progress will be discussed in this paper. Syllabi and textbooks were reviewed to document the intended curriculum. Research assistants in each classroom recorded instructional episodes to document the implemented curriculum. One American teacher and one Russian teacher provided biographical information about their teaching credentials and philosophy. Complete sets of data were collected from 29 American children and 20 Russian children. The Russian curriculum is more structured and grounded in theory than the American curriculum. Content differences include a pronounced treatment of basic facts and the inclusion of probability and statistics in the American curriculum. Students' mathematical growth was documented for both groups.

Educational reform in the United States and Russia is having an impact on the teaching and learning of mathematics. Current reform in the American educational system is characterized by the need for more and better mathematics for all, emphasizing inquiry-based instruction and student-centered activities (NCTM, 1989, 1991). Current reform in the Russian system is focusing on providing an appropriate mathematics curriculum for all students emphasizing free inquiry and critical thinking (Dorofeyev et al., 1993), features traditionally emphasized by mathematics teachers of gifted and talented students in the former Soviet Union (Toom, 1993). How mathematics curriculum and instruction are reflecting these changes was the subject of this study.

The Research Questions

1. What are the differences and similarities between the content of the intended and curriculum in grade 5 classes in both cities?
2. What are the differences and similarities between the level of preparation of mathematics of the grade 5 mathematics teachers?
3. How do students progress in their development of mathematics concepts during the academic year in both cities?

The Method

Although on a very small scale in comparison to depth and breadth, this study attempted to model the work of Stevenson et al. (1990) who analyzed the mathematics achievement of first and fifth graders in Taiwan, Japan, and the United States. The first research question was addressed by reviewing and examining curriculum materials and reviewing protocols of teachers' lessons. The second question was addressed by interviewing the teachers and having them describe their preservice preparation. The third question was addressed by designing a mathematics inventory that reflected the intended grade 5 curriculum and administering it at the beginning and at the end of the school year. Operational definitions and a description of the participants follow.

Operational Definitions

The intended curriculum is the grade-level-specific curriculum mandated by boards and ministries of education. The intended curriculum is communicated to teachers and administrators in the form of recommended syllabi and commercial textbooks selected and purchased by the school system to support the recommended syllabi. For this study, the intended mathematics curriculum for grade 5 in New York City (NYC) was operationally defined by syllabi (NYC, 1987; NYS, 1980), and a commercial textbook available in the target classroom (Scott Foresman, 1985). The intended curriculum for grade 5 in St. Petersburg was defined as a Ministry-approved textbook (Nupk & Telgmaa, 1990).

The implemented curriculum, that is, the curriculum as it is interpreted by teachers and delivered to students during formal instruction, was defined by what mathematics content was actually presented by the grade 5 teachers This was documented by American and Russian research assistants who spent several days each week recording the events of their respective classrooms.

Level of teacher preparation is defined as the formal mathematics studied in preservice and in-service experiences. Teachers were interviewed to obtained this information.

The development of mathematics concepts is defined as a comparison between results of pre- and post-test measures of a researcher-designed inventory. The "Mathematics Inventory" contained nine items: reading a nine-digit numeral; comparing fractions; solving a multi-step rate problem; calculating a complex whole number expression; rounding; calculating area, perimeter, and volume; and, solving one-variable equations.

The Participants

St. Petersburg, an urban center in Russia, cited as a hub of recent innovative work in mathematics education (Curcio, Evans, & Plotkin, 1997), was selected as one of the study sites. NYC, an urban center, recognized as the most multi-ethnic community in America (Sontag, 1992), was another site. Grade 5

was selected as the target grade because it is the beginning of the critical, highly formative middle school years, during which students develop an appreciation of self and others, critical reasoning skills, and build a firm foundation for adolescence and adult life (Carnegie Council, 1989). Complete sets of pre- and post- "mathematics concepts" data were collected from 29 American and 20 Russian grade 5 children.

The American teacher was selected because she is an exemplary, self-contained classroom teacher who subscribes to the reform in mathematics education, taking an investigative, student-centered approach. The Russian teacher was selected because she is exemplary, believing in student-centered instruction, and differentiating tasks to meet the diverse needs of the learners.

Results

The Intended and Implemented Curriculum

An examination of the intended Russian and American grade 5 mathematics curriculum reveal that for the most part, the content is the same, including the four fundamental operations with whole numbers, and common and decimal fractions; geometry and measurement. The American curriculum also includes probability and statistics. American curriculum recommendations for grade 5 are saturated with low level basic facts, computation, and place value review (NYC, 1987; Scott Foresman, 1985). These low level skills do not appear in the Russian text because students are expected to have mastered all of these by the end of grade 4. Major differences appear in the depth of the content and in the way it is structured and organized. For example, throughout the Russian curriculum relationships and rules are explicitly generalized.

During the academic year, research assistants documented the mathematics activities and tasks in the target classrooms. In the American classroom, the observations supported an investigative, inquiry-based, student-centered approach, employing the use of cooperative learning, learning centers, and calculators. The level of mathematics went beyond the recommendations of NYC and NYS guidelines. In the Russian classroom, the observations supported a structured, rigorous, in-depth analysis of the underlying mathematics being discussed. Throughout the year, students were actively involved in writing their solutions on the chalkboard and explaining their work.

Teacher Preparation and Philosophy

The American grade 5 teacher is representative of NYC public school teachers in that she was prepared to teach all the required content of the elementary school curriculum (i.e., reading, social studies, science, and mathematics). She earned bachelor's and master's degrees in elementary education, and had been teaching for nine years at the time of the study. She described herself as having a "hands-on" teaching style, employing investigative techniques in student-cen-

tered activities. She uses cooperative learning and learning centers. Calculators are available for students whenever they need or wish to use them. She also uses an overhead projector and video when she feels they are appropriate. Her students record problem-solving strategies and thoughts about the problems they solve in journals. Using the textbook only to supplement problems for homework, she goes beyond the intended curriculum by providing students with nonroutine, nontraditional problems that build algebraic thinking (Curcio, Nimerofsky, Perez, & Yaloz, 1997). The problems are assigned and discussed as a regular part of her curriculum. She stays professionally active by attending conferences and workshops, sometimes conducting sessions at conferences.

The Russian teacher is typical of Russian teachers in that beginning in grade 5, teachers must have a degree in mathematics to teach it. She earned a degree in mathematics and physics from a pedagogical institute, now called a university. At the time of the study she had been teaching for fourteen years. It was her first year in the school where the study was conducted. During the study, she taught 23 mathematics lessons per week: 18 lessons for three groups of grade 5; 5 lessons for grade 10. Typically her lesson begins with a warm-up to build students' mental readiness. Then, she explains new material, giving time for understanding. She has children solving problems by themselves, then checking the solutions at the chalkboard together. She does not use electronic equipment because there is none in the school. She emphasizes correct mathematical language and symbols. She follows the content of the textbook but she uses her own logic for lesson development and explanations. She refers to supplemental resources for problems. She differentiates instruction based on students' ability. Homework assignments have obligatory tasks for everyone and non-obligatory tasks for advanced students. Periodically, she attends professional meetings and workshops. As is typically done in Russia, this teacher will teach mathematics to the children in this study until they complete grade 11.

Development of Mathematics Concepts

The development of mathematics concepts was measured by a researcher-designed pre- and post-assessment. As expected, pre- and post-tests results for American and Russian children yielded a significant improvement ($t = 8.86$, $p < 0.001$; $t = 292.5$, $p < 0.001$; respectively). Means and standard deviations are reported in Table 1.

For the pretest, the problem that was the easiest both for the American and Russian children was being able to read a nine-digit numeral (i.e., 47% and 97% answered correctly, respectively). The most difficult items for the American and Russian children were solving one-variable equations, and the multi-step rate word problem. American children also had difficulty with comparing fractions.

For the posttest, American students improved in their ability in comparing fractions (i.e., 34% answered correctly, and 38% got the item partially correct),

Table 1 *Means and Standard Deviations of Mathematics Inventory for Complete Data Sets, Pre- and Post-Test Administrations, New York City and St. Petersburg*

	Mean	Standard Deviation
New York City (n = 29)		
Pretest	4.3	2.2
Posttest	10.5	2.57
St. Petersburg (n= 20)		
Pretest	7.5	2.46
Posttest	13.35	2.53

N.B. Pretests were administered during the first part of September 1994 for both groups.
Posttests were administered at the end of the school year which was in May 1995, in St. Petersburg, and June 1995, in New York City.

and in solving one-variable equations (i.e., 17% answered correctly, and 69% got the item partially correct). American students, who used calculators regularly, did poorly on the complex whole-number calculation item (i.e., none answered correctly), and on the multi-step rate word problem (i.e., only one child got the item correct).

For the posttest, Russian students improved in their ability to solve one-variable equations (i.e., 45% answered correctly, and 39% got the item partially correct); and to solve the multi-step rate word problem (i.e., 39% answered correctly, and 13% got it partially correct). Russian students did poorly on comparing fractions (i.e., none answered correctly, but 97% got it partially correct).

Concluding Comments

When we planned this study, our goal was not to make statements about who is "better" or who is "right" (Davis et al., 1979). "Better" and "right" can only be judged in the context of culture and environment. However, although our cultures and environments are different, we can learn from alternative conceptualizations of curriculum and instruction, and perhaps, "move on to new and improved conceptualizations" (Davis et al., 1979, p. 2). Keeping this in mind, we would like to highlight what we have learned.

A strength of the Russians' approach to mathematics curriculum is attributed to the active involvement of such well known mathematicians as Kolmogorov and Pontrigin, whose influence is still evident in Russian schools

(Sossinsky, 1993). A characteristic of Russian mathematics curriculum is that it evolves, preserving the strengths and addressing the problems without radically discarding major components of the curriculum (Prof. Izaak Wirszup, University of Chicago, personal communication, 19 May 1995) and relies "on contributions of continuous and original psychological research" (Keitel, 1982, p. 110). The structure, organization, and theoretical foundation of the grade 5 curriculum exemplifies these characteristics. Furthermore, expecting students to master basic computational facts prior to grade 5 sets the stage for more "serious" mathematics in grade 5 and beyond.

Statistics and probability are tools for decision making. It is never too early to prepare children for being critical consumers of data. Mathematics programs in the U. S. include these areas. But, being exposed to a lot of concepts that may appear to be discrete, unrelated topics has earned many American mathematics curricula the reputation of being "'a mile wide and an inch deep'" (Shanker, 1996, p. E7).

The American teacher in this study is atypical—although she did not major in it, she admits to "loving" mathematics and she conveys this to her students. She did exceed the expectations of the NYC and NYS recommended course of study, but for the most part the "curriculum" was presented as a set of activities.

This is a "snapshot" of learning and teaching mathematics in only two grade 5 classes, with a limited focus. The mathematics curriculum and expectations of students in the previous and subsequent grades were not examined in this small-scale, exploratory study, but our work has provided us with ideas for developing a more extensive, comprehensive, longitudinal study.

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SOME RESULTS IN THE INTERNATIONAL COMPARISON OF PUPILS' MATHEMATICAL VIEWS

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The report deals with the realization of an international comparison project on pupils' mathematical beliefs. Today, the project is still in the first stage of the pilot study, i.e. the preliminary data has been gathered with a questionnaire from eight countries (Estonia, Finland, Germany, Hungary, Italy, Russia, Sweden, the USA), with N ≈ 200 13-year-old pupils in each. The number of the differences between the countries is big. Only in 4–15 items of the 32, the differences are not statistically significant (on the 95 % level). The chart of similarities between the countries shows that the European countries form a cluster, whereas the US is situated totally alone.

Introduction

Although beliefs are popular as a topic of study, the theoretical concept of "belief" has not yet been dealt with thoroughly. The main difficulty has been the inability to distinguish beliefs from knowledge, and the question is still unclarified (e.g., Abelson, 1979; Thompson, 1992).

Here, we understand *beliefs* as one's stable subjective knowledge (which also includes his feelings) of a certain object or concern to which tenable grounds may not always be found in objective considerations. The reasons why a belief is adopted are defined by the individual self—usually unconsciously. The adoption of a belief may be based on some generally known facts (and beliefs) and on logical conclusions made from them. But each time, the individual makes his own choice of the facts (and beliefs) to be used as reasons, and his own evaluation on the acceptability of the belief in question. Thus, a belief, in addition to knowledge, also always contains an affective dimension. This dimension influences the role and meaning of each belief in the individual's belief structure. (For more on the concept belief see e.g., Pehkonen, 1995 or Pehkonen & Törner, 1996.)

In one's belief system, beliefs are usually held with a different degree of conviction (Abelson, 1979). For example, Kaplan (1991) refers to the concepts "deep belief" and "surface belief" which could be understood as unconscious beliefs and conscious beliefs. One interpretation here could be that unconscious beliefs are primitive (basic) beliefs, and conscious beliefs are conceptions. As a matter of fact, we explain here *conceptions* as conscious beliefs, i.e., we understand conceptions as a subset of beliefs. Thus for us, conceptions are higher order beliefs which are based on such reasoning processes for which the premises are conscious. Therefore, there seems to be an

argument basis for conceptions, at least they are justified and accepted by the person himself.

One variation of conceptions are *views*. They are very near conceptions, but they are more spontaneous, and the affective component is more emphasized in them. Conceptions are more considered than views, and the cognitive component will be more stressed in them.

The Focus of the Research Project

The purpose of the research project "International comparison of pupils' mathematical beliefs" (Pehkonen, 1995) is to clarify pupils' views of mathematics. But the focus lies in the comparison of pupils' mathematical views: *Are there essential differences and/or similarities in pupils' views of mathematics in different countries?* And in the pilot study results of which we are presenting here, we try to provide answers to the research problem with the aid of the questionnaire data.

The Realization of the Pilot Study

In the pilot study of the research project "International comparison of pupils' mathematical beliefs", data was gathered with the help of a questionnaire. The questionnaire used was developed for another research project, "Open Tasks in Mathematics" (Pehkonen & Zimmermann, 1990). The purpose of the questionnaire was to clarify pupils' views of mathematics teaching. In the questionnaire, there are 32 structured statements about mathematics teaching for which pupils were asked to rate their views on a 5-step scale (1 = fully agree, . . . , 5 = fully disagree), and three unstructured questions inquiring pupils' experiences and wishes.

Countries in Question

The first part of the pilot study consisted of collecting data from about 200 seventh-graders in each country. The questionnaire has been administered in the following eight countries (the name of the local coordinator and the number of pupils' questionnaire answers in each country are given in brackets): *Estonia* (Dr. Lea Lepmann, University of Tartu; N = 257), *Finland* (Dr. Erkki Pehkonen, University of Helsinki; N = 260), *Germany* (Nordrhein-Westphalen; Prof. Günter Graumann, University of Bielefeld; N = 258), *Hungary* (Dr. Klara Tompa, Institute of Public Education, Budapest; N = 196), *Italy* (Prof. Fulvia Furinghetti, University of Genova; N = 246), *Russia* (Prof. Ildar Safuanov, University of Tatarstan; N = 206), *Sweden* (Arne Engström, University of Lund; N = 196), and the *USA* (Georgia; Prof. Tom Cooney, University of Georgia; N = 203).

Administration of the Questionnaire

This stage of the pilot study was running in 1989–94. The original questionnaire was translated from German into English by the author, and the translation was check by the USA coordinator. The versions in other languages are translated and checked by the local coordinators.

Each national representative has organized the data gathering in their own country. Usually, there are about 100 pupils from the capital city and the same amount from a smaller town in the neighborhood of the capital. The questionnaire was filled in during a mathematics lesson, and conducted by the mathematics teacher.

In large countries, the data collection has happened only in one state, e.g., in Georgia, USA, in order to be comparable to smaller countries. And in the countries with parallel school system, such as Germany, the data has been gathered from all school forms.

Research Results on International Comparison

The question of the international comparison of pupils' mathematical beliefs still seems to be an almost unexplored field. The main question here is: "Are there essential differences in conceptions of mathematics teaching in different countries?" We know that mathematics can be understood as a universal discipline. So, the question arises whether pupils' conceptions on mathematics and on mathematics teaching and learning are also universal, or whether they are, perhaps, culture-bound.

About six years ago, an international project on comparison of pupils' mathematical beliefs was started (Pehkonen, 1995). Before the project "International comparison of pupils' mathematical beliefs" from which some preliminary results are published (Graumann & Pehkonen, 1993; Pehkonen, 1993, 1994; Lepmann, 1994; Pehkonen & Tompa, 1994; Pehkonen, 1995a; Pehkonen & Safuanov, 1996a, 1996b) and many others are under elaboration, there was almost no research into variations between pupils' beliefs on an international scale. Only in the Second International Mathematics Study (Kifer & Robitaille, 1989) were pupils' responses to some questions on the affective domain dealt with in a background questionnaire. The study indicates that there are large differences between countries on measures of mathematical beliefs and attitudes.

The Concept of Consensus Level

People differ in expressing their position regarding a statement: Some like to take an extreme position, whereas others tend to respond carefully. But usually their attitude (positive or negative) is clear. Therefore, for further analysis of the responses, we reduced the original response scale (1-2-3-4-5) by com-

bining the two response values at the extreme ends of the scale, which yields a three-step scale of *agree* (1 or 2), *neutral* (3), and *disagree* (4 or 5). This might diminish some of the tendencies in the data, but on the other hand it offers us a solid base to begin with.

In the analysis and interpretation of the responses, the terminology for the consensus level was used as follows: We say that the responses to a statement are in *complete consensus*, if at least 95% of the test subjects' views were on the same extreme end of the scale; *consensus*, if at least 85% but less than 95% of the test subjects' views were on the same extreme end of the scale; *almost consensus*, if at least 75% but less than 85% of the test subjects' views were on the same extreme end of the scale; *lack of consensus*, if less than 75% of the test subjects' views were on the either extreme end of the scale.

Comparing Results from the Questionnaire

Here, we are looking for similarities and differences between countries in question concerning all items. Consensus levels (Table 1) give a good measure for agreement within a country. Since there are so many statistically significant differences between countries, we will focus on similarities, i.e. items with no significant differences (Table 2).

In the following, we will use the following abbreviations: EST = Estonia, FIN = Finland, GER = Germany, HUN = Hungary, ITA = Italy, RUS = Russia, SWE = Sweden, and USA = the United States.

Consensus Levels of Responses

Here, we consider agreement percentages of the responses in each country separately, and check, whether they have reached any of the consensus levels. In Table 1, each item is given with its consensus level.

20: only ... talented pupils can solve (disagreement percentages)

In three items (1, 19, 24), consensus levels were reached by seven countries, and in three further items (11,15, 31), six countries resulted consensus. Furthermore, there was a lack of consensus in each country in six items (2, 7, 12, 17, 21, 23).

Similarities in Pupils' Views

When checking the differences between the country means with the Mann-Whitney U test, we found that there were more items with statistically significant differences than those without such a difference. Hence, we decided to concentrate on similarities. Table 2 shows the amount of similarities between the countries, i.e. the amount of items where the Mann-Whitney U test was not showing a statistically significant difference (on the 95 % level).

The number of the similarities, i.e. items without a statistically significant difference, varies between 4...15. The biggest number of similarities (15) is

Table 1 *The Level of Consensus on Pupils' Responses to the Questionnaire Statements (cc = complete consensus, c = consensus, ac = almost consensus, and Σ = the number of items in consensus)*

Items	FIN	HUN	SWE	EST	USA	GER	ITA	RUS	Σ
1: mental calculations	c	c	c	c	-	ac	ac	ac	7
2: right answer ... more important than the way		-	-	-	-	-	-	-	-
3: mechanical calculations	ac	-	-	-	-	c	-	-	2
4: pupil ... guess and ponder	ac	-	-	ac	-	-	-	ac	3
5: everything ... expressed ... exactly	-	-	-	-	-	-	-	c	1
6: drawing figures	-	c	-	-	-	-	ac	ac	3
7: right answer ... quickly	-	-	-	-	-	-	-	-	-
8: strict discipline	c	-	-	c	-	-	-	-	2
9: word problems	ac	ac	-	-	-	ac	c	ac	5
10: there is ... procedure ... to exactly follow	-	ac	-	ac	-	-	-	-	2
11: all pupils understandac	cc	cc	c	-	c	-	ac	6	
12: learned by heart	-	-	-	-	-	-	-	-	-
13: pupils ... put forward their own questions	ac	c	-	ac	ac	-	c	-	5
14: pocket calculators	-	-	-	cc	ac	-	-	-	2
15: teacher helps ... when ... difficulties	ac	-	ac	c	ac	ac	-	ac	6
16: everything .. reasoned exactly	-	ac	-	c	-	-	ac	c	4
17: different topics... taught separately	-	-	-	-	-	-	-	-	-
18: repetition as much as possible	-	-	-	-	-	-	-	ac	1
19: tasks ... have practical benefit	c	ac	ac	cc	ac	-	ac	c	7
20: only ... talented pupils can solve (disagreement percentages)	(ac)	-	-	-	(ac)	(ac)	-	-	3
21: it could not always be fun	-	-	-	-	-	-	-	-	-
22: calculations of areas and volumes	ac	ac	ac	-	-	-	ac	-	4
23: it demands much effort	-	-	-	-	-	-	-	-	-
24: there is ... more than one way	c	ac	ac	ac	c	ac	ac	-	7
25: learning games	-	-	-	ac	c	-	-	ac	3
26: teacher explains every stage exactly	-	ac	ac	ac	-	c	-	-	4
27: pupils solve tasks ... independently	-	-	-	-	-	ac	-	-	1
28: constructing of ... concrete objects	-	-	-	-	-	-	ac	-	1
29: as much practice as possible	-	ac	-	-	-	ac	-	ac	3
30: all ... will be understood	ac	c	-	c	-	c	-	ac	5
31: pupils are working in small groups	c	ac	-	ac	ac	ac	ac	-	6
32: teacher ... tells ... exactly what ... to do	-	-	-	c	-	-	-	-	1

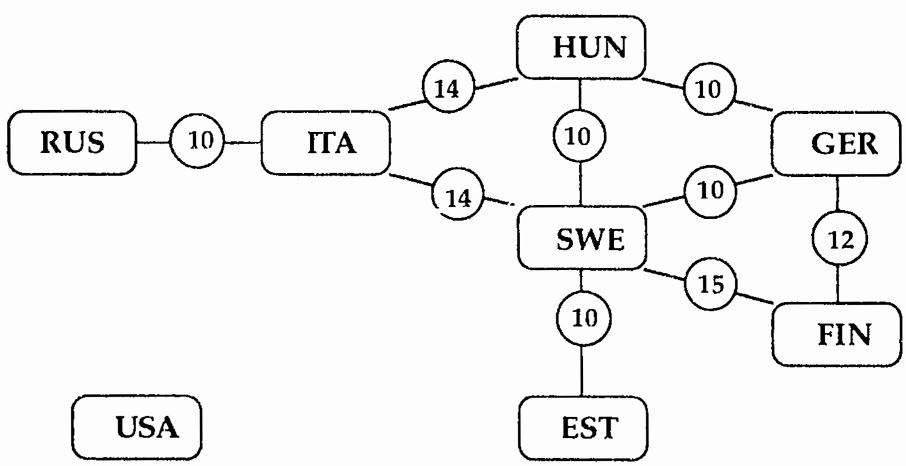
Table 2 *The Number of Similar Items Between the Countries*

	FIN	HUN	SWE	EST	USA	GER	ITA
HUN	9						
SWE	15	10					
EST	9	7	10				
USA	8	7	7	7			
GER	12	10	10	7	7		
ITA	9	14	14	4	7	5	
RUS	6	8	8	7	7	8	10

between Finland and Sweden which is not surprising, since these two countries have long time been developing their systems according to similar ideals. Instead of that, the big numbers of similarities (14) between Italy and Hungary as well as between Italy and Sweden are cumbersome. If we consider only the biggest numbers of similarities (≥ 10), we may construct Chart 3.

The chart shows that the European countries form a cluster, whereas the US is situated totally alone. In addition, we see that Sweden has the most of the similarities with other countries. Will this show that Sweden has taken (and amalgamated) many ideas from other countries? Finland had a lot of similarities with Sweden, but also with Germany, and these could be explained with

Table 3 *The numbers of the biggest similarities (≥ 10) between the countries.*



common history and culture. Instead of that, a surprising point is that Russia, Hungary and Estonia do not have many similarities, although they had a long period of common politics, also in education. It is worthwhile noting that if we change in Table 3 the limit of acceptance (number of similarities ≥ 10), the chart will change drastically.

Some questions which arise automatically are: In which items of the questionnaire are the similarities? Are there some items in which there are more similarities than in others? When answering these questions, we might try to sketch a common view of mathematics teaching for pupils from these eight countries: During mathematics lessons, there should be also small group working, and the teacher should help when there are difficulties. In doing mathematics, the right answer is not more important than the way of solving. Tasks in school mathematics are not only for talented pupils, and doing mathematics could not always be fun.

Endnote

The number of the differences between the countries is big. Only in 4–15 items of the 32, are the differences not statistically significant (on the 95 % level). When checking, in comparison, the differences between boys and girls in these countries, there are, as a rule, only about the same number of items with a statistically-significant difference. Thus, the differences between countries are much bigger than within a country (e.g., between boys and girls).

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VIEWS OF GERMAN MATHEMATICS TEACHERS ON MATHEMATICS

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In a questionnaire with 77 items, more than 300 mathematics teachers of secondary schools in Germany were asked for details on their views on mathematics. Thus, we were questioning their mathematical beliefs by which we understand the teachers' attitudes towards mathematics. We were particularly interested in whether their views could be recognized as a structure. The aspects of "formalism," "scheme," "process" and "application," which are known from former research, were central dimensions of attitudes in the teachers' answers. These four global dimensions formed a global part structure which we derived as a graph through the significant partial correlations. Thus, the purpose of the talk is two-fold: first of all, to present the results of such an investigation for which only scattered results are available in Germany; and secondly, to provide a more precise insight into the structure of beliefs and to reveal their interrelationships using the multivariate methods.

The Relevance of the Investigation

The central role of beliefs for the successful teaching and learning of mathematics has been pointed out again and again by numerous educational researchers. Pehkonen / Törner (1996) mentioned four aspects, in particular, which justified a close investigation of beliefs and belief systems: (i) mathematical beliefs as a regulating system, (ii) mathematical beliefs as an indicator, (iii) mathematical beliefs as an inertia force and (iv) mathematical beliefs as a prognostic tool. With this in mind, the teachers' beliefs play a key role in the teaching and learning process so that priority is given to this in research.

Theoretical Framework

In the German language there does not exist an adequate translation for the widely-used word *beliefs*, because each translation is inhibited with limited associations. On the other hand, many publications avoid laying a clear, understandable foundation. We will identify *beliefs* in the following with *attitudes*, whereby we understand attitudes as being conceptional constructions which we assume *a priori* to contain a certain consistency among cognitive, emotional and behavioral components.

Mathematics as a world of experience and action can be assumed to be an extremely complex field. This also applies to the corresponding attitudes. On the *cognitive level* we can assume that the subjective knowledge of mathematics and teaching mathematics contains ideas in several different categories: (a) beliefs about mathematics, (b) beliefs about learning mathematics, (c) beliefs

about teaching mathematics and (d) beliefs about ourselves as practitioners of mathematics. At the same time, category (a), "beliefs about mathematics," comprises a wide spectrum of beliefs which, at least, contains the following components: (a1) beliefs about the nature of mathematics as such, (a2) the subject of mathematics (as taught in school or at the university), (a3) beliefs on the nature of mathematical tasks and problems, (a4) beliefs on the origin of mathematical knowledge and (a5) beliefs on the relationship between mathematics and empiricism (particularly on the applicability and utility of mathematics) and so on. It is obvious that there are easily understood *affections* as well as *behavioral dispositions* and intentions associated with each component (a) to (d) and its subcategories.

It is therefore evident that there cannot exist an absolute smallest attitude unit which cannot be broken down and analyzed further, just as an atom may not broken down into its respective sub-particles. In this context there will always exist a smaller particle. In most of the cases it was impossible to differentiate between beliefs and "belief systems." With this in mind, we define a belief system as a hypothetical attitude construction which, concerning particular attitudes towards mathematics, is yet to be proven and is, therefore, of no empirical, but rather of heuristic value. In the German language the expression *belief system* is often replaced by a term occasionally used in analogy by Schoenfeld and referred to as a "mathematical world view." With this in mind, the expression, "mathematical world view" will be understood as a synonym for a "belief system" with respect to mathematics.

Thus, information gained from two levels is significant to a definite expression of a 'mathematical world view' or belief system respectively: on the one hand, expressions of single beliefs; on the other hand, the relationships between different beliefs within the 'world view.' The relationships between single beliefs form a structure which is probably more important to the representation of a 'mathematical world view' and its relevance to action than to all the beliefs it contains.

The following question then arises: *Which structural parameters do mathematical world views possess?* It may be that these structures offer better explanations and predictions of certain ways of acting rather than single beliefs. Furthermore, changing a belief system requires a detailed knowledge of the interfering parameters as well as the number and strength of the connections which are intricately woven into a net.

Research literature on this subject offers some approaches to a possible structuring of belief systems. For example, Rokeach (1960) organized beliefs along a dimension of *centrality* to the individual. The beliefs most centralized were those on which there was a complete consensus; beliefs about which there were some disagreement would be less central. In contrast to this idea, Green (1971) discusses three dimensions of belief systems: *quasi-logicalness, psy-*

chological centrality, and *cluster structure*, which will be considered here more closely (see also Pehkonen 1994). We prefer a *multivariate method* in order to visualize belief systems, which will be described later. This approach is a central theme of the in-depth examination of more than 1,600 students in Grigutsch's dissertation (1996).

The Investigation

Approximately 400 teachers participating in the German Annual Mathematics Education Conference in Duisburg were asked to fill out a questionnaire containing 77 items. A total of 310 questionnaires were filled out and returned. By conducting this investigation, Grigutsch, Raatz & Törner (1995) tried to explore teachers' beliefs concerning mathematics. This spot check cannot, however, be classified as fundamentally representative of mathematics teachers at the secondary level, because participation in this continued teacher training conference was voluntary. It can be assumed that the returned questionnaires were filled out more or less by innovative teachers.

Methodology

The 'attitude' concept is, among other things, signified by a consistency in reactions. Thus, the existence of an attitude may only be inferred and empirically recorded if a class of similar stimuli reacts to similar situations.

In relation to our survey, this means (even if the situation of the survey is no real situation of action) at least the following: it is not enough to try to infer an attitude from a certain reaction to an single item during one observation. A group of statements of similar content must be answered in a similar way. Only in this case can the existence of a characteristic (i.e. the object of an attitude) be assumed in mathematics, producing a necessary but not sufficient prerequisite.

We decided upon a method of designing a questionnaire which reflected more of an antagonistical idea of mathematics, on the one hand, as a static product and mathematics as a dynamic process on the other. Our presumptions were based on investigating mathematical world views of mathematics students (Törner, Grigutsch 1994). In the teachers' questionnaires we grouped the questions under three headings: (i) my experiences with the normal, classroom, teaching situation in the school; (ii) mathematics as a field, according to my perspective; (iii) the origin of mathematics; and (iv) mathematics and reality.

First of all, we did a factor analysis to form groups of statements which were part of the questionnaire. The items were scaled as follows: 5 = totally agree, 4 = agree for the most part, 3 = undecided, 2 = partly agree, 1 = do not agree. Furthermore, we used listwise deletion provided a person did not have a positive value in one of the items. Thus, 207 persons were left for the statistical procedures through the factor analysis.

Each factor analysis consisted of 75 items. Items 1 to 77 were included except for items 22 and 23. As for items 22 and 23, the large number of refusals to answer them resulted in too many observations (subjects) being excluded from the factor analysis. Furthermore, it was questionable in principle whether items to which many subjects refused to respond could be taken into consideration during the evaluation of the questionnaire.

The data was analyzed through factor analysis. First of all, an analysis of the principal components was calculated in order to determine the eigenvalues and to carry out the Scree-test. There were 25 eigenvalues which exceeded 1. When taking data from the scree-plot, an analysis based on *four* factors seemed to be recommended. In performing principal component analysis, the four factors were assumed, and then the varimax-rotation as a transformation method was applied (using StatView Software 4.5). As to the orthogonal solution, each factor was determined by items whose loadings exceeded .39.

We were able to prove that the components represented different aspects or views on mathematics, namely the *tool* or *schematic orientation aspect* (S) resp., the aspect of *formalism* (F), the aspect of *process* (P) and the aspect of *application* (A). According to statistical analysis, these aspects were independent dimensions of attitudes toward school mathematics.

Each of these four aspects is operationalized through 8 to 13 items: the methodological and statistical approach demanded this.

Results

In the mathematical world views of teachers, these four global dimensions formed a global partial structure. Using the matrix of partial correlations (n = 253) in Table 1 we obtained the following diagram (Figure 2).

The partial correlations resulted in a partial structure of the 'mathematical world view' or the belief system, respectively, which corresponded to our theoretic assumption of antagonistic ideals. The formalism and scheme (= tool aspect) scale represents both aspects of the static view of mathematics as system and intercorrelate highly. Both parts of the static paradigm correlate with the process scale in a significantly negative way. This confirmed our original hypothesis

Table 1 *The Intercorrelation Matrix of the Four Factors: Tool aspect (S), Formalism (F), Process (P), and Application (A)*

	Formalism (F)	Application (A)	Process (P)
Application (A)	,042		
Process (P)	–,127*	,127*	
Tool aspect (S)	,364***	,087	-,146*

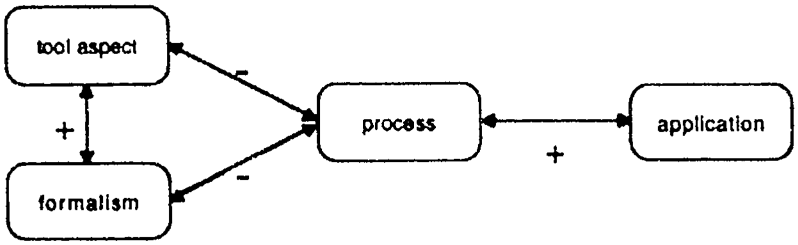

Figure 2. *Structure of teachers' belief system defined by the significant partial correlations* (Grigutsch, Raatz & Törner 1995)

that both views are in direct opposition of one another (at least if a paradigmatic analysis is carried out). The application aspect of mathematics correlated only significantly with the process aspect of mathematics. This corresponded to our pre-theoretic assumptions in that scheme and formalism express a static property, which does not include, however, that solving problems of reality is not a primary aim. From a formalist point of view, mathematics largely refers to itself, a precise conceptualization, a purely formal-logical verification of statements and to its logical-systematic structure. From a schematic point of view, mathematics is a collection of calculation techniques and algorithms which (the non-connection with the application scale has to be interpreted this way) are considered suitable for mathematics-related routine rather than for concrete applications and solutions to problems of reality. On the other hand, the process aspect aims at developing knowledge through a problem-related cognitive process, emphasizing the importance of seeing connections of ideas and of intuition. This dynamic concept of mathematics is more likely to be suitable for application, and this is expressed by the teachers' attitudes. On account of the sample-related findings, the 'mathematical world view' is not uniformly, but differently marked. As to all four belief objects, the frequency distribution covers certain values. For this reason, there are individually different ways in which teachers look at mathematics, ranging from rejection to agreement. Furthermore, those differences are supported or stabilized by the structure formed by these attitude objects. While they are negatively connected to the process aspect, the attitudes toward the scheme aspect and the formalism aspect show mutual support. Looking at mathematics as being dominated by schemes corresponds with this attitude, expressing the idea that formalism is of great importance whereas a process; a like view of mathematics is less significant.

Each attitude of a certain dimension, therefore, supports other attitudes belonging to other dimensions. As for these four dimensions, the "mathematical world view," in the very least, is highly consistent and stable. But, then, this only scarcely implies that any changes of the 'mathematical world view' con-

cerning these dimensions come about. Thus, every attempt must manipulate effects on all four dimensions simultaneously. For this very reason, a change of beliefs will be carried out most effectively if it imparts experiences to all four dimensions which may cause a change.

Further Observations — Comparison of the Means

We shall briefly mention some more evaluations on the basis of our data. Scale values referring to each of the four dimensions were set for each person involved. These dimensions were operationally defined by 8 to 13 items, respectively. The score concerning the statements of each dimension was added up for each teacher involved. A transformation and stretching of the scale resulted in each teacher having a scale value in each dimension ranging from 0 to 50; 0 to 10 represent utter rejection; 40 to 50 in full agreement.

There was no unique view of mathematics in the sample. Teachers had different individual attitudes towards each of the four dimensions, ranging from rejection to approval. The attitudes were 'normally distributed'. Because the distributions in each scale were "normal," there was no preference of a specific value except for the mean which the distribution optimally characterizes. Because the variances are principally the same, we cannot neglect them when comparing their characteristics or underlying features. The overall attitudes of the teachers were (with respect to formalism, schema, process and application) represented by their mean and marked as single points on the scale. The average attitude of the teacher in each dimension was optimally identified by its mean. Due to the fact that the means were different, their were also varying attitudes, which contrasted to the four attitude objects. In the eyes of the mathematics teacher, these four factors were not of equal importance. As a result, in an accented mathematical view there exists much more in that some elements were found to display more emphasis while others, less. According to the average view of the teacher on mathematics, the aspect of scheme was estimated rather low and, in a way, somewhat refused, with formalism ranking in the upper middle sector of a scale. In contrast, the aspects of application and process were valued relatively high. The views relating to the application and process aspects were considered as being quite meaningful and were not distinguishable by their average estimations.

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Language, Discourse, and Socio-Cultural Issues

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TEACHER CHANGE: DEVELOPING AN UNDERSTANDING OF MEANINGFUL MATHEMATICAL DISCOURSE

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This paper reports how a sixth grade middle school mathematics teacher has learned to use discourse in her instruction. The first year of data shows the teacher as central in the development of classroom discussions even though much of her intention was to use students' participation as a means for mathematical learning. An intervention was constructed during the summer between the two years of research to examine the teacher's perceptions of her classroom discourse. The intervention, grounded in the teacher's classroom data, precipitated a change in instruction during the following year. Results suggest that the teacher is no longer the central figure in the mathematical discussion. Instead, students are working to construct their own understandings. This new practice, while honoring the students' understanding, has subtracted the teacher's mathematical knowledge from the classroom discussion. The teacher is a facilitator of the social structure of the discourse without adequately scaffolding the substantive mathematical structure.

Introduction

Many researchers define discourse as the core of mathematics instruction. Understanding the nature of discourse and how teachers construct mathematical communication in practice is the focus of this paper. As teachers attempt to enact the Standards and construct practices in mathematics education that embrace mathematical reform many find the journey an arduous process. This study focuses on a middle school mathematics teacher's facilitation of a discourse community of sixth grade mathematics students. We report on this teacher's understanding as it has evolved over the last two years in the practice of her classroom and through discussions with the research team.

Over the past ten years the mathematics education community has seen an increasing awareness of the importance of communication in mathematics classrooms (NCTM, 1989, 1991, 1995; Hiebert, 1992; Ball, 1993; Elliott & Kenney, 1996; Williams & Baxter, 1996). This increased attention points to the salient issues of mathematical communication and the role the teacher plays in establishing substantive discourse. According to the *Professional Standards for Teaching Mathematics* we see mathematical discourse as "the ways of representing, thinking, talking, agreeing, and disagreeing" (p. 34). Furthermore, mathematical discourse is an integral part of doing mathematics—agreeing on assumptions, making assertions about relationships, and checking if the assertions are

reasonable (Hiebert, 1992). The teachers' role is to orchestrate such discourse by "posing questions, engaging and challenging student' thinking, listening to students, and deciding when to provide information, when to clarify an issue, when to model, when to lead, and when to let a student struggle with a difficulty" (NCTM, 1991, p. 35). When teachers undertake a facilitative role in classroom discourse, students not only express their mathematical thinking but also conceptualize situations in a variety of ways (Yackel, Cobb, Wood, & Merkel, 1990).

It is through the work of Williams and Baxter (1996) we begin to examine the nature of this theory and rhetoric on discourse in practice. They propose an examination of discourse oriented teaching which deconstructs mathematical communication into two component parts, each necessary to scaffold discussion. Furthermore, the scaffolding of mathematical discourse must include a social component—norms for social behavior and expectations—and an analytic component—the scaffolding of students' mathematical ideas, and support of mathematical content. These two forms of scaffolding come into tension as teachers perceive of their role in instruction as "inculcat[ing] knowledge while apparently eliciting it" (p. 24).

Our research further explores the complexity of the balance between the social and analytic scaffolding of mathematical discourse in classroom. The following questions framed the first year of our research: Who was participating in the classroom discussions? How did the teacher make decisions about classroom activity? How was the curriculum was chose? Base-line data were gathered in order to typify the teacher's practice and provide substantive information for the teacher/ researcher summer meetings. Data from the first academic year revealed that the teacher developed mathematical understanding through small group, whole class, and individual activities. Each of these forms of instruction would culminate in whole class discussions of the mathematics where the teacher was the central vehicle for constructing mathematical understanding.

During the summer meetings the teacher and researchers reviewed video from the classroom. Typical classroom activity was discussed in order to understand the teacher's decision making while in action, as well as her reflections on her actions. After a number of class activities were reviewed, the teacher undertook a series of investigations to determine the terrain of the mathematical discourse. Classroom conversations were analyzed to determine who spoke, in what order, and what patterns developed in the discourse. Based on the continuing analysis of this discourse, video models of other teachers were also examined. During this examination and in further reflection, the teacher determined that she was consistently central in the conversation, students where addressing her in discussion and not other students, and the mathematical understanding that was developed in discourse was consistently her own and not

the students. To address these findings the teacher and researchers investigated and designed a beginning of the year curricula to shift the focus of discourse from the teacher's understanding to the students. Tasks were examined, the classroom was reorganized, and the teacher informally role played activities with the researchers sharing possible questions, structure, and follow-up activities.

Year Two

The second year of the study the teacher implemented these changes. While large changes occurred in the nature of who was speaking, to whom the discussion/questions were addressed, and the authorship of the mathematical understanding—students were now the central figures of the mathematical discourse—the mathematical content of the discussions remained solely focused on the students' nascent understandings. Whereas during the first year the teacher was the central interpreter of the mathematics, during the second year the teacher's expertise was noticeably absent from the discussion.

Methodology

The data collected from the first and second year consisted of video taping the class one to two times a week and informal discussions which were audio taped. The focus of the video taping was on the speaker in the class during whole class instruction and selected small groups during group activities. Informal interviews investigated the teacher's decision making and issues which arose in class. A sampling process was used for this research in which the first author identified various forms of discourse used in class, selected tapes equally distributed across the various forms, and reviewed at least one tape from each month of data collection. Extensive notes were taken on each of these video tapes and selected excerpts were transcribed. The second year's data is in preliminary stages of analysis and will inform a second set of summer meetings with the teacher and researchers.

Findings

The following is a typical dialogue sequence in which the teacher, Ann, facilitates the discussion of a task. Ann's questions were designed to foster the development of a conceptual understanding of decimal multiplication through a comparison of place value. Furthermore, her questions provide the students with an overview of what Ann knows to be typically confusing mathematics for her students. "[I am] presenting them with something that I know they have seen before, but they never explored. They've just accepted as God given (6/96)." The known-answer questions that Ann poses lead her into an explanation of decimal multiplication.

Year One

Ann begins her math class with a warm-up, places a problem on the overhead, and walks around looking at students' papers.

The problem:

A tent is on sale at 30% off. What is the sale price ? Choose a decimal or fractional method to solve.

Ann adds orally, "When you finish try the other method."

After looking at a number of papers, Ann moves to the overhead to begin her lesson. Her discussion of the problem is led by *her* questioning the students and eliciting as many different solutions methods as she has noticed on students' papers. She wants students to see more efficient approaches and also recognize that other methods provide the same solution.

Ann:	The tent is $150 and at 30% off.
	Who chose a decimal method? "C", What did it look like?
C:	150 times .3 . . . multiplied by .30.
Ann:	Okay, Did anyone set it up differently, still using a decimal method?
	Did anyone do this? Just asking, right. [She writes 150 x .3]
	Did anybody do this? [waits about 3 sec.] Could you have done that?
Student Chorus:	Yeah.
Ann:	Is this solution going to be the same as that solution? [Points to the first and then the second problem]
Student Chorus:	Yeah.
Ann:	Right, Why?... Why? (1/96)
Ann:	I want to talk about these [two problems]. How did you know it was 45? [waits] M?
M:	Cause you count to the right of the decimal. And if it was 30 there are two places to the right so you count two over.
Ann:	Oh yeah? How many people heard that before? Why does that work? [waits] It's a great trick, don't get me wrong. I don't want to discourage you from using this trick. But you need to know why tricks work mathematically. Because you know what can happen . . .

Ann socially scaffolds the discourse by calling on students. She establishes the norms of response that students provide by leading the content of the discussion. The "known-answer" questioning format limits students' responses thus directing them toward Ann's instructional goal of recognizing multiple solution methods. The analytic scaffolding of the discourse is subsumed by Ann's instructional goal for students to see the connection between the two solution methods. While the understanding Ann wants to ensure for her students is important, the locus of meaning resides with her. Furthermore, the students' construction of the mathematics is not part of the analytic scaffolding Ann provides. Williams and Baxter (1996) suggest this construction of discourse is understandable in that teachers "are still under the obligation to have students embrace a more or less fixed body of knowledge, and students still see their job as getting through class. An attempt to have students participate in the social negotiation of meaning may or may not be meaningful for them" (p. 36).

During the second year of the study, the students' participation became the center of instruction, yet the meaningful nature of the discourse remained in question. Class typically started with a problem, similar to the first year, in which students would work either individually or in groups. The difference in instruction arose when the whole class discussion took place. The students' mathematics was now the focus of the discourse. Students would present their solutions to the whole class. Typically, Ann would sit at her desk in the back of class, unlike the previous year in which she sat at the overhead. She often would provide reminders for the structure of classroom interactions. "Don't look at me, you are in charge of this. Remember, you have to call on someone after you have presented your solution." The structure of the discussion was facilitated by Ann's instructions for students to always re-word questions, respond to them, and then provide their own position. Students would do so by saying, "I agree with or disagree with student # 1's solution. I did . . ." Discussions would continue in this way until time was called or another activity was assigned. Ann's intention was for students to begin understanding other positions in hopes that they would deepen their own understanding. The mathematical content was driven by the students' presentations of solutions.

Ann socially scaffolds discussions by asking, "What do think? Does anyone else have another way to explain this?" The established norm, which required students to first comment on the last speaker's solutions and then argue their own position, was intended to support the students in connecting the different solution methods. However, what was noticeably absent was Ann's interpretation of the mathematics, and as a result, the students were left to construct the substantive mathematical connections. Her role in providing closure or evaluative explanation to the discussion was left to the students' individual interpretations. Ann's role had shifted from the interpreter of the mathematics to the coach in the discussion. Students' construction of meaning was the fo-

cus, yet there remained an imbalance in the analytic and social scaffolding of discourse. Ann socially scaffolded the discourse through the established participation structure, but during the second year, the students were primarily responsible for the analytic scaffolding. In this case, the analytic scaffolding was limited by the strongest students' understanding of mathematics.

Implications

We have seen Ann promote very different forms of mathematical discourse over the last two years, each situating the locus of mathematical understanding with different participants. This work suggests that the social and the analytical aspects of discourse may serve as a lens by which to view instructional discourse and examine the nature of the students' and teacher's roles in that discourse. Further, this study highlights the importance of balancing the social and analytic scaffolding of discourse. This balance is crucial for establishing patterns of interaction which lead to substantive mathematical discussion.

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GROUP CASE STUDIES OF SECOND GRADERS INVENTING MULTIDIGIT SUBTRACTION METHODS

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Four groups of second graders explored the subtraction of horizontally presented 4-digit subtraction problems using base-ten blocks. The blocks afforded children's inventions of several variants of separate-multiunit methods involving trading or putting one multiunit in the next-right adjacent position to make ten of that multiunit. When dominant group members had good mathematical knowledge and were good rather than bossy leaders, the groups made better mathematical progress. Language describing subtraction is complex because different phrases reverse the direction of subtraction. Trading is also complex because some children focus only on one part of the trade. Descriptions of block and numeric operations were often too general to follow, but children could explain more clearly when asked. Important functions of a teacher are to increase such specific full descriptions using quantitative language, to facilitate linking of numerical and block (or other referent) operations, to help overcome impasses, and to create a simpler problem or use another method to focus children on available mathematical meanings.

This research was undertaken to increase our understanding of how individual conceptual and social competences affect individual learning within social learning settings. This was addressed by studying children's invention of multidigit subtraction methods within a small-group setting. Brief case studies of four groups of second graders overview interactions of individual personalities and varying mathematical understandings of group members that created different patterns of group interaction and different group and individual learning paths.

Perspectives and Theoretical Framework

The authors of this paper take a constructivist view of learning as individual meaning making by each participant and a Vygotskiian view of teaching as assisting the performance of learners by adapting to the perspective of the learner while helping the learner move toward more culturally adapted conceptions. This study was designed to allow teaching to arise mainly from the group interactions of the children, though the adult group supervisors did some scaffolding. The analysis of Fuson (1990), Fuson and Kwon (1992), and of Fuson, Smith, and Lo Cicero (in press) concerning conceptual structures children use

in multidigit situations is used to analyze children's multidigit thinking. The base-ten blocks used by the groups afforded primarily separate-multiunit conceptions of subtraction as requiring trading from the adjacent-left position to get more of a given multiunit. The two conceptions identified by Fuson and Kwon (1992) in Korean children's subtraction thinking were the conceptions used in most verbalizations of children's thinking in this study. The multiunit quantities conception used block words or multiunit quantity words (e.g., "I put this Big Mac [or thousand] here to make ten plates [or hundreds]"). The regular one/ten trades conception was used for all places; it described the action above as "I took this one and made ten here." The resulting teen number was viewed as consisting of one ten and some ones and not just as concatenated single digits (e.g., 13 as a one and a three). As in the Fuson and Kwon (1992) study, some children in this study also integrated these two views for a full conception that could use the multiunit values along with the tens/ones numeral view of each column.

Methods and Data Sources

Four groups of four or five children were asked to subtract horizontally presented 3- and 4-digit numbers using base-ten blocks and to make numeral recordings of their block methods. The children were from the highest-achieving of three second-grade classes. These subtraction sessions followed a 2- to 3-day introductory period with the blocks, and a 5- to 8-day period solving 4-digit addition problems (Fuson & Burghardt, 1993). Groups spent between three and eight days on subtraction. The subtraction time was not sufficient, but was limited by the total time the school was willing for the groups to participate in the study.

Each group session was videotaped. Each group had an adult who oversaw the videotaping and took live notes. Each videotape was transcribed and then checked by a second transcriber. Mathematical and group interaction verbalizations were transcribed verbatim. Mathematical and social/emotional actions were described in the transcript. A third person made separate drawings of all operations with blocks and all writing of problems in numerals either on the group writing pad or on individual papers; these records were related to transcript line numbers.

Results

Group A

The group did not discuss or have difficulties with the direction of subtraction or of words describing subtraction. U immediately saw and verbalized the difficulty: "We have a problem, though. We can't take 2 away from 6. Take 6

away from two." (this was the required hundreds subtraction). He correctly inferred directionality from the horizontal problem. T repeatedly suggested "borrowing" and even described a ten/ones borrow with the blocks more than once. But her English was not very good, and she never used the blocks to explain or justify her suggestion. She was not an assertive group member, so this suggestion was ignored repeatedly. The children did not resolve their posed problem (how to take 6 from 2) on that day. On the second day T again suggested borrowing and finally put in 10 teeth [ones] and took out 1 licorice [1 ten]. Da objected that she was adding (focusing just on the *adding in* of ten teeth). U used a 2-digit tens and ones conception to describe a thousand/hundreds trade, "Take one ten away from this (5 in the thousands column)." This trading attempt was cut short by Da's insistence that you have to go from right to left in math. T showed the licorice/teeth [tens/ones] trade again, but never explained it. Da again objected that "it is adding." Finally T said that she was not adding but taking away "because I have to take ten of these here (pointing to tens)" (focusing Da on the taking-away part of the trade but not discussing the whole trade). Da said, "That's a good idea. Let's do the same with the pancakes (immediately generalizing the trading method to the hundreds)." N finally articulated the whole trade in block words, "She took one licorice away, and she put on ten teeth." This seemed to reduce objections, and the three girls (Da, T, and N) made trades with the blocks linked to numeric recordings.

The group then showed their acceptance of the norm to explain actions in order for everyone to understand by a long period of repeated explanation by the girls to the boys. However, these lacked clarity because of the lack of full multiunit or block words (lots of "here" and "there") and no explicit justification of the fairness of the trade. Also, U throughout wanted to subtract from left to right, and all of the others insisted on subtracting from right to left. This group for addition had worked in either direction, and even started in the middle, in their method of adding everything and then fixing the answer (Fuson & Burghardt, 1993). They might have invented a "fix everything first and then subtract" inverse of this procedure if some of them had not heard the "math rule" to work from the right in subtraction.

On the next day U again tried to begin from the left and do a thousand/hundreds trade; he was stopped by T. He never got a chance to try out a whole problem from left to right. This difference in approach, combined with lack of clear language, continued to plague the group's discussions and eventually led to some withdrawal by U. Most of the time the trades were not fully articulated. But in the final two days Dh did explain both aspects of a trading action, "She took one of these (pointing to the three remaining licorices [tens]) away and put ten teeth [ones]." and Da explained about T that "She is putting the same thing out but in different places."

Group B

This group had with addition developed block and digit-card methods in which the blocks and digit cards for each column were removed and replaced by answer blocks or digit cards (index cards each with a number used to show the numeric problem). The adding was usually done mentally or with fingers. Children did show trades with blocks (e.g., removing ten tinies [ones] and adding in another rectangle [ten]), but they never evolved a method for showing this trade with numerals: Three of the four children added in the trade mentally and did not write it. This group developed similar methods for subtraction. They placed blocks and digit cards vertically and then moved from right to left subtracting mentally or by counting up. Problem blocks and digit cards were removed, and answer blocks and digit cards were put into that column. Until the third and last day when the adult suggested that they relate their block and digit-card trades, children predominantly used a method of making mental teens in order to subtract and did not really focus much on the blocks.

They began by adding the first problem, and then M did smaller from larger with the blocks. N said that the smaller from larger (e.g., 2 - 8) columns should all be 0. Three of the children then did smaller from larger on their paper, but N subtracted to give zeroes. The adult then focused them to look just at the rightmost two columns, which showed 62 - 38. D said, "Maybe you can take a ten from the 6 column and put it with two. You get twelve minus eight." The adult asked D if she could do that with the blocks; D put a long from the 6 longs with the 2 tinies. No one was paying any attention, so the adult asked D to do everything again. Then each child solved the problem again numerically on individual sheets. D and X showed the thousand/hundreds trade and the tens/ones trade, i.e., they immediately generalized the trading. M wrote no trades but did them mentally; she forgot to reduce the thousands by 1. N probably just copied the answer from M, but he might have done part of the problem himself.

D was absent the next day. M worked hard all day and did try to explain issues to both boys. A couple of times M took out one of the next-left top blocks when removing all blocks of one kind in order to put in the answer blocks (e.g., took out one rectangle and all of the tinies before putting back the answer in tinies). But she never said what she was doing (taking out the top 16 and bottom 9) or explained, so this subtle version of putting a larger multiunit with the next smaller column was not noticed or understood by the boys. All three children did help correct each other on various columns about both parts of the group's strategy: They each initiated the statement of the correct subtraction in a column requiring more (e.g., said 3 - 6 as thirteen minus six), and they each initiated or corrected the subtraction in a column from which putting had occurred. Therefore, the boys each showed understanding at some points. Children evolved two different methods for subtracting a traded-from column. M usually first subtracted the numbers and then took away one if a teen had

been made to the right. N added one to the bottom number and then subtracted. The children never used multiunit words and hardly ever used block words; their brief and rare descriptions or discussions used 2-digit tens and ones or concatenated single-digit language. They were all focused on the written numeral problem and worked orally and mentally from this. Each frequently understood their mental "make a teens" method, but each also sometimes made an error (usually not decreasing the traded from column by the one traded).

Group C

The group set up blocks and numbers vertically. N said, "You have to borrow because you can't take 2 away from 8, I mean take 8 away from 2." When asked what to do with the blocks to show borrowing, she said, "Ask Mr. Ten if you could borrow a pretzel [a ten] and then you would take it - one of them - and you would take ten sugar cubes [ones] and put them down." This was one of the fullest descriptions of the next four days spent on subtraction; block words were rarely used after this. Other children did pick up the theme of "Ask Mr. X" and the language "take one." Block trades were done for all columns. The language describing the trades usually used the word "take" or "take away" for both the one taken and the new 10 blocks put in ("Now take ten of these."), but occasionally the word "put" was used for the 10 blocks put in. The equivalence of these block actions was never discussed. Some children clearly indicated by various comments that they understood this equivalence, while others did not indicate so clearly. Children occasionally said the subtraction words backwards, but everyone understood the direction of subtraction (the bottom number was being subtracted from the top number). J once said, "2 take away 6 is minus 4, negative 4," but this was never pursued.

T dominated the block trading over the first three days, either doing it or telling other people what to do before they could think it out for themselves. N continued to show her understanding throughout. B initiated block trades and numeric recordings for various columns, but she was slow and T often told her what to do. K did initiate some block moves and numeric recordings, but he had difficulty explaining what he had done. He finally did so at the adult's insistence. On the next-to-final day, they all wrote on individual papers as the problems were done with blocks. Everyone wrote both problems correctly, often before the block trading. Everyone by the end could do accurately and could record numerical borrowing, but K and J were not so consistently clear about the block actions for borrowing. J at one point argued that he should add <u>one</u> pretzel [one ten] to the tens rather than <u>ten</u> pretzels from the 1 bread [hundred] taken away; this seemed to reflect thinking (at least at that moment) that they were always trading <u>a ten</u> to a given column. The group's tens and ones language suggested this, and their failure to use block words or multiunit names facilitated this view. T, at that time in response to J, said the only clear multi-

unit description of trading, "You need to borrow one hundred; and hundred's are ten tens."

N kept wanting to subtract from the left, and the others from the right. This was the one group that did addition problems from both directions and had discussed relative advantages of each. They had decided that going from the right was faster because they did not to cross out their first answer in a column. T used this in arguing that they should subtract from the right, "Remember, it is faster." On the final problem on the final day, at the suggestion of the adult, they did all of the trades first (from the right as usual) and then subtracted from each direction. They saw that they got the same answer.

Group D

This group especially at the beginning had great difficulty with, and long controversies about, correct language to express the direction of subtraction. The work of this group was heavily led by one girl C, who knew the standard U.S. subtraction algorithm and a standard Chinese algorithm. Her blocks subtraction methods looked like these algorithms, with a block put into the next column to the right when trading was necessary. C did little explaining, though her occasional comments indicated that she was using both multiunit values and a 2-digit tens and ones conception. The two boys E and N had only partial understanding of her method, initially doing a smaller-from-larger method on some columns when they wrote their own problem. L several times throughout the subtraction proposed that the difference of a column be a "minus 6" or "negative 6," but this idea was never pursued by the group. C also introduced her Chinese subtraction method, which was to put a dot above a column from which a multiunit was borrowed. Usually, nothing else was written. But in the first use of this method, she recorded each aspect: A dot was put above a column, a 10 was written to the right of the dot (i.e., it was then 10 of the next-right multiunit), and the top number was reduced by one.

On the last three days problems with zeroes on the top were introduced. C and N were absent for two days. E and C struggled with how to get more ones, coming up with several unsatisfactory ideas (e.g., putting a thousand block in the ones, putting a thousand block in each column), and E was his usual unpleasant and aggressive self. On the second day the experimenter asked them where the thousand block usually was traded. This was all the two needed, and E worked productively and collaboratively for the rest of the day, even saying, "This is fun." when they had solved all of the top trades. Their block method was to make all of the usual trades (put a thousand above the hundreds, a hundred block above the tens, and a tens block above the ones) and then to compensate for the trades from the hundreds and the tens (the thousands block in the hundreds was changed to 9 hundred blocks and the hundreds block in the tens was changed to 9 tens blocks). Then all of the subtraction was done right to left. This method was done with the numerals and the blocks.

On the final day a regular problem without zeroes was done because the other two children returned. N did with the digit cards a fix-everything-first version in which he crossed out the top digit cards with carrots, but he did not put new digit cards to show the changes. The changes were said in multiunit values (e.g., "We have thirteen hundreds."), and the subtraction was then done right to left. All children then individually wrote that problem on their papers. The boys did the traditional algorithm correctly. C did the Chinese method writing only dots. L invented a new dots method in which she wrote ten dots above each of the traded to columns and also made dots for the top number. This method conceptually shows a 2-digit tens and ones conception underlying her method.

This group would have been much more productive if E had been both encouraged and constrained, and if appropriate support had been given to elicit full explanations. These children could probably have articulated a fully integrated multiunit and 2-digit tens and ones conceptually-based subtraction method, even for zeroes in the top number. Furthermore, adult support of L's negative-number idea might have led to a negative-number method (get positive and negative multiunit differences, then fix the answer). The group lacked a strong leader. Neither girl was strong enough to stand up to E's negative behavior, precipitating frequent boy-against-girl battles. N would have been a good collaborator in a group without E's negative influence.

Conclusions

Personality factors combined with the mathematical strength of individual children to create different group learning paths and different subtraction methods with the blocks and the written numerals. Official leader and checker roles rotated daily among children in a group. Most children were adequate leaders, but "natural" leaders also emerged in all groups. When dominant members had good mathematical knowledge and were good rather than bossy leaders, the groups made better mathematical progress. These groups exemplify groups at the beginning of learning to work together or groups whose teacher has not worked to establish powerful social norms or group interaction competence. They indicate that good mathematical ideas (e.g., negative number approaches) can get lost in group processes if the ideas do not come from dominant children.

Directionality in the language of subtraction was a difficulty for some groups but not for others. Different English ways to describe subtraction are opposite to each other: for example, 2 take away 8, take 8 from 2, 2 minus 8, 8 from 2, 2 subtract 8. Children frequently said such subtraction phrases backwards (e.g., 8 take away 14). Sometimes everyone seemed to know what was actually meant, and sometimes such reversals confused discussion or operations.

Several groups did not adopt very well the social norm to assist all group members to understand or were not very good at such assistance. There were in some groups sustained efforts to help everyone understand, and there were isolated incidents of effective explanation in other groups. However, overall the spontaneous explanations were quite limited. Most explanations did not use multiunit quantity language ("thirteen hundreds minus six hundreds is seven hundreds") or block language. Many explanations did not even use numbers ("We took away these from those."). This lack of verbal clarity made it difficult to follow what a child was saying.

Children's objections and misunderstandings of trading illustrate its complexity. Children need to be able to see both parts of the trade: the taking from one place—and the subsequent reduction of those multiunits by one—and the adding to the adjacent-right place—and the subsequent increase of those multiunits by ten. Explanations and demonstrations that focus on both these parts—and the numerical consequences of each—are necessary. Given the paucity of full verbal explanations in most groups, and the too brief learning period, it perhaps is surprising that most children came to understand trading and to use some numerical method of recording such a trade.

Several vital functions of a teacher are clear from the above case studies: Supporting children to describe their mathematical actions using quantity language, to link numeric and block (or other referent) operations, to explore deeper aspects of an operation, to focus on meaning (e.g., in looking at 62-38), and to overcome an impasse. It is also important for teachers to help the voices of non-dominant children be heard because they may contain productive mathematical ideas.

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TEACHING MATHEMATICAL PROCEDURES MINDFULLY: EXPLORING THE CONDITIONAL PRESENTATION OF INFORMATION IN MATHEMATICS

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The present study explored the effects of conditional instruction on the learning of an invented mathematical procedure called "pairwise." 53 female undergraduates were randomly assigned to either a conditional, absolute, or no-instruction group. Following instruction, participants' performance on pairwise problems was assessed for accuracy, procedural workability, and understanding. Also, their ability to creatively adapt their pairwise procedure and their potential for procedural misapplication was assessed. In contrast to previous research findings, analysis revealed that simply modifying the language of instruction alone is insufficient to produce conditional instruction in mathematics. However, when the conditionality was salient, the conditional group outperformed the absolute group in all areas as hypothesized. The no-instruction group's procedural workability was comparable to that of the conditional group while their understanding and avoidance of mindless misapplication was superior to other groups. The implications of these findings for further developing conditional instruction and mindfulness in mathematics are discussed.

Through the presentation of information in an open-ended manner, conditional instruction acknowledges the uncertain and shifting nature of knowledge and information—thus, encouraging personal meaning making and mindfulness. In this way, conditional instruction supports constructivist teaching. However, the unique conditions of conditional instruction's application to mathematics have yet to be fully investigated. Is conditional instruction effective in learning new mathematical procedures? As a domain, what unique demands and constraints does math place on conditional instruction? The present experimental study seeks to investigate these questions.

Theoretical Framework

Mindfulness theory (Langer, 1989) seeks to explain the creation of open and creative states of consciousness (mindfulness) and the power of such states to influence human behavior, affect, and health. According to theory, mindfulness results from drawing novel distinctions, exploring new perspectives, and being sensitive to context while mindlessness is fostered through the premature formation of fixed mindsets, overgeneralizations, automaticity and acting from a single perspective. Mindfulness is a facilitative state, promoting increased creativity, flexibility, memory, and spontaneous use of information while mindlessness may lead to loss of control, narrow thinking, and a loss of opportunity.

Mindfulness theory is broadly applicable to issues of education, challenging traditional assumptions concerning the focusing of attention, memory, the nature of intelligence, and the assumption of absolute truths (Langer, 1997).

Research on "conditional" instruction represents an application of mindfulness theory particularly applicable to issues of instructional design. Drawing on research on the formation of mindsets (Duncker, 1945; Lucians, 1942), this line of inquiry has focused on the form and context in which new information is presented. Conditional instruction presents information in an open-ended, opportunistic, and inconstant manner that encourages a sense of possibility and introduces a degree of uncertainty. Conditional instruction is in direct contrast to absolute instruction in which information is presented as fixed, absolute truths lacking in ambiguity. In its simplest form, conditional instruction may entail stating that something "could be" rather than that it "is"—a pencil, case of injustice, example of impressionism, etc. Research has shown that, in contrast to more traditional absolute instruction, conditional instruction is more effective in producing mindfulness by encouraging a sense of possibility and inhibiting the formation of fixed mindsets (Eck, 1994; Langer & Piper, 1986; Mueller & Langer, 1995; Salomon & Globerson, 1987). What conditional learning offers is an explicitly-focused design principal for adapting conventional didactic instruction. Furthermore, mindfulness theory and conditional learning research provide an explanatory mechanism for why traditional didactic instruction can be inhibiting and lead to mindlessness.

Methods

Overview. To investigate the effects of conditional instruction in learning mathematical procedures, a mathematical "operation" called "pairwise" was invented. The pairwise concept was introduced to all participants through the following written instruction:

> When you *Pairwise* a number, you figure out how many pairs can be made from that number of objects and how many singles are left remaining. For example, to pairwise the number 6, you would ask yourself how many pairs can be made from 6 objects? The answer would be 3. You then ask yourself how many singles are remaining? The answer would be 0. You would write your answer to *Pairwise* 6 as 3/0. P/S 6 = 3/0.

As an invented operation, pairwise represented new content for all participants and made it possible for us to assess the effects of different forms of instruction rather than prior knowledge[1].

[1]While an invented operation may not capture the contextual nature of "real" mathematics, the many ways the pairwise operation and instructional sequence used is

Subsequent instruction focused on learning a procedural algorithm for applying the pairwise procedure to two 2-digit numbers. Four instructional conditions were devised: absolute instruction, no instruction, and two different forms of conditional instruction. Having two types of conditional instructional allowed us to explore the most effective aspects of conditional instruction in mathematics. It was hypothesized that participants receiving conditional instruction would demonstrate the greatest mindfulness as evidenced by facility and flexibility with the procedure.

Participants. Fifty-three, female undergraduates from a small private college in the Northeast participated in the experiment as part of their mathematics classes. Three classes, taught by the same instructor, were used. Two thirds of the participants were enrolled in a basic math course focusing on numeration, graphing, and logic. The remaining students were enrolled in a general geometry course. The content of both courses was comparable to pre-algebra and sought to provide a basic level of mathematical proficiency for math phobic students majoring in the social sciences.

Procedure. Participants were told they were participating in a study of the effects of different types of instruction in mathematics. In each class, participants were randomly assigned to one of four conditions: no instruction (n=14), absolute instruction (n=13), 1-example conditional (n=13), and 2-example conditional instruction (n=13). All instruction was provided in written form, and the instructional packets were collected prior to giving participants four problems to solve.

All participants received the same introduction to the pairwise concept (stated above). Participants in the no-instruction condition received no further instruction. In the absolute-instruction condition, participants were provided with a four step procedure and accompanying example for applying the pairwise operation to two 2-digit numbers. The algorithm involved: 1) arranging the two numbers vertically, 2) multiplying the sum of the ten's column by 5 to determine "the number of pairs so far," 3) adding the one's column and "figuring out" the number of pairs and singles in this sum, and 4) adding the pairs and singles from steps two and three. This algorithm was introduced in absolute terms with the statement: " Mathematicians have invented *a method* to allow them to quickly find the answer to these pairwise problems. This method contains four steps."

representative of mathematics and mathematics instruction should be considered. In mathematics instruction learners frequently encounter "invented" procedures in the form of formulas or functions that they must seek to understand and master. Like these procedures, the pairwise operation builds on previously learned mathematics and can be contextualized, allowing participants to draw on their prior mathematical knowledge in learning or devising a pairwise algorithm.

In the 1-example conditional group, participants were provided with the exact same four-step algorithm and example as in the absolute condition but with a conditional introduction: "Mathematicians have invented *several ways* to quickly find the answer to these pairwise problems. *One possible method* contains four steps." In the 2-example conditional group, participants were provided with the four-step algorithm introduced with conditional language as well as a second five-step procedure. The presentation of more than one model is conditional because it breaks the set of a single correct method, solution, or answer.

Measures. Following instruction, participants were given four problems to solve. The first two were 2-digit pairwise problems similar to the instructional example. Participants were asked to solve the problems, describe their solution method, and provide an explanation for why their method works. Participants' solutions were evaluated for accuracy, their procedures were judged on workability, and understanding of the method employed was evaluated. The third problem was designed to measure the creative use of instruction. In this problem, participants were asked to solve another 2-digit pairwise problem using a method different from any they had been shown.

Participants' potential for the mindless misapplication of the pairwise procedure was assessed in the fourth problem. This problem involved a description of an optometrist who sold single and pairs of contact lenses and information on two shipments of contact lenses. This problem looked like it might have involved application of the pairwise procedure, but the actual question only asked how many contact lenses were available. Participants' solutions were evaluated for accuracy.

Analysis and Results

Initial analysis led us to question whether our 1-example group actually represented a form of conditional instruction. While previous research had shown the effectiveness of written conditional language (Langer et al, 1989), we speculated that in mathematics learners may have a tendency to absolutize instruction. A follow-up study was designed to test this hypothesis. Ten students, enrolled in another of the same instructor's sections, were asked to read and then paraphrase the 1-example conditional-instruction information. Only 2 of the ten used conditional language—"there are a *number of ways*" and "mathematicians have invented *methods*". The remaining students stated they did not read the introductory statement or that it introduced a single method, procedure, or formula. Therefore, the 1-example conditional group received essentially the same instruction as the absolute group. In our subsequent analysis, we combined the 1-example and absolute groups into a single absolute-instruction group to obtain more statistical power.

Accuracy. The first two pairwise problems were scored together. Accurate solutions (=2) provided the correct number of pairs and singles for both problems. Partially accurate solutions (=1) either solved only one of the two problems correctly or provided the number of pairs and singles for each number as opposed to a single, combined total (i.e. P/S 18 and 12 = 9/0 and 6/0 rather than 15/0). Inaccurate solutions to both problems were scored as 0.

The conditional-instruction group demonstrated greater accuracy than either the absolute (M_{con} =1.53, M_{abs} =.96; z =1.72, p<.05) or the no-instruction group (M_{con} =1.53, M_{no} =.64; z =2.57, p<.01)[2]. One hypothesis for this finding is that the second procedure provided to the conditional group was easier to understand and use, providing them with an advantage. However, analysis of the methods used revealed that only 2 out of 13 participants used the second method.

Procedural Workability. Participants' solution methods were evaluated as either workable (=1) or unworkable/limited workability (=0). Workable methods were those that had the potential for yielding a correct solution. Methods of limited workability worked in this particular instance but lacked generalizability. For example, in the problem P/S 25 and 36, some participants said to add the ten's column and square it. Unworkable solutions were those that could not possibly yield a correct answer. For example, for P/S 18 and 12 a participant wrote, " Take 18 and divide it by 12, [then] put it [6] over 12." Only 65% (17/26) of the absolute group produced workable solutions compared to 85% (11/13) of the conditional group (M_{con} =.85, M_{abs} =.65; t =1.40, p<.10) and 86% (12/14) of the no-instruction group (M_{no}=.86, M_{abs} =.65; t =1.50, p<.10)[3].

Understanding. Participants' explanations of the method they employed to solve the pairwise problems was assessed to determine their level of understanding. Explanations were rated as acceptable (=2), weak (=1), or no understanding (=0). Acceptable explanations provided a mathematical rationale for some aspect of a workable procedure. For example, "this works because dividing a number by two will 'pair' any number." Weak explanations often reiterated the method in new language without actually justifying it. Explanations for procedures that were unworkable or stated that "I followed the steps" were rated as no understanding.

The no-instruction group showed understanding superior to both the conditional group (M_{no} =1.29, M_{con} =.31; z= -2.67, p<.01) and the absolute group

[2]Unless otherwise noted, the Wilcoxon Rank Sum test, was used to compare group means. All p-values are for one-tailed tests. Only z scores that approach at least the .10 level of statistical significance are reported.

[3]An arc sine transformation was performed on these data to permit the use of a more powerful t-test.

(M_{no} =1.29, M_{abs} =.19; z = 3.81, p<.0001). Since the no-instruction group had to invent their own procedure, they were unlikely to invent something that they didn't understand or could not explain. Thus, it makes sense for this group to demonstrate a higher level of understanding.

Creative Use of Instruction. The ability of participants to go beyond the instruction given to creatively produce a novel and workable procedure for solving pairwise problems was assessed in the third problem. Participants' solutions methods were coded as either novel and workable (=1) or the same and/or unworkable (=0). 82% (9/11) of the conditional group and 64% (9/14)of the no-instruction group produced novel solutions compared to only 46% (11/24) of the absolute group (M_{con} =.82, M_{abs} =.46; z =1.95, p<.05). Even though showing the conditional group a second example eliminated one possible response from their potential repertoire of responses, the instruction appears to have had the effect of increasing flexibility and creative thinking.

Mindless Misapplication. The tendency to misapply the pairwise procedure based on surface features of a situation was assessed in the fourth problem. Participants' solutions were judged as either providing a correct answer (=1) or providing only an incorrect answer (=0). Many of the participants provided both the requested solution and a pairwise answer. These were judged as being correct solutions. 73% (16/22) of the absolute group mindlessly misapplied the pairwise procedure compared to 55% (6/11) of the conditional group (M_{con} =.45, M_{abs} =.27; t =1.71, p<.05)[4] and 38% (5/13) of the no-instruction group (M_{no} =.62, M_{abs} =.27; z =1.95, p<.05).

Discussion

The greater facility, creativity, and decreased mindlessness of the conditional-instruction group in this study supports previous findings about the power of conditional learning to promote mindfulness. Thus, conditional instruction may be useful in reframing traditional didactic, including textbook, instruction in mathematics. However, our research found that the use of conditional language alone was insufficient to produce effects in the mathematics learning of this population, leading us to conclude that individuals' past experiences in mathematics may lead them to absolutize their instruction. Therefore, to be effective conditional instruction must be made more salient to the learner in mathematics. In addition, the performance of the no-instruction group indicates that building on the learners' intuitive understanding and encouraging personal agency also support mindfulness—a premise emphasized in constructivism. Additional research is needed to further investigate what com-

[4]An arc sine transformation was performed on these data to permit the use of a more powerful t-test.

binations of features of conditional learning are most effective in producing mindfulness in mathematics. Furthermore, this research needs to be extended to various other populations and mathematical contexts.

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THOSE WHO TALK, THOSE WHO LISTEN, EVER THE TWAIN SHALL MEET: FURTHER EXAMINING THE ROLE OF DISCOURSE IN THE PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS

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Mathematics education reform efforts in the United States are setting ambitious goals for schools, teachers, and students. Central to many of these initiatives in the reform of school mathematics are linkages between mathematical reasoning and communication. Accordingly, discourse plays a vital role in mathematics teachers' efforts to increase their content and pedagogical knowledge. In this article, we use a conceptual framework built around the "functional dualism" of discourse (Lotman, 1988; Wertsch & Toma, 1995) to closely analyze the discourse in a university mathematics education classroom and a high school mathematics classroom. In particular, we examine the discourse and reflection central to one teacher's (Jim) involvement in a year long professional development project for secondary mathematics teachers—a project that has as its goal the integration of discrete mathematics content into secondary school curricula.

As we focused on the nature of the discourse in Jim's classroom it became apparent that, for the most part, Jim was the locus of mathematical authority in the classroom setting. Yet at the same time we were impressed with Jim's flexibility as a teacher and how he strived to listen to the students, in a dialogic fashion, and make sense of what they were saying and the thinking that grounded their mathematical discourse. However, the students in Jim's classroom did not appear to listen to Jim in the same fashion. Instead, they heard Jim in a univocal nature as they relied on him to correct their thinking and convey to them a proper understanding of mathematics. As we look to the professional development activity where Jim acquired his new found knowledge of discrete mathematics—the DMP—we see similarities and differences, with respect to discourse, between Jim's classroom and the DMP classroom. We conclude by exploring possible factors that influence the nature of the discourse in each setting.

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THE BENEFITS OF TUTORING PROGRAMS FOR UNDERACHIEVING STUDENTS

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Background of the Study

This study involves one hundred Latino families who attended a family math tutoring program consisting of three, two-hour sessions. The tutoring materials are a component of the Children Math World's curriculum developed at Northwestern University. This study draws from data in the family questionnaires and follow-up interviews with families. This paper will focus on how and why these families benefited from the family math tutoring program. In particular, we explore the role that tutoring provides for these families.

Why Families Attended

Families felt unprepared to help their children with mathematics homework in grades K–3. They attended the tutoring sessions because of the limited resources within their community that prepare them to work with their children in mathematics. Schools very seldom offer instruction that can benefit the students by working with the whole family. These Latino parents all had children who were getting below average grades in mathematics. Parents needed tutoring on how to be more effective math helpers. They viewed the math classes as an investment of time and effort in shared activities with their children.

How These Families Benefited

Families reported that they no longer focused on *just* getting the correct answer. They looked at errors as a way of helping them understand where children had gaps in understanding. Parents came to understand the problem solving strategies they had learned in other countries. Families found that even with limited math understanding, they could be better tutors or math helpers for their children. They were provided with many practice math activities that reinforced what was being learned in the classroom using quality language. Parents who were too busy to provide help every day saw the importance of having a home math helper to make sure the practice activities were being done at home. Math eventually was viewed as a subject that could be learned with practice and guidance.

Conclusions

The role of tutoring programs can be very helpful to families as well as

individual students. By training families as tutors for their children, gaps in understanding can be filled and higher levels of math can attained.

The research reported in this paper was supported by the National Science Foundation under Grant No. RED 935373. The opinions expressed in this paper are those of the author and do not necessarily reflect the views of NSF.

TEACHING MATHEMATICAL PROBLEM SOLVING TO LANGUAGE MINORITY STUDENTS

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Among its position statements, the National Council of Teachers of Mathematics has indicated its commitment to equity in mathematics education for all students including language minority students. However, research on the achievement of language minority students tells us that although it takes only two years to achieve conversational competence in a second language, it takes up to seven years to attain sufficient second language proficiency to achieve in academic areas at the level of native speakers. The result is that language minority students fall increasingly behind native speaking peers in mathematics achievement. Logic dictates that to take the NCTM's recommendation for mathematics for everyone seriously, we need some specific adaptations in instructional methods in American school systems. In general, adaptations might come from the fields of English as a second language or bilingual education and be blended with current practices in mathematics education that focus on communication and problem solving. The purpose of this study was to develop and assess the effectiveness of adapting methods normally employed by educators for promoting second language proficiency to the context of teaching mathematical problem solving to sixth grade students with limited English proficiency.

The research methodology began with an observation of techniques regularly utilized by a bilingual teacher in teaching mathematics to his 30 sixth grade Spanish/English bilingual students. Fourteen of these students were selected as target subjects because they were close to being mainstreamed and performed at an intermediate level in English language skills. After several weeks of observation, an English-only instructional model with five central components emerged and was utilized during an 8-week instructional period that included English and Spanish pretest and posttest assessments of target students' problem solving performance. The instructional model included: 1) providing a content-based linguistic warm-up to problems, 2) breaking down problems into natural grammatical phrases, 3) having students work out problems in pairs, 4) having students present their own solutions to the group, and 5) having students create problems with similar structures which were subsequently shared and solved by the rest of the class.

Performance on pretests and posttests was compared for accuracy and for quality of explanations, each on a 4-point scale. Utilizing t-tests for related samples, comparisons were made between scores within each language and across the languages. Results indicated that students significantly increased their accuracy scores in both English and Spanish from pretest to posttest times.

However, they did not significantly increase their explanation scores in either language, although there was a trend toward increased scores in both. No significant differences were found between performance in English and Spanish on any measure, but students consistently tended to be somewhat stronger in Spanish.

The results of this study suggest the potential effectiveness of utilizing ESL/bilingual teaching techniques for teaching students with limited English proficiency to become more successful mathematical problem solvers. It is expected that these techniques could provide monolingual teachers with an improved method for communicating with language minority students and may have value in mathematics education with native speakers who have limited literacy skills. Further investigation using modifications the study's methods is recommended to develop procedures for increasing students' abilities to reflect and report on their own mathematical thinking.

STUDENT INTERACTIONS AND MATHEMATICS DISCOURSE: A STUDY OF THE DEVELOPMENT OF DISCUSSIONS IN A FIFTH GRADE CLASSROOM

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The current trends in the reform of mathematics education suggest a shift toward interactive, discussion-based classrooms (National Council of Teachers of Mathematics, 1989, 1991). This calls for the mathematics teacher and students to work together to establish a community for investigating and sharing mathematical ideas and concepts.

This research project looked specifically at the development of mathematical discourse in a fifth grade classroom. Through extended observations, documentation, and collaboration with the classroom teacher, various aspects of the classroom, the mathematics, and the participants interactions were investigated to determine those characteristics that played a part in the development of the mathematics discourse. A social constructivist perspective of learning (Cobb, Yackel, & Wood, 1993) guided the research and analysis, as did theories of discourse and communication (Cazden, 1988).

Visible changes in the discourse and interactions over the course of the school year were shown to be at least partially linked to the teacher's role and the nature of the tasks she chose. In fact, the teacher's role was the most important factor in the development of the mathematics discourse. Even so, the students' contributions to discussions and their roles in establishing the classroom environment were also important factors in the development of the mathematics discourse. The research results suggest many implications for teacher education and for continued research in the area of mathematics discourse.

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LATINO URBAN STUDENTS AND THE MEANING OF REFORM IN MATHEMATICS EDUCATION

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The reform movement in mathematics education is slowly gaining momentum and reports of successful student learning are commonly presented at educational conferences. It seems clear that the introduction of legitimate problems which are investigated collaboratively has appeal to many students and does have the potential to involve students who have previously rejected school mathematics. In a 7-8th grade class taught by one of the authors, changes in teaching in line with the NCTM Standards had changed the atmosphere in the classroom and broadened the participation of members of the class in mathematical activities. However, despite its broader appeal, we saw a significant proportion of the class who showed no interest and had poor achievement. Rather than seeing these students as having a deficit, we began to ask ourselves why, from the perspective of students outside the dominant culture and socio-economic classes, this 'new' mathematics would have appeal or seem relevant.

Analysis of NCTM and other reform documents indicated that a dominant justification for achievement in mathematics is future economic success. Although it is easy to see how such a justification could appeal to mainstream students, it was less clear to us how it would be interpreted by low SES, ESL, first generation immigrant students who had little personal connection with the kinds of success being promised. With this in mind, we interviewed students from this classroom in January and again in June about their attitudes towards mathematics, school and their own future expectations. Results indicate mathematics was not directly connected to the achievement of future goals as envisioned by many of these students. In addition, for students who did articulate such a connection, the emphasis on future benefits often supported a focus on rote learning. In our analysis, we discuss the relevance of current educational goals and propose alternatives. One implication is that more emphasis needs to be placed on intrinsic values of learning mathematics.

COLLABORATIVE PROBLEM-SOLVING IN MIXED-LANGUAGE GROUPS

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The research reported here was carried out as part of a teacher-researcher collaboration aimed at investigating effective strategies for mathematics instruction in classrooms including native speakers of both English and Spanish. The central instructional strategy selected for the research was the use of small, collaborative groups, with materials specifically designed for cooperative problem-solving in mathematics (Erickson, 1989). From a theoretical perspective, students in such groups would be expected to provide peer support and complementary perspectives in the problem-solving process (Cohen, 1986; Forman, 1992). The composition of the groups was heterogeneous in terms of mathematics and also in terms of English language proficiency, in order to investigate cross-language mathematical communication.

The subjects were 140 students in 5th and 6th grades (ages 10-12), most of whom spoke Spanish as a first language. The groups engaged in 20 minutes of collaborative problem-solving four days a week for four weeks, with clues provided in both English and Spanish. The students were given written pre- and post-tests on mathematical problem-solving, English language skills, and written mathematical communication (an individual write-up of a group solution). In addition, videotapes were collected during 12 sessions in one 6th grade class. The poster will present the results of the pre- and post-tests, and will focus on an analysis of selected problem-solving episodes from the videotapes. Issues to be discussed include the nature of peer support and expertise within the groups; cross-language mathematical communication; barriers to participation in problem-solving; and the construction of joint representations for solutions.

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MATHEMATICS ATTRIBUTION DIFFERENCES BY ETHNICITY AND SOCIO-ECONOMIC STATUS

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The differences in attributions for success and failure in mathematics between African-American and Caucasian students and between students from low and higher socio-economic status (SES) were examined. 529 seventh grade students from 26 classrooms in a mid-size urban school district were surveyed to analyze attribution differences. A stratified purposeful sample of 12 focus students, representing different ethnicity and socio-economic combinations, were selected for follow-up interviews. Initial results indicated that students from all groups provided similar ratings on five attributes related to mathematics success—with effort and ability rated higher than luck and rapport with teacher. Caucasians attributed success significantly more to ability than did African-Americans. Similarly, low SES students attributed success significantly more to ability than did higher SES students. Failure in mathematics was most commonly attributed to a lack of effort and ability. African-American students mentioned these two attributes significantly more often than did their Caucasian counterparts.

BUILD YOUR DREAM HOME: AN ETHNO-MATHEMATICAL APPROACH TO MOTIVATE THE LEARNING OF MATHEMATICS

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In recent years a variety of researchers have investigated informal mathematics which is learned in the Social Cultural context and how to design Instructional Treatments utilizing said math backgrounds as a pedagogical resource in the process of teaching and learning of mathematics.

The purpose of this work is to share the experience of designing and implementing a program for teaching mathematics using previous skills and knowledge. Our framework uses the theory "Funds of Knowledge" (Moll, 1992; Gonzalez et al, 1995.) and attends the ideas of Social Constructivism of Vygotsky concerning learning as a result of social interaction. Our project, Build your Dream Home: A Strategy to Motivate the Learning of Mathematics, was initiated by a previous study, conducted at a bilingual middle school. This study showed that 60% of the students involved had prior skills in construction through the process of assisting an adult family member on home projects and/or construction sites. This finding led to the design of an instructional treatment in which students developed model houses (designing, drawing and constructing model houses) as a method for learning mathematics in context. Along with this process, the school math curriculum was included. This work presents the methodology used to implement classroom activities involved and an analysis of how the students experiences were connected to the learning of academic mathematics.

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APPLYING SOCIO-CULTURAL THEORIES TO RESEARCH IN MATHEMATICS EDUCATION

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The focus of this discussion group will be the connection between socio-cultural theories and research in mathematics education. Although socio-cultural theories have recently received attention in the literature on mathematics education many important questions remain regarding how to apply these theories to research design, data analysis, and teacher professional development. Since this theoretical perspective encompasses several theorists, starting with Vygotsky and continuing with Leontiev, Luria, and Bakhtin, it is not always clear which specific aspects or versions of socio-cultural theory are being invoked. Also, some aspects of this theoretical perspective may be more useful than others for framing research in mathematics education.

This discussion group will server as a forum where participants can discuss these issues in detail, learn about several projects in mathematics education using a socio-cultural perspective, reflect on selected readings representing a spectrum of socio-cultural theories, and discuss how this perspective might inform their own research projects.

Plan for the Discussion Sessions

The discussion group will meet for two sessions. During the first session several researchers will present brief (5-10 minutes each) overviews or examples of how they have used socio-cultural theory in their research work. The central purpose for these short presentations is to provide concrete examples of how mathematics education research has used socio-cultural theory and show several different perspectives in a structured way. After the presentations, participants will begin discussions in small groups which will be continued in the second session. The discussion in small groups will focus on the following questions:

1. What aspects of socio-cultural theory have been used for math education research?

2. What areas might socio-cultural theory be useful for studying that have not yet been linked to this theory?
3. What are some of the basic characteristics of a study conducted from a socio-cultural perspective?

Participant Reflection Activities

After the first session participants will be asked to read and reflect on a short (3-5 pages) excerpt taken from the writings of Vygotsky, Leontiev, Bakhtin, or Luria. During the first hour of the second session, participants will continue the small group discussions. The last hour of session two will be used to generate a collective summary and synthesis of the small group discussions. Participants will use their reflections on the texts, presentations, and small group discussions to generate some of the following:

1. A collective list of basic characteristics of studies conducted from a socio-cultural perspective.
2. Plans for research projects applying socio-cultural theory.
3. An outline of how these theoretical issues function in the classroom.
4. Suggestions for how we can study these issues in the classroom.
5. Suggestions for how socio-cultural theory might inform teacher professional development.

LANGUAGE AND CULTURAL ISSUES IN THE MATHEMATICS CLASSROOM: A DISCUSSION GROUP

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The purpose of this discussion group is to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics among linguistic and ethnic minority students within the United States. The underachievement among these students has been clearly documented. However, recent reform documents in mathematics education have addressed this concern only peripherally, stating in general terms a commitment to improving mathematics learning for all students but making almost no specific recommendations in multilingual and multicultural contexts. Teachers do not see their own students' needs reflected in current reform movements, even though the population of linguistic and ethnic minority students is more than two million nationwide and one in four teachers has limited English proficient students in his or her classroom.

Researchers have recently begun to fill this gap in mathematics education literature, but there is clearly a need for more research that investigates the day-to day reality of teachers and students in diverse classrooms. Constraints that prevent mathematics learning must first be addressed before improvements to present practices can be made.

Discussion leaders will share their research studies that report methods and strategies that have proven to be successful among underachievers.

The first topic, Teaching Mathematics in the Spanish Speaking Classroom, will be led by discussion leader, Ana Lo Cicero, Northwestern University.

The second topic, Teaching Mathematics in English Speaking Classrooms, will be led by discussion leader, Lena Licón Khisty, University of Illinois at Chicago.

The third topic, After-School and Community Programs that Make a Difference, will be led by discussion leader, Yolanda De La Cruz, Arizona State University West.

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The research reported in this paper was supported by the National Science Foundation under Grant No. RED 935373. The opinions expressed in this paper are those of the author and do not necessarily reflect the views of NSF.

Problem Solving

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GENERATING MULTIPLE SOLUTIONS TO MATHEMATICAL PROBLEMS BY PROSPECTIVE SECONDARY TEACHERS

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This study analyzed multiple solutions and explored the process of generating multiple solutions to each of the five routine and nonroutine high school algebraic problems by eight prospective secondary teachers. The results of this study showed that prospective teachers were able to generate multiple solutions to each of the problems and representing relationships from different perspectives was one of the mechanisms that they used to generate multiple solutions. This study suggests the value and feasibility of further research about teaching school students to generate multiple solutions and teaching mathematics via generating multiple solutions to problems.

Problem solving is a process which provides students with an opportunity to experience the power of mathematics in the world around them. It is also an instructional approach which provides a context for students to learn and apply mathematics. Recently, the National Council of Teachers of Mathematics [NCTM] (1989, 1991) suggested that students should be encouraged to generate multiple solutions to a mathematical problem. Generating multiple solutions to a problem is one of the regular features in Chinese and Japanese classrooms (Cai, 1995; Stigler & Perry, 1988). Asian teachers have experienced success with such teaching strategy in the classroom. It is also consistent with current constructivist view of learning mathematics. Generating and experiencing multiple solutions to a problem can provide students with the opportunity to actively construct their own mathematical knowledge. Students will also learn to value the process of solving problems as much as they value the solutions (Cai, Moyer, & Laughlin, in press).

During the past several decades, great progress has been made in understanding affective, cognitive, and metacognitive aspects of complex mathematical problem solving (e.g., Lester 1994; McLeod, & Adams, 1989; Schoenfeld, 1992; Silver, 1985). Researchers have extensively analyzed many aspects of the solution to mathematical problems which normally require solvers to provide only one solution to each of the problems. Despite the extensive educational interest in the activity of generating multiple solutions to mathematical problems, little or no research has been conducted to examine how subjects generate multiple solutions to a problem. Although a few researchers have incorporated generating-multiple-solutions approach to examine creativity (e.g., Jausovec, 1993), mathematical ability (e.g., Krutetskii, 1976), and U.S. and Japanese students' problem solving behaviors (Becker, 1992) and these research-

ers have successfully used such an approach and studied related phenomenon, their focus was not on the understanding of the mechanism in which subjects generate multiple solutions to a mathematical problem.

The purpose of this study was to analyze multiple solutions generated by eight prospective secondary mathematics teachers to a series of eight problems and to explore the process in which these prospective teachers generated these multiple solutions. In particular, this study was an attempt to examine products of multiple solutions and understand processes of generating multiple solutions to mathematical problems. In order to implement the activity of generating multiple solutions into the classroom, teachers themselves should have personal capability of generating multiple solutions to a mathematical problem. The decision of using prospective secondary mathematics teachers as subjects in this study was based on this consideration.

Method

Subjects

Subjects were eight prospective secondary mathematics teachers who were in their junior or senior years. They were chosen on a volunteer basis. They had completed most of their required math and math education courses by the time this study was conducted.

Tasks and Administration

Each subject was asked to solve eight problems, which were adopted or modified from various sources (e.g., Heid, 1995; Krulik, 1980). Five of them were high school algebraic problems, shown in Figure 1, and the other three were geometric problems. However, the results for the geometric tasks were not reported here. For each of the problems, the subjects were asked to "list all possible different solutions that he/she thinks a secondary school student might provide to the problem. In each of the solutions, full justification or explanation should be included." Each of the tasks was administered to subjects individually. At the end of each interview session, each subject was asked to explain why the solutions to a problem were different. After each subject had completed all eight tasks, he or she was asked to comment on the educational value of providing students with opportunities to generate multiple solutions to mathematical problems.

Coding of Solutions

Each solution was coded as using an algebraic, arithmetic, graphical, or tabular approach or strategy. For example, six different solutions to Task 1 were identified and described in Figure 2. In this case, solutions 1 and 2 were coded as using an arithmetic approach, while solutions 3 and 4 were coded as using an algebraic approach. Solution 5 was coded as using a graphical approach, and solution 6 was coded as using a tabular approach.

1. Given the two job offers below, determine the better-paying summer job. Justify your answer.

   Offer 1: At Timmy's Tacos you will earn $4.50 an hour. However, you will be required to purchase a uniform for $45.00. You will be expected to work 20 hours each week.

   Offer 2: At Kelly's Car Wash you will earn $3.50 an hour. No special attire is required. You must agree to work 20 hours each week.

2. Yolanda has $1.35 in nickels and dimes. She has a total of 15 coins. How many of each kind of coin are there?

3. A man is placing rabbits into cages. He notices that if he places 4 rabbits in each cage, he has 2 rabbits left over, and if he places 6 in each cage, he has 3 cages left over. How many cages and how many rabbits does the man have?

4. Two candles are 10 feet high, but are made of different material. One of the candles will burn up in 4 hours, and the other in 5 hours. How long will they have to burn before one candle is three times the length of the other?

5. An office manager must decide between two options to fill the copying needs of his department. He wants to find the more economical option. Help the manager to make his decision.

   AAA copiers, the first company contacted, offers to lease a copy machine for a fixed fee of $50.00 and an additional charge of 2.1¢ for each copy.

   For the same machine and comparable service, a second company, Speedy Print, offers a fixed charge of $180.00 a week with an additional charge of 5¢ for each copy.

Figure 1 *Five Algebraic Tasks*

Results

Table 1 shows the kind of solutions that were generated by each subject for each of the five algebraic tasks. Solutions in each cell of Table 1 were presented in the order each subject generated. Overall, subjects were able to generate multiple solutions to most of the problems. Six (Subjects 1, 2, 4, 5, 6, and 8) of the subjects were able to generate multiple solutions to all of the problems. Subject 3 was able to generate more than one solution to all problems except for problem 3. Subject 7 generated multiple solutions to two of the problems, but only one solution to the remaining three. Three of the subjects

SOLUTION 1: In a 20 hr. week, Offer 1 will pay \$4.50 5 20 =\$90.00. Offer 2 will pay \$3.50 5 20 = \$70.00. Since the difference is \$20 per week and the uniform for Offer 1 costs \$45.00, it will take (\$45.00 ÷ \$20/week =) 2.25 weeks to pay for the uniform and break even. If you keep the job for three weeks or more, you should take offer 1.

SOLUTION 2: At Timmy's you make \$1.00 more for each hour of work. After 45 hours of work, you'd make \$45 more at Timmy's than Kelly's. This extra money would pay for the uniform. From that point on, you'd make \$1 more an hour at Timmy's than Kelly's. 45 hrs ÷ 20 hrs/week = 2.25 weeks.

SOLUTION 3: Let x be the number of weeks you intend to work. The total amount for Offer 1 = 4.5 5 20 x - 45 and the total amount for Offer 2 = 3.5 5 20 x. If $(90x - 45) < 70x$, then $x < 2.25$. So if you work less than 3 weeks, you should take Offer 2, otherwise take Offer 1.

SOLUTION 4: Let x be the number of weeks you intend to work. The total amount for Offer 1 = 90x - 45 and the total amount for Offer 2 = 70x. If $90x - 45 = 70x$, then $x = 2.25$. So if you work less than 3 weeks, you should take Offer 2, otherwise take Offer 1.

SOLUTION 5: Let x be the number of weeks you intend to work, y_1 be the total amount for Offer 1 after working x weeks, and y_2 be the total amount for Offer 2 after working x weeks. Therefore, $y_1 = 90x - 45$ and $y_2 = 70x$. Using a graphing calculator to graph them, you will see they intersect at (2.25, 157.5). From the graph, you will see that If you have the job for three weeks or more, you take Offer 1.

SOLUTION 6: Construct a table to show the amount of income for Offers 1 and 2 for one week, two weeks, and three weeks..., and then compare the information from the table to determine which offer you will take.

Figure 2 *Sample Solutions to Task 1*

generated four solutions to at least one of the problems. Algebraic and tabular approaches were most frequently used strategies in solving these problems. In fact, algebraic approaches were used 38 times and tabular approaches were used 36 times. Graphical approaches were used 20 times. On a few occasions, students used an arithmetic approach.

Table 1 Number and Type of Solutions by Subject by Task

	Task 1	Task 2	Task 3	Task 4	Task 5
Subject 1	Ta,Gr	Ag,Ta	Ag,Ta, Ar	Ag,Gr,Ta	Gr,Ag,Ta
Subject 2	Ta,Ag,Gr,Ar	Ag,Ta,Gr,Ar	Ag,Ar,Ta	Ta,Ag,Gr	Ag,Gr,Ta
Subject 3	Ag,Gr,Ta,Ar	Ta,Ag,Gr	Ta	Ag,Ta	Ag,Gr
Subject 4	Ta,Ag,Gr	Ta,Ag	Ta,Ag	Ta,Ag,Gr,	Ta,Ag,Gr,Ag
Subject 5	Gr,Ta,Ag	Ta,Ag	Ag, Ta	Ta, Ag	Ag,Gr,Ta
Subject 6	Ar,Gr,Ta	Ta,Ag,Ta	Ag,Ta	Ag,Ta	Gr,Ag,Ta
Subject 7	Ag	Ag	Ag	Ag, Ta	Ag,Gr,Ta
Subject 8	Ta,Gr,Ag	Ta,Ag	Ag,Ta	Ta,Ag	Ag,Gr,Ta

Note. Gr = Graphical approach, Ag = Algebraic approach, Ta = Tabular approach, and Ar = Arithmetic approach

After a subject generated a solution to a problem, they often transformed one representation to another for an alternative solution. In solving Problem 1, for example, Subject 2 started to represent the information using a table. Then she commented that " the problem is really about figuring out which job is better in terms of # of weeks working there." So, she let x be number of weeks when offer 1 would exceed offer 2 and set up an inequality to solve the problem: $90x - 45 \geq 70x$. She set up this inequality instead of $90x - 45 \leq 70x$ because she already knew Offer 1 is better if you work three or more than three weeks. After the second solution, her third solution became very natural. She said: "Offer 1 gets $(90x - 45)$ dollars and Offer 2 gets $70x$ dollars after working x weeks. If I graph them I will see the break-even point." She, then, graphed two functions ($y = 90x - 45$ and $y = 70x$) using TI-82. From graphs, she observed that "Offer 1 increases faster than Offer 2, but Offer 1 starts from -45 because of the uniform." With such a constructive observation, she provided her fourth solution to the problem: *At Timmy's you make $1.00 more for each hour of work. After 45 hours of work, you'd make $45 more at Timmy's than Kelly's. This extra money would pay for the uniform. From that point on, you'd make $1 more an hour at Timmy's than Kelly's. 45 hrs ÷ (20 hrs/week) = 2.25 weeks.*

Prospective teachers' subsequent solutions were influenced by their previous solutions. Therefore, their first solution seems to be critical. Twenty out of 40 times subjects used an algebraic approach as their first strategy. 16 out of 40 times subjects selected a tabular approach as their first strategy. Occasionally, subjects used an arithmetic or a graphical approach to solve these problems as

their first choice. Although subjects were more likely to use an algebraic or a tabular approach in their very first solution, it is not clear why a subject chose one over another. Six of eight subjects did not show consistency in their first solution across five tasks. Only two of the subjects consistently used the same approach in their first solution, with Subject 4 using a tabular approach and Subject 7 using an algebraic approach.

Subject 4 chose a tabular approach as her first strategy for solving all five problems. For her, she "likes to have something concrete," so that she "can grab it." The tabular approach enabled her to see relationships directly and make sense of them. However, she commented that a tabular approach "is not always the most effective or simple method it can work." She approached all five problems in the same way in her first attempt. She was trying to find a "variable" which she can based on to organizing and presenting information. For example, in solving Problem 1, she organized information based on "the number of weeks you work," and then compared the earnings at Timmy's Tacos and Kelly's Car Wash. In solving Problem 2, she organized information based on "the number of nickels", and determined the number of dimes. Then, she checked if the total amount of money is $1.35. In contrast, Subject 7 seemed to be in favor of using an algebraic approach as his first attempt. After Subject 7 was presented a problem, he immediately tried to set up equations to solve it. It is almost effortless for him to attempt these problems algebraically. However, he made tremendous effort trying to provide an alternative solution and failed to provide more than one solution to each of the first three tasks.

Discussion

Each of the eight prospective secondary teachers was asked to generate multiple solutions to each of the five routine and nonroutine high school algebraic problems. The results of this study suggest that prospective teachers were able to generate multiple solutions to these problems. In fact, the majority of them generated multiple solutions to each of the problems and all of them generated three or more solutions to at least one of the problems. These results suggest that prospective secondary teachers are capable of solving a mathematical problem in multiple ways. Tasks used in this study all contain linear relationships between variables. For each of the tasks, subjects generated multiple solutions through representing these linear relationships arithmetically, algebraically, graphically, and tabularily. After they generated their first solution, they tried to generate alternatives through transforming one representation to the other. Thus, the findings of this study suggest that representing relationships from different perspectives is one of the mechanisms that subjects used to generate multiple solutions. However, generalization of these findings is limited because the findings were based on the analysis of multiple solutions to

one type of algebraic problem. Nevertheless, these findings suggest the value and feasibility of further studies to investigate processes of generating multiple solutions using various mathematical tasks.

As we all know, all too often students hold the misconception that there is only one "right" way to approach a problem. The misconception might be largely due to their lack of experience of using multiple ways to approach a problem. The findings of this exploratory study suggest that teachers should feel comfortable in guiding their students to engage in activities that generate multiple solutions to problems, since they have the personal capability to generate multiple solutions. Tasks used in this study are from existing resources with a minor modification and some of the tasks are pretty standard. Therefore, teachers are not lacking in resources to locate problems which allow for multiple solutions. On the other hand, creating a classroom environment that emphasize the generation of multiple solutions to a single problem is a challenging, but important goal for classroom teachers. More research is needed to examine the role of teachers when they are trying to engage students in activities of generating multiple solutions to problems.

Solving an algebraic problem arithmetically, algebraically, graphically, and tabularily provides students with a great opportunity to experience how mathematical relationships can be represented from various perspectives. Experiencing multiple solutions to a problem may not only provide students with the opportunity to actively construct their own mathematical knowledge and see how mathematical concepts and representations are connected, but may also change students' attitudes toward mathematics (Cai, Magone, Wang, & Lane, 1996; Cai et al., in press). The results from national and international assessments show that many students view mathematics as a set of rules and procedures that they must memorize in order to follow the single correct way rapidly to obtain the single correct answer. After exposure to multiple solutions, students may realize that doing mathematics can be fun, creative, and intellectually engaging. However, experimental studies are needed to examine instructional impacts on students' attitude toward mathematics, knowledge acquisition, and thinking and reasoning through engaging students in activities of generating multiple solutions to mathematical problems.

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A PROBLEM SOLVING SESSION DESIGNED TO EXPLORE THE EFFICACY OF SIMILES OF LEARNING AND TEACHING MATHEMATICS

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This paper describes a problem-solving session by contrasting some characteristics we predicted it might have with those we observed. The session was part of a study of two preservice mathematics teachers. They had previously been interviewed about their responses to a list of similes for learning mathematics and for being a mathematics teacher. We wanted to find out how well their responses to the similes predicted their participation in contexts other than an interview (in this case, a problem-solving context). In this paper we discuss how the choices we made in our a priori analysis allowed us to plan a session in which certain solution strategies would emerge. Although the a priori specification of characteristics was not enough to ensure the achievement of optimal solutions to the problems, the lack of specification shaped the situation so as to promote the participants' reflection on their views on mathematics, its learning, and its teaching.

Background and Purpose

This paper reports part of a study with two preservice teachers' responses to similes[2] for learning mathematics and for being a mathematics teacher. The participants—Jack and Jill[3]— were interviewed separately and asked to comment on how learning mathematics is like . . . (working on an assembly line, watching a movie, cooking with a recipe, picking fruit from a tree, working a jigsaw puzzle, conducting an experiment, building a house, creating a clay sculpture) and how a mathematics teacher is like . . . (news broadcaster, entertainer, doctor, orchestra conductor, gardener, coach, missionary, social worker). The questions had originally been designed "to gather information on beliefs about mathematics and its teaching" (Cooney, Shealy, & Arvold, in press).

The participants had different reactions to the similes. Jill had said that learning mathematics is like working a *jigsaw puzzle*, because "you have to play with them to try to figure them out" and also, "I kinda like conducting an experiment because you don't always know what's going to happen." Jack com-

[1]We thank Jeremy Kilpatrick for his valuable comments on a previous draft.

[2]These similes were adapted from a survey from the RADIATE project (DUE 9254475), University of Georgia.

[3]Jack and Jill were preservice secondary teachers in their junior year and were majoring in mathematics education.

pared learning mathematics to *building a house* as "you start at the bottom with something simple and build up; you need a concrete understanding before moving up," and to *watching a movie* as sometimes "when we start off . . . it [is] hard to see where's he going with this [but] by the end of the movie, you figured out why that was important"

We devised and used a problem-solving context as one way to address the question of how the participants' responses to the similes would predict what they do when facing an actual mathematical context. The problem-solving session also provided elements for a second discussion of the similes in which the participants were asked to discuss how the experience related to their ideas about teaching. The purpose of this paper is to present a contrast between a priori and a posteriori analyses of the problem-solving session.[4]

A Priori Description of the Problem Session

Brousseau (1988) has developed the notion of *milieu* (of a situation) as "the system antagonistic to the player . . . and a model of the universe of reference for the knowledge that is at stake and for the interactions that this knowledge determines." (p. 320).[5] The *didactic contract* is the system of conditions and constraints that regulates each participant's cognitive obligations and choices in a given milieu. Those notions were instrumental for us in designing the problem session and in organizing our analysis of what could be expected from the participants' engagement in it. The participants were to solve two problems that could be perceived as representing the current opposition, in pedagogical discourse between routine and nonroutine problems, although they were not to be represented that way to the participants. We chose calculus problems so as to evoke recent learning experiences from these participants.

For Problem 1, we wanted a "traditional calculus problem" whose formulation was as close as possible to the usual formulations of the exercises that the participants had been exposed to at school or college. The problem would not explicitly demand the application of a routine algorithm, but in any case it should evoke a standard school experience for which they were used to providing a *right or wrong* answer by a *correct or incorrect* procedure. For Problem 2, we wanted a problem that would resemble an invitation to use "alternative methods of solution," one that would hint at several possible explorations, even outside the usual practice in college calculus and that would not have a clearly foreseeable right or wrong answer. We set up the room so that the participants would work together and provided them with graphing calculators, a calculus book, paper, rulers, and pencils. We used what we knew from the participants

[4]A complete report of this study by Herbst, Mesa, and Gober is forthcoming. Other aspects are discussed in Gober (in press) and Herbst (in press).
[5]Our translation. Here, knowledge is our choice to translate the French *connaissance*.

in specifying the characteristics of the problems so that we could anticipate how they would act when faced with these problems.

Problem 1 was: Determine the maximum and minimum values of the function $f(x) = x^4 - 8x^2 + 16$ on the interval [-3, 3]. In principle, at least two strategies could have been used for solving this problem: an analytic approach (solving $f'(x) = 0$ for x and analyzing the solutions) and a graphic approach (graphing the function and estimating extreme values using a mix of algebraic and visual procedures). The task was formulated, however, so that it evoked the use of the analytic approach. Moreover, the derivative of the function was easy to obtain, and the equation $f'(x) = 0$ was easy to solve (making the exercise straightforward and seemingly ruling out any need for alternative methods). Still, the function was defined on a closed interval sufficiently large to include all local extremes and to have its maximum values at the endpoints. While an interval of the form $[-a, b]$ would likely demand participants' attention to the endpoints, the chosen form $[-a, a]$ was intended to mask that issue. For the same reason, we chose small integers in defining the interval. We selected $a = 3$ because that was the first integer for which the necessary condition did not solve the problem.

We anticipated that both participants would recall how to begin to use the analytic approach, but that Jill might prefer to graph the function on the graphing calculator as a first option. Jack would work individually and would use the calculator to verify the decision but not as a tool for obtaining the solution. He would assign relatively minor status to the informat n from the graphing calculator in comparison with the application of derivatives, although he might not be able to justify why the derivative appro h was better.

These predictions were suggested by the interview data. Jill had mentioned it was important for her to figure things out and, when commenting on the experiment simile, had said "you know what basic steps to take . . . but sometimes it's going to be something totally opposite of what you thought." When Jack was asked to comment on the cooking-with-a-recipe simile, he had said that "there's just certain steps you have to take to get the final product. . . . If you just follow that path, you will always get the right answer." Jack did not like this simile in general, but he associated it with some specific school practices: Problem 1 would likely evoke such a recipe for him.

We designed Problem 2 (see Figure 1) so that it would allow for alternative solution methods but would still be seen as a calculus problem. Items A and B were stated so as to leave the activity open for radically different strategies. To ask for pairwise comparisons of particular areas did not rule out a *modeling strategy* (obtaining the equations and integrating), but it also invited other approaches. The functions whose graphs were involved were not given algebraically, but some of them were familiar. Moreover, we placed the graphs on equally scaled coordinate systems and provided some *indexical references*[5] (such

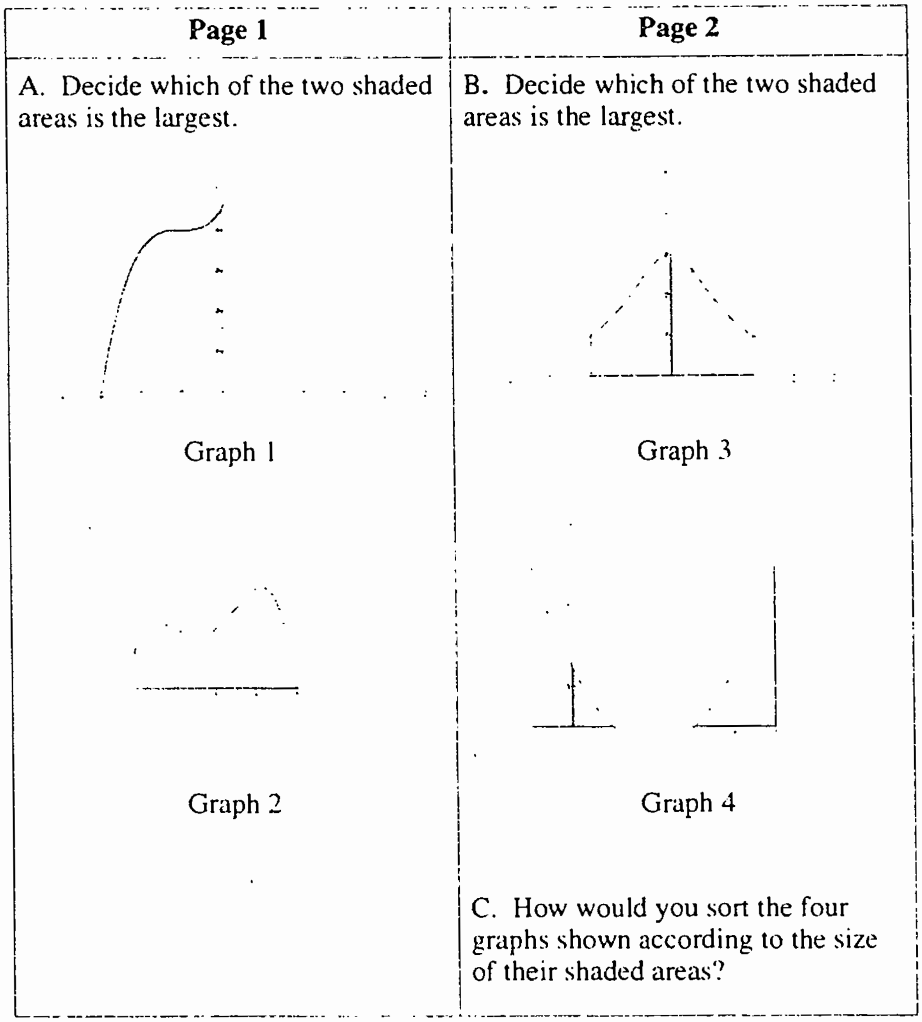

Figure 1 Text for problem 2.

as the interval on the x-axis where the shaded area was based, with all of the references chosen to be integers). These characteristics supported the legitimacy of a modeling strategy within that milieu. However, other variables were set up to suggest that the modeling strategy might be ineffective: Indexical

[6] By indexical references we mean those indications that may accompany a figure so as to indicate what it stands for (i.e., the presence of these references usually determines whether a figure is to be taken at face value or as a representation of something else).

references were not enough to allow one to obtain the correct equations and the algebraic expressions of Graphs 2 and 3 did not have integer coefficients.

We anticipated possible comparison strategies that would not necessarily involve algebraic modeling. Graphs were given on separate coordinate systems so as to discourage a solution based on visual estimation. Still, it was possible to approach the problem by cutting the pages and superimposing the graphs or even by cutting out one shaded area and rearranging the pieces on top of the other. The graphs were given on plain paper rather than graph paper so as to conceal the alternative of counting squares, but we had confidence in the emergence of a *grid strategy* (draw a grid, count whole blocks contained in the target area, and compensate for the remaining parts). Items A and B, asking simply for a comparison, allowed for a variety of strategies to provide a legitimate and effective solution. But Item C required the solver to bring into question the partial results from A and B and likely the methods by which those results had been obtained. The areas were chosen so that for each set of two areas deciding which one was bigger would not be difficult, but across the sets the areas of Graphs 2 and 4 would require either the complete process of modeling (taking some visual risks) and integrating, a refinement of the grid (as the whole unit blocks inside the areas would be few), or a significant use of compensating practices.

We expected Jack to attempt a modeling strategy. He had strongly advocated for the building-the-house simile ("Mathematics is like nothing else"), which entailed that "it starts at the bottom with something that's simple, you take that idea and make something else. It keeps building up." Also, when confronted with the missionary simile, he had said, "They go and teach something people have never heard of. . . . Until I got into algebra, I didn't know what it was. . . . You're showing them, how to help them figure it out." It seemed to us that when facing a variety of strategies, Jack would lean toward one that would look most like what he had learned in school.

As for Jill, we expected her to be open to a variety of possibilities and even to look forward to being creative. When reacting against the watching-a-movie simile, she said, "[In] mathematics, you're always . . . active . . . in learning. . . I'm always playing around with different stuff to see what I can come up with . . . on my own." Also, she spoke against the teacher as a missionary: "They try to help people, but . . . it's always trying to impose their beliefs. . . . Mathematics teachers shouldn't throw something at you just because that's what they believe [to be] right [but instead they should be] saying you come up with whatever . . . ideas you think works." It seemed to us that she would try to use alternative methods if only to be consistent with what she had advocated.

The Session and Some Observations About Its Implementation

The participants were invited to work together on the problems. One of us administered the problems and conducted a short joint interview immediately after they finished the problems. Later, they were interviewed separately about the similes and the problem-solving session. The session was videotaped and then transcribed. Additional data consisted of field notes from three observers, the participants' written work, and transcriptions of the pre- and post-session interviews.

For Problem 1, the two participants quickly identified the problem as demanding an analytic approach and worked out the first steps together—Jack taking the lead and Jill helping him. Once they solved $f'(x) = 0$, they evaluated the solutions in the original polynomial and announced that there was a maximum at 0 and minimum values at -2 and 2. While Jack started graphing the function with the graphing calculator, Jill began to make a table of the integer values of the variable. After evaluating $f(3)$, Jill said "This is not right. . . . You're right" suggesting an error in her evaluation. At that point Jack could see the whole graph of the function in the calculator and was able to use her comment as a prompt to re-elaborate the solution. He mentioned that he had been confronted by those tricky problems before and showed her that both her computation of $f(3)$ and their previous algorithm were right and that the function indeed had its maximum values at the endpoints.

In the interview afterwards, Jill explained that her idea had been to plug integers into the equation so as to make a table and then to graph the function by hand. Her finding that $f(3)$ was greater than $f(0)$ was not related to the need to check the endpoints of the interval so as to challenge the analytic solution but to the usual school practice associated with making a table.

After providing an agreed-upon solution for Problem 1, Jack and Jill started looking independently at Problem 2. Jill began to draw a grid for each graph of Item A, although qualifying her work as "just playing." After restlessly watching what Jill was doing for a while, Jack grabbed a graphing calculator and began to look for a function to model Graph 1. He identified the corresponding function as a cubic and was successful in obtaining an algebraic expression. Although his success with Graph 1 encouraged him to look at Graph 2 in a similar way, he was not successful in getting Jill's collaboration to work on his strategy. On the contrary, his failure to find quickly an adequate expression for Graph 2 and Jill's success in answering Item A with her grids were enough to draw his attention to her strategy. They used it to respond to Item B, but faced an ambiguity when attempting Item C. Instead of refining the grid, Jack used that ambiguity as an opportunity to draw Jill's attention to the modeling strategy. They were not able to model Graph 3, and when time ran out they resorted

to Jill's grids and gave a final answer (which happened to be correct) based on a new visual compensation of the squares.[7]

The characteristics of their work on Problem 2 seem to agree with what we would expect from their answers to the similes. For Jill, the problem was an occasion to play, and it was never as important to get an elegant or general solution as it was to just solve the problem; it was like working on a jigsaw puzzle. For Jack, solving the problem was a matter of carrying out a long-term project, like building a house.

Discussion

The intent of the session was that both participants would work together and produce an agreed-upon solution. Problem 1 established an initial unbalanced division of labor in which Jack was the architect and Jill the worker. The fact that Jack delayed starting to work Problem 2 encouraged Jill to hold on to her strategy (even when Jack had started work and had tried to involve her in his modeling strategy) as a way to balance the distribution of power. However, instead of breaking the working agreement, they engaged in a negotiation of what their agreed-upon strategy would be.

The number and size of areas that were to be compared was effective in incorporating a perturbation into the strategy, but it was not enough to warrant a qualitative refinement of the tool. Jill did not change her visual compensation of incomplete blocks into a refinement of the grid. The difficulty in finding correct expressions for the functions depicted was not enough to induce Jack to participate in the use of the grid strategy, although it led him to accept Jill's as providing the solution to the problem. In order to preserve the work agreement, Jack did not risk arguing against Jill's strategy. Also, in the interview afterwards, Jack implied that he had been deceived by the difficulty of getting the equation for Graph 2. Although Jill defended the fact that "[her] way [had] worked," she also understood her dependency on the grid strategy as an indication that she had to "revise [her] math" and recognized their solution as a compromise.

The specifications of the milieu we had made on the basis of the participants' responses to the similes were not enough to warrant the achievement of an optimal solution. This lack of specification allowed for the emergence and development of diametrically opposed strategies. The situation was, therefore, one in which social and mathematical working conditions were developed so as to maintain an initial social agreement. However, for our purposes the milieu seemed adequately specified. We were able to observe some confirmation of

[7]For a detailed account of the norms that regulated the emergence of a group strategy, see Herbst (in press).

the participants' stated approaches to learning mathematics and to provide a controversial topic for a discussion about teaching in the succeeding interview.

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AN EXPERT'S APPROACH TO MATHEMATICAL PROBLEM-SOLVING INSTRUCTION

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What learning activities play an important role during the implementation of mathematical problem-solving instruction? This is a fundamental question that needs to be analyzed in terms of what research in the area has shown lately. The use of the term problem-solving itself has produced different forms of interpretation and implementation in mathematical instruction. In this context, this study documents some instructional activities that have shown to be successful in a mathematical problem solving course taught at university level. A conception of mathematics that resembles aspects of the practice of doing mathematics appears as a framework to discuss and solve series of problems. The problems are used by the students as platforms to discuss their ideas and search for other forms of solutions or connections. All the students' approaches are supported by mathematical arguments and the students themselves value the potential of their solutions.

Introduction

Recent proposals in mathematical problem solving instruction have suggested that students in their learning experiences should be engaged in activities that are related to the practice of doing mathematics. As Schoenfeld (1988) pointed out "doing mathematics is fundamentally an act of sense-making, an act of taking things apart (mathematically) and seeing what makes them tick" (p. 87). Thus, learning mathematics goes beyond studying rules, procedures, or algorithms; it involves the use of both heuristics and metacognitive strategies to solve problems, the use of various representations to make sense of information, and the search for mathematical connections or applications in different contexts. Here, it becomes important that students develop a mathematical disposition, and a set of beliefs and attitudes consistent with the practice of doing mathematics. What type of learning activities tend to promote mathematical values during the implementation of problem solving instruction? What kind of tasks help students to engage in mathematical discussion in the classroom? What type of evaluation could be used to assess students progress in mathematical problem solving? These are issues that need to be addressed in order to evaluate the potential use of mathematical problem solving instruction.

An important stage in the research on problem solving has been to document what students do while working on mathematical tasks. For example, Schoenfeld (1992) found that the process of doing mathematics includes the use of resources or basic mathematical knowledge (facts, procedures, algo-

rithms), the use of heuristic strategies, the presence of metacognitive activities (monitoring and control), and an understanding of the nature of the mathematical practice (conception of the discipline). As a consequence, it is necessary to investigate to what extent the students' problem solving behaviors could be improved when the instruction they receive takes into account learning activities related to those dimensions.

The purpose of this paper is to discuss aspects related to the implementation of problem solving activities in the classroom. The discussion will be based on the analysis of segments from a mathematical problem solving course taught at the university level.[1] This course has been part of a research program in problem solving for about 20 years. The results have shown that students who take the course make significant progress in the development of their problem solving abilities (Schoenfeld, 1985, 1992, 1994). Thus, in this paper there is interest in discussing specific components of the course that illustrate what and how some instructional activities are implemented. There will be special attention given to the identification of mathematical messages or morals associated with the problems used during the course.

Approaches to Mathematical Problem Solving Instruction

The idea that a problem solving course may focus on various aspects of mathematical practice (heuristics, resources, metacognition, etc.) during the implementation period, makes it important to identify some important features related to several approaches. For example, in the early 80's several of the problem solving approaches relied on the Polya's four phases model as the main structure of the class. Later, there was more attention to the role of nonroutine problems and the presence of metacognitive strategies as an important component of instruction. Here, an important line of research was to study fundamental differences between mathematicians and students while working on mathematical problems (expert, novice studies). Recently there has been interest in considering the role of social factors during the learning of mathematics. "Learning occurs as people engage in activities and the meaning and significance of objects and information derive from their roles in the activities that people are engaged in" (Greeno, Smith, & Moore, 1993, p.100). Indeed, one important feature of Schoenfeld's problem-solving approach is that the students act as a mathematical community while working on mathematical tasks. However, few studies have analyzed the effects of the implementation of these types of instruction (Lester, 1994).

Schoenfeld, who has been teaching a mathematical problem solving course for many years, does not address a content oriented course directly; but through-

[1]The author observed the development of the course during one semester.

out the development of the course, his students deal with several examples in which specific content is discussed. For example, the basic concepts of geometry (constructions), number theory, combinatory, and calculus are topics that frequently appear as a context in the problem solving class. It may be that the course helps students to reconceptualize their ideas about mathematics and deal with problems without focusing on specific content. However, that fact that basic mathematical ideas are addressed consistently during the course seems to suggest a new vision for the organization of mathematical curriculum. That is, rather than addressing a specific sequence of content, it is important to deal with fundamental ideas of mathematics (including heuristics) that students could use to deal with problems from different areas and contexts. This idea is consistent with some curriculum proposals in which emphasis is given to the study of the essential or key mathematical concepts (Steen, 1990).

Schoenfeld teaches the course at the university level. Perhaps, this course is among the few courses that have been attached to a research program in mathematical problem solving. The fact that Schoenfeld himself teaches the course might offer some advantages during the implementation, however, the analysis and discussion of what happens during instruction could help other instructors to transfer or apply some of the activities related to the implementation of this approach. It is important to mention that the features or aspects of the problem solving course addressed in this paper were identified by watching videotapes of the class development and observing the class directly.

Attention to the Solution Process

What makes the type of problems that are part of Schoenfeld's problem solving class interesting is not only the variety of mathematical ideas involved in the solution process, but also that the majority of the problems are accessible to the students. In fact, many of the problems might be familiar to the students, nevertheless, the careful discussion of different methods of solution, connections to other situations, and extensions to more general cases are aspects in which students are exposed to new challenges. It seems that a basic property of the problems chosen for the course is that they offer the opportunity for the students to become engaged in mathematical discussions.

Another important aspect that appears consistently during the implementation of problem solving activities is that students should pay attention to the process involved in reaching the solution(s) of the problem. This challenges the idea that the main goal for students while working on a problem is to find the solution. Paying attention to the process gives students the opportunity to analyze and compare diverse qualities of methods of solutions, and to look for applications and extension of the problem. Some features of the course in which the students examine the solution process of the problem include actions in which the students are aware that: (i) The solution of a problem is an initial

point to launch new mathematical ideas. Thus, students are encouraged to work on different types of problems and to search for connections and extensions of the original problem. An important part of the students' approaches is to conceptualize that finding a solution of a problem is just the beginning of a process in which they have the opportunity to think of other methods of solution, to pose more questions or related problems, to extend the problem by changing the original conditions, and evaluate new relations among other contexts. Thus, other problems emerge and students spend time discussing the qualities of different approaches used to solve those problems. This approach challenges the idea that students normally work on problems that are given to them by their instructors and rarely have the opportunity to go beyond a specific solution. In this context, the students could also explore ways in which the statement of the problem is changed. That is, they may analyze what happens if one or more parts of the original statement are contradicted. Brown and Walter (1983) called this activity the "what-if-not" strategy and have used it extensively in their courses.

(ii) The analysis of the quality of different approaches offer the students to compare and value aspects in which it is important to think of what methods are more efficient than others. This activity is present throughout the development of the course. It is common that students show approaches that include the use of particular and general methods. For instance, when the students were dealing with the problem "prove BBCl(tan(x) + cot(x)) ≥ 2", a first approach was to use trigonometric identities to transform the right side in a manageable terms to show that it the inequality was true. That is, expressing the right side as BBCl(f(sinx, cosx) + f(cosx, sinx)) =

BBCl(f(1, cosxsinx)) = BBCl(f(2, sin2x)) which is always greater or equal than 2. In addition, another more general approach to the problem was shown. That is, representing the right side as

BBCl(tan(x) + f(1,tan(x))) ≥ 2 or BBCl(z + f(1,z)) ≥ 2 . Then, the inequality could be represented as

$z^2 + 1 \geq 2z$ or $(z - 1)^2 \geq 0$. This approach appeals more to an abstract form to represent the original expression rather than the specific content of the problem. This general approach is what Polya calls "inventor's paradox" in which a more general problem may on occasion be easier to solve than the given problem.

Developing a Sense of Confidence in Students' Behavior

The development of mathematical disposition seems to be an important aspect of the problem solving course. During the class development, students are encouraged to present their ideas to the class. They are aware that a clear mathematical argument is what counts in their presentation. In the discussion

of their ideas, they expect criticism and other challenges from other students. These activities are part of the process of dealing with any mathematical tasks. "Becoming a good thinker in any domain -may be as much a matter of acquiring the habits and dispositions of interpretation and sense-making as of acquiring any particular set of skills, strategies, or knowledge (Resnick, 1988, p.58).

A class environment in which students are constantly asked to explain and communicate their ideas to other students is an important feature of the class. For example, students work in small groups of three or four students during a significant portion of the class. The small groups are formed randomly and during the session, Schoenfeld makes sure that the students' interactions involve all the participants. Questions that help to frame the students' interactions include: What are you doing? Why are you doing that? and Where will that lead you? which are similar in spirit to those that Halmos (1994) asks when teaching: What is true? What do the examples we can look at suggest? and How can it be done? The discussion of these questions encourages students to elaborate on what they are thinking, organize their ideas, and provide convincing arguments to defend their conjectures. Thus, students' ideas are normally challenged during the students' interaction based on examining other ways to solve a task, analyzing connections, or refuting a counterexample. In addition, there is a set of expressions that become part of the classroom culture. For example, are we done?; do you know a related problem?; can you think of a special case?; can this be solve geometrically? etc., are questions that appear while working on any task.

> If you understand how things fit together in mathematics, there is very little to memorize. That is, the important thing in mathematics is to see connections and see what makes things tick and how they fit together. Doing the mathematics is putting together the connections and making sense of the structure. Writing down the results -the formal statement that codify your understanding–is the end product, rather than the starting place. (Schoenfeld, 1991, p 328)

Students are exposed to the challenge of explaining why their ideas might work while dealing with the tasks. Comments and feedback that Schoenfeld provides to the students often involve examples in which the students have to rely on their own mathematical arguments to support their work. That is, the students should not expect the instructor to give the final word about the correctness of specific results; rather, the students have to construct and present their arguments to the rest of the class for discussion and judgement. A typical Schoenfeld's response when a student asks for his approval to a mathematical work is:

Don't look to me for approval, because I'm not going to provide it. I'm sure the class knows more than enough to say whether what's on the board is right. So (turning to class) what do you folks think? (Schoenfeld, 1994, p.62).

It is clear that an important value in the learning of mathematics is that students have to provide a mathematical support to their work. The idea that students are always asked to look for more than one way of solution or to explore other connections of the problems help them see the importance of this activity.

Conclusions

There isn't a recipe that will always produce good instructional results in mathematical problem solving. However, there are various ingredients that could be identified as essential during a problem solving approach. For example, an important instructional goal that could be used as a framework is that students need to develop their mathematical disposition to the study of mathematics. This include that students share a set of values that are consistent with the practice of doing mathematics. Here, the type of problems used during the class discussion, the type of student participation, and the evaluation of the students' work are aspects that play an important role in achieving that goal. Thus, learning activities should include tasks in which the students are encouraged to present different ways of solution and search for new extensions or connections of the original problems. In addition, the students should discuss their ideas with other students and use mathematical arguments to support their approaches to the problem.

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RELEVANCE JUDGEMENTS IN MATHEMATICAL PROBLEM SOLVING

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Two groups of Grade 4 students were given word problems containing a numerical statement unrelated to the solution of the problem; one group was directed to underline the irrelevant statement and solve the problem the way they had been shown in class, the attention of the second was not drawn to the presence of any irrelevant statement in the problem. A third, control group received the same problems without the irrelevant numerical statement. Students experiencing irrelevant numerical statements performed better than those not experiencing them. Exploratory analysis indicated that students aware of the presence of irrelevant numerical statements perform better than those not experiencing such statements. Performance on two-step, transfer problems showed the same results, except that in this case, students receiving irrelevant numerical statements tended to perform better than those not experiencing such statements.

Background

The comprehension of mathematical word problems can be considered as different from that of the reading and comprehension of a regular text passage in that the word problem reading task is domain specific. Kintsch and Greeno (1985) cite three factors as evidence for this. The first involves presuppositions relating to problem integrity. Consider the following story: *Matthew had three marbles. Then Chris gave him five more marbles. How many marbles does Matthew have now?* The solver assumes that Matthew didn't lose any of the original marbles and that "three marbles" means exactly three because it is not defined by a modifier such as "least" or "most".

The second factor is a reference to sets. The terms *glass* marbles and *five* marbles may be linguistically similar, but a competent problem solver would treat them differently, classing the latter term as a finite set. Thirdly, a direct consequence of the second factor is the reader's concern with mathematical sets, their quantities and the relations between them. Unlike the reading of a text passage, the reader is not interested in *why* Chris gave Matthew five marbles, only that he now has five more.

If the reading of mathematical word problems is domain specific, an important implication is that language ability *per se* cannot be a reliable determinant of success at solving word problems. Much experimental evidence, eg., Silver (1981), Paul, Nibbelink and Hoover (1986), supports this notion. More recently, Hembree (1992a), in a review of research into mathematical word-problem solving, found (p. 259) that at all grade levels the presence of "extraneous data" (not elaborated on in the review) resulted in lower problem-solving performance. This finding is related to one important concern of this study,

the problem-solver's focusing on those aspects of the problem relevant to its solution.

For any given word problem, certain material can be considered relevant to the problem's solution, and other material irrelevant. To consider our original simple problem: it doesn't matter whether the problem reads, *John had three marbles. Then Bill gave him five more marbles. How many marbles does John have now?* The name changes are irrelevant to the problem's solution. We can also introduce irrelevant numerical data: *John had three marbles. Then Bill, who was twelve years old, gave him five more marbles. How many marbles does John have now?*

Krutetskii (1976), Hayes, Waterman and Robinson (1977) and Robinson and Hayes (1978) have all noted the importance of schemata for making relevance judgements in solving familiar problems (for example, problems regularly encountered in a school mathematics program). A schema is a cognitive structure which specifies both the category to which the problem belongs (eg., addition, single-step) and the most appropriate solution steps for that category. It aids the solver in directing his or her attention to important problem elements, making relevance judgements and, where necessary, retrieving information concerning relevant equations.

So effective are these schemata, that Krutetskii (1976), working in the Soviet Union with capable grade six and seven students, observed that they were not impeded by the presence of irrelevant numerical data in standard problems. Instead, they "singled out the complex of interrelated quantities that constituted the backbone of the problem" (p. 228). This observation had important implications for the present study, the purpose of which was to investigate two seemingly contradictory findings which have arisen in the review of the literature relating to mathematical word-problem solving. On the one hand, for more familiar problems, problem solvers possess schemata which reflect the underlying mathematical structure of the problem and are thus enabled to direct their attention to the relevant material in the problem (Krutetskii, 1976; Hinsley, Hayes and Simon, 1977; Robinson and Hayes, 1978). Conversely, the presence of extraneous data in word problems results in lower problem-solving performance at all grade levels (Hembree, 1992a).

It was expected that emphasizing the process of selecting relevant material would be beneficial to problem-solving performance, this emphasis being achieved by the inclusion of irrelevant numerical material in a set of training problems and drawing the students' attention to this as part of the solution process. To what extent this needs to be a conscious process was investigated by the inclusion of an experimental group who were unaware of the presence of the irrelevant material. To test the efficacy of the schemata in reducing working-memory load, transfer test-problems involving two operations were included. The administration of a second set of test problems containing irrelevant items

allowed a comparison within the groups of the effects of standard problems and "extraneous item" problems.

Procedure

Twenty-six grade 4 students from a Sydney primary school, judged to be competent problem solvers on the basis of their half-yearly exam results, were randomly allocated to one of three groups: Aware (treatment group 1), Unaware (treatment group 2) and Control.

A pre-treatment test consisting of six problems (one addition, two subtraction, two multiplication and one division) was administered to the students. Subjects were then given five training sessions, each requiring the solution, without time restraint, of four problems (one for each of the four arithmetic operations, in order to give equal exposure). The main degree of difficulty was imposed by setting a conversion-of-metric-units question (Loftus and Suppes, 1972) for each of the operations over the various sessions.

The problems given to the Aware Group contained a numerical statement unrelated to the solution of the problem, for example, *There were 20 children at a party. They were seated at 4 tables. They were each given 5 balloons to blow up. How many balloons were there?* The students were directed to underline the irrelevant statement and solve the problem the way they had been shown in class. The Unaware Group was given the same problems as the Aware group, but their attention was *not* drawn to the presence of any irrelevant statement in the problem. They were simply directed to solve the problem the way they had been shown in class. The Control Group received the same problems without the irrelevant numerical statement, for example, *There were 20 children at a party. They were each given 5 balloons to blow up. How many balloons were there?* They were instructed to solve the problems the way they had been shown in class.

Every student in each group received twenty training problems. Pupil feedback sessions, in which the teacher went over problem solutions while the children marked their work, occurred at regular intervals. The Aware group was told which was the irrelevant statement in each of the problems. No mention of the irrelevant statement was made to the Unaware group.

Following the training sessions and associated feedback sessions, a set of six post-treatment problems was administered. In this set of problems, the degree of difficulty was again imposed using a conversion-of-metric-units problem. In addition, two 2-step problems were included (division-subtraction; addition-multiplication). These had been shown previously (Loftus and Suppes, 1972) to be difficult for this age group. The second pairing involved the calculation of a perimeter, thus imposing a further degree of difficulty. The students had not previously encoun-tered problems of this sort, and they could

thus be considered as transfer problems. A second set of six post-treatment problems, involving similar operations but containing irrelevant material, was given to all subjects the following day.

The four one-step problems were scored in the same way as those in the pre-test. The two-step transfer problems, however, were scored out of four marks, allowing for the extra step involved.

Results

The means and standard deviations for the pre-treatment test are shown below.

Group	Mean	Standard Deviation
Aware	15.875	2.800
Unaware	15.750	2.659
Control	15.100	3.035

A one-way analysis of variance indicated no significant differences among the pre-treatment means:

	SS	ndf	MS	
Treatments	3.187	2	1.593	$F(2,23) = 0.196$
Error	87.275	23	8.142	

After treatments, the means and standard deviations were as follows:

Group	Mean	Standard Deviation
Aware	16.250 (μ_A)	3.059
Unaware	15.500 (μ_U)	3.338
Control	11.300 (μ_C)	4.347

The following planned contrasts were tested using *a priori* Scheffé tests, with the following results:

Contrast	Contrast Estimate	F
$0.5(\mu_A + \mu_U) - \mu_C$	4.575	9.42*
$\mu_A - \mu_U$	0.750	0.17

The mean for the combined treatment groups was significantly greater than that of the control group, but no difference between the treatment group means

was apparent. Students experiencing irrelevant numerical statements therefore performed better than those not experiencing such statements.

An overall test and exploratory *post hoc* Scheffé tests were also conducted. As shown below, the hypothesis of homogeneity of means was rejected at the 0.05 level:

	SS	ndf	MS	
Treatments	131.054	2	65.527	$F(2,23) = 4.806^*$
Error	13.600	23	13.635	

and the *post hoc* analyses gave the following results:

Contrast	Estimate	F
$\mu_A - \mu_C$	4.950	8.01*
$\mu_U - \mu_C$	4.200	5.76

Thus, students aware of the presence of irrelevant numerical statements performed better than those not experiencing such statements. While students unaware of the presence of irrelevant numerical statements did not perform significantly better than those not experiencing such statements, the difference was in the expected direction.

Our second concern was to examine the students' performance on the transfer problems in isolation. The maximum possible marks for these problems was eight, the respective means and standard deviations being

Group	Mean	Standard Deviation
Aware	5.375	2.200
Unaware	5.375	1.923
Control	2.300	2.983

Again, an *a priori* Scheffé test was conducted, with the following result:

Contrast	Contrast Estimate	F
$0.5(\mu_A + \mu_U) - \mu_C$	3.075	9.55*

indicating that, as for the non-transfer items, students experiencing irrelevant numerical statements therefore performed better than those not experiencing such statements.

	SS	ndf	MS	
Treatment	58.188	2	29.094	$F(2,23) = 4.785$
Error	139.850	23	6.080	

Again, *post hoc* Scheffé tests were conducted, with the following results:

Contrast	Contrast Estimate	F
$\mu_A - \mu_C$	3.075	6.91*
$\mu_U - \mu_C$	3.075	6.91*

indicating that students experiencing each type of irrelevant numerical statements performed better than those not experiencing such statements.

Finally, the completion by students of a set of post-test problems which contained material which were irrelevant to the problem's solution allowed a comparison of means for the two post-tests, within the groups. The means and standard deviations are shown below.

	Post-Test 1		Post-Test 2	
Group	Mean	SD	Mean	SD
Aware	16.250	3.059	15.500	2.828
Unaware	15.500	3.338	13.750	3.700
Control	11.300	4.347	11.200	4.895

No significant differences exist between post-test means, within groups.

Conclusions

Generally, students experiencing irrelevant numerical statements performed better than those not experiencing such statements, indicating that training in having to select relevant material is beneficial to problem-solving performance, but the effect is more evident for students who are aware of the presence of irrelevant numerical statements than for those who are unaware of them. Performance on two-step transfer problems showed the same results, except that in this case all students receiving irrelevant numerical statements tend to perform better than those not experiencing such statements. No within-group difference in mean performance on a second set of non-transfer post-treatment problems which contained material which was irrelevant to the problem's solution and the first set, which did not, indicating that differences are engendered

at the training stage and are not a function of the type of test problems used.

The results support the hypothesis that students with some background in word-problem solving possess schemata for the various problem types. These enable the student to make judgements about the appropriateness of various parts of the text for the problem's solution. The grade 4 students in this study had been exposed to a range of problems involving the four operations throughout the year. For them, word problem schemata would be in a state of formation, particularly for those problems involving multiplication and division.

On the basis of these findings, it seems that the presence of the irrelevant statement has acted to clarify the relations within a problem form and in turn for the problem schemata based on that form. Referring to the modification of the model proposed by Hayes, Waterman and Robinson (1977), an effective schema, when triggered, directs the solver's attention toward the relevant sets, key words and the underlying relations between these problem elements. This serves to reduce the load on working memory.

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ASPECTS OF WORD-PROBLEM CONTEXT THAT INFLUENCE CHILDREN'S PROBLEM-SOLVING PERFORMANCE

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In this study, 127 fourth graders and 146 sixth graders solved word problems constructed to have parallel mathematical structure but different problem contexts (the nonmathematical, verbal aspect of the problem, such as the story line). Preferences for solving problems and problem-solving performance varied across problems. Students provided written and oral comments that afforded insight into which aspects of word-problem context influenced their problem solving.

Word-problem context elements that appeared to impact students' problem solving include *readability*, *verbal structure*, *story concepts*, and *personal factors*. *Readability* refers to comprehensibility of problem context and relates to such elements as vocabulary and wording. *Verbal structure* includes physical proximity of numbers to each other, number shape (e.g., visual similarity or dissimilarity of different numbers), physical proximity of set names (e.g., "miles") to each other and to related information, number position in relation to its set name (i.e., preceding or following it), and whether or not set names that are the same are repeated for each set or are merely implied. *Story concepts* involve activity level of story line, imagery fostered by story elements, number and complexity of concepts, degree of distinctiveness between and among sets (e.g., those bearing relevant versus extraneous information), whether or not subsets of a set bear the same name as each other, whether or not subset and superset names are the same, strength of association of a set of unknown quantity with other pertinent information (e.g., its superset), and number size. *Personal factors* include interest, personalization (e.g., using a child's name), and familiarity. Individual response to problem context might vary according to a problem solver's gender, age, community type and geographical location, family background (including values, preferred activities, race/ethnicity, socioeconomic status, and religion), individual personality (which includes interests), and academic ability, in addition to random and coincidental factors.

Problem-context variables are many and interact in highly complex ways, so that no individual factor can account in itself for differential responses to various problems.