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ABSTRACT

Problem solving is one of the main components of mathematics education. This document treats problem solving as a process to be used for learning skills across the mathematics curriculum. Definition of problem solving, the environments promoting problem solving, teaching behaviors and attitudes that enhance problem solving, the criteria for good problems, and how to assess problem solving are emphasized in his document. Three types of mathematics problems for each grade level from grade 3 through grade 8 are also presented. (ASK)

SE

The Problem Solving Strand

Grades 3-8 • Mathematics

TEACHER GUIDE

Linking Curriculum, Instruction, and Assessment

Public Schools of North Carolina
State Board of Education/Jay Robinson, Chairman
Department of Public Instruction/Michael E. Ward, State Superintendent
Accountability Services/Testing
Instructional Services/Mathematics
Raleigh, NC 27601-2825

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Introduction

"Problem solving should be the central focus of the mathematics curriculum. As such, it is a primary goal of all mathematics instruction and an integral part of all mathematical activity. Problem solving is not a distinct topic but a process that should permeate the entire program and provide the context in which concepts and skills can be learned" (NCTM *Curriculum and Evaluation Standards for School Mathematics,* 1989, p. 23). In 1989, after much deliberation, the writers of the North Carolina Revised Standard Course of Study for Mathematics K–8 decided to include as a separate goal that "The student will solve problems and reason mathematically." This treatment of problem solving was not a result of disagreeing with the NCTM Standards, but was a result of the concern that the instruction of problem solving might be omitted if not included as a specific goal. This document will treat problem solving more as a process to be used for learning skills across the mathematics curriculum. Thus, the format will differ from those written for previous goals. The general sections will be expanded and answer the following questions:

- **What is problem solving?**
- **How do children learn to solve problems?**
- **What kind of environment promotes problem solving?**
- **What are the teaching behaviors and attitudes that enhance problem solving?**
- **What are the criteria for good problems?**
- **How does one assess problem solving?**
- **What about giving answers?**

Then, for each grade level, some examples of worthwhile problems or tasks will be presented. This section will also include a list of the specific objectives from the Standard Course of Study addressed by the problems, and sample multiple choice and open-ended questions that might be asked about the problems.

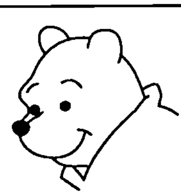
What is problem solving?

Problem solving is basically what we do when we don't know what to do. If we know what to do, then we are performing an exercise, not solving a problem. What constitutes a problem for one person may not be a problem for another. Here are several criteria for defining mathematical problems:

- ✏ The student recognizes a perplexing situation.
- ✏ The student is interested in finding a solution.
- ✏ The student is not clear about how to proceed in order to find a solution.
- ✏ The solution requires use of mathematical ideas.

If a student knows how to proceed in order to find a solution, then this student is performing an exercise, not solving a problem. The most important consideration when determining whether or not something presents a problem is whether or not the path to the solution is clear for whoever is attempting to solve the problem.

Pooh's Principle:



By definition, when you are investigating the unknown, you do not know what you will find or even when you have found it.

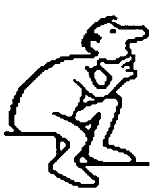
Algorithm is a rule or procedure for solving a recurrent mathematical problem.

Heuristics are general procedures or strategies. They are something to try when solving problems, and it is important not to treat them as algorithms.

Heuristics used in mathematics include:

- Guess & Check
- Make a Table
- Act It Out
- Work Backwards
- Use Objects
- Draw A Picture
- Use Smaller Numbers
- Look For A Pattern
- Make An Organized List
- Sleep On It

<u>All</u> students can learn to think mathematically ... every student can and should learn to reason and solve problems, to make connections across a rich web of topics and experiences, and to communicate mathematical ideas.



"Students learn best when they are encouraged to construct meaning for themselves rather than having fixed procedures imposed upon them."

Grayson Wheatley

Problem solving situations require that students develop and execute a plan. Many approaches exist, sometimes called heuristics, but these are general approaches and not algorithms that can be routinely applied. These general approaches include working backwards, making a table, looking for a pattern, using smaller numbers, acting it out, drawing a picture, guessing and testing, making an organized list, using objects, etc. A student's plan for solving a problem may require using arithmetic; however, worthwhile tasks also will require other types of mathematical thinking. Problem solving presents many opportunities for demonstrating that math is much more than arithmetic and that all the strands of mathematics, and other curricula, are interrelated.

How do children learn to solve problems?

The potential for learning exists when there is an experience which does not fit into students existing ways of thinking. Then students need experiences upon which they can reflect, or opportunities to construct their own patterns, relationships, meanings, and concepts. Jean Piaget's work has long informed us that several basic conditions need to exist for learning to occur:

- ✏ *maturation:* a task must be "developmentally appropriate" for the potential learner ... learning to ride a bicycle is not developmentally appropriate for a three-month-old but is for a nine-year-old;
- ✏ *concrete experiences:* the learner needs plenty of opportunities to interact with his/her environment ... learning to ride a bicycle requires many opportunities to get on a bike and try to ride it ... all the telling and lecturing in the universe will not cause a child to be able to ride a bike without the chance to actually ride one;
- ✏ *socializing:* the learner needs plenty of time to talk with others about the task ... sharing ideas, strategies, successes, etc.

Thus, ideas and concepts are meaningful and retainable to the extent that they are part of self-constructed frames of reference.

"Students are more likely to take risks in proposing their conjectures, strategies, and solutions in an environment in which the teacher respects students' ideas, whether conventional or nonstandard, whether valid or invalid. Teachers convey this kind of respect by probing students' thinking, by showing interest in understanding students' approaches and ideas, and by refraining from ridiculing students. Furthermore, and equally important, teachers must teach students to respect and be interested in one another's ideas."

NCTM *Professional Standards for Teaching Mathematics, 1991, p. 57*

"Serious mathematical thinking takes time ... A learning environment that supports problem solving must allow time for students to puzzle, to be stuck, to try alternative approaches, and to confer with one another and with the teacher. Furthermore, for many worthwhile tasks, tasks that require reasoning and problem solving, the speed, pace, and quantity of students' work are inappropriate criteria for 'doing well.'"

NCTM *Professional Standards for Teaching Mathematics*, 1991, p. 58

What kind of an environment promotes problem solving?

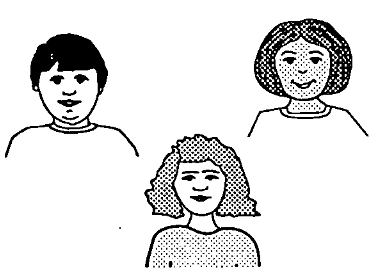
In order for students to be engaged in solving problems, they need to feel free to take risks. This requires an environment in which all thoughts are accepted and valued. The teacher must ask questions which probe students' thinking and strategies, rather than questions which require regurgitation of facts presented in lectures or textbooks. This also means that teachers are genuinely interested in hearing students' thoughts, not answers already established in the teacher's mind. This environment also promotes and requires that students listen to each other and try to understand and justify the thinking of others. Students are required to question and challenge the ideas of others with respect and in ways which avoid any kind of put down.

In order for students to be engaged in solving problems, they need rich problems to solve. These problems might require work over several days or even longer. *One of the characteristics of good problem solvers is persistence.* Students need opportunities to develop persistence by working on some "big hairy problems" over longer periods of time than just a few minutes or one class period. The characteristics of rich problems or worthwhile tasks will be outlined in a later section.

In order for students to be engaged in solving problems, they need to participate in defining the problem or task. This requires some initial negotiating at the total class level. For example: consider the task of determining how many different rectangles can be created with the pieces of a tangram puzzle. The class must negotiate in order to define the word "different". Will different relate to area, perimeter, number of pieces, number and size or shape of pieces, various arrangements of the same set of pieces, etc.? If every group in the class is working with the same definition for the word different, the total group discussion will be more meaningful and valuable. This problem solving environment also involves students working in small groups. During this group problem solving, students should be asking questions, giving information, giving suggestions, sharing ideas, helping others explain, pulling ideas together, showing interest and kindness toward each other, and showing a willingness to change one's own ideas if someone makes a good argument. Tracking these behaviors can be included in assessment. Then students must be pulled together into a total group for presenting, justifying and defending their problem solutions and strategies. During this time, students are obliged

"Demonstrating respect for students' ideas does not imply, however, that teachers or students accept all ideas as reasonable or valid. ... Therefore, the central focus of the classroom environment must be on sense-making. ... Teachers should consistently expect students to explain their ideas, to justify their solutions, and to persevere when they are stuck. Teachers must also help students learn to expect and ask for justification and explanations from one another. The teachers' own explanations must similarly focus on underlying meanings; something a teacher says is not true simply because s/he 'said so.'"

NCTM *Professional Standards For Teaching Mathematics*, 1991, p. 57



Which one has the math gene?

to listen and try to make sense of other class members' explanations. They should indicate agreement, disagreement, or failure to understand others' solutions. In the case of conflicting ideas, students should attempt to justify a solution, examine alternatives, and work toward consensus.

This problem solving environment must include a teacher who models problem solving behaviors. If "problem solving is what we do when we don't know what to do," how does one model problem solving if one always knows the answers or exactly how to proceed? Students need to have opportunities to observe teachers solving problems, and teachers need to have the courage to pose problems that are problems for themselves. While working on a problem in class, teachers can model their own processes by thinking out loud. This kind of "self talk aloud" also helps students who are having difficulty expressing their own thought processes.

Basically, a problem solving environment sends a consistent message that student thinking is required and valued, provides opportunities for students to take risks and spend time solving problems and then present and justify solution strategies, and includes important others who model the problem solving process.

What are the teaching behaviors and attitudes that enhance problem solving?

Some very basic beliefs or attitudes are necessary in order to promote problem solving. These include:

- all students are capable of solving problems and increasing their mathematical understandings;
- teaching is not telling or imposing methods upon students, but rather creating an appropriate environment in which to coach students as they construct their own understandings;
- confusion is a natural part of the process of learning; and
- students learn best in a cooperative rather than a competitive setting.

All students are capable of solving problems and learning mathematics. Our culture is the only one that promotes the myth that the ability to learn mathematics is inherited. We push this myth to an even further extreme by suggesting that if one has this "math gene," this person is some kind of "nerd or weirdo." Thus, it is perfectly acceptable in our culture to

"One aspect of the teacher's role is to provoke students' reasoning about mathematics. Teachers must do this through the tasks they provide and the questions they ask. For example, teachers should regularly follow students' statements with, 'Why?' or by asking them to explain. Doing this consistently, irrespective of the correctness of students' statements, is an important part of establishing a discourse centered on mathematical reasoning. ...Teachers also stimulate discourse by asking students to write explanations for their solutions and provide justifications for their ideas."
NCTM *Professional Standards for Teaching Mathematics*, 1991, p. 35

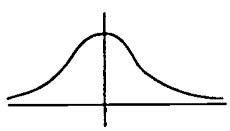
actually brag about not being able to perform even the simplest of mathematical tasks like balancing a checkbook. There is no such thing as a "math gene." We must stop sending the message that it is okay not to be able to do math. We must, instead, establish high expectations for achievement in math by all students. "The evidence from other nations shows overwhelmingly that if more is expected in mathematics education, more will be achieved. Clear expectations of success by parents, by schools, and by society can promote success by students" *(EVERYBODY COUNTS A Report on the Future of Mathematics Education,* Mathematics and Science Education Board, p. 2).

Teaching is not telling or imposing methods upon students. Concepts and relationships are constructed by people only in their minds. This requires opportunities to interact continuously with concrete experiences and discuss ideas with peers. Students cannot be talked into learning or told what to understand. Learning is an internal process that happens in individual ways.

Learning often begins with the recognition of a problem. When one is faced with a problem, teachers can promote problem solving in many ways:

- send consistent messages that all students are capable of learning,
- establish clear expectations for success,
- provide a rich intellectual environment for students to construct their own meanings and methods by choosing problematical tasks,
- accept confusion as an opportunity for learning,
- establish a cooperative setting, and
- model problem solving.

"Some tasks, although they deal nicely with the concepts and procedures, involve students in simply producing right answers. Others require students to speculate, to pursue alternatives, and to face decisions about whether or not their approaches are valid.

$\text{mean} = \frac{\Sigma}{n}$ 

For example, one task might require students to find means, medians, and modes for given sets of data. Another might require them to decide whether to calculate means, medians, or modes as the best measures of central tendency, given particular sets of data and particular claims they would like to make about the data, then to calculate those statistics, and finally to explain and defend their decisions. Like the first task, the second would offer students the opportunity to practice finding means, medians, and modes. Only the second, however, conveys the important point that summarizing data involves decisions related to data and the purposes for which the analysis is being used."

NCTM *Professional Standards for Teaching Mathematics*, 1991, p. 26

What are the criteria for good problems?

A central responsibility of teachers is to select and develop worthwhile tasks and materials that create opportunities for students to develop mathematical understanding, competence, and interest. A worthwhile task:

- includes sound and significant mathematics;
- accounts for students' understanding, interests, and experiences;
- accounts for the range of ways diverse students learn;
- engages students' intellects and encourages "what if" questions;
- invites students to make decisions and use own methods;
- promotes communication and discussion;
- nests skill development in the context of problem solving;
- calls for problem forming as well as problem solving;
- is accessible to all students at the start;
- is extendable in order to challenge those with high levels of understanding;
- stimulates students to make connections (math strand to math strand and math to other curricula); and
- engages the majority of students.

In addition to the adopted math textbook, many resources containing worthwhile tasks are available for teacher use. This document contains a bibliography listing some of these resources and an appendix which includes a "Worthwhile Task Analysis" checklist (on page 1) which might be helpful for listing tasks and noting which criteria are met by each task.

The act of teaching should be founded on dialogues between teachers and students. Assessment is the process of trying to understand what meanings students assign to ideas.

Assessment should produce a biography of the learning of a student ... a basis for improving the quality of instruction.

Student self-assessment is also an important part of this process. Provide students with opportunities to assess their own progress and establish some learning goals. Journal entries related to answering questions such as, "What do you understand about fractions?" or "What more would you like to know about fractions, or what is confusing at this point in time?" might be a useful approach.

How does one assess problem solving?

Teachers have always used their own informed judgment to evaluate student progress and understanding. This judgment needs to be trusted and validated as much as a score on a test. The teacher is the best resource for providing information about students' strengths, thinking processes, and "needs." This information can be documented in a variety of ways.

To document this information, first establish a list of desired problem solving behaviors. Involve students in developing this list by asking them to brainstorm answers to the question, "How might you describe a good problem solver?" This list might include some of the following:

- persistent (stays with a problem to great extent);
- confident;
- competent (successful);
- cooperative;
- communicates clearly;
- flexible (willing to try many different strategies and ideas).

This list, or Student Checklist (on page 2 of the Appendix), could then be used in several ways. Periodically, for example every four weeks, use a general code to indicate to what extent these attributes have been observed for each student. This code might be M = most of the time, S = some of the time, and N = not yet. Another way to use this list might be to write anecdotal examples which illustrate students' levels of development in relationship to these criteria.

Choose a maximum of five students to focus on for a class period or a day. Write these students' names at the top of the checklist. Attach the lists to a clipboard for easy access. When a student's behavior illustrates one of the attributes, or lack thereof, note the behavior on the list next to that criterion. For example, perhaps Becky has been observed working to find different solutions to a problem even after finding one correct solution. This might be noted next to "persistent" on Becky's list.

Similar lists might be developed for small group and total class discussion obligations.

"How might an effective group member behave?"

- asks and gives information;
- offers new ideas or suggestions;
- stays on task;
- does part of the work;
- uses and shares tools appropriately;
- explains math ideas;
- organizes the group;
- encourages others;
- shows kindness and interest in others; and
- is willing to change ideas based on input from others.

"How might an effective total class member behave?"

- explains ideas;
- listens and tries to make sense of others' explanations;
- indicates agreement, disagreement, or confusion;
- attempts to justify;
- questions appropriately; and
- is courteous and considerate.

PROBLEM
PROCESS
SOLUTION
EXTENSION
EVALUATION

Holistic scoring involves judging student work based on overall impressions rather than small bits of knowledge. This method encourages looking for students' thinking and minimizes the need for structuring questions to elicit predetermined answers.

When judging a piece of work as a whole, ask questions such as:

1. Does it contain some outstanding feature?
2. Does it show a spark of originality?
3. Has the problem been extended?
4. Is the presentation coherent?

Students will also learn from looking at and discussing top level papers.

Student products can also be evaluated in several ways. Ask students to write about their problem solving process by addressing the following:

- *Problem:* State the problem in your own words. Be as clear and complete as possible. Anyone reading your paper should be able to understand what you've been asked to do.
- *Process:* Explain what you did to work on this problem. How did you get started? Where did you get stuck? What did you do when stuck? If you have a solution, what makes you think it is correct?
- *Solution:* Explain your solution in a way that will be clear and convincing to someone else. Do you know that your solution is unique (the only one)? Explain.
- *Extension:* Write a new problem that is related to the given problem. This can be done by finding a variable to change or by looking at the problem from a new perspective. You do not have to solve this new problem.
- *Evaluation:* Was this problem too easy, too hard, or about right? Explain why.
  (adapted by the Interactive Math Project from "*Mathematics: Problem Solving Through Recreational Mathematics*" by B. Averback and O. Chein, Freeman, 1980)

These problem write-ups can then be sorted in a holistic way by creating three groups: those with something special or impressive, those that meet basic requirements, and those that need reworking. These groups might be sorted even further if desired. Grades and/or number scores can be assigned to these groups if necessary for record keeping purposes. However, the most beneficial feedback for students and their parents is in the form of specific comments related to the thinking and processing reflected in the write-up. These comments might also be in the form of questions designed to keep students' minds in motion. "Have you thought about ...?" "What if ...?" "How would you convince me that ...?"

A rubric can be thought of as a more specific description of standards for evaluation. A special rubric can be written for a specific task based upon reviewing student responses to this task. Or, a more general rubric can be applied which lists more global factors to be considered.

A general rubric can also be used to evaluate student products. The following is a sample four-part rubric:

Level 4: Outstanding, insightful, exceptional, goes beyond, demonstrates in-depth understanding of concepts and content, communicates effectively and clearly to various audiences using dynamic and diverse means.

Level 3: Essentially complete, effective, clear, accomplishes the purposes of the task, shows clear understanding of concepts, communicates effectively.

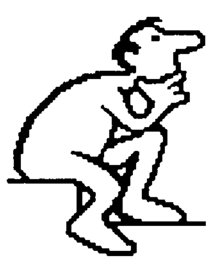
Level 2: Rework, revise, partially effective, purpose of the task not fully achieved, needs elaboration, gaps in conceptual understandings are evident, communication limited to some important ideas, may be incomplete or not clear.

Level 1: Reteach, restart, major flaws, purposes of the task not accomplished, fragmented or little understanding of concepts, communication not successful.

Consider involving students in the process of creating rubrics. Good work is good work and as such is always recognizable by teachers, parents, and students. One advantage to using a rubric like the one cited here is that students are expected to continue working until completing a quality product. Teachers need to insist upon quality work.

"What you have been obliged to discover for yourself leaves a path in your mind which you can use again when the need arises."
G. C. Lichtenberg

Whoever explains and elaborates is the one who learns!



Another way to assess student problem solving is the following scoring guide from *How to Evaluate Progress in Problem Solving* by Charles et al., NCTM, 1987:

Analytic Scoring Scale

Understanding the Problem	0:	Complete misunderstanding
	1:	Partly misunderstood or misinterpreted
	2:	Complete understanding
Planning Solution	0:	No attempt, or totally inappropriate
	1:	Partially correct plan based on part of problem being interpreted correctly
	2:	Plan could have lead to correct solution if implemented properly
Answer	0:	No answer or wrong answer based on inappropriate plan
	1:	Copying error, computational error, partial answer for problem with multiple answers
	2:	Correct answer with correct label

Again, these points could be converted to grades or percentage scores if necessary for record keeping purposes. Above all, keep in mind that assessment is a much broader and basic task than that of assigning grades. Assessment should be used to determine what students know and how they think about mathematics. Assessment needs to be the basis for improving the quality of instruction. "Teaching is effective to the degree that it takes student thinking into consideration. Without ongoing communication, a teacher's instructional strategies can only randomly enhance learning." (NCTM *Curriculum and Evaluation Standards for School Mathematics,* 1989, p. 203.)

One of the greatest challenges for teachers who engage their students in problem solving is deciding when to assist and when to wait. When a student is feeling frustration, it is difficult to avoid stepping in ... after all, we don't want the student to give up! Yet, we also don't want to "eat the student's banana split"* for her/him. This is exactly what we do if we step in with too much help too soon. Students who experience solving a difficult problem also experience great satisfaction and joy.

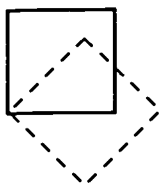
*a Grayson Wheatley expression

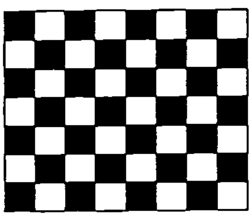
What about giving answers?

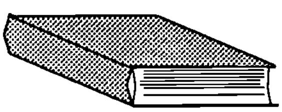
The main purpose for engaging students in the problem solving process is to engage them in thinking. As soon as answers are given, thinking stops. One of the important goals established by the NCTM *Curriculum and Evaluation Standards for School Mathematics* includes that of justifying thinking. As adults in the real world, we are always validating our own problem solutions. "Have I made the right career choice?" "Have I purchased the right car?" Students need opportunities to validate their own problem solutions and to justify their thinking. Students very quickly find out if the teacher is going to do this for them, rather than expecting them to justify and validate for themselves, and they will act accordingly.

Types of Problems in Document:

- Spatial Visualization
- Strategy Game
- Literature Based







Counting on Frank
"Meet Frank's owner, the narrator of *Counting on Frank*. Call him a measuring maniac or a fool for figuring or a demon for detail, but this young man surely has a unique way of looking at the world. From peas to ball-point pens, everything takes on a new —and hilarious—significance when he examines it."
from back book cover

More About the Problems in This Handbook

Three types of problems are presented for each grade level. The first of these focuses on some kind of spatial thinking or visual imagery. Mathematics gets hard around the point that students have difficulty visualizing what is happening. Therefore, they need plenty of opportunity to practice this skill, especially in today's society where most of a student's visual images are supplied. In fact, the National Reading Association recommends not showing illustrations to students while reading aloud to them. This provides another opportunity for students to visualize.

The second type of problem at each grade level is related to finding a strategy for winning or getting optimal scores while playing a game. Games can be very engaging. Many students will persist with searching for game strategies who would not otherwise engage in problem solving. It is very important to stress searching for strategies, not just keeping track of who is winning. If game competition presents problems in a classroom, change the focus to cooperative searches for "best strategies". Partners might work together and discuss alternatives ... what would happen if the first person did ... etc.

The third type of problem at each grade is based on a piece of literature. Recently, much attention has been paid to connecting literature and mathematics. Many of these problems also lend themselves to the development of thematic approaches to teaching, another approach which has received a lot of emphasis. Following is a list of the books used for these problems. A media specialist should be able to help supply these books. The book inspiring the problem is listed by * while others mentioned in the problem are also listed.

GRADE 3

*Clement, Rod. *Counting on Frank*. Milwaukee, Wisconsin: Gareth Stevens Publishing, 1991.

Feinsilber, Mike and Mead, William. *America Averages*. Garden City, New York: Doubleday & Company, Inc., 1980.

Anno's Hat Tricks
"Look out! Don't be fooled! This man can pull some amazing tricks from his marvelous hat box. Keep your wits about you as the hatter puts a red or white hat on Tom and Hannah and even you, the reader. You can beat the hatter at his own game if you just think carefully."
from book jacket

The King's Chessboard
A king, too vain to admit what he does not know, learns a lesson when he grants his wise man a special request: one grain of rice on the first square of a chessboard on the first day, two on the second square on the second day, four on the third and so on. The king soon realizes that there is not enough rice in the world to fulfill the wise man's request.

Fractals, Googols and Other Mathematical Tales
This collection of tales offers an amusing and entertaining way to explore mathematical ideas and make them come to life.

Socrates and the Three Little Pigs
This humorous version of the three little pigs introduces combinatorial analysis, permutations, and probabilities as the wolf attempts to figure out in which of five houses he is most likely to find one of the pigs.

The Phantom Tollbooth
Milo finds a mysterious package in his room. This package contains what looks like a genuine turnpike. When he drives his electric car through the tollbooth gate, he finds himself in The Lands Beyond where he meets some of the craziest creatures ever imagined. This witty fantasy has long been a favorite with readers of all ages.

GRADE 4

Geringer, Laura. *A Three Hat Day*. New York: HarperCollins Publishers, 1987.

Kallevig, Christine Petrell. *Folding Stories ,Storytelling Origami Together As One*. Newburgh, Indiana: Storytime Ink International, 1991.

*Nozaki, Akihiro, and Anno, Mitsumasa. *Anno's Hat Tricks*. New York: The Putman & Grosset Book Group, 1985.

GRADE 5

*Birch, David. *The King's Chessboard*. New York: Penguin Books, 1993.

Pentagram, Compiler. *Pentagames*. New York: Simon & Schuster Inc., 1990.

Pittman, Helena C. *A Grain of Rice*. New York: Bantam Books, 1992.

Sackson, Sid, Collector. *The Book of Classic Board Games*. Palo Alto, California: Klutz Press, 1991.

Slocum, Jerry, and Botermans, Jack. *Puzzles Old & New How to Make and Solve Them*. Seattle, Washington: University of Washington Press, 1992.

GRADE 6

Burns, Marilyn. *The I Hate Mathematics Book*. Boston, Massachusetts: Little, Brown, and Company, 1975.

*Pappas, Theoni. *Fractals, Googols and Other Mathematical Tales*. San Carlos, California: Wide World Publishing Tetra, 1993.

GRADE 7

Anno, Masaichiro. *Anno's Mysterious Multiplying Jar*. New York: The Putnam & Grosset Book Group, 1983.

Messenger, Norman. *Making Faces*. New York: Dorling Kindersley, Inc., 1992.

*Mori, Tuyosi. *Socrates and the Three Little Pigs*. New York: The Putnam & Grosset Book Group, 1986.

GRADE 8

*Juster, Norton. *The Phantom Tollbooth*. New York: Random House, 1972.

Grade 3 Standard Course of Study Goals and Objectives

Competency Goal 5: The learner will use mathematical reasoning and solve problems.

5.1 Identify and describe problems in given situations.

5.2 Develop stories to illustrate problem situations and number sentences.

5.3 Solve routine and non-routine problems using a variety of strategies, such as use models and "act out," use drawings, diagrams, and organized lists, use spatial visualization, logical thinking, estimation, guess and check and patterns.

5.4 Explore different methods of solving problems, including using manipulatives, pencil and paper, mental computation, calculators, and computers.

5.5 Describe processes used in finding solutions; suggest alternate strategies/methods.

5.6 Discuss reasonableness of solutions and completeness of answers.

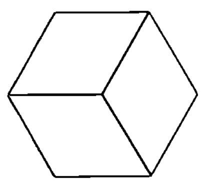
PROBLEM: How many different shapes can be made using four of the blue parallelograms (rhombi) from the set of pattern blocks?

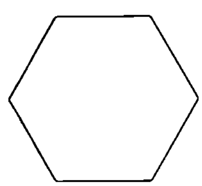
Standard Course of Study Goals and Objectives addressed:
5.1, 5.3, 5.4, 5.5, 5.6, also 2.1, 2.4, 2.5, 3.1, 3.3, 3.4, 3.5

Before working with this problem, students should have some prior experience working with pattern blocks. Students might make a design with pattern blocks on a piece of paper. They could trace around the outside of this design to form a shape and then record the number of blocks used. Then students could exchange papers with the idea of challenging another to cover the shape with the same number of pattern blocks. Students might also explore how different numbers of blocks can be used to cover each shape. What's the smallest possible number? What's the greatest possible number? Is it possible to get all the numbers in between?

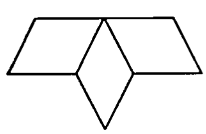
To introduce this problem, use 3 blue rhombi to create a hexagon congruent to the yellow hexagon on the overhead projector. Ask students if they think that other shapes might be created using these 3 rhombi placed side by side so that complete sides touch. Invite a student to come to the overhead to show another possibility. Repeat this process a few more times. Then ask students if they think all the shapes created so far are different from each other.

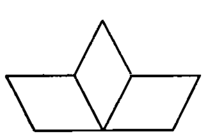
Basic Definitions:
RHOMBUS: a quadrilateral having one pair of equal parallel opposite sides and two equal adjacent sides. plural: rhombi or rhombuses





Are these two the same or different?





How will the shapes be recorded so that they can be compared to each other? Tracing around pattern block shapes or coloring on Pattern Block paper are two possibilities. How will "different" be defined? If a second shape is a flip (reflection) or turn (rotation) of another already recorded, will it be considered different or the same? Once the class has agreed upon how to define "different," continue working and recording as a group. Discuss answers to the question, "How will we know when all the different shapes have been found?"

Now students can be put to work in small groups attacking the problem of how many different shapes can be created using 4 rhombi. They should understand that they will need to present their solutions and justify their thinking. If some groups finish working ahead of others, ask them to create new and related problems to investigate. These might include: How many different shapes can be created with 5 rhombi? What other pattern block shapes can be used for similar investigations? How about using combinations of shapes; for example, two rhombi and one triangle? Can any of these shapes be cut out and folded into three dimensional shapes using the sides of the polygons as edges?

SAMPLE QUESTIONS

Multiple Choice (Objective 5.3):

A group of students has decided to investigate how many different shapes can be created using 2 blue rhombi and one square. The group has agreed that any shape that is a flip or a turn of another already found shape is considered the same. How many different shapes will this group of students find?

A less than 6

B between 6 and 9

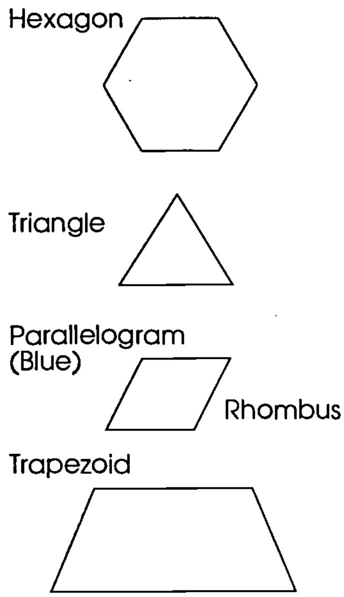
C between 9 and 12*

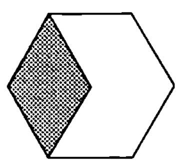
D more than 12

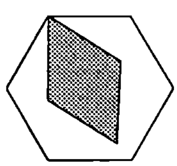
Open Ended:

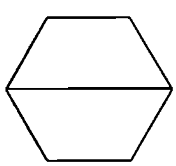
Create a new problem which is related to the four rhombi-shape problem. Define terms like "shape" and "different" if they are part of your problem. Estimate the answer to your problem and explain how you would go about getting started.

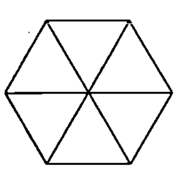
In order to learn how and what a student is actually thinking, it is often necessary to ask for explanations. "How did you get that answer?" should be asked regularly, even when answers are correct.




Yes


No


Fewest


Most

PROBLEM: Find a winning strategy for HEX NIM.

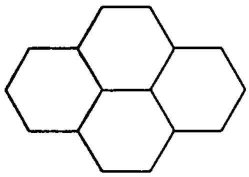
Standard Course of Study goals and objectives addressed:
5.1, 5.3, 5.4, 5.5, 5.6, also 2.4, 2.5

This strategy game uses the following pattern blocks: hexagons, triangles, blue parallelograms (rhombi), and trapezoids (Pattern Block paper is located in the Appendix). The goal of this game is to be the last one to place a pattern block. A yellow hexagon is placed between the two players. Players take turns placing any one of the other three kinds of blocks (triangles, trapezoids or blue parallelograms) on top of the hexagon. When placing another block on top of the hexagon, at least one side of the block must coincide with a side of the hexagon.

Play this game with the entire class on the overhead a few times. Make sure students understand the placement rules and how to win. Ask students if they think there might be a way to always win this game. Does it matter if a player goes first or second? Is there a strategic first move? Then set pairs of students to work, playing and looking for winning strategies. After a while, stop play and ask students to share their theories about how to win. When they resume playing, ask them to try some of the theories other students shared. How will they know when they've found a way to always win? What kind of "proof" will convince others? Students might be encouraged to keep track of what is happening. Is there a relationship between the number of blocks used and whether the first or second person won? When covering the hexagon with the other three blocks, what is the fewest number of blocks possible? What is the greatest number possible? Is every number in between possible?

Once students have found a winning strategy for this game, change the rules to create a new game. This might be accomplished by increasing the number of hexagons used ... 2, 3, or more. This might also be accomplished by redefining the winner ... you win by forcing your partner to place the last piece. What other variations are possible?

Extension: Make a game board similar to the following:



SAMPLE QUESTIONS

Multiple Choice (Objective 5.6):

When playing HEX NIM, which of the following are true at the end of the game?

1. If 6 blocks have been placed on the hexagon, the person who went first was the winner.
2. If there is an even number of blocks on the hexagon, the person who went first won.
3. If there is an odd number of blocks on the hexagon, the person who went first won.
4. It is possible to win by going second and ending up with an even number of blocks.

A Numbers 1 and 3 only

B Numbers 1, 3, and 4 only*

C Numbers 2 and 3 only

D Numbers 1 and 2 only

Open Ended:

Make up a new version of HEX NIM. Carefully and completely explain the rules and materials for playing. Explain how your new game is the same as HEX NIM and how it is different.

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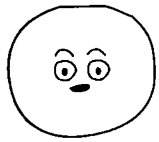


A pea that can see

"I don't mind taking a bath—it gives me time to think. For example, I calculate it would take eleven hours and forty-five minutes to fill the entire bathroom with water. That's with both faucets running."

from *Counting on Frank*

Extensions

- How many peas in a pod?
- What is a split pea?
- How many different kinds of peas exist?
- How do peas grow?
- What are some favorite ways to fix peas?
- How about compiling a pea recipe book? Would anybody buy it?
- Find poems and literature about peas ...

I eat my peas with honey;
I've done it all my life.
It makes the peas taste funny,
But it keeps them on my knife.
Anonymous

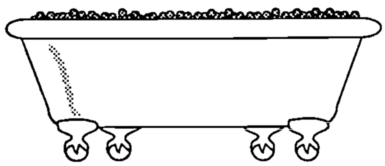
PROBLEM: At the rate of 15 peas on the floor every night for eight years, would the peas really be level with the table top?

This problem is based on the book *Counting on Frank* by Rod Clement.

Standard Course of Study goals and objectives addressed: 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, also 1.3, 1.4, 3.5, 4.3, 4.13, 7.1, 7.3, 7.5, 7.8 and others depending upon the extent to which this book is explored.

In this book, a young boy and his dog, Frank, compare and count various things. For example, the boy calculates that if he had knocked 15 peas on the floor each night for the last eight years, they would now be level with the table top. "Maybe then Mom would understand that her son does **not** like peas." Many other situations in the book can be used to pose a problem for students to investigate. Begin by reading the book aloud. Ask students what they think about the peas. How many peas would this be? How much space would all those peas "consume"? Ask students to plan some strategies for working on this problem. After discussion time, ask students to report their plans. These plans may require bringing in some special supplies ... perhaps some peas, measuring tools, etc. This would also be an opportunity to involve parents by having students gather some data at home. What other questions might be asked about the peas? How much would the peas weigh? How much would they cost? What is a serving size of peas and how many servings might this be? This might also lead to some class surveys related to vegetable likes and dislikes. This data could be represented in various ways, bringing in more of the objectives from Goal 6.

Much more could be done with this book. Students could be encouraged to pose their own problems to investigate. How about the claim that the average pen draws a line seven thousand feet long before it runs out of ink? What about the claim that the average pencil draws a line 35 miles long before running out of lead, see *American Averages* by Mike Feinsilver and William B. Mead (1980). How do these two distances compare? Perhaps students might be inspired to write their own version of *Counting On...*



How many peas would fill the bath tub?

SAMPLE QUESTIONS

Multiple Choice (Objective 5.1):

A group of students has decided to investigate whether or not putting 15 peas on the floor each night for eight years would reach the table top. Which of the following questions do they need to answer in order to decide?

1. How many peas would that be altogether?
2. How big is the room with the table?
3. How high is the table top from the floor?
4. What is the average pea size?
5. How much space is taken by other things in the room?

A 1, 2, 3, 4, and 5*

B 1 and 2 only

C 1, 2, and 3 only

D 1, 2, 3, and 4 only

Open Ended:

Write your own problem based on something from *Counting on Frank*. Write a list of the questions you would need to answer in order to solve your new problem.

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Grade 4 Standard Course of Study
Goals and Objectives

Competency Goal 5: The student will solve problems and reason mathematically.

5.1 Develop an organized approach to solving problems involving patterns, relations computation, measurement, geometry, numeration, graphing, probability and statistics.

5.2 Communicate an understanding of a problem through oral and written discussion.

5.3 Determine if there is sufficient data to solve a problem.

5.4 In solving problems, select appropriate strategies such as: act it out, make a model, draw a picture, make a chart or graph, look for patterns, make a simpler problem, use logic, work backwards, guess and check, break into parts.

5.5 Estimate solutions to problems and justify.

5.6 Solve problems by observation and/or computation, using calculators and computers when appropriate.

5.7 Verify and interpret results with respect to the original problem. Discuss alternate methods for solutions.

5.8 Formulate engaging problems including ones from every day situations.

Basic Definition

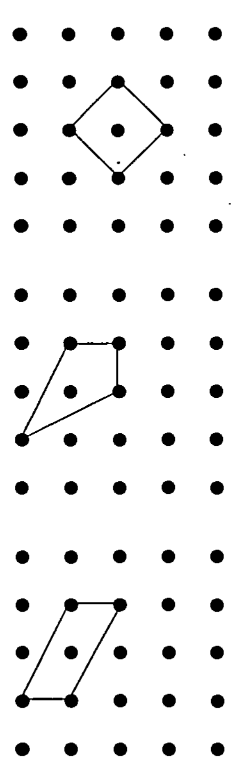
TRIANGLE: A closed plane geometric figure (polygon) having three sides. Triangles are classified in two ways:

by sides—
- scalene has all three sides different in length
- isosceles has two equal sides
- equilateral has all three sides equal

by angles—
- acute has every angle measuring less than 90°
- right has one right angle, 90°
- obtuse has one angle greater than 90°

These classifications can be combined. For example, a right isosceles triangle is one with a right angle and two equal sides called legs.

Here are several solutions for a fence with 4 posts and 1 tree:



PROBLEM: Given a standard 5 by 5 geoboard, how many different triangles can be created?

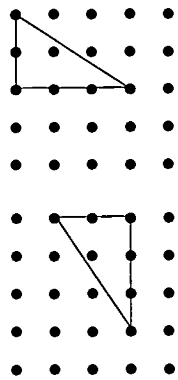
Standard Course of Study goals and objectives addressed: 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, also 2.1, 2.3, 2.5, 3.2, 3.4, 4.10, 4.11

Before working on this problem, students should have some prior experience using geoboards. (Geoboard paper is located in the Appendix.) They might spend some time investigating "fences and trees." A fence is created by stretching rubber bands around a given number of pegs, or fenceposts. Trees are defined as those pegs inside the fence which are not touched by the rubber band. For example: create a fence with 4 fence posts and 1 tree (see examples in sidebar). Students should be encouraged to find as many different answers as possible. Next, by creating fences with only 3 posts, students will begin looking for a variety of triangles on the geoboard.

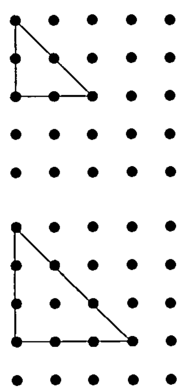
Introduce the problem by reviewing the definition of triangle. Then lead a total class discussion about how to define "different" when searching for different triangles. Visualize two congruent triangles. The second of this pair is either a turn, a flip, or a slide of the first. Are these triangles the same or different? Visualize two similar triangles; both are right isosceles triangles. One of these triangles has legs of 1 unit in length, and the second has legs of 2 units. Are these the same or different? In general, are triangles different because they have different areas, perimeters, placements on the geoboard, or different because they have different shapes? The class needs to agree upon a definition for different.

After the defintion of "different" has been established, students should estimate the answer to the problem, even write this estimate in a journal or special problem solving book. Students should understand that they will be expected to present their solutions to the class and justify their thinking. Students could use geoboard dot paper for recording the various triangles. If the class has decided to consider congruent triangles regardless of placement on the geoboard as the same, they might be encouraged to cut out each triangle in order to check for congruency. Pairs of students, or small groups of three or four should be given plenty of time to work on this problem. If a group finishes working ahead of others, encourage them to write extensions and continue working on these. Some of these extensions might include extending the size of the geoboard, redefining "different," searching for the

Are these different?



How about these?



"Life is not easy for any of us. But what of that? We must have perseverance and, above all, confidence in ourselves."
Marie Curie

number of different quadrilaterals, or looking for a pattern in the number of different triangles based upon the number of pegs on the geoboard.

SAMPLE QUESTIONS

Multiple Choice (Objective 5.1):

A group of students is searching for the number of different triangles that can be found on a 3 by 3 geoboard. If a triangle is congruent to another already found, then it is not different regardless of placement on the geoboard. How many different triangles should this group of students find?

A 6

B 7*

C 8

D 9

Open Ended:

Write a new but related problem to the one about finding all the different triangles on a 5 by 5 geoboard. Then solve this problem and justify your thinking.

ADD UP
Game board

4	4	4	4
3	3	3	3
2	2	2	2
1	1	1	1

"I see certain order in the universe, and math is one way to make it visible."

May Sarton
As We Are Now

PROBLEM: Find a winning strategy for the game "ADD UP."

Standard Course of Study goals and objectives addressed:
5.1, 5.2, 5.4, 5.6, 5.7, 5.8, also 3.2, 3.3, 3.6, 7.1, 7.2

The playing board for this game is a 4 by 4 grid numbered as follows: the bottom row of square units each contains the number 1, the next row up contains 2's, the next row up contains 3's, and the top row contains 4's. Each game board also needs to be accompanied by 16 counters, or markers, of some kind; such as plastic chips or beans.

To begin, players choose a target number between 20 and 35. Players take turns placing a counter on a number and announcing the sum of all the numbers covered to that point. For example, if the first player places a marker on one of the 2's, then he announces 2, then if the second player places a marker on one of the 3's, she announces 5, and so on. The winner is the player who says the target number.

Play this game on the overhead a few times to make sure everyone understands the rules. Then ask the class if they think it matters whether one plays first or second. Does the target number have anything to do with deciding to play first or second? Then ask the class if they think that it is possible to figure out a way to always win this game. Have students begin playing and searching for winning strategies. They might begin by taking turns being first and keeping track of what is happening ... is the first or second player winning more often? Does the target number affect the strategy? Perhaps everyone should play with the same target number for awhile until some "data" have been gathered.

After students have had an opportunity to play and formulate some theories about how to win, lead a total class discussion. Ask students to explain their theories and ask others to react to these theories. Then have students continue playing and testing theories, either their own or someone elses. How will students know for sure when they have a winning strategy? What kind of "proof" will convince others?

When students are convinced that a winning strategy has been found, it's time to create a new game. For example, the following aspects of the game can be changed:

✓ The size of the gameboard; how about a 5 by 5, or a 6 by 6, or a 4 by 5?

For addition examples related to Goal 5 examine *Strategies for Instruction in Mathematics* developed by the NCDPI—Mathematics Section.

- ✔ The numbers placed on the gameboard; how about using 1's, 2's, 3's, 4's, & 5's, or how about some fractions or decimals?
- ✔ The range of target numbers; how is this related to the gameboard?
- ✔ The operation; how about multiplying instead of adding? How about subtracting from the target and the winner says "zero"?

When students state their strategies, they should be so complete that a player knows exactly what to do on each turn. For example: always go second and place your marker on a number which is equal to your opponent's choice (or always go second and mirror your opponent's move). This may or may not be a "good" strategy, but it is complete since a player knows exactly what to do. Notice that this strategy ignores the target number. A complete strategy that takes into account the target number might state: if the target number is even, go second and mirror your opponent's move; but if the target number is odd, go first and keep the sum odd. Again, this may or may not be a successful strategy.

SAMPLE QUESTIONS

Multiple Choice (Objective 5.2):

A group of students is designing a new version of ADD UP. They are using a 3 by 3 game board. The bottom row of numbers is 3, the middle row is 4 and the top row is 5. When playing this game, the rules are the same as for regular ADD UP. Which of the following represents a reasonable range of target numbers?

A any number between and including 12 and 36

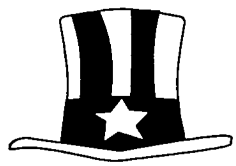
B any even number between and including 12 and 36

C any multiple of 3, 4, 5, 7, 8, or 12 which falls between and includes12 and 36

D any number between and including 12 and 36 except 34 and 35*

Open Ended:

Create a new version of ADD UP. Draw the game board and clearly explain the playing rules. Justify your choice of range for the target numbers.



Hats can tell a lot about a person: where the person lives, what the person does. Who do we usually associate with this "stove pipe" hat?

PROBLEM: What color hat is Shadowchild wearing in various situations?

This problem is based on *Anno's Hat Tricks* by Akihiro Nozaki and Mitsumasa Anno.

Standard Course of Study goals and objectives addressed: 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, also 3.1, 3.2, 3.3, 3.6, 6.7

Introduce this problem by asking, "What is a hatter?" The book *Folding Stories, Storytelling and Origami Together As One* by Christine Petrell Kallevig begins with a story called "For Each A Hat". In this story, Mattie Matter the Hatter believes that there is a perfect hat for every person and a perfect person for every hat. While telling this story, one folds and refolds a paper square into various hats and the final result is a bird.

Next, read the first 15 pages of *Anno's Hat Tricks*. Ask students to talk to each other in small groups about the color of Shadowchild's hat. How can they justify their thinking? At this point, it might be helpful to stop and act out the possibilities. In order for students to work on the more difficult problems, they really need to be comfortable with solving these simpler ones at the beginning of the book. Continue reading, pausing at appropriate places to give students an opportunity to discuss and justify their answers to the question, "What color is Shadowchild's hat?" Probably the first significant pause should come at page 35. Let groups of students work a long time on this one. The final trick starts on page 40. After this, students can create their own problems or "tricks" by varying the number of children and keeping the number of white hats at one less than the number of children.

After students have had an opportunity to use and explain their own strategies for working on these problems, it might be appropriate to draw a tree diagram which shows all the possibilities for any one of the problems. See the "Note to Parents and Other Older Readers" at the end of the book for some ideas about this strategy.

A Three Hat Day by Laura Geringer provides an opportunity to pose another problem. This book is about R. R. Pottle who collects and loves hats. R. R. Pottle leaves the house one day wearing three hats; a bathing cap, a fireman's helmet, and a sailor's hat. How many different ways, or orders, can R. R. Pottle wear these three hats? What about a four-hat day? What about a five-hat day? Challenge students to pose even more problems related to hats. Some measurement problems might come to mind.

SAMPLE QUESTIONS

Multiple Choice (Objective 5.4):

The hatter has three children, two white hats and three red hats. If the hatter puts a hat on each child, how many different arrangements are possible?

A 5

B 6

C 7*

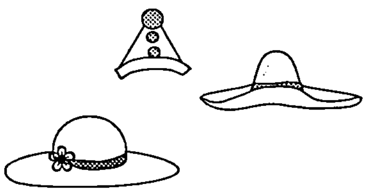
D 8

Open Ended:

Think about the hatter in *Anno's Hat Tricks*. Create a new "hat trick" which is similar, yet different, from any in the book. Give examples of how your new trick is similar and yet different.

"We especially need imagination in science. It is not all mathematics, nor all logic, but it is somewhat beauty and poetry."

Maria Mitchell



This problem could be developed into a thematic unit.

- Make different hats with origami folding or through other methods.
- Collect hats and then use for categorizing or ordering in various ways.
- Who invented or created the first hat? Why?
- What are all the purposes for wearing hats... warmth, shade, style, advertisement, safety, hygiene, occupation, school affiliation, ...
- What superstitions or sayings are associated with hats? For example: "Throw your hat in the ring."
- What songs are about hats?
- What etiquette is associated with hat wearing?
- What are all the synonyms for "hat"?
- What is a hat? How about visors, sweatbands, babushkas, balaclavas???
- How are hats sized?
- What's the average number of hats owned by students in this class, this school's 4th grade, this school?
- What if there were no hats?
- Write a book all about hats.
- How does your head circumference compare to your height? What's the result when you divide this circumference by 3?

Grade 5 Standard Course of Study Goals and Objectives

Competency Goal 5: The student will solve problems and reason mathematically.

5.1 Use an organized approach to solve multi-step problems involving numeration, geometry, measurement, patterns, relations, graphing, computation, probability and statistics.

5.2 Communicate an understanding of a problem using models, known facts, properties, and relationships.

5.3 Determine if there is sufficient information to solve a problem; identify missing and extraneous data.

5.4 Use appropriate strategies to solve problems such as restate problems, use models, patterns, classify, sketches, simpler problems, lists, number sentences, guess and check.

5.5 In problem solving situations, use calculators and computers as appropriate.

5.6 Verify and interpret the results with respect to the original problem. Identify several strategies for solving a problem.

5.7 Make generalizations and apply them to new problem situations.

Basic Definitions

QUADRILATERAL: a polygon with 4 sides.

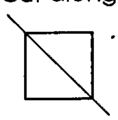
PARALLELOGRAM: a quadrilateral in which each pair of opposite sides is parallel.

RECTANGLE: a parallelogram all of whose angles are right.

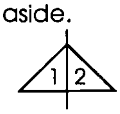
SQUARE: a rectangle all of whose sides are equal.

Directions for cutting a square into a set of tans:

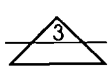
1) Fold the square in half along the diagonal to create two right triangles. Cut along the fold.



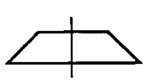
2) Fold one of these triangles in half to create two congruent right triangles and cut along the fold. These two triangles are the two large tans ... set aside.



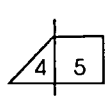
3) Take the other half of the original square and pinch the midpoint of its hypotenuse or longest side. Fold the right angle to this midpoint and cut off the resulting top right triangle. Set this tan aside ... the middle sized right triangle.



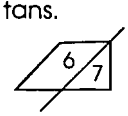
4) Fold the resulting trapezoid in half and cut along the fold. Two smaller trapezoids result.



5) Fold one of these trapezoids to create a square and a small right triangle. Cut and set these two tans aside.



6) Fold the remaining trapezoid to create a parallelogram and a small right triangle. Cut apart to create the final two tans.



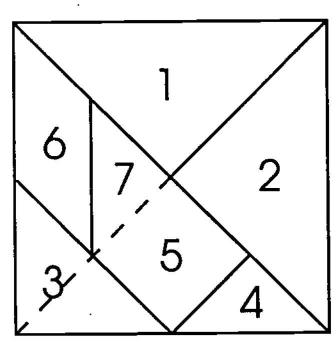
PROBLEM: Given one set of tangrams, how many different rectangles can be created?

Standard Course of Study goals and objectives addressed: 5.1, 5.2, 5.3, 5.4, 5.6, 5.7, also 2.1, 2.3, 2.9, 2.10, 3.1, 3.3, 4.1, 4.2, 4.4, 4.6

Begin with a reminder of the definition of a rectangle. Be sure students accept that a square is a special kind of rectangle and, therefore, fits in this category.

Students should also have some opportunity to explore with the tangrams before beginning this task. A tangram begins with a square that is cut into seven polygons called "tans." When creating the over 1600 possible designs, all seven tans are used and must touch without overlapping. Among the many people who have enjoyed solving tangram puzzles were John Quincy Adams, Edgar Allen Poe, Lewis Carroll, Gustave Dore, and Napoleon Bonaparte. The earliest reference to the tangram appears in a Chinese book dated 1813, and there are many theories about the origin of the name *tangram*.

A good introductory activity might include cutting a construction paper square apart into the seven tangram pieces. Instructions for this are included in the sidebar.



Students might also be asked to fit the seven "tans" back into the original square.

If students have had an opportunity to explore with tangrams, they will be better prepared to discuss what "different" means. Before starting to work on this problem, the class must negotiate and agree upon a definition for "different." The following questions might be asked:

- ✔ Will different mean a different number of tans?
- ✔ Will different mean different areas?
- ✔ Will different mean different perimeters?
- ✔ Will different mean a different set of tans?
- ✔ Will different mean different arrangements of the same set of tans?

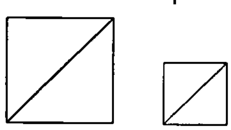
The two small triangles, as well as the two large triangles, can be used to create a rectangle. The number of tans in both cases is two, but the area and perimeter are different. Are these different rectangles? Two congruent rectangles can be created, one with the small square and two small triangles and the second with the parallelogram and the two small triangles. The number of tans used, the area, and the perimeter of these two rectangles are the same, but the particular tans used are different. Are these two rectangles different? Consider a rectangle which is created with four tans: for example, the parallelogram, two small triangles, and the middle sized triangle. If a second congruent rectangle is created by rotating or reflecting this arrangement of tans, is this one different from the first? It is possible to arrange the middle sized triangle and the two small triangles into two rectangles with different perimeters. Are these two rectangles different?

Once the definition of "different" is established, students should be encouraged to devise a systematic plan for finding and recording all possible rectangles. They should be prepared to justify their answers, including "proof" that they have found all the possibilities without repeating any. Their recording process might be that of tracing around tans to create drawings of the possible rectangles. Or, they might cut more sets of tans from squares and glue pieces into the possible rectangles.

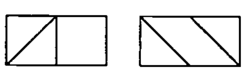
Another part of the plan may involve listing areas and perimeters of various rectangles. Thus, it might be helpful to suggest possible units for measuring. One approach is to use the small triangle as one unit of area. As an introductory activity, students might be asked to determine the area of each tan if the small triangle is used as the unit of measure. By comparing and arranging the tans, students can discover that the large triangles have an area of 4

Are these rectangles different?

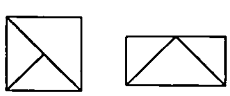
- same number of tans but different areas & perimeters from different tan pieces:



- same number of tans, same area and perimeter, but a different set of tans:



- same set of tans but different arrangement creating two rectangles with different perimeters and same area:



square units; the middle sized triangle, the square, and the parrallelogram each have an area of 2 square units. Thus, the entire set of pieces has an area of 16 square units. Measuring the perimeter of any rectangle might be accomplished by using one of the sides of the small triangle. Students will need to decide whether to use the hypotenuse or one of the legs as a unit of linear measure.

Many tangram activity books are available from school publishers.

SAMPLE QUESTIONS

Multiple Choice (Objective 5.4):

A group of students has been asked to determine how many different rectangles can be created with a set of tangrams. Imagine that they have decided that "different" refers to the number of pieces only, regardless of which pieces or size or arrangement. How many different rectangles will they find?

A 4

B 5

C 6

D 7*

Open Ended:

Consider the task of finding all the possible rectangles that can be created with a set of tangrams. Write a new but related problem. Explain why you think this is a worthwhile task.

Grandfather Tang's Story by Ann Tompert tells a story which uses some of the many tangram animals. While telling the story, one could arrange the tangrams on an overhead or felt board to create the animals in the story. Students might like to pair up and one could read the story while the other creates the animals. They might enjoy doing this for classrooms of younger students.

"It is common sense to take a method and try it. If it fails, admit it frankly and try another. But above all, try something."
Franklin Delano Roosevelt

VARIABLES:
- matrix design
- data generator
- discard option
- change one decision option
- choose operation

PROBLEM: Finding a strategy to get as close as possible to a target number when placing digits in a matrix which includes places for numbers and operations.

Standard Course of Study goals and objectives addressed: 5.2, 5.4, 5.5, 5.6, 5.7, also 1.1, 1.8, 1.9, 7.1, 7.3, 7.6, 7.8, 7.10, 7.11, 7.14

This game involves looking for strategies to get as close as possible to a target number. The particular matrix established will determine which basic computational skills are employed by students. This example uses two-digit multiplication. Have each student draw this matrix on his/her paper:

$$\begin{array}{r} \Delta\,\Delta \\ \underline{\times\,\Delta\,\Delta} \end{array}$$

Next choose a target number and write it on the board. This should be a round number between 120 and 4,350. Why this range?

Roll a die and announce the result. Each student writes the number in one of the triangles on his or her matrix. Once a number has been placed, it cannot be moved to another box. Repeat this three more times, a total of four rolls. Students should then find the answer to the problem they have generated. The results are reported in order to determine who is closest to the predetermined target, either over or under. Students should also discuss: "How many different problems and answers are possible?" "If it were possible to exchange two numbers in order to get closer to the target, which would they be?" "What kinds of strategies were used to make decisions about where to place numbers?" "Would it be possible to get the exact target number as an answer?" "Why or why not?"

A wide range of "new" game challenges can be created by varying several aspects of this activity. Different kinds of data generators can be used. These include decahedra (10-sided dice) with the numbers 0 through 9. Of course, spinners or numbers "in a hat" could be used also. The matrix and computational operation can also be changed. Consider the following fraction matrix:

$$\frac{\Delta}{\Delta} + \frac{\Delta}{\Delta}$$

Students need to state complete strategies:
"If possible, place 1, 2, and 3 in the ten's place and 4, 5, and 6 in the one's place. Otherwise place a number in any vacant place."
This is an example of a complete strategy because the player knows exactly how to proceed after each roll of the die. This may or may not be a successful strategy. To determine whether or not this is successful, one would need to:

- play with this strategy over many trials
- record the distance from the target number
- determine a way to define a "successful" strategy ... for example; produces the smallest average difference from the target after 100 trials

Students might be asked to extend their strategies to include how to proceed based on a variety of target numbers. For example, if the target number is between 120 and 1000, then ...; if the target is between 1000 and 2000 then ... etc.

With this matrix, the die is rolled only three times. One of the rolls becomes the common denominator for both fractions. Different target numbers could be established ranging from something like 1/3 to 12 with a regular die. Why? Students might also be given a target number and the opportunity to choose any of the four basic operations after placing their numbers. Another variation might include a "discard" box. Thus, students can choose during the game to discard one of the rolls of the die.

While working with these various matrix challenges, the focus must always be on finding an optimal strategy for placing the numbers. Students might be asked to write a "rule" which would reflect their own thinking about the best strategy for deciding where to place numbers in any matrix. For example, a complete strategy rule might say: "If the number rolled on the die is 4 or greater, place it in the one's place if possible. If the number rolled is less than four, place it in the ten's place if possible." This may or may not be a "good" strategy for consistently getting as close to a target number as possible. However, this strategy is complete because the student knows what to do in each situation.

Consider the following strategy: "Place big numbers in the one's place and smaller numbers in the ten's place." This strategy is not considered complete because the player must decide for him/herself how to define "big" and "small" numbers. Have students test their strategies over many trials to determine "success." Which strategies create the smallest variance from target numbers over many trials? Without this focus on strategies, these matrix games will become exercises and not problematic tasks for most students.

SAMPLE QUESTIONS

Multiple Choice (Objective 5.1):

Consider the following "Matrix Game": The target number is 1,000 and the matrix is:

$$\begin{array}{r} \Delta\,\Delta \\ \underline{\times\,\Delta\,\Delta} \end{array}$$

A student has already placed the numbers 5 and 6 in the one's places and a 2 in one of the ten's places. Which of the following numbers on the die will give this student the *closest* possible answer at this point?

A 3*

B 4

C 5

D 6

Open Ended:

You are going to play the following "Matrix Game" using a regular decahedron die (numbers 0 through 9):

$$0.\Delta\Delta + 0.\Delta\Delta = ?$$

What range of numbers represent reasonable target numbers? Explain. Choose a particular target number and explain your strategy for getting as close to this target as possible over many trials. Explain why you think your strategy will accomplish this purpose.

Solution:

By using a decahedron, the largest number possible is 9. Thus

$$\begin{array}{r} 0.99 \\ \underline{+0.99} \end{array}$$

or 1.98 would be the greatest possible result. Since 0 is the smallest possibilty,

$$\begin{array}{r} 0.00 \\ \underline{+0.00} \end{array}$$

0 is the smallest possible answer. Thus an appropriate range for this matrix game is 0 to 2.

As students play these strategy games and search for "winning strategies," they are engaged in solving non-routine problems. It is important to point this out to students and emphasize the search for strategies. Otherwise, students fail to connect mathematics to the playing of these games.

For more information see *Puzzles Old & New* by Jerry Slocum and Jack Botermans. This book also explains another puzzle called "The Problem of Eight Queens." *Pentagames* explains the following games which all use this same playing board: Pyramid, Reversi, Fox & Geese, Maharajah and the Sepoys, Go-Bang, Knights, Wold and Goats, Chess, Checkers, Diagonal Checkers (Draughts), and Checkers (Draughts). *The Book of Classic Board Games*, collected by Sid Sackson, also includes several games using this 8 by 8 board: Checkers, Go, Brax, and Cats and Dogs. Many of these games also provide an opportunity to learn more about the mathematical contributions of a variety of cultures.

The King's Chessboard
A king insists upon rewarding one of his wise men. The wise man finally tells the king to give him one grain of rice which will double each day after that for 64 days, the number of squares on the king's chessboard. After several days, the counting of rice grains gives way to weighing, then the weighing gives way to counting sackfuls, then to wagonfuls. The king finally realizes that there is not enough rice in the entire world to grant the wise man's request.

Problem: How much rice will the king be giving to the wise man?

This problem (or set of problems) is based on the book *The King's Chessboard* by David Birch.

Standard Course of Study goals and objectives addressed: 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, also 1.1, 1.3, 1.4, 1.5, 2.9, 2.10, 3.1, 3.3, 4.5, 4.6, 7.1

Before introducing the problem, engage students in any of the many games or puzzles which use the 8 by 8 board known as a chess or checkerboard. The checkerboard puzzle is created by dividing a checkerboard into pieces of different shapes, each shape consisting of a whole number of squares. A range of 8 to 15 different shapes make "good" puzzles. Simply color an 8 by 8 square grid in the alternating color pattern of a checkerboard; red, black, red, black, etc. Then cut the checkerboard apart into the different shapes desired for the puzzle. The puzzle is to re-assemble the pieces into the original checkerboard.

At this point, students should be familiar with the chessboard and know that there are 64 small (1 unit) squares. Introduce the problem by reading the first 6 pages of *The King's Chessboard.* Ask students to imagine that they are in charge of the king's granaries, and it is their task to figure out how much rice the wise man will be receiving. Lead a class discussion/brainstorming session about how to get started. Then read the next 3 pages. Stop at the point in the story where the weigher decides to send an ounce of rice rather than counting out 2,048 grains. Ask students what they think about this idea. How long might it take someone to count out this many grains of rice? Also ask students which day this is (in the story) and how many squares on the chessboard have been accounted for. Ask students to estimate how much rice will be given to the wise man on the 64th day. They might even be asked to record these estimates in a journal or special "problem solving" book related to this problem. Have available some of the following for student use: calculators, uncooked rice, measuring spoons and cups, baggies, and scales calibrated in ounces and pounds. Then ask students, working in groups, to devise a plan and attack the problem of finding out how much rice the wise man will receive on the 64th day. Numbers will get large very quickly. Ask students if they can find ways to deal with this. What did the weigher do on the 12th day? Use this opportunity to introduce or review exponential notation; $2^4 = 2 \times 2 \times 2 \times 2 = 16$. Allow plenty of time for working on this problem, perhaps days. After groups have

One might devise a new measurement system based on this story:
1 bag = 16 ounces
1 sack = 128 bags
1 ton = 16 sacks
1 wagon = 1 ton

When reading the story to students, pause at the point that the Grand Superintendent sees the four granary workers carrying sacks of rice from the granary. Ask them to determine how much each sack weighed based on the information given so far in the story.

Lead a discussion about the impact of dropping 48 pounds when rounding the 2,048 pounds to 2,000 pounds or 1 ton on day 27. (This amounts to about 3-1/3 billion tons). According to the book, the wise man was promised 549, 755, 830, 887 tons of rice. Does another 3 billion tons make a difference when talking about 549 billion?

Another interesting book with a similar problem is:
A Grain of Rice by Helena C. Pittman published by Bantam Books, 1992. In this book, a farmer's son wins the hand of the Emperor's daughter by outwitting the father. Pong Lo's strategy is to ask for a grain of rice that is to be doubled every day for one hundred days. How much rice is that?????

decided how much rice the wise man received on the 64th day, ask them to find out how much rice was promised to the wise man in total and to be ready to present their solutions and justify their thinking. Students might also be asked to complete a problem write-up (see section on "How does one assess problem solving?"). If some groups complete this problem before everyone is ready for presenting solutions, ask them to write some other problems related to this one and work on them. For example: How much would this rice cost at today's prices? Is there this much rice in the world today? What other "rewards" of a similar nature might the wise man ask for? This "nature" being something seemingly simple which turns out to be next to impossible. How much space would be required to store this amount of rice? How many people could this amount of rice feed over the period of one year? This last problem needs even further definition ... how much rice might one eat in a day?

After students have presented their solutions and justified their thinking, read the rest of the story, *The King's Chessboard*. Ask students what they think about the solution in the book. Is it possible for this answer to be correct? Explain why or why not.

The computer can be a helpful tool while working on this problem. When using the calculator available with most computers, larger numbers can be calculated than those on a hand held calculator. Most hand held calculators will not compute higher than 99,999,999 while it's possible to compute as high as 9,999,999,999,999 on the computer's calculator. One could go even higher by writing a special program which would compute powers of 2.

There is also a MECC (Minnesota Educational Computing Consortium) software called "Problem-Solving Strategies #784" that leads students through different strategies for working on the squares problem: how many squares on a chessboard?

SOME FACTS & QUESTIONS

- It's interesting to note that rice might cost about 50 cents a pound; probably less if purchased in larger quantities.
- What is the per capita consumption of rice in the United States?

  According to the U.S. Department of Agriculture, the average American ate 7.2 pounds of rice in 1976.

  The 1989 Almanac reports the following estimated populations:

  United States: 250,000,000

  China: 1,104,000,000

  Japan: 123,200,000

  World: 5,234,000,000

  How does the U.S. rate compare to that of Japan and China?
- How and where is rice grown?
- How many different kinds of rice exist and what are they?
- Which dietary needs are supplied by rice?
- What is everyone's (those students in your class) favorite rice recipe? How about writing a rice recipe book?
- Are rice crispies and puffed rice made from rice? What about rice paper? How is this all done? How about writing a trivia book about rice?

SAMPLE QUESTIONS

Multiple Choice (Objective 5.5):

Which of the following options would result in the greatest amount of money?

A Someone gives you $1,000,000.00

B Someone gives you a penny on day 1, two pennies on day 2, four pennies on day 3, eight pennies on day 4, and so on for 30 days.*

C Someone gives you $1.00 on day 1, $2.00 on day 2, $4.00 on day 3, $8.00 on day 4 and so on for 15 days.

D Someone gives you $10.00 a day for 100 days.

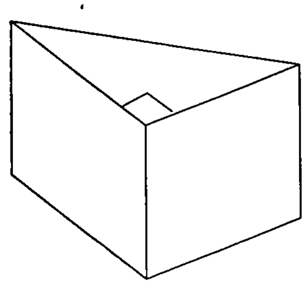
Open Ended:

Write a new problem related to The King's Chessboard. Explain why you think this new problem is worthwhile and suggest a strategy for getting started.

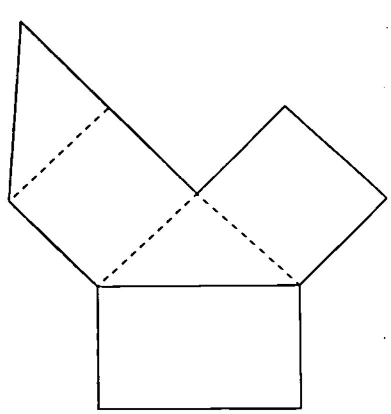
Basic Definitions

PRISM: A solid (polyhedron) whose bases are parallel congruent polygons and sides are parallelograms.

Thus, an isosceles right triangular prism has bases which are right isosceles triangles.



This is a sample net which would create the prisms used in this investigation:



Grade 6 Standard Course of Study Goals and Objectives

Competency Goal 5: The student will solve problems and reason mathematically.

5.1 Use an organized approach to solve non-routine and increasingly complex problems involving numeration, geometry, pre-algebra, measurement, graphing, computation, probability and statistics.

5.2 Analyze problem situations and apply appropriate strategies for solving them.

5.3 Use inductive and deductive reasoning to solving problems.

5.4 Select an appropriate method for solving problems including estimation, observation, formulas, mental math, paper and pencil culation, calculator and computers.

5.5 Make conjectures and arguments and identify various points of view.

PROBLEM: How many different three dimensional shapes can be made using 4 right triangular prisms?

Standard Course of Study goals and objectives addressed: 5.1, 5.2, 5.3, 5.4, 5.5, also 2.1, 2.3, 2.4, 2.5, 3.4

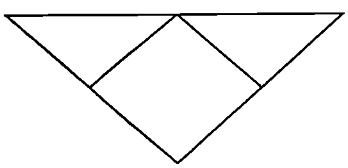
Any set of congruent, right isosceles triangular prisms can be used to work on this problem. The geoblocks include several sizes of these prisms. Students might begin addressing this problem by creating a set of congruent right triangular prisms from cardstock. They can draw a net as follows: draw or construct a right isosceles triangle, then construct a square on the side of each leg, a rectangle along the hypotenuse, and a right triangle on one of the squares opposite the original triangle. See the sample net to the left. Students will need to add tabs at appropriate places in order to glue their prisms together. By creating many of these prisms, students will be able to glue them together as they work on the problem.

Students might draw either orthographic or isometric representations of their structures. Orthographic drawings are based on three perpendicular views of the same object: top, end and front. The literal meaning of orthographic is perpendicular. Isometric drawings are the typical 3-D representations which many artists utilize. Isometric means equal measure and refers to the equal angles used in such a drawing.

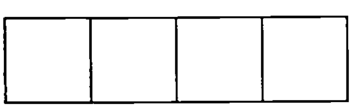
Below are two different drawings of one shape which can be created with one cube and two right isosceles triangular prisms.

Orthographic Drawing

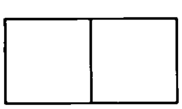
TOP VIEW



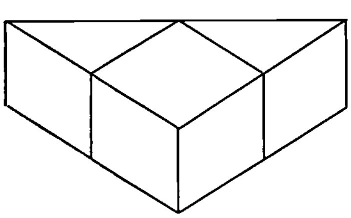
FRONT VIEW



END VIEW



Isometric Drawing



Using two of these prisms, work as a class to find different shapes. The prisms should be fit together on congruent faces. How many different faces exist? (There are two congruent right isosceles triangles, two congruent squares, and one rectangle ... 3 different types of faces.) The class should be able to find four different solids with these two prisms if "different" means that any solid which is congruent to one already found is the same. The class should agree upon how to define "different." Briefly explore the number of different shapes for 3 prisms and start a table which relates the number of shapes to the number of prisms used.

Now students are ready to work in small groups on the stated problem using 4 prisms. Ask students to consider how to record their answers and how to "prove" they have found all the possibilities without repeating. If groups complete working on this problem before other groups, ask them to create new related problems. These might include working with sets of different polyhedra, different numbers of polyhedra (try 5 instead of 4), combinations of polyhedra like a cube and two triangular prisms where the prisms have square faces congruent to those of the cube. Are there any patterns in the numbers of different shapes which might make it possible to predict how many different shapes will result when given a specific number of congruent prisms? Students might also begin exploring the surface areas and volumes of the various shapes. What kinds of patterns emerge?

SAMPLE QUESTIONS

Multiple Choice (Objective 5.1):

A group of students is investigating how many different solid shapes can be created when using a cube and two isosceles right triangular prisms whose square faces are congruent to those of the cube. They have agreed that any solid shape that can be rotated and/or reflected and made congruent to any previously found shape is the same. How many different solid shapes should this group of students find?

A less than 6

B between 5 and 8

C between 7 and 12

D more than 12 *

Open Ended:

A group of students is investigating how many different solid shapes can be created when using a cube and two isosceles right triangular prisms whose square faces are congruent to those of the cube. They have agreed that any solid shape that can be rotated and/or reflected and made congruent to any previously found shape is the same. Explain all the strategies you can think of for recording the different solid shapes. Which of these strategies would you use and why?

PROBLEM: Find a winning strategy for DECIMAL NIM.

Standard Course of Study goals and objectives addressed:
5.1, 5.2, 5.3, 5.4, 5.5, also 1.1, 1.3, 1.6, 3.1, 3.4, 7.4, 7.11

One of the oldest and most popular mathematical strategy games is NIM. There are many NIM games and the rules are simple while finding a winning strategy is quite challenging. In NIM, a certain number of objects are arranged in some specific way, or a target number is established. Each player takes turns removing, or adding, up to a specific number per turn. The player taking the last object or reaching the target either wins or loses, depending upon how winning is defined.

If students have little or no experience with NIM games, they might begin by exploring with the MECC* software "Problem-Solving with NIM" #A-257.
*Minnesota Educational Computer Consortium

Playing DECIMAL NIM might begin by setting out base ten blocks equivalent to 2.1. In this case, the longs can represent units while the cubes represent tenths. Or, flats could represent units while longs represent tenths. Students take turns removing up to 3 tenths (that is 0.1, 0.2, or 0.3). The winner takes the last 0.1, 0.2, or 0.3. This game could also be played by entering 2.1 on a calculator. Partners take turns subtracting up to 0.3, and the winner subtracts the last 0.1, 0.2, or 0.3, thus making the display show zero. Use overhead base ten materials to introduce this game and play several times with the entire class. Ask students to consider the following: Does it matter whether one goes first or second? How can one keep track of what is happening as games are played? Set students to work searching for a winning strategy. After playing a while, ask students to share their theories for winning strategies. Then, as play resumes, ask students to test either their own or others' strategies.

Teachers can't evaluate and lecture simultaneously.

We foster communication when we ask students to clarify their answers. This includes oral and written clarification.

As students continue to search for strategies, suggest they also try "smaller" versions of the problem. What happens if the game begins with four tenths? Is the winner the first or second to play? What about five tenths? What about six tenths, and so on? How will students know they have a winning strategy? What kind of "proof" will convince others?

Number	Winner
0.4	go second
0.5	go first, take 0.1
0.6	go first, take 0.2
0.7	go first, take 0.3
0.8	go second, leave 0.4

Is there any kind of pattern developing?

When students are convinced, and can convince others, they have a winning strategy, then it's time to create a new game. Change the target number, change the number that can be taken on a turn (how about up to nine tenths), change who wins (the winner forces the other person to take the last tenth), etc. As students search for winning strategies for these various games, they will have an opportunity to develop number sense and an understanding of numeration.

SAMPLE QUESTIONS

Multiple Choice (Objective 5.2):

Two students are playing the following version of DECIMAL NIM: the target number is 1.0 and up to three tenths can be removed on any turn. The winner takes the last one, two, or three tenths. Which of the following is a complete winning strategy for this game?

A Go first and take two tenths.

B Go second and take the number of tenths which will always leave an even number of tenths.

C Go first and always leave your partner with a multiple of four tenths.*

D Go second and mirror your opponents move; if he or she takes two tenths, you take two tenths, etc.

Open Ended:

Create a new DECIMAL NIM game. Be sure to define the winner and a complete set of rules. Explain your theory for winning this new game.

TOPOLOGY

There is an entire class of puzzles, sometimes called tavern puzzles, that require topological thinking for solving. The "Ox-Yoke" and "Button Hole" puzzles are classic examples of this type. They are actually a kind of network. "The Great Loop and Jacket Trick", described by Marilyn Burns in *THE I HATE MATHEMATICS BOOK*, p. 79, is another example of this kind of thinking. Put on a loose jacket, place a 2-yard length loop of rope or heavy yarn over your right arm and place your right hand in your jacket pocket. Then remove the loop without removing your hand from your pocket or untying the loop. (see pages 22 - 24, 28 - 29, 68 - 69, 78, and 91 also)

Fractals, Googols and Other Mathematical Tales by Theoni Pappas offers an amusing and entertaining way to explore and share mathematical ideas. She goes beyond the usual mechanical presentation by linking math to vital, everyday life and adding cultural and historical information.

MOEBIUS STRIP: A term of topology denoting a one-sided surface, a model of which can be made by pasting together the ends of a long strip of paper after putting in a half twist.

PROBLEM: What does Leonhard the Magic Turtle know about paths?

This problem is presented in "Leonhard the Magic Turtle" from *Fractals, Googols and Other Mathematical Tales* by Theoni Pappas.

Standard Course of Study goals and objectives addressed: 5.1, 5.2, 5.3, 5.4, 5.5, also 2.6, 3.1, 3.4, 7.1, 7.2

The sixth grade curriculum introduces students to the study of basic topolgy. Topology is a branch of mathematics that deals not with shape or size but with much more fundamental properties of objects and of space. When a cartoon character follows along a tangled hose from a tap and finds it leads back to the tap again, why does the audience laugh? It is a topological truth that, regardless of its length or curvative, a hose has two ends. Topology takes such intuitive matters and formalizes them into mathematical logic.

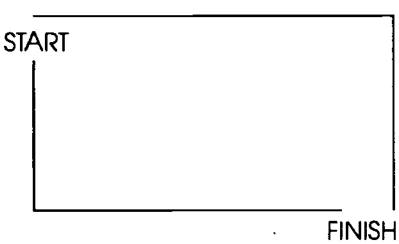
This story about Leonhard, the magic turtle, introduces networks, and the next story in Theoni Pappas' book, "Penrose discovers the Mobius strip" introduces the effects of twisting. Leonhard is a turtle that can determine before taking a walk whether or not he can accomplish this by never doubling back over any part of his path. This story about Leonhard can be read aloud to introduce the problem of using networks. Students might enjoy drawing their own paths that either can or cannot be traced without doubling back. Students might also investigate practical applications of networks.

Keeping track of moebius strip experiments
half turns
of sides & edges
Kind of cut (center or 1/3)
Result of cut (# twists or knots)

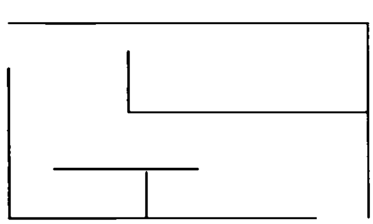
Another interesting aspect of topological thinking includes mazes. Students might bring in mazes and create their own. In order to create mazes with exactly one route, follow two basic rules:

1. Draw two lines which will form the outline or outside of the maze. These two lines also form two openings ... the start and the finish. As soon as a line is drawn, it becomes an old line.

2. Attach any new lines inside this outline so that each new line touches or crosses a previous line at exactly one point.



First two lines



Adding new lines

The first two lines do not have to be rectangular in shape ... they might form a rocket, a pocket or just about anything!

Students can create a great variety of moebius strips using adding machine tape, scissors, and cellophane or masking tape. Broad tipped marking pens are helpful when counting sides and edges on each strip. By holding the strip in one hand, place the marking pen on a side and anchor with the other hand. Then gently pull the strip along under the pen. If the pen is never raised and ends up back at the beginning point, the strip has one side! Students should be encouraged to perform various experiments with strips, keep track of results somehow, and make some generalizations.

SAMPLE QUESTIONS

Multiple Choice (Objective 5.3):

Which of these statements are true of all networks? Remember, *traversible* means that each part of the path is traversed exactly once.

1. If a network has only even vertices, it is traversible. Any vertex can be a starting point, and this same vertex must be the ending point.

2. If a network has two odd vertices, it is traversible. One odd vertex must be the starting point, and the other must be the ending point.

3. If a network has more than two odd vertices, it is not traversible.

4. There is no network with exactly one odd vertex.

A only 3

B only 2 and 3

C only 1, 2, and 3

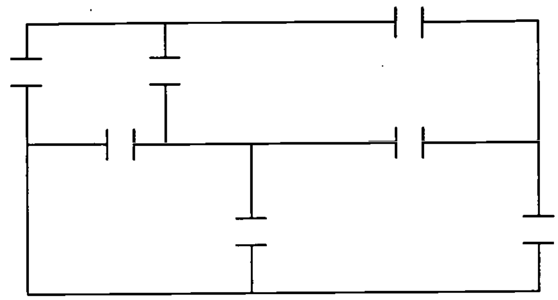
D all of these*

"There is no branch of mathematics, however abstract, which may not someday be applied to the phenomena of the real world."

Lobachevsky

Open Ended:

Here is an application of networks: Below is the floor plan of a house. Use your knowledge of networks to determine if it possible for a security guard to go through all the rooms on the house and pass through each door exactly once. Explain your process clearly. Then write another example for applying networks.



Grade 7 Standard Course of Study Goals and Objectives

Competency Goal 5: The student will solve problems and reason mathematically.

5.1 Use an organized approach and a variety of strategies to solve increasingly complex non-routine problems.

5.2 Use calculators and computers in problem solving situations as appropriate.

5.3 Discuss alternate strategies, evaluate outcomes, and make conjectures and generalizations based on problem situations.

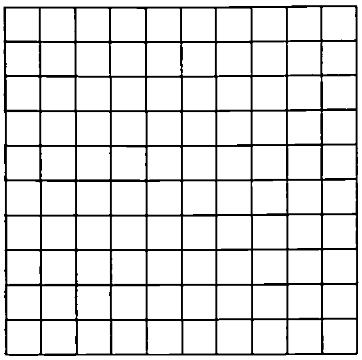
5.4 Use concrete or pictorial models involving spatial visualization to solve problems.

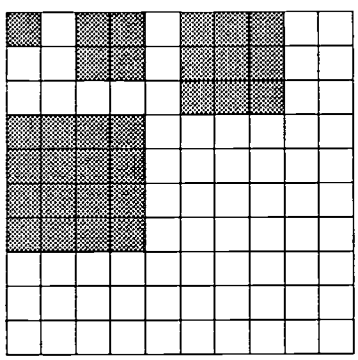
5.5 Solve problems involving interpretation of graphs, including inferences and conjectures.

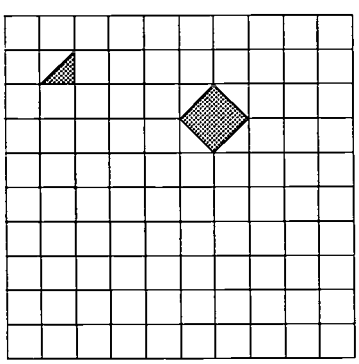
PROBLEM: On a 10 by 10 centimeter grid, how many different squares can be drawn using intersecting points as vertices?

Standard Course of Study goals and objectives addressed:
1.6, 2.2, 2.3, 4.1, 4.2, 5.1, 5.2, 5.3, 5.4

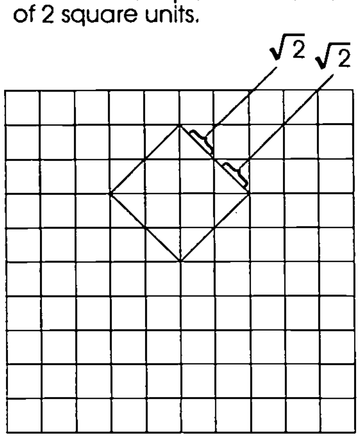
Display a 10 by 10 centimeter grid on the overhead. (Grid located in the Appendix.) One unit of area is defined here as one square on the grid or 1 cm^2. Shade in various squares with areas that are square numbers, such as, 1 cm^2, 4 cm^2, 9 cm^2, 16 cm^2, etc. How far can this pattern be extended on this particular grid? How many squares does this represent so far? Distribute grid paper and challenge students to create squares with areas of 2 cm^2, 5 cm^2, 8 cm^2, and 10 cm^2. If nobody finds a way to do this, ask these questions: "If a square unit is divided in half by drawing a diagonal through the square, what is the area of one of the triangles? What happens if 4 of these right triangles are positioned in such a way that the right angles are adjacent? Draw this square on the overhead grid and ask students to report its area. Ask students what the largest possible square will be on this grid. Will they be able to find squares with areas for all the numbers from 1 through this largest square? How will they prove that a shape is indeed a square and they've found an accurate area for that square? How will they organize their search?







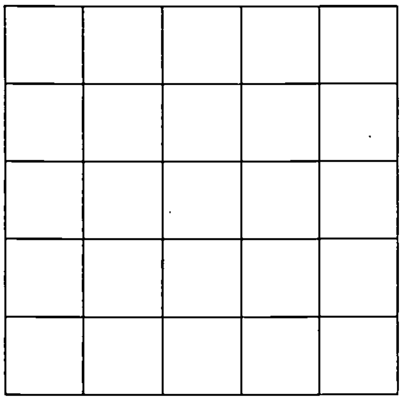
This shows a square with area of 2 square units.



This square with an area of 8 units has a side of $2\sqrt{2}$.

Small groups or partners should be ready to attack the problem. Be sure students understand that they will be reporting their results and justifying their thinking. Any groups finishing early might investigate how to report the lengths of the sides of all the squares. If the area of a square is 4 cm^2, then its side length is $\sqrt{4}$, or 2 cm. If the area of a square is 2 cm^2, then what is the side length? Students might find a way to measure its length and thus estimate something between 1 and 2. Whatever the estimate, assume 1.5, then 1.5^2 should equal 2. When checking, students will find that 1.5^2 equals 2.25 which is too big. If the side of this square is recorded as $\sqrt{2}$, students can find this on a calculator. What about the length of a side of a square whose area is equal to 8 cm^2? Students can see on the grid that this length is twice that of a side of the 2 cm^2, or $2\sqrt{2}$, or $\sqrt{8}$. Early finishing groups might also be encouraged to create new problems related to the squares problem. These might include changing the shape under investigation. What if the grid became three dimensional and the question becomes how many different cubes exist in a 10 by 10 by 10 space?

Students might also be asked to complete individual problem write-ups describing their group process. See **"How does one assess problem solving?"** in the Introduction.



SAMPLE QUESTIONS

Multiple Choice (Objective 5.3):

A group of students is investigating a simpler version of the squares problem: "How many different squares can be found on a 5 by 5 centimeter grid?" Which of the following statements will they find to be true?

1. The largest possible square has an area of 25 cm^2.

2. Squares with areas of 1 cm^2, 2 cm^2, 4 cm^2, 5 cm^2, 8 cm^2, 9 cm^2, 13 cm^2, 16 cm^2, 17 cm^2, and 25 cm^2 are possible.

3. It is not possible to draw a square whose area is 20 cm^2, because this would require a grid of at least 6 by 6.

A all of these*

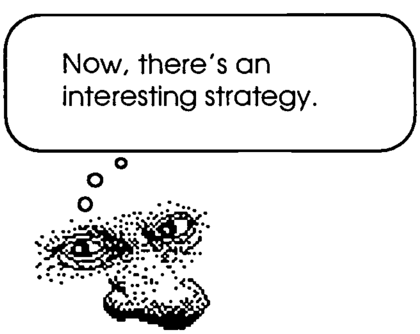
B 1 only

C 1 and 2 only

D 1 and 3 only

Open Ended:

Write an explanation which makes clear, even to someone who does not understand, the relationship between the area of a square and the length of its side. Then explain how this relationship might be used in a real life application.



Another one of Murphy's Laws:

If mathematically you end up with the incorrect answer, try multiplying by the page number.

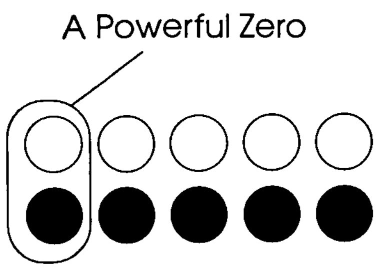
PROBLEM: Find a winning strategy for INTEGER NIM.

Standard Course of Study goals and objectives addressed:
5.1, 5.3, 5.4, also 1.1, 3.1, 7.5

This NIM variation gives students an opportunity to use a model for integers while looking for NIM strategies. There are many NIM games and the rules are simple while finding a winning strategy is quite challenging. In NIM, a certain number of objects are arranged in some specific way, or a target number is established. Each player takes turns removing, or adding, up to a specific number per turn. The player taking the last object or reaching the target either wins or loses, depending upon how winning is defined.

Playing INTEGER NIM also involves understanding the concept of "The Powerful Zero." The Powerful Zero is formed when –1 and +1 are combined. Two different colors of markers, or playing pieces, are needed. One of these colors represents the +1's and the other color represents the –1's. Something like glass beads used in crafts or checkers, or colored plastic chips work well. Visualize a set of glass beads. White stands for –1's and blue stands for +1's, while a white and blue together stand for The Powerful Zero. When using glass beads or transparent colored plastic chips, the following can be modeled on the overhead. Place five Powerful Zeros, five +1's and five –1's, on the playing area. Take turns removing anything from –2 through +2; i.e., –2, –1, Powerful Zero, +1, or +2. (This means, remove one or two markers.) The winner removes the last piece or pieces <u>and</u> ends up with a total of ±1 or zero when all of his/her pieces are totaled.

Play this game as a total class a few times so that everyone has an opportunity to understand the rules. Ask students to consider whether playing first or second makes a difference. Is there a strategic first move? After students have played with a partner for a while, ask them to share their ideas about how to win. When students go back to play, ask them to test their own or other's theories. How will they know when they have found a winning strategy? How will they convince others? One helpful way to go about looking for NIM strategies is to play with a smaller number. What happens with two Powerful Zeros, or two each of +1's and –1's. When students have found a winning strategy for this game, it's time to change the rules in order to create a new challenge. This can be accomplished by changing the number of playing pieces, the number that can be taken on any turn, or the definition of winning.



Half of the markers represent positive integers, or charges, while the other half represent negative integers. By combining a positive and a negative, a POWERFUL ZERO is created.

Sometimes this model helps students visualize adding and subtracting integers: There is a town in the West. In this town are good guys and bad guys. It is good for the town when the good guys enter and the bad guys leave. It is bad for the town when the good guys leave and the bad guys enter.

	enter/+	leave/-
good guys/+	+	-
bad guys/ -	-	+

±1 is read plus or minus 1

Maier's Laws:
1. If any facts do not conform to the theory, they must be disposed of.
2. The bigger the theory, the better.

RATIONAL NUMBER: A number which can be expressed as a fraction, that is the quotient of two integers. Non-terminating and non-periodic decimals are irrational, such as $\sqrt{2}$ and π.

SAMPLE QUESTIONS

Multiple Choice (Objective 5.3):

A group of students is looking for a winning strategy for a version of INTEGER NIM. They are using six playing pieces, three each of +1's and –1's, or three Powerful Zeros. On any turn, a player may take one or two markers, –2, –1, a Powerful Zero, +1, or +2. The winner takes the last piece or pieces and has a total of zero or ±1. Which of the following is a complete winning strategy for this game?

A Go second.

B Go second and mirror your partner's play. If your partner takes a Powerful Zero, you take a Powerful Zero, if your partner takes –2, you take –2, etc.

C Go second and leave your partner with three markers. Take markers in such a way that balances +1's and –1's that have been taken as much as possible.*

D Go first and always take Powerful Zeros.

Open Ended:

Create a new NIM game which uses the set of rational numbers. Write a complete set of rules including the number and type of "playing pieces" or numbers, the number which may be played on any turn, and how to win. "Prove" that your game is indeed a NIM game.

BASIC DEFINITIONS

PERMUTATION: any arrangement of the elements of a set in a definite order. AB is different from BA.

COMBINATION: any selection of the elements of a set. Any arrangements having the same members are equivalent ... order is not a consideration. AB is the same as BA.

The best way to have a good idea is to have a lot of ideas!

Extension Activity

Since Pythagorus is one of the charac ters in this book, this might be an opportune time to introduce the Pythagorean Theorem using models, objective 2.3.

Three Little Pigs' Law:

There is no safety in numbers, or anything else.

Can one read a book like this to seventh graders? Absolutely!!!! Seventh graders and kindergarteners have a great deal in common. Among these commonalities is enjoying listening to a good story ... especially when the reader "hams it up" and enjoys the humor in the story also.

PROBLEM: In how many ways can the three little pigs arrange themselves in five houses?

This problem is introduced in the book *Socrates and the Three Little Pigs* by Tuyosi Mori.

Standard Course of Study goals and objectives addressed: 5.1, 5.2, 5.3,5.4, also 3.1, 6.6, 6.7, 6.8, 6.9

Many interesting and engaging problems have been posed using the theme of this old fairy tale. In this version of The Three Little Pigs, Socrates is the wolf who spends a lot of time thinking and talking with his friend Pythagoras the frog. Xanthippe (pronounced zan - 'tip - e) is Socrates' wife and has no patience for thinking. She would rather eat! When Socrates decides that he might catch one of the three little pigs for dinner, he is faced with the problem of deciding in which of the five houses in the meadow he should look for one of the pigs. This sparks a lengthy discussion between Socrates and Pythagoras about all the possibilities ... some being permutations and some being combinations. Pythagoras, being a mathematician, introduces one way to think about this problem by using a tree diagram, and an opportunity for exponential notation, p. 9 ($5 \times 5 \times 5$ or 5^3). When Xanthippe expresses her impatience, Socrates decides to think about a simpler version of the problem, one in which there are no roommates. This introduces a new kind of tree diagram (see p. 17 in the book).

Begin by reading the first 8 pages of this book. Stop reading at this point and ask students to work in pairs or small groups to "clear up the confusion." After allowing plenty of working time, ask students to present their "solutions" and justify their thinking. Then read pages 9 through 13 and stop. Ask students to attack this new problem. After working time and student presentations, continue reading. Be sure to stop at appropriate points for student investigation (p. 21, p. 25 [note 3! on p. 24], p. 29, p. 35). If Socrates had finished working on this problem before daybreak, what would his chances have been of finding a pig? This entire process might take some class time over a period of days, and be combined with some homework time.

At the end of *Socrates and the Three Little Pigs*, the "Note to Parents and Other Older Readers" introduces the problem of finding how many different ways seating can be arranged in the classroom and the probability of best friends sitting together. Challenge students to write new problems related to that of Socrates. Encourage students to search for real life applications.

Librarian's Motto:
Information is where you find it.

If students are not familiar with factorial, read *Anno's Mysterious Multiplying Jar*, by Masaichiro and Mitsumasa Anno. Another interesting way to approach these same concepts would be to introduce students to *Making Faces* by Norman Messenger. This book is a series of faces divided into 5 sections. By mixing these various sections, it is possible to create new faces. How many different faces are possible? Are these combinations or permutations? Students could create their own versions of *Making Faces* which might even include word forms such as prefixes, roots, and suffixes. How many different words could be created?

SAMPLE QUESTIONS

Multiple Choice (Objective 5.1):

The Three Little Pigs are becoming quite affluent and have purchased yet another house in the meadow. Socrates wants to know how many possible ways the pigs might live in these 6 houses if there are no roommates. Xanthippe, being very impatient, wants Socrates to ignore which pig is where. After all, they all taste the same in the dark. What is the answer to Socrates' problem?

A 18

B 120*

C 216

D 720

Open Ended:

Using *Socrates and the Three Little Pigs* as a starting point, write a new yet related problem. Suggest a strategy for working on a solution to your problem.

Extension Activity:
Who was Socrates? Was he really married to Xanthippe? What is the socratic method? Who was Pythagoras? What's a theorem?

POLYOMINO: a shape created by joining a set of equal-sized squares edge to edge so that the corners match.

The following are names for some of these polyominoes:

# squares	shape names
1	Monomino
2	Domino
3	Trominoes
4	Tetrominoes
5	Pentominoes
6	Hexominoes
7	Heptominoes

How might others be named?

POLYOMINO ORDER: the smallest number of copies, or repeats, of a specific polyomino that will fit together to fill a rectangle. The rectangle can have any size or shape as long as it fits the definition of a rectangle. For some polyominoes, this is not possible. For these, the order is not defined.

Grade 8 Standard Course of Study Goals and Objectives

Competency Goal 5: The student will solve problems and reason mathematically.

5.1 Use an organized approach and a variety of strategies to solve increasingly complex non-routine problems.

5.2 Use calculators and computers in problem solving situations as appropriate.

5.3 Make and evaluate conjectures and arguments, using deductive and inductive reasoning.

5.4 Investigate open-ended problems, formulate questions, and extend problem solving situations.

5.5 Represent situations verbally, numerically, graphically, geometrically, or symbolically.

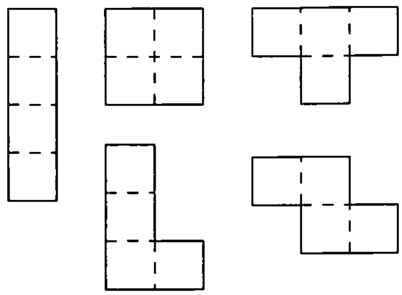
5.6 Use proportional reasoning to solve problems.

PROBLEM: Find the "order" of each pentomino in the set of pentominoes.

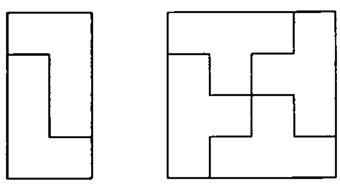
Standard Course of Study goals and objectives addressed:
5.1, 5.2, 5.3, 5.4, 5.5, also 2.7, 3.4, 3.6, 7.1, 7.2

Ian Stewart presents this problem in the book *Another Fine Math You've Got Me Into*. Stewart introduces mathematical curiosities and puzzles through imaginative stories with curious characters and dozens of jokes and puns. In one of these stories, "Tile and Error", Henry Worm, his wife, Anne-Lida, and baby Wermetrude have just moved into a modern bungahole. Anne-Lida wants the new bathroom fully tiled. When Henry calls the builder, who had promised this complete tiling, he's told that the fancy tiles ordered by Mrs. Worm wouldn't fit together. Young Tyler the tiler said he kept getting gaps. Well, Henry decides to finish the tiling himself and after several frustrating days decides to seek advice from his friend Albert Wormstein. Albert observes that this fancy tile is a kind of polyomino, a plane figure formed by joining a set of equal-sized squares edge to edge so that the corners match. Albert further explains that the order, m, of any polyomino is the smallest number of copies of that polyomino, or tiles, that will fit together to fill a rectangle, assuming this is possible, of course. Students will probably enjoy hearing this tale as an introduction to this problem; however, this isn't absolutely necessary.

Here is the set of Tetrominoes



Tetromino Orders



Order = 2 Order = 4

n = number of squares used to create the polyomino.
P(n) = number of different polyominoes when counting reflections and rotations of a polyomino already found as the same.
Q(n) = number of different polyominoes counting reflections and rotations as different.

n	P(n)	Q(n)
1	1	1
2	1	2
3	2	6
4	5	19
5	12	63
6	35	216
7	108	760
8	369	2,725

What patterns and relationships might exist between n, P(n), and Q(n)? Can a formula be written which will provide P(n) and/or Q(n) based on n?

To begin, students need to either have had prior experience with searching for pentominoes or need to create a set of pentominoes. If rotated or reflected copies of a pentomino are considered the same, there are 12 different pentominoes. Students might be asked to create this set of pentominoes as a homework assignment. Before students work in pairs or small groups to attack this problem, work a smaller version as a whole group. Consider the set of tetrominoes, shapes made with 4 squares. What is the order of each tetromino?

- Two of these have an order of 1, the rectangle and the square, since these are already rectangles. This leads to a general principle: "A polyomino has order 1 if and only if it is a rectangle."
- The "L" shaped tetromino requires 2 copies for a rectangle, thus has an order of 2. A polyomino of order 2, by definition, is obtained by cutting a rectangle into two equal pieces. Thus, the cutting line must be symmetric under 180° rotation. This states an effective characterization of polyominoes with order 2.
- The "T" shaped tetromino has an order 4. There is also a pattern to how order 4 polyominoes are formed. Challenge students to find this.
- Now, what about the order of the "Z" shaped tetromino?

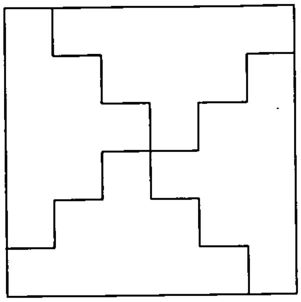
In order to prove that a polyomino has a specific order, order m, one must do two things:

- Find a way to tile a rectangle with m copies, or repeats
- Prove that no number of copies or repeats smaller than m will also tile a rectangle.

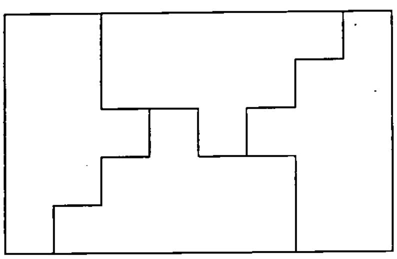
If one suspects that a polyomino has no order, how does one go about proving this? It turns out that this "Z" tetromino does not tile a rectangle. The proof is obtained by considering what happens at the corners of a presumed rectangle.

As students begin working, they should understand that they will be required to report their results and justify their thinking. When a summary list of the orders of various polyominoes is examined, some interesting patterns emerge. Challenge students to find these patterns. Students might also be challenged to investigate the following: Is there a pattern, and/or general formula, which states the relationship between the number of squares used in a set of polyominoes and the number of different shapes possible?

Here are two ways to get an order of 2:



Rotating same shape 4 times in 90 degree increments



Rotate 90 degrees and translate, repeat 4 times

There are several ways to think about the possible number of different shapes. One of these was stated earlier. The number of different shapes counts reflections and rotations of already found polyominoes as being the same. A second answer to finding different shapes counts rotations or reflections as different. The accompanying table provides a partial summary of these numbers to help students get started. Encourage students to write new and related problems, especially with real world applications.

SAMPLE QUESTIONS

Multiple Choice (Objective 5.3):

A group of students has been investigating the order of various polyominoes. They have written the following generalizatons. Which are true?

1. A polyomino has order 1 if and only if it is a rectangle.
2. A polyomino has order 2 if it can be created by dividing a rectangle into two equal pieces.
3. A polyomino of order 3 does not exist.
4. A polyomino of order 4 can exist in two ways. One by rotating the polyomino in 90° increments about a center point and a second by rotating in 90° increments along with a translation.
5. Every multiple of the number 4 can be an order for some polyomino.

A 1 and 2 only

B 1, 2, and 3 only

C 1, 2, 3, and 4 only

D All of these*

Open Ended:

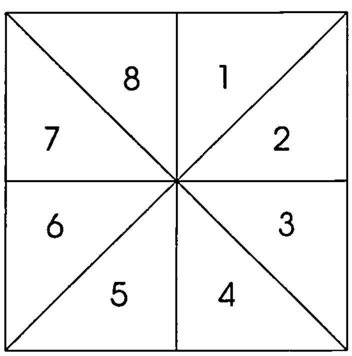
Explain how knowing the order of a polyomino could be helpful in a real life situation.

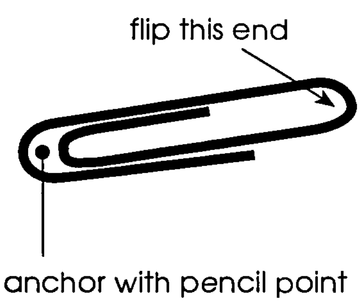
How about a little pig humor????

Q: Where does a pig go to get a new tail?
A: To a retail store.

Pigs make good cheerleaders because they root for the home team!

Q: What happens when two pigs get engaged?
A: They are betroughed.





Complete strategies leave no question in the player's mind about what to do at each step of the game.

PROBLEM: Find a winning strategy for the game Porker.

Standard Course of Study goals and objectives addressed:
5.1, 5.2, 5.3, 5.4, 5.5, 5.6, also 6.1, 6.2, 6.3, 6.4, 6.5, 6.6

Porker is a variation of the games "PIG" and "HOG" with which students may already have some familiarity and experience. This variation of the game will provide many opportunities for new investigations as students create new types of spinners. First, students need to create spinners for playing the game. Provide each student with a piece of cardboard and a paperclip. A 4" × 6" or 5" × 8" index card could be trimmed to create a square. Using a straight edge, students should draw the two diagonals and the vertical and horizontal bisectors of the square. This divides the square into 8 equal sections. Have students number these sections 1 through 8. Anchor a jumbo paper clip to the center of the square by placing a pencil point in the closed loop section of the paper clip. The paper clip can be spun around the pencil point by flicking the other end.

Students play as a team of two against another team of two. Both members of a team spin their spinner. Their score for that spin is the sum of the two numbers showing on the spinners. This team continues to spin and add to their score until they either decide to stop or a 1 comes up on one of the spinners. If a 1 should come up before the team decides to stop, they lose all their points for that turn. If a 1 should come up on both spinners, their total score goes back to 0 for all turns. Then the other team of two takes their turn.

This game should be modeled on the overhead to make sure everyone understands the rules. Students will be playing this game and searching for a strategy which will give them the highest possible score over a period of many turns. They need to write complete strategies and be ready to justify their thinking or convince others they have found an optimal strategy.

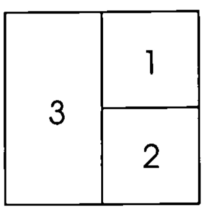
A complete strategy leaves no question in the player's mind about what to do. A sample complete strategy might state: "Spin both spinners three times for each turn and stop." This may or may not be an optimum strategy, but it is complete since the players know exactly how to proceed. A sample incomplete strategy might state: "Spin until the total score is over 15 and stop." This is incomplete because the players still need to decide when to stop; "over 15" could be interpreted many different ways.

While searching for optimal strategies for playing Porker, students might be encouraged to consider the probability of getting a 1 on one spinner. They might also be encouraged to consider the probability of getting a 1 on both spinners on the same spin. They should be encouraged to keep track of their spins and scores. They might represent these data with scatter plots, or some other kind of graph, and find measures of central tendency. How does any of this help form optimal strategies?

When students have found an optimal strategy for this game, they can change the design of their spinners and begin working on a new game strategy. What if the sections on the spinner are not equal? What if each partner has a different spinner? This could become quite challenging.

SAMPLE QUESTIONS

Multiple Choice (Objective 5.5):



A group of students have designed a new version of Porker. Their spinners are shown to the left. Otherwise, this version is played the same as the original game. They have written some generalizations to help them find an optimal playing strategy. Which of these generalizations are true?

1. The highest possible score for any spin is 6.
2. The sum most likely to occur is 4.
3. The probability of getting a 1 on one spinner is 1/4.
4. The probability of getting a 1 on both spinners at the same time is 1/16.

A all of these*

B 1 and 3 only

C 1 only

D 1, 3, and 4 only

Open Ended:

Create a new version of Porker. Draw a picture of the spinner or spinners used in your new game. Write a list of questions that players might need to answer as they search for an optimal playing strategy.

The Phantom Tollbooth is a witty fantasy that has been a favorite with readers of all ages for many years. In this story, Milo finds a mysterious package in his room. This package contains what looks like a genuine turnpike. When he drives his electric car through the tollbooth, he finds himself in The Lands Beyond where he meets some of the craziest creatures ever imagined. One of these creatures introduces itself by saying:

"My angles are many.
My sides are not few.
I'm the Dodecahedron.
Who are you?"

Of course, what the mathemagician ignores is the order of operations.

"To see a world in a grain of sand,
And a heaven in a wild flower,
To hold infinity in the palm of your hand,
And eternity in an hour."
Blake
Auguries of Innocence

PROBLEM: How can one explain "infinity", especially to someone who doesn't understand?

This problem is inspired by the book *The Phantom Tollbooth* by Norton Juster.

Standard Course of Study goals and objectives addressed: 5.3, 5.4, 5.5, also 1.1, 7.1

This book is too long to read aloud in a class period, or even over several class periods. However, one might summarize the events up to the chapter titled "This Way to Infinity" and then begin reading aloud at this point in order to introduce the problem. Actually, the story begins dealing with numbers in the chapter before this, "The Dodecahedron Leads the Way". If students have not yet read *The Phantom Tollbooth*, the language arts teacher might like to include this in her or his required reading for students.

Students might be asked, "What is the biggest number there is?" Then explain that Milo asked the same question in DIGITOPOLIS and here is what the mathemagician replied. Then read that section of the book. Direct students to work in pairs to create as many different models or explanations as possible related to explaining infinity or "the biggest number there is". Of course, students will be required to present these to the class. Perhaps the class could compile everything into a book about infinity. This might be a good opportunity to use the computer for creating the text and illustrations.

Also, challenge students to find other problems that might be posed in *The Phantom Tollbooth*. For example, the equation at the beginning of "This Way to Infinity" poses an interesting problem. The mathemagician claims that $4 + 9 - 2 \times 16 + 1 \div 3 \times 6 - 67 + 8 \times 2 - 3 + 26 - 1 \div 34 + 3 \div 7 + 2 - 5 = 0$. What do students think about this? What can be done to this equation in order to have it equal zero? Can any other problems be posed or created based on this? This book is full of puns and wordplay. Students might be challenged to write a book of puns related to mathematics.

Could the set of "social security numbers" be infinite? What happens to your social security number when you die? Well, your number gets retired! The first number was issued in 1936. If social security numbers are never reassigned, is it possible that we'll eventually run out of numbers to assign to people in the US? It's comforting to know that you can take something with you ... but what about the future? Challenge students to figure out how many possible social security numbers are possible. When, if ever, will we run out?

Currently, 41 million Americans receive monthly social security checks totaling 20% of the entire federal budget. Each decade these benefits will double. By this decade's end, $500 billion will be spent, $1 trillion by 2010, and $20 trillion will be necessary when today's schoolchildren retire, starting in 2050.

facts from: *America by the Numbers* by Les Krantz, Houghton Mifflin, 1993

SAMPLE QUESTIONS

Multiple Choice (Objective 5.3):

Which of the following sets of numbers are examples of infinite sets?

1. The set of counting or natural numbers.
2. The set of integers.
3. The set of integers between 0 and one billion.
4. The set of rational numbers between 0 and 1.

A 1 and 2 only

B 1, 2, and 3 only

C 1, 2, and 4 only*

D all of these

Open Ended:

Write a list of all the things you can think of that come in infinite sets and explain how you know each set is infinite.

BIBLIOGRAPHY for PROBLEM RESOURCES

CODE: E = appropriate for grades 3-5; M = appropriate for grades 6-8

EM Burns, Marilyn. *About Teaching Mathematics*. Sausalito, CA: Math Solutions Publications, 1992.

E Burns, Marilyn. *A Collection of Math Lessons From Grades 3 Through 6*. Sausalito, CA: Math Solutions Publications, 1987.

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EM Burns, Marilyn. *Every Kid's Book of Things to Do*. New York: Bantam Books, 1979.

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EM Charles, Linda, and Brummett, Micaelia. *Connections: Linking Manipulatives to Mathematics* (Grades 1-8). Sunnyvale, CA: Creative Publications, 1989.

EM Cook, Marcy. *Justify Your Thinking*. Balboa Island, CA: Cook, 1993. (Note: Marcy Cook publishes a long list of problem solving materials. Write for catalog: P.O. Box 5840, Balboa Island, CA 92662)

EM Dolan, Daniel, and Williamson, James. *Teaching Problem-Solving Strategies*. Menlo Park, CA: Addison-Wesley Publishing Company, 1983.

E Downie, Diane, Slesnick, Twila, and Stenmark, Jean K. *Math For Girls and other Problem Solvers*. Berkeley, CA: Lawrence Hall of Science, University of California, 1981.

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(Note: Creative Publications has published a series of *Cooperative Problem Solving With* ... books)

M Seymour, Dale. *Favorite Problems.* Palo Alto, CA: Dale Seymour Publications, 1981.

EM Seymour, Dale. *Problem Parade.* Palo Alto, CA: Dale Seymour Publications, 1984.

EM Souviney, Randall J. *Solving Problems Kids Care About.* Santa Monica, CA: Goodyear Publishing Company, 1981.

Worthwhile Task Analysis

	includes sound and significant mathematics	accounts for students' understandings, interests, and experiences	accounts for the range of ways diverse students learn	engages students' intellects and encourages "what if" questions	invites students to make decisions and use own methods	promotes communication and discussion	nests skill development in the content of problem solving	calls for problem forming as well as problem solving	is accessible to all students at the start	is extendable in order to challenge those students with high levels of understanding	stimulates students to make connections	engages the majority of students

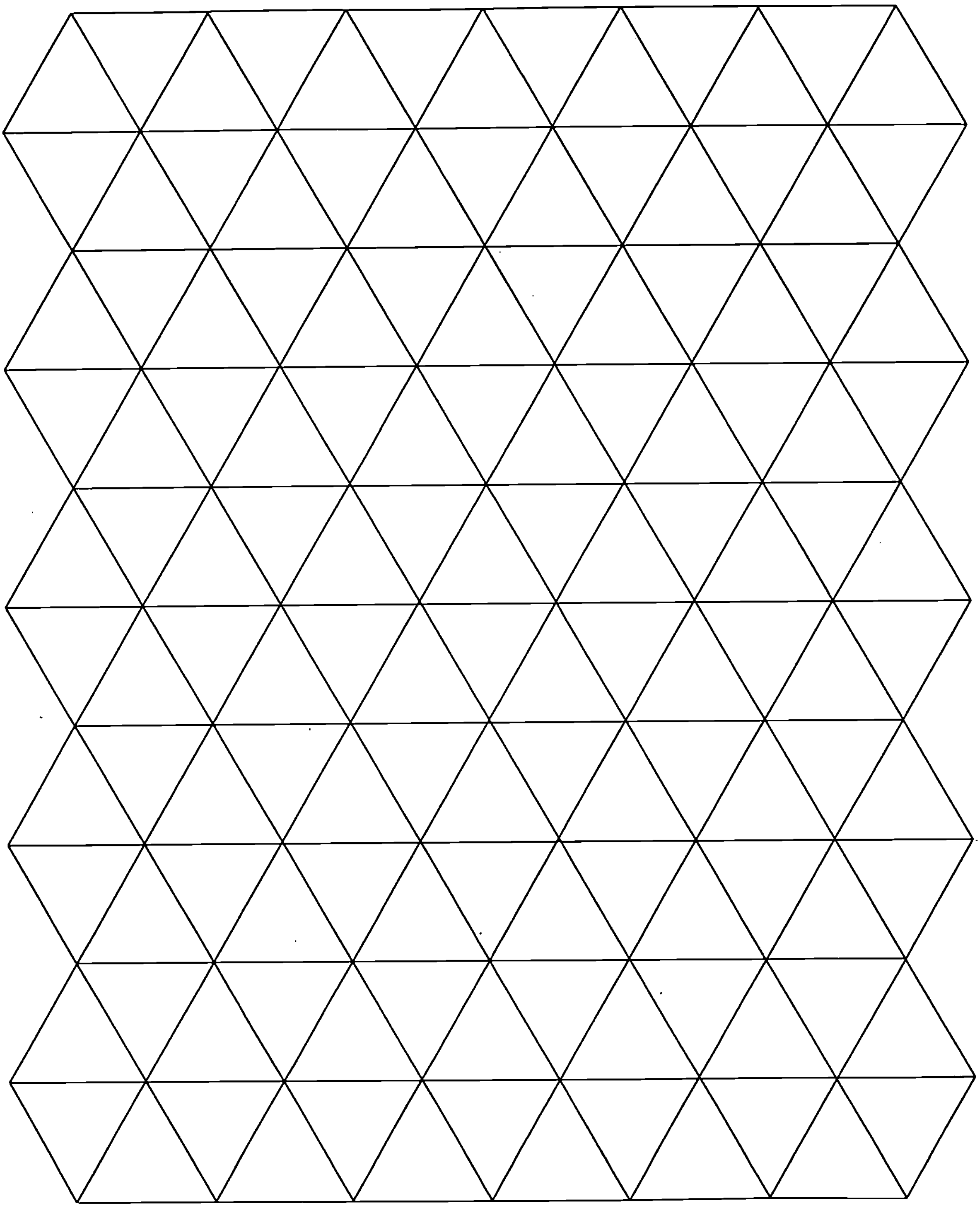
65A

When using the "Student Checklist," write all notes on one sheet. Then, go to the copier and make 5 copies. White-out the names of students 2, 3, 4, and 5 on one sheet. This becomes the copy that goes into student number one's portfolio. On a second sheet, white-out the names of students 1, 3, 4, and 5. This copy goes into student number two's portfolio, and so on. The advantage to this method is avoiding having to rewrite notes, and each student's checklist also shows how others in the group were "functioning" in comparison.

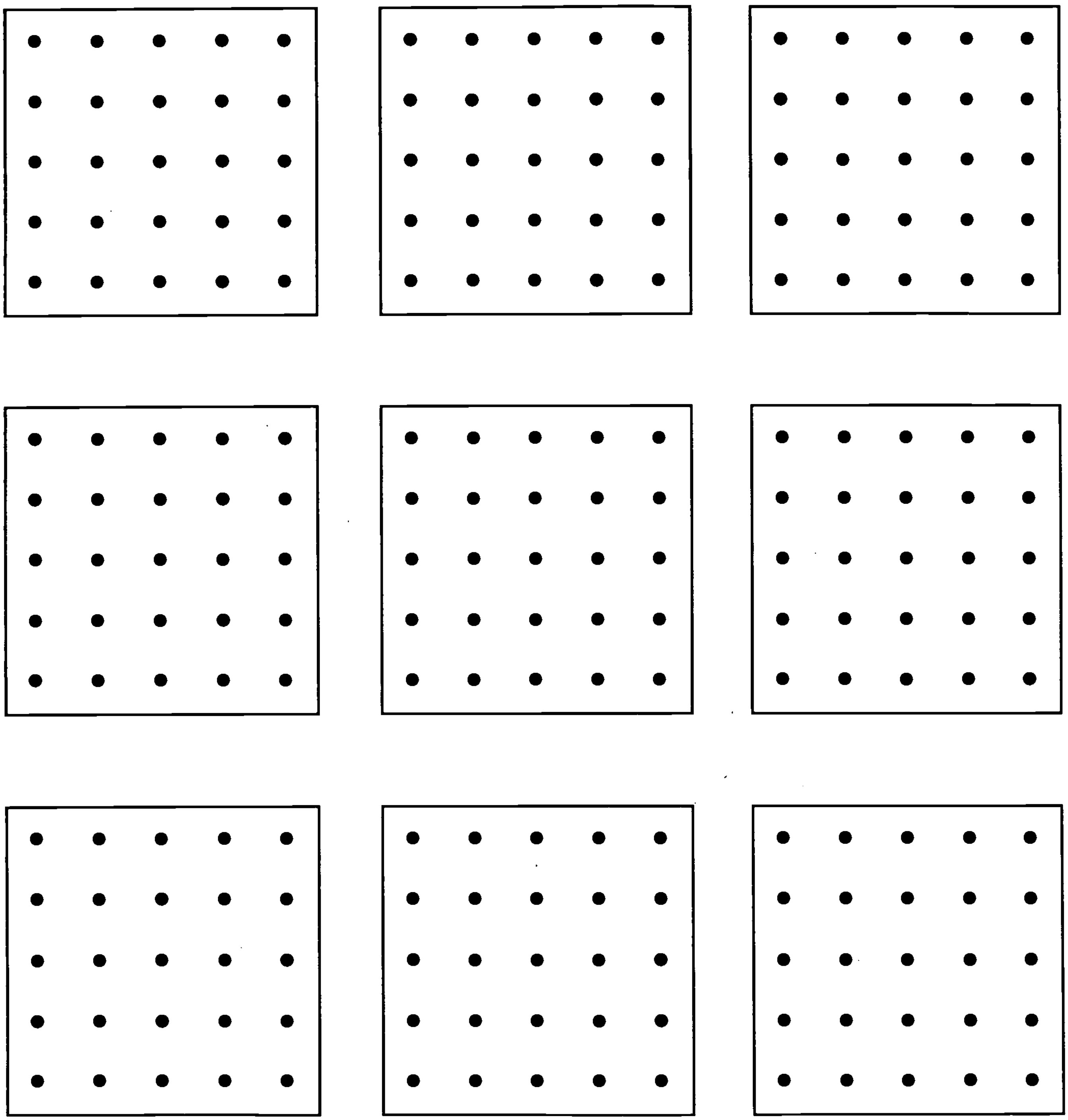
STUDENT CHECKLIST

Date: **Name:** **Action Observed**	**Student 1**	**Student 2**	**Student 3**	**Student 4**	**Student 5**
remained on task					
offered theories about winning strategies					
participated in group discussion without prompting					
encouraged others to participate					
responded to others' theories and ideas					
other:					

PATTERN BLOCK PAPER



GEOBOARD PAPER



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