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ABSTRACT

This report contains papers from the Nordic Conference on Mathematics Teaching (NORMA-94). The first three papers are plenary talks aimed at giving the participants an opportunity to form a coherent view of the new theories of learning. The themes of the paper sessions addressed a variety of topics on different levels from elementary school to university. There were also workshops in which the practical solutions of teaching were discussed in detail. Poster sessions gave other insights into teaching practice. Papers include: (1) "The Use of Concrete Material as a Means To Support the Learning of Mathematics" (Joop van Dormolen); (2) "Constructing Mathematics, Learning and Teaching" (Barbara Jaworski); (3) "Constructive Teaching in Mathematics" (Jarkko Leino); (4) "Control Activities at Problem Solving" (Andras Ambrus); (5) "Interactive Exercises as a Part of Mathematical Hypermedia" (Kostadin Antchev, Seppo Pohjolainen, and Jari Multisilta); (6) "Assessment in Mathematics from a Social Constructivist Viewpoint" (Ole Bjorkqvist); (7) "Responsibility for Own Learning in Mathematics" (Trygve Breiteig); (8) "A Comparison of School Leaving Math Exams" (Gard Brekke and Algirdas Zabulionis); (8) "The General Education of Children in Mathematics Lessons" (Gunter Graumann); (9) "Gender and Mathematics Education. Theory into Practice" (Barbro Grevholm); (10) "The Geometry behind the Cupola of the Cathedral of Florence" (Ivan Tafteberg Jakobsen); (11) "Theory into Practice of Teaching Mathematics" (Tunde Kantor); (12) "A Heuristic Alternative To Formalistic Teaching in School Geometry: Theoretical Scrutiny" (Tapio Keranto); (13) "DiffEqLab A MATLAB Based Package for Studying Ordinary Differential Equations" (Simo K. Kivela); (14) "Structural Teaching" (Ricardas Kudzma); (15) "Mathematics Teachers' Beliefs and Conceptions of Mathematics Education" (Pekka Kupari); (16) "Learning and Teaching Geometry" (Frantisek Kurina); (17) "On Mathematical Education--Content, Meaning and Application" (Anna Lothman); (18) "Study on Affective Factors in Mathematics Learning" (Marja-Liisa Malmivuori); (19) "Teaching Geometry, A Constructivist Approach" (Hartwig Meissner); (20) "Five Fingers on One Hand and Ten on the Other: The Teacher as a Mediator in

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Interactive, Playful Teaching" (Dagmar Neuman); (21) "The 1994 Finnish Summer Math Camp and The Edutech Mathematics and Physics Enrichment Programme for Gifted High Schoolers 1993-1995" (Kullervo Nieminen and Robert Piche); (22) "On the Role of the Hypermedia in Mathematics Education" (Seppo Pohjolainen, Jari Multisilta, and Kostadin Antchev); (23) "Some New Geometrical Ideas in Teaching Mathematics" (Maido Rahula and Kaarin Riives); (24) "New Components for the Study and Evaluation of Mathematics" (Maarit Rossi); (25) "The Use of Computers in Mathematics" (Rui J.B. Soares); (26) "Calculus and General Secondary Education" (Vitaly. V. Tsuckerman); (27) "Factors for Change in Teachers' Conceptions about Mathematics" (Gunter Torner); (28) "Teaching of Measurements--Experiences from Primary Teacher Education" (Maija Ahtee and Hellevi Putkonen); (29) "The Use of Concrete Material as a Means To Support the Learning of Mathematics" (Joop van Dormolen); (30) "Symmetry in Primary School" (Gunter Graumann); (31) "Computer Based Geometric Ideas for Construction of Mathematics" (Lenni Haapasalo and Roland J.K. Stowasser); (32) "Hypermedia as Motivation Medium in Mathematics Learning" (Anna Loimulahti); (33) "Open Approach To Mathematics in Lower Secondary Level" (Erkki Pehkonen); (34) "Verbal Arithmetical Problems in Exercise Books Printed for Children" (Zbigniew Semadeni); (35) "Mathematical Activities for High School Students in Estonia" (Elts Abel); (36) "Economical Problems as the Contexts for Constructivist Approach To School Mathematics" (Juri Afanasjev); and (37) "Teaching Geometry in the Czech Republic" (Frantisek Kurina). (ASK)

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Erkki Pehkonen (ed.)

NORMA-94 Conference. Proceedings of the Nordic Conference on Mathematics Teaching (NORMA-94) in Lahti 1994. 207 pp.

Abstract

A Nordic Conference on Mathematics Teaching (NORMA-94) took place in the Lahti Research and Training Centre at the University of Helsinki from the 2nd of September to the 6th of September 1994. This report contains most of the papers given in the conference. The conference language was English.

There were four plenary talks (Cooney, van Dormolen, Jaworski, Leino). These are the three first papers in the report. With the plenaries, we were trying to give the participants an opportunity to form a coherent view of the new theories of learning.

The themes of the paper sessions (Ambrus, Antchev & Pohjolainen & Multisilta, Björkqvist, Breiteig, Brekke & Zabulionis, Graumann, Grevholm, Jakobsen, Kántor, Keranto, Kivelä, Kudzma, Kupari, Kurina, Löthman, Malmivuori, Meissner, Neuman, Nieminen & Piché, Pohjolainen & Multisilta & Antchev, Rahula & Riives, Rossi, Soares, Tsuckerman, Törner) dealt with a varied range of topics, and on different levels, from elementary school to university. Furthermore in the conference, there were workshops (Ahtee & Putkonen, van Dormolen, Graumann, Haapasalo & Stowasser, Loimulahti, Pehkonen, Semadeni) where the practical solutions of teaching were discussed in detail. A poster session with many posters (here are only three of them: Abel, Afanasjev, Kurina) gave another insight into teaching practice.

Keywords: mathematics teaching, pedagogy of mathematics, teaching practice

HELSINGIN YLIOPISTO
OPETTAJANKOULUTUSLAITOS

Tutkimuksia n:o 141, 1995

Erkki Pehkonen (toim.)

NORMA-94 konferenssi. Raportti pohjoismaisesta matematiikanopetuksen konferenssista (NORMA-94) Lahdessa 1994. 207 s.

Tiivistelmä

Pohjoismainen matematiikanopetuksen konferenssi (NORMA-94) pidettiin Helsingin yliopiston Lahden tutkimus- ja koulutuskeskuksessa syyskuun 2. päivästä syyskuun 6. päivään 1994. Tämä raportti sisältää useimmat konferenssin esityksistä. Konferenssikieli oli englanti.

Kokouksessa oli yhteensä neljä pääesitelmää (Cooney, van Dormolen, Jaworski, Leino). Nämä ovat raportin kolme ensimmäistä artikkelia. Pääesitelmillä pyrittiin antaa osallistujille mahdollisuus muodostaa yhtenäinen kuva uusista oppimisteorioista.

Sektioesitelmät (Ambrus, Antchev & Pohjolainen & Multisilta, Björkqvist, Breiteig, Brekke & Zabulionis, Graumann, Grevholm, Jakobsen, Kántor, Keranto, Kivelä, Kudzma, Kupari, Kurina, Löthman, Malmivuori, Meissner, Neuman, Nieminen & Piché, Pohjolainen & Multisilta & Antchev, Rahula & Riives, Rossi, Soares, Tsuckerman, Törner) käsittelivät laaja-alaisesti matematiikanopetuksen eri osa-alueita, ja eri tasojen opetuksesta – alkuopetuksesta yliopistoon. Lisäksi kokouksessa työskenneltiin erilaisissa työryhmissä (Ahtee & Putkonen, van Dormolen, Graumann, Haapasalo & Stowasser, Loimulahti, Pehkonen, Semadeni), joissa keskusteltiin opetuksen käytännöllisistä ratkaisuista. Posteri-istunto monine postereineen (joista tässä teoksessa on vain kolme: Abel, Afanasjev, Kurina) antoi toisenlaisen kuvan opetuskäytäntöön.

Avainsanat: matematiikanopetus, matematiikan didaktiikka, opetuskäytäntö

Contents

Preface

Lahti Research and Training Centre at the University of Helsinki organised, on the initiation of the editor, the first Nordic Conference on Mathematics Teaching NORMA-94 in Lahti from Friday the 2nd of September to Tuesday the 6th of September 1994. There were about 120 participants from sixteen countries. Most of the participants were from the Nordic and Baltic Countries. Additionally, some came from The Czech Republic, England, Germany, Israel, Malaysia, Poland, Portugal, and USA. The conference language was English.

About two years ago, we discussed informally on the need of such a conference in the Nordic countries, and consequently put up the programme committee for the NORMA-94 conference. The members of the programme committee were, as follows: Dr. Erkki Pehkonen (University of Helsinki, chairman), Dr. Jüri Afanasjev (University of Tartu), Prof. Ole Björkqvist (University of Åbo Akademi), Dr. Gard Brekke (Telemark Teacher Training College), Dr. Bengt Johannson (University of Göteborg), Prof. Jarkko Leino (University of Tampere), Dr. Ole Skovsmose (University of Aarhus), and Ms. M.-L. Neuvonen-Rauhala (University of Helsinki, secretary).

The main purpose of the NORMA-94 conference was to study the influence of the latest learning theories on the practice of teaching mathematics, and to give examples how to realise these ideas in school practice. In addition, the aim of the conference was to give a forum for researchers in the Nordic and Baltic countries. Much of the recent research done on mathematics education has benefited the conception of learning which the new learning theories are offering. But an open question seems still to be, how much of that has been and can be changed into teaching practice. The focus of the current conference was just to help to bridge the gap between the theory and practice. Therefore, we used as a subtitle for the conference the words: Theory Into Practice.

The conference comprised four kinds of official activities: plenaries, workshops, paper sessions, and posters. Each of these was followed by a group discussion. Furthermore, informal discussions be-

tween the participants formed a very important part. In the conference, the role of discussions was emphasised, since according to the constructivist view, learning demands a learner's active participation in the process of constructing knowledge.

There were four plenary talks (Tom Cooney, Joop van Dormolen, Barbara Jaworski, Jarkko Leino). And these form the three first papers in the proceedings; unfortunately prof. Cooney has found no time to elaborate his talk in the paper form. With the plenaries, we were trying to give the participants an opportunity to form a coherent view of the new theories of learning.

The themes of the 33 paper sessions dealt with a varied range of topics, and on different levels, from elementary school to university. The point of emphasis was on teaching practice, e.g. on pupil activities. Furthermore in the conference, there were 11 workshops where the practical solutions of teaching were discussed in detail. A poster session with 9 posters gave another insight into teaching practice. This report contains almost all the papers given in the conference. But regrettably, only few of the workshops and posters are described here.

In this conference report, every author is responsible for his/her own text. These are neither proof-read by the editor, nor their language is checked. Addresses of the contributors can be found in the appendix.

In this place, I want to thank again the programme committee for work it did for the conference. Especially, my thanks are due to Ms. Marja-Liisa Neuvonen-Rauhala who made everything to run, and even to run admirably smoothly. Also I want to express my gratitude to our Head of Department, prof. Irina Koskinen for allowing me to publish these proceedings in the Research Report -series of the Department of Teacher Education.

The conference was financially supported by the Ministry of Education in Finland.

Helsinki, March 1995

Erkki Pehkonen

PLENARIES

The use of concrete material as a means to support the learning of mathematics

Joop van Dormolen

Introduction

Two essential ideas in this talk are *the use of concrete material* and *the use of metaphors*. My main purpose is to give good reasons why (a) metaphors are essential for learning new concepts and (b) to lay a link between that kind of learning and the manipulation of concrete and adequate material.

For those of you who are not used to the concept of metaphor, and certainly not in the context of science and mathematics, let me explain shortly what it is about. Later I shall explore this concept more.

My dictionary tells me that a metaphor is a kind of figurative speech that uses analogy, and it gives as an example: 'ship of the desert'. When we hear the expression 'ship of the desert', we not only know that it is about a camel, but at the same time we know a lot more about the camel, because we know a lot about ships that sail over the oceans of the world.

I want to show you that in mathematics teaching we can do the same thing, and in fact, do so often without realizing it: tell a lot about one thing by speaking about another thing. In our knowledge we know much about that other thing. That is the reason why we use it.

First I shall give some examples where metaphors play an important role and then I shall generalize.

Examples

The first example consists of a cord on which I can hang cards with numbers. Imagine me as a teacher, where I tell my students that this is a number line, I used hang two cards with numbers on it, for example 1 and 2. I then asked them where to hang other numbers. Whenever I did this, my students answered me correctly and they could give me good mathematical arguments for it.

You can ask yourself already now (I shall do that later in this talk) what I told my students when I spoke about the cord and when I asked them where to hang minus the square root of 30.

Second example. In my early days as a teacher, when I tried to explain what a function is, I used examples like: x ---> $2x + 3$ and I used to say that the function assigns to each number x the number $2x + 3$. For example: it assigns 2 x 1 + 3 to 1, 2 x 2 + 3 to 2, 2 x 3 + 3 to 3, etc Then I asked the students to calculate some results.

Very often students came with results like 1 -----> 6, 2 -----> 9, 3 -----> 12, 4 -----> 15. It took me some while to discover the 'error'. In the Dutch language the verb 'to assign' (= 'toevoegen') means something like 'to add to', but in the sense of 'to make belong to', like in: John was assigned as a guide to the group of tourists. This was the intended meaning of 'to assign a number to a certain number'. There is however another meaning in Dutch. You also can use it in the sense of adding sugar to tea. In that sense 'toevoegen' is solemn language, but it is not rare. Now you can see what happened: 1 becomes 5, assign (= add) 5 to 1 and you get 6. Etc.

Here again, I told my students something about functions, by talking about assigning. That my students did not hear what I intended them to hear, does not detracts anything from the principle of what a metaphor is.

Third example. As a teacher of 7-th graders I wanted once to stress that two points coincide with one and only one line. I did not want to use this kind of sophisticated language. Pure intuitively I switched to a more tangible and active language and I said: "You can only draw one line through two points". I was surprised that one of my most gifted students told me that this was not true. She said, one can draw numerous lines through two points. To show that she was right, she drew two dots on a piece of paper, laid her ruler along them and drew with her pencil a line along it. Then she did the same, drawing another line over the first one, and then again, and again ...

My last example comes from an article of Kirsch [1978] on what he called pre-mathematical proofs. Imagine a figure that consists of points and lines.

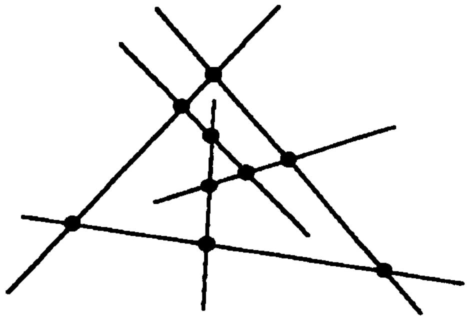

Fig. 1

This figure has a particular property:

- Every point coincides with exactly two lines (of the figure).
- On every line lie exactly three points (of the figure).

Figures like these are called configurations.

In general: A configuration is a figure, consisting of l lines and p points, each line coincides with exactly m points, each point coincides with exactly q lines.

These configurations have a property: $l \times m = p \times q$. I shall prove that with a model. It consists of six sticks with nine balls, arranged as in the figure. Every ball can be cut into q pieces (in this special case q=2), such that though each piece goes exactly one stick. We have l *(=6)* sticks, each with m (=3) pieces, so that we have $l \times m$ (=18) pieces. On the other hand we had p (=9) balls, each cut into q (=2) pieces, so that we have $p \times q$ *(=18)* of these pieces.

So: $l \times m = p \times q$.

A quick look back at the examples

I was successful in my first example. All my students understood immediately that this cord was not to be taken literally as something to hang things on it. They all translated questions like "Where shall I hang this card?" to the mathematical context, that was meant.

The second example shows how figurative speech was recognized as such, but the metaphoric interpretation was not the intended one.

In the third example I was not successful at all in using figurative speech. "One can draw only one line through two point", was meant as a metaphor for "There exists one and only one line that coincides with two points". But my speech was taken literally. The content of what I said fitted too much into the current context, that the possibility that it might be taken figuratively did not occur.

I am sure that I was almost successful with the last example. You knew without doubt, that the whole operation was metaphoric for a genuine mathematical proof, although not every one of you might already see at this very moment what the mathematical equivalent is for the cutting of the balls, and how to translate 'pieces of the balls' into genuine mathematical concepts.

General remarks

Several remarks can be induced from the examples:

- Metaphors are a strong interface between one's own experiences and a newly to learn concept.

- Metaphors have a materialistic background.

- In using metaphors one has to be very careful to create the right context and to assure that the other party catches the intended meaning.

I now want so say something more about each of these remarks.

Metaphors are a strong interface between one's own experiences and a newly to learn concept

Otte [in Keitel, Otte, Seeger 1980] used to talk about the paradox of learning: We learned a lot since we are born, and yet learning psycologists and educationalists tel us that one can only learn through one's own experiences.The solution of the paradox is, that we use experiences to understand new concepts. This learning does not come from reading or hearing abstract descriptions, but from taking our experiences figuratively. A good example is the use of examples: adequate examples act like a *pars pro toto* (a part acting

for the whole) for the intended concept. Another example is the metaphor, like was shown in examples.

Here are some more metaphors, or at least expressions that sometime ago were metaphors:

the night is falling, we grasp a meaning, laying down one's arguments in a row, this shows what I mean, he is so far away when somebody tries to talk to him, take this example

and some from mathematics itself:

the legs of an angle, block, function, degree, group, field, power, mid, diamond, equation, set, root, to prove, to divide, to rotate, to invert, to solve, to mirror, to assign, to draw, to express, to move, to translate, to find (the root of an equation).

The consequence for teaching is that

(a) the teacher (and the school book author) should provide the student with adequate experiences;

(b) the teacher (and the school book author) should provide a kind of interface between the student's experiences and the new concept that the student must learn.

We cannot provide as experiences the concept itself, because that is too abstract. So we must provide material that, in some way, has similarities with the new concept. In some cases this concept is of a lower order than some concept that the student already has [Skemp 1973] In that case logic explanation is sufficient. Like explaining what a circle of unity is, when the student already has the abstract concept of a circle.

In most cases we have to refer to similarities that have more or less the same structure as the new concept. Here we touch on the use of metaphors, as it is essentially the metaphor that is used to show similar structures. Look at a metaphor as an expression, that says something about an unknown concept *A*, by reference to a known concept *B*: "*A* is a *B* ...". *A* and *B* have a totally different *content*. like we have seen in many examples before, while they have a similar *structure* [Bauersfeld and Zawadowski 1981].

This argumentation already has much in itself to justify the use of tangible materials, as these materials belong to the student's experiences. The materials act like metaphors.

But there is more. As Sfard [1994] points out, metaphors always refer to something materialistic. When we look at the examples, they all have this materialistic property. In most cases they refer to bodily experiences. This might legalise and explain why the manipulation of concrete, tangible materials can be such a successful means of understanding new concepts. In particular really manipulating, not only in one's imagination.

It is interesting to observe that, at conferences of processional mathematicians, those lectures always draw big audiences of those that are known for their use of tangible material.

In using metaphors one has to be very careful to create the right context and to assure that the other party catches the intended meaning

Let us look at what can happen when somebody receives some language sign. First there is the perception of the sign. If the literary meaning does not fit in the actual context, then the receiver looks for another interpretation. In most cases the receiver is not consciously aware of this process.

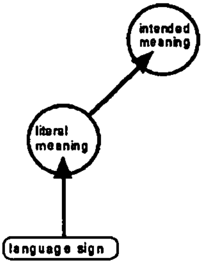

Fig. 2

We have now the following possibilities.

- The interpretation is just the intended meaning.

- The receiver sees no cause to look for another than the intended meaning, like in the example of the many lines through two points.

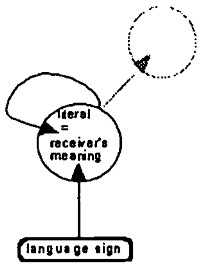

Fig. 3

- The receiver interpretation of the sign differs from intended meaning.

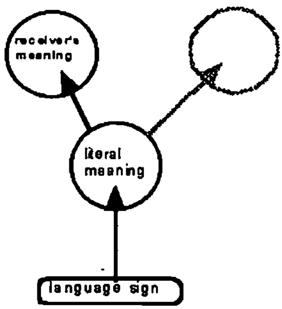

Fig. 4

This is what happened with the assigning function in the third example.

Next we have the possibility that the receiver understands that the literal meaning cannot be intended, but cannot find any figurative meaning.

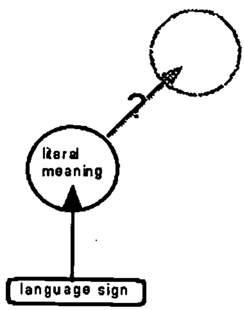

Fig. 5

One can imagine what is happening to a person who suddenly, during a mathematics lesson, hears about the square root of a number. And even if it is not the first time, there still might be a mental blockade if the receiver is not yet familiarized with the new words. In the best case the receiver asks the teacher what is meant. More often than not the receiver starts to copy what he or she supposes what is expected.

This concludes the possibilities of understanding an intended figurative speech.

There is, however, one other situation that is most important to know, because in teaching we use it all the time, but mostly without being aware of it.

Fig. 6

More often than not, we use words that are completely normal for us, while the receiver cannot recognized the intended meaning. This is because we do not realize that the words were once metaphors. Like in the night that *falls*, the *grasping* of an idea, a line *through* a point, the *edge* of a figure, *negative* numbers, ... For us, they are dead metaphors, but not for those who still have to learn these words in their new meaning.

Summarizing: we, as teachers have to take into account that the following can happen when we use figurative speech:

- Receiver's meaning coincides with intended meaning
- Receiver takes the language literally.
- Receiver assigns another meaning than was intended.
- Receiver is just bewildered.

Final remarks

I have tried to argument that, in communicating with others and in particular when we teach, we cannot always tell the other person directly what we mean. We then have to recourse the other person's experiences by using figurative speech. More often than not this speech is a metaphor.

Metaphors have a materialistic background, which is why we can use concrete material as metaphors.

We always have to be careful with our language when we teach, in particular when we use figurative speech, as that can easily be misunderstood.

I am sure that every one of you uses figurative speech, in particular metaphors, in mathematics. It may be, that not every one of you was conscious of this, and, now that you are, might feel a little uneasy. It seems to be a *contradictio in terminis*, as mathematics has the reputation of being exact, unambiguous, formal ... This, however, refers to the ultimate end product (if such a thing exists), not to learning, discovering, inventing mathematics. The main purpose of this talk was to clarify how we, as teachers of mathematics, can use metaphors, in particular in the form of tangible material, as an advantage to help our students to not only do mathematics, but do it with understanding. If you would agree to that now, I would feel happy that I was able to enable you, through all the metaphors I used, to use your own experiences towards relatively new concepts.

And talking about learning through one's own experiences: I sincerely hope that you were like monsieur Jourdain in Moliere's Bourgois gentilhomme, who receives lessons in literature and then discovers with enjoyment, that all his life he has already spoken prose. But now that he realises it, he will do it consciously in the future. If that will be the case with you in using metaphors in your teaching, then my talk was fruitful.

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Constructing mathematics, learning and teaching

Barbara Jaworski

Summary

This paper is about the constructing of teaching for students' effective learning of mathematics, by teachers who might be seen to employ an investigative classroom approach. It draws on research involving close observation of mathematics lessons and a study of the thinking processes of selected teachers. An investigative approach was seen theoretically to be embedded within a constructivist perspective of knowledge and learning. The research sought to characterise 'investigative' teaching and to focus on issues raised for the teachers involved. The effects of teachers' critical reflection on their teaching development, and the constructive nature of the research itself were significant outcomes of the study.

The first part of the paper will comprise a theoretical overview and the second part will focus on specific details to highlight issues and outcomes.

Theoretical Overview

Background

How do mathematics teachers construct their teaching? What is the essence of such construction? How does construction of teaching relate to construction of mathematics and construction of learning? How can mathematics teaching develop as a result of this constructive process?

In the research I discuss here, I found myself addressing the above questions as a result of exploring the nature and outcomes of 'an investigative approach to the teaching of mathematics' (for brevity, to be labelled 'investigative teaching'). Investigative teaching, although not well-defined[1], was seen as an alternative to what Celia Hoyles (1988) describes as 'a transmission model of teaching and learning [mathematics] where knowledge and expertise is assumed to reside with the teacher'. Hoyles, reflects on classroom constraints which obviate effective teaching:

[1] See Jaworski, 1994, for a history of the development of investigative work in mathematics classrooms in the UK.

We know that teachers and pupils tend not to search together in a genuine and open way to uncover mathematical meaning. We know, for example, that pupils want teachers to "make it easy" or "tell them the way" and we have to recognise the powerful influences on teacher practice which almost compel an algorithmic approach. We need to find a significantly different mode of education and practice in our classrooms, new roles for teachers which they value and which they see as significant for the mathematics learning of their pupils. (p 162)

My motivation for undertaking the research resulted from dissatisfaction with the effects of the teaching to which Hoyles refers, and from a personal excitement with the potential of investigative approaches to teaching mathematics to provide for the needs she highlights. The field work, which took place between 1985 and 1989, was designed to explore classroom manifestations of an investigative approach and the related thinking of the teachers who implemented it. There was no intention of reforming or changing practice.

The mathematics teachers who were subjects of my study were chosen because they or I considered their teaching to be in some way investigative. The fieldwork for the study involved extensive participant observation of lessons of the chosen teachers; long hours of discussion with the teachers about their thinking, their planning for lessons and their reflections on what occurred; some informal interviewing of students; some use of video-tape for stimulated response; and a small amount of questionnaire data. It drew methodologically on the research literature in classroom ethnography (eg. Stubbs and Delamont, 1976; Burgess, 1985, Hammersley, 1986) and on sociological traditions in interpreting social interactions (eg. Schutz, 1964; Blumer, 1966; Cicourel, 1973). At the time, there was little research of this form in mathematics education itself. Highly significant to the research outcomes was the involvement of the researcher (myself) and the extent to which this involvement coloured the research and its findings (Jaworski, 1991).

Constructivism

Investigative teaching might be seen to be motivated by a desire to develop students' conceptual frameworks in mathematics for

which a metaphor of individual *construction* of understanding and its mediation through social interaction is powerful. The research took place during growth in the mathematics education community of a belief in *constructivism* as a theory of mathematical cognition and learning (eg von Glasersfeld 1983, 1984, 1990, Cobb and Steffe, 1983, Ernest 1991, Taylor and Campbell-Williams, 1993)[2]. Mathematics educators were starting to explore the implications of a constructivist perspective for the effective teaching of mathematics. Effective teaching might be conceptualised as providing an environment in which students develop a wide *relational* understanding of mathematics (Skemp, 1976) and *higher-order thinking skills* in doing mathematics (Peterson, 1988). It was part of my research to consider how constructivism might inform a study of investigative approaches in classroom teaching and to evaluate critically in what ways the teaching might be considered effective.

It became important, early, to recognise that constructivism was a theory, or perspective (Ernest, 1991b), of cognition and learning, and that it was dangerous, albeit seductive, to draw inferences for teaching (Kilpatrick, 1987). In studying closely the thinking of the teachers, I was able both to recognise aspects of their thinking which I would label as constructivist [One teacher said, wryly, after reading one of my accounts "Oh, so I was a constructivist before I knew what constructivism was"!], and to recognise tensions in and issues arising from their teaching in accordance with a constructivist philsophy. What became most significant for the research and its outcomes was seeing the research itself and its insights into teaching development in constructivist terms. The relationship between *radical* and *social* constructivism(Ernest, 1991, Jaworski, 1994) was central to the outcomes of the research.

Research outcomes

An important objective of my research was to characterise an investigative approach to mathematics teaching through the study of the teachers involved, seeking for common features and issues. Any attempt at generalisation from ethnographic or interpretive research is problematic. Delamont and Hamilton (1984) have

[2] von Glasersfeld, 1983, offered two principles of radical constructivism which have formed the basis of much work subsequently.

pointed out, however, that it is possible to 'clarify relationships, pinpoint critical processes and identify common phenomena' from which 'abstracted summaries and general concepts can be formulated, which may, upon further investigation be found to be germane to a wider variety of settings' . This was very much the case in my research, and one of its results was to highlight teaching actions and outcomes which seemed to pertain to the classroom environments studied and might be seen as characteristic of an investigative approach to teaching mathematics (Jaworski, 1991).

For example, teachers' management of the learning situation involved fostering an ethos to support mathematical talk, collaborative working, negotiation of ideas and justification of conceptions and strategies. This resulted in students sharing responsibility for their own learning with much evidence of high-level thinking. Teachers were sensitive to students' individual propensities and needs in judging appropriate levels of cognitive demand. These characteristics demonstrated in practice recommendations from the 1982 Committee of Enquiry into the Teaching of Mathematics in Schools in England and Wales (The Cockroft Report): for example, that mathematics lessons should incorporate aspects of problem-solving, practical and investigational work, and mathematical discussion (Para 243). Hoyles (1988) suggested that such 'exhortations' were affecting, at least superficially, what was going on in UK classrooms. However, she continued:

> ... but will mathematics teaching be more effective? Encouraging pupils to talk about their methods and perceptions, to justify their strategies of exploration and proof, and to learn from each other implies a major shift in the social relations in the classroom which may not be acceptable to some teachers ...(p 159)

It seemed that there were many respects in which the teaching I observed could be considered effective, both in terms of its embodiment of the characteristics observed and in terms of students' mathematical thinking and concept formation. This did not mean that the teaching was unproblematic. The investigative nature of the teaching approach resulted in many questions about teaching as well as raising issues and tensions for the teachers.

However, perhaps even more significant than these characteristics were the insights which the research provided into the development of mathematics teaching and its relationship to the research itself. It became clear that the teachers concerned,

influenced perhaps by being the subjects of research, were actively reflecting on and developing their own teaching through tackling questions and issues. How was this related to the characteristics observed? Could it be that the major shift of which Hoyles talks is related to the reflective activity of the teachers, and that this, in its turn, is consequent on constructivist-related thinking on the part of the teachers?

Throughout the research, issues and tensions arose which guided the progress of research and substantiated or initiated theory of teaching[3]. In building a story of effective teaching – resulting from the reflective practice of teachers who act from a constructivist perspective of cognition and learning – I have to be aware of my own position and personal theories. As I worked with the teachers, my interactions with them and their classes were significant to research data. Interpretations were always through the lens of the researcher, although I sought verification by triangulation and respondent validation. Rigour lay in a reflexive accounting process where I had to examine critically my interpretations and judgments, taking into account their full situation and context. My own thinking and perspective were a part of this context.

In the second half of this paper I shall support this theoretical position with discussion of examples of significant episodes in the data and their analysis, and an indication of the nature of teachers' developing thinking with respect to the mathematical learning and understanding of students. Space in these proceedings has required that this next section be cut very considerably from that presented at the NORMA conference. A copy of the fuller paper can be obtained from the author.

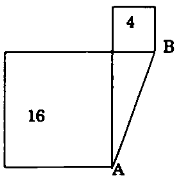
[3] For example, as a result of categorisation of data from one teacher, a theoretical construct, the teaching triad – management of learning, sensitivity to students and mathematical challenge, emerged as characteristic of the observed teaching. This was subsequently tested against the practice and thinking of other teachers. Details can be found in Jaworski (1991 and 1994).

Research episodes providing theoretical insights

Mathematical Learning: issues for teaching

A teacher, Ben, planned a lesson on *vectors*. He expected that in working on vectors, students would need to review their work on Pythagoras which had taken place some weeks earlier. At that time, Ben had said to me that he felt one student, Luke, had not yet really understood the Pythagorean relationship. I shall refer to Luke's development of knowledge related to Pythagoras' Theorem to address some of the above issues from a practical perspective.

During the vectors lesson, I sat alongside Luke and his partner, Danny. They had been asked to make up their own questions about finding the length of a vector, and to try out the questions on their partner. Luke explained to Danny what he thought they had to do. He wrote down the vector AB, as below, placed points A and B on a grid, drew the triangle around them, drew squares on two sides of the triangle as shown.



$\vec{AB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Luke's explanation to Danny

He wrote the square numbers in the squares, and then worked out mentally, talking aloud: "16 plus 4, that's 20; square root (pause) about 4.5". Danny seemed to follow what he had done, and each of the pair, independently, set about inventing vectors and finding lengths. Luke, in each case, drew a diagram similar to the one above, writing the square numbers into the squares. He then performed the calculation mentally and wrote down the result.

Another person in the class invented a vector with a negative number in it and asked others how to find its length. It was Luke, eventually, who said that a negative number did not make any difference – his method would still work because you just had to multiply the number by itself, and then you had a positive number as before.

It seemed to me, from my position as observer, that Luke's knowledge of Pythagoras' Theorem was at least adequate for his needs in finding lengths of vectors. There was some sense, then, in which he 'knew' Pythagoras' Theorem. This was particularly significant in the light of the teacher's comment about Luke's earlier lack of understanding. What had happened in the meantime? What learning had taken place? What growth of knowledge? And how had it happened?

Addressing such questions can help us become more knowledgeable about the pedagogy of mathematics. Teachers, constructing teaching, not only have to recognise learning and understanding in students, but have to create opportunities for their knowledge to develop, and evaluate the situations and interactions which result. The management of this learning environment is a highly complex process. Caleb Gattegno (1960) said the following:

When we know why we do something in the classroom and what effect it has on our students, we shall be able to claim that we are contributing to the clarification of our activity as if it were a science. (Gattegno, 1960)

Luke's teacher, Ben, was aware of Luke's developing understanding. He had planned the vectors' tasks knowing that consolidation of understanding of the Pythagorean relationship was necessary. In Luke's case it seemed that his teaching had been, to some degree, effective. In our research together, Ben and I questioned the processes involved.

Reflecting on Teaching and Learning

It was Ben's declared aim to use *an investigative approach* to his teaching of mathematics. On a number of occasions, he referred to his teaching as being 'more *didactic* than usual'. He seemed to suggest that to be 'didactic' implied being less 'investigative'. It was in exploring what he meant by the term 'didactic' that I gained insight to important issues for Ben. This led to a recognition of links between a constructivist philosophy and classroom manifestations of an investigative approach.

I shall refer to a conversation which took place just before the vectors lesson. Ben was talking about his plans for the lesson, and seemed to be apologising because he felt it would be less investigative in spirit than he would like, or I would expect. My

degree of influence here is significant for the outcomes of the research. I tried to make clear that my purpose was not to judge what I saw, but sincerely to find out as much as possible about what motivated it and its implications. However, my focus and questions, may have prompted Ben to justify his actions in terms of his perception of my expectations. There seemed to be some way in which he did not regard what he was about to do in the vectors lesson as *investigative*. He referred to it as *didactic*, and other than his normal style.

The same day, I had observed Ben take a lesson for an absent teacher involving the topic of probability. Students had experienced difficulty in using certain formulae relating to probability. This is mentioned in the conversation between Ben and myself from which Extract 1 is taken.

<u>"Very didactic"</u>

(1) Ben: [The lesson will be] very didactic, I've got to say, compared to my normal style. But we'll see what comes out. There's still a way of working though, isn't there?

BJ: That's something that I would like to follow up because you say it almost apologetically.

Ben: Yeah, cos I / Yeah, I do. Erm We're back to this management of learning, aren't we?

BJ: Are we?

(5) Ben: Can I read what I put here? [He refers to his own written words about a concept which we called 'management of learning'] I put here, "I like to be a manager of learning as opposed to a manager of knowledge", and I suppose that's what I mean by didactic – giving the knowledge out.

BJ: Mm. What does,'giving the knowledge' mean, or imply?

Ben: Sharing my knowledge with people. I'm not sure you can share knowledge. Mathematical knowledge is something you have to fit into your own mathematical model. I've told you about what I feel mathematics is?

BJ: Go on.

(9) Ben: I feel in my head I have a system of mathematics. I don't know what it looks like but it's there, and whenever I learn a new bit of mathematics I have to find somewhere that that fits in. It might not just fit in one place, it might actually connect up a lot of places as well. When I share things it's very difficult because I can't actually share my mathematical model or whatever you want to call it, because

> that's special to me. It's special to me because of my experiences. So, I suppose I'm not a giver of knowledge because I like to let people fit their knowledge into their model because only then does it make sense to them. Maybe that's why if you actually say, 'Well probability is easy. It's just this over this.', it doesn't make sense because it's got nowhere to fit. That's what I feel didactic teaching is a lot about, isn't it? Giving this knowledge, sharing your knowledge with people, which is not possible?

Extract (1) from transcript of discussion with Ben before the vectors lesson (23.11.88)

Ben seemed to be saying that if we offer probability to students as simply a formula, "this over this" it is likely to have little meaning for students because they have no means of 'fitting' it into their experience. He seems to use the term 'fit' in a radical constructivist sense (von Glasersfeld, 1984). He speaks of knowledge in terms of personal construction. He seems to eschew any absolute sense of knowledge (*giving* knowledge is not possible). He also makes problematic the communication of knowledge (*sharing* knowledge is not possible).

Ben seemed to offer a spontaneous articulation of a radical constructivist philosophy, emphasising the individual nature of knowledge. He saw a didactic style of teaching as being inappropriate to individual construction. I questioned what I saw as being a tension for him: that of didactic teaching approach versus constructivist philosophy. The above conversation continued as follows:

<u>"A conjecture which I agree with"</u>

(10) BJ: I'm going to push you by choosing an example. *Pythagoras* keeps popping up, and Pythagoras is something that you want all the kids in your group to know about. Now, in a sense there's some knowledge there that's referred to by the term 'Pythagoras'. And, I could pin you down even further to say what it is, you know, what *is* this thing called Pythagoras that you want them to know about?

Ben: My kids have made a conjecture about Pythagoras which I agree with. So, it's not my knowledge. It's their knowledge.

BJ: How did they come to that?

Ben: Because I set up a set of activities leading in that direction.

(15) BJ: Right, now what if they'd never got to what you class as being Pythagoras? Is it important enough to pursue it in some other way if they never actually get there?

Ben: Yeah.

BJ: What other ways are there of doing that?

Ben: ... I've always found in a group of students if I've given them an activity to lead somewhere there are some students who got there. It sounds horrible that. *Came up with a conjecture which is going to be useful for the future if I got there,* yes? And then you can start sharing it because students can then relate it to their experiences.

BJ: So, it's alright for them to share with each other, but not alright for you to share with them?

20 Ben: If I share with them I've got to be careful because I've got to share what I know *within those experiences.*

BJ: OK. So, if we come back to didactic teaching then, if you feel they're at a stage that you can fit – whatever it is that you want them to know about – into their experience, isn't it then alright? You know, take the probability example this morning. If you felt they'd got to a stage

(22) Ben: That is nearly a definition, isn't it? That is, I suppose that's one area I'm still sorting out in my own mind. Because things like $\overrightarrow{AB}$ and *vector* is a definition. What work do you do up to that definition?

Extract (2) from transcript of discussion with Ben before the vectors lesson (23.11.88)

The tension seemed to be between having some particular knowledge which he wanted students to gain, and the belief that he could not *give* them the knowledge. However, he encouraged students to share with each other in classroom discussion and negiotiation. Ben seemed, epistemologically, to be grappling with both radical and social constructivist positions while to some extent bound by absolutism. (See Ernest, 1991, for an account of these terms.)

The above conversation seemed to summarise his pedagogical approach – the presentation of activities through which the students could construct knowledge, and his monitoring of this construction, "My kids have made a conjecture about Pythagoras which I agree with. So, it's not my knowledge. It's their knowledge." Implicit in this is his need to know about their construction, to gain access to their construal. Students have to be able to express their thoughts in a coherent way for the teacher to make this assessment, so he has to manage the learning situation to encourage such expression – he distinguished between being a *manager of learning* and a *manager of knowledge* (statement 5).

In statement 22, he referred to a *definition*. The probability example involved a definition, as did the notion of vector and its representation as $\overrightarrow{AB}$. His remark, "I'm still sorting out in my own mind" seemed to refer to the status of a definition in terms of knowledge conveyance or construction, and indeed the nature of knowledge itself. There seemed to be some sense in which you could only *give* a definition. If this is the case, what preparation needs to be done so that the student is able to fit that definition meaningfully into their own experience? The tension here seems to be an example of what Edwards & Mercer (1987) call the *teacher's dilemma* . There is some concept which the teacher needs to elicit or to inculcate. However, inculcation is likely to result in lack of meaning, and eliciting of what the teacher wants may never occur. Statements 21 and 22 provide examples of teacher and researcher actively working on epistemological questions and the teacher overtly acknowledging his own developing thinking.

We see, therefore, a picture of students grappling with mathematics, working on tasks set by the teacher. The teacher's construction of tasks is designed to support the student's construction of mathematics. The teacher's personal philosophy of knowledge and learning is central to the construction of teaching, and the teacher actively questions his own theoretical position.

The social construction of knowledge

One characteristic of an investigative classroom approach was that students frequently worked in groups, that mathematical ideas were discussed between students and with the teacher, and that lessons involved active negotiation of meanings and understandings. I believe that it is through such overt negotiation of meaning that the teachers' dilemma is to some extent resolved. My perception was that groups developed a high level of common, or *taken-as-shared* meaning (Voigt, 1992). There was evidence of meanings growing through the activity and discourse of the classroom of which the teacher was a part. Individually students were developing their own meanings, but their understanding was strongly dependent on the classroom discourse. When it came to giving an account to the teacher, various students contributed, supporting or modifying what someone else had said. Within such an atmosphere of sharing, the teacher could offer his perspective

as just another voice in the social discourse. In practice, of course, the teacher's power being greater than that of individual students, the teacher had to be aware of the relative influence of his words when contributing explanation. However, there were many situations where a teacher's contribution seemed to be effective in enabling a student to make progress as evidenced by articulation of thinking.

As a result of such reflection on developing understanding through interaction, negotiation and sharing of meanings, I came to see that radical constructivism alone is an incomplete, and sometimes unhelpful theory to underpin classroom learning. It leaves unanswered many questions about the construction of knowledge and how students come to know the mathematical ideas the teacher wants them to address. Through a consideration of social construction, it is possible to perceive of knowledge growing within the social discourse with the consequent development of individual understanding. Neither radical nor social constructivism deals with the status of knowledge, either in terms of where the knowledge exists or its validity (Noddings, 1990). However, it seemed that, in an atmosphere of negotiation, understandings could be articulated and challenged, so that knowledge grew in a dynamic way as part of the socially-constructed classroom. A crucial part of the activity of the mathematics teacher was the creation of an environment in which mathematical tasks could be tackled and an atmosphere conducive to sharing and negotiation could flourish.

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Constructive teaching in mathematics

Jarkko Leino

Summary

One of the main trends in mathematics education is the shift from behaviorist and cognitivist to constructivist learning theory. Constructivist taching attempts to make learning constructive, i.e. to base teaching on the learner's previous experiences in forming new knowledge. The emphasis is on process rather than products and, hence, on student-centred methods.

Introduction

Constructivist learning theory is based on the now commonplace idea that knowledge is actively constructed by the learner. While behaviorist learning theory emphasised a suitable reinforcement on he learner's reactions and cognitivist theory the learner's information processes and strategies, the constructivist theory attempts to find out what kind of previous experiences and beliefs the learner already has and takes these to be the basis of teaching. The constructivist notion of learning leads to calls for a dramatic shift in classroom focus away from the traditional transmission model of teaching towards one that is much more complex and interactive (Prawat & Floden 1994). In response to these calls, various approaches have been suggested of how best to facilitate the knowledge construction process.

There have been intensive discussions among mathematics education researchers about the variations in constructivism (Leino 1993). In radical constructivism the experimental reality is irrevocably subjective, always constructed by the learner (Glaserfeld 1990). Thus, ulterior reality is inherently unknowable which makes all claims on learners' representations of something inaccessible. Learning something means that the learner experiences various conflicts of which some seem to be solved. On the viable knowledge the learner builds, at least temporarily, his/her understanding of the exterior world. This leads to competence based teaching in mathematics as well as in other subjects.

Knowledge cannot be transmitted from a person to another. This is very difficult for mathematicians to accept. They see that the true mathematics of mankind is the basis of all mathematics teaching. In social constructivism knowledge is shared rather than individual experience, a dialectical interplay of many minds, not just one mind. This fits better mathematics educators as could be heard in the discussion during ICME 7, Quebec. Though mathematical knowledge cannot be transmitted as such from teacher to student there are at least shared outcomes and longstanding truths which form the basis of subject matter centered teaching. It leads in teaching to a process of negotiation within classroom. However, there is little agreement about basic process: what aspects of knowledge best lend themselves to negotiation, and what it means to negotiate this knowledge (Prawat & Floden 1994). Social constructivism has strong links to Dewyan philosophy which attempted to balance subject-centered and naive learner-centred teaching. The structure of knowledge e.g. in mathematics was the goal of teaching but this goal was not attained by the teacher presenting mathematics. Learning is the learners' process and only well-organized activities can be educative (Dewey 1938).

From behaviorism to constructivism

In traditional teaching the teacher is responsible for providing the information to the learner and covering the content according to the curriculum. Textbooks, lectures, and drill-and-practice exercises suited well for the purpose. As an objective learning theory behaviorism (cf. Skinner 1954) gives the basis for instruction: the learner is supposed to show his/her learning in terms of changed behavior so that the instructor can reinforce it. If instructional materials or modern technology are used in teaching they have to serve the same information providing function, i.e. special books or software programs loaded with information and short control tasks are favoured. Knowledge is seen as fixed and external to the learner who has to memorize the facts and skills included. As a result the student may learn the information but not usually in a personal or deep way. The information has seldom any other meaning to the student than that to be memorized if asked.This kind of learning easily becomes boring and unmotivating. Knowledge in school is also separated from the learner's everyday knowledge at home and from other information sources.

During the past two decades the educational scientists have increasingly taken constructivism as the basic theory for meaningful learning. Constructivism (e.g. Magoon 1977) takes the learner's previous experiences as the necessary basis for all new learning, The learner constructs the meaning through experience and develops personal theories about the physical and social life. Knowledge is an internal personal representation of objective reality (Stodddart & Niederhauser 1993). The learner brings his/her experiences and beliefs also to the content to be learned in school. These preconceptions form a filter through which he/she approaches the new topic. Learning means a change in the learner's conceptul perspective through which the facts, principles and practice in teaching-learning process are personally understood. Authentic experiences are more important and powerful in learning than the teacher's speech or scientific theories. Because the learner's experiences are individual and situated, so is also the learning process. A good teaching has to be contextual which means that the learner's active engagement with real world situations must be provided. Hence hand-on inquiry-oriented instruction is advocated.

The cognitivist theory of learning (cf. e.g. Neisser 1967) can be regarded as a version of constructivist approach. However, the usual notion of knowledge of cognitivist researchers seems to be traditional, based on the correspondense criterion of truth. The main concern has been in how the learner processes external stimuli presented to him/her and what kind of strategies and metacognitions he/she uses. Individual basis for knowledge construction has had a minimal role. Thus the cognitivist learning theory has not come into the view as a genuine constructivist theory and it has had a very little influence on mathematics teaching. Mathematics curriculum, with its traditional syllabus, concepts and operations, has kept its previous importance and position in teaching.

I would like to repeat the main differences between the learning theories mentioned above which still are influential in classroom. Behavioral learning theory has no interest in the learner's internal processing because no perceivable evidense of that is available. Behaviorists and cognitivists see reality as external to the knower with the mind acting as a processor of input from reality. Cognitivists are interested in the learner's internal processing to

the extent to which it explains how reality is understood while constructivists see the reality determined by the experiences of the learner. Meaning is derived from the structure of reality. Learning is for constructivists a problem solving process based on the learner's personal discoveries and intrinsic motivation.

A proper constructivist view gives an alternative to the traditional information transmission if taken seriously. It sets demands on teaching very different from those of behaviorism or cognitivism. Goals and objectives cannot any more be set as fixed and static but rather to be negotiated with the learners. There is no best way for sequencing instruction, content-independent knowledge or skill. Neither can constructivistic evaluation be criterion-referenced to the extent it was in behaviorism and cognivism.

My somewhat hesitating attitude to advocate constructivism to mathematics teachers at the present comes from the difficulties of applying it and the radical changes it suggests or demands on teaching. Changes are always difficult in teaching. In order to get knowledge meaningful and viable to the learner teaching has to be based on the learners' thinking and to have references to their worlds. On the other hand, mathematics is a scientific field of its own, with the generally accepted structure. Mathematics teaching also has its old and strong tradition as to the method and content which means that mathematics educators know what is possible, important and necessary to teach on each grade and how this can be done in the prevailing conditions. Teaching is at least locally cumulative in the meaning that there are some basic concepts and operations, such as lines, numbers, adding, subtracting etc., which are rather close to the learners' reality and easy to learn, and then there are very general levels of mathematics which have only indirect references to reality and harder to learn through the easier mathematics. The learner's individual world of experiences and the world of important mathematics have to meet and fit each other. The main facilitator of learning is the teacher who has to fit the learners' individual worlds and the world of mathematics together in the classroom. There are also minor helpers, such as school mates and parents but they are not professional. If we have only good professional mathematics teachers I do not hesitate to advocate constructivist learning theory to be applied in classroom. I'll come back to the issue later on.

Constructive teaching

I call teaching constructive if it is based on the constructivist learning theory. Teaching constructively involves building models of learner thinking, an enterprise that demands evidense from multiple data sources, solving problems, working in groups, participating is class discussions, and performing suitable assessment (Confrey 1992). We all know how difficult it sometimes is to understand another person; and the teacher has a large group of learners at the same time. We also know what the parents and school authorities expect mathematics teaching to be and what materials have to be covered. These form the environment in which the teacher has to work. I am sure that many teachers want to be constructivist but because of the prevailing conditions they choose the traditional model. At the moment the constructive teaching and research on it are in a starting phase: gathering data about experimenting some approaches in practice and general conditions for constructive teaching.

In Finland several scientist of mathematics education have used constructivism as the basic theory and given their contributions to constructive teaching (see Ahtee & Pehkonen 1994). Leino introduced several variations of constructivism, Pehkonen penetrated into teachers' and students' belief systems, Keranto emphasised the importance of contextual mathematics teaching and learning, Björkqvist stressed the necessary change in evaluation, Haapasalo experimented one systematic approach in several topics and grades, Häggblom made materials within the framework of constructivism, to mention only a few of those interested.

Internationally the group of constructivist researchers is increasing but for instance in the world congress ICME 7 in 1992 the discussion focussed on the very basic level of the theory and philosophy of teaching. I dealt the question earlier and now to be brief I only say that there are several variations of constructivism and, hence, a number of good approaches to constructive teaching. Constructive teaching starts with the learners' experiences about the topic, theme, concept, operation or any other object which may be the focus of learning. To the teacher, the constructive teaching is an attitude to accept this starting point. It is followed by various possibilities to organise teaching in a way which gives room for the learners' experiences. There are numerous ways to carry it out.

One teacher can first organise discussions about the object, another negotiate with the learners a project work. Sometimes students can verbally express their beliefs and preconceptions about the object as the start for the teacher, sometimes students' work reveals these better. Teachers have always used students products as the basis for guessing how students have understood the teaching object and applied it in their exercises. Participating in discussions and working with the students give the teacher a more complete view and an opportunity to help students when needed. The constructive teaching favours students' activities and other ways which reveal students beliefs and pre-conceptions.

Of vital importance for good constructive teaching is providing access to several resources from which students can get the information and the new impulses they need to solve their acute problems. Textbooks can be used in various ways to start a project or other student centred activities but also modern technology gives magnificient new tools for the same purpose because it is process orienting.

I had an opportunity to organize a topic group of Teaching mathematics through project work in ICME 7, Quebec. Though these contributions (see Leino 1992) seldom explicitly included any learning theory, with the exception of two papers, they all presented experiments which can be said to base on constructivism. Motivating students was the most frequently mentioned argument to use project work but also the fact that through project work mathematics became more meaningful, process-orienting and open to questions than in the traditional teaching. Meaningfulness seemed to need invividual approaches where the students' previous experiences could be used and combined with the mathematical concepts and operations. This feature also moves the view of knowledge in mathematics away from absolute to human-made, ever-changing, imperfect and corrigible.

What was also mentioned in several contributions was the need to move from the traditional product assessment to process assessment. E.g. Haines & Izard (1992) emphasized assessment strategies which cover knowledge, understanding, skills and personal qualities such as the development of initiative, taking responsibility for learning, and applying problem solving strategies. They had developed a checklist of important aspects in learning so that the students did not omit these issues in project work. These aspects

were such as how clearly objectives were stated, how the main features of the task were identified to meet the objectives, what kind of simplifying assumptions were made, what relevant variables were selected in the model, how the projects were presented, etc. (See also Berry1990.)

There are several examples and models of constructive approaches in mathematics teaching. Thus the teachers need not start with zero. However, these models are meant to be for the teacher a starting point of personal reflection. Every teacher, new student group, new topic and new conditions need a new reflection. As Ormell (1992) put it: if a project has been successful this year it does not guarantee success next year to the same teacher, same grade, same topic, and same environment. Though the teacher has conceptualised the idea of the constructive teaching he/she has to be ready to meet surprises. The basic idea of constructive teaching is that one focusses the students' minds by posing a specific problem, a set of problems or an area of problems, sufficiently hard to be worth solving, and sufficiently easy to look as if it could be successfully investigated by them. The problem or task the students perceive at the beginning is not a fixed constant; having found a good problem and started to work on it, individually or in small groups, the students need to feel that they are making some sort of progress on it. I would like to recommend small group work because then students learn from each other and the process presentations are few enough to be seriously discussed.

How to choose the topic, problem area, or other educative task is a question which demands creativity from the teacher. As a principle I recommend the curriculum, whatever it may be, to be flexibly followed.

Also the curriculum itself should be flexible. It can be seen as a network of ideas, larger topics and projects which form flexible sequences for promoting students' understanding and educational development. The teacher's task is to plan what a part of the curriculum means within an individual school, and how to work it out together with the students. The teacher is more a coach or guide than an information presenter. This does not mean that scientific concepts can be forgotten. The focus of teaching is on practical activities, but the goals are in the growth of understanding. To solve an acute problem the learner needs tools; and scientific concepts can serve as such, introduced at the right moment, i.e.

when needed. Prawat has described in detail this process and used Vygotsky's concept of "zone of proximal development" for the purpose (Vygotsky 1978, 86; Prawat 1993).

As I have emphasised there is no special way to teach constructively. As a consequence of this statement I can only give examples of the attempts of how the constructive teaching has been organized. A collection of these attempts is already published (e.g. Leino 1992; Ahtee & Pehkonen 1994).

Difficulties in constructive teaching

In the beginning of my presentation I hesitated in advocating constructivism as the basic learning theory of mathematics teaching. Last year I even asked if constructivist teaching is possible at all (Leino 1994). I would like to come back to this theme.

One of main argument to keep the traditional transmission of mathematics alive is the fact that it is simple and clear. The teacher only covers the fixed content of the curriculum and uses the traditional means to evaluate student learning. Everything can be organised in advance, curricular content, materials, teacher training, evaluation, and control. Objectives are public, parents understand the system because they have gone it through etc. The minor problems are that mathematics teaching is boring, learning results are not as good as they could be, almost nothing changes in school though the society with its new demands does change all the time; and teachers, students and parents can stick to their absolute view of knowledge in mathematics.

The constructive teaching is difficult, and it continuously demands the teacher to search for new solutions as the basis of a new group of individual students. Teaching is teacher and student dependent, materials have to be used flexibly, different information sources are recommended, teacher training has to move to teacher education to emphasize professional reflection, and besides the product criterion the process assessments are to be used. The curriculum cannot be a fixed collection of content to be covered but only a general frame of topics, and a network of important concepts and ideas. The teachers of an individual school have to plan the curriculum of their school on the basis of the foundations by the national authorities and each individual teacher has to plan how best to organise teaching accordingly in his/her classroom.

Constructive teaching is difficult, needs good reflective teachers and the teacher's whole personality, creativity, and a view of what mathematics is important in changing society and educative for his/her students. Mathematical knowledge, pedagogical knowledge, student characteristics and the environmental context of learning have to be integrated. But when all these conditions are fulfilled teaching is rewarding and creates enthusiasm both in the teacher and the students alike. Research on constructive teaching has not reached the level of proper help to the teacher. However, performance assessment is one interesting topic of study in which open problems are frequently used, and, in addition to students' products, also their verbal descriptions as well as justifications are included. Though these studies have often been carried out within the frame of cognitivist learning theory they have already shown that all performance assessments do not measure the same thing, "achievement", and are not equally exchangeable (Baxter & Shavelson 1994). Students' verbal descriptions are especially revealing to the researcher as well as to the teacher. They also give students opportunities to link the topic to be learnt with their everyday experiences and beliefs. This is just what the constructivist teacher needs.

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PAPER SESSIONS

Control activities at problem solving

András Ambrus

One of the most neglected phase on problemsolving process is the control. The questions posed by Polya, Mason, Schoenfeld are too general, they do not give enough help for the teacher. After a short theoretical overview we will discuss in this paper some concrete control activities (methods) with the aim to help for the teaching practice.

Theoretical overview

In the problemsolving phases Carrying out the plan and Looking back poses Polya (1973) the following questions, recommendations: Carrying out your solution, check each step. Can you see clearly that the step is correct? Can you prove that it is correct? Can you check the result? Can you check the argument? Can you derive the result differently? Can you see it at a glance? Mason proposes in the Review phase the next activities: check calculations, arguments to ensure that computations are appropriate, consequences of conclusions to see if they are reasonable, that the resolution fits the question. According Schoenfeld (1985) the global decisions regarding the select and implementation of resources and strategies (Planning, monitoring and assesment, decision making, conscious metacognitive acts) belong to the control activities. These models seem to be linear: Read - Decide - Solve - Examine. But these models do not capture the dynamic inquiry involved in problemsolving, including the managarial processes of self-monitoring, self-regulating and self-assesment. Wilson (1993) presents a dynamic and cyclic interpretation of Polyas stages. In this model the Looking back stage is the main part of the control activities, but we can see from figur below that this stage has connections with the whole problemsolving process.

Mathematics teacher-students about the control methods used in the school

I asked 5 mathematics teacher-students (4th year) at the Eotvos Lorand University about control methods. The task of the students was to collect the control methods they know. The results: The numbers show how many students have chosen the mentioned method.

Substitution at solving of equations: 10

Construction at geometrical computational tasks: 6

Estimation: 2

Use of the computer vs. pocket calculator: 1

Graphical solution (equalities, inequalities): 1

Investigation of special cases: 2

Comparison with the result of an analogous problem: 1

Does the result belong to the definitions domain?: 1

The result of the neighbour: 1

Mental computation: 1

Is the result reasonable (practical problems)?: 1

The use of equivalent equations: 1

Solving the problem in a different way: 1

We can see from the datas, that the mathematics didactics courses must pay more attention to control methods.

Some control methods for the teaching practice

1. The use of inverse operations
2. Substitution of the result (equations)
3. Estimation. At realistic problems: is the result reasonable?
4. Complex exercices. The task of the students is to fill in the table and then to choose 3 from the 9 results, so that from each column and row we have chosen one number. Multiplying these 3 numbers every student must have the same result. Why?
5. Control with a geometrical construction. At the geometrical computation tasks often reasonable to try to constrct the intended

figure. Analysing the construction we may find that he problem has more solutions or does not have any solution.

Example: In ABCD trapezoid (AB parallel to BC) we can describe a circle with radius r. The angle BAD is 60 The diagonal AC builds with the side CD an angle 90 and bisect the angle BAD. What is the area of the trapezoid?

Most of the students assume: There is a solution, and with help of trigonometry they get the solution. But if we try to construct the trapezoid, we will see there is no solution.

6. Dimension test. At quantity-formulas the same unit must stand on both sides.

7. Symmetry-principle. If in an algebraic expression the variables are symmetrical to each other, then after the transformations of this expression these variables must remain symmetrical.

8. Control with argumentation. The use of the indirect arguments is often useful.

9. Control of the solutions of stereometrical problems with help of concrete models.

10. Control with help of a graphical representation of the solution. Example: the solution of the algebraical problems with help of the analytical geometry. (Graphical solution of a system of equations.)

This method is a special case of the control through another solution.

11. The visual representation often helps to control the whole solving process.

Example: The sum of the first n positiv integer.

$$\begin{array}{c} 1+2+3+4+5+6+7 \\ \underline{7+6+5+4+3+2+1} \\ 8+8+8+8+8+8+8 \end{array}$$

Concluding remarks

Most of the Hungarian students often begin performing calculations without giving much thought to the problem. Students often pay to much attention to the surface features of the problem (context or type and the size of the given numbers). A lot of

students often pursue unfruitful directions without monitoring or assessing their knowledge or activities. The main aim of the control activities in the problemsolving process is to convince ourselves, the result indeed correctly answers the question, in the solution-process we took the right steps. The control gives security for the problemsolver. But most of the Hungarian students are not interested in possessing greater security. It is a great task of the teacher to make students feel more responsible for their work.

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Interactive Exercises as a Part of Mathematical Hypermedia

Kostadin Antchev, Seppo Pohjolainen, Jari Multisilta

Summary

We discuss about computer aided exercises and present Exercise Maker-program which has been designed as a part of hypermedia learning environment for mathematics at Tampere University of Technology. Exercise Maker (EM) is an interactive exercise environment for students. Questions, several hint levels, and the answers are given in the ordinary mathematical notation. We use Mathematica as computational engine both to generate exercises and to check the answers.

Introduction

Mathematical hypermedia enables creation of mathematical virtual reality on a computer where mathematics can be studied with aid of hypertext, graphics, animation, digitised videos etc. Particular design and implementation of software that enables creation of hypermedia based learning environment for mathematics is discussed in Multisilta et al. (1994) and Pohjolainen et al. (1994). The software has been used to develop pilot hypermedia course on Matrix Algebra. One of the most attractive features of the hypermedia is that students can follow their own way while studying. The more freedom they have the greater is the need for feedback about their progress and competence with essential skills. That is a good reason for including exercises as a part of the hypermedia. Indeed, we are going to use the answers to some of those exercises as input data for an intelligent tutoring system which will guide students throughout the materials.

In general, a hypermedia database contains two different types of information, namely nodes and links between nodes. There are many different types of nodes in our hypermedia database. EM is the program that handles exercise-nodes. Links to exercise-nodes can be made from words or icons within text nodes or a number of exercises can be grouped together in exercise-collection-nodes.

Exercise Maker was designed using object oriented approach and implemented with Think Class Library, part of Symantec C++ development environment for Macintosh computers [Symantec

(1993a) and (1993b)]. It supports two different types of exercises referred to as theoretical and computer-verified exercises, respectively.

Theoretical Exercises

Most of the exercises in the Matrix Algebra hyper-course are theoretical, where students are asked to show or to prove something. To help them two kinds of hints are given. The first hint level explains the problem in more detail, the second reveals the structure and results needed in the proof. Finally, complete solutions are given to some exercises.

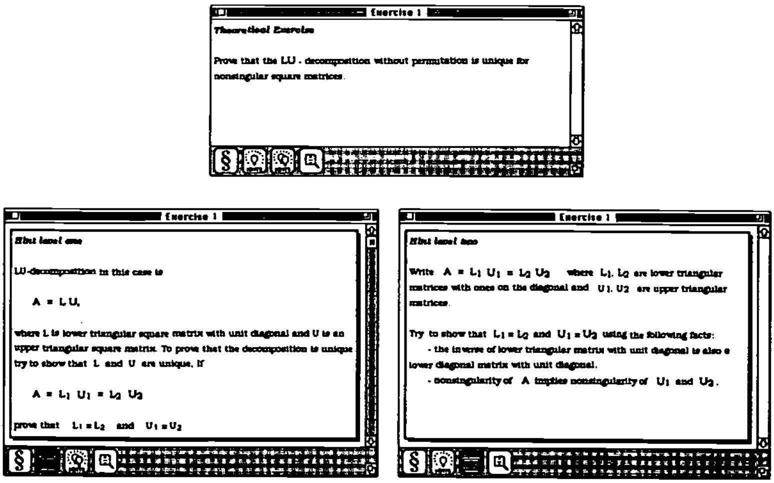
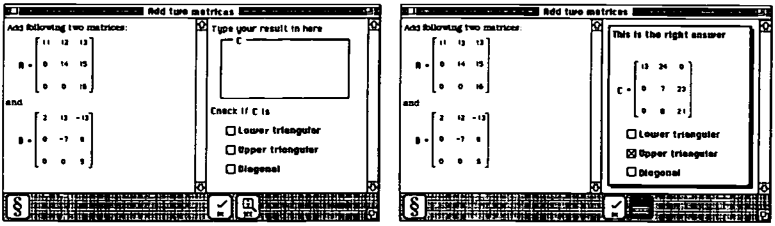


Fig. 1. A theoretical exercise. The first and second hint levels are obtained by pressing corresponding buttons

Each hint level is accessible by pressing a button in the bottom toolbar of the window (see Fig. 1). The question and all hint levels are originally written in separate sections of a MS Word document. They may include pictures, formulas, etc., as needed.

Computer-verified exercises

The second kind of exercises addressed by EM are those that could be formally verified by computer. For instance, basic computing with vectors and matrices, various decompositions, etc. It is also possible to pose questions that should be answered by making appropriate choices. An example is given in Fig. 2.



a. students can give their answer into the template and have it checked

b. the same window after SEE button has been pressed

Fig. 2. A computer-verified exercise displayed by EM

In general, those exercises include the following actions

- Generating values
- Handling student's interactions
- Checking the answer

The main characteristic of our approach is the use of Mathematica [Wolfram (1988)] to "randomly" generate values for the exercises and then to check the answer. In fact, we have infinite number of exercises on a topic. For instance, by giving "New Values" menu command students can reopen the same exercise with a new set of values (matrices) as many times as needed. The use of Mathematica makes it possible to verify exercises which have numerical as well as symbolic answer. Also, to some extent it is possible to cope with different mathematically equivalent expressions in order to decide whether an answer is correct.

The role of EM is to isolate students from the "complexities" of Mathematica and to provide consistent, self-explanatory user interface. The answer should be given by students into answer templates that explicitly show what the answer is expected to be (on the right-hand side of the window in Fig. 2.a). An author of

computer-verified exercise, to be run by EM, has to design it as collection of GUI objects. The types of those objects have to be among the types "known" to EM. For Matrix Algebra the "known" types are *Matrix, MatrixInputBox, CheckBoxes, and RadioControls* [Pohjolainen et al. (1994)]. For instance, in Fig. 2 there are two objects of type Matrix, one of type MatrixInputBox, and one of type CheckBoxes. The later two objects form the answer template for this exercise.

EM defines a few simple conventions that has to be followed by the author, so that the collection of objects for particular exercise can be defined as Mathematica expression, and to ensure that the answers are properly transferred to Mathematica for checking. A computer verified exercise is created as an ordinary Mathematica notebook, that contains two programs written according to the set of conventions. The first program is executed by EM whenever the exercise is opened or "New Values" menu command is given. Its purpose is to define the objects involved in the exercise and to provide values for them according to certain "random" generation scheme. The second program is executed by EM whenever checking of the answer is initiated. Its purpose is to verify the answer, and if it is not correct to return relevant message to be displayed to students. For Matrix Algebra we have developed a Mathematica package, that significantly simplifies the task of writing those two programs.

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Assessment in mathematics from a social constructivist viewpoint

Ole Björkqvist

Summary

Assessment in mathematics has a specific character when seen from a social constructivist viewpoint. Viability of knowledge is a criterion for quality both at the individual and at the social level. Goals and their relationship with the modes of assessment are made explicit. The role of the teacher is emphasized.

On social constructivism

Many a scientific theory has existed in several different varieties during its first years of existence. Such is the case for constructivism, being a broad theory with an epistemological as well as a psychological component. Its application to mathematics and science education has entailed consideration of the nature of those sciences, and, particularly for the variety called *social constructivism,* the nature of the interaction between different individuals engaged in an educational enterprise.

It appears that all varities of constructivism agree that knowledge is something that is actively *constructed* by the student - it is not passively received from the environment - and that the process is adaptive, organizing the experiences of the student. *Transmission* of knowledge from one person to another is an impossibility or a contradiction in terms.

Regardless which variety of constructivism a mathematics educator adheres to, assessment in mathematics constitutes a problem. Existing curricula and teaching methods now reflect quite a bit of consciously constructivistic philosophy, while at the same time assessment methods, as they are implemented in classroom, are traditional and not far from a behavioristic philosophy. There is thus a problem of alignment between curricula, teaching methods, and assessment. And, if a teacher should want to achieve such an alignment, there remains the problem of formulating criteria to be used in formal assessment, *acceptable to society.* In what way can you e.g. compare students with each other with respect to their proficiency in mathematics without resort to behavioral objectives?

Social constructivism is concerned with the interaction between individual knowledge and collective knowledge. Neither will be defined here - the reader is referred to Björkqvist (1993) for a discussion that also refers to Ernest's (1991) view of individual and "objective" knowledge. It is assumed that collective knowledge is built up from contributions by individuals, and that the construction of individual knowledge is influenced by collective knowledge.

For a discussion of assessment, it is important to note that collective knowledge *is carried by* a group or a specific society. If the group is shattered, the collective knowledge may disappear. On the other hand, collective knowledge *supports* a group or a specific society. If the collective knowledge disappears, the group may go extinct.

The interdependence can be expressed in terms of survival. Collective knowledge is for human survival - and collective knowledge itself survives only if it is available again and again for new individuals as they construct their personal knowledge. This "Darwinian" view of knowledge implies that the knowledge that is the most viable is the knowledge that will survive the longest.

A conclusion to be drawn from this is that, in assessment, *viability of knowledge is a criterion for quality.* This is true for individual as well as collective knowledge.

Mathematics education and its goals.

Social constructivism emphasizes the unique capacity of man to receive abstract information through oral or written communication. That kind of information can be efficiently used in the personal construction of knowledge. Abstract knowledge allows for a large variety of applications to problems that have to be solved in life. This does not diminish the value of context-bound knowledge, but the versatility of abstract knowledge has given mathematics a high status in society. It epitomizes the unique capability of man to invest in future knowledge which will help him as a species in a variety of ways.

Mathematics is a collection of concepts and methods that have proved viable in society, and it has made immense contributions to the shaping of society. It cannot neglect its responsibility for the future shaping of it.

> We face more and more cases of life under fear, hatred and violations of the basic principles upon which civilization rests. A return to barbarism and even the inviability of our species are possibilities for the future. This has everything to do with Education, in particular with Mathematics Education. We recognize that Mathematics has so much to do with the modern world, with everything that is identified with advanced science and high technology, that Mathematics indeed created the possibilities of such development. Besides, Mathematics has understood standards of precision, rigor and truth which permeate every facet of modern thought. (D'Ambrosio, 1993, p.31)

The goals of education, including mathematical education, are derived from survival of the individual and society. It means considering, e.g., the role of mathematics for science and technology, the construction of new mathematics in society, mathematics and democracy, gender and mathematics, etc. Mathematics education, just like mathematics itself, should be *dynamic* and respond to changes in society.

In particular, there is a rising need to assess the dynamic aspects of the personal mathematics of the students. This includes assessing the processes by which students create knowledge that is new to them and assessing the quality of their constructs with respect to viability in foreseeable and unforeseeable future situations.

Assessment of the relatively static aspects of the mathematics of the students should not disappear. However, the aim is not to assess reproductive ability, but rather to promote communicability and interaction between individual and collective mathematical knowledge.

Assessment is not only directed towards the achievement of educational goals. Awareness of the modes of assessment also steers the actions of all the persons involved: *You get what you assess, and you do not get what you do not assess* (Resnick & Resnick, 1991).

Assessing mathematics at the individual level.

If a student is to develop a realistic picture of his or her proficiency in mathematics, self-assessment should be encouraged. Self-assessment, however, may be viewed as predominantly social. The picture one has of one's own knowledge is largely derived from communication with important others.

In general, the personal beliefs with regard to mathematics are shaped in a process that involves much more than "pure" mathematics. There is, e.g., a considerable amount of pragmatism at the individual level. The social context very often is such as to present mathematics as a useful subject, a tool. Investment in mathematical knowledge then is a personal investment in the future. Mathematics may be just a means towards other ends. Self-assessment and formal assessment of mathematical knowledge must acknowledge that.

This also affects personal conceptions of what is meant by understanding mathematics. It appears to favor a non-objectivist view of understanding (Sfard, 1994). According to such a view, symbols or experiences do not carry meaning. Symbols or experiences are rather interpreted (given meaning) through understanding.

Mental schemata organize and prepare "for future use", for future interpretation, the essence of our previous experiences. Schemata are usually non-propositional, e.g. *metaphors*.

In fact, Sfard prefers to see metaphors as central to the construction of all mathematical concepts, the *metaphor of an object* being the end result of a process of *reification:*

1. A sequence of operations is repeated on numerous occasions.
2. The sequence is condensed as an operational concept.
3. The operational concept is thought of as an object, a structural concept.

Assessing individual knowledge then involves finding out what metaphors an individual uses to construct meaning for symbols and new experiences. In which way do mathematical structural concepts, operational concepts, and metaphors from everyday life interact? What metaphors are the most viable ones for the individual, now and in the future?

The role of the teacher

Social constructivism puts much emphasis on the teacher, not only as a facilitator of learning, but as a person with a social responsibility. The teacher is the representative selected by society

to keep the culture of mathematics available to the students, as they individually construct their mathematical knowledge.

Considering the changing nature of collective knowledge and the conditions of society, this is not a definite, but a provisional position of authority - a matter of confidence in a professional.

The teacher is expected to have visions of the social development of mathematics and to actually put those visions into use in the planning of educational activities and in the assessment of the outcomes of them.

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Responsibility for own learning in mathematics

Trygve Breiteig

Summary

This paper is a preliminary report from investigations based on tests and observations of four classes of secondary school over a three years period, 13-16 years. The aim is to investigate conceptual development, and describe the progress by observing conceptual obstacles and misconceptions. In their teaching, the students have used some special tactics to stimulate metacognitive activities, as means to develop responsibility of own learning.

The research task

Research on teaching and learning mathematics during the last decades has extensively studied students' development of central concepts like numbers, operations on numbers, probability, variable, function, symmetry etc. Typical development is described by characteristic levels of understanding and misconceptions are identified. Interrelationships between learning, teaching, social, emotional, cognitive and metacognitive factors are brought into focus.

Such studies have shown that the effect on the learning process is clearly influenced by the students' active cooperation in the learning process. The aim of this study is to investigate the students' learning in mathematics, and focus on their cognitive development, their awareness of learning, their metacognitive abilities, like monitoring and control of their own learning as individuals and as a group.

It is a hypothesis that in order to obtain these abilities, it is necessary to have tools to control own learning and to be engaged in metacognitive reflections.

We have tested and observed about 110 students over a three years period of time, from their grade 7 to 9 in Norwegian schools, i e from the age 13 to 16. We have investigated their learning, with respect to cognitive, metacognitive and affective aspects, attitudes and beliefs towards mathematics. Special tactics are used in some of the lessons, in order to let them do metacognitive reflections, and we will study the effect of these.

Written tests on decimal numbers, functions and graphs, algebra and probability have been used in collecting informations on conceptual understanding in these areas, and questionnaires has been developed – which focus on thoughts on mathematics and on the learning of mathematics.

Diagnostic tasks – some results

The tests used are *diagnostic*, and common misconceptions will be exposed during the solving process. Diagnostic tasks are developed on our knowledge of common misconceptions and thus depend heavily on research on learning of mathematics. A large body of research material is available. See for example overviews of research Dickson *et al* (1991) or Grouwes (1992).

Each diagnostic task is aimed at a conceptual problem or a common misconception. A comparison between ordinary and diagnostic tasks may illustrate the special function of the diagnostic tasks. They are critical questions in the study of the development of mathematical ideas.

"Ordinary" task	*Diagnostic task*
0.4 x 0.3	0.4 x 0.2
What number is the largest of 0.45 0.68 0.31	What number is the largest of 0.274 0.6 0.85
Write a story to fit 18 : 3 = 6	Write a story to fit 13 : 3.25 = 4

Answers to some diagnostic tasks at age 13 is shown in the following table.

Task		*Answer*	*Percentage*
1	Write the answer to 0.12 : 2	0.06	*36*
		0.6	52
2	Write the answer to 0.6 : 0.2	3	*21*
		0.3	54
		0.4	10
3	What number is the largest of	38 : 0.17	*46*
	38 x 0.17 and 38 : 0.17	38 x 0.17	45
4	A box with 25 necklaces weights	3 : 25	*35*
	3 kg. How can we find the	25 : 3	42
	weight of one necklace?	25 x 3	9
		3 : 25 and 25 : 3	9
5	The price of the bananas is	10,50 : 13,50	*13*
	13.50 kr per kg. How much	13,50 : 10,50	38
	can Ann buy for 10.50 kr?	13,50 – 10,50	17
		13,50 x 10,50	8
6	What number is the largest of	0.65	*47*
	0.236 0.4 0.65	0.236	31
		0.4	21
7	1 kg sausages costs kr 49.50. How	49,50 x 1.7	*81*
	can we find the price of 1.7 kg?	49,50 : 1.7	12
8	1 kg minced meat costs kr 65.50.	65.50 x 0,76	*35*
	How can we find the price	65.50 : 0,76	42
	of 0.76 kg?		

The idea that a decimal number is a *pair of integers*, separated by a point, is obvious in 1 and 2, of which the latter (quotition type) has lowest facility. Similarly the "rule" that *multiplication makes bigger* is exposed in 3. The idea that you *divide the larger number by the smaller* also impacts the choice of operation, as is shown in 4 and 5. We also notice the characteristic drop in facility when the multiplier becomes smaller than 1, see 7 and 8. Misconceptions like *division makes smaller* thus guide the choice of operation in many cases.

The partitive model is the only model for division for many students, as can be seen from the following story, to fit the task 17 : 4.25 = 4.

> Frank played 4 whole matches and 22.5 minutes in another match. He got a total of 17 points on the player's score. He had 4 in average.

For some the essential feature of the operation is to *do* it. The following example may be typical, for the case 5.25 x 3.28 = 17.22.

> Arne had a calculation going like this: 5.25 x 3.28. Help Arne to find the answer to this task.

Teaching built on knowledge or misconceptions

The teacher may use misconceptions constructively, as vehicle for promoting understanding. The diagnostic tasks may promote reflective discussion in the class, where conflict may be exposed and resolved. It may bridge procedural and conceptional knowledge. The focus upon concepts may also give the students a tool in planning their own work. An awareness of own learning, own progress and own concepts is developed.

Mathematical competencies constitutes a complex body of knowledge: skills, conceptual structures, ability to do investigative creative work, to do problem solving using heuristic strategies, ability to mathematize a situation and be critical to models and arguments, building proper attitudes and confidence. At this point however, we emphasize the dichotomy of *procedural* knowledge versus *conceptual* knowledge, as discussed by Hiebert and Wearne (1986). The procedural knowledge is concerning the formal language, the symbols, the syntactic rules. Contrary, conceptual knowledge shows many relationships and connections between components of knowlegde, like a spiders web.

This distinction can clearly be seen in the students' responses on decimals. Students construct erroneous rules without reference to the conceptual content or the meaning of the symbols. They have a tendency to overgeneralize concepts from a familiar domain in order to interpret a new domain. When the number domain is extended to decimals, multiplication and division have to be conceptually rebuilt and revised, to assimilate new knowledge.

Metacognition – students' attitudes and beliefs

Metacognition refers to knowledge about own thinking, concepts about learning, reflection upon actions, – self monitoring, self instruction and control. During the last years research on metacognition has grown rapidly, which is documented e g by Schoenfeld (1992) or by Tanner & Jones (1993).

In some of the lessons in our project the students did activities that aimed to stimulate metacognitive reflections. Such tactics include

- writing log (What is the most important idea you have learned in this topic? How did you learn it? Make a task to see if a friend has learned it?...)

- student made tests, students mark a anonymus or a fictive student's test

- conceptual maps

- constructive use of misconceptions, cognitiv conflict and discussion

- variation of teaching methods. Change of point of view

- open ended investigations, including a written 'journal'

We wanted to investigate thoughts on learning by posing questions on attitudes and beliefs. The students were invited to react to different statements on attitude aspects by grading from "strongly disagree" to "strongly agree". As an example here is given the results from one of the classes.

- I can put my own ideas into maths. I sometimes feel as if I'm inventing something	1.6
- I usually find maths easy	2.6
- Maths is very useful for me	3.2
- I usually enjoy maths	2.1
- I feel as if I'm really making progress in maths	2.3
- I work hard in maths lessons	2.2
- It is my own responsibility to learn maths	2.1

(0 = strongly disagree, 4 = strongly agree).

In the final report we will study this further, also gender differences. It is also a further task to study the effect of these means of initiating metacognitive reflections, raise the awareness of own learning, and how they might influence the students' mathematical competencies.

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A comparison of school leaving math exams

Gard Brekke, Algirdas Zabulionis

Summary

Some comparisons of secondary school leaving mathematics examination in countries surrounding the Baltic Sea area are presented. The paper includes analysis of the system of examination and description of structure. The main goal is to examine the mathematical content and performance expectations of the single exam paper. Two frameworks are developed for this purpose, one for mathematical content and one for performance expectations. Some diagrams on the structure of examination and tables showing how the examinations cover these frameworks are presented.

Background

To examine something means to inspect it closely. Examination or exam in school/university is in dictionaries defined as testing of knowledge or ability, to examine individual's knowledge and skills. Examination is linked to documenting, which means gathering tangible evidence to demonstrate the existence of knowledge or what occurred, and testing means to create an ordeal to inform decisions about acceptance or rejection, passing or failing.

Historically exams in this manner were known in China more 2000 years ago. Notes about oral tests are also found in the Old Testament. In the medieval Europe it is used equivalent to disputation, and by the end of the last century it has become synonymous with the written test, and later also oral tests. The first exams in mathematics in Norway were introduce at the university at the mineralogical study, (see Froyland 1965). A complicated item of the 1835 exam was:

> How can the function Log (1+x) be expressed as series to the power of x, and how be it possible to use this series to develop other to which "with easier calculate every numbers logarithm"?

It is important to emphasize that every exam task shows some level of knowledge set by the society to be fulfilled by the individual. Such levels should nowadays be passed by a large number of individuals. Exam tasks will therefore give us much information even without knowing the exam results.

Currently there are many comparative studies of the systems of education, involving many different countries. One of the largest of such studies is TIMSS, which deals with curriculum and achievement in mathematics and science of different levels of the schooling. That study does not touch the important aspect about requirements set by different country at the final stage of mathematical education. The influence of high stake assessment on the content of teaching and learning is well known. Our understanding of what is important in mathematical competence is changing, resulting in curricular developments. The current emphasis appears to be on encouraging flexibility and adaptability in thinking, which should imply that teaching needs to concentrate on how students plan, make decisions and solve problems, rather than whether they are able to reproduce chunks of mathematics. This requires that we evaluate students' thought processes as well as their knowledge of facts and skills. Generally though, assessment is not keeping pace with such changes in mathematical education. In this perspective it is important to consider which parts of mathematical competence school leaving exams addresses.

The framework

Two frameworks are used for the analysis. One is a modified version of the TIMSS mathematics performance expectation framework, and the another one is a mathematical content framework. Every question which students are invited to answer is in this paper named an item, and coded according to these two frameworks. The content is classified into eight main categories:

MATHEMATICAL CONTENT FRAMEWORK (basis only)	
1. Numbers	5. Functions
2. Plane Geometry	6. Calculus
3. Solid Geometry	7. Stochastic
4. Algebra	8. Data Representation

Each of these main categories is separated into subcategorizes, and several of the subcategorizes are again split into categories on a third level. To give an idea of the system, the framework for Calculus (number 6) is presented bellow:

*6. **CALCULUS**: 61 Limits; 62 Continuity; 63 Differentiation (631 Calculating, 632 Applications); 64 Integration (641 Indefinite integral - calculating, 642 Definite integral - calculating, 643 Applications); 65 Differential equations.*

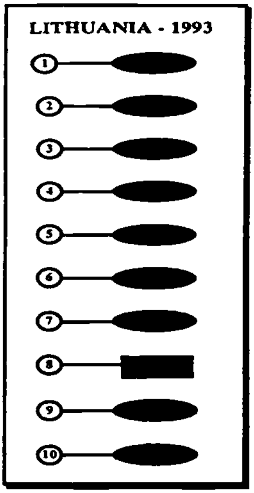
The framework for the performance expectation has four main codes, which are subdivided into categories on a second level:

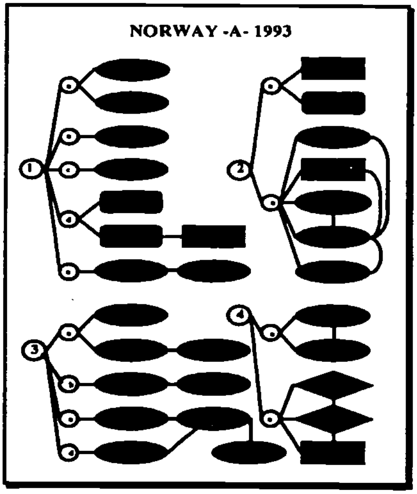
PERFORMANCE EXPECTATION FRAMEWORK (basis only)

1. Definitions, recalling and knowing mathematical objects (with subcods for investigating, discussing mathematical results);
2. Using routine mathematical procedures (with subcods: 21 Single procedure - solving, calculating and transforming; 22 Combining procedures; 23 Graphing);
3. Mathematical modeling;
4. Mathematical reasoning, proving, generalization.

Some examples of results

I. First the structure of exam papers could be studied. In the figures below Lithuania and Norway are given as two examples. Each item from the exam paper in this figures represented by a geometrical symbol according to performance expectations. Let say all items (except 8th) from Lithuanian exam paper are asking to use a single routine mathematical procedure (21) and only the one 8th is for proving. A few last items from Norway (4b) are about math modeling. Lines linking symbols show that one item is dependent on the solution of the previous one. We notice the difference in complexity of these two papers. It should be emphasized that this does not tell us anything of the difficulty of the paper.





Next table shows the part of items that falls into the different content categories (in percentages of total number of items in the exam paper, the first row indicates the numbers from the math content framework):

	1	2	3	4	5	6	7	8
Norway A			31	8	36	25		
Norway G		10	8	49	33			
Latvia	8	23		54	15			
Lithuania	10		10	70	10	10		
Finland A	12		8	20	36	2	4	
Finland G	24	10	3	17	28	10	7	
Denmark A		26	17		26	30		
Denmark G	3	26		10	39	10	13	
Russia G				50	12	38		
Russia H				50	12	38		
Russia A	12			38	12	38		
Poland A			17	8	33	33	8	
Poland G		27	9	27	27		9	
Sweden		29	4	17	25	25		

Note. The letters A, G and H stand for the type of math examination (if here are any in the country): A - advanced, G - general, H - humanitarian streams.

The diversity of the content in the exam papers is quite large: from Finland with nearly all main categories to Russia with only algebra, functions and calculus. We also observe that all these countries have something about functions in their papers.

II. In the next table the performance expectations are summarized in the same way. From this table it is obvious that most of the items are about routine procedures, for some countries all the time is spent on such items. It's quite difficult to comment this table - everything is evident. The mathematical examination is testing what is teaching in the school. Therefore we could get something about the need of school math education reforms (and the directions for that) from this information as well. The first row of the table linked with the math performance expectation framework, all other figures are the percentages of total number of items in the whole exam paper)

	1	21	22	23	3	4
Norway A	10	67		3	6	13
Norway G	9	75	9	3		
Latvia	9	36	45	9		
Lithuania		95				5
Finland A	10	66	3		14	7
Finland G	12	63	6		19	
Denmark A		68	9		18	5
Denmark G	4	61	4	13	4	13
Russia G		67	33			
Russia H		67	33			
Russia A		50	33	17		
Poland A	12	33	33	22		
Poland G		67	17			17
Sweden	7	33	33	13	7	7

It's difficult to show a lot of intertesting information about math exams in these five pages. The main goal of this project is to

compare the exams papers, and the project is not finished. This is only first attempt to analyze information. The report on some selected countries of this project will be published by the US Department of Education, Office of Educational Research and Improvement in 1995. This office has made this study possible by a grant. The joint Lithuanian-Norwegian project will publish a detailed analysis of the system of examination in the countries around the Baltic Sea and the Nordic countries in 1995.

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The General Education of Children in Mathematics Lessons

Günter Graumann

Summary

Mathematic subjects by themselves are not the goals of mathematics lessons. Rather the general education children are learning by working with mathematics is the sense of mathematics teaching. Therefore reflections on mathematics and mathematical thinking are necessary.

Remarkable points are: Mathematics as system of concepts which give us clarifications and orientations, mathematical thinking as special way to work on problems, mathematical knowledge and skills as tools, the method of simulation, limitations of mathematical modelling and the aesthetical effects of mathematical objects.

Mathematical teaching then can be reflected under these point in five dimensions: Pragmatic dim., Enlightment dim., Social dim., Dim. of personal factors and Critical dim.

Since the middle of the 80th in Germany general education is discussed again because the science-orientated teaching in the 70th gave no orientation and no midpoint to integrate the lot of learned details. So also in didactics of mathematics the discussion about general education was forced again.

In the following I first want to present some reflections on mathematics and mathematical thinking and then later on give some more hints for general education in mathematics lessons sorted into five dimensions of goals.

1. A fundamental characteristic of human beings is that they organize life with help of concepts. We know of course that with concepts and modellings we never catch our empirical knowledge totally, but we have no other chance to get orientation for our life. In addition we get the possibility to reflect about experiences of the past and to make plans for the future. *Mathematics* now is to be seen as a specially elaborated system of concepts which put us clarifications and orientations at our disposal.

2. Traditionally mathematics concentrated on the aspects of quantification and spatial form. However in modern mathematics the abstract forms, structural aspects and funtional connections are

more important. With this we also touch a point of everyday life. Just in our world of today the structuring of fields of experiences is a necessary task to master life in an appropriate way. Therefore *mathematics education* can supply us a special way of thinking for better understanding of our world and better ability to act. Just mathematics built a wide field for experiences of connections and funtional relationships. Moreover alive mathematics demand not only linear, algorithmic thinking (as it is often to be seen in school unfortunately) but also creativity, problem-solving, more-dimensional thinking, looking out for connections and structures as well as finding and holding out till the end systems to solve problems. I think this to learn especially today is a very important task for everybody because of the lot of things we get presented bit by bit.

3. *Mathematical* knowledge and skills as widely known can be used as tools to solve problems of everyday. How broad normal people should learn this has to be discussed more precisely. At the moment I only want to point out that the modelling of world and the application of mathematics are things that have to be learned extra, in addition to mathematical knowledge and skills. Moreover in a certain degree it is also possible to train *general abilities* of problem solving, creativity and multi-dimentional thinking already by acting with pure mathematical problems.

4. Recently a new practice of solving problems in science and technology - the method of simulation - gains importance more and more. Actually at ancient times it was possible to predict certain effects of the realisation of special plans with help of mathematics, but with the modern computers we can simulate complex events which can't be calculated. This simulations then give us ideas of the possible consequences. So we can think about consequences without real experiences which may have bad unrepairable effects. In *mathematics lessons* we can learn such methods already on a smale scale e.g. by discussing different systematic varations in a given word problem (even without using a computer) or by simulating lotteries (with a random generator). In upper grades we could simulate a more complecated situation with a computer e.g. different predicted growths of mankind.

5. Not for all problems respectively all situations a solution with help of mathematics is appropriate. Therefore the reflection on senseful limitation of mathematical modelling is also a necessary topic of learning mathematics.

This begins in grade 1 with dicussing that calculating money does not constitute the whole life or with clarifying that the problem "5 - 7" is unsolvable and can end with reflecting on chaos-theory.

6. Furthermore in mathematics especially you can make experiences that human mind starting with everyday problems can come to pure theoretical questions and that pure theoretical questions can fascinate men. This only speculativ aspect of human mind is also one part of our life and should not be left behind in the education. In *mathematics* we find this aspect first and foremost with looking out for rules, structures and mathematical aesthetics.

7. Last but not least in respect to our reflections about mathematics and mathematical thinking it has to be mentioned that *parts of mathematics*, especially in geometry, have normal aestetical effects. The feeling of aethetics is also one aspect of life which should be developed in school.

I hope you got already an idea of the points at which we have to pay attention in respect to general education in mathematics education.

Now I want to give you some more hints which I sorted into *five dimensions of general goals of school*, namely

1. Pracmatic dimension (that is: what we directly need or what can help us in everyday world)

2. Enlightment dimension (that is: for better understanding of our world)

3. Social dimension (that is: becomming an active member in groups and working with on social problems)

4. Dimension of personal factors (that is: all those abilities we need to become an integrated person)

5. Dimension of critical reflection (that is: reflection of our acting and the limits techniques we are using)

In the following for each dimension I will give a short explanation and some suitable examples of mathematics education.

Ad 1 (Pracmatic dimension): The first task of school is to equip every person with knowledge, skills and abilities so that he or she can master his or her present and predicted future life in an apropiate way. Here the terms "life" and "master life" have to be seen in a wide sense. Thus the pragamtic dimension of general education

doesn't include only solving problems of everyday life. Also special ways of thinking as already mentioned and abilities of communication have to be included. But for all of that we have to aquire clear and funded concepts. For *mathematics teaching* therefore first of all it is important to learn concepts not only by one definition but also by using them in connection with problem solving and underlining different characterisations as well as clearing their different use in everyday world. In addition the pragmatic dimension includes skills of calculation and drawing as well as abilities of modelling and of applying mathematics in problems of reality as already mentioned.

Ad 2 (Enlightment dimension): If we want to use mathematical concepts, structures and techniques properly we have to know them in a good way. But to open the world for somebody also knowledge of objects and connections which can not be used in everyday world directly are necessary. For *mathematics teaching* only some headwords may explain this: Symmetry and ornaments, different growth of special functions, behaviar of dynamic systems, principles of computers, basic ideas of probality and the role real analysis plays in physics and technology.

Furthermore the so-called "cultural coherence" belong to the dimension of enlightment. For mathematics education this means that we also have to teach e.g. about the genesis of our calendar, numerals and ornaments in prehistoric times, the origin of proofs and its connection with religion or philosophy in the ancient world, the way of positional notations of numbers from Babylon to India to Arabia and finally to Europe, the development of algebra during the renaissance and its influence for music, and last but not least the discovery of non-euclidian geometry and the arise of modern structural thinking in mathematics. Also pure theoretical questions like distribution of prime numbers, the pythagoreen tripels, the proofs of irrationality of special roots, the mightness of real numbers and the extension of euclidian geometry to projective geometry belong to here.

Ad 3 (Social dimension): First of all social objectives concern the way of company with each other and paying respect to any other person (children included). These objectives are not directly related to mathematics learning but to the the atmosphere in mathematics lessons. Secondly in everyday world more and more problems are solved in a team. Therefore in school we have to develop abilities of

communication and coorperation and to train teamwork also during solving problems of mathematics and especially problems of everyday world as mentioned above.

Ad 4 (Dimension of personal factors): A fundamental task of school is to develop the personality of each child. Especially in mathematics lessons one should pay attention to the following points: Spatial imagination, visualisation and intution, ability of abstracting and structuring as well as finding out connections, logical thinking respectively ability of organisating abstract things, finding structures and holding out systems till the end, creativity and ability of discovery, playful handlings with abstract things, ability to find appropiate actings for complex situations.

Ad. 5 (Dimension of critical reflection): First of all we have to be aware of the limitation of human beings and their thinking. Especially we have to reflect on the danger of the extensive use of special tools, techniques and methods. In respect to *mathematics education* therefore the reflection on the limitation of mathematical modellings and tools are important. This can start with discussing the solutions of word problems in respect to reality in primary school and end with the treatment of mathematical expansion during history in upper secondary school.

Concluding remark

I hope I could stimulate you for more discussion about general education in mathematics lessons and fasten the opinion that the sense of mathematics teaching must be seen in the points of general education. Of course this has to be discussed more detailed especially in connection with concrete instructions.

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Gender and mathematics education. Theory into practice.

Barbro Grevholm

Summary

How do teachers construct their beliefs about gender and mathematics? Do teachers' beliefs and perceptions have any connection with accepted research results in the area? Swedish and international research results will be presented in an overview. An investigation about teachers' beliefs in relation to these research results will be presented.

Background

Research on gender and mathematics education has been accomplished for about two decades. Some results are common for many studies, others are more controversial. An important question, when discussing theory into practice, is to what degree the accepted research results are known to practitioners, teachers. Another question is what teachers can change, when they know about the results. I have carried out a study to find out about the beliefs and perceptions of teachers concerning gender and mathematics. Some results from this study will be presented.

Some of the most frequently researched areas on gender and mathematics education include students' differential participation in mathematics, gender differences in performance and attainment, teacher-student interactions in classrooms, students' beliefs and selfconfidence, students' attitudes to mathematics, students' expectations of results in mathematics and confidence in their ability. Several areas of research will be discussed below.

Performance

In Sweden girls obtain better marks in mathematics than boys both in compulsory school and in upper secondary school (Grevholm & Nilsson, 1994). Boys have slightly better results on written standardised tests in year 9 in compulsory school, but in spite of this girls get better end of school marks. This discrepancy between results in the form of marks and results on written standardised tests has been focused on by Kimball (1989). Ljung (1989) has

studied gender differences in mathematics in national, standardised tests in upper secondary school in Sweden. In the most advanced mathematics course girls were found to be better or as good as boys on average.

Gender differences in achievement are very small. They seem to be dissappearing over time (Emanuelsson & Fischbein, 1986; Feingold, 1988) and are presumably not worth the attention they have received in the debate. According to Hanna's (1989) reports in the International Evaluation of Achievement from 20 countries the differences in results between countries are more serious than between results of the genders.

Participation

Girls' participation in mathematics studies differs from boys' as soon as it is no longer a compulsory subject. In Sweden only one third of the students in the most advanced course in mathematics in upper secondary school are girls. The distribution is the same at university level in undergraduate courses. At graduate level very few women are found in mathematics (Grevholm, 1994; in press). The same situation is found in many other countries.

The problem with the low participation of girls in mathematics studies is serious because of its consequences for their choice of profession and career possibilities. Sweden, as many other countries, has severe problems of attracting enough students to science and technology (Grevholm, 1993). In this perspective girls are an important group, a potential group for such studies. Why girls are so reluctant is still not obvious, in spite of many attempts to change the situation.

Teacher-student interactions

Teacher-student interactions in classrooms have been studied by many researchers. An early report in Sweden of differential treatment in the classroom was made by Kilborn in 1979. He found that the girls as a group systematically get less contact from the teacher during mathematics lessons. Molloy (1990) has studied what happens when the teacher attempts to distribute her contacts equally to boys and girls. Engström (1994) found that the pattern that girls get much less time and attention from the teacher has not

disappeared. Fennema & Leder (1990) report the same pattern in international research. They give many references to earlier research that has indicated that teachers often interact differently with their male and female students, with males attracting more and qualitatively deeper interactions.

Pupils' selfconfidence

Pupils' selfconcept or selfconfidence in mathematics is considered an important factor, that influences the learning and attainments of mathematics. Linnanmäki (1994) has reported that girls in school-year 8 with low or average selfconfidence have considerably better results on average in mathematics tests than boys. Girls with high selfconfidence on the contrary have somewhat lower average results than boys. Leder (1988) reported gender differences in attitudes about mathematics, including confidence in ability to do mathematics, without attendant differences in performance for seventh-grade males and females. Engström (1994) found that more boys have a more positive selfconcept, although the teachers estimate girls and boys as equally good in mathematics.

Mathematics textbooks

Textbooks in mathematics for all levels in Sweden show a world that consists of at least twice as many men as women. Areskoug and Grevholm (1987) published an investigation of textbooks for years 6 to 9 that aroused intense discussions. The ensuing debate brought about revisions of most textbook series. However, renewed investigation in 1994 shows that this gender bias in pictures and names still exists in textbooks. The context of problems are also gender biased, being more often contexts with which boys are familiar (Areskoug & Grevholm 1987; Rönnbäck, 1992).

From many other countries the same situation concerning textbooks has been reported. No studies are known about the consequences of this differential picture of men and women in textbooks. Working with teacher education means that you are interested in how theory and research results can influence practice. The investigation I will discuss here is a result of this perspective.

The investigation

Teachers in randomly chosen schools were asked to respond to a set of statements. It was of course voluntary for teachers to fill in their answers. The contact person at each school has reported the degree of participation, which varies from 50 % to 90 %.

The findings for the first 9 statements follow in table 1 below.

The existence of certain teacher profiles in the material have not yet been investigated and there is much to be said about teachers written comments to the statements. Future steps must be to continue the research on gender and mathematics, find better ways to disseminate knowledge about the research results, find good intervention programs and implement them in different countries and introduce gender and education as an important field of knowledge in teacher education. The achievements will be important not only for girls, but for all pupils and for the subject mathematics as such.

Table 1. Answers from 68 teachers in upper secondary school

Statement	Yes	No	?	Omit
1 Mathematics is an equally important subject for girls and boys	67	1	-	-
2 Girls are as well suited for studies in mathematics as boys	65	1	1	1
3 Girls have as good marks as boys in mathematics	49	7	10	2
4 Mathematics teachers treat boys and girls in equal ways	47	16	4	1
5 Girls and boys have equally high expectations of their possibilities in mathematics	30	24	11	3
6 Girls have as good selfconfidence in mathematics as boys	21	38	6	3
7 In mathematics textbooks girls are shown and mentioned as often as boys	21	27	11	9
8 Contexts in mathematics problems deal with environments that girls are familiar with as often as they deal with environments that boys are familiar with	19	30	10	9
9 Girls are better at routine problems	20	25	19	4

Statement 1&2. The socially acceptable answer is yes. Not much can be said to this fact. Interviews may reveal variations in views.

The answers to statement 3 show that 28 % of the teachers are not aware of or have not accepted the facts that girls have as good marks as boys, maybe even a little bit better on average.

In the answers to statement 4 as many as 69 % of the teachers show that they have not internalised the fact that teachers treat girls and boys in different ways, although quite a lot has been written about it.

Statement 5. 44 % of the teachers do not know that girls often have lower expectations of their possibilities in mathematics education.

Statement 6. 31 % of the teachers do not know that girls tend to have less selfconfidence than boys in relation to mathematics.

Statement 7. Here 40 % of the teachers have knowledge about facts. The rest have a false picture or no decided opinion.

Statement 8. 28 % of the teachers have not seen that the contexts in problems are mirroring boys' interests more than girls' and the same number have no decided opinion.

Statement 9. "No" is the typical answer, but the choices "yes" and "don't know" have about the same number. This statement is special because here it is not so obvious how things really are. There are researchers who claim that girls are better on routine problems, but there are also those who say this is not true. My hypothesis is that the latter statement will prove to hold in the long run, but we can't say for sure now. It is good that this statement gets the highest number of "don't know" answers. Teachers beliefs' seem to mirror researchers' opinions.

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The Geometry behind the Cupola of the Cathedral of Florence

Ivan Tafteberg Jakobsen

Summary

An outline is given of a school project on the title theme, aimed at secondary school pupils in their last year but one. The project is explained as a specific case study of applications of geometry in a historical framework, and it puts special emphasis on the use of models of several kinds.

Motivation

One of the pressing problems in modern teaching of mathematics is to convey some understanding of the interaction between mathematics and other disciplines. The more recent and exciting applications of mathematics often require mathematical skills far above the secondary school level to be understood properly. The more modest and elementary applications on the other hand - often applications that were formerly used as pedagogical tools in the teaching of mathematical skills - are increasingly becoming "black box" applications because of the technological development. In surveying, for instance, smart electronical devices are nowadays able to perform surveying tasks that used to require some mastering of trigonometry. Inside mathematics itself the use of graphical pocket calculators to draw graphs is another example of an application that formerly required some practice in the methods of calculus.

Another problem in modern teaching of mathematics - a problem which seems equally pressing at least to me - is to make the students know and understand some of the history and evolution of mathematics. Words, methods, and problems are parts of a long tradition and often appear meaningless without knowledge of that tradition. More important, the knowledge of the interaction between mathematics and other disciplines as it took place yesterday may help the students to understand better the interaction of to-day. This is not only because our generation has the historical general view over the studied period, but also because the mathematics in question is often simpler and easier to grasp than the mathematics applied to-day.

The Renaissance is a period that is exceptionally suitable for case studies of this kind: the mathematics involved is not yet too intricate for secondary school students (although some of it is difficult enough, like the methods used to solve cubical equations), and there is a long and well-documented development in the scientific curiosity extending from the early experiments with mathematical constructions of perspective in the 1400s to Galileo Galilei's fundamental physical experiments in the early 1600s.

The purpose of this paper is to present an example of a Renaissance case study used as a school project in the secondary school, penultimate year (17-18 year olds). The detailed contents of the study can be found in my school project paper, Jakobsen (1994).

Contents of the project

The building of the cupola of the cathedral of Florence (the Santa Maria del Fiore) is one of the most ambitious Renaissance projects, and the story of its construction became a legend already shortly after it was finished. The large cathedral had been under construction since just before 1300, and as early as in the 1360s there were definite plans to cover the octagonal crossing with a huge cupola. Cupolas covering the crossing of a big church were not unknown in the area (they existed already in both Pisa and Siena), but they had never been built in that size since the construction of the enormous hemisphere of Pantheon in Rome.

It was only in 1420, after many years of heated discussions, that the vaulting of the octagon of the crossing was finally started, now with the rising architectural star Filippo Brunelleschi in charge. With Brunelleschi the new Renaissance emphasis on the individual genius, here the master mason par excellence, is coming into focus, and the stories that were told about him reflect this very clearly. For instance, the famous story about getting an egg to stand on its top, a story which is usually told about Columbus, is also told about Brunelleschi.

The Middle Ages had seen a great accumulation of em-pirical knowledge of building technology and this knowledge was available to Brunelleschi, but it was also exactly this knowledge which had made his predecessors shrink back from the task. They simply did not know how to go about constructing a cupola of such dimensions and being placed so high up above the ground.

It was known from experience that a cupola would exercise a considerable horizontal thrust on its understructure. The constructors were not able to calculate the size of this thrust at all, (it was only in the 1700s that the engineers could begin making calculations of statics), but they knew it could be detrimental to the understructure and make the whole thing collapse.

The cupola was nevertheless constructed and it still stands, so apparently the problems were solved satisfactorily. Brunelleschi cannot possibly have calculated how to build safely, but he must have had some qualitative ideas concerning the solution of the statical problems. Unfortunately he did not write down his thoughts about this, but we do actually possess his written description of the cupola building programme, written in 1420, i.e. before the construction. This programme contains a not very detailed geometrical description together with some directions concerning materials and technical details (Saalman (1980) p. 70ff, reproduced in my school project paper p. 45ff). Furthermore we possess a drawing from 1426 by one Gherardo Gherardi on a piece of parchment giving an outline of the profile of the cupola with some geometrical explanations (Saalman (1959) or Ricci (1983), reproduced in my school project paper p.52ff).

If one combines the information from these two early sources one gets a rather clear picture of the cupola profile, where the corner ribs follow a so-called "pointed fifth arc", i.e. a circle having four fifths of the cupola basis diameter (corner to corner) as radius (see fig.1).

Investigations of the brick-laying pattern of the cupola reveal three aspects of interest: 1) the lateral joints of the beds are oriented toward the vertical axis of the cupola, 2) the masonry beds have a radial slope inwards, and 3) the masonry appears to rise from the center of each side toward the corners. These aspects combined can be understood to indicate that the bricks were laid such that the working surface was always part of some inverted cone with its center on the vertical axis of the cupola (see fig.2).

The content of the project is then to follow the uncovering of the geometry behind the cupola in detail. On the way it becomes obvious to the students that the cupola could not be constructed without a precise preknowledge of the geometrical set-up. It also becomes clear that although the geometry is not all that intricate, it

is not at all obvious how the constructors on the scaffolding were able to ensure that the right pattern was followed.

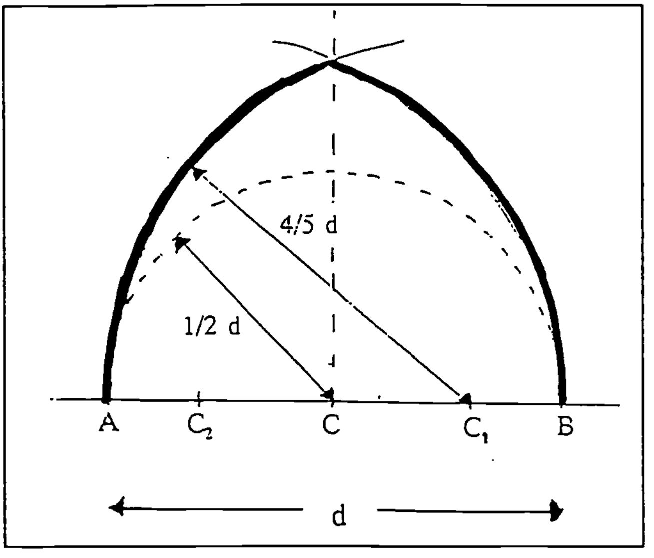

Fig. 1

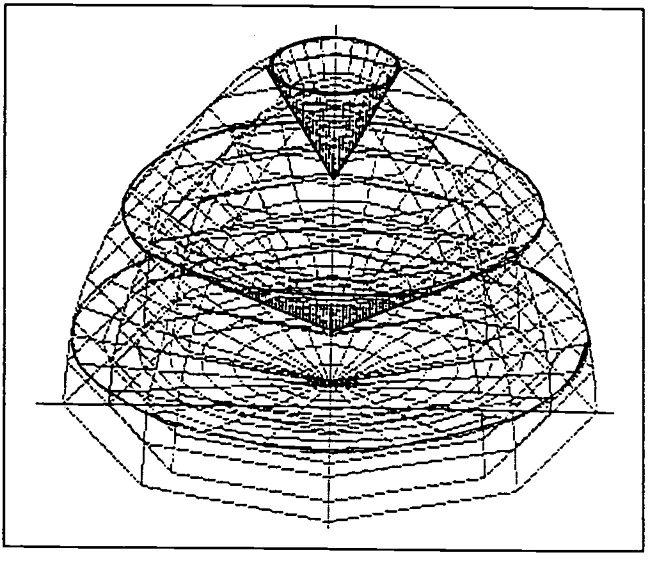

Fig. 2

Use of models then and now

It is known that 3-dimensional models of both wood and brick were made at several stages in order to make the geometry of the building clear. The drawing of the profile was mentioned earlier. This use of solid models in the construction process is supplemented with the modern use of models in order to understand the geometry and the statics of the building. Here the computer generated models offer a great help in making the 3-dimensional cupola geometry transparent and vivid. Thus the use of models is illustrated in a very natural way on a problem which is mathematically manageable for the students, but certainly not trivial, and a selection of models is taken into consideration, ranging from the bare geometric outline drawing to small scale solid models.

The computer models have to be constructed by means of 3-dimensional analytical geometry, which in Denmark is only part of the curriculum at the advanced level (last year). Once they are there, however, one can visualize very clearly facts that otherwise would have required a good many calculations (e.g. the intersection curve between a cone with a vertical axis and an elliptical cylinder with a sloping axis). Apart from this the analysis rests on elementary geometry and trigonometry. An analysis in keeping with the time of Brunelleschi would probably have to skip the trigonometry and instead only use similar right-angled triangles. This illustrates another often-encountered fact, namely that the mathematics actually applied is far less developed than the front-line mathematics of the time.

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Theory Into Practice of Teaching Mathematics

Tünde Kántor

Summary

Problems of the teaching of mathematics have psychological features too. The teaching of mathematics should acquaint the pupils with all aspects of mathematical activity, it should give opportunity for independent creative work as far as possible. We developed some resources for pupils aged 12-14. The topic of these resources for mathematics teaching is sport.

Resources for Mathematics Teaching : Olympic Games

The pack consists of Topic Sheets and of Teachers'Notes. (Tóth, Kántor, 1993) We have hoped that this material will provide pupils with motivation and stimulation to enhance effect of our teaching.

Children aged 12-14 are interested in many things, they want to develop their ego identity. Sport is loved by a lot of children. It is evident that sport is based on results, and results are based on numbers, so in this manner sport and mathematics are closely linked. The main goal in choosing this topic was that we would like if all pupils understood and came to like mathematics , became acquainted with its beauty and importance. The authors hoped that this approach will yield a very important and effective aspect of understanding mathematics. Children can easily assimilate mathematical problems in connection with real life situations. The authors constant aime was to encourage children to find out themselves, to understand facts, to investigate problems of everyday life and to give their mathematization. Long calculations have been avoided. It should not be necessary for the pupils to complete all exercises but the brighter and diligent pupils did it. The worksheets were designed with actuel pictures and with tables of data to challenge childrens curiosity and intelligence.

We followed the aspects which were developed by Tamás Varga in the OPI (Klein, 1987, pp.44)

In our worksheets we presented problems, but we encouraged the students to state their problems too. We did not use the question approach only, we wanted to teach them how to use reference work, how to gain any experiences with other ways of finding information. We asked our pupils to report on their out-of-class in-

vestigations (for example to make public opinion research about favourite sports, to measure their own results in different sports, etc.)

Treated Mathematical Topics Are

Measurement, change of units of measure, averages, percentage, ratio and proportion (direct, inverse), calculation of circumference and area, calculation of surface and volume, elements of combinatorics, elements of graphs, elements of functions.

Details from the booklet (Tóth - Kántor, 1993)

Introductory page :

What do you know about sports?

Do you know that the first modern Olympic Games were held in 1896 in Athens? From then the Olympics have been held every fourth year. Can you count how many Olympic Games have been held ?	Have you noticed that in 1996 it will be the 100th anniversary of the 1st Olympic Games? How can this fact help you in counting ?

If you get your result substract 3 from it, as in 1916, 1940 and 1944 the Olympic Games were because of the first and second world wars.

For which sport do you think we use the most muscles at the same time ?	In which sport do you think there exists the longest distance ?	Which sport do you think is the so called Queen of the sports ?

The students' answer for this introductory page was not as good as we hoped. They know better mathematics then sports. The majority of pupils gave wrong answers to two questions :

1. For which sport do you think we use the most muscles at the same time?

The wrong answers were: Pentathlon, wrestling, running. The wright answer is: swimming.

2. Which do you think is the so-called Queen of the sports?

The wrong answers were: Chess, football, fencing. The wright answer is: athletics.

We were surprised to see that the girls answered better than the boys.The best boy did not like sports.He told us that he want to deal only with mathematics.

B: Sport Survey

Choose 10 branches of sport from the Olympic sports (the Table will help you), and make a questionnaire. Ask your classmates how much they like those sports you have chosen. They can give points from 0-5, where 0 means that they do not like it at all.

The pupils were fond of the sheet B. To make a survey - was a totally new task for them. The exercises gave them an opportunity to make and evaluate a table by themselves which they will certainly need later in everyday life. Discussing the possibilities some of the pupils suggested and directed their attention to the fact that one table can tell us many different things. They got a possibility to read and understand tables , but these tables contained their own experiences, therefore they were understandable to them.

Comparing their tables they could evaluate other facts too:

Which was the most popular sport in the class? The whole class compared the results here. The answer was: swimming.

Which was the least popular sport in the class? Conclusion was that they have quite different opinion.

How many pupils like sports in the class ? Who likes sports the most?

Besides these we encountered the calculation of averages and we could practise the concept of the arithmetical mean.Since these mathematical concepts had a practical and understandable background it was easy to generalize them on the basis of exercises.

I: Who will be the next?

At the Olympic Games in Barcelona the National American basketball team, which consisted of professional players (eg. Larry Bird), was in one group with the following countries: Angola, Brasil,

Croatia,Germany and Spain. All of these groups play matches with all groups to find out who the first,the second, etc. will be in the group. On the seventh day of the Olympics the situation was the following:

America has played with Angola, Croatia, Germany. Angola with America, Brazil, Croatia, Germany. Brazil with Angola, Germany. Spain. Germany with Angola, America, Brazil. Croatia with America, Angola,Spain. Spain with Brazil, Croatia.

Questions:

a. How can you substitute this long list with a figure ?

b. In another figure represent those matches that have not been played yet. Compare the two figures. What do you notice ?

c. Draw the lines of the second figure into the first figure with a colour pencil. Are there any two teams in the figure you got which are not connected? What is the conditioin of getting this type of figure ?

The sheet I helps the teachers with introducing graph theory. The following basic concepts can be taught in connection with the exercises: graphs, vertices, edges of graphs, complementary graphs, complete graphs. If the children did not illustrate the matches with a graph then the teacher or the clever students can suggest that there is a very simple solution with the help of marking the teams by little circles, if it is not enough then go on with connecting those teams which have already played with each other. If we ask the students to draw their different shaped figures onto the blackboard then we prepare the concept of isomorphism of graphs. The children enjoyed these problems, they had very good ideas (see the students' solutions)

Results

We observed that on the basis of real and interesting problems and applying motivated and creative learning methods it is easier to make pupils understand mathematical problems and mathematical concepts. We brought activity into mathematics and it gave a higher motivation. We constructed concrete situations for our pupils and with their interaction the understanding of mathematics became more effectively.

Solution of a boy (age 13)

Notations: America-USA, Angola-Ang, Brazil-Bra, Germany-Ger, Croatia-Cro, Spain-Spa.

a./-c./

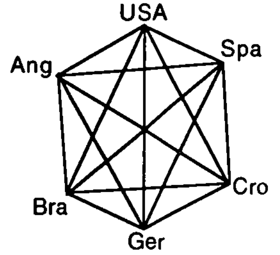

Fig. 1.

b./

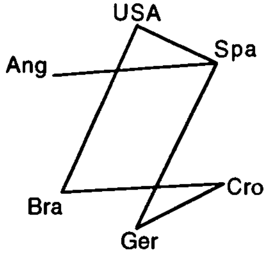

Fig. 2.

Solution of a girl (age 14)

x	USA	Spa	Bra	Ger	Cro	Ang
USA	x	-	-	+	+	+
Spa	-	x	+	-	+	-
Bra	-	+	x	+	-	+
Ger	+	-	+	x	-	+
Croa	+	+	-	-	x	+
Ang	+	-	+	+	+	x

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A heuristic alternative to formalistic teaching in school geometry: theoretical scrutiny

Tapio Keranto

Summary

The paper criticizes the present-day school geometry in Finland and outlines the opportunities which a heuristic approach could offer to the learners of geometry. The analysis is based on Imre Lakatos' ideas of the historical development of mathematics and of the nature of creative mathematical activity. The theoretical discussion is illustrated by examples.

Some critical notes

As we know, geometry and its teaching have a tradition of thousands of years. In fact, we have to go back to the times of Plato (about 400 B.C.) in the history of mankind to find its origins. Plato realized that the Pythagorean idea of the purely arithmetical essence of nature – with all things being in essence numbers – was defeated by the discovery of irrationals, and that a new mathematical method for description and explanation of the world was needed. Thus he encouraged the development of an autonomous geometrical method. Plato's aspiration found its fulfilment in the 'Elements' of the Platonist Euclid (about 300 B.C.) (for further details, see e.g. Kitcher 1984, Kline 1980).

Although Euclid's book was originally intended as an organon (instrument) of a theory of the world and not – as it is often assumed – as an exercise in pure geometry (see Kaila 1939, p. 36; Popper 1969, pp. 88–89), it has been adopted as a 'textbook of geometry' and a model for the planning of geometry teaching also in school mathematics.

In fact, the teaching of geometrical contents in the comprehensive school proceeds globally from points, lines and figures to the solids. To crown it all, the chapter dealing with space geometry (pp. 56–67) in a new Finnish mathematics textbook entitled Taso 9 (1991), for instance, ends up – just like the 'Elements' – with the regular polyhedra (see, pp. 64–65). It is also typical that the teaching of geometrical knowledge and skills has been designed locally according to the

"formalistic" pattern: first an example, then "theory" followed by "practice" (e.g. Taso 9; double pages, pp. 56–67).

It is not very difficult to infer that this kind of geometry teaching reduces the possibilities to discover or invent things by one's own self. It is not surprising that many pupils feel that school mathematics is meaningless and boring. This kind of ready-made mathematics deprives them of any opportunities to learn to organize subject matter, to learn to conceptualize, to learn to define, and to learn what the role and function of definition is in creative mathematical activity. It is not surprising that mathematics is mostly thought of in terms of calculation, as a pre-worked, infallible discipline in which even the discovery of new things proceeds deductively from basic propositions or axioms to theorems (for further details, see e.g. de Villiers 1986, Frank 1988, Thompson 1992).

Lakatos is in my opinion very informative in this respect (p. 142, note 2): *"It has not yet been sufficiently realised that present mathematical and scientific education is a hotbed of authoritarianism and is the worst enemy of independent and critical thought. While in mathematics this authoritarianism follows the deductivist pattern just described, in science it operates through the inductivist pattern."*.

One thing that I would add to this critique is that the transition from the first phase (a visual example) to the second phase (definition) in the "formalistic pattern" mentioned above is inductivist. According to the research of the "Vygotskyan school", this kind of generalization seems to lead to empirical concepts (e.g. "roundness" abstracted from some "round" objects) rather than theoretical concepts, the origin of which is in the action of their construction (e.g. the construction of the circle with a string) (for further details, see e.g. Davidov 1982, Engeström 1983, Schmittau 1992, Vygotsky 1962). Besides that one cannot get the second from the first: a leap is required (Davidov 1990).

A heuristic approach

On the basis of what has been said so far, there are weighty reasons to change the teaching of geometry towards creative mathematical activity. In his book 'Proofs and Refutations. The Logic of Mathematical Discovery' Lakatos recommends like many others (e.g. Freudenthal 1983, Polya 1957) that heuristic elements should be in-

cluded in mathematics education (p. 144): "*Heuristic style on the contrary highlights these factors. It emphasises the problem situation: it emphasises the 'logic' which gave birth to the new concept.*". Here Lakatos speaks about a heuristic approach which is obviously parallel with open ended problem solving outlined for instance by Hans Erik Borgersen (1994).

In fact much of Lakatos' book (pp. 1–126) has been written in the spirit of open problem solving in geometry and constructed around the historical development of the famous Descartes-Euler problem. Lakatos' aim is to show how mathematical knowledge evolves through a rich, in places dramatic process of successive improvements of creative conjectures, by attempts to 'prove' them and by criticism of these attempts – to put it briefly, by the logic of proofs and refutations. Because much of the book takes the form of a discussion between a very qualified and tolerant teacher and his students, it shows in a very vivid way how mathematical knowledge grows and how it might be taught heuristically. In a nutshell, the teaching-learning process in an imaginary classroom proceeds as follows.

In the spirit of discovery-learning, the students first try to find the relationship between vertices, faces and edges in the particular case of the regular polyhedra. After testing many kind formulas against the background of the calculated numerical data, they notice that for all these 'platonic' or 'cosmic' polyhedra $V+F=E+2$ (so called Descartes-Euler formula, first noticed in an explicit way by Euler 1758; see Lakatos 1976, p. 6, note 1). It it good to mention that in the textbook Taso 9 this formula – as expected – has been given in a ready-made form. The pupils' task is to "confirm" that the formula fits in with the numerical data calculated in the previous task (p. 65).

Back to Lakatos' imaginary class. One of the students guesses that the formula found for the regular polyhedra may apply for any polyhedron whatsoever. The others' task is to try to falsify the generalized conjecture by studying for instance prisms and pyramids just like Euler did in the 18th century (for further details, see Lakatos, p. 7, note 1). Because these tests corroborate the conjecture, students "dare" to try to prove it. New problems and proof-generated concepts, such as convexity, simply connectedness etc., are emerging incessantly from these efforts to provide a proof to the naive conjecture and from the analysis and criticism of the steps of proving by counterexmples (local and global).

Conclusions

Hence at least the students in Lakatos' imaginary classroom examined above were very eager to study mathematics in a heuristic way and learned a lot about the nature of discovery and the way of thinking in mathematics. They had the opportunity to learn the attitude of tolerance: we are all fallible, prone to error. They had opportunities to learn to discuss critically conjectures and proofs (cf. scientific or critical attitude towards myths, Popper 1969, 126–127). At the same time these students became familiar with a mathematical disposition: be ready to defend and refute your own and others' conjectures and arguments at the same time.

They also learned to recognize the complexity of overcoming contradiction in mathematics due to the diversity of the possible ways of dealing with counterexamples. For instance, a conjecture could be rejected straight away ("the strategy of surrender"). But other approaches are – of course – possible, such as trying to reject the counterexamples by using ad hoc redefinitions ("monster-barring" method; for further details, see Lakatos 1976; pp. 14–19), trying to improve the conjecture by using restrictive clauses or piecemeal exclusions (the so-called "exception-barring" method; pp. 24–30). Perhaps the most instructive and fruitful method from the mathematical point of view is to analyse carefully the proof in order to uncover possible 'guilty' or 'hidden lemmas' and then to incorporate them as new conditions in the primitive conjecture (pp. 33-42).

I have no doubts that such heuristic processes could also be implemented in the real life. I refer to the articles of Balacheff (1991), de Villiers (1991) and Borgersen (1994), among others. Of course there are many problems and challenges in the planning and implementation of such heuristic or open-ended problem solving processes in school geometry. For instance, an attitude of tolerance, mutual acceptance, safety and trust is necessary to solve problems and to discuss their solutions. Besides that the heuristic approach in elementary geometry, quite as emphasized by Borgersen (pp. 6–7), requires lots of work with concrete materials. It also takes much more time than ready-made mathematics (see Keranto 1994). But as I have tried to convince above, doing so is necessary and worth the while.

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DiffEqLab — A MATLAB Based Package for Studying Ordinary Differential Equations

Simo K. Kivelä

Abstract

DiffEqLab is a software package based on the interactive mathematical program Matlab and intended to the study of ordinary differential equtions at the level of basic mathematics courses in universities. The main ideas are discussed and the structure of the package is represented.

General background

The use of computers in the teaching of the basic mathematics courses at the university level has been very much discussed during the last years. There are different viewpoints on the subject:

- Computers and interactive mathematical programs can be used as tools in mathematical problem solving, very much like pocket calculators, except that they are much more powerful. Good examples of such programs are Matlab, Mathematica, Maple, etc. It has been customary to divide the programs in two groups, numeric and symbolic, but this difference is now vanishing.
- Computers can be a platform for mathematical demonstration and experimentation. The necessary programs can be separate pieces of software for the demonstration of one topic or they can be larger packages for the study of a whole course. They can be of stand-alone type or have general use programs (e.g. Matlab, Mathematica) as the basis.
- Computers can form the platform for courseware. They are used as electronic textbooks with demonstrations, interactive exercises, etc. or as encyclopedias with hypermedia type links. Media of different types are integrated: computing, pictures, sound, animation, etc. Software pieces or packages of the previous type can be used as parts of the courseware.

The package DiffEqLab falls in the second category. It is a Matlab based package for studying the solutions of ordinary differential equations of the form $y' = f(t,y)$, where y can be a vector valued

function. The objective is to understand the basic features of the behaviour of the solution curves.

The most important reasons for using a large interactive mathematical program as the basis are the following:

- Many good and well tested mathematical algorithms are available in the basis program and the programmer of the package need not write the code himself. For example, in the package DiffEgLab the Matlab algorithm for solving differential equations (initial value problems) numerically and testing possible singularities is used. Matlab has also tools for designing user interfaces.
- Except that the students of science or engineering must learn mathematics, they have to learn the main features of widely used mathematical software.
- At the beginning, when the students are not yet familiarized with the features of the basis software, it is reasonable to use menu driven demonstration packages. Later, when students know more about the software, they can utilize all features even inside the package.
- When the package is based on a widely used program, it can easily be ported to any platform where the program is available.

On the other hand, the use of a licenced program as basis brings one serious disadvantage: Although the package itself can be distributed freely, it has no use without the basis program. At the campus, this does not form any problem, because the basis program usually is available in the network of the university, but for own machines, students have to buy the program.

The structure of DiffEqLab

The package DiffEqLab is based on Matlab version 4.0 or later. It consists of 19 m-files and 10 predefined differential equations or systems of differential equations (m-files, too). The files take only 50 kB of disk space. The package is started by running one of the files, the Matlab script diffeq; all other files are called internally.

The kernel of the package is the file ode.m which computes the numerical solution of an initial value problem; it is a slight modification of the Matlab file ode23.m.

The current version works in PC-Windows and Unix/X-Windows environments.

The package is divided in two parts: the case of a single (scalar) equation of the first order and the case of a system of first order equations (including the scalar equations of higher order). At the beginning, the user has to choose which case he wants to study. Then he must select the equation or system, either a predefined one or he can write his own.

In the single equation case, the direction field is plotted in a window and the user must give the initial condition by mouse or numerically. Then, the solution both forward and backward is computed and plotted. Several solution curves (corresponding to different initial conditions) can be plotted in the same picture, which makes the comparisons easy. For giving the initial conditions and controlling the lay-out of the window, there is an additional window (called control panel) with mouse operated buttons etc.

In the case of a system of equations, two windows are opened: one for the time (the independent variable) dependence graphs of the unknown functions, and the other for the phase plane graph. Any two of the unknown functions can be put on the axes of the phase plane figure. The initial condition is given by mouse in a window or numerically. The solution curves are plotted in both of the windows; several solutions can be seen at the same time. Again, there is a control panel window, too.

If the order of the system is 2 (i.e. there are two equations and two unknown functions), the direction field is plotted in the phase plane figure. This is computed at the time of the initial condition. It gives information about the behaviour of the solution, if the system is autonomous, otherwise not.

In addition to the figure windows, the usual Matlab command window is open on the screen. This is used for two purposes:

First, it gives information to the user. When the user selects the equation or system he wants to study, the corresponding file is typed in the command window for checking. If the user clicks on some help button in a control panel, the help text is found in the command window. The Matlab error and warning messages are displayed in the command window.

Second, the command window can be used in the usual way: for giving Matlab commands. If the user wants to study the situation

more closely, in some way which is not covered by the tools of the package, he can do this in the command window.

The differential equation may also contain parameters whose values are changed during the study. If they are declared to be global variables of Matlab, their values can be changed in the command window.

More detailed information can be found in the reference [1].

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Computer programs

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Structural teaching [4]

Ricardas Kudzma

One can distinguish two methods of proofs, i.e. a linear and a structural. These notions may be extended for a presentation of a subject. Most textbooks on Calculus are examples of a linear presentation. We think that the idea of structural proof provides a lot of new opportunities for a teacher. Our goal is to rearrange the structure of Calculus course and to make it a collection of strongly motivated and interesting problems. We illustrate our attempts on the theme about sequences.

Introduction

The last two decades were distinguished by a too abstract teaching of mathematics both in high schools and universities. Textbooks became a long and boring series of theorems. Our main goal is the teaching of Calculus at university, but at first these ideas were realized for high school by Kudzmienë & Kudzma (1994).

We can mention here two resources which made some influence on our work.

The first resource is the paper by Leron (1983), where the idea of linear and structural proofs is found. "The basic idea of the structural method is to arrange the proof in levels, proceeding from the topdown; the levels themselves consist of short autonomous "modules", each embodying one major idea of the proof." We think that the idea of structural proof, maybe in slightly different sense, provides a lot of new opportunities for a teacher and fits well for our goals.

The second resource is the book by Griffits & Oldknow (1993). The first part of this book is a collection of amazing problems and is very impressive. We are going to illustrate our ideas by an example. It will be "Sequences", the first chapter of the Calculus course which author had at the Faculty of Mathematics, Vilnius

[4] The initial title of this presentation was Teaching of proofs. During the discusssions it was pointed out present name, which fits better.

University in 1993-94. These considerations are valid for high school as well.

An example of a linear presentation

Let us take any book on Calculus. For example: Zorich (1981). Let us take some topics from this book:

...

II Real Numbers

§1 Axioms of real numbers

1. Definition of a set of real numbers

(IV) Axiom of completeness (continuity) (1)

3. The axiom of completeness and existence of supremum (2)

§3. The main lemmas connected with the axiom of completeness

1. Lemma on imbedded intervals (Cauchy-Cantor principle) (3)

2. Lemma (Borel-Lebesque) on finite covering of an interval (4)

3. Lemma (Bolzano-Weierstrass) on a limit point (5)

III. Limit

§1. Limit of a sequence

3. Existence of a limit

a) Cauchy criteria (6)

b) Criteria of convergence for monotonic sequences (7)

...

This is a typical presentation of Calculus in contemporary text-book.

Structural presentation

At first we suggest the levels of presentation for sequences:

(i) problems;

(ii) mathematical models and simple finite formulas;

(iii) geometrical interpretations and arithmetical calculations;

(iv) strict definition of a limit and the simplest theorems on limits;

(v) existence theorems.

Then we take a problem and wander with it through these levels which one can imagine as the parallel planes.

Population growth

We start with the problem of population growth. We choose the Malthus geometric sequence model, i.e.

$$b_{n+1} = qb_n.$$

On this level we derive some finite formulas for b_n and partial sum s_n. We can use graphical interpretations ("cobwebs", for example) and arithmetical calculations. This leads to the intuitive notion of a limit. Later on we can give the strict definition of a limit. Applying Bernoulli inequality

$$(1 + a)^n > 1 + na \ , a > 0,$$

we can prove $\lim_{n\to\infty} q^n = 0 \quad \text{if} \quad 0 < q < 1 \quad \text{or} \quad \infty \quad \text{if} \quad q > 1.$

One can obtain the above result using the convergence of monotonic sequences (7), too.

Square root algorithm

Let us formulate the square root finding problem as a solution of the equation

$$x^2 = c, \quad c > 0.$$

Applying the Newton method we get the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right). \tag{8}$$

Using the statement (7) of the level (v), the monotonicity of the sequence $\{x_n\}$ and some arithmetical properties of limits we can prove that $\{x_n\}$ converges to the $\sqrt{c}$.

Here we can develop a more general theory. We can consider the relation (8) from the fixed-point theorem view. The Cauchy sequence notion and the property (6) do appear.

There is another interpretation of the formula (8). It comes from Heron's work. Separating the right of (8) into two parts

$$b_{n+1} = \frac{a_n + b_n}{2}, \quad a_{n+1} = \frac{c}{b_{n+1}},$$

we get that $\{b_n\}$ is decreasing and $\{a_n\}$ is increasing. Moreover, it is easy to prove the inequality

$$b_n - a_n < \frac{b_0 - a_0}{2^n}, \quad n \geq 1.$$

For completing the proof we need the lemma on imbedded intervals (3) of the level (v).

Population growth again

Malthus geometric law doesn't serve well in many cases. J.F.Verhulst in 1840 suggested another one which restricts the unbounded growth

$$x_{n+1} = a \cdot x_n - c \cdot x_n^2, \quad a > 0, \ c > 0.$$

After simplification we can get recurrence relation with one parameter only

$$x_{n+1} = b - x_n^2.$$

Very interesting effect of modern mathematics, such as attractors, bifurcations and chaos emerge. "Cobwebs" becomes a very important tool for investigating the behavior of the sequence and fits well for some values of parameter *b*. For other values of the parameter *b* powerful PC must be used because the convergence at the points of bifurcation is very slow. But the strict convergence should be proved using the classical tools of the level (v).

The of end of the section on sequences

After considering the above and other problems we return to the theory. Now we prove the equivalence of all existence statements (1) - (7). There is a need to prove one of them. Then one can introduce a new topic

(vi) construction of real numbers.

<u>Home work problem for students.</u> Take some books on Calculus. Which one of the statements (1) - (7) is taken as an axiom or it is proved at first? Find the order of proving the statements (1) - (7).

Concluding remarks

What do we win by such approach?

1. Calculus becomes more interesting both for teacher and for students.
2. Interesting problems increase motivation among students.
3. The structure of Calculus becomes more spatial. We can walk vertically through the different levels when solving a problem.
4. We do not need to prove everything at first. We can return to the same topic descending to the deeper level each time. Thus the teaching becomes similar to a spiral.

What do we lose?

1. Such method takes more time and the conflict with the curriculum may arise.
2. We lose some classical (the way we were taught) structure of Calculus.

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Mathematics teachers' beliefs and conceptions of mathematics education

Pekka Kupari

Summary

Mathematics as a branch of science as well as a school subject will arouse very different feelings, images and beliefs. In its utmost forms it can even be perceived as mysterious and magical by nature. Likewise, mathematics teachers's beliefs are of various kinds, and it is understood that they have a powerful impact on matematics teaching in practice.

In the present paper the author discusses the content and structure of mathematics teachers' conceptions, partly based on empirical research data from the Finnish comprehensive school in the spring of 1990. Some connections between the conceptions and the teaching practices are also examined.

Introduction

There exist important studies of teachers' thinking and decision making, that demonstrate how teachers' knowledge and beliefs influence the way they interpret and implement the curriculum (e.g. Clark & Peterson 1986). For this reason, the importance of studying teachers' beliefs and conceptions has been universally recognized.

In this paper I will briefly discuss the nature of mathematics teachers' beliefs and conceptions. In addition, drawing on empirical data collected in the Assessment 90-study, I will examine the relationship between comprehensive school teachers' conceptions of mathematics and their teaching practice.

Theoretical considerations

As the discussion framework of mathematics teachers' beliefs and conceptions I use Ernest's model (1991) which at the same time introduces the great importance of beliefs in mathematics education. First of all, I have to briefly introduce two central concepts of the topic.

The concept of *a belief system* can be used as a means or metaphor to examine and describe the way an individual's beliefs are orga-

nized (Green 1971). It is a dynamic system, which undergoes changes and reorganization, when individuals assess their beliefs on the basis of their own experiences.

The important characteristic of belief systems is the tendency of beliefs to form clusters, which are more or less isolated from other clusters and thus have no connection with other belief groups. The clustering prevents contradictions between beliefs and makes it possible to maintain conflicting beliefs. This property to form clusters may help in explaining certain inconsistencies related to teachers' beliefs, which have been reported in several studies (Cooney 1985, Thompson 1984).

Conceptions can be defined as higher-level beliefs. Conceptions have a more general structure - they include beliefs, meanings, concepts, arguments, rules, images, preferences, etc.

The main components of teachers' beliefs and conceptions can be presented in the form of the following three questions: (1) *What is mathematics as a science and a school subject?*, (2) *How is mathematics taught?*, and (3) *How does the learning of mathematics take place?* Teachers have accumulated and continue to accumulate these components during their own time at school, in teacher education, and while teaching in everyday school life.

Ernest's model (1991) shows, how teachers' view of the nature of mathematics provides a basis for the teachers' mental models of teaching and learning mathematics. The mental or espoused models of teaching and learning mathematics are transformed into classroom practices. These are the enacted models of teaching and learning mathematics, and the use of mathematics texts or materials. The espoused-enacted distinction is necessary because there can be a great disparity between a teacher's espoused and enacted models of teaching and learning mathematics (e.g. Cooney 1985).

During their transformation into practice two factors affect these beliefs: the level of teacher's thought, and the constraints and opportunities of the social context of teaching. Higher level thinking enables the teacher to reflect on the gap between beliefs and practice, and to narrow it down. The autonomy of the mathematics teacher depends on all three factors: beliefs, social context and level of thought. The social context clearly constrains

the teacher's freedom of choice and action, restricting the field of the teacher's autonomy.(Ernest 1991)

Finnish research of this topic has been quite scarce. Since the late 1980s, Erkki Pehkonen has investigated teachers' and teacher educators' beliefs and conceptions of mathematics teaching (e.g. Pehkonen 1994). He has also headed a research project that compares students' beliefs of mathematics in different countries. In 1991, Sinikka Lindgren studied Japanese elementary school teachers' beliefs of mathematics teaching in Tokyo and this provides a possibility to make some comparisons with the Finnish data.

Empirical research part

The Institute for Educational Research at the University of Jyväskylä conducted an extensive study *Assessment 90 of the Comprehensive School* in 1990-91. As a part of the mathematics study, teachers' conceptions of mathematics teaching were also examined. This part of the study was, however, fairly restricted and inventory-oriented. Answers were sought for the following research questions:

1. How can teachers' conceptions of mathematics and mathematics teaching be described?
2. How are the teachers' background factors related to their conceptions?
3. How are the differences of teachers' conceptions manifested in the accentuations or practices they apply in their mathematics teaching?

Teachers' conceptions were measured with the teacher questionnaire which consisted of 27 statements using the Likert scale. It was based on the questionnaire constructed by Pehkonen and Zimmermann, which they used in their Hamburg-Helsinki-project during 1989-92. Both elementary class teachers (grade 4: 56 teachers; grade 6: 52 teachers) and subject teachers from the lower stage of the comprehensive school (grade 9: 73 teachers) participated in the study in spring 1990.

Some results concerning teachers' conceptions

In the following, I focus my interest only on research question 3, discussing the relationship between teachers' conceptions of mathematics teaching and their instructional practices.

On the basis of the factorization of the conception variables and the earlier theoretical and empirical results (e.g. Dionne 1984), three conception components of mathematics teaching could be formed: 1) *the constructivist view of teaching*, 2) *the traditional view of teaching* and 3) *the formalist view of teaching*.

Next, we tried to find teacher groups on each grade which could be distinguished from the others either on the basis of their background or instructional style. Cluster analysis (CLUSTAN) was used and we found 5-6 teacher groups with different backgrounds (the so-called *background groups*) and 3-6 teacher groups with different instructional practices (the so-called *instruction groups*). The relationships between conceptions and instructional practices were then examined by analyzing the conceptions of these teacher groups.

The results demonstrated that the relationship was not at all clear. Concerning the class teachers, some kind of relationship could be observed only in a few subgroups, but on the ninth grade more relationships were demonstrated. There were three groups on the ninth grade where teachers' constructivist conceptions differed from each other. The teachers whose conceptions were more constructivist than in the comparison group

- placed clearly more emphasis on thinking skills, application and problem-solving,
- used in class distinctly more often mental arithmetic, pair or group work, explanations of students' own solutions and classroom discussions,
- gave to their students fewer problems for homework.

Summary and conclusions

The results related to teachers' conceptions were similar to the international results (cf. Thompson 1992). The conceptions of Finnish mathematics teachers could be described using the constructivist, traditional and formalist components of conceptions.

Both class teachers' and subject teachers' conceptions contained various aspects of all three components and no subgroups could be found whose conceptions would have been, for instance, purely traditional or constructivist.

No strong relationships could be found between teachers' conceptions of mathematics teaching and their instructional practices. Some connections, however, could be demontrated, but on the other hand there were inconsistencies which were also analogous with earlier research literature. Some kind of inconsistency came up, among other things, in the fact that although teachers' professed conceptions were mostly constructivist, many of the instructional practices they regularly used were very traditional (cf. Kupari 1993). This can, however, be interpreted by Ernest's model (1991) where the significance of social context is extremely high. Because during the last few years problem-solving and constructivism have continually been brought up both in teachers' in-service training and teaching discussions, they have become familiar to teachers. Teachers have, in principle, accepted these ideas and objectives but because of different (also social) reasons and pressures they have not yet reached the level of implementation.

Nevertheless, we have to take a rather critical stand to these results, partly due to the method teachers' conceptions were measured in the study. Earlier studies in research literature have namely revealed that in observing and measuring beliefs and conceptions it does not suffice to rely solely on verbal or written answers. In future research, it should also examine the instructional setting, the practices characteristic of the teacher and the relationship between the teachers' professed view and actual practice. And this should be done by utilizing several complementary techniques.

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Learning and teaching geometry

Frantisek Kurina

Summary

This paper is about the role of constructions in school geometry for the students aged 6 to 15 years. Constructions are very important both in the teaching process (representation of knowledge) and in the learning process (counting, proving and problem solving).

Although "construction" is one of those words that can mean everything and even its own opposite and constructivism seems to be no more then a new slogan (Freudenthal, 1991), one of the main trends in mathematics education today is the shift from behaviorist and cognitivist to constructivist learning theory (Leino, 1994).

Constructions as the process of creation of something new are traditionally the parts of school geometry, and geometry is the main object of our interest in this paper. Some aspects of constructive dimension I studied in the paper Kurina (1993). I am convinced with R. B. Davies (1990) that "learning mathematics required constructions, not passive reception, and to know mathematics requires constructive work with mathematical objects in a mathematical community. Mathematics teachers, therefore, need to accept as a major task the responsibility for establishing a mathematical environment in their classrooms."

Geometrical forms of representation of knowledge

In this part I would like to show how to use some constructions in the teaching process. Geometrical models of quantity are very important and may be also interesting. Some examples from the primary school are shown in the figure 1. The pupils are constructing here quadruples from various elements. The constructions are free - the final patterns are not exactly determined. It is useful to introduce some geometrical properties by means of various "geometrical structures" of the plane (figure 2). In figure 2a is geoboard whose numerous constructive exploitations are known: construction of various polygons (figure 3), broken lines (figure 4) and symmetrical figures (figure 5). On the checkerboard or grid it is

possible to study a lot of counting problems. It is also a very good means for the preparation of area notion. Examples of pattern "with given basis" constructed by 14 years old pupils are in figure 6.

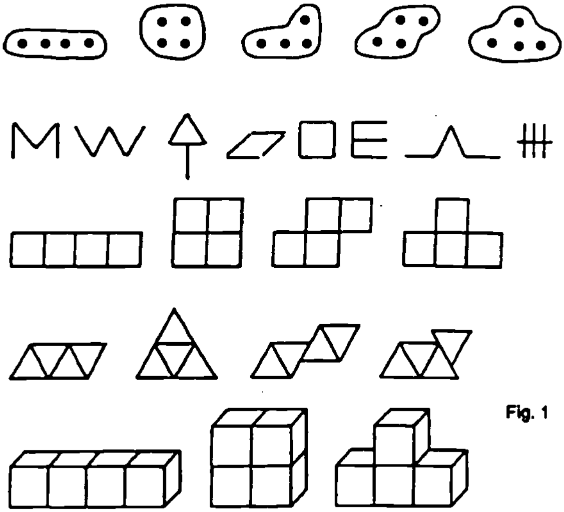


Fig. 1

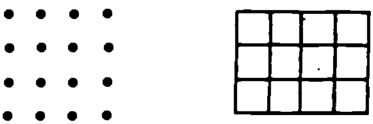
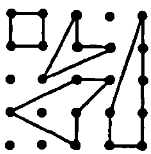
a.  b. Fig. 2

Fig. 3 

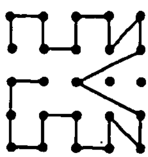
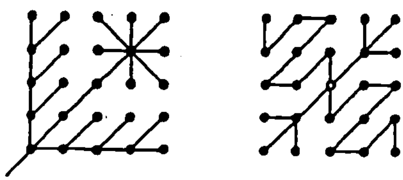
 Fig. 4

Fig. 5 

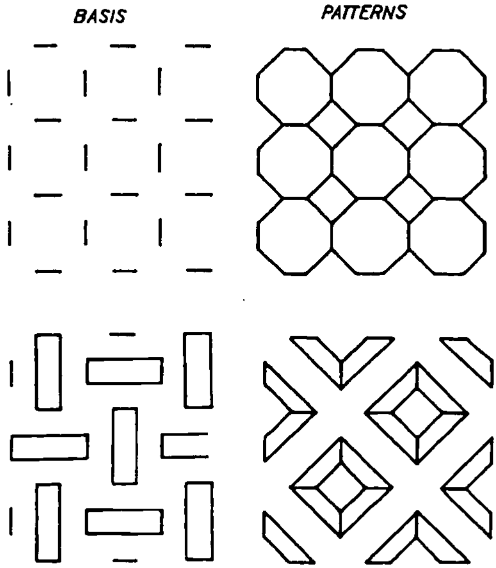


Fig. 6

Constructions and problem solving

Many calculations, proofs and problem solving activities require constructions too. You can verify it by the solutions of these problems.

Example 1. There are given points A = 1, B = 7, C = 2, D = 5 on the dial. Find the measure of the angle between straight lines AB and CD.

This problem was surprisingly difficult for the pupils: only 7 from 29 succeeded in solving it. The basis for the solution was a suitable construction. There occurred five types of them.

Example 2. Estimate exactly the moment between 15.00 and 16.00 in which the hands on the clock are coinciding.

Nobody from our group of pupils formulated this problem as an equation and its solution. All who succeeded constructed a sequence of the instants which "converged" to the studied moment.

The constructions are important parts of the proving of mathematical theorems. Let us consider the task of looking for the proof of the theorem:

The straight lines containing three altitudes of a triangle intersect at one points.

The well known proof of our theorem is based on the following construction. If the sides in the triangle A'B'C' are parallel with the sides of triangle ABC, then the umcenter of the triangle A'B'C' is the orthocenter of the triangle ABC. In the acute triangle ABC are the altitudes AA_1, BB_1, CC_1 the angle bisectors of the triangle $A_1B_1C_1$. The incenter of the triangle $A_1B_1C_1$ is the orthocenter of the triangle ABC. If we construct the images of the middle perpendiculars of the sides of the triangle ABC in the similitude with center T (centroid of the triangle ABC) and ratio - 2 is the image of the circumcenter S the orthocenter H (Coxeter).

For all mentioned proofs and of course also for number of other the suitable construction is the basis for discovering of the proof. I recommend to check it on this problem:

Example 3. In the triangle ABC which is not isosceles the straight line v = CE is the altitude and t = CC' is the median. Prove: If the angles ACE and BCC' are congruent, then ABC is right triangle.

Conclusions

In the learning and teaching geometry we can distinguish two levels of constructions. The "soft" constructions which are the parts of man's mental activity and the "hard" constructions which are the parts of solution of mathematical problems and which can be described by means of number of certain steps. Examples of soft constructions: idea, image, representation, notion, hypothesis, proof, theory, ... Examples of hard constructions: construction of image that a given function (mapping) assign to an element of its domain, construction of number, set, geometrical figure, ... with given property. The hard constructions are often the parts of school curricula. Soft constructions are components of every human mental activity, especially components of the problem solving. Soft constructions are connected with hard ones, hard constructions are unperformable without soft ones.

One of my didactical problems is: Do we pay enough attention to the constructive points of view in teaching and learning mathematics? In my opinion the answer is negative.

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On mathematical education - content, meaning and application

Anna Löthman

Summary

This is an explorativ study of mathematical education in municipal adult education and in the compulsary school, senior level, elucidated from student's and teacher's perspectives. In mathematical education you may think about the students' real thinking of mathematical concepts. What is their own thinking and what is a reproduction of the teacher 's thoughts? Students solve the problems correctly but on which level is the comprehension? In my case the actors gave expression to different conceptions, which I then described in the form of patterns and structures. These descriptions had been put into a number of results pictures, showing the teaching and learning as an entirety.

Aim

The object of research in this study is mathematical education and its actors. The aim is to describe students'and teachers'conceptions of mathematical education in connection with concrete educational course. In these conceptions I try tò find patterns and structures and catch their importance for the actors and the education.

The aim of the empirical study in the concrete situation in the classroom is to investigate the actors'conceptions of the education. In the study I also intend to identify the students'knowledge of mathematical concepts used outside school. The participants in my study consist of students and teachers in two classes, one from secondary school (16 years old, grade 9) and one from municipal adult education (first part, stage 1). Both classes are studying the equivalent mathematical courses on this occasion.

The main study of students' and teachers'perspectives on mathematical education.

According to the aim of the study I was going to describe students'and teachers'conceptions of mathematical education in connection with a concrete educational course.The main study was of explorative character and consisted of two empirical parts, partly one study which dealt with the selection of a mathematical

section, partly one study that caught the concrete education and elucidated the situation from the actors´ perspectives. To define my selection of mathematical sections and mathematical problems I raised the following criteria.

-The content of the mathematical problems should be descended from the students´ experiences.

-The problems should contain text and have alternative solvings.

-The problems should demand more than one calculation in order to get the answer.

-The results of the problems should give a possibility of discussion, analysis and valuation.

These criteria have a frequent representation in problem solvings in different moments.

The empirical study of education.

The participants in my study consists of students and teachers in two classes. One class from grade 9 (16 years old) with 13 students, seven girls and six boys, and one class from municipal adult education first part, stage1, with 7 students, two women and five men. As I was interested in how students and teachers conceive the education I chose groups with different backgrounds and ages. Both classes were studying the equivalent theoretical course in percentage calculation for the compulsary school, senior level, and the education had the following content.

-A repetition of concept percentage from basic.

-Calculation with raising and reduction of prices expressed in percentage.

-Calculation with problems where the percentage number are asked for

-Practical application of percentage calculation like value added tax, interest, discount, rates and taxes.

Procedure

I planned my study on the basis of my experiences from methods and techniques in an earlier study (Löthman,1990) and my received knowledge from research literature. The interviews had appeared

suitable according to the stimulated recall technique and I intended to use that together with questionnaires to students and teachers. I also planned to study and analyse the content of the vidoefilms. Consequently I was going to use partly a perspective from inside, partly a perspective from outside in order to catch the actors´ conceptions of the mathematical education. In grade 9 I videofilmed seven lesseons, together 280 minutes, and in the adult group I videofilmed six lessons, together 270 minutes. This camera was turned to the students. Another camera was turned to the teacher and videofilmed during the teacher´s work at the blackboard, in grade 9 about 100 minutes and in the adult group about 150 minutes. In connection with the lessons I used the stimulated recall technique and interviewed the teachers and the students. The students were interviewed individually or in groups.

The interviews emanated from the actors´descriptions of what they had seen on the videofilms. Then I gave complementary questions from their descriptions about the mathematical content and the lesson. I also used the students´textbooks in mathematics. The teachers were interviewed in the same way 2,5 hours each, total 5 hours. All interviews were transcibed verbatim. The students had to answer some questions about the education at the end of every lesson. Before the lesson the teachers had to account for their planning and afterwards they had to write down the results from their perspectives.

Findings

After successive workings up and analyses of the interviews I got a number of descriptions of the conceptions of the education in mathematics. The adults´ discussions of mathematical knowledge, particularly percentage calculation, touched upon their experiences from private lives and work. It dealt with money like bank loans, wages, prices of dwellings and living costs. The boys in grade 9 on the other hand had quite different conceptions of percentage calculation and where you can find it in everyday life. They were talking about finances like shares and sharecourses and about saving in banks. The girls in grade 9 moved within a limited mathematical sphere with more immediate commonplaces. Their conceptions of percentage calculation in my study for example circulated around the purchase of clothes and sales. Perhaps the social background of the students in the current study had a special

role here. The boys came from homes with traditions of education contrary to the adults and the girls, who in common were without such traditions. I maintain that the boys *searched for* the mathematical problems. The girls in return *caught* the mathematical problems that appeared to them.

Teachers and students were aiming at teaching and learning mathematics, but the meaning of these activities were different for the actors. It appears to me that this was one of the most representative and essential aspects of my empirical study. The meanings of the teaching and learning activities were pointing in two directions. One was aiming at comprehension and another was aiming at procedure. Naturally these two directions were not quite separate. They occurred side by side. But the actors showed some distinctive marks.

Table 1. Comprehension and Procedure orientation

	Grade 9	Stage 1 Adult
Student perspective	Procedure	Comprehension
Teacher perspective	Comprehension	Procedure

Result pictures

In the phenomenographic approach the descriptions of the phenomenon from the second order perspectives are the result. In my case the actors gave expression to different conceptions, which I then described in the form of patterns and structures. These descriptions had been put into a number of result pictures, showing the teaching and learning as an entirety.

The four pictures are

- *Mathematical traditions*, consisting of the conventional dwelling of mathematical problems in relation to the students' experiences

- *Mathematical strategies*, consisting of the students' way of understanding, reflecting and solving mathematical problems.

- *Mathematical reasonings*, consisting of the students' way of - discussing, analysing and judging mathematical information.

- *Mathematical applications*, consisting of the students' way of understanding and practising mathematical concepts in situations outside school.

These are not isolated, instead they complete and support each other and are important components in the education. They catch some pictures of the complicated situation in teaching and learning.

Mathematical traditions are an integral part in all the pictures. In my two groups I found different kinds of traditions. In the adult group the students and the teacher both agreed with strong rules and formal dispositions of problems. In grade 9 the students also wished the same things of rules and dispositions. The teacher in return thought that an informal and creative calculation was sometimes preferable, but the students did not accept that.

There were similar problems with the stratgies in the adult students. They have been taught percentage calculation earlier through dividing by 100 and now they had to use decimal fractions in the calculation like 5% = 0.05. In the beginning they were rather scattered in relation to new solving strategies. It appeared as if they used the old strategies as a mental calculation and the new strategies in the book at the same time. The teacher tried to convince them that the decimal fraction method was very important for mathematical studies in the future. The following problem was given as a mental calculation during a lesson.

---How much is 25% of 300 kr?

At the interviews the students had to explain their solving strategies. One of the adults argued in this way. -- Oh, it is 25 kr on every one-hundred note and there were 3. It will be 75 kr. --

The teacher in the adult class asserted that the students would probably have to solve this problem according to the decimal fractions method. He did not mention anything about the "practical" method. With the aid of the interviews and my studies of filmsequences from lessons I could make a number of strategies visible. Some of them exposed lack of mathematical understanding of percentage concepts and the using of decimal fractions. Other strategies proved to be practical, but not so useful and desirable in the perspective of mathematical development. These differences and lack of mathematical understanding were not discovered during the lessons as the students should often only give the right answer.

Mathematical applications in a concrete situation elucidated the importance of experiences.The adult had a knowledge of the percentage concept outside school, for example interest, but they were uncertain about calculations in practice sometimes. The young students, particularly the girls, had a vague knowledge of interest for example. They did not know how the bank could pay interest to them every year.

This showed the young students´ difficulties to connect mathematical problems with concrete situation outside school. It appear to me that the ruled based mathematical education in school in my study had lost contact with the world outside school. The practical function of the knowledge had not been followed up in the education. The teacher admitted that this was a shortcoming and also the textbooks did not talk about this very much.

Discussion

In the four pictures I presented and analysed the importance of the structures for the education. I maintain, that as a teacher in a class you are not conscious of these structures and the meaning of them for the education. This study can be seen as a classification and systematizing of what the actors considered as important in a education course, in this case percentage calculation.The work had also answered some questions about the mathematical knowledge . For example some parts of the students´ knowledge seemed to be a reproduction of the teacher´s report and the understanding was not so good by young students, but as a adult student you wished a deeper understanding of the problems. As a whole I think that my study will confirm the traditions in mathematical education and the difficulties to have a "dynamic" teaching and learning in mathematics.

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Study on Affective Factors in Mathematics Learning

Marja-Liisa Malmivuori

Summary

Mathematics attitudes represent terms that most frequently have been connected to affective factors in mathematics learning. The present study on affect in mathematics deals with several different aspects in it including i.a. pupils´ beliefs and emotions, as well as connections to cognitive and social variables in mathematics learning situations. The influence of affect on pupils´ behaviour has been widely recognized, and this increasing interest in pupils´ affective factors is seen to expand the cognitive learning theories more appropriate for daily situations in mathematics classrooms.

Characterizing the Study on Affect

Some general differences can be seen between the earlier research on pupils´ affective factors and the studies carried out in 1980´s and at the beginning of 1990´s. Attitudes toward mathematics and its learning have formed the basis for studying affect in learning situations. Often these were refered to as a kind of unidimensional affective variable that affected mathematics learning outcomes, and was measured by scales of pupils´ liking or disliking for mathematics mixed with a variety of other perceptions or feelings of pupils´ (e.g. Hart 1989). These were founded on the behaviouristic educational paradigm and quantitative measurement including paper-and-pencil tests, large scales and extensive data (e.g. McLeod 1992), and were used to account for (mainly gender) differences in mathematics achievements.

The present study on affect in mathematics deals with distinct aspects of mathematics attitudes that have separate impacts on pupils´ behaviour and performances. Pupils´ beliefs about mathematics or themselves, their unique emotions and experiences in mathematics classroom are viewed as significant characteristics and meaningful outcomes of mathematics learning (see e.g. McLeod & Adams 1989). Further, affect is perceived to be intertwined with cognitive and social aspects of learning in a way that sometimes makes it even the most important feature of a mathematics learning situation.

This new approach to and interest in pupils´ affective factors derive both from the more advanced measurements in the field and the research done in mathematical problem solving and among cultural aspects of mathematics learning in 1980´s. By the former is refered to multiple research resources (interviews, observations, physiological measurements etc. associated with questionaires), and questionaires with detailed and carefully defined items. The latter again points to individual pupils, their active and holistic behaviour during mathematics performances, and the social and cultural impacts acting in these performances.

Affective Topics in Mathematics Learning

The present study on affect includes topics that clearly can be classified as affective factors but also concepts that relate both to affect and e.g. to cognition in mathematics learning. Mathematics attitudes shape one catecory for examining pupils´ affective factors, but more and more often instead is used expressions as pupils´ beliefs, emotions, feelings or affective responses.

Attitudes. Mathematics attitudes are now studied in regard to theoretical considerations done in the affective domain and with accurate definitions and scale constructions. Attitude toward mathematics is no longer considered as unidimensional, but instead pupils are seen to adhere to attitudes toward different aspects of mathematics, as mathematics as a scientific domain, various mathematical topics, mathematics learning and teaching or social context in mathematics classroom. They are viewed as firmly stable affective responses with rather low intensity and long duration, developed through repeated emotional responses in mathematics learning situations or as transferences from already adopted mathematics attitudes (e.g. Lester et al. 1989, Marshall 1989). Most of them can be considered as mathematical beliefs or emotions.

Beliefs. The interest in pupils´ mathematical beliefs originates mainly from studies on mathematical problem solving (e.g. Schoenfeld 1985) but is related also to social and cultural aspects of learning, studied as mathematics enculturation or contextual factors of learning (e.g. Lave 1988). Pupils´ beliefs can be viewed as slowly developing affective responses toward various aspects of mathematics learning, with low intensity, high stability and a lot of cognitive involvement in them (e.g. McLeod 1992). Pupils´ beliefs

about mathematics, about mathematics learning and teaching, about themselves and about the social context shape the basic categories of these, the first two coming from studies on mathematical problem solving, the third from considerations among mathematics gender differences and the last one e.g. from crosscultural studies in mathematics.

Emotions. Instead of all kinds of feelings, present investigation of pupils´ emotions includes reactions that are classified as short-cut affective responses resulting from cognitive appraisals of a situation and the autonomic nervous system (e.g. Mandler 1984). These all-inclusive affective states are involved in the processes of pupils´ learning experiences and may contribute to pupils´ learning, but usually are found as harmful by interfearing pupils´ cognitive performances especially in high level mathematics problems. The effects of the emotions depend on pupils´ attitudes and beliefs, as well as their possibilities to control these emotions.

Relations of Affective Factors to Cognitive and Social Aspect of Mathematics Learning

As affect can not be totally distinguished from cognition in learning situations, purely affective topics are unusual in the studies of mathematics learning. Rather they are related both to pupils´ affective and cognitive domains, and even to social or cultural variables. Anxiety toward mathematics, feelings of helplessness or insecurity are examples of pupils´ clearly affective factors that are developed through personal features and individual experiences, but also in connection to socialization or enculturation in mathematics.

Respectively, it can be difficult to place e.g. pupils´ beliefs about mathematical problem solving or about the nature of mathematical tasks among affective factors. But investigations show that also these, mainly cognitive, beliefs are based on pupils´ personal experiences and interpretations in mathematics classrooms, and on things like mathematics teacher´s individual beliefs or generally adopted beliefs of mathematics learning and teaching culture. This means that socialization has much to do with these beliefs and that different amounts of affect is inevitably involved in their formation and maintenance.

Besides pupils´ mathematical beliefs, recent studies consider principally cognitive topics as pupils´ autonomy, intuition or metacognition in mathematics learning, which all have relations to the affective domain. By autonomy is refered to pupils´ independency in working with mathematics and mathematical tasks (Fennema 1989) which is contributed by pupils´ beliefs and social influences, and affects the acquisition and the utilization of mathematical knowledge as well as e.g. pupils´ willingness to persist in mathematical problem solving. Intuition means pupils´ implicit or self-evident mathematical knowledge, which has no direct connection to available facts or conscious thinking at hand but involves feelings of assurance.

Metacognition represents pupils´ cognition and cognitive processes directed toward themselves or their cognitive actions during mathematics learning (see e.g. Garofalo & Lester 1985). It includes i.a. pupils´ conscious beliefs about themselves and their learning strategies, as well as their capacity to direct or regulate e.g. their own learning, their cognitive actions during problem solving episode or their own behaviour while seeking help in mathematics. And these again are closely related to clearly affective factors as e.g. pupils´ confidence and interest in mathematics, their willigness to self-reflection or their emotional states while involved in mathematics.

The Dynamics of Pupils´ Affective Factors

One of the few learning patterns that expound the functioning of pupils´ affective factors is the ALB model developed by Fennema and her colleagues (see e.g. Fennema 1989), especially for interpretations of gender differences in mathematics. It relates pupils´ internal belief systems and external influences (e.g.teacher actions) to mathematics learning outcomes through pupils´ participation in autonomous learning behaviours (e.g. persisting at problem solving or achieving success in it). Thus e.g. pupils´ beliefs about themselves affect their willingness to persist in a problem and further their performance in that problem. Also few other suggestions has been made to characterize mathematics learning in a larger context including pupils´ affective domain as a meaninful and consistent component in them (see e.g. Bereiter 1990). However more research should and will be done to fully understand the functioning of

pupils´ affect and its dependence on cognitive and contextual features of mathematics learning.

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Teaching Geometry, A Constructivist Approach

Hartwig Meissner

Summary

We will report about a project in teaching geometry to introduce basic concepts like polyhedra, plane surfaces, straight edges, vertexes, pyramids, cubes, cuboids, rectangular, parallel (2-dim. and 3-dim.), perpendicular, etc. This sequence of about 7 lessons can be taught in grades 3 upwards (= age 9 years upwards) without any preceding experiences in "geometry". We start with the environmental knowledge and experiences to discover the properties of about 35 solids and to compare them with solids in our environment. The students sort and classify, draw, cut, fold, construct developments, use plasticine, build houses, ... Finally we design a whole village.

Constructivism

Like a sudden wave a keyword has appeared to describe proper teaching-learning situations which are practiced in Europe since more than 100 years. "Constructivism" is the label now and it is an advantage having got that label. Because a label is something you can concentrate on. And this concentration leads to a reflection, to discussions and to a conscious application of rules and methods of teaching. Thus non-constructivistic thinking teachers can be led to constructivism and constructivist teaching may become reflective constructivist teaching.

Constructivism is a theory of knowledge acquisition. It is not a teaching method, but constructivism can lead to teaching methods. Its basic idea is that all knowledge is actively constructed by the learner, knowledge is not passively received by hearing, seeing, listening, etc. "Learners do not simply add new information to their store of knowledge. Instead, they must connect the new information to already established knowledge structures and construct new relationships among those structures. This process of building new relationships is essential to learning. It means that mathematical knowledge - both the procedural knowledge of how to carry out mathematical manipulations and the conceptual knowledge of mathematical concepts and relationships - is always at least partly *invented* by each individual learner" (Resnick-Ford 1981, p. 249 f).

Thus, knowledge is something individual. Coming to know is organizing one's own experiences. There is no matching of knowledge to "reality" or no pre-existing "knowledge" to acquire. Knowledge can be constructed by active processes of the individuum like thinking, computing, analysing, reflecting, writing, ... or by the interaction with the environment: Via assimilation and accomodation (in the meaning of Piaget) we face a new challenge and develop our own experiences and adapt our already existing knowledge to these experiences.

The American use of the word teaching mainly has a transitive meaning like handing over or "reading" a paper in a conference. The German language has two words for teaching, namely the not often used "lehren" (transitive) and the commonly used "unterrichten", which is intransitive like learning ("lernen") or understanding. "Teaching Geometry - A Constructivist Approach" is meant in the German sense of "unterrichten". The students shall learn geometry and understand. Also here the German language has a characteristic word, *begreifen*, which describes the constructivist approach. "Begreifen" on the one hand means touching, feeling, taking in your hands. And the same word "begreifen" also means understanding. The essence of our teaching unit therefore can be described in two words: Geometry begreifen. This is the base upon which we developed our teaching unit in geometry.

We build a village

There is not much geometry teaching in Germany, arithmetic and arithmetic skills are the favourites in primary schools. Very often there are less than 5-10 school lessons taught in geometry during a whole school year and the students' geometry concepts and spatial abilities are very limited. Thus we decided to design a teaching unit, which can be taught independently from other curriculum units and which can be used in many grades. (We practiced the unit in grades 2 - 7.) Here we only can give a summary about the content. For more details see Tonn (1990).

In the *first lesson* the students become acquainted with about 35 solids, made by styroform, by wood, by paper-developments, or by plastics. We concentrate on daily life experiences which are three-dimensional: "You see these solids on the floor. Which objects from

your environment can you discover?" Possible answers: hat, bank, tent, ball, orange, pyramide, ... Children have much more fantasy and creativity than adults. On a worksheet they also have to write down names for given solids. The results of the worksheets then are discussed in the classroom. The children use their natural language. To identify the different solids, each solid got a specific letter (A, B, C, ...). Thus the solid is in the center and not a complicated name of a solid, and also many *no-name-solids* can be used having equal rights to their famous mathematical peers like *cuboid, tetrahedron, cylinder*, ... We speak about a coin, a pencil, or a chimney of a factory, and no mathematical vocabulary is used.

In the *second lesson* the students shall learn the terms vertex, edge, and surface (and in addition to this common goal they shall expand their individual experiences with the solids). The teacher puts the set of solids on the floor in the classroom: "I have a rule in mind. These two solids follow my rule". He places two of the solids separately: "Who can find other solids, which follow my rule?" From the already given examples the children make their guess and select the next solid. There is no explanation by the teacher but only the answer *yes* or *no*. By the *no's* we form a second subset. Thus the whole set of solids is split into two subsets, a set of polyhedra and a set of counterexamples. Sorting the solids only by yes-or-no-answers the students themselves start discussions about their assumptions, their guesses, and their trials. They use their fingers and the solids for arguing. In their language they detect the properties of polyhedra, they compare different solids in each subset and solids from different subsets. Intuitively they develop the concept of *polyhedra* like they did with the concept of *dog* or *table* or *flower*.

It is now that the teacher introduces the terms *edge, vertex* and *(plane) surface* and that they start counting these things at the polyhedra by touching each vertex, sliding along each edge or putting the flat hand on each surface. Each child gets two or three solids and a table to write down the results. Our goal: "polyhedra begreifen".

Cuboids are the topic of the *third lesson*. Again we start with a big set of solids, *a rule in mind* of the teacher, and *yes-or-no-answers* to split the set of solids into the set of cuboids and the set of counterexamples. Discussions while sorting and discussions at the end. Why does that solid fit? Why this solid? And why not that solid? Properties are discovered, demonstrated, shown, and described. A

global intuitive view becomes partly conscious and reflective. To make *begreifen* even stronger each child has to form a cuboid by plasticine. After this all solids are placed on an exhibition table.

The last part of this lesson is devoted to the surfaces of cuboids. Different sets of polygons are sorted and discussed. At the overhead projector all those quadrangles get coloured, when they can be interpreted as surfaces of a *rectangular solid*. A relational understanding develops of *cuboid, rectangle* and *right angle*.

The *fourth lesson* starts with a question: "Who knows that solid?" The teacher shows a pyramid and the students report about their own knowledge. But how to build such a pyramid? Discussions in the class lead to the idea of developments. We place the pyramid on a big sheet of paper and draw the shape of the base. Then the pyramide is turned down on one side and we draw the first side surface. Turn up to the base, turn down to the next side surface and draw also this surface. Back to the base, down to the next surface, draw, etc. At the end we cut out the development and fold it up. It covers properly our original pyramid.

After this demonstration done by one pair of students under the guidance of the teacher in front of the class every two students form a pair to construct their own development of a given pyramid. With adhesive tape they stick it together. We get another display on our exhibition table.

In the *fifth lesson* we construct the development of a cuboid. Two students use a brick and very quick they draw the base and the four adjacent surfaces. But where is the top surface? The question already gives the answer. They turn the brick again, and once again, and they get the missing part. Again, each two students then get a cuboid as a model to construct their own development. (We use match-boxes, boxes of video- or audiotape, and other daily life solids as models.)

In the *sixth lesson* the students have to build houses. Folding up and down in a continuous change of two-dimensional and three-dimensional views the students learn to imagine the front of the house while painting doors and windows. Painting the fronts of the houses, each front in an appropriate relation to the next front, is a big task, which in our experiments was solved excellently by the students. Each house was made individually. The students tried to rebuild a very specific house of their own imagination. Some of the

houses, we know, were the homes of the students. The result of this lesson again represents the importance of the connection of mathematics with the environment.

The *seventh lesson* combines mathematical ideas and topics with environmental applications. The mathematical knowledge and techniques developed in the course will be used to build a village. We start with a set of about 20 developments (one for each student). They all are put on the black board and we discuss for many of them which type of building this might be. Our students discover a school, a church, a tower, a hotel, and many different types of other houses. (There are no paintings or drawings on the developments, only the shapes are given). Each student then selects a development of his choice. They fold and draw windows, fold and draw doors, fold and draw, fold and draw, ... Individual buildings develop, glued together then with adhesive tape by the help of a classmate. On a big paper we then draw streets and parking lots, each house gets its place, the church in the middle, a creek nearby, ...

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Five fingers on one hand and ten on the other: The teacher as a mediator in interactive, playful teaching

Dagmar Neuman

Summary

The topic of this presen-tation is to illustrate how the teacher can act as a mediator in playful interactive teaching, that ends up in conceptual development. A phenomenographic cross-sectional interview study, carried out with the goal of understanding how scool beginners experience part-whole relations of numbers became the starting-point of an interactive, problem-oriented two-year teaching experiment with two classes. A child, who had been seen to have hardly any understanding of number, took part in individual interview lessons twice a week during these two years.

Background

In my dissertation (Neuman, 1987), I have described a phenomenographic study in which 105 children starting school at the age of 7 in Sweden, were interviewed immediately after school started. The motive for the study was to identify how children experience number and counting before they have received formal instruction in mathematics. It became the starting-point for a teaching experiment with two classes from which all pupils had been interviewed. The study showed that certain ways of understanding the part-whole relations within the number range 1–10 seemed to result in knowledge of the kind which is usually referred to as 'retrieved number facts'. The two year long teaching experiment aimed to examine the possibility of bringing about such understanding. The teachers, who through the interviews at school start were aware of the pupils' conceptions of the topic, acted as mediators arranging play and games through which the children could invent this knowledge (see Vygotsky, 1979).

Three of the pupils in the interview study did not seem to understand yet what counting means. One of these pupils was Jenny (J). She came to me for individual tape recorded and transcribed interview lessons twice a week during these two years.

The teaching experiment as related to the interview study

The teaching experiment J's class was involved in set out from knowledge acquired in the interview study, which showed that many school beginners mix the ordinal and the cardinal meaning of the counting words. Therefore, during the first months, the children played a game called 'the Long Ago land'. (This game is described elsewhere, e g Neuman, 1987, 1993). In the 'Long Ago land' there were no counting words and no maths at all. But from time to time problems popped up, demanding different kinds of 'maths'. At one occasion, for instance, the children felt that they needed number symbols as 'keeping track tools' for 'disappearing' units. Some proposed they put up one finger for each unit, others that they should draw one stroke for each. Later, when the children discovered that it was impossible to perceive the number of strokes written for large numbers, the teacher illustrated that the number became perceptible if 'V' was used as a symbol for one hand (with four fingers 'glued' together), and 'X' as a symbol for two hands (with the thumbs crossed). The reason for introducing Roman numerals – which were used in their ancient form, once pictures of our hands, where 4 was IIII and 9 was VIIII – was not only that they were part of the story. Several children in the interview study had shown how they avoided splitting up the four fingers of their first hand, which thus resembled the symbol V. It was seen that this idea of an 'undivided 5' was responsible for creating the powerful part-whole structure of the 'basic numbers' which would eventually enable answers to word problems to be given spontaneously, without counting. The fingers used in this way lent a structural isomorphism to all part-whole relations of the 'basic numbers' larger than five: the larger part became the first one, since the 'undivided hand' had to be a sub-part of it. Thus, it could always be ordinally experienced through the last finger and cardinally through the known finger pattern '5+some finger(s)'. The last part then became small enough to subitize. Some children 'thought already with their hands' in this way. Others did not refer to fingers at all any more, but illustrated a similar 'biggest first' structure in their abstract experiences of number.

The Roman numerals were later translated to our modern digits, and the counting words were introduced. The ordinal counting numbers were introduced as well, since several children had used

the cardinal words sometimes with an ordinal and sometimes with a cardinal meaning in the interview study.

Which counting word denotes the last part of a number?

The first interview lesson with J began at this point. The theme of the part-whole relations within the basic numbers was taken up in her first individual lesson. I took the experience of 5 plus 5 fingers making 10 fingers altogether as being a suitable starting-point for introducing this theme. In J's first interview lesson (28/10) – she had then taken part in the teaching experiment for about two months – I therefore ask her how many fingers she has, and she answers: 'Five.' – 'But on both hands?' I add. J has to count her fingers before she can answer this question. I then point to one of J's hands, saying: 'On that one you had ... ?' and J answers directly: 'Five.' After that I point to her other hand asking: 'And there?' J sits quietly for a while, before she answers: 'Ten.' – 'Then you got how many there are on *both* hands, see?' I say. After that I ask J to sit on the hand to which I pointed when she said 'five,' and add: 'Now let's just count here', pointing to the hand she is not sitting on. But J makes no attempt to count. Finally I prompt her to count with me: 'one, two, three, four, five'. J did not seem to understand that the fingers on her last hand could be counted by other words than '6, 7, 8, 9, 10', and she let the number word related to her last finger denote all her fingers as well as the fingers of her last hand.

There were fourteen pupils in the interview study who in several contexts, e g talking about buttons or pencils, had used the counting words in the way J used them for her fingers. Now, two months after school start J seems to experience numbers in the same way as these children did then. This is an improvement. At school start J's counting did not seem to have had any quantitative meaning at all.

We had a 'maths train' in which there were ten dolls, placed in five pairs, five dolls in each of two rows. J talked about these dolls in the same way as she talked about her fingers: there were five dolls in one row, ten in the other and 'ten' also in the two rows together. She used the number words so in lots of contexts.

Once J and I gave the dolls paper labels with ordinal numbers on – 1a, 2a, 3e, and so on (equivalent to 1st, 2nd, 3rd in English) – and put them into a queue, pretending that they were going to enter a skiing competition. While J looks away I take two dolls, and ask

'How many are there here?' – 'Two,' J says. 'Can you write Roman numerals under the dolls to say how many there are?' I go on. She writes II, one stroke for each doll. Then I ask her to translate 'II' to 'our own numerals' and she writes '2'. After that I show J the numbers on the dolls' labels, asking: 'Which numbers are they wearing?' 'A seven and a two' J answers. 'But 'the two' and 'the seven' they're two dolls aren't they?' I add, pointing to the dolls.

The next lesson the first question again concerns how many dolls there are in the 'maths train'. J counts under her breath, and answers: 'Ten,' and I go on as usual: 'And how many are there in that row?' – 'Five' (at once). – 'How many in this row, then, J?' After an endlessly long pause J finally says: 'There are five here and, then ... nearly the same.' – 'Only *nearly*?' I ask astonished. 'Yes.' – 'Aren't there *exactly* the same?!' My question convinces J that her new idea is not very good, and hesitantly she answers: 'Mm?' – 'Yes, *exactly* five! *Exactly* the same as there!' I add decidedly. J says nothing. Yet, when I some minutes later ask 'How many fingers have you got on this hand then?' pointing to one of J's hands she immediately answers 'Five!' On my question 'How many on the other one then?' there is, however, once more an endless pause, before J gives her usual answer: 'Ten'.

In the last of the seven lessons to be described here (22/11) I put all the dolls pell-mell on the table, and put a piece of paper in front of J on which I write en equals sign. The children had used the equals sign in the classroom from the very first lesson, meaning at first that two people had been fairly treated, and J knows that there must always be a similar number of units on each side of this sign. 'Now' I say, 'let's share out these dolls, you and me, ... so we get just as many each. ... Do you think you can do that?... Share them between us?' But J just sits quietly, looking at the dolls. 'We can take it in turns to take one' I say. 'First I'll take one ... How many have I got now?' – 'One.' – 'Now you can take one!' J takes *my* doll. 'Are you going to take *that* one?!' I cry, astonished, but I understand when I see '1a' written on its label. J's idea – which she had repeatedly illustrated in many contexts not possible to describe here – is that each counting word is firmly related to each counted object as its name, which cannot be changed. Now, however, I tell J that this is the doll *I* have chosen, and that *I want to keep it*. She has to take another doll. Finally she takes one from the pile. We establish that we both have the same number of dolls: one doll on each side of

the equals sign. J writes 'I' below each doll. After that we both take another doll, and I ask: 'How many have you got now?' – 'Two.' – 'And me ...?' – 'Two.' – 'I took the first and the third. And which did you get?' – 'One ...' and 'number ten', J says. 'And there are two of them!' I say pointing first to my doll and then to hers. J has now written II below the dolls on both sides of the equals sign.

We go on in this way until we have five dolls each. J also tells me about the numbers on the paper slips of her and of my dolls every time, and I ask again and again if we have got as many dolls each. J answers: 'Yes.' Finally I ask her to write an equals sign between the two groups of five strokes (fig 1). She writes an enormous one. After that, pointing to the strokes, I say: 'This is five as well, isn't it', showing my hand, and J writes 'V' below 'IIIII'. 'Can you write how many there are with another numeral as well, like those we have on telephones?' I say, and J writes '5' below each 'V'. 'Now ... how many have we got all together?' I ask. J counts quietly to 'Ten' and writes 'X' on her own initiative. 'Yes, there're ten dolls ... five dolls and five dolls make ten dolls. Can you write ten here with the other numerals?' I ask, and J writes '10', meaning: V + V = X.

After that I ask the usual question: 'How many fingers have you got on this hand? – 'Five (at once). – 'And on this hand?' Now something unforeseen happens. Almost before I have posed the question J shouts 'Five!' – 'You know it!!' I exclaim, astonished and very happy. Before the lesson ends I say, just to see J's reaction: 'How many are five plus five, then?' To my surprise J nearly immediately answers 'ten'. I decide to pose the same question in a new context, asking how many toes J has. She takes the shoe and sock off one foot and begins to count: '(mumbling)... four five ...' – 'How many toes in your other sock, then?' I ask. Once again, J immediately shouts 'Five!!' happily and with great confidence. 'And how many are there on both feet?' Again – happily and almost immediately – she exclaims: 'Ten!!' After that J understands how to use the number words for the whole number and its parts.

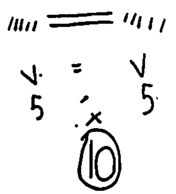

Fig 1

Conclusion

My starting point for designing the teaching experiment, and also for communicating with J, was my knowledge of the conceptions of number expressed by the children in the interviews at school start. The goal was to change early experiences to functional ones, through a negotiative process, where children in play and games could invent their own knowledge. In the evaluation at the end of grade 2 all pupils in the experiment classes – even J – had invented knowledge about the part-whole relations of the 'basic numbers'. They could also – without the help of any kind of concrete aid – use their understanding of how to divide the ten basic numbers when they added and subtracted over the 10-border within the range 1–100. In the control group only few children could do that, and some children in this group still denoted the last part of numbers with the counting word related to the last object of the whole.

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The 1994 Finnish Summer Math Camp and The Edutech Mathematics and Physics Enrichment Programme for Gifted High Schoolers 1993-1995

Kullervo Nieminen, Robert Piché

Summary

A mathematics and physics enrichment programme for gifted high schoolers was started last year by Tampere University of Technology's Continuing Education division Edutech. A high point of the programme was this summer's ten-day math camp at Päivölä college in the heart of Finnish countryside. The camp was a success thanks to the prestige and competence of the professional mathematicians who lectured, to the use of advanced computer software to reinforce and enrich the teaching, and to the cordial social atmosphere provided by the residential facilities of Päivölä College.

The need for programmes for capable high schoolers

High schoolers who are interested in mathematics are not sufficiently challenged in their classes. In Finland, the number of students admitted to the advanced high school mathematics courses is relatively large, approximately one quarter of the entire age-group. Class sizes are often large, as many as 38 students. In such a large group there are usually 10-15 who are weak, and the teaching has to be tailored to their speed and level. The textbooks are old-fashioned: there has not been any significant curriculum change in the last decade. In high schools there is little or no use of mathematical software, there are no physics labs, and improvements in these regards are hampered by the budget restrictions of the current economic recession.

With all this, it is no wonder that young people don't develop an interest in mathematics. This is bad news for Finland, because the future information society will need people with strong mathematical skills.

The authors have decided to do something about it. Last Fall the first author (Kullervo Nieminen) proposed a two-year teaching

project to the Tampere University of Technology where talented and capable high schoolers could be exposed to high-level training in mathematics and physics. This paper describes the first year of this project and tells about its summer math camp that has attracted national media attention.

Two-year project at Tampere University of Technology

The project started in the Fall of 1993. It was administered by Edutech, the Continuing Education Department.of Tampere University of Technology. 21 high school students aged 16-17, of which 4 were girls, came from high schools in the Tampere region. They were recruited by sending letters to all the math teachers in the area.

The group meets at Tampere University of Technology on Friday evenings and on Saturdays at 2-3 week intervals. During the weekends at the University of Technology the students listen to lectures on university-level mathematics and physics, participate in computer exercises, do physics laboratory projects, and visit research labs. There are no exams.

Kullervo Nieminen does most of the lecturing. The lectures follow three textbooks. Two textbooks are modern American university textbooks: a 1000-page Calculus book and a 1400-page Physics book. These books are in English and give the students much more information than the much shorter (but more expensive) Finnish counterparts. The third textbook is the set of Finnish-language course notes for the Tampere University of Technology linear algebra course. Linear algebra is not taught in Finnish high schools.

The math teaching is enriched by use of modern mathematical software (MATLAB and MAPLE) on the math department's macintosh laboratory. The computer sessions are taught in English by the second author. The students especially enjoy the real-life problems they are led to explore, such as cryptography, Monte-Carlo integration, and simulation of a bungee jump.

The physics teaching is enriched by doing projects in the Physics laboratory under the supervision of teaching assistants from the Physics department. The students also receiv guest lectures by

professors. They have had lectures in chaos theory, aircraft aerodynamics, and astrophysics. Finally, the students are given guided tours of various research labs such as the plasma physics lab and the computer centre's multimedia lab.

As part of this project, a 4-day intensive course was organised at Päivölä college in Valkeakoski (which is 30 minutes drive south of Tampere) in October. The teaching was done in English by Professor Piché. The students were intrigued by the variety and novelty of the topics (many from discrete mathematics, not covered in Finnish high schools), and by the challenge of axiomatic thinking (which was entirely new to them).

According to the students, one of the best things about the intensive course was the opportunity to get to know other young people with similar interests. This social aspect was helped by the excellent residential facilities provided by Päivölä College.

Summer Math Camp

Because of the success of the October intensive course, it was decided to organise a 10-day summer camp at Päivöla college in June of this year.

In addition to the group involved in the Edutech project in Tampere, the math camp attracted students from all over Finland. The camp was marketed by sending letters to all the high school math teachers in the country. The 32 participants included 5 girls, 1 member of the Finnish Math Olympiad team, and 1 high school teacher.

The two teachers at the camp wereProf. Piché and Dr. Mike Raugh from the USA. Dr. Raugh is an eminent applied mathematician who has worked in senior positions for the US Geological Survey, Hewlett Packard, Stanford University, and most recently as chief scientist for a NASA computer research institute in California. He responded enthusiastically to Prof. Piché's invitation to come teach at the summer camp because he had himself been gifted in mathematics as a teenager, and he wanted to share his experience with young people.

Raugh and Piché gave mathematically rigorous presentations of many concepts from the high school Calculus curriculum, as well as

some discrete mathematics topics not covered in Finnish high schools. The students gave both teachers top marks for clarity of English, patience, and ability to inspire and motivate the students. After every lecture there would be a large group of students around Raugh discussing his lecture.

In addition to lectures, there were computer exercises using the MAPLE symbolic computation program. Also, Vicki Johnson (Raugh's wife) provided expert advice on the use of the Internet.

The summer camp activities also included sports and socialising. The students were encouraged to form a musical group, to make a newspaper, and to make a video documentary. The 10-minute video was made by the students during a 14 hour session in the editing studio with help from Vicki Johnson and from Valkeakoski Community Television.

Aftermath

The summer math camp was a high point of the project, but it was certainly not the end. This summer, Nieminen organised two-week work terms in the information industry for the students. He was able to get 5 places at the Nokia research centre, 2 at the Nokia telecommunications factory, 3 at ICL Data, and he organised Excel programming contracts for 2 more.(Nokia is a leading maker of cellular phones and ICL makes computers.)

Some of the students have participated in summer courses offered under the Open University programme and have gained university credits in linear algebra and in physics. The linear algebra summer course was organised by Prof. Piché with the high school students in mind, and was taught in English by a teacher brought in from Scotland.

The programme at Tampere University of Technology continues. Already this Fall the students have visited laboratories studying such things as semiconductors, sound reproduction, and paper tearing.

The summer camp attracted attention from several newspapers. Helsingin Sanomat, the largest newspaper in Finland, printed a

half-page article with a colour photograph. There was also an article in the nation's second largest newspaper, Aamulehti.

This media visibility has been very helpful in securing support from the Ministry of Education and from industrial sponsors for the plans for the future of this project.

The authors hope that the successful experience can be repeated next summer, and look forward to participation by students from outside Finland as well.

On the Role of Hypermedia in Mathematics Education

Seppo Pohjolainen, Jari Multisilta, Kostadin Antchev

Summary

Hypermedia enables creation of mathematical virtual reality on computer, where mathematics can be studied with the aid of hypertext, graphics, animation, digitised videos, interactive exercises etc. The discussions are based on Matrix Algebra hypercourse at Tampere University of Technology (TUT). Essential features, in addition to the lecture notes, are hypertext, computer aided exercises, connections with existing mathematical software, animations, and graphics. Different learning strategies will be discussed and compared. The concrete outcome, however is not only a single course, but software which enables creating other hypermedia courses in mathematical sciences.

Introduction

The concept hypertext in this context means database where information (text) has been organised nonsequentially, or nonlinearly [Conklin (1987)]. The database consists of nodes and links between nodes. The nodes and links form a network structure to the database. Hypermedia is a database, which in addition to hypertext may contain pictures, digitised videos, animations and sound and where computer is used to organise the material to make studying easier.

Mathematical text consists typically of definitions, axioms and theorems which may be deduced from the axioms. A lot of definitions should be well understood to be able to read the proofs. Hypertext provides a good database for modelling the structure of mathematical courses. The basic form of mathematical hypertext is an electronic book with dictionary of definitions, which may be called and studied always when needed. The main advantage of hypertext from the users' point of view is that it is able to adapt to their needs. Since everybody need not read all the material, hypertext supports reading and studying in different ways and on different levels depending on the user. For the author this means that he has to write the text to support reading on different levels. This is usually not difficult, since hypertext may organise the writing process, too.

A hypermedia-based course on Matrix Algebra at Tampere University of Technology (TUT), was our first trial. For more details see Multisilta & Pohjolainen (1993a) and (1993b), Pohjolainen et al. (1993b). Lecture notes, exercises, and examples [Pohjolainen (1993a)] have been written as hypertext and links to Matlab and other mathematical programs have been made to enable numerical experiments and graphics.

The achievement is not only a single hypermedia based course on mathematics, but also a set of programs for transforming other lecture notes on mathematical sciences into hypermedia. The hypermedia learning environment (HMLE), developed at TUT, has some advantages compared with existing hypermedia software, e.g.

- Implicit linking is a default . This feature is very useful for the authors. They do not need to think about links when writing mathematical text.
- Every word can be an anchor word. Whether or not there are links, depends on whether the associated files, animations, stacks etc. exist.
- The anchor words are not marked, coloured or underlined. This makes reading and studying mathematics easier.

Implementation issues of mathematical hypermedia for Matrix Algebra are discussed in Pohjolainen et al. (1994). The HMLE has been programmed in Macintosh environment, but PC and UNIX versions are under development.

The Structure of Mathematical Hypermedia

Mathematical hypertext. The structure of the hypertext should reflect the structure of the subject matter. That is why it has been structured as

- concept database,
- theorem database, and
- problem database.

Concept database consists of definitions, theorem database of theorems, lemmas, and corollaries, and problem database of problems, exercises, and examples. By concepts we mean mathematical definitions and concepts. In the course of matrix

algebra there are about 70 different concepts, and the database consists of hypertext definition files. They form a hierarchical structure where elementary concepts are used to explain higher level concepts. Thus concept maps can be used as navigation maps in the hyperspace of concepts.

Concepts, theorems and problems are related. In order to understand a 'new' theorem or concept the student should understand the necessary 'input' concepts and theorems (Fig. 1). From this point of view, studying a 'new' theorem or concept is a process starting from 'old' concepts and theorems, going through a problem, or list of problems, to a 'new' theorem or concept.

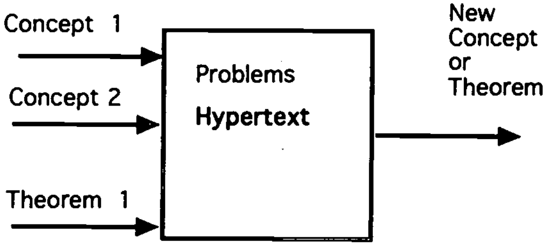

Fig. 1. Relation between 'old' and 'new' theorems and concepts.

Interactive exercises. Most concepts should be studied not only as definitions, but as examples, exercises, and in numerical and symbolic way. One can write them in separate files to the problem database that can be opened when needed. The implementation of interactive exercises in hypertext is interesting because the computer can be used both to generate these exercises and to check the results. ExerciseMaker program has been developed for that purpose, which uses Mathematica in background to generate problems of given type and to check the answers [Antchev et al. (1994)].

Orientation basis. Some mathematical results can be given as algorithms or computational processes, which can be presented as a flow diagrams. Flow diagrams are example of orientation basis. Every block in the orientation basis may be opened to have definitions or information about what to do.

Visual elements. Many essential ideas in mathematics are geometrical or visual by nature and they can be shown and studied to some extent using computer graphics. Animations are used to clarify mathematical concepts, methods applications etc. Iterative methods, convergence, and time-dependent behaviour are easy to present as animations. Video presentations can be shown on a small video window. So far only popping heads and 3D-graphics have been used.

Studying with hypermedia

Linear way. The course of matrix algebra can be studied in the normal, 'linear' way. The hypertext can be read from the beginning to the end and all the exercises can be done in the regular order. In this case the full power of hypermedia is not used since it can organise the material for the user's needs and knowledge.

Goal-oriented way. In the case that the student knows what he wants to study, he can make concept and theorem maps for himself and study the concepts and theorems in the maps.

Marketplace-way. Here the student gathers information from the mathematical database the way that hypermedia recommends. In contrast to the above two methods, this part of the HMLE has not been implemented yet, but it will be available in the next software version. The implementation can be done in the existing HMLE-framework.

To begin the HMLE should be personalised, so that it knows the student personally. Then it is easy to keep record on the student's database of concepts, theorems, and problems. In this setting studying can be seen as student's activity to expand his knowledgebase. Interactive exercises can be used, to examine that student understands particular 'new' concept. It will be included in his knowledgebase, only if he is able to solve correctly certain problems/questions. Though the examination may sound a bit mechanical, there are lot of concepts for which reasonable set of numerical exercises and questions can be designed to detect whether the concept has been understood. For example, to understand the concept vector, it would be sufficient to be able to add vectors and multiply vector with scalar.

There are several advantages in this setting. First, every new subject is based on the student's personal knowledge, since the

computer knows it. Second, the student knows that, in principle, he should be able to understand the new subject. Normally, there are several problems, theorems or concepts that he might try to take. The path of study totally depends on the order he is able or wants to solve the problems and study the definitions.

Conclusions

In the design of HMLE for mathematics, special attention has been paid to the hypermedia knowledgebase in order to support different ways of learning. Students may study all the concepts with the aid of definitions, examples, exercises, animations and with numeric and symbolic programs. Concept maps, theorem maps and orientation bases help them in navigating in the hyperspace. The material may be studied on different levels and on different paths depending on the user. Different learning strategies: linear, goal-oriented, and marketplace style, have been discussed.

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Some new geometrical ideas in teaching mathematics

Maido Rahula, Kaarin Riives

Summary

Some mathematical topics which can be illustrated by geometrical concepts are described.

In teaching mathematics many topics become clearer and more interesting when the teacher uses geometrical thinking. Here we give three such examples.

Topic A. The concept of convergence of number sequences and the corresponding theorems about limits are among the most difficult themes in school mathematics. It is not easy for students to understand the abstract "language of $\varepsilon - \delta$" which leads into classical calculus. However, starting the discussion with vector sequences allows us to use schemes of greater lucidity. Thus let a vector sequence be generated by a mapping of natural numbers into space vectors

$$N = \{1, 2, ..., n, ...\} \Rightarrow \{\mathrm{r}_1, \mathrm{r}_2, ..., \mathrm{r}_n, ...\}.$$

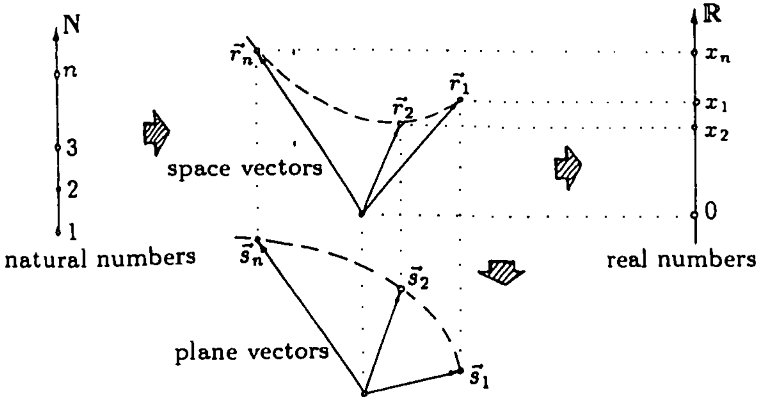


Fig. 1

Here the following new aspects arise:

1. By projecting the vector sequence on the line the number sequence $x_n = \text{pr } \mathbf{r}_n$ is generated, so the concept of a vector sequence is more general than that of a number sequence.

The convergence and limit of a vector sequence are formally defined as in the usual calculus:

$$\mathbf{S} = \lim_{n\to\infty} \mathbf{s}_n \iff \forall > \varepsilon\, \exists N : n > N \Rightarrow |\mathbf{S} - \mathbf{s}_n| < \varepsilon.$$

Geometrically this means that if $\mathbf{s}_{N+1}$ enters the ε-neighborhood (interior of sphere), then all following vectors never leave it.

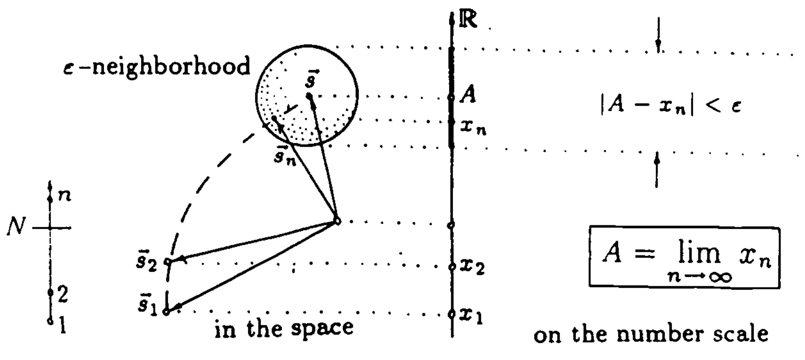


Fig. 2

Now the relevant theorems can be proved and illustrated geometrically. Here is one as an example.

THEOREM. If the sequences $\{\mathbf{a}_n\}$ and $\{\mathbf{b}_n\}$ converge, then their sum $\{\mathbf{c}_n = \mathbf{a}_n + \mathbf{b}_n\}$ also converges and the limit of the sum is the sum of the limits:

$$\lim_{n\to\infty} \mathbf{c}_n = \lim_{n\to\infty} \mathbf{a}_n + \lim_{n\to\infty} \mathbf{b}_n .$$

Proof. Let $\mathbf{A} = \lim_{n\to\infty} \mathbf{a}_n$, $\mathbf{B} = \lim_{n\to\infty} \mathbf{b}_n$, $\mathbf{C} = \mathbf{A} + \mathbf{B}$.

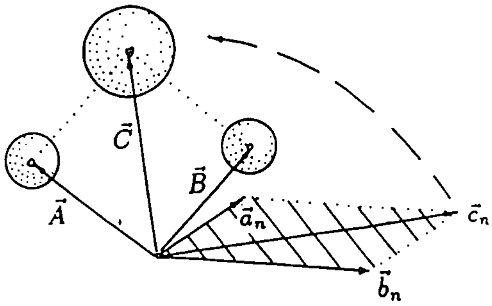


Fig. 3

$(|\mathbf{A} - \mathbf{a}_n| < \varepsilon/2, |\mathbf{B} - \mathbf{b}_n| < \varepsilon/2) \Rightarrow (|\mathbf{C} - \mathbf{c}_n| \leq (|\mathbf{A} - \mathbf{a}_n| + |\mathbf{B} - \mathbf{b}_n| < \varepsilon)$

and $\mathbf{C} = \lim\limits_{n\to\infty} \mathbf{c}_n$.

We also have the following

THEOREM.

A. The projection of a convergent sequence is convergent.

B. From the convergence of the projection it does not follow that the sequence converges.

C. From the divergence of the projection follows the divergence of the sequence.

EXAMPLE.

Let $\mathbf{r}_n = x_n\mathbf{i} + y_n\mathbf{j}$, where $x_n = 1 + 1/n$, $y_n = n$. Then $\lim\limits_{n\to\infty} x_n = 1$, $\lim\limits_{n\to\infty} y_n = \infty$ and the sequence $\{x_n\}$ is convergent, but the sequences $\{y_n\}$ and $\{\mathbf{r}_n\}$ are divergent.

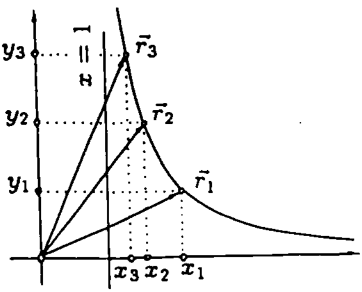
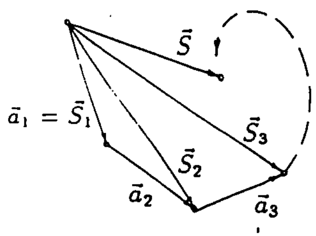


Fig. 4

2. Now it is easy to interpret notion of a vector series, which is the



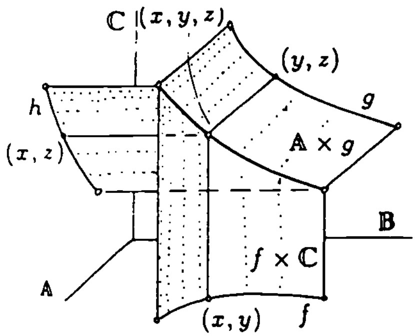
infinite sum of the vectors $\sum_{n=1}^{\infty} \mathbf{a}_n$, and is said to be convergent if the sequence of its partial sums is convergent:

$$\mathbf{s}_n = \sum_{k=1}^{\infty} \mathbf{a}_k \rightarrow \mathbf{S} = \lim_{n\rightarrow\infty} \mathbf{s}_n .$$

Fig. 5

3. Vector functions are introduced by mappings of the set of real numbers R (or a subset $U \subset$ R) into the elements of a vector space: $t \rightarrow \mathbf{r}_t$. This allows us to deal with higher order arithmetical progressions. Indeed, the linear function $\mathbf{r}_t = \mathbf{a}t + \mathbf{b}$ ($\mathbf{a} \neq \mathbf{0}$) has a line as its hodograph and the values $\mathbf{r}_n = n\mathbf{a} + \mathbf{b}$ generate an arithmetical progression (of 1st order): $\mathbf{r}_{n+1} - \mathbf{r}_n = \mathbf{a} = const$. The quadratic function $\mathbf{r}_t = \mathbf{a}\, t^2/2 + \mathbf{b}t + \mathbf{c}$ (a and b are not collinear) has a parabola as its hodograph and the values $\mathbf{r}_n = n^2/2\ \mathbf{a} + n\mathbf{b} + \mathbf{c}$ generate a 2nd order arithmetical progression, because the differences $\mathbf{r}_{n+1} - \mathbf{r}_n = n\mathbf{a} + 1/2\ \mathbf{a} + \mathbf{b}$ generate a 1st order arithmetical progression.

Topic B. Compositions of two binary relations are generalizations of ordinary composite functions and give a good geometrical interpretation to these functions and their first-order derivatives. Actually the graph of a composite function can be sketched proceeding from the graphs of the inner and outer functions by the "rule of the third projection": let A, B and C be the x, y and z axes, and let f and g be the figures (curves) on xy and yz planes, respectively. Then $h = gf$ is the projection of intersection of the bodies (surfaces) $f \times$ C and A $\times g$ onto the xz plane:



$$h = \mathrm{Pr}_{A \times C}\{(A \times g) \cap (f \times C)\}.$$

EXAMPLE 1. A tangent of the graph of the composite function $z = g(f(x))$ is determined by the tangents of the graphs of inner and outer functions, because the composition of the lines

$$y - y_0 = f'(x_0)(x - x_0) \text{ and } z - z_0 = g'(y_0)(y - y_0)$$

as binary relations is

$$z - z_0 = g'(y_0)f'(x_0)(x - x_0),$$

where $h'(x_0) = g'(y_0)f'(x_0)$ is the derivative of the composite function.

Many applications can be found in descriptive geometry.

EXAMPLE 2. The composition of the circle

$$f = \{(x, y) \mid x^2 + y^2 = r^2\}$$

and the disc $g = \{(x, y) \mid (y - r/2)^2 + z^2 \leq r^2/4\}$ is the figure "eight" with its interior:

$$h = \{(x, z) \mid (r^2 - x^2 - z^2) \leq r^2(r^2 - x^2)\}.$$

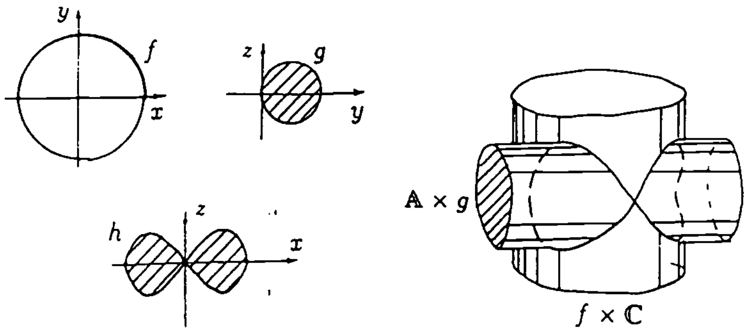


Fig. 7

Topic C. Traditional school curricula do not contain interpolation problems. But their introduction may be very worthwhile, because this gives rise to the interesting possibility of deducing the formulas which transform the roots of the 2nd order equation $x^2 - 1 = 0$ (or the 3rd order equation $x^3 - 1 = 0$) into the roots of general equations of the 2nd (or 3rd) order. In fact, in solving the interpolation problem we look for a polynomial $y = f(x)$ whose

values on the finite sequence $\{x_n\}$ generate the sequence $\{y_n\}$, where $y_n = f(x_n)$.

EXAMPLE 1. For $n = 1, 2$ the solution is a linear function $y = ax + b$ and its graph is a straight line. If x_1, x_2 and y_1, y_2 are the roots of the quadratic equations $x^2 + Px + Q = 0$ and $y^2 + py + q = 0$ respectively, then the solution is

$$y = -\frac{p}{2} + (x + \frac{P}{2})\sqrt{\frac{p^2 - 4q}{P^2 - 4Q}}.$$

In the special case $P = 0$, $Q = -1$ this generalizes the known formula for the roots of the quadratic equation $y^2 + py + q = 0$ (for $x_{1,2} = \pm 1$) and describes the mapping of the segment $[-1, 1]$ onto $[y_1, y_2]$ on the complex plane.

EXAMPLE 2. For $n = 1, 2, 3$ the solution is $y = ax^2 + bx + c$ and its graph is a parabola. In the special case when x_1, x_2, x_3 and y_1, y_2, y_3 are the solutions of the cubic equations $x^3 - 1 = 0$ and $y^3 + pq + q = 0$, we get the Cardano formula, which determines geometrically a certain mapping of point triplets into each other. This latter mapping can be regarded as a "root picking machine" on the complex plane (see Rahula (1993)).

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New Components for the Study and Evaluation of Mathematics

Maarit Rossi

Summary

All the lessons in our comprehensive school have been divided into six-week periods. In every grade of the secondary school there are three periods of mathematics during a school year. Mathematics is studied 6 lessons a week for six weeks (a total of 36 lessons). The lessons have been double lessons (2x45 min). The division of the lessons into periods has created a situation in which changes in mathematics teaching have been natural but also a necessity.

Future Changes

At the same time in the new outline of the comprehensive school curriculum the pupil is seen in an active role acquiring, processing and recording information. In the teaching situations he learns new concepts by study, experiment and concrete examples. The focus will be changed from teaching to a balance between learning, teaching and evaluation.

The study of mathematics will include new and more versatile components.The ability of the pupil to classify, analyze and model new situations cannot be evaluated merely with a traditional written exam. We need new components for the assessment of mathematics studies.

Functions of Evaluation

In the active learning, the learning process and the total performance of the pupil get more emphasis. One task of assessment is also to give useful information to the learner and help him to develop his own performance. As the pupil is now regarded as an active learner who takes the responsibility for his own learning, he has to be regarded in an active role, evaluating how he has learned and, eventually, evaluating the teaching he has received, too.

The purpose of assessment, in addition to giving the teacher information on learning and his own work, is also to encourage and positively direct the learner.

Self-Evaluation

The pupil sets himself goals in the entity studied, and later evaluates how he has reached these goals. Here is an example of how three pupils set their goals and how they reached them. The subject was the concept of power.

My goals and expectations	My strongest points	My estimate of how the goals were achieved	The period positive: negative:
to get a better grade, to behave properly	*I'm good with numbers*	*I've behaved well I've worked on many problems*	*I don't know*
I want to learn as much as possible about power	*I'm rather active because I'm interested in maths*	*I've reached my goals well, the exam went well, and the project work, too*	*The planets were presented in an interesting way and they were even i nteresting. I'm not usually interested in planets*
to get at least the same grade, to learn about power	*I understand easily*	*I think I have studied rather well*	*the exercises were versatile, we could have studied about power a bit quicker*

Fig. 1. The goals of three pupils and how they have reached them

Experience has shown that the goals the pupils set themselves are on very different levels, but very realistic. On the other hand, we believe that the pupil learns to take responsibility for his studies this way.

A pupil may evaluate his learning in other ways. At the end of the study period, when there are a few minutes left, the pupils may be asked to write down concisely what they have learned. When a pupil evaluates his own work, the teacher always gets valuable information.

Portfolio

The division of the mathematics lessons into periods has made it possible to carry out more comprehensive entities and work. The pupils have collected their work in secondary school in a portfolio (evaluation folder), where they have the project and research papers, essays, exams, tests and old notebooks. The work done in one year provides the teacher with a broader view of each pupil and his contribution to his studies.

The teacher has evaluated part of the project papers in the portfolio by using the evaluation forms he has developed. When the pupils do project work for the first time, it is worthwhile to show them what parts are included in a written presentation. This makes it easier for the pupils to evaluate their own work and other pupils' performance in the future.

A pupil may evaluate his own work and learning by discussing with his teacher. The following is an example of a conversation taken from a videotape. The pupil take his portfolio with him up for discussion with the teacher. "Ville, which of your assignments has been the most interesting?" Ville:" *The project work on the number of the diagonals in a polygon.*" "Why?" Ville:"*Well these diagonals are a new thing, how they increase when the polygon grows.*" " Which was more interesting in the work, geometric drawing or the final result?" Ville:" *The final result, the formula for calculating the diagonals.*" The pupils were also asked which task had been the most difficult, easiest etc. When a pupil evaluates his own work, the teacher gets again valuable information.

The teacher does not always have to do the assessment. Another group may evaluate a groupwork. Also, the group itself may evaluate the efficiency of the group and the success in reaching the goals. We have also experimented with situations where a pupil assesses another pupils performance.

The teacher assesses his work

Quite a few of the evaluation methods presented above give the teacher feedback on his own work. Professor Gilah Leder, from Australia, has said that one way to evaluate one's teaching is to ask the pupils to draw a picture of a mathematics lesson. The pupils

who get teacher-centered teaching often draw the blackboard and the teacher. The pictures rarely indicate communication between pupils. As an example, here is one picture of a mathematics lesson, drawn by a pupil whose teaching had been developed towards active learning.

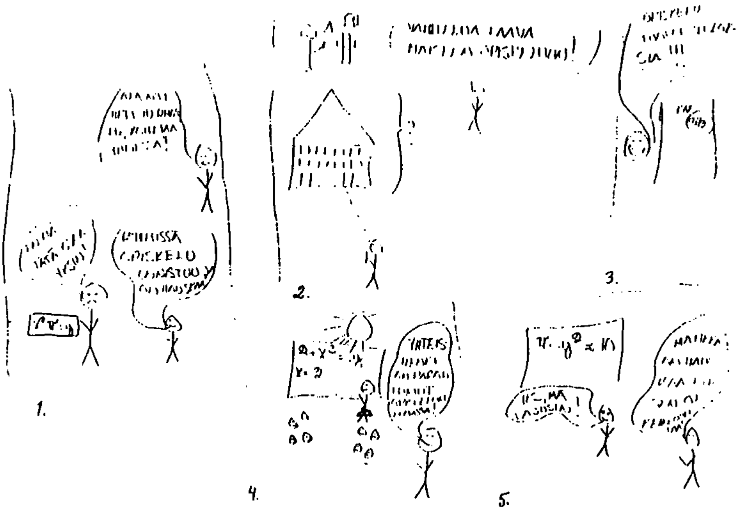


Fig. 2. 1. *Don't get upset, lets try together! I can't do it alone! Learning in groups is successful and fun!* 2. *A change from the routine.* 3. *Studying produced good results.* 4. *The atmosphere has improved during my studies.* 5. *Yes, I got it. Maths is fun when you get to use your brain.*

To conclude

Evaluation, which includes several different components, supports development in the teaching of mathematics, correspondingly, with the diversification of mathematics teaching the expectations concerning the diversification of evaluation also become greater.

The use of computers in mathematics

Rui J.B. Soares

Summary

In this paper we shall present a brief description of the Portuguese educational system and of some specific goals of Mathematics at the different levels of non-higher education.

Being aware that the main task of Mathematics teachers is to create the conditions and dynamics necessary for the students to learn to like this subject, and to encourage them to participate in creative activities, we shall present in this paper educational software to be integrated in teacher training programmes as well as in the classroom.

The Portuguese educational system

General Structure

According to the Portuguese fundamental law all the citizens have the right to access the education and the culture with equal opportunities. To realise this aim the teaching and learning will take place in three types of schools: public, private and cooperative. Public schools are connected in a network that covers all the country; private schools as well as cooperative schools constitute a complementary network, based on private initiative, and they follow the same structure and the same programmes of public schools, with total respect to freedom in learning and teaching.

The Education Act n° 46/86 defines the educational system as the set of resources used to fulfil the right to education, which is expressed through the guarantee of a continuing training process, designed to encourage personal development of the personality, social progress and a more democratic society ... with a guaranteed right to a fair and effective equality of opportunity for access to education and for academic achievement.

The educational system is implemented by initiative and under the responsibility of different institutions and entities in an organised set of varied structures and actions. The coordination of the policies concerned with the educational system is made by a ministry, specially concerned with this area, no matters the *type of*

education (pre-school, school and extra-school) and the *level of teaching* (basic, secondary and higher).

The educational system is applied equally to all the country, in a flexible way, in order to include the generality of the countries and places where live Portuguese communities or in everywhere we meet some kind of interest to know about Portuguese culture.

Mathematics syllabus

The specific goals of mathematics' learning are integrated into the school education aims. The understanding of society needs and the Information and Communication Technology (ICT) spread have originated new areas and diminished the importance of other ones; consequently, in the pedagogical work, it is covered a great spectrum of objectives dealing with attitudes/values, capacities/ skills and knowledge.

The specific goals for the 1st cycle are:

M01 - to become Mathematics an enjoyable subject;

M03 - to promote a self-domain on the essential notions of arithmetic and computation,

M05 - to enlarge number and proportion concepts;

M07 - to develop reasoning and solving problem skills;

M09 - to develop the capacity to use Mathematics in real situations,

M10 - to develop understanding about other cultures and international cooperation;

T01 - numbers and operations;

T05 - numbers and computation;

T10 - probability and statistics;

T16 - functions and calculus;

Teacher attitudes towards ICT

Since seventies, teachers have been discussing on the technological impact on the educational process. The ICT have caused a true

revolution in society in general (Toffler' s Third Wave) and in some sectors of the educational system in particular.

However, and due to the fact that educational systems lack in general a dynamic attitude and are resistant to change, the adoption of ICT in the non-higher education curricula has been just a token of modernity.

The enormous potential of ICT as a mean of support to the learning process has been neglected in the teaching of Mathematics. Teachers generally have a favourable attitude towards the inclusion of the computer in the classroom, and voice to need of having the necessary training to use it with professionalism and as a means to conceive, apply and evaluate the learning process globally. The results of an inquiry, conduced at national level, shows that teachers are very interested to use computers in their disciplines. The responses to the following nine statements are given in a five point Linkert' scale, and they try to measure teacher' attitudes towards the use of computers in the classroom.

A06 - Students that can use computers will learn better some topics.

A11 - I should like that my students could work with computers in my discipline.

A18 - I should be able to allow my students to use computers in some topics of the course.

A26 - To learn how to use computers will become essential in students' education.

A32 - In my opinion, computers cannot be left permanently available for students.

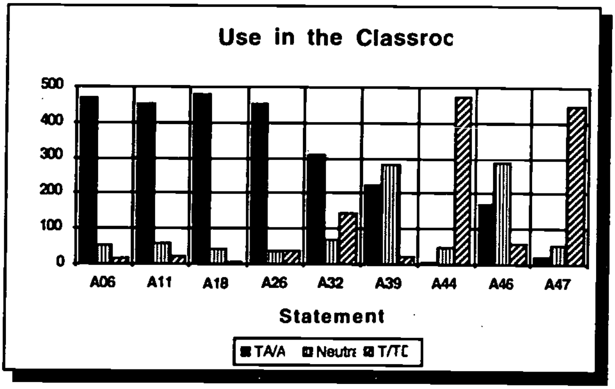
A39 - I appreciate those students that bring their own software to the classroom.

A44 - I will not allow the use of computers at my classes.

A46 - I appreciate very much that my students use computers in their classroom activities.

A47 - I would be clearly in favour of a non ambiguous law that forbid computer's use at school.

The chart below has three categories: total agreement/agreement (TA/A), neutral (N) and disagreement/total disagreement (D/TD).



Educational software

Having in mind the positive attitude clearly shown in the chart, and the opinion given to the following statements: "I am in favour of training teachers courses that shows how to use computers in the classroom" (with 92.5% TA/A) and "Pedagogical material must be created to show us how to use computers in different learning situations" (with 92.3% TA/A), we present some small programs to help teachers in their main concerns. According to the specific goals and the themes given before, we indicate where they can be used (see table below).

L01 - build "Rúdico" square.

L02 - puzzle to find digits in a multiplication.

L06 - play with absolute values.

L09 - harmonic series (rounding errors - Srinivasa).

L11 - divisibility by 7.

L17 - solving equations (1st/2nd order in one unknown).

L20 - problems about Pythagorean theorem.

L22 - computation of GCF an LCM.

L25 - change of numeration system.

L28 - Mathematics' Olympiad: systems of non linear equations.

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Calculus and general secondary education

Vitaly. V. Tsuckerman

Summary

Major issues concerned with the introduction of the fundamentals of calculus into the contents of secondary school education have been considered.

In a natural manner, the search for a solution of the undoubtedly hard problem "*How should the fundamentals of calculus be taught at secondary school* ?" must be preceded by an explication of "*Why should this be done at all* ?". In doing so, the argumentation that the school mathematics should be brought closer to the actual state-of-the-art is not sufficient for the mere reason that the classical calculus at present cannot claim to be the cutting edge of mathematical science. Compelling reasons are needed to show that a knowledge of calculus will contribute felicitously to the cultural background of an educated personality of present day (be even he or she a pure humanist). The importance of such a contribution having been acknowledged, one will inevitably have to face a challenging question "*How* ?".

The most important concepts of calculus are the *derivative* , the *integral* (definite integral), and the *differential equation*. They convey a profound significance pervasive of the intellectual, spiritual and cultural achievements of our civilization. They also bear a direct relevance to the universal problems of motion and development, to the characterization of complex entities and the prognostication of events. One will recall, coming over a century back, as Leo Tolstoy was speaking in his "War and Peace" about the differential of history and the integral of history. The great novelist and thinker had intuitively grasped the profound conceptual significance inherent in calculus as a generalizing means of obtaining local and global characteristics of the current events. Let us take a closer look of these three fundamental concepts of the calculus.

The *derivative* serves to characterize the tendency of a process to change relative to a particular instant in time or to a particular

point in space. Still, the very necessity and possibility to characterize the tendency to change (for example, velocity) at a given instant are not by any means self-evident. One may visualize the situation in such a manner that a moving body at a particular instant is simply found somewhere, and for this reason it apparently makes little sense to seek for a characteristic of motion (velocity) at this instant of time. Indeed, if the distance travelled at a given point in time is zero, then the temporal length of this particular instant is also zero, and the conventional definition of velocity leads to an absurd operation of dividing zero by zero. This kind of reasoning lies within the mainstream of the arguments to which the Greek philosopher Zeno appealed to substantiate the infeasibility of motion. If all the instants of time are employed to show that a body is indeed existent somewhere, than the motion proper will simply fall short of time to be implemented. A solution to this paradox resides in the fact that motion should never be divorced from location. At any instant of time the body is moving and is simultaneously localized somewhere. If this turns out to be the case, then one may attempt to specify the speed of motion at the given instant. The derivative of a function that describes motion creates such a possibility (and implements the notion of an instantaneous velocity). It should be kept in mind though that the derivative at a given value of the argument is merely a number – but, considering that the derivative is defined on a certain set of values of the argument, it may be regarded as a function on this set; consequently, a derivative of the first derivative (second derivative) may be used to characterize the tendency of second order in the change of a process at a given value of the argument. Thus, the function which possesses derivatives of different order is capable of modelling a process through characterizing simultaneously instantaneous tendencies of the process to change. Thus, the properties of the function turn out to be linked intimately to its derivatives. Staying within the theory of differential calculus, we can explore a function (and, simultaneously, a process this function models) through determining its derivatives.

The concept of an *integral* (definite integral) is instrumental in obtaining overall characteristics of a nonuniform process (in what follows the term "integral" will denote a definite integral only. Historically, such was the evolvement of the concept of an integral as formulated by Leibnitz). Thus, the area of a rectangular or a trapezoid can be calculated in a straightforward manner. However,

in a curvilinear trapezoid the ordinate of a point as a function of its abscissa does not obey a linear law, and one stands in need of employing the definite integral concept to find the area of this trapezoid. In much the same manner, if in a rectilinear motion the velocity (and, consequently, the instantaneous velocity) is a constant, then the distance is obtained through multiplying the time elapsed by the value of velocity. With the instantaneous velocity varied in time, the travelled distance is calculated using the familiar instantaneous velocity change law and the concept of an integral.

The *differential equation* is seen as a relationship between the argument, the value of a function (which are major characteristics of a modelled process) and the derivatives of a definite order (reflective of the tendency of the process to alter) for a certain set of argument values. The representation of laws of nature in the form of differential equations, first suggested by Newton, has proved to be an exceptionally fruitful. Simultaneously, Newton outlined methods for solving such equations, which has in fact provided a means of reaching sophisticated conclusions from the discovered relationships, in particular, predicting the change of processes in time. The differential equations have come to be a very effective tool for mathematical modelling of cause-and-effect relations of the real-life world. The spectacular achievements accomplished in this field of scientific reasearch have engendered, by the third part of the 19th century, an illusory belief that the differential equations represented an all-embracing and versatile form for expressing scientifically the laws of nature. Further progress in science and technology have constrained the differential equation formalism to renounce its monopoly in describing the laws of nature; nonetheless, the differential equations continue to remain a powerful and effective technique in the study of various processes.

We have outlined here in a general form the major concepts which in fact argue for the expediency of introducing the fundamentals of calculus into the content of secondary education. The concepts are unalienable from the history of the human civilization development. The calculus has grown up to about three hundred years of its existence. Projected onto the sum total of the actual scientific and technological knowledge, the human experience in this domain dating back for three centuries can hardly amount to a hundredth of a per cent. The tremendous progress in science and technology accomplished within this period of history has inseparably been

linked with the productive use of the ideas and techniques practices in the calculus.

The said above can hardly be a revelation for anyone with a solid background in calculus. On the other hand, a straightforward explication of the basic concepts of calculus does not appear typical for advanced courses in this discipline. Such courses are most commonly taught in an emotionally restrained manner. In a general sense, the comprehension of the cultural values borne out by the calculus concepts is arrived at *mediately*, through in-depth study of a logically consistent theory and through solving a sufficiently large number of problems based on this theory. At school, such an approach is unrealistic, and one must therefore look out for an alternative way that would enable *a formulation of the humanistic content of basic concepts of the calculus expressed in an explicit form*. In the author's opinion, other topics of calculus (except for those emphasized above) can be dealt with in the school program as an obligatory minimum needed to complete the learned material to a consistent whole.

One will have easily observed that from the question "*Why?*" we have actually come to a question "*How?* ". In this context we shall focus on two other issues that appear to be relevant to the matter at hand.

1. Of prime importance is to have *a printed textbook* accessible to use, in its *basic core specifically concerned with the calculus issues as outlined above ; on the whole, such a textbook must represent a minimal extension of the basic core as to make possible an argumentative exposition of the theory.* For a long time, the author has been nursing the idea of writing such a book. For a number of years, the author was lecturing a special course "The Principles of Calculus and Elementary Functions" (name was varied occasionally) to university students (trained to be teachers of mathematics in the future). Two years ago, the author delivered a paper on the subject in question at the 7th International Congress of Mathematical Education (August 1992, Quebec, Canada). The issue concerning the elementary functions was not an accidental choice. For one thing, the elementary functions provide a starting basic set of functions suited for studying the general functional relations, and for another thing, they are in fact the basis from which the calculus has evolved.

2. *In teaching the basic school curriculum, the teacher will intentionally omit all essential technicalities which are commonly made use of in proving a particular material and reformulate them as problems that are to be solved.* In well-grounded pupils, such an approach (the exact opposite to evade difficulties) will not fail to elicit interest in the subject. *The pupils are thus invited to find a way to the intentionally omitted proofs – either attending special lessons, or learning individually without assistance, with the above referred to textbook for guidance.* This circumstance imposes heavy demands on the availability of such a textbook for the pupil within his hand's reach. Currently, the author is planning to have his book "The Principles of Calculus and Elementary Functions" completed in 1995.

Factors for change in teachers' conceptions about mathematics

Günter Törner

Summary

This is a short report on joint work with E. Pehkonen which was realized during spring and summer 1994 in Germany. A more detailed version will appear somewhere else. We investigated questions concerning the development of teachers' conceptions about the role of teaching of mathematics as part of their career. Particular emphasis was given to changes of this conception. We were interested in the factors which caused changes, especially when some discontinuity arose in these changes. For gathering the data of the 13 teachers, we used two types of methods: a brief questionnaire and interviews. During the interviews, altogether 49 statements were made about the change. In all the given statements, the test person refered to himself as a teacher in 72 % of the cases and as a learner in 28 %.

On the conditions for teacher change

First we refer to the model of Shaw et al. (1991): In order to affect a successful and positive change, (1) teachers need to be perturbed in their thinking and actions, and (2) they need to commit to do something about the perturbance. In addition, (3) they should have a vision of what they would like to see in their classrooms, and (4) develop a plan to realize their vision. --- It was the school adminstration which provided us with the teachers' adresses, however under our constraints: the teachers ought to teach innovatively at least according to the view of the school administrators and should have taught at least 10 years. So we were allowed to assume in our first approach that the teachers participating in this investigation should have some*vision* and felt some *commitment*. In particular, it, thus, was important to focus on the factors of their pertubance.

In addition, we wanted to include teachers from all German school forms (Gymnasium, Realschule, Hauptschule, Gesamtschule) in this research since different teaching traditions in these schools might cause the teachers to think differently.

The design of the research

During his career, each teacher changes all the time as new experiences are gained. Typically, such a change is continuous since beliefs do not change radically, but evolve through extensive, extended experience. However, every now and then there are some bigger steps, points of discontinuity. This research, among other problems, is particularly interested in these discontinuities: Which were the key experiences which changed these conceptions (see Pehkonen 1994)? --- For this, we generated the following four main questions to be used in the video-taped interview:

- Tell your "history" as a mathematics teacher.
- How have you taught at the very beginning?
- How are you teaching today?
- Can you name some factors which have, perhaps, had an influence on your change?

The teachers' responses to these questions was aided by some additional questions depending on the situation, until we thought that we have gotten the answers to the question set (theme interview methodology (Lincoln & Guba 1985). For example, while discussing the change factors, we might ask for them using the list of possible sources for the perturbance given by Shaw et al. At the end of each interview, there were additional questions about the questionnaire which will not reported here.

Results

Change factors. During the interviews, according to our interpretations, 49 statements about the change were made. We have classified these statements into fifteen different change factors. Furthermore, these fifteen factors could be classified into two groups: experiences as a teacher, and experiences as a learner. And these both again into two groups: experiences with individuals, and experiences with institutions, respectively authorities. Finally, we have four groups of change factors:

Experiences as a teacher • with individuals

(1) experiences with pupils in school

(2) experiences with own children at home

(3) experiences with children of relatives

(4) parents of pupils

- <u>with institutions resp. authorities</u>

(5) experiences from other school forms

(6) experiences with the school administrations

(7) experiences as a class teacher

(8) cooperative teaching with colleagues

(9) changes in society

<u>Experiences as a learner</u> • <u>with individuals</u>

(10) excellent teacher tutor

(11) a working group of volantary teachers

- <u>with institutions resp. authorities</u>

(12) in-service course

(13) further studies at the university

(14) mathematics education conferences

(15) literature

<u>Description of the change factors</u>. Each factor has caused the teacher to begin a critical reflection of his own teaching. We will briefly comment on the factors.

(1) However, teachers do not only report changes concerning pupils as mathematics learners, but do point out the individual and personal dimensions of their pupils as part of a complex social environment. (2) Nearly every teacher who is also a parent mentioned the difficulties of teaching mathematics observed in his own children. In many cases the topics of the mathematics lessons cannot be reconstructed on the basis of the informations given by the children. The same applies to hidden goals of the critizised teaching. However, it is very difficult to influence this process in order to support the own child. (3) A mathematics teacher is regarded among his relatives as a "special authority". It seems to be self-evident that such an expert should be able to give therapy and cure in the case of any difficulty. (4) Today, a teachers' relation to the parents of his pupils appears to have changed into a more formal one. A few teachers are frightened by the legal democratic power

given parents by law. Hence, this relation can no longer, in general, be understood as a cooperative personal relation.

(5) Some teachers had come into contact with other methodological conceptions, e.g. the teaching of mathematics in primary or vocational schools. The proposed and realized solutions for teaching in those school forms lead to a critical review of one's own teaching.(6) Many teachers claim that the situation with respect to administrative regulations have become worse in the last five years. The average number of pupils per class has risen although problems with the pupils have emerged. (7) The responsibilities of a class teacher cause the instructor to concentrate on pupils' personal problems. As a result, mathematics often stays in the second place. This enlargens the horizont of the teacher, but at the same time, relativates the role of mathematics as a dominant subject. (8) Especially in Gesamtschulen, there was a strong trend to realise reform conceptions in teaching, such as co-teaching. This heavily influenced the teachers' later teaching style. (9) Nearly every teacher reported a drastic change within the society which influenced the situations within families. The changing society includes also changes in the organisations of schools (e.g. large classes, restricted financial resources) which have an impact on the teaching of mathematics. Some keywords were: the crisis of family in our western society, materialism, TV-consumption, lack of interest in school achievement, deficiency in the potential to concencrate, computerization may lead to a lack of ability to communicate, the 'electronisation of children's rooms, lack of professional perspectives, deficiency in critisms, manipulation by media, politication of schools ...

(10) In many cases, experiences with and influences of teacher educators during teacher pre-service education had a great impact. Sometimes these experiences with inspiring personalities could lead to a life-long contact and friendship. (11) The activities of these local groups usually consisted of developing and producing teaching materials for the mathematics class, e.g. working sheets. There was also a group of teachers who discussed interesting actual mathematical questions, e.g. the introduction of fractal mathematics into school.

(12) In Germany, there are govermental institutions which offer in-service courses. However, administrative arguments restrict the attendance of these courses is restricted to at most one participant

from each school. Thus, cooperative learning with colleagues from the same school is not possible in general, however. (13) Some teachers reported on further voluntary studies at the university. (14) Mathematics teacher conferences were quoted as a place to come in contact with new ideas. However, the attendance of conferences is restricted for teachers by administrative orders, as is the access to in-service courses – see factor (11). (15) A few teachers refered to the book of Freudenthal's "Mathematical Education as an pedagogical task" or to many books of Polya, as well as to some wellknown German journals on mathematics education.

<u>A summary of the interview data</u>. In Table 3.1, there are given the data obtained with interviews.

	(a)	(b)	(c)	(d)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Mr. A	Gy	II	#	10	x	x	x			x			x	x					x
Mr. B	Re	I	-	10												x	x		
Mrs. C	Ge	I	-	10	x								x						
Mr. D	Ha	I	-	17	x					x			x		x	x			
Mr. D	Ge	I	#	13	x								x			x			
Mr. E	Ge	I	-	15	x					x	x		x						
Mr. F	Re	I	-	12	x					x			x						
Mr. G	Gy	II	#	22	x				x	x			x	x	x				
Mr. H	Gy	I	-	20				x	x			x							
Mr. I	Gy	II	#	30								x	x				x	x	
Mrs. J	Gy	I	-	18	x	x							x						x
Mrs. K	Ge	I	-	12		x			x	x					x				x
Mr. L	Ge	I	#	28									x						
Sum					8	3	1	1	3	6	1	2	10	2	3	3	2	1	3

Table 3.1. The data obtained with interviews: (a) = school form (Gy = Gymnasium, Re = Realschule, Ha = Hauptschule, Ge = Gesamtschule); (b) = school level (I = lower, II upper secondary); (c) = other activities (director, mentor, supervisor); (d) = teaching experience (in years). The rest are the fifteen change factors mentioned above.

From all 49 given statements about the change, the test persons refered to themselves in 72 % as a teacher and only in 28 % as a learner.

Conclusion

In this short report we can only present some preliminary consequences as the following ones: Let the teacher ...

- change roles (let the teacher become a learner, exercise problem-solving, force the teacher to identify with a student, improve the system of in-service learning ...)
- change point of views (encourage the teacher to observe himself from outside, use video-taping of teaching, let the teacher observe teaching in other school types, accept students as individuals since mathematics is only part of their life, be critical with the general role of mathematics in life, make the teacher acquaintant with 'nonstandard' reflections on mathematics, create groups for fruitful communication, give the teacher access to qualified material and literature ...).

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WORKSHOPS

Teaching of measurements – experiences from primary teacher education

Maija Ahtee, Hellevi Putkonen

Introduction

Measurement is the process used to answer the questions: how many? and how much? Measurements can be made by human senses, for example estimating distances, weights and temperatures and evaluating roughness by touch. Human capabilities are, however, both extended and refined by instruments. The information is always a comparison of the measurand with a reference quantity of the same kind. There are many basic ideas which have to be perceived before one can fully understand all the aspects in measurement of different concepts (Ahtee 1994; Dickinson *et al.* 1984). According to Piaget's work (Piaget *et al.* 1960, Piaget & Inhelder, 1974) the three fundamental operations,which the measurement process depends on, are conservation, reversibility and transitivity.

Teaching project of measurement in primary teacher education

At the department of teacher education the primary teacher trainees study mathematics and its teaching methods during their first academic year. Nearly one fifth (20 hours from the total of 110 hours per student including lectures, practices and individual work) of the mathematics program was taken for the integrated teaching project of measurement at the end of the first year.

In the project the following ideas were central:

- to integrate mathematics with pedagogy, science and oral communication skills
- to promote teacher trainees' broad professional competence
- to improve teacher trainees' knowledge of quantities and units
- to develop co-operation between the teacher educators in different subjects.

After eight lessons each group of 16 students was given a total of 10 hours to concentrate on one of the following six concepts: length,

area, capacity, volume, weight (mass) and time. Then in each group the students were divided into five teams to plan and organize a teaching session of two hours about their specific concept for another student group. One of the main ideas was that the students taught each other. By teaching in teams of threes the students also got an experience of team teaching. The role of the lecturer was to support and supervise the working of the groups.

At the beginning of the project the students discussed what aspects should be taken into account when teaching a quantity and its units. As a result of the discussion the following stages were introduced:

1. The introduction of the quantity
2. Measuring with primitive, personal or improvised units like handspans, sticks and strings
3. Measuring with standardized units
4. Estimation
5. The system of standardized units
6. Applications and problem solving

Although the student teams were to plan their own teaching around the frame of the stages above, they were completely free to choose teaching methods as they preferred. The main idea in presenting the concept was to put in practice the concrete learning activities for measurement and estimation of the quantity outside the classroom and in the open air. The students were encouraged to use pedagogic drama, songs and performances as well.

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Workshop: The use of concrete material as a means to support the learning of mathematics

Joop van Dormolen

In these exercises you must discover one or more mathematical rules or/and concepts. It would be nice if you proved the rules too.

During the exercise you are supposed to register your own experiences and your reactions to those experiences. The mathematical rule(s) is (are) not essential for this exercise, the experiences and your reactions are more important now.

After each exercise you might discuss what experiences and reactions you had and how you can use them for teaching mathematics to young students:

What did you think, feel, discover, ... – while doing the exercise? – while looking for and finding a rule? – while trying to prove that rule?

In particular reflect on how the concrete material and your handling with it influenced your findings:

- In what way did it help to learn more (about) mathematics?
- In what way was it a hindrance?

Can you think of consequences for your own teaching methods?

Can you think of other ways to present the material to young students?

Can you think of more practical or more adequate material?

50 (or more) numbers

Lay the number cards in their natural sequence and face up.

Starting from number 1, turn every second card face down.

Then, starting again from number 1, turn every 3rd card: count every card, irrespective whether it is lying face up or face down.

Then, starting again from number 1, turn every 4th card.

Then every 5th, then every 6th, and so on until you have had the last card in the sequence.

During the process you might be able to conjecture what the final result will be:

Which cards will finally be face up and which will be face down?.

Then you should find the mathematical equivalent for 'turning a card'.

You can then formulate a mathematical rule, using that equivalent expression.

You should, of course, be able to prove the rule.

Squares and other rectangles

1 Play around with the material for a minute in order to get acquainted with it. As a preparation for what is coming next, build some squares and some other rectangles.

2a Take 1 yellow, 6 reds, 9 blacks and make a square.

Do the same with 4 yellows, 20 reds, 25 blacks.

b Take 4 yellows, 9 blacks and as many reds as you need to make a square.

c Take 12 reds and as many yellows and blacks as you need to build a square.

Do the same with 16 reds.

d Think of some more exercises like the ones in a, b and c.

e If you did not already found a rule, find one now.

f If you did not already found a mathematical equivalent for the cards and for the making of squares, find one now.

3a Take 1 yellows, 8 reds, 15 blacks and make a rectangle.

Do the same with 16 yellows, 10 reds, 1 blacks.

b Take 1 yellow, 5 reds and as many blacks as you need to make a rectangle.

c Take 9 reds and as many yellows and blacks as you need to make a rectangle.

d Think of some more exercises like the ones in a, b and c.

e If you did not already found a rule, find one now.

f If you did not already found a mathematical equivalent for the cards and for the making of rectangles, find one now.

World of neutrals and values

Take many yellow squares and the same amount of red squares (leave about 20 of each colour apart as reserve).

Put what you took in a heap, in a box, in an envelope, under a handkerchief, or whatever, as long as they are together.

Together they form a neutral world.

1a Take away 5 yellows.
Now the world is no longer neutral: it has a value of +5.

b Put the 5 yellows back. The world is neutral again.

c Take away 8 reds.
Now the world is no longer neutral: it has a value of -8.

d Put the 8 reds back. The world is neutral again.

e What is the rule for the values of the world?

2a Give the world a value of +9. Then take away 6 reds.
What is the value now?

b Give the world a value of +9. Then take away 13 reds.
What is the value now?

c Give the world a value of -6. Then take away 5 reds.
What is the value now?

d Give the world a value of -6. Then add 5 reds.
What is the value now?

e Give the world a value of -12. Then add 15 reds.
What is the value now?

f Make some more exercises like the ones before if you want.

g If you did not think about the mathematical equivalent for the world, its values and the taking/adding of reds, find it now.

3 In exercise 2a to 2g, change 'red' for 'yellow' and then do that exercise again.

Symmetry in Primary School

Günter Graumann

Summary

Geometry in primary school is very important to elaborate geometrical concepts and imagination of space in a concrete and acting way. Symmetry is a fundamental idea in everyday world that can be seen under three aspects: Balance, Optimality, Regularity. Symmetry thus is a topic that can connect different subjects of school.The bilateral symmetry is a standard theme for the third grade in Germany. A unit for this and a video about its performance taught in an integration class where normal and hanicaped childrens learn together were presented and discussed. Also aims and other topics of geometry in primary school have been discussed in the workshop.

In many domains of life symmetry plays an important role

Many plants and animals have a symmetric physique whereat the symmetry probaly arose because of equality of special directions (for example left-right with walking or flying animals). Also I heard (in TV) that symmetry is important for some animals by looking for a partner (e.g. birds whose tail is not equal because of an accident are getting no partner). Perhaps our aesthetic feeling has its roots in this. Any way symmetry was important for arts and architecture since ancient times.

Besides these examples where symmetry in its geometric form appears we find **symmetry in a generalized form** in all areas of life. Three aspects can be pointed out: **1.** BALANCE **(a)** in a physical way (e.g. with an aeroplane, a swing, a bridge or the erected walk of men) **(b)** as balance of power (e.g. in parliament or a round of discussion, between the authorities of the state, between principles of justice or the balance of darkness in a picture) **(c)** as principle of harmony, completeness and beauty (in arts, architecture and music but also in the sense of physiology and phylosophy). **2.** OPTIMALTY **(a)** with regard to physical or mathematical conditions or relations (e.g. states of energy in an electromagnetic field or a gravitation field, the distribution of heat or the relation between surface area and volume with solids) **(b)** with regard to the functionality of forms in nature and technique (e.g. with leafs, liquid drops, tins, balls etc.) **(c)** with regard to the economy of problem solving,

constuctions and principles of ordering in mathematics, sciences and everyday world. **3.** REGULARITY **(a)** in the sense of repetitions and coincidence with geometric forms (e.g. ornaments, parquets, patterns and self-similarity) **(b)** in the sense of general formings and processes (e.g. rhythm in Music, rhythm of the year, palindromes, oscillations, computer-loops or winding engines) **(c)** in regard to mathematical regularities and structures (e.g. commutativity, symmetry of a relation and patterns of matrices).

As already seen is symmetry an aspect we find in all areas of life. Especially the subjects in primary school can be connected during teaching symmetry. For thus I will now list several aspects of symmetry for each subject:

1. GENERAL KNOWLEDGE: The symmetric physique of men, animals, leafs (in first view) as already mentioned, snow crystals and other crystals, symmetric road signs, several buildings respectively their fronts, a lot of packings and technical apparatuses and most of all vehicles. **2.** SPORTS: Sports equipment, courts and other fields for sports, balance acts and formations during gymnastics or swimming. **3.** ARTS AND MUSIC: Ornaments, parquets and patterns, church windows or facades of houses, commercial designs, knitting pattern and string pictures, forms of music instruments, dances with formations, rythm, a melody and its inversion. **4.** LANGUAGE Symmetry of letters, words and sentances which mean the same read forwards and backwards, double words and rhymes. **5.** MATHEMATICS: Besides symmetric figures and patterns in geometry symmetric letters and digits - the so-called reflection-numbers (palindromes) - are examples for generalized symmetry. The commutativity of arithmetic operations and symmetry of relations as already mentioned are other examples for generalized symmetry in mathematics.

Two years ago I planned **a unit about symmetry for grade three** together with two teachers and also made a video of it. 24 children visited this class, five of them with special handidcaps: 3 with Down-Syndrom, 1 with MCD (Multi-Celebral-Dysfunction) and 1 child is multiple handicaped. The class is lead by the two teachers, a primary school teacher and a special school teacher. Mostly the two teachers make the instruction together for all children and differentiate in the classroom. Sometimes one teacher workes together with a small group - mostly the handicaped children - in an extra room. The unit was taught in <u>5 double-periods of 90 minutes</u>

each. Besides this in an extra hour for sports and one hour for music a special dance was exercised. In the 5 double-periods the following themes have been treated: **1.** Introdution with blot-pictures and folding-exercises. **2.** Balance-exercises, the symmetry of our body and the symmetry of paper-aeroplanes. **3.** Looking out for axis of symmetry and completing figures to symmetric figures. Consolidating the body-schema with the handidcaped children. **4.** The concepts "symmetry", "symmetry-axis", "reflection-axis", "folding-axis" and different exercises for deepening symmetry. **5.** Generalized symmetry in language and mathematics and closure of the unit by presenting a symmetric dance.

The final discussion dealed with: The general aspects of symmetry, the unit and its video and geometry in primary school in general.

Computer based geometric ideas for construction of mathematics

Lenni Haapasalo, Roland J. K. Stowasser

Summary

Modern views on teaching and learning call for a comprehensive reform of mathematics in the whole offending the main assumptions that knowledge can be filled into the mind of the learner and that children optimally learn at about the same age the same content in the same order. With respect to constructivist views on the nature of mathematical knowledge and the genesis of mathematical ideas in the mind of the learner, geometry in a broad sense with its abundance of problems, its richness of ideas and its power to develop intuition has to be redefined. Why not with a computer?

The first part of the workshop

consisted of introducing the idea of the 'systematic constructivism' of the *MODEM* project (*Mo*del Construction for*D*idactic and *E*mpirical Problems of*M*athematics Education; e.g. Haapasalo 1994): in the process of concept building the phases of orientation, definition, identification, production and reinforcement are systematically involved with the different modes of information representation. In the MODEM1 studies (Haapasalo 1991) the learning of the slope of a straight line was investigated with the computer program LINE*. In this program each of the phases of concept building is programmed as a submodule and can also be used for computer-based learning as a stand alone program. The verbal, symbolic and graphic cocept attributes (like proportionality, linear dependence, etc.) are dealt simultaneously, opposite to conventional school teaching, which separates them into several different school grade courses. The use of this learning program was introduced with some research results, also. In the MODEM 5 the construction of the concept 'slope' was studiet by gathering the data about the dependence of distance and time when an object was moving with a constant velocity through light gates. A microcomputer, equipped with a measuring interface can visualize this physical pehenomena making the depence develop step by step.

Second part of the workshop

The study of *curves* has degenerated into treating them as graphs of functions or for illustrative purposes only. Mathematics teachers are not aware of the educational potency of them as a fruitful field for exploration with geometric, kinematic, algebraic and other precalculus tools - methods of the 17th century. Computer technology allows efficient revitalising of geometric and analytic model building and makes available the extensive drawing tools which are necessary from an aesthetic appreciation of curves. The very dynamic nature becomes manifest when step by step geometric or kinematic generation of the curve takes place on the screen. The rich material on curves in Newton's Lucasian Lectures on Algebra, or the problems in the Calculus books of J. Bernoulli can in fact be exploited properly using kinematic representations (Stowasser 1976-1979, 1994, Stowasser & Mohry 1978, Haapasalo & Stowasser 1993, 1994). DEMODISK [5] represented in the workshop offers examples of the art of visual communication as a by-product of a European project supported by Stowasser.

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Hypermedia as motivation medium in mathematics learning

Anna Loimulahti

Summary

The mathematical skills of the new university students vary widely and this may cause some difficulties in the learning and also in the teaching processes. After the quite uncomplicated high school mathematics studies the freshmen often find the university mathematics theoretical, difficult and even dispensable. The attitude towards mathematics studies can be very perfunctory. This attitude can be influenced with a new learning environment, a hypermedia learning package for mathematics. The aim of the learning package, introduced in this paper, is to motivate the students to study mathematics more efficiently right from the begining of the university studies. The aspire is to make the students appreciate the fact, that mathematics has potentiality not only in science, but as well in the everyday life.

Introduction

The new learning package is called HypeMATH. This package has been designed in accordance with the principles of hypermedia. Hypermedia combines various media, ie. text, graphics, animation, nonsequently /Nielsen, 1990/. HypeMATH is created in UNIX/X-Windows environment and the hypermedia tool used in designing is MetaCard© /MetaCard, 1992/.

HypeMATH

It is commonly accepted, that we learn by building upon the knowledge we already have. Methaphors can be used quite successfully to teach new concepts in terms of which the students are already accustomed with. The learning package, HypeMATH, introduces several topics e.g. geography, economics, biology, which contain problems that can be solved mathematically. The topics and the problems are not 'mathematical', but cases which are familiar to the students from their daily life.

One can use hints, mathematical theories and tool programs to help working out the problems. The results of the problem solving process are evaluated. This means that for a right answer the

student will score points. The score depends on the amount of the hints and the answer attempts used. The scoring feature is not the most essential part of the system, but it is still an important aspect, because it will give some feedback to the student of the problem solving process.

Everyone, who has used the system before, has a personal score record. When the student uses the system again, the setup of the previous session is restored. The personal score record is updated during the session, and is saved when exiting the system. The basic idea of the learning package, HypeMATH, is shown in Fig. 1.

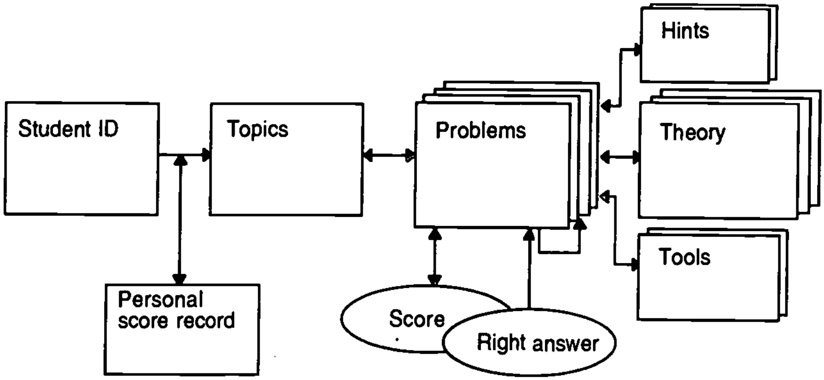

Fig. 1. The structure of the hypermedia learning package, HypeMATH

Development scenarios

The system demonstrated at the workshop was a prototype of a learning environment designed. The intention is, that when the project on this specific product is concluded the same idea described here will be transfered to the WWW environment (World Wide Web) /Hughes, 1993/ using HTML (Hypertext Mark-up Language) /Berners-Lee, 1993/ and also to the PC/Windows environment.

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Open approach to mathematics in lower secondary level

Erkki Pehkonen

The method of using open-ended problems in classroom for promoting mathematical discussion, the so-called "open-approach" method, was developed in Japan in the 1970's (Shimada 1977, Nohda 1988). About at the same time in England, the use of investigations, a kind of open-ended problems, became popular in mathematics teaching (Wiliam 1994). In the 1980's, the idea to use some form of open-ended problems in classroom spread all over the world, and research on its possibilities is still very vivid in many countries (see references e.g. in the paper of Pehkonen 1995a).

Some examples of the problem fields

For about the concept of open-ended problems see e.g. Pehkonen (1995a). Furthermore, many examples of different types of open-ended problems can be found e.g. in the paper of Stacey (1995). Here, we will deal with one realisation of the open-ended approach – the use of problem fields.

Problem fields were used in grades 7–9 within the research project "Open tasks in mathematics", carried out during the years 1989–1992 in Helsinki (Finland) and sponsored by the Finnish Academy. The project tried to clarify the effect of open-ended problems on pupils' motivation, the methods and how to use them. Thus, open-ended problems were used as a method for change in mathematics teaching. In realising the experiment, we tried to stay within the frame of the "normal" teaching, i.e. in the frame of the valid curriculum, and to take account the teaching style of the teachers when using open-ended problems. For more about the research project see e.g. Pehkonen (1993).

In the workshop, the following problem fields were used: Nought and Crosses, Folding Polygons, and Dividing a Square. Nought and Crosses belongs to the group "four-in-a-row" of learning games. The aims of the game are beside the repetition of multiplication tables (also in reversed form) strategic and complex thinking. The second activity, Folding Polygons deals with a paper (DIN A4) folding task: One should try to fold from the paper the

largest possible (according to the area) isosceles triangle, equilateral triangle, right-angled triangle, rectangle, square, parallelogram (not right-angled), trapezium (not right-angled), and rhombus. In the last problem field, Dividing a Square, the problem is to divide a square in four congruent pieces, firstly in five different ways, secondly in infinitely many different ways, thirdly in three times infinitely many different ways, ...

The research results suggest that the open-ended approach, used parallel to the conventional teaching methods, seems to be promising. The pupils preferred this kind of mathematics where one important factor was the freedom let to pupils to decided their learning tempo. They experienced the problem fields used as an interesting form of learning mathematics. They liked most of them very much, and were motivated and activated also during other parts of mathematics lessons. (Pehkonen 1995b)

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Verbal arithmetical problems in exercise books printed for children

Zbigniew Semadeni

Summary

The paper briefly describes some questions concerning the way verbal problems (= word problems) are taught in early grades in Poland.

When I watched lessons in primary grades in Poland, the following pattern of solving problems was manifest over and over again: First, the text of the problem was read (at least twice) by the teacher and/or by selected pupils. Then the problem was analysed verbally: ``What we know?", ``What are we looking for?".

Children performed suitable arithmetical operation(s) and, finally, formulated an answer (necessarily as a full sentence!). Two elements were obviously lacking. 1) I could hardly see any attempt to simulate the situation with manipulatives (this fact is well known and will not be discussed here). 2) No data (given in the problem) were recorded to help children concentrate on the problem. I had expected that teachers would make short records of the data (``15 apples",

``3 boys" etc.). However, on the blackboard and in pupils' notebooks only pure formulas (e.g., 12-5=7) were written. Teachers appeared very consistent in their beliefs that only formulas are worth writing.

No words of explanations were written either. For years I had similar experience with my own children: when any of them was doing her homework and I advised her to write down a word of comment, the answer always was: ``The teacher does not allow this. Only computations should be written."

These kind of observations were taken into account when a new set of exercise books (Lankiewicz & Semadeni, 1994) was written for children in the second grade (8-9 years old). At the beginning, the text of the problem was followed only with blank spaces for computation and/or an answer (with a comment: ``You always give an answer to the problem. You write it if a space is provided for it;

otherwise your answer should be oral or in thought). Then gradually an analysis of the problem is developed, e.g.: ``There were ..." (a number or something meaning `unknown', e.g., the question mark, should be filled in), ``Gain ..." (or ``Increase", ``Loss" etc.), ``There are now ... ". This becomes particularly important when inverse problems are considered (i.e., when the original quantity or the change is unknown).

Equations of the type 5+[box]=12 are used as a means to develop both: children's arithmetical skills and their understanding of operations. They are also used to analyse verbal problems; children write an equation, find the unknown number and write it in the box. According to the Polish curriculum, very simple equations of the type 5+x=12 (with unknown x) should be also introduced.

At a certain stage, students learn to verify whether the number obtained by them is really a solution. We asked teachers whether, in their opinion, verification should precede formulating an answer or vice versa. Teachers argued that it is not a good habit to write an answer before verifying it. However, we should also consider two other points: 1) A second grader may encounter difficulty in understanding what is looked for. Formulating an answer may help the child understand what is to be verified. 2) It is desirable that solvers check whether the number satisfies the original problem. Substituting it to the equation or verifying the arithmetical operation is less preferable because of the risk that the equation or the operation was not the right one. Checking the solution before writing an answer decreases the chance that the child would think of the original problem while verifying the solution. Therefore our advice is: ``Always verify your solution. If a space is left for this, you write; otherwise you check the correctness mentally. You may verify your solution before writing your answer or after; preferably twice (before and after)."

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POSTERS

Mathematical activities for high school students in Estonia

Elts Abel

Curriculum procedures in many Western and Nordic countries indicate that the implementation of curriculum changes is very slow and difficult. Traditionally in Estonia the process of enriching the curriculum and developing the mathematical abilities of young people has been realized through optional lessons, specialized classes and out-of-classes/school activities (OoCA).

OoCA is systematic and organized work with students outside of school lessons (regular or/and optional) under the supervision of good specialists in mathematics. Its main aims are:

1) to present a range of stimulating and enjoyable activities to enrich the mathematical experiences of students;

2) to stimulate interest in mathematics by encouraging young people to deal with this subject in their free time;

3) to establish friendly contacts between students, teachers and scientists;

4) to discover talented project-type students in mathematics and give them extra problems to improve their knowledge;

5) to identify olympiad-type gifted students and prepare them for international competitions;

6) to foster the development of creativity in mathematics teachers.

Math circles, clubs and competitions are popular and have very long traditions in Estonian schools (Abel, 1993).

The studies in the State Advanced Training School in exact sciences (by correspondence) last three years. Learning materials and problems are mailed to students 5 times each school year. In recent years not only most national competitions, lectures and workshops for students and teachers, but also training sessions for candidates of Estonian teams have been organized and run by this school.

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Economical problems as the contexts for constructivist approach to school mathematics

Jüri Afanasjev

Finding the contexts for constructivist teaching from microeconomics is observed.

Traditionally physical and geometrical contexts are used as the problems for teaching school mathematics. One of the richest fields for finding the contexts for constructivist teaching is also economics, especially the field of microeconomics. The using of economical problems in teaching mathematics is very important, particulary for young students in the countries now building the society based on market economy.

We are working out the preliminary basic course in form of a booklet (Afanasjev,1994) for the teachers and upper secondary level students. In that we have used some ideas of microeconomics as contexts for the teaching and learning school mathematics. The functions of demand, supply, costs, revenue, profit and price elasticity are observed as the contexts for approach to such mathematics terms and problems as increasing and decreasing functions, intercrossing the lines, increment and derivate of function, solving the linear and quadratics equations, systems of linear equations and linear inequalities.

For example, if you know the cost function of production instruments, $C(x) = 0.1x^2 - x + 100$ we can find the cost of production for one instrument on the different amounts of production $C(101)-C(100) = 19.1$, $C(201)-C(200) = 39.1$. Now we can suppose that these costs will be increasing with the increasing of amount. It's clear, that derivative function $C'(x)$ (the marginal function of costs) is approximately equal to the cost of production of one instrument, $C'(x)= 0.2x - 1$ and $C'(100) = 19.0$, $C'(200) = 39.0$. As the marginal function $C'(x) = 0.2x - 1$ is an increasing function, our previous presumption about the increasing of costs (after the minimum of $C(x)$!) is true, too.

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Teaching Geometry in the Czech Republic

Frantisek Kurina

In this poster we give a piece of information about a new conception of geometry for the students aged 6 to 15 years in the Czech Republic.

Using manual activities with models of geometrical shapes (building sets, mosaics, ...) students become acquainted with properties of geometrical shapes in plane as well as in the space. Geometry is not a deductive, but much more a "constructive" and a practical one. Along with modeling, sketching and drawing are also cultivated in geometry. We do not come out from any axiomatic system with the sequence of primitive concepts but we build up a didactical structure. On the basis of experience with the space the students live in we learn that it is possible to divide and fill up the space and plane in a proper manner. It is possible to model the division of plane by folding a sheet of paper; closed broken line which does not intersect itself leads to the notion of a polygon, etc. Experiments with measurements of length, filling up the parts of plane using stones of mosaic or space using cubes leads to understanding of the principle of filling up which is the basis of the idea of length, area and volumen. These pieces of knowledge are on one hand connected with technical practice and study of nature, on the other hand they put arithmetic and geometry together. The poster contains 15 illustration of the pupils' work.

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