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ABSTRACT

The fourth volume of the proceedings of 21st annual meeting of the International Group for the Psychology of Mathematics Education contains the following papers: (1) "A Model for Discriminating amongst Student Teachers of Mathematics" (P. Perks); (2) "A Study of Teachers' Conceptions about Mathematics" (G.N. Philippou and C. Christou); (3) "Emily and the Supercalculator" (D. Pitta and E. Gray); (4) "Knowing v. Knowledge-in-Action" (S. Prestage); (5) "Readability of Verbal Problems in Mathematics: Some Evidence from Secondary Schools in South Africa" (E.D. Prins); (6) "Images and Definitions for the Concept of Even/Odd Function" (S. Rasslan and S. Vinner); (7) "Constraints and Opportunities in Teaching Proving" (D.A. Reid); (8) "Assessment of Teacher-Student Interpersonal Behavior in Secondary Mathematics Classrooms: A Seed for Change" (T. Rickards and D. Fisher); (9) "Beliefs and Their Warrants in Mathematics Learning" (M.M. Rodd); (10) "Pupils' Strategies and the Cartesian Method for Solving Problems: The Role of Spreadsheets" (T. Rojano and R. Sutherland); (11) "Fallibilism and the Zone of Conjectural Neutrality" (T. Rowland); (12) "The Experience of Mathematics Teaching" (U. Runesson); (13) "Calculator Use by Upper Primary Pupils Tackling a Realistic Number Problem" (K. Ruthven); (14) "Spatial Abilities, van Hiele Levels and Language Use in Three Dimensional Geometry" (S. Saads and G. Davis); (15) "Algebraic Expressions and Equations: An Example of the Evolution of the Notions" (C. Sackur and J.P. Drouhard); (16) "Inferential Processes in Michael's Mathematical Thinking" (A. Saenz-Ludlow); (17) "Students Appropriation of Mathematical Artifacts during Their Participation in a Practice: 2 propos de A. Sfard..." (M. Pinto dos Santos and J.F. Matos); (18) "Does Teaching Mathematics as a Thoughtful Subject Influence the Problem-Solving Behaviors of Urban Students?" (R.Y. Schorr, C.A. Maher and R.B. Davis); (19) "Framing in Mathematical Discourse" (A. Sfard); (20) "Theory of Global and Local Coherence and Applications To Geometry" (A. Shriki and E. Bar-on); (21) "Generating Theoretical Accounts of Mathematics Teachers' Practices" (M.A. Simon and R. Tzur); (22) "Towards a New Theory of Understanding" (J. Duffin and A. Simpson); (23) "How Far Can You Go with Block Towers?" (C.A. Maher and R. Speiser); (24) "Chance Estimates by Young Children: Strategies Used in an Ordering Chance Task"

(A.G. Spinillo); (25) "Multiple Referents and Shifting Meanings of Unknowns in Students' Use of Algebra" (K. Stacey and M. MacGregor); (26) "Elementary Components of Problem Solving Behaviour" (M. Stein); (27) "Changing Teaching and Teachers Change" (P. Sztajn); (28) "Is the Length of the Sum of Three Sides of a Pentagon Longer than the Sum of the Other Two Sides?" (P. Tsamir, D. Tirosh and R. Stavy); (29) "Investigating Change in a Primary Mathematics Classroom: Valuing the Students' Perspective" (D. Tomazos); (30) "Studying Children's Argumentation by Incorporating Different Representational Media" (D. Potari and T.A. Triadafillidis); (31) "Perceptions of Unfamiliar Random Generators--Links between Research and Teaching" (K. Truran and R. Ritson); (32) "Representations of Points" (P. Tsamir); (33) "Understanding of Different Uses of Variable: A Study with Starting College Students" (S. Ursini and M. Trigueros); (34) "Gender Differences in Cognitive and Affective Variables during Two Types of Mathematics Tasks" (H. Vermeer, M. Boekaerts, and G. Seegers); (35) "Coming To Know Pupils: A Study of Informal Teacher Assessment of Mathematics" (A. Watson); (36) "Supporting Elementary Teachers' Exploration of Children's Arithmetical Understanding: A Case for CD-ROM Technology" (J.W. Whitenack, N. Knipping, L. Coutts, and S. Standifer); (37) "Numbers versus Letters in Algebraic Manipulation: Which is More Difficult?" (M.P.H. Wong); (38) "'You Have To Prove as Wrong': Proof at the Elementary School Level" (V. Zack); (39) "Constructing Knowledge by Constructing Examples for Mathematical Concepts" (O. Hazzan and R. Zazkis); and (40) "Changes that Computer Algebra Systems Bring To Teacher Professional Development" (N. Zehavi). (ASK)

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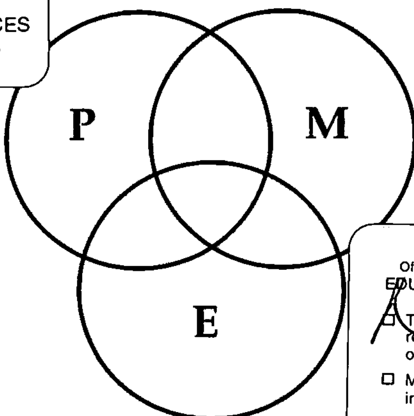

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Zehavi Nurit

Changes that computer algebra systems bring to teacher professional development

RESEARCH REPORTS

(continuation)

A MODEL FOR DISCRIMINATING AMONGST STUDENT TEACHERS OF MATHEMATICS.

Pat Perks, The University of Birmingham, UK.

During my research into the effects on mathematics students teachers of a lesson planning structure, I developed a two-dimensional model to describe students' potential as good teachers. The image presented the continua for analysis and performance as axes and describes the qualities of students within five regions.

Introduction.

In my current research on the effects of a particular planning model (Perks & Prestage, 1994) used with mathematics students on a one year post-graduate course in teacher education, I looked for ways to structure the judgements made about students' potential to become good teachers. Existing models tend to be uni-dimensional and focus on performance. I believe that performance itself is insufficient. Investigating the evidence on the use of the planning model amongst thirty-eight students over one year, the need emerged to differentiate the levels of analysis in pre-service teachers.

Methodology.

The research data included students' lesson plans and evaluations, observation of lessons, tapes of debrief sessions, videotapes of lessons and interviews of a sample students at the end of the course. The analysis is based within the interpretive research paradigm (Bassey, 1995), and integral to it is my role as a participant observer (Eisenhart, 1988), my reflection on my professional role as a tutor (Schön, 1987) and the discipline of noticing (Mason, 1994).

The Performance Continuum.

Performance is what 'natural' teachers, be they liberal or authoritarian will be deemed to show during their first class lessons. It includes the most commonly described features of student teachers, and is often the only aspect of one-dimensional student growth pictures. Performance is used here in the sense of technical skills, which can be whole substance of competencies (e.g. DfE 1992). Students may start anywhere on the continuum, but the aim is that they should become better performers.

Performance is often related to confidence about standing up in front of a class and getting all pupils to work. Some students are extremely confident from the beginning, some only need a few experiences, but for others their performance will improve or regress in response to different classes and situations.

The continuum is divided into three sections (fig 1), negative, neutral, positive. The

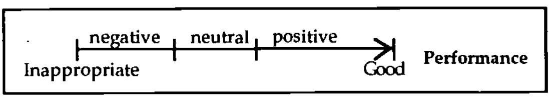

Figure 1: The Performance Continuum.

negative zone describes performance which is not successful in the classroom. It is the performance of those who may avoid eye contact, continue to speak when pupils are not listening, the type of student teacher who appears unaware of the 'audience' i.e. the pupils. The student will complete tasks related to teaching, but often only in response to someone asking for them to be done. These are the students who may be late, who miss deadlines, whose time management is poor. They seem unaware of strategies necessary for classroom management and teacher/pupil interaction. Students in this group fall into two categories, those who are unaware of the need to demonstrate such attributes and those who are so nervous that they cannot demonstrate them.

The neutral zone describes the starting point in performance of most students, they are aware of the strategies for teaching, the professional demands of the role and are trying to improve. Different individuals will have different aspects to develop.

The third zone describes those who quickly develop a wide range of skills, they speak well, demand the attention of their classes, they deal with the everyday jobs of teaching and volunteer to take on extra responsibility. The routines of teaching are learned quickly. They are efficient and resources, time etc., are all well managed.

Competencies, such as those in Circular 9/92 (DfE, 1992), tend to highlight the performance continuum. Even attributes related to planning good lessons or motivating pupils, which should be more than technical skills, can often be judged on appearance.

I want my students' to be confident about their performance, I do want them to be good actors in the role of teacher, but I also want them to be more than technicians, something else is required. Emerging from my data is a further division, which is related to the relativist/absolutist dichotomy (Burton, 1994, p 209), i.e. those who enjoy discussing mathematics and its teaching, and those who feel that the teaching of mathematics is well defined and unproblematic. There is a need for another way of differentiating between the students, beyond performance.

The Analysis Continuum.

Students on the course have to consider many aspects of analysis, analysis of mathematics, analysis of lesson styles, analysis of teaching and of learning, reflection on and evaluation of lessons, and analysis of their own development. It is in the area of analysis where it is possible to see behind the facade of performance.

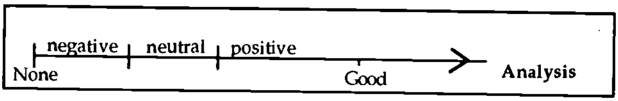

Figure 2: The Analysis Continuum.

The continuum is split into three main zones (fig 2), where the neutral zone is where I would place the analytical skills of the majority of students. Many of the students initial reactions to the analysis expected of them is affected by their views of pedagogy. The students' previous experience of schooling is an important factor. They use the worked example and practice from text books, because that is what they believe worked for them. Many believe that teaching is repetition of facts and that by repeating the same thing the children will learn as a result. As a result, questioning the authority of the text book and working on different ways of presenting mathematics to children can seem irrelevant. In order to analyse the mathematics, they have to accept that some children fail to learn and that this is not necessarily the children's fault. It is my contention that it is only by working on the analysis of mathematics and the teaching of mathematics that the students can challenge the conservatism of their own experience and begin to work on the mathematical experiences which will suit their pupils. It is a view compatible with the idea of good teachers as learners (Sotto, 1994).

Those who are poor at analysis are uninterested in lesson planning, other than as something they have to do for others, it is not seen to contribute to their development as teachers. Students in the neutral zone follow the expected routines for lesson planning, but they do not really know why they are doing it, they do not have their own questions, although they will willingly try to engage with those of others. They only see the importance of analysis in terms of other people's expectations, it is required, so they do their best.

It is only in the positive area that analysis is valued for its own sake, as an important feature of teaching and learning to teach. Students further along the continuum become interested in analysing the mathematics, the lesson, their roles, the pupils' roles. Those who are good at analysis are moving to be articulate about their practice, they can reflect on their reflection-in-action (Schön, 1987), they can record those incidents where noticing them helps to improve their teaching, they see planning and evaluation as critical to their learning, they are beginning to see how many ways one can question the mathematics.

It is the analysis as evidenced in lesson plans and evaluations, the dialogues of debriefing sessions and the work in method sessions which is more important to me in discriminating between students. I value those who analyse more than those who perform. This may be because our students are successful products of a system of mathematical teaching which they should not model, because it currently fails the learning of many. I view my job as challenging my students' views of the ways others may come to know mathematics.

The Analysis/Performance Axes.

In considering the interrelation between performance and analysis, figure 3 shows the two continua as orthogonal axes. The top right hand area represents those who are my best students. The axes create four quadrants, with the neutral zones to the right and above the origin, with the shaded area representing the majority of students.

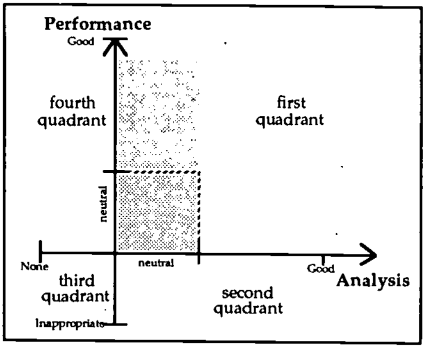

Figure 3: The Analysis/Performance Continua as Orthogonal Axes

The First Quadrant.

The Shaded Region:

When placing students on figure 3, the majority begin in the shaded region. They know where they are going, they know what they want to do, and their confidence to perform improves over time. They think about the subject and try to analyse, but whether these aspects will continue to develop will depend much on their first appointments or their own attitude towards professional development. They tend to be "self-referencing" (Ellwein et al., 1990, p 9), all their success is ascribed to their planning and practice, but then so is the failure of lessons. They are as yet unable to view their role, except in a few instances, in terms of the pupils' learning. They most often believe that teaching is skill-based (Applegate, 1989, p 84), they are concerned about their performance in terms of lists of competencies, and these can become to be seen as more important than the thinking about the why of teaching rather than being complementary to it.

The Upper Right Sector:

In upper right sector of the first quadrant lie those students I would label as "good", I give them the best references and I would employ them. They are still not yet good teachers, but the quality of their analysis is far superior to anything which was expected of me at the end of my training. Their performance is good, they are confident yet have a "self-forgetful ease" (van Manen, 1995, p 46). Their classroom management is nearly always appropriate, they are helpful to the department in which they work and the questions they ask show a depth of interest which is challenging and satisfying. They are deeply interested in their pupils' work and reactions, they have "pedagogical tact" (ibid., p 43). They believe that teaching is knowledge-based (Applegate, 1989, p 86) as well as being developmental (ibid., p 82) in that they expect to go on learning throughout their careers. They have an "open concept of self" (Quicke, 1996, p 15), they can identify opportunities for change, aspects of others' learning and teaching which they can use to help their teaching. Their analysis is not self-conscious, they accept the need to develop routines but recognise the need to question automatic reactions, (van Manen, 1995, p 40). They can be described as "The Teacher on the Grow" who is "strongly committed to being a learner" (Lambdin & Preston, 1995, p 135). They enjoy discussing mathematics and want to challenge traditional approaches to its teaching. They seek alternative lesson styles and see the need for variety in their teaching approaches.

There are variations of quality within this sector but most of these students should be the heads of department of the future. They are likely to become teachers who are "The Standards Bearer" where "student inquiry is the predominant mode of learning" (Lambdin & Preston, 1995,p 136) the type of teacher who "builds classroom discourse around students' ideas, whether right or wrong" (ibid.). There are not many students who lie in this part of the quadrant, and the majority, in my experience, have been women.

The Second Quadrant.

The second quadrant is more problematic. This usually describes a small group of students, of whom the majority are women. From their analysis these students have the potential to become good teachers, but their performance gives cause for some concern. They may improve once they are not meeting the demands of several teachers. They know what mathematics they would like to teach and how, but trying to adapt to the norms of the class with which they are working can act against their developing more confidence. They wish to please and find it difficult to put their own viewpoint to the class teacher. Sometimes they feel that they are loosing control, when they are, in fact, over-controlling the class. They can be afraid to offer a little freedom in case chaos ensues, but the tight rein can provoke rebellion. However, there is often not a control problem with all classes. The potential of the student can often be seen with a 'difficult' class, it is almost as if knowing that others have failed with the class allows the student to experiment with his or her own methods. These students listen to advice and act upon it, but this can lead to them feeling pulled in different directions. In this quadrant are those students who in describing lessons tend to talk about success in terms of their pupils, Ellwein et al. (1990,p 8) would call them "self-effacing", in that the 'I' of their recall was low in comparison to the 'they' of their pupils. The contribution of the pupils is highly valued, their learning is noticed and their participation analysed for how it can be even further improved. Any failure is seen to be the responsibility of the student, in that the material offered was too difficult or too varied or ... They see teaching as developmental (Applegate, 1989, p 82) they expect their expertise to pass through a series of stages, during their teaching career. For me these students are also Lambdin and Preston's "teachers on the grow" for they are "open to change and anxious to learn" (ibid., 1995, p 136). These students have difficulty in getting through interviews for jobs, because they are more nervous with peers and superiors than they are with pupils. They also get limited references from their schools because of the slow development of technical performance.

The Third Quadrant.

The third quadrant contains a number of students at the beginning of their first teaching practice. They may need a lot of nurturing if they are to succeed. However, if by the later stages of teaching practices they are not in the top right hand corner, they should be those who fail or are counselled to withdraw from the course.

For those who fail there are groups of factors, personal, professional, contextual, identified by Sudzina & Knowles (1993,) some relating to a "sense of development of self-as-teacher" (p 255), which describe how they want or do not want to act in classroom, personality traits, levels of participation, unwillingness to ask for help, or lack of time and resource management. In lesson planning theirs is an "inability to select and relate goals to objectives" (ibid., p 256). They generally lack the technical skills of teaching, and they have problems with evaluation and assessment procedures. They may not accept the nature of their role in school, they have little understanding of the school as institution.

The majority of students who fail or withdraw from our course are male, and there may be issues to explore related to perceptions of teaching and gender which need to be considered.

The Fourth Quadrant.

The fourth quadrant contains, for me, the most frustrating students, they have the performance of teachers and on first glance would be labelled as good, but they are not interested in analysis. They perceive themselves as 'experts', they have been in school and seen mathematics taught, so 'know' how to pass on that expertise. They believe that teaching is acquired naturally (Applegate, 1989, p 80) and are confident that they are ready to teach. They see the methods course as irrelevant, it is only the practice in schools which interests them. They could be considered "The Frustrated Methodologist" (Lambdin & Preston, 1995, p 133), they know how to follow traditional methods and do not see why they should have to engage in discussion of alternative practices. In their classes there is

> *tight control over discussions, discouragement or ignoring of most statements that may be incorrect or that could lead to confusion ... more concern with procedural facility than with conceptual understanding. (Lambdin & Preston, 1995, pp 133, 134)*

These students can be described as "ego-enhancing" (Ellwein et al. 1990, p 7), all success in lessons is due to the student, all failure in lessons is due to the pupils. The 'I' is very strong in their reporting, in the sense that the 'I' is always right. Their reflection is tightly restricted by being "grounded in a reified view of the self" (Quicke, 1996, p 15), everything is related to their own personality, there is no perception of any need for change. These students receive very limited references from me; I would not want to employ them, as they appear to have little real interest in their pupils. However, they are articulate and often get jobs in competition with those I would consider good students. Many of them will be labelled as good teachers, because they will deliver examination results. They teach by force of personality rather than thinking about the pupils and their needs. This is, fortunately, a group which rarely contains many students, and they have been mostly male.

There is an issue of recognising the limitations of such students. Teachers in their practice schools often value these students highly, because their classes are well-disciplined. Teachers often see classroom control as being the first pre-requisite of the new teachers, they do not necessarily relate this to the content of the lesson. Students who can control are often valued more highly than those with better lesson ideas and emergent classroom control.

The Four Quadrants.

The division of the students within the four quadrants offers a way of categorising students and it can be reconciled with other work, for example a classification from Transactional Analysis, (cf. Harris, 1973), (figure 4).

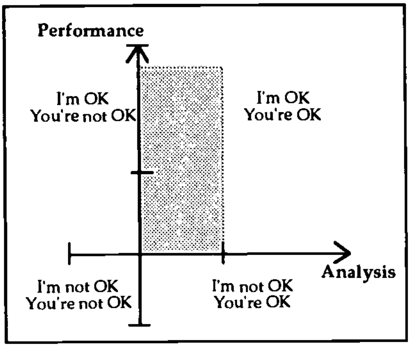

Figure 4: The language of Transactional Analysis and the Analysis/Performance axes.

The four states describe the relationship of the student to their pupils, with those in the shaded region moving between all of the four different states. The use of Parent, Adult, Child in this approach to psychotherapy can be used to describe some behavioural traits. In the first quadrant are those individuals who enjoy the creativity of their Child, know the power pull of their Parent, but use their Adult to select the appropriate transaction with pupils, in accepting them as reasoning, reasonable beings. In the second quadrant, the Child is active, but the pull of the Parent may make the students behave in ways in which they would, in retrospect not admire. In the fourth quadrant, the Child is unappreciated, creativity and questioning seem irrelevant, the Parent is right and the only transactions with pupils are Parent-Child. In the third quadrant, the transactions with pupils are confused, they will sometimes be Parent-Child, but will often be Child-Adult, or Child-Parent or even Child-Child, the Adult seems unavailable in this scenario, as the students behaviour is inappropriate.

Conclusion.

The development of the analysis/performance continua as diagrams offers an on-going appraisal of student development. I know where each student is on the diagram, I know where I would like them to be and this allows me to offer appropriate support. Clarifying the relationship between analysis and performance has offered me a greater insight into my judgements, my criteria are more explicit, my criticism of some students has a much stronger rationale.

I now see my role as helping students to move from the neutral zone, along the diagonal to the top right hand corner. By plotting an individual's movement on the diagram over time, there is a clear picture to help students to understand how they are developing. As partnership increases with our teaching practice schools and more is expected of teachers in schools to help students to develop, the analysis/performance continua offer a valuable adjunct to a list of competencies in discussing teaching.

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A STUDY OF TEACHERS' CONCEPTIONS ABOUT MATHEMATICS

On Results from the Third International Mathematics and Science Study (TIMSS)

George N. Philippou & Constantinos Christou
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Abstract: *This study focuses on teachers' conceptions about mathematics and its pedagogy in relation to students' achievement. Three groups of countries were formed according to students performance in mathematics and responses of teachers on 20 items were analyzed; those items referred to conceptions about the nature of mathematics, mathematics teaching, and mathematics learning. Substantial differences were found in teachers' conceptions among groups of countries, indicating culture bias and a relation to student achievement.*

Introduction and aims

Despite the volume of related research, the term "conceptions" is still loosely defined, currying various meanings mainly within the affective domain. Conceptions about mathematics have been defined as "conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning mathematics" (Thompson 1992. p. 132). In the German language, the same construct is met as *subjective theories*, or as *one's mathematical world views* (Pehkonnen, 1994), while in Italian the terms *conception, image, opinion, view*, and *belief* are used interchangeably (Furinghetti, 1997).

Two major aspects of mathematical conceptions have been distinguished: conceptions of mathematics as a discipline-what mathematics is really all about- and conceptions of pedagogical knowledge of mathematics. Pedagogical knowledge refers to teachers' learning of theoretic principles and focuses on questions such as: what is the meaning of mathematical knowing, what does it mean to teach and learn mathematics, and how does one come to know mathematics? It also includes such topics as didactical models, planning instruction, student motivation, classroom management, classroom environments, etc. Teachers' conceptions about the *teaching*, and *learning* of mathematics are the two aspects of pedagogical knowledge which are presently discussed.

Conceptions that teachers hold about mathematical knowledge, teaching and learning have profound effect on the selection of classroom activities (Greeno, 1989). A teacher's epistemology or philosophy of mathematics and its teaching (if articulated as a coherent philosophical system) functions as a filter and regulator determining one's teaching style and decisions made before, during and after instruction (Shierpinska, 1994). The relationship between conceptions of pedagogical knowledge and teaching actions was extensively studied (Grossman, 1990) and significant differences were found between classroom actions adopted by teachers with and without formal pedagogical knowledge. Those actions are influenced by beliefs about the subject matter, which "contribute to the ways in which teachers think about their subject matter and the choices they make in their teaching" (Grossman, Wilson, & Shulman 1989, p.27).

Knowing mathematics is currently considered as equivalent to *understanding* and *doing mathematics,* where the latter is associated to problem solving. Mathematical understanding has been defined in terms of mental representations, which constitute part of wider networks of representations: "a mathematical idea, procedure or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections" (Hiebert & Carpenter 1992, p. 67). Understanding and teaching mathematics involves "metaphysical" parameters which lead to epistemological obstacles: "beliefs about the nature of scientific knowledge, our views, images that we hold and that are imprinted in the language that we use, schemes of thinking-all form the starting point for our dealing with scientific problems as much as they bias our approaches and solutions" (Shierpinska, 1994, p. 126). Positive interactions were found between teachers knowledge and affective factors, since what teachers believe about understanding, knowing, and using mathematics, inevitably affects their teaching style, and hence the outcome of a long and laborious process. Though there is a widely felt need to relate and integrate cognitive and affective elements into empirical research (McLeod, 1992), there has been hardly any progress towards the combination of those variables.

Past international studies have demonstrated that Japan and other countries of the Far East figured at the top of the list of participant countries (Robitaille & Travers, 1992). Among the various interpretations for this phenomenon, cultural and factual factors were mentioned, such as the length of the working week and/or the academic year. Presently, we investigate teachers′ conceptions about mathematics and its pedagogy in an international context, and at the same time, we compare the conceptions of teachers from different groups of countries, in terms of culture and student achievement. Specifically, the following questions were formulated:

1. What are the conceptions of teachers about the nature of mathematics, the learning and teaching of mathematics, and how do these conceptions vary by culture?
2. Are there significant differences between teachers' conceptions about mathematics according to students' achievement?

Methodology

By design IEA studies (International Association for the Study of Educational Achievement) provide a wealth of standardized data, which are available for further consideration and analyses, facilitating international comparisons. The teacher questionnaire of the TIMMS included items which examined teachers' conceptions about a wide spectrum of mathematics instruction, while students' achievement was assessed by other tests. To investigate the research questions, we monitored 20 items from the questionnaire completed by 7-Grade teachers (see Appendix). These Likert-type items, were classified into three categories i.e., conceptions: a) about the *nature* of mathematics (five items), b) about the process of *teaching* mathematics (nine items), and c) about the process of *learning* mathematics (ten items). (Note that four items were classified in two categories). The items in the first category involved three points and those in the other two involved four points. Teachers were expected to react on the idea expressed by the statement by selecting the relevant alternative. By splitting items according to a general interpretation of mathematics and its pedagogy as either "a set of rules and procedures" or as "a connected and coherent body of knowledge" (see headings in the Appendix), two more categories were formed, which were also analyzed and studied.

The subjects were 7-Grade teachers from three groups of countries (out of 48 participants in the TIMMS), consisting of four countries each, according to students' achievement in mathematics. The "Top Group" (TG) consisted of four Eastern Asian countries (Singapore, N=137; Hong Kong, N=86; Korea, N=149; and Japan, N=151), whose students appeared at the top of participant countries. The "Middle Group" (MG) consisted of four European countries (England, N=201; Germany, N=178; Belgium, N=123; and Sweden, N=187), whose students' performance was about the median of the success list, and the "Bottom Group" (BG) of four low achievement countries from three different continents (Greece, N=162; Cyprus, N=115; Colombia, N=146; and Iran, N=192). The responses of teachers in each group of countries (drawn from the report made available by our national representative) were combined together (weighed average) to form one entity i.e., the agreement/disagreement proportion for each group of countries.

The *Median Polishing Analysis* (Velleman and Hoaglin, 1981), was employed. This method partitions two-way tables into four interpretable parts: the grand or overall effect (GE), the row effect (RE), the column effect (CE), and the interaction of rows by columns. The GE indicates the typical response of the subjects across the total set of items, that is the extent to which they endorse the content of the items. The RE tests for differences between responses among rows- in the present case among groups of countries. The CE reveals relative differences among items, and the cells contain the Residuals (rows x columns interactions), which indicate the extent to which RE and CE cannot explain the levels of endorsement that represent unique patterns of responses by specific subsets of the subjects to particular items.

Results

The results are presented in three sections corresponding to each category of teachers' conceptions under study, on the basis of the Median Polishing Analysis. According to this method the numerical values indicate significant differences when they exceed 10 in absolute value. A fourth section presents the results of the analysis of conceptions in the two broad categories i.e., the "algorithmic", and the "coherent" interpretation of mathematics.

Conceptions about the Nature of mathematics. Table 1 shows that the GE of this category was 66%, meaning that teachers as a whole endorsed the ideas portrayed by these items. However, due to variability within items, the overall effect should be interpreted in the light of CE, which showed striking differences among items. Teachers felt negatively about the importance of the ideas in the first four items- the first three describe mathematics as a fragmented body of knowledge, (CE:

-20.8, -41.3, -19.8, and -20.8, respectively). On the other hand, teachers endorsed strongly the content of item 5 which emphasizes the coherent view of mathematics (CE: 20.8).

The RE represent differences between the groups of countries, denoted by TG, MG, and BG. The main finding of this comparison concerns MG teachers, who were found to reject the content of the items of this category (RE: -20). The residuals, however, indicate that this rejection was mainly due to disagreement on items N1 and N3, while MG teachers endorsed N4 and N5, which refer to mathematics as a coherent body of knowledge. On the other hand, TG teachers (RE: -4), endorsed items N2 and N3, which underline the sufficiency of computational skills and mathematics as an abstract subject, and rejected N4 which connects mathematics to real world. The latter item was endorsed by the BG teachers who consistently rejected the fragmented concept of mathematics (N1, N2, and N3).

Table 1

Median polishing Analysis of Conceptions about the nature of mathematics

Countries/Item	N1	N2	N3	N4	N5	Row Effects
TG	0.75	4.2	5.8	-31.8	-0.75	-4
M G	-10.3	6.2	-3.2	21.2	10.3	-20
BG	-0.75	-4.2	-5.8	31.8	0.75	2.5
Column Effects	-20.8	-41.3	-19.8	-20.3	20.8	Grand Effect=66

Conceptions about the Learning of mathematics. The first four items in Table 2 indicate an inclination for algorithmic skills and knowledge reproduction, while the next five items draw attention to active strategies, relational and structural understanding of mathematics. The overall effect of 46 shows that in general the subjects endorsed the importance of these ideas.

Table 2

Median Polishing Analysis: Conceptions about the learning of Mathematics

Count./Item	L1	L2	L3	L4	L5	L6	L7	L8	L9	Row effect
TG	2.3	-0.8	-10.3	-2.2	9.8	-15.8	-1.2	-9.2	-9.8	-2.5
MG	7.8	0.8	31.2	-7.8	1.2	23.8	3.2	1.2	13.8	-24
BG	-2.2	0.8	10.2	2.2	9.8	15.8	1.2	9.2	13.8	-2
Col. Eff	-0.8	37.2	0.8	0.8	-13.3	10.2	39.8	28.8	-29.2	Gr. effect = 46

The CE showed that teachers strongly endorsed the idea that students need to think sequentially and procedurally, understand concepts, think creatively, and understand real world use (L2, L7, L8, and L6: CE = 37.2, 39.8, 28.8 and 10.2, respectively), while they do not consider writing of equations and reasoning as crucial in learning mathematics (L5, and L9: CE =-13.3 and -29.2, respectively).

The RE showed that as a total TG and BG teachers reacted in a rather identical manner to these items (both small but negative RE), while MG teachers were found to be particularly negative (RE: -24). The residuals indicate that the overall position of the MG teachers cannot explain their reactions to items L1, L3, L6, and L9, i.e., they are not negative about recall of formulas, practicing, applications and reasoning (residuals: 7.8, 31.2, 23.8, and 13.8 respectively).

Conceptions about mathematics teaching. The items of this scale refer to instruction related to: practicing as a means to overcome difficulties, computational skills, writing of equations, use of textbooks, understanding, reasoning, group work, problem solving, and use of multiple representations. The GE of 31 (Table 3) showed high level of general endorsement of the ideas expressed by these items. High negative CE were observed on items T2, and T9, concerning the sufficiency of computational skills and the solution of non-routine problems (CE:-34.8 and -29.3 respectively). On the other hand, particularly positive views were expressed on items T1, T6, and T7, concerning the emphasis on recall of formulas, on reasoning, and the of liking and understanding of students (CE: 29.2, 19.8 and 40.8).

Table 3

Median Polishing Analysis of Conceptions about the teaching of mathematics

Count/Item	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	R. Ef.
TG.	-10.3	7.8	16.8	-3.2	12.8	-0.8	-2.8	3.8	10.3	8.2.	-10
MG.	-11.3	29.8	-6.2	23.8	-2.2	9.2	3.2	14.8	11.2	1.2	-14.5
BG.	10.2	-7.8	-16.8	3.2	-12.8	0.8	2.8	-3.8	-10.3	-8.2	27.5
Col. Ef.	29.2	-34.8	-10.8	3.2	6.2	19.8	40.8	7.9	-29.3	-17.3	Gr. Ef=31

The row effects indicate a rather similar negative overall reaction to these statements by the TG and MG teachers, while BG teachers were found positive (RE: -10, -14.5 and 27.5, respectively). It seems that the positive GE was due to BG teachers rather than to the total teacher population of the study. The similarity between TG and MG teachers becomes also evident from the residuals, where they primarily differ on T4 (-3.2 compared to 23.8).

Algorithmic vs. Coherent interpretation of mathematics. The GE of the algorithmic and the coherent nature of mathematics showed that teachers endorsed both those conflicting interpretations, with an increased acceptance of the second one (GE: 42 vs. 51). TG teachers were positive on the algorithmic items and negative on the coherent (RE: 5.5, and -10), MG teachers rejected the algorithmic and endorsed the coherent interpretation (RE: -19, and 8), whereas BG teachers were slightly positive towards the algorithmic and most positive towards the coherent interpretation of mathematics (RE: 2.5, and 35). The CE showed that the overall reaction to the algorithmic interpretation was due to strong endorsement of items T1 and L2 (CE: 53.5, and 43), which emphasize drill and practice, and sequential and procedural thinking. The coherent interpretation was adopted mainly due to endorsement of N5, L7, T7, and T10, despite the rejection of items N4, T6, and T9.

This means that the subjects appreciated mathematics as a practical guide to represent real world, concept understanding, positive attitudes toward students, and multiple representations, while they did not value the formal feature of mathematics, they did not ask students to justify their solutions, and they did not assign students non-routine problems.

Discussion

The aim of this study was to search for understanding of implicit theories on teachers' conceptions about mathematics and its pedagogy, within a framework of cross-cultural comparison. According to Robitaille (1993), ranking of countries on any measure is less important than the interpretation of differences in terms of cultural and curricular variation; hence, the main emphasis was to interpret differences among groups of countries in teachers' conceptions as related to culture and students' achievement.

The analysis of responses produced evidence of significant differences among the three groups of countries concerning teachers' conceptions about mathematics and its pedagogy. The results seem to confirm the claim that teachers' conceptions are directly related to their teaching style. Much of the contrast in teachers' instructional emphases could be explained by differences in conceptions. In particular the observed consistency between professed conceptions and classroom activities, which was evident in all countries, suggests that there is a relationship between the two constructs. For instance, MG and BG teachers were found to view mathematics as a coherent subject consisting of connected topics. In line with their conceptions, they seemed to adopt conceptual teaching approaches and emphasize mathematical meaning and understanding. Conversely, TG teachers tended to regard mathematics as a fragmented set of rules and algorithms, as an abstract subject, which is in essence prescriptive and deterministic in nature. As a consequence, they rather stressed algorithmic and computational skills.

The relationship between teachers' conceptions and students' achievement is so complex as to defy any simplistic cause-effect interpretation, and one should be cautious against drawing universal conclusions. Yet, the findings seem to support the original assumption of this investigation that teachers' conceptions constitute a major factor affecting student's mathematics learning. However, the basic question of which conceptions are likely to produce better long run results, remains to be seen. The didactic approach based on the fragmented conception of mathematics was found to be associated with higher student's performance. Despite reasonable reservations that may be raised about the comparability of "non-similar", about differences in goals etc., the superiority of Eastern Asian countries in mathematics achievement seems to be undeniable.

The teachers from Eastern Asia seemed to endorse views presenting mathematics as a fragmented body of knowledge, in general, they tended to accept the algorithmic interpretation of mathematics, they made extensive use of textbooks and paid little attention to developing creative thinking. Those views were rather steadily expressed in all three dimensions of the research questions and particularly on the final specific categorization of the items into two interpretations of mathematics. MG teachers showed a clear tendency to endorse items along the coherent nature of mathematics and they claimed to emphasize creativity and non-routine problem solving. BG teachers were found somewhere in the middle of the other two groups, and in the final analysis they endorsed both the algorithmic and the coherent interpretation of mathematics.

Why do the teachers from each of the tree groups of countries hold the specified conceptions is a question non addressed by this study. It is, however, evident that there is a prominent cultural element influencing teachers' conceptions, teaching behavior, and student achievement. Traditional social values, ethics and philosophies, individual motives and aspirations are quite different in Europe than in the Easter Asian countries. So, teacher education and professional training, and consequently practicing and learning, is by all means different. The cultural variability within the group of low achievement countries-two European, one South American and one Arab country-is probably the main reason of noticed differences and some inconstancies.

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APPENDIX

The statements used in the study

Conceptions about the Nature of Mathematics

Mathematics as a set of rules and procedures

N1. Mathematics should be learned as sets of algorithms that cover all possibilities.

N2. Basic computational skills are sufficient for teaching primary school mathematics.

N3. Mathematics is primarily an abstract subject

Mathematics conceived as a connected and coherent body of knowledge

N4. Mathematics is primarily a formal way of representing the real world.

N5. Mathematics is primarily a practical and structured guide for addressing real situations.

Conceptions about the learning of mathematics

Mathematics conceived as a set of rules and procedures

L1. To be good in Mathematics how important is to remember formulas and procedures

L2. To be good in Mathematics how important is to think in a sequential and procedural manner.

L3. To be good in Mathematics how important is to practice on calculation and skills?

L4. Mathematics should be learned as sets of algorithms that cover all possibilities.

L5. How often do you ask students to write equations to represent relationships?

Mathematics conceived as a connected and coherent body of knowledge

L6. To be good in Mathematics how important is to understand real world use?

L7. To be good in Mathematics how important is to understand mathematical concepts?

L8. To be good in Mathematics how important is to think creatively?

L9. To be good in Mathematics how important is to be able to provide reasons to support solutions?

Conceptions about the Teaching of Mathematics

Mathematics conceived as a set of rules and procedures

T1. If students have difficulty, they should be given more practice for themselves.

T2. Basic computational skills are sufficient for teaching primary school mathematics

T3. How often do you ask students to write equations to represent relationships?

T4. How often do you ask students to practice computational skills?

T5. What percentage of your teaching time is based on the textbook? (76% -100%)

Mathematics conceived as a connected and coherent body of knowledge

T6. How often do you ask students to explain the reasoning behind an idea?

T7. A liking for and understanding of students are essential for teaching mathematics and d science.

T8. How often do the students work in small groups?

T9. How often do you ask students to work on problems with no obvious method of solution?

T10. More than one representation should be used in teaching a mathematical topic.

EMILY AND THE SUPERCALCULATOR

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This paper reviews a short learning programme devised for a low achiever in elementary arithmetic. Using a graphic calculator, the programme was designed to change the quality of imagery associated with numerical symbolism. Earlier observations had shown that the child's symbolic images were episodic and active, representing mental procedures that were analogues of physical ones. By providing an alternative, non-counting dependent procedure, it was hypothesised that the calculator would encourage the formation of semantic and generic images which used symbols as objects of thought. Positive indications suggest that continuing to encourage most low achievers to count when they experience difficulty in elementary arithmetic may need reappraisal.

Introduction

Symbolism has the power to dually and ambiguously represent computational procedures and the results of these procedures (Gray & Tall, 1994). To benefit from the flexibility provided by such ambiguity the young child's conception of arithmetic must progress through several phases of compression: lengthy counting procedures which are interpretations of processes to *do* must eventually become concepts to *know*. It is through procedural compression that symbols may become objects of thought.

This is a story of one eight year old whose efforts to progress through stages of procedural compression had not provided her with the flexibility to use the power of symbols. It was hypothesised that if the 'procedural clutter' associated with the perceptual and figural items that dominated her interpretation of mathematical symbols could be removed, she too may focus on the power of symbols. To do this we provided a graphic calculator, the 'supercalculator'. The paper adds a further dimension to notions that the use of calculators not only does not harm computational ability but supports concept development (Shuard, Walsh, Goodwin and Worcester, 1991; Shumway, 1990). Research on the use of graphic calculators (Ruthven, 1993; Dunham & Dick, 1994) had indicated that there was potential for this resource within the classroom although the outcomes did not always give positive results (Ruthven, 1995). We consider the changes in the child's use of symbolism during a period using the calculator. Our focus is the opportunity that the resource may give for stimulating the construction of mental imagery associated directly with arithmetical symbols as opposed to imagery that is an analogical transformation of them.

What can imagery tell us about success and failure in arithmetic?

Pitta and Gray (1997) describe how children at extreme levels of achievement in elementary arithmetic focus on imagery which is of different qualities. Imagery identified by 'high achievers' tended to be symbolic, used to support the production of known facts and/or numeric transformations which produce derived facts. Imagery reported by 'low achievers' was usually based on analogical representations of physical objects. These images appear to be clear imitations of actions that could have taken place with real objects. Pitta & Gray went on to suggest that the essential differences between the imagery of 'high achievers' and that of 'low achievers' was that the imagery of the former was semantic and generic whilst that of the latter was episodic and active. The terms 'episodic' and 'semantic' were used to draw a distinction between those images that arise from memory associated with the recollection of personal happenings and events, and those images linked to organised knowledge associated with meaning and relationships but independent of an event. Such distinctions lead to the conjecture that the images of 'low achievers' are essential to thought. In contrast, those of 'high achievers' appear to act as thought generators. They 'flash' as memory reminders, momentarily coming to the fore so that new actions or transformations may take place. In the belief that the former is a factor of the procedural thinking associated with the proceptual divide, the issue for this paper is whether an alternative 'procedure' may discourage a 'low achiever's' need to use manipulatives in the mind but stimulate the creation and construction of symbolic images that help to generate thought.

An Alternative Procedure: Focusing on Symbols

There is a tendency within pedagogy to provide practice to confirm "understanding". For children who have difficulty with elementary arithmetic such practice is usually based upon the use of counting. It is suggested that such experience may confirm the understanding that arithmetical symbols can be transformed into physical objects, or mental analogues of these objects. These then form the basis of counting procedures. It would seem reasonable that if the learner puts effort into this solution to problems it is perhaps the case that the more procedures are remembered and the more likely they are to be used but, paradoxically, the learner may possess less understanding. However, it has been recognised for some time that calculators can give children an insight into numerical patterns (Shuard *et al.* 1991). To identify relationships between numbers children need no longer be constrained by the use of lengthy counting procedures. The supercalculator seems to have an added advantage. Combinations can be recorded and displayed in their entirety, equivalent outcomes from different procedures may also be seen at the same time (Ruthven, 1993) and the child can control the form of display on the screen. Additionally, for our attempt to minimise a focus on counting the supercalculator offered two strengths; it provided an alternative procedure which had the potential to provide an alternative representation for numbers, and it could display all symbols and operations at the same time. It was conjectured that this would offer firstly an opportunity to concentrate on numerical symbols as objects of thought, and secondly provide a stimulus which would support mental organisation. It had the potential to support the creation and use of symbolic images. It did not support analogical transformations of them.

A calculator provides and opportunity to create a number by pressing a button. It also permits a particular number to be created using the combination of a composite series of button pressing. Thus, by asking the child to create 9, this could be done by pressing 4+5=, by pressing 6+3= or it could be formed from 2+3+4 or 13–4 etc. By eliminating a counting procedure the 'alternative' procedure had the potential to create a "wholeness" about number. This may be seen at two levels; a specific one in which the focus could be on number triples, and a more generic one during which it is possible to identify the relationships between numbers and simple operations. It is unfortunately the case that many "low achievers" find it hard to switch from harder to easier methods if the first is habitual and unfamiliar (Krutetskii, 1976; Steinberg, 1985). The "button pressing" procedure had the potenial to overcome this difficulty since the child may not regard it as a mathematical activity which should become a focus of attention.

Emily

We first met Emily in February 1995. She had considerable difficulty with elementary arithmetic. Articulate and highly motivated, she was identified as one of the lowest achievers within her year group of 119 children. Test results (SEAC, 1994) placed her amongst the bottom four children. Our initial conversations with her were about the numbers 1 to 10. Her responses were dominated by descriptions of images that were analogues of physical objects. Over a series of four interviews, during which she was given elementary addition and subtraction combinations and asked to talk about her approach to each one she indicated that she relied extensively on active mental images. As the items began to involve combinations greater than ten Emily made considerable use of her fingers. She was representative of the group of low achievers who concretise symbols and focus on mental or physical manipulation (Gray & Pitta, 1996).

Verbal and written symbols of the numbers one to six were seen as mental arrays of dots in the mind. Those between seven and ten were mental images of fingers arranged in a linear fashion. Emily manipulated her mental images of dots, her preferred image, relatively easily. The solution to 4–3 was explained as:

> As I see it there's two dots above each other and then there's.... the first one, the one below and the one next to it are being taken away and there is only one left up at the top. *(Emily, 1995)*

It may well be that extensive experience with board games provided the episodic background for this frame of images:

> *When I was young, when it was winter, we often played board games because we were not allowed outside. We were using dice. We were playing all of the time using dice. (Emily, 1995)*

She recognised that there was greater difficulty associated with finger like images. Using these meant doing two things at once, counting and concentrating on the sequence in which each finger was used:

> *I am trying to think out the answer as well as use all of my fingers–this is confusing... with the dots it is easier [than with fingers] because you don't have to keep thinking, 'No it's that one I need to move, no, its that one, or that one... [with the dots] it doesn't matter which you move.* (Emily, 1995)

It was as if Emily recognised that if she used fingers she had to count particular fingers, whereas by using mental images of the dots she could use any dots. For relatively more difficult combinations such as 'nine take away six' Emily used her fingers in an indirect way by 'feeling' them without looking at them, touching them or moving them. This was no surprise since evidence had shown that it is more likely that an individual will move from a mental episode to a real episode as things become more difficult (Pitta & Gray, 1997). But this too caused problems:

> *I find it easier not to do it with my fingers at times because sometimes I get into a big muddle with them because I find it much harder to add up because I am not concentrating on the sum. I am concentrating on getting my fingers right...which takes a while. I can take longer to work out the. sum than it does to work out the sum in my head.* (Emily, 1995)

But there was a third problem for Emily. Her perception was that any procedure that was not overt could place her in a position of conflict with the teacher:

> *If we don't [use our fingers] the teacher is going to think, 'why isn't she using her fingers–it is meant to be the easiest way–and they are just sitting there thinking. It is like, ...because we are thinking that...we are meant to be using our fingers because it is easier....which it is **not**.* (Emily, 1995)

Unlike most of the other children who formed part of the study into children's use of imagery (Pitta & Gray 1996, 1997) Emily appeared to recognise that there was a qualitatively difference between using perceptual items and mental representations of these items. It was not only that she believed the later was easier but to her it also made a difference between 'doing' arithmetic and 'thinking' about arithmetic:

> *I try not to use my hands much... I don't bother looking because I am too busy thinking so... when I am not using my hands I am trying to work the sum out.* (Emily, 1995)

Emily appeared to have come to some conclusions. First, it was easier to do the sum in her head and secondly, some images were better than others. It seemed to her that it was easier to see a number and remember it if it was recognised by some form of pattern like the array on a die. It was harder to think about if the representation was based upon a line of finger like objects, each being focused upon at a separate point in the counting procedure. Thirdly, arithmetic involved being seen to be 'doing', but this was unsettling because she was trying to 'think'. Unfortunately however, she was not thinking with the tools her more able peers were using, the arithmetical symbols. Her tools were analogical images of real objects manipulated in accordance with her recollections of former experiences. Numerical symbols were concretised to form objects which supported the use of mental imagery that was episodic and active. Her focus was on an action which could be simplified by the nature of the representation that she gave to the objects. However, whether or not she used dots, fingers or finger like objects the intrinsic quality of the object did not change. Her perception of quantity represented by the symbols influenced her choice of objects and the way the objects were used, so the focus turned to the nature of the action. Though it was evident that her procedural competence was sound it had not supported the encapsulation of numerical processes into concepts. She was not filtering out unnecessary information and making the cognitive shift that would lead to the realisation that symbols could become objects of thought. The longer term prognosis was that the qualitative difference between Emily's thinking and that of her more able peers would widen into a gulf.

A Programme with the Supercalculator

Emily was introduced to the supercalculator after the first series of interviews. Directed work with it extended over a period of three months, April 1995 to July 1995. The programme build around its use was not seen as simply another way of doing things. The calculator was not a means for completing the result of arithmetical combinations but a way of seeking different combinations that made a particular number. Thus she started with the number and considered different routes to it. Four phases were established to support the development:

Working with nine......

9

Making nine 1. ....
2. .... 3. ....
4. .... 5. ....

Working with the calculator

Ways to make nine
1. .... 2. ....
3. .... 4. ....
5. .... 6. ....

Ways to make nine starting with 5
1. .... 2. ....
3. .... 4. ....
5. .... 6. ....

Ways to make nine starting with 10
1. .... 2. ....
3. .... 4. ....
5. .... 6. ....

An interesting thing I have discovered ....

1. Emily was given an opportunity to think about numbers without using the calculator
2. During this phase Emily used the calculator to support her thinking not simply to check answers. She could control the form of the numbers and seeing one combination maintained in display could try the same numbers in a different procedure. Memory usually associated with holding quantities and carrying out procedures could be directed towards thinking about number combinations.
3. At the end of each activity she consider interesting things that had been discovered during the activity.
4. She was given and opportunity to talk about individual numerals and associated combinations.

To accompany her work a specially personalised booklet was designed with each page following a pattern similar to that in the adjacent figure.

The programme called for Emily to try to complete a page of her booklet each week. Each week she discussed her work with the programme designers. During this time she was asked to talk about her numbers without access to the calculator or to her written responses.

Programme Development

Initially Emily had to overcome some reluctance to use the calculator. This stemmed largely from her perception of what others may think. However, by the end of the first week she had established that there were many ways in which she could make nine, the first number in the booklet. There were of course standard addition combinations such as 4 + 5, 3 + 6 etc. but she also provided others, 4 + 4 +1, 3 + 4 + 2, and using the starting points of 5 and 10 she now provided solutions such as, 5+1+1+2, 5 + 5 – 1, 5 + 6 – 2, 10 – 1. Emily admitted that she wouldn't have thought of these sorts of combinations earlier but her outstanding discovery for the week was that she had found out that she could add larger numbers and then take away.

> *I didn't know that you could add larger numbers and then take away. I didn't know you could go up and down.* Emily, 1995

As she worked through the programme written evidence of Emily's use of standard triples during the non-calculator phase tended to decline. It became noticeable that for the first four numerals in her sequence, 9, 7, 8, and 6 she gave at most two but then she provided other 'non standard' combinations. When working with 7 for example she provided 10+10+10–20–3, with eight she provided 99–91 and 34–32+6. Working with the calculator she provided written evidence of combinations such as 90–80–4=6, 2+9+1–6=6, 30–15–9=6, 40–30–5=5, 10+30–30–2=8, 5+20–19=6.

Ruthven's (1993) suggestion that different rules established through the use of the super-calculator could provide a highly motivating context for discussion formed the basis for the interviews that followed Emily's written work. This discussion provided a platform for the necessary stages of reflection. Her use of the calculator not only removed the need to focus on counting procedures but also provided an opportunity to see different descriptions of addition and subtraction procedures leading to the same results. Furthermore, from the interviews it became evident that Emily's understanding of the relationship between numbers was beginning to change.

> *Well,... before I would have found it harder with nine, but...um...its not that hard because I know that ten is really easy so nine is really easy because you just take away one from ten...* (Emily)

In contrast to her earlier comments in which she had indicated that she found subtraction difficult, Emily was now beginning to see a different framework for working with numbers:

> *It was easier to take away from eight than I thought it would be. Before I found it a bit hard with the other numbers. I thought eight would be a bit hard. But in the end it wasn't as hard as I thought it would be.* (Emily)

I have discovered it is much easier to use multiplication in sums (Emily, 1995)

Inevitably pattern became a feature of Emily's discovery. When talking about 8 the following exchange took place:

Int.	*What about 30–22.*
Emily.	*Well, 20–12, add another ten to twenty, then if you take away, instead of twelve... it can't be twelve, because that is much too low to take away from 30. So, I would have thought it would have been one of the twenty's, so if it was twelve it would be 22.*
Int.	*Let me give you one to do now. If you started at 40 how would you make eight?*
Emily:	*It would be....take away....32.*
Int:	*If you started at 50....*
Emily	*...take away 42.*
Int:	*....and 60?*
Emily:	*52.*
Int.	*Why has it all become so easy all of a sudden.*
Emily:	*Well, it wasn't very easy when I did the first one here, but then, if it was 40 it would be 32, and then it would be 42 and then 52.....*

By this time it was common for Emily's written work to extensively include any numbers up to 100 and at times she included numbers over 100 in her combinations. She was beginning to realise that:

It is a lot easier to work with big numbers than I thought... I thought that big numbers would be very hard because they are so big... but it isn't. It is just the same as low numbers. (Emily, 1995)

It was evident from our discussions that Emily was now talking about numbers as objects. During all of the interviews that followed work with the calculator only on one occasion did she volunteer information about her dots. However was left until a series of follow-up interviews in January 1996 for us to begin to obtain some evidence that her imagery may be changing. When asked to think about numbers that make seven Emily's first comment was:

I just see the symbol 7 flashing in my mind waiting as if I was about to add it up... (Emily, 1995)

During our investigations into children's imagery no other low achiever had associated the word 'flashing' with symbolism to describe imagery (Pitta & Gray, 1997). The word had dominated descriptions of imagery by high achievers. Other numbers were also associated with this notion of flashing and when directly asked to talk about what she could see when she heard the word "Four" Emily responded by saying

4 flashes through my mind, and then I see, two two's like on a dice, 2 + 2, 100-96, four pounds...

Discussion

In contrast to interaction with concrete objects which requires the individual to interpret what is going on, interaction with the supercalculator offers a system in which the individual could build and test concepts first by observing and then by predicting and testing what happens. The form of presentation could be directly controlled by the child. What was becoming clear from our interactions with Emily was that she was building a different range of meanings associated with numbers and numerical symbolism – she was beginning to build a new image, a symbolic one that could stand on its own or be part of the options that would give flexibility. It seems as if her imagery was beginning to be associated with the notion of 'thought generator'.

Super calculators can carry out the evaluation of numerical expressions whilst the child can concentrate on the meaning of the symbolism that remains evident throughout. The evidence would seem to indicate that if practical activities dually focus on the process of evaluation and the meaning of the symbolism they may offer a way into arithmetic that helps those children who are experiencing difficulty develop a more powerful understanding of symbols. However, belated emphasis on the ambiguous meaning of symbolism, when the greater proportion of previous experience has emphasised procedural and manipulative aspects, is embraced with difficulty. We may need to reappraise our purpose in emphasising counting procedures with the "low achievers". It may be too late once the die is cast.

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KNOWING v KNOWLEDGE-IN -ACTION

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In researching teachers knowledge and beliefs about progression within the mathematics National Curriculum for England and Wales, the tension between teachers' knowledge of mathematics and their knowledge-in-action of the classroom becomes apparent in their discussion on number and generalisation.

> *A lot of what I decide to teach is almost an inertia of what has always been taught. We always taught about angle so angle is an important thing and so we'll teach it again and it isn't very often that I'll actually sit down and ask myself questions like that like why am I teaching this.*
>
> *head of secondary mathematics department*

Introduction

Mathematics is a constantly growing and developing subject with a history that goes back over hundreds of years. As a discipline it is well defined, as a form of knowledge it has structures and agreed styles of justification and truth. As part of a school curriculum it has unassailable pride of place. Becher (1989) in his study of the disciplines claims that mathematics is a discipline of 'inherent order, neatness and regularity' and by and large has subject matter that is 'simple and orderly'. Such are the myths that abound. The history of school mathematics, in contrast, is relatively recent and the definition and imposition of a compulsory curriculum across the 5 - 16 years age range even more so. What understandings do teachers have of the school curriculum? How do they select from the field of knowledge called mathematics? How do they organise and present the knowledge to pupils? How are these decisions legitimated?

Forms of knowledge

Stemming from Schon's 'knowledge-in-action', (Schon, 1983), research into what teachers know and do has been carried out by Shulman (1986), Wilson et al (1987), Brown and McIntyre (1993) and Cooper and McIntyre (1996). This research defines the attributes of teaching in order to discover better what it means to be an effective teacher. Shulman uses the phrase 'pedagogical content knowledge' to summarise his findings and refers to the kinds of knowledge that teachers need in order to transform content knowledge to make it accessible to those they are teaching. Brown and McIntyre (1993) and, later, Cooper and McIntyre (1996), use the phrase 'professional craft knowledge' to summarise the different knowledge that teachers have. However, there is an interesting tension in all this research about the need to know about subject knowledge:

> *While one can infer from studies of teacher thinking that teachers have knowledge of their students, of their curriculum, of the learning process that is used to make decisions, it remains unclear what teachers know about their subject matter ...* *Wilson et al , 1987, p.108*

and results, which focus on the representation of that knowledge in classroom action. From my reading it would appear that the researchers believe that 'knowing' about the subject and their 'knowledge-in-action' are synonymous. Examples about subject knowledge are couched in terms of pupil and/or teacher activity. Is it possible to elicit from teachers what they know about their subject without them speaking about what they do?

I find support for my own work in the ways in which data from teachers about their knowledge was collected, mainly through interview and discussion, and in the analyses which were carried out via interpretation by the researchers and the teachers involved in the project. All conclusions carry a warning of uncertainty. In fact a major theme running through all the research in this area is that of 'complexity'. To elicit understanding about teachers' knowledge is to do so in full awareness of the complexity of the job. To ignore a variable is not to understand of the whole picture. To work with all the variables makes understanding the whole picture virtually impossible.

The project

The research presented here is part of a larger, nationally funded two year project, (the whole of the project results are published in SCAA, 1993). I worked with a group of eight teachers working on the given sequencing and defined progression in the mathematics National Curriculum for England and Wales (DES, 1991). We met monthly over the two years collecting data in a variety of ways to provide evidence for our final report. At the end of the project I carried out a lengthy interviews discussing several curriculum topics with six of them, 3 primary mathematics co-ordinators and 3 secondary heads of mathematics departments. It is the findings from the interviews that I present here, using the data from the discussion on 'number' and on 'generalisation'.

Methods

The work with the teacher group provided an opportunity for them to be involved in some action research. The cycle of deliberation, decision-making, action and deliberation on that action, was a major part of their work. My role was to analyse and synthesise the interpretations and findings of the teachers of the definitions of progression within the different attainment targets. The role of participant observer was therefore crucial to my research.

> *Participant observation is a kind of schizophrenic activity in which, on the one hand, the researcher tries to learn to be a member of the group by becoming part of it and, on the other tries to look on the scene as an outsider in order to gain a perspective not ordinarily held by someone who is a participant only.*
>
> *Eisenhart, 1988, p 105*

Over the two years I was involved in a 'progressive focusing' (Hammersley &Atkinson, 1983) upon the data from the teachers in order to describe and account for teachers' understanding of the curriculum, to make their 'tacit knowledge explicit'. House (1980) describes the evaluator as a hunter, a detective, an investigator armed with a variety of techniques. Oja and Smulyan (1989) use the word 'messy' to describe the possible processes involved. The interviews were a 'final conversation' about progression.

Support for analysis comes from social psychologists. Mead (1934) believed that we form a concept of ourselves through the eyes of others, which he called the reflected or 'looking-glass' self. It had three components "how we imagine others see us, how we imagine others judge us, and our emotional reactions to those judgements" (Levin, 1992, p. 128). This latter part is important. It is this interactive reflective process through which the self is built. Possibly, therefore, as knowledge-in-action will be 'reflected' more often than knowing so a teacher's 'self' will be built upon beliefs about their knowledge-n-action.

The interviews

I decided, during the final phase of the project, that as the teachers were trying to accommodate *personally interpreted* givens of the National Curriculum with their *own frameworks* for teaching and learning mathematics, I would try and determine what 'own frameworks' and 'personally interpreted' meant. If, what is important are the beliefs that a teachers holds and brings to bear upon the curriculum, I sought to discover what forms and informs such beliefs, what teachers are able to make choices about and what role their knowledge of mathematics has to bear on such beliefs.

With permission from the group I sought to stand back and, by re-engaging in conversation with each member of the group, to try to track and explain the roots of their histories and how these exert an influence on the decisions of the teachers. I wanted to gain some further insight into their ideas and, I suppose to reassert the individual back out of the group. I sought to build up a picture of influences on teachers' ideas about progression and gain further insight into their perceptions and understandings about mathematics. I was interested in:

- individual responses;
- comparing responses;
- describing how ideas seem to form;
- influences upon choices.

The particular questions that I want to explore concern choices in teaching. What are the conditions that influence a teachers' decision making? To what extent are individual perspectives static or dynamic. How does a teacher form ideas about what to teach, when and how to teach? Do teachers have a sense of order of the curriculum? What influences this order? What is it in mathematics that pupils come to know and then come to know better?

Knowledge about number

All the teachers were eager to talk about number. They all had plenty to say but with warning comments ranging from 'its hard' to 'I teach it all the time'. I asked them what they thought that number meant and what they wanted the pupils to know. Most of the replies were about knowledge-in-action as opposed to knowing the mathematics. The primary teachers talked about how to teach or not teach the four rules with the four operations happening at all stages through the primary phase; ' the actual practical experiences suggest that they seem to be able to cope with all four...'. Recording and algorithms, they all suggested often happen too early. One of the primary teachers (an ex secondary teacher) also talked about the 4 rules but in relation to the scheme of work that he was using.

As to number sets which might be appropriate, one suggested that content emerged from the classroom action and talked about using whatever came up including fractions and decimals and negative number:

> *We even talked a lot about minus sums and things because that's quite interesting.*

Another thought that decimals were hard and anyway : ' When do we use decimals in the real world ... it's always in measurement of some sort'. The same teacher did comment that she was trying to move her teachers away from the seemingly logical progression found in many schemes, i.e. first learn about numbers 1 to 10 , then move onto 10 to 20 , then 20 to 100 and so on.

The secondary teachers all talked about fractions, decimals and percentages, putting the four rules of number first, all offering a more linear and hierarchical models of thinking about number. Is this a consequence of teaching about mathematics across the age and attainment range rather than teaching the whole curriculum to a class. One teacher was adamant that, by the time they get to secondary school, they should know the 4 rules, then onto fraction and decimals : ' I suppose **you** [my emphasis] would start with fractions rather than decimals'. (She often displaced the responsibility through the interview in this way). Another felt that the 3 rules of arithmetic should come first, excluding division, and then decimals next in the meeting of different numbers, as: 'fractions are going out of fashion'. The third teacher was very firm about what he thought number was about and in what order things should be presented to pupils, shifting sets and numbers in a particular order with the four operations

No-one talked about the consequences of technology, of calculators and spreadsheets. Was this because they had no knowledge-in-action to reflect upon? Also, no-one talked about pupils getting to know and getting to know better, though maybe this is implicit in their practice as one primary teacher said:

> *... not all children need the same sequence, some children go through it a lot quicker. some children might need to double back, some children could be solving things out, some children detour and come back at a different point.*

The sources of the teachers' knowledge-in-action and the ordering of that knowledge are given below.

Primary Teachers
" No I don't think there is a hierarchy ... mostly I react to interest".
" from the scheme and the scheme matches the National Curriculum. I assumed that the text books were right, what I have discovered is that it is not really like that. I share my experience with colleagues and they will follow it line by line whereas I'm kind of darting around because I am confident.".

Secondary teachers
"....from my experience of how pupils seem to be able to do ...the way schemes have done it in the past ...".
"..... from my head ... no from my experience of teaching pupils ... I think experiences have dictated more of that hierarchy".
" within my own mind I am so convinced that there is this order and would argue with people ... there are some things that I believe are ordered".

Experiences, schemes and the National Curriculum figure as the main players in the arena. For the secondary teachers an order exists that matches experiences and schemes and these in turn match what then pupils can do. Implicit justification for these orders lies in the mathematical development of the pupils that the teachers observe happening. What is very strong in the explanations for the orders is the use of the word 'I' though my conjecture for the 'I' is that the teacher tells me what he/she believes to be so and, when these beliefs are reflected externally, finds agreement with others, for example, texts and colleagues and the National Curriculum.

Knowledge about generalisation

The National Curriculum for England and Wales has an attainment target including process aspects of mathematics. The word 'generalisation' occurs several times in this attainment target. When I asked about generalisation, most were uncomfortable with the word. Defensive reactions began most of the talk around this aspect of mathematics. Most said that they did not know about it or they did not teach it or that they did not really do enough of it. I tried to explore the 'it' in each of these statements but I was not very successful. I was able to gather comments about the given attainment target and spotting patterns and doing investigations. Progression was offered in terms of getting better at spotting patterns. Algebraic generalisation, as in external examination requirements, was offered as a progression from word generalisations. One of the secondary teachers suggested that: 'Generalisation is trying to find patterns in results that go beyond particular examples ... it's a hard one'. One primary teacher however said that the whole of her work in mathematics was about finding generalisations. Is there any agreement among teachers on what is meant by the word? The graduate mathematician and head of department thought that generalisation was hard, the infant teacher and ex-ballet dancer thought that it was about what mathematics was all about.

Summaries for their knowledge-in-action show minimal understanding of this aspect of mathematics.

> Primary teachers
>
> "My role would be in a sense listening and watching, watching what is going on and beginning to raise specific ideas with them about what they are trying to do".
>
> "I have made an effort to change the experiences at school because I hated maths at school, I feared it terribly,... my maths has developed from teaching it ... [its about] investigating".
>
> "I would look at the teachers book that I trusted and compare with my experience the other strand is attainment target 1 and investigation work".

> Secondary teachers
>
> " we were never taught generalisation ... the big influence is GCSE and the National Curriculum".
>
> :...based on coursework activities in year 10 and 11..".
>
> "... its part of it [the first attainment target]".

Generalisation then is not part of the teachers knowledge-in action, nor was it an explicit part of their own mathematics education .They had few activities to describe the word and therefore were neither able to discuss their knowledge-in-action nor their knowing. A sense of order for learning in this aspect of mathematics becomes more tentative and wary. Experience is still called upon but this time it is through the National Curriculum levelling and through examination coursework. The word holds an interesting tension of being something elusive, something ill-defined to being a description, in words or algebraically, for the nth term of a number pattern. Maybe this is the way that we control parts of the curriculum that we are not sure about, to impose an simplistic interpretation that we can see happening in the classroom.

In conclusion

What is noticeable throughout each interview is the brief references to the work of the two years of the project. In response to questions about their own teaching externals were used. High on the list of other referents that they do use are text books, their own experience of learning mathematics, the National Curriculum and departmental decisions. My conjecture that progression is problematic is barely shared by the interviewees, despite our two years working together. Just as those who teach, learn more than their pupils, so too the researcher learns more than the researched. Defining progression other than accommodating the National Curriculum was not perceived as their task and I suspect that they engaged in conversations to please rather than to explore. This has echoes of a comment from Barrs (1994), that progression is an attractive idea educationally and anybody who queries it is liable to look unreasonable.

The dominant theme is existing and/or traditional patterns of behaviour. I suspect that the social psychologists would find these results unsurprising. In the words of Mower-White, (1982): 'Many of our beliefs are founded in social reality and we need other people's opinions to validate our own'. The identification of, and justifications for, progression in the teachers' interviews were predictable. Typically, they offered the orders of the National Curriculum or orders found in many text books. They offered justification for orders from their own learning, (stronger in most cases than other justifications), from the way that syllabuses are written, and from 'this is what I always do'. It may be that there is a certain truth in the tradition or that the tradition is rarely challenged. What is noticeable is that all the teachers found talking about progression difficult, difficult to articulate, difficult to be clear about, difficult to validate. Since tradition is one of the main justifications for decisions about progression, it may be that making sense of someone else's reasoning is, in itself, quite challenging, or that we think that we ought to know what the progression is, when in fact there is no one progression.

Logical orders are tempting, and, maybe the fact that there appears to be comfortable logical orders in mathematics, offers a tidy structure for teaching

> *A third meaning of thought is belief .... Such thoughts grow up unconsciously. They are picked up - we know not how. From obscure sources and by unnoticed channels they insinuate themselves into the mind and become unconsciously part of our mental furniture. Tradition, instruction, imitation - all of which depend upon authority in some form, or appeal to our own advantage, or fall in with a strong passion - are responsible for them.... Such thoughts are pre-judgements, not conclusions reached as the result of personal mental activity.* *Dewey (1933). p.7*

The teachers rarely talked about their knowledge of mathematics in the interviews. It may be that the impact of external sources is so strong in determining beliefs and hence knowledge-in-action, that we are prevented from accessing 'knowing' about the subject. But what about the future if tradition is the main justification for action about the curriculum?

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READABILITY OF VERBAL PROBLEMS IN MATHEMATICS: SOME EVIDENCE FROM SECONDARY SCHOOLS IN SOUTH AFRICA

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Abstract: *Currently many problems in mathematics are posed in the context of real-life situations. This not only introduces more language, but also more issues related to culture. There is a surmise that readability factors in the ordinary language of mathematics texts often cause unnecessary comprehension difficulties to students. If this were true for first language students the impediment experienced by second language students could be even greater. This research used protocol analysis to identify different types of readability problems experienced by first and second language readers. Identified problems are reported in five categories. The hypothesis that improved readability will improve achievement was tested by a composite test using questions with varying degrees of readability. A few results are reported.*

1. The research problem.

Miscomprehension in mathematics has more far-reaching consequences than in most other subjects. Exact reasoning necessitates exact understanding. During the last few years educators have become increasingly aware of the important role reading and language play in the successful accomplishment of mathematics tasks - especially amongst second language learners (Adetula, 1990; Lagerwerf, 1992:36). More and more ordinary language is presently being used in school textbooks and therefore also in assessment tasks. Whereas a non-verbal mathematical problem can be posed in an international, precise language - a language students are expected to learn - a verbal problem has to be set in a language that takes the linguistic and cultural aspects of the reading audience into consideration. When writing for large audiences this is a difficult commission. There is a surmise that readability factors in the ordinary language of mathematics texts often cause unnecessary comprehension difficulties to students. (Weerman, 1994:167). If this were true for first language readers, the impediment experienced by second language readers could be even greater.

The issue of second language readers is especially relevant to a country like South Africa. More than 80% of all secondary school students are black and receive their secondary education in a language which is not their mother tongue. Primarily this research was concerned with one aspect of the language issue, i.e. readability problems related to ordinary English in mathematics texts. For a writer it would be important to know what makes one text more readable than another. Research questions like the following arose:

- What type of readability factors in the ordinary language of mathematics texts prevents a clear understanding of a mathematics problem?
- What is the influence of readability factors on achievement levels of African pupils whose home and school cultures are often different?

2. Readability: a brief theoretical background

2.1 Defining readability

Readability needs to be defined because the term is used in different senses. The majority of researchers have identified readability with comprehensibility (Selzer, 1983:73). For the sake of this study readability is defined as *the ability of the text to communicate the intention of the writer to the intended reader*. Literature suggests that readability be approached from a cognitive and psycholinguistic point of view. The cognitive approach focuses on the reading *process* and investigates the mind of the reader whereas in psycholinguistic theory, *language* is the dominant factor and interaction between reader, writer and text plays an important role.

2.2 Psycholinguistic factors influencing readability

2.2.1 The reader

A reader's background knowledge as well as his/her language proficiency have proved to be two factors that have an important influence on readability. Background knowledge is the linguistic equivalent for what cognitive theorists call schemata or mental constructs. Successful communication between reader and writer is based on shared schemata - on those available to the non-specialist partner of the communication process. It is clear that for second language readers, comprehension problems, due to a lack of background knowledge, will be intensified by a weak language proficiency.

2.2.2 The text

Mathematics text normally contains a certain amount of ordinary English together with portions of the mathematics register. Textual issues influencing readability can therefore be related to either.

2.2.3 The writer

Readability of text depends to a large extent on writers' choices. School mathematics texts could easily lead to miscomprehension because the reader and writer are not equal. Comprehension problems could be aggravated even more if differences between reader and writer were extended to the cultural level (Bishop, 1993).

2.3 Cultural factors influencing readability

Reading in a second language is not only a matter of language acquisition. It is also a matter of learning another culture. Differences in cultural behaviour have proved to cause problems in written and oral communication (Hall & Hall, 1989; Wierzbicka, 1991). When considering cultural influences on the comprehension of mathematics text, two manifestations of culture seem especially relevant. The one is the relationship between *culture and the structure of a language* and the other, the influence of *cultural experiences.*

2.4 The two experiments

After all is said and done it is not quite clear how students themselves experience the readability of mathematics texts. The best way to find out seemed to be to go to the students themselves.

Two experiments were initiated. First a protocol study was launched to find answers to the research questions. Analysis of students' protocols not only revealed various readability problems, but also generated the hypothesis which was tested in the second experiment. This paper will report mainly on results from the first experiment

3. The protocol study

Students were 17-18-years old. Three different language groups were involved: a first language group (the E1 group) and two second language groups. One second language group, the E2 group, had Afrikaans as first language while the other second language group comprised African students, the E3 group. Afrikaans is a language with Germanic roots and therefore related to English, whereas the African languages are in no way related to English.

There were six students in each language group and all of them were high achievers of mathematics. Students did the think-aloud protocols individually. They were asked to read and think aloud as they solved nine previous examination questions. The questions represented so-called word problems. All think-alouds were captured on tapes. After students had completed the think-aloud experiment, they were asked to adapt the nine questions to a more comprehensible form (cf. Appendix for the original and adapted versions of Questions 3 and 5).

3.1 Results and discussion of think-aloud protocols.

Although the talk-alouds revealed that all three groups encountered readability problems, African readers (the E3 group) experienced most problems more intensely. Increased anxiety was clearly audible on the tapes of E3 readers and readability problems at times even caused communication breakdowns - something that never happened to the other two groups. As the analysis of protocols progressed, it became clear that E3 readers found the mathematics text less accessible than their E1 and E2 counterparts not only because of linguistic reasons, but also because of cultural issues. Cultural thought patterns, culturally biased contexts and cultural issues related to reading behaviour had a definite impact on how readers experienced the readability of questions.

Readability problems were identified by carefully listening to the think-alouds. Factors causing comprehension difficulties were grouped according to the following five categories: Difficult vocabulary; Text structure; Obscure information; Visualization difficulties and Non-verbal factors. A variety of readability problems were identified in all nine questions. For the sake of this paper the report will focus mainly on results generated by the analysis of African readers' protocols. Only a superficial report is possible. The first two categories will be discussed in a bit more detail.

3.1.1 Difficult vocabulary

Although difficult vocabulary is problematic for all kinds of readers, in mathematics it causes more problems for second language readers. The protocols confirmed that second language readers are often unable to discern whether the meaning of a difficult word is absolutely necessary for solving a mathematical problem. Listening to the think-alouds also confirmed that most E3 readers process information bottom-up and a difficult word often hinders a global conceptual analysis or recognition of relationships between variables.

Question 5 had a few words and phrases that caused comprehension problems even for first language readers namely, *utilized, profit margins, optimal search line, daily capacity*. Question 5.3 caused much anxiety and to some E3 readers, even a complete comprehension breakdown. One reader took 47 minutes to do question 5 and could not get further than section 5.3. She kept on saying: *"I don't understand the question.. Oh, I don't know... Oh, I'm taking too much time"* After another 13 minutes she despondently said, *" I'll come back to this one later if I have enough time."* Needless to say she never came back.

The think-alouds not only confirmed the need of the E3 group for more plain language, but also the need for more time to read and process information. Need for more time could be because of the relative weaker language proficiency in English, but the need could also have a cultural basis. Some of the E3 talk-alouds took more than twice as long to complete than those of the other second language group.

3.1.2 Text structure

This category refers to problems related to the overall organization of text, whether in sentences or overall discourse. The talk-alouds verified the importance of structural issues for second language readers (Kieras, D. 1978). In Question 3 the composition of the text has in a sense violated the principle of hierarchical progression by inverting the order of importance. The irrelevant, redundant information concerning 8000 calculators is given in the prominent first sentence, whereas the crucial information that the selling price refers to only *one* calculator is reserved for the inferior last position of the text. More E3 readers tripped over this hurdle.

Another important structural issue causing comprehension difficulties, was the issue of cultural thought patterns (Kaplan, R. 1980). When comparing the analysis of the think-alouds with the students' adaptations it became clear that often the linear thought patterns of predominantly English or Afrikaans writers, made the text less accessible for E3 readers. One noticed that the adapted formulations of E3 readers were inclined to have a circular structure resembling the cultural thought patterns of their mother tongue. Not only was the information more descriptive and more explicit, but it was also more repetitive and had a recurring nature.

3.1.3 Obscure information

Information of this kind is not clearly understood and causes uncertainty within the reader. Different reasons for obscurity were identified like: *confusing information, culturally biased contexts, contradictory* and *senseless information.*

3.1.4 Visualization difficulties

Information that is too abstract or too condensed often makes it difficult to form an image of the communicated information. Readers find sentences that are easy to visualize, easier to understand. For example, the rather abstract information in Question 5 made it difficult for students to form an idea of the situation.

3.1.5 Non-verbal factors

This category refers to letter symbols or mathematical formulae used in such a way that it interferes with the processing of information. This happened in various ways: *inappropriate* or *ambiguous use of the letter symbol; entangled verbal/non-verbal information* and *artificial functions.*

3.2 Formulating the hypothesis

During the protocol experiment, students also wrote down their solutions. On average, test scores proved 26% lower than students' school performance. Although readability factors could have been responsible for low test scores, one was not sure whether the level of mathematical difficulty was not the main reason for the students' poor performance. It seemed necessary to test the following hypothesis:

Improved readability of the ordinary language in mathematics examination questions will improve achievement.

4 The experiment to test the hypothesis.

A composite test was used to test the hypothesis. The test contained the same nine questions, but were set in different versions: original, adapted and non-verbal. More than 300 students, representing all language groups, wrote the test. The hypothesis was confirmed in a number of important cases. To form a general idea of the effect of improved readability on achievement one could consider the total improvement of test scores across all nine questions.

Average percentage score of all nine questions for the original and adapted versions

Group	Average % for nine original versions	Average % for nine adapted versions	% Gain:	p-value
E1 (n=108)	49	61	12	$p < 0.02$**
E2 (n=108)	51	70	19	$p < 0.001$**
E3 (n=108)	41	55	14	$p < 0.002$**
E1+E2+E3 (n=324)	47	62	15	$p < 0.001$**

Differences in scores between the original and adapted versions were tested by subjecting the differences to the Mann-Whitney U-test.

5. Closing remark

Various conclusions were drawn from this study (Prins, 1995). Whereas writers of mathematics text can do very little to improve the language proficiency of their reading audience, the results of this study do emphasize the responsibility of writers to make their text as comprehensible as possible. In fact, the first step towards successful problem solving is to fully understand the problem (Polya, G. 1946). One should also keep in mind that readability problems not only have the potential of affecting achievement. Issues like mathematics anxiety and attitudes toward mathematics are also likely to be influenced. If one were to apply the advice of Ausubel to writing one could conclude by saying, *"Ascertain the reading needs of your audience and write accordingly"*

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IMAGES AND DEFINITIONS FOR THE CONCEPT OF EVEN / ODD FUNCTION

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ABSTRACT

Definitions and images, as well as the relation between them of the even / odd function concept, were examined in 184 Arab and Jewish Israeli high school students. A questionnaire was designed to exhibit the cognitive schemes for the even / odd function concept that becomes active in identification problems. One of the research questions aimed to check whether the students know to define the concept of even / odd function. Another question was whether the students know how to apply the definition of the concept for specific functions. A third question was whether the students know to link the graphical aspect with the algebraic aspect of the concept. The results show that 54% of our sample knew the definition, but only between 14% to 50% of the students knew how to implement the definition.

The study examined several aspects of the images and definitions that junior high school students have regarding the even / odd function. Concept images and concept definitions (henceforth called *images* and *definitions*) have been discussed in detail in several papers (Tall & Vinner, 1981; Vinner, 1983; Vinner & Hershkowits, 1980; Vinner & Dreyfus, 1989). We will therefore introduce them here very briefly. All mathematical concepts except the primitive ones have formal definitions. Many of these definitions have been introduced to high school or college students at one time or another. The student, on the other hand, does not necessarily use the definition when deciding whether a given mathematical object is an example or a nonexample of the concept. In most cases, he or she decides on the basis of a concept image, that is, the set of all the mental pictures associated in his / her mind with the name of the concept, together with all the properties characterizing them.

The concepts of even as well as odd function are central in the chapter about functions and their graphs. In many countries, including Israel, the chapter on functions and their graphs is taught in the tenth grade. The topic is mentioned again and again in high school courses and elementary college courses (pre-calculus and calculus). In most mathematical textbooks one can find definitions such as the following: A function is said to be *even* if $f(x) = f(-x)$ for all x. A function is said to be *odd* if $f(x) = -f(-x)$ for all x. (Lang, 1973, p.16).These definitions are algebraic, formal, rigorous and general. The even / odd function concept has graphical aspects according to *symmetry* (The graph of an *even* function is symmetrical according to y axis, and the graph of an *odd* one is symmetrical according to the origin of the coordinate system). Sometimes, in order to present a new concept, authors of mathematics textbooks limit themselves first to a "special case" in which they state a rigorous definition at the early learning stage. Obvious examples of concepts can be found in these textbooks such as derivative, increasing or decreasing function, and even / odd functions as well. The "special case" in our instance was the power function of the form $f(x) = x^n$ where n is natural. The definitions of the *even* function or the *odd* function concepts were stated rigorously by observing the even power function of the form $f(x) = x^n$ where n is even and the odd power functions of the form $f(x) = x^n$ where n is odd. The "special case" approach frequently causes serious difficulties in the formulation and the application of concept definitions (Vinner and Dreyfus, 1989; Rasslan, 1996). In a previous study (Ben David, 1986), where a close observation was made on 10^{th} grade class, it was found that there seemed to be a tendency among high school students to relate the concept of even / odd function with the even / odd exponent of a polynomial function. It was found that 7 students out of 26 had this tendency. One of the tendencies the first auther found during his experience in teaching mathematics is that for a certain number of high school students the misconception that, in general, a function must be even or odd, therefore; a function which is not even was understood as being an odd function. This misconception was not reported earlier.

Many of the difficulties students have with mathematics are a result of communication failure. The pseudo-conceptual behavior phenomenon discussed in detail in several papers (Vinner, 1994; Rasslan & Vinner, 1995) are example of such failure.The *pseudo-conceptual* behavior is a behavior which might give the impression that it is based on conceptual thinking but, in fact, it is not.

The *inconsistent behavior* is a specific case of the *compartmentalization phenomenon* mentioned in Vinner, Hershkowitz, and Bruckheimer (1981). This phenomenon occurs when a person has two different, potentially conflicting schemes in his or her cognitive structure. Certain situations stimulate one scheme, and other situations stimulate the other.

This study investigated the following:

1. What are the common definitions of the even / odd function concept given by high school students?
2. What are the main images of the even / odd function concept that these students use in identification tasks?
3. What are the main misconceptions that these students have according to the even / odd function concept?
4. How frequently do students compartmentalize their formal definition of an even / odd function and their image of this concept?

METHOD

Sample

Our sample comprised three classes of Israeli Arabic students and three classes of Israeli Jewish students, all 11th graders. The total number of students was 184.

The Questionnaire

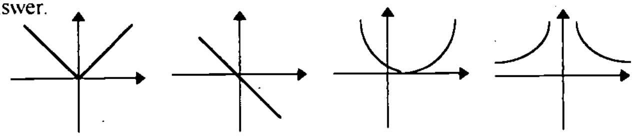
The questionnaire in figure 1 was administered to all subjects in the sample. Questions 1 through 5 were designed to examine some aspects of the even / odd function image of the respondents, whereas Question 6 was designed to examine their definitions. Questions 1 and 3 were designed to examine firstly, the ability to reason and the ability to apply the definition of the even / odd and secondly, to examine the tendency to relate the even / odd function concept to the exponent of x in polynomial functions (Question 1) and in another functions (Question 3) which our students knew. Question 2 was designed to examine whether the students realized the graphical issue of the even / odd function concept. Questions 4 and 5 were designed to examine two tendencies; the first is the tendency of students in high school to relate the concept of even / odd function to the even / odd exponent of x; the second, the tendency to conclude that not even function, is an odd function. We believe that special preparation is required in order to answer these questions, only an understanding of the basic mathematical language is needed. Such an understanding is a necessary condition to any mathematics lesson.

Procedure

The questionnaire was administered to the students in their classes. They were not asked to fill in their names, only their background information. It took them 30 minutes at most to complete the questionnaire. About 40 randomly chosen questionnaires were analyzed in detail by the authors. On the basis of this analysis the remainder of the questionnaires were analyzed.

1. Which of the following functions is: even, odd? Explain your answer.
   (a) $f(x) = 1/x$. (c) $f(x) = x^2 + x^4$ (e) $f(x) = x^3 + 3$
   (b) $f(x) = x^3 + x$ (d) $f(x) = 1/x^2$ (f) $f(x) = x^4 + 5$

2. Graphs of function are given. Which of the functions is even, odd? Explain your answer.



(a) (b) (c) (d)

3. Which of the following functions is: even, odd? Explain your answer.
   (a) $y = x$ (c) $y = x^2 + |x|$ (e) $y = (x+1)^{20}$
   (b) $y = |x|$ (d) $y = |x^2 + x|$
4. Joseph claimed that every even function is a function of the form $f(x) = x^n$ (n is even). What is your opinion about Joseph's claim? Is it correct? Is it incorrect? Explain your answer.

5. Is it true to say that if a function is not even, it must be odd? Explain your answer.

6. What is an "even function" in your opinion?

***Figure 1.* The questionnaire.**

RESULTS

Definition Category

We categorized the students answers according to methods described elsewhere (Vinner, 1983; Vinner & Dreyfus, 1989; Rasslan & Vinner, 1995) when dealing with other concepts (function, slope). We illustrated each category using a number of sample responses.

Question 6

Category I: An algebraic definition where the universal quantifiers are missing. (42%). Examples: 1. $f(-x) = f(x)$. 2. $f(-a) = f(a)$.

Category II: An algebraic definition with a graphical aspect (symmetry according to y axis).(12%). Example: An even function is symmetrical according to y axis as well as one which satisfies the condition $f(-x) = f(x)$.

Category III: A definition which has the right element as well as an erroneous element and a use of special case as a universal definition of the concept (16%)

Category IV: Wrong definition (18%). Examples: 1. The even function is an even power function as well as a symmetrical drawing. 2. $f(x) = (x)$.

12% of the students did not answer the question.

From the above categorization it emerged that only about 54% of the students (categories I, II) knew the definition of the even function concept. The remainder (46%) did not know it.

Concept Images

Various aspect of the even / odd function concept, as conceived by the students, were expressed in their answers to Questions 1 to 5. Some of the major aspects of the even / odd function concept that played a crucial role in the explanations given by the respondents are as follows:

Question 1.a ($f(x) = 1/x$)

Category I: The student uses correctly the algebraic definition of the odd function ($f(-x) = -f(x)$), he uses the letter **a** or substitutes a number instead of x (34%)

Examples: 1. $f(-x) = -1/x = -f(x)$, the function is odd. 2. $f(-a) = -1/a$, $f(a) = 1/a$, then the function is odd. 3. $f(-2) = -1/2$, $f(2) = 1/2$, then the function is odd.

Category II: Ritual reasoning. The student recites the definition of the odd function concept ($f(-x) = -f(x)$) correctly but he does not try to show that the above function satisfied the definition (10%). Example: The function is odd because $f(-x) = -f(-x)$.

Category III: Right answer without reasoning (10%). Example: The function is odd.

Category IV: A wrong answer as a result of different and inexplicable reasons (40%)

Category IV_a: The function is odd because $f(-x) \neq f(x)$ (14%)

Category IV_b: Failure in the definition's application (5%)

Category IV_c: Determining the oddity on the basis of the exponent of x (7%)

Category IV_d: A failure in substitution, meaningless answers or a correct answer but an incorrect explanation. (14%)

5% of the students did not answer the question (1.a.)

From the above categorization it emerged that only about 44% of the students (categories I, II) we can claim with certainty know to apply the definition of the concept of odd function according to the function $f(x) = 1/x$. Regarding another 10% of the students (category III) we cannot claim it, but we also cannot claim the opposite. About all the rest (category IV, and those who did not answer), 46% of our sample we can claim that the concept of odd function is not clear to them. The analysis of the parts (b - f) of Question 1 was done in the same way as in part a. Table 1 provides information about the percentages of the correct answers. It turns out that only 14% of the students' know how to apply the definition to the function $f(x)= x^3+3$ which is not even and not odd.

Table 1: Distribution (percentages) of Correct Answers to Question 1 (Parts a - f) (N = 184)

Part	a	b	c	d	e	f
Function	$f(x)=1/x$	$f(x)=x^3+x$	$f(x)= x^4+x^2$	$f(x)=1/x^2$	$f(x)= x^3+3$	$f(x)= x^4+5$
Percentage	44	37	50	47	14	39

The last result was especially bad with the function $f(x) = x^3 + 3$. Maybe it is a result of the idea that every function is even or odd. The concept of not even and not odd function is unreasonable to many students. It turns out for example that 76% of the students who answer correctly regarding the function $f(x) = 1/x$ and claim that it is an odd function, failed in the question about $f(x) = x^3 +3$, and 72% of the students who answer correctly regarding the function $f(x) = 1/x^2$, failed in the question about $f(x) = x^3 +3$. In the categorization analysis of Question 1.a ($f(x) = 1/x$) above, it turns out that 7% of the students link the odd function concept with the exponent (category IV_c). From table 2, it turns out that this approach was found in higher percentages according to another functions. For example 16% of the students claimed that the function $f(x) = x^4+x^2$ is even because the exponent of x is even. Here it can be *even* considered as a correct answer, and there is something new: what guides the students was the thought that an even function is a "linear combination" of even powers (or the "sum of even powers").

Table 2: Distribution (percentages) of Respondents Linking the Even or Odd Function with the Exponent of x in Question 1 (parts a - f) (N = 184)

Part	a	b	c	d	e	f
Function	$f(x)=1/x$	$f(x)=x^3+x$	$f(x)=x^4+x^2$	$f(x)=1/x^2$	$f(x)=x^3+3$	$f(x)=x^4+5$
Percentage	7	10	16	14	12	11

One of the objectives of this research was to examine the ability of the students to apply the definition of the concept even / odd function. Table 3 provides information about our sample according to Question 1 (parts a - f). The obvious result from table 3 is that between 31% - 79% of our sample who defined the even function correctly (Question 6) answered incorrectly Question 1 (parts a - f).

Table 3: Distribution (percentage) of Correct Definition of Even Function Concept and Incorrect Answers to Question 1 (parts a - f) (N = 100)

Part	a	b	c	d	e	f
Function	$f(x)=1/x$	$f(x)=x^3+x$	$f(x)=x^4+x^2$	$f(x)=1/x^2$	$f(x)=x^3+3$	$f(x)=x^4+5$
Percentage	37	44	32	31	79	49

QUESTION 2.a

Category I: The student used the algebraic definition of the even function $f(-x) = f(x)$ in various ways: (He identified the function formula and used the above definition, or was helped by a scheme in order to show that the condition in the definition was satisfied) (9%). Examples: 1. $y(-x) = |-x| = |x|$, the function is even. 2. $f(-a) = f(a)$ (A correct graph was included), the function is even. 3. $f(-2) = f(2)$ (A correct graph was included), even, because the y is equal.

Category II: Right answer, the student mentioned the definition $f(-x) = f(x)$ but he did not link it to the specific graph. Thus there was no indication that the student know to implement the definition. There was no indication of the opposite either. (17%). Examples: 1. The function is even $f(-x) = f(x)$. 2. For every x, -x there is the same y. Therefore, the function is even. 3. Even, because for every two opposite numbers the same picture exists.

Category III: The student used the graphical property (symmetry to y axis) or used the detailed definition (the algebraic definition $f(-x) = f(x)$ and the graphical property) (24%). This category included two subcategories as follows:

Category III_a : The student used the graphical property correctly (19%). Example: The function was even because it was symmetrical according to y axis.

Category III_b : The student used the detailed definition (5%). Example: The function was even because it was symmetrical according to y axis, and $f(-x) = f(x)$.

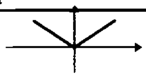
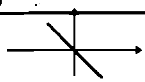
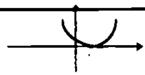
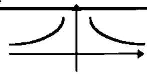
Category IV: Incorrect answers were based on pseudo-conceptual behaviour, or a confusion between even and odd (29%). Examples: 1. Even function because its graph look like parabola. 2. Even because the line equation is $y = x^n$, n is even. 3. Even because the function is up to x axis. 4. Odd. Because it is the absolute function. 5. Odd, $f(3) = f(-3)$. It is not even $f(3) = -(-3)$.

Category V: Right answer without reasoning (13%). Example: The function is even.

7% of the students did not answer the question.

From the above categorization it turned out that at least 14% of the students (categories I, III_b) knew the graphical aspect and knew to link it with the algebraic one. About another 10% of the students (category III_a) we cannot claim it because they refered to the graphical aspect only. Another 17% (category II) mentioned the algebraic definition without any link to the given graph. Those are "suspicted by ritual behavior" which was not based on real knowledge. A further 13% of the students (category V) gave a right answer without reasoning. About them we cannot claim that they knew how to link the graphical to the algebraic aspect of the concept; nor can we claim the opposite. About the rest (category IV, and those who did not answer), which are 37% of the students we can claim that they did not know how to link the graphical aspect with the algebraic one of the even function concept. The analysis of the other parts (b-d) of Question 2 was similar to part a. Table 4 shows the distribution of correct answers to Question 2. The obvious result is that only 14% of our sample knew that the function in 2.c. was not even and not odd function.

Table 4: Distribution (percentages) of correct answers to question 2 (parts a - d) (N = 184)

Part	a	b	c	d
Function				
Percentages	50	29	14	41

QUESTION 3.a (y = x)

Category I: The student used correctly the algebraic definition of the odd function concept f(-x) = -f(x) and correctly. He used the letter a or substituted a number instead of x. (27%). Examples: 1. Odd, y(-x) = -x, y(x) = x. 2. f(a) = a, f(-a) = -a, - f(-a) = -(-a), y = x is odd function. 3. y = 2, y = -2, -(-2) = 2, odd.

Category II: Correct answer. The reasoning is ritual. The student repeats the algebraic definition of the odd function f(-x) = - f(x) (10%). Example: The function is odd, f(-x) = -f(x).

Category III: Correct answer without explanation (11%).Example: The function is odd.

Category IV: Nonsensical answers (47%)

Category IV_a : The function is odd because it does not satisfy the even function definition (13%). Examples: 1. The function is odd $f(a) \neq f(-a)$. 2. $-x \neq x$, the function is odd.

Category IV_b: Meaningless quotation of the even function definition as a condition of odd function without any examination (3%). Example: The function is odd f(-x) = f(x).

Category IV_c: The determination of the oddity of the function based on the exponent of x (3%). Examples: 1. $y = x^1$, odd. 2. The function is odd, its power is one.

Category IV_d : Correct answer with a wrong explanation, or incorrect answer (with or without explanation (27%). Examples: 1. The function is even. 2. The function is even and odd. 3. Not even and not odd. 4. The function is odd f(-y) = -x. 5. The function is odd because it is constant. 7% of the student did not answer the question (3.a)

From the above categorization it turns out that for only about 38% of the students (categories I, II) we can claim with certainty that they know to apply the definition of the odd function to the function y = x. For about another 10% of the students (category III) we cannot claim it but we also cannot claim the opposite. About all the rest (category IV, and those who did not answer), which are 52% of our sample we can claim that the concept of odd function is not clear to them. The analysis of parts (b - f) of Question 3 was done in the same way as in part 3.a. Table 5 shows the distribution of correct answers to all parts (a - e) of question 3. The results were similar to those of Questions 1 and 2 mentioned above.

Table 5: Distribution (percentages) of correct answers to question 3 (parts a - e) (N = 184)

Part	a	b	c	d	e
Function	$y = x$	$y = \|x\|$	$y = x^2 + \|x\|$	$y = \|x^2 + x\|$	$y = (x+1)^{20}$
Percentage	38	40	42	10	11

Table 6 shows the distribution of students who made a link between the even / odd function concept and the exponent. The obvious result from table 6 was that 11% of the students linked the concept of even function with the exponent of the function $y = (x + 1)^{20}$. The even exponent of the function $y = (x + 1)^{20}$ was especially obvious. Those students did not distinguish between polynomial function of an even exponent of the form $f(x) = (x + c)^{2k}$ (when k is natural and $c \neq 0$) and between an even power function of the form $f(x) = x^{2k}$ (when k is natural).

Table 6: Distribution (percentages) of Respondents Linking of the Even or Odd Function with the Exponent of x in Question 3 (parts a - e) (N = 184)

Part	a	b	c	d	e
Function	$y = x$	$y = \|x\|$	$y = x^2 + \|x\|$	$y = \|x^2 + x\|$	$y = (x+1)^{20}$
Percentage	3	1	2	1	11

Question 4

Category I: Right answer supported by a counter example (23%)

Category II: Right answer but the explanation was meaningless (5%)

Category III: Right answer without an explanation (4%)

Category IV: Right answer with a wrong explanation (10%)

Category V: Wrong answer (with or without explanation) based on the idea that the form of every even function is determined by an even exponent (49%)

8% of the student did not answer the question.

QUESTION 5

Category I: Right answer supported by a counter example (18%)

Category II: Right answer based on the definitions of even functions and odd functions (7%)

Category III: Meaningless right answers or right answers without explanations (41%)

Category IV: Wrong answer because of different as well as inexplicable reasons (19%)

Category IV_a: Right answer with a wrong explanation (14%)

Category IV_b: Pseudo-conceptual answers (5%)

Category V: A not even function is an odd function (7%)

8% of the student did not answer the question.

From the analysis of the last two questions, two conclusions emerge:

a) A remarkable percentage of the students linked the concept of even function to an even exponent appearing in the algebraic representation of the function.

b) Althouh, only 7% accepted erroneous claim in Question 5. The fact that 60% failed to explain why it should be rejected, is worrying.

Compartmentalization is an interesting aspect of this study. It turns out that 17 students from those who were mentioned in **(a)** above claimed correctly the function $y = |x|$ (Question 3. b) is even function based on the formal definition of the even function concept. It turns out that also 22 students claimed correctly that the function $y = x^2 + |x|$ (Question 3. c) is an even function.

DISCUSSION

One of the goals of this study was to expose some common images of the even / odd function concept held by high school students. This has a direct implication for teaching. If one wants to teach even / odd function to a group similar to our sample, it is important to know the starting point of its members (Vinner & Dreyfus, 1989). Taking into account the difficulties mentioned in this study, at least a doubt should be raised whether the "special case" approach to the even / odd concept is the best way for teaching such a concept. If simple functions, polynomial functions, absolute value functions, or other strange functions are needed, we think that they should be introduced as cases extending the students' previous experience. The formal definition should be only a conclusion of various examples introduced to the students. A similar conclusion was mentioned also in Vinner and Dreyfus (1989) according to Dirichlet-Bourbaki's approach to the function concept, and in Rasslan (1996) according to other concepts, such as increasing (decreasing) function concept.

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CONSTRAINTS AND OPPORTUNITIES IN TEACHING PROVING

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Abstract: *Analysis of reasoning taking place in classrooms involves more than consideration of the forms reasoning takes and the needs which motivate it. The curriculum, didactic contracts, and culture of the classroom constrain what reasoning can occur. At the same time, discovery activities and opportunities for discourse can provide occasions for reasoning. This paper briefly considers the interplay of these constraints and occasions in a Canadian grade 10 classroom.*

In previous research (Reid 1995 a,b) I developed a vocabulary for describing deductive reasoning. This research was based on clinical interviews with secondary school and university students engaged in open ended problem solving. Recently I have been attempting to apply this vocabulary to describing the deductive reasoning of secondary students in their classrooms. In so doing I have also seen the effects of applying classroom cultures to my vocabulary. In the following I will describe these experiences in terms of occasions for, and constraints on, deductive reasoning in a secondary school classroom.

Background on enactivism and evolution

Over the past few years I have been exploring, with other members of the enactivism research group, the ways in which ideas derived from Maturana and Varela's enactivist theory of cognition (Maturana & Varela 1992; Varela, Thompson & Rosch 1991) can be applied to the learning of mathematics. We have addressed such ideas as the coemergence of learners in a problems solving situation (Kieren, Gordon Calvert, Reid & Simmt 1995), ways of describing proving (Reid 1995 a, b), the nature of teaching (Kieren, Gordon Calvert, Reid & Simmt 1996), and research methodology (Reid 1996a). I will be using enactivist ideas here to describe students' proving in classroom contexts.

An important idea in enactivism is that of *satisficing*. This idea is derived from theories of evolution which posit that organisms do not evolve to optimal states, but rather any form which is not fatally detrimental continues to be propagated. In such a view the organism's environment does not *determine* the form of the organism, but it does offer *constraints* which shape the organism by disallowing certain forms.

In classroom teaching the idea of satisficing can be applied to mathematical behaviours. The teacher's actions and the classroom context do not determine what mathematical behaviours the students will evolve, but they offer constraints which disallow certain forms. In practice the forms of behaviour which are allowed are likely to exceed the bounds intended by the teacher, because of the interplay of factors involved in classroom contexts.

In describing what teachers do in classrooms we consider teaching to be providing opportunities for learning. Because students' learning is determined by their own structures the opportunities created by a teacher cannot "cause" students to learn. When learning does take place we call the opportunities created by the teacher "occasions."

Proving as explaining

The classroom context which I will describe here is that of a Canadian grade 10 ($\approx$15 years old) middle stream mathematics class. My observations covered three months of instruction, involving problem solving and group work on graphs of linear equations and Euclidean geometry. The class used materials which I prepared in consultation with their classroom teacher, which had been tested in a pilot study the previous year (Blackmore, Cluett & **** 1996). The materials encouraged students to explain aspects of graphing linear functions, and theorems in Euclidean geometry. The teaching methods employed provided students with opportunities to learn to communicate mathematically, and to work cooperatively in mathematical situations.

In my descriptive vocabulary proving is considered to be in response to one of four *needs*: to explain, to explore, to verify, and as part of a social process. As the importance of proving to explain has been emphasized in recent research (Hanna 1989, 1995; de Villiers 1991, 1992), explaining was presented as the primary motive for proving in the activities used in the class. Students were encouraged to use deductive reasoning to explain mathematical propositions, and proving in this context meant explaining deductively.

Opportunities: discourse, discovery, debate.

Previous research has suggested that it is through guided exploration and class discussions that students can best learn to reason deductively (Balacheff, 1991; Lampert 1990; Fawcett 1938). On most days the class's activities allowed for a great deal of exploration and discussion. Prompts were given which presented a situation and asked for an explanation. These prompts were worked on in small groups (2-5 students per group) and group conclusions were presented to the whole class. The assumption was made that this pattern would be an opportunity for students to explain deductively, and to express and clarify their reasoning through social interaction.

The students' discussions often occasioned short explanations, which were sometimes deductive. One prompt which occasioned some extended reasoning was: "In general two lines which are perpendicular have slopes which are negative reciprocals. Why? The following example may help. [The graphs of $y=\frac{3}{2}x$ and $y=-\frac{2}{3}x$ were given]" In one group the students observed that the triangle they had drawn to show the slope was rotated 90 degrees. They then used this observation to conclude that because the triangles were rotated the rise indicated by one would be equal to the run of the other, and vice versa.

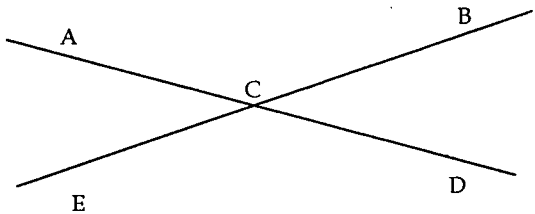
Many discoveries were occasioned by the prompts given the students. These ranged from procedures to determine equations of lines, to definitions of geometrical terms, to congruencies related to transversals of parallel lines. In some cases students made discoveries which had not been anticipated by their teacher and me. For example in explaining why two triangles were congruent in a particular diagram, they observed that the triangles formed a parallelogram and asserted their congruence based on the congruency of the opposite sides.

The process of social debate was especially valuable during the unit on deductive geometry. In the course of addressing the questions of their peers, their teacher, and myself, the students clarified and formulated their arguments. One pair presented the following to the class:

A) Which angles are congruent? Why?

$\angle ACB \cong \angle DCE$

$\angle ACE \cong \angle DCB$



— Two angles that have the same measures are congruent.

In the ensuing debate they were asked how they knew the angles had the same measure. Other members of the class asserted that they had the same measure because they were "vertically opposite." This was something they remembered from the previous year.

As "vertically opposite" had not been defined previously, the class paused to discuss a suitable definition, arriving at "Two angles that have the same vertex and the opposite rays from the opposite angle form a straight line." While this is not the most elegant definition, it has the advantage that everyone in the class understood it and accepted it, and it had been carefully thought about by the students themselves.

Once this was settled the pair presenting amended their statement to read: "— Two angles that have the same measures are congruent. because they are vertically opposite."

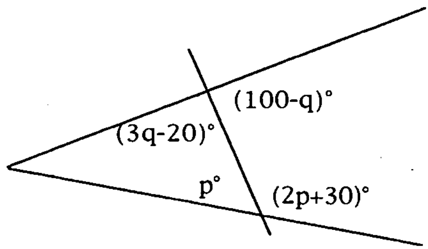
One of the roles I played in the class was to ask "Why?" In this case this occasioned a debate concerning why vertically opposite angles have the same measure. Several members of the class were actively engaged in this debate, contributing suggestions for ways of expressing ideas and new approaches. One student suggested several verbal explanations, and with the help of several classmates arrived at this written formulation: "two angles that have a common angle to make them supplementary have to be equal." Again the language is strained, but the students understood its meaning, and accepted it because it embodied the reasoning they had gone through.

Constraints: curriculum, contracts, culture.

Although the students' use of opportunities as occasions for learning was encouraging, even more evident were the constraints which kept them from reasoning deductively. These included the official and traditional curricula, the roles of the teacher and students, and the classroom culture.

Some objectives in Unit I: Linear Sentences of the provincial curriculum (Government of Newfoundland and Labrador, 1993) acted as a constraint on the students' reasoning. Objectives 1.6: "Students will be expected to rewrite linear equations." (p. 17) and 1.7: Students will be expected to graph a linear equation using various methods" (p. 18) are typical in their focus on algebraic manipulation and skill. The amount of time needed to develop the students' algebraic skill to the required level limited the amount of time which could be devoted to developing a deductive structure for the coordinate plane. The short time allotted to the deductive geometry unit in the provincial curriculum also acted as a constraint as it precluded a thorough discussion of the properties the students discovered in their explorations.

In addition to the official curriculum, there is also a traditional curriculum which acts as a constraint on reasoning. The official curriculum objectives for the Deductive Geometry unit explicitly state that "the overall objective is for students to be able to produce proofs" (p. 26). The text used for the course, however, adds an additional objective, that students be able to reason in geometric contexts using algebraic language. A typical* textbook exercise is this:



(Ebos, Morrison, Spinney & Dalton, 1992, p.319)

Similar exercises are also traditionally a part of the year end examinations. In such exercises students are expected to determine the measures of the angles by solving equations. For the most part the students had no trouble setting up the equations based on the geometric properties of the figure. Solving the equations and making sense of the result were their main difficulties. The experience of struggling through

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* This exercise is only atypical in that the diagram contradicts the information given. This led to a useful discussion.

solving an equation was frustrating to most of the students, and limited the time and mental energy they had to engage in more sophisticated reasoning.

Laborde (1989) describes *didactic contracts* (originally named by Brousseau, 1980): the asymmetrical roles assigned to teachers and students as a result of the differing knowledge they bring to the classroom. As in many classrooms the contract tacitly agreed at the beginning of the school year called for the teacher to demonstrate mathematical procedures and for the students to copy them. In expecting the students to discover mathematical procedures and principles their teacher violated this contract, and the students reacted by being uncooperative for the first week. As the students came to understand their new roles the contract was effectively renegotiated, and the students became more cooperative. In times of stress, however, both teacher and students returned to more familiar roles and patterns of activity. The original contract continued to influence what was possible in the classroom.

Lampert (1990) has emphasized the importance of developing a classroom culture in which students feel at ease proposing conjectures and defending them. For many of the students in the class involved in my study, they had not experienced such classroom cultures in previous years, and did not expect to be a part of one. The tension between the teachers' use of her authority to maintain order in the class room and the intent to provide students with autonomy served to undermine the development of a culture of mathematical respect. Some were willing to participate in debates and discussions in a way which fit the developing culture, but others felt intimidated, and were not willing to state a conjecture unless they knew they were correct.

Teaching proving: ideas for the future.

It is not a new observation that mathematical reasoning in classrooms depends as much on social factors as on psychology. In looking for ways to expand my descriptions of reasoning I was not surprised to find myself considering the social dynamics of the classroom — only the extent to which constraints kept deductive reasoning from happening. The claims I had heard teachers make that their students *did not* reason deductively began to make more sense to me.

Considering the social context from the pint of view of reasoning *satisficing* needs to explain, explore and verify (as opposed to *satisfying* those needs) let me connect the enactivist ideas of constraints and occasions to my existing structure of needs to and forms of reasoning mathematically.

I am now engaged in elaborating these connection and preparing to add the new perspective this gives me to those through which I have researched reasoning in the past. what I hope for is a multifaceted perspective including a multiplicity of mutually intelligible but distinct points of view, and the consideration of constraints and opportunities in making sense of students' mathematical behaviour.

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ASSESSMENT OF TEACHER-STUDENT INTERPERSONAL BEHAVIOUR IN SECONDARY MATHEMATICS CLASSROOMS: A SEED FOR CHANGE

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PROPOSAL ABSTRACT

This paper reports on current research using a convenient questionnaire designed to allow mathematics teachers to assess teacher-student interpersonal behaviour in their classrooms. The paper discusses the various forms of the Questionnaire on Teacher Interaction (QTI), and reports its use in past research. The paper provides validation data for the first use of the QTI with a large sample of mathematics classrooms. It also describes how mathematics teachers can and have used the questionnaire to assess perceptions of their own teacher-student interpersonal behaviour and used this as a basis for reflecting on their own teaching and thus providing a basis for guiding systematic attempts to improve their teaching practice.

INTRODUCTION

Most mathematics teachers believe that good relationships with their students are important. But are the students' perceptions of teacher-student interpersonal behaviour the same as their teachers? Is there a difference in mathematics teachers' perceptions of their actual teacher-student interpersonal behaviour in the classroom and what they perceive to be ideal?

The purposes of this paper are to outline a convenient questionnaire designed to assess teacher-student interpersonal behaviour and to report its use in answering such questions as these. The paper describes various forms of the *Questionnaire on Teacher Interaction (QTI)* and reports its use in past research. Finally, the paper describes how mathematics teachers have used the questionnaire to assess perceptions of their own teacher-student interpersonal behaviour and used this as a basis for reflecting on their own teaching.

THEORETICAL FRAMEWORK

International research efforts involving the conceptualisation, assessment and investigation of perceptions of psychosocial aspects of the classroom environment have firmly established classroom environment as a thriving field of study (Fraser, 1994; Fraser & Walberg, 1991). Recent classroom environment research has focused on mathematics laboratory classroom environments (McRobbie & Fraser, 1993), constructivist classroom environments (Taylor, Dawson & Fraser, 1995) and computer-assisted instruction classrooms (Teh & Fraser, 1994).

Researchers in The Netherlands extended this research by focusing specifically on the interpersonal relationships between teachers and their students as assessed by the QTI (Wubbels, Créton & Hoomayers, 1992; Wubbels & Levy, 1993). The Dutch researchers (Wubbels, Créton & Holvast, 1988) investigated teacher behaviour in a classroom from a systems perspective, adapting a theory on communications processes developed by Waltzlawick, Beavin and Jackson (1967). Within the systems perspective of communication, it is assumed that the behaviours of participants mutually influence each other. The behaviour of the teacher is influenced by the behaviour of the students and in turn influences the student behaviour. Thus, a circular communication process develops which not only consists of behaviour, but determines behaviour as well.

With the systems perspective in mind, Wubbels, Créton and Hooymayers (1985) developed a model to map interpersonal teacher behaviour using an adaptation of the work of Leary (1957). In the adaptation of the Leary model, teacher behaviour is mapped with a Proximity dimension (Cooperation, C - Opposition, O) and an Influence dimension (Dominance, D, - Submission, S) to form eight sectors, each describing different behaviour aspects: Leadership, Helpful/Friendly, Understanding, Student Responsibility and Freedom, Uncertain, Dissatisfied, Admonishing and Strict behaviour. Figure 1 shows typical behaviours for each sector. The Questionnaire on Teacher Interaction (QTI) is based on this model.

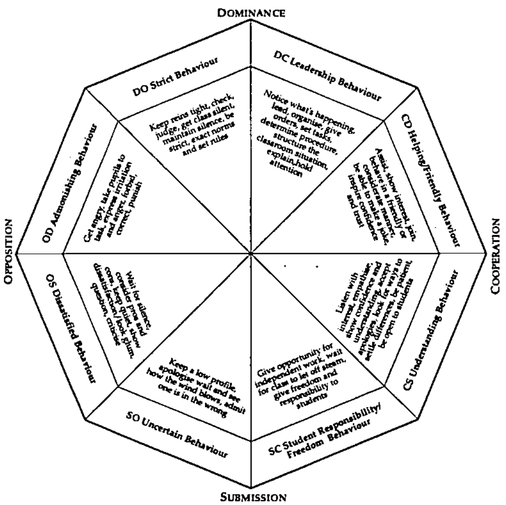

Figure 1. The model for interpersonal teacher behaviour

METHODOLOGY

The study described in this paper is distinctive in that it is centred on students in mathematics classes, whereas previous research using the QTI has focused largely on students in science classes. The study involved students in grades 8, 9 and 10 mathematics classes in Australia and was composed of 405 students in 9 schools with their 21 teachers.

Associations between students' perceptions of their interpersonal relationships with their teachers and their attitudinal outcomes were examined in this study. The 48-item version of the QTI (Wubbels, 1993) was used to gauge students' perceptions of student-teacher interpersonal behaviour and student attitudes were assessed with a seven-item *Attitude To This Class* scale, which was based on the the *Test of Science-Related Attitudes* [TOSRA] (Fraser, 1981).

Using the scales of the QTI as independent variables, associations were computed with attitude to the class. Simple correlations were calculated between each QTI scale and each student attitude. Also a multiple regression analysis, involving the set of QTI scales, was conducted to provide a more conservative test of the association between each QTI scale and attitude when all other QTI scales were mutually controlled.

RESULTS

Validity of the QTI

Table 1 provides some cross-validation information for the QTI when used specifically in the present sample of mathematics classes. Statistics are reported for two units of analysis, namely, the student's score and the class mean score. As expected, reliabilities for class means were higher than those where the individual student was used as the unit of analysis. Table 1 shows that the alpha reliability figures for different QTI scales ranged from 0.62 to 0.88 when the individual student was used as the unit of analysis, and from 0.60 to 0.96 when the class mean was used as the unit of analysis. The values presented in Table 1 for the present sample provide further cross-validation information supporting the internal consistency of the QTI, with either the individual student or the class mean as the unit of analysis.

Another desirable characteristic of any instrument like the QTI is that it is capable of differentiating between the perceptions of students in different classrooms. That is, students within the same class should perceive it relatively similarly, while mean within-class perceptions should vary from class to class. This characteristic was explored for each scale of the QTI using one-way ANOVA, with class membership as the main effect. It was found that each QTI scale differentiated significantly ($p<0.001$) between classes and that the eta^2 statistic, representing the proportion of variance explained by class membership, ranged from 0.14 to 0.43 for different classes.

Table 1
Internal Consistency (Cronbach Alpha Coefficient) and Ability to Differentiate between Classrooms of the QTI

Scale		Alpha Reliability		ANOVA Results
		Student	Class	Eta²
DC	Leadership	0.86	0.93	0.43*
CD	Helping/friendly	0.88	0.94	0.29*
CS	Understanding	0.88	0.96	0.36*
SC	Student responsibility/ freedom	0.69	0.79	0.23*
SO	Uncertain	0.78	0.87	0.29*
OS	Dissatisfied	0.83	0.91	0.28*
OD	Admonishing	0.84	0.89	0.36*
DO	Strict	0.62	0.60	0.14*

*$p < 0.001$

Associations between Interpersonal Teacher Behaviour and Student Outcomes

Table 2 reports the results for associations between students' perceptions of teacher/student interpersonal behaviour and students' attitudinal outcomes when the data were analysed using both simple and multiple correlations.

Table 2
Associations between QTI Scales and Students' Attitudinal Outcomes in terms of Simple Correlations (r) and Standardized Regression Coefficients (β).

QTI Scale	Strength of Environment – Outcome Association Attitude to Class r	β
Leadership	0.53**	0.24**
Helpful/friendly	0.64**	0.19*
Understanding	0.61**	0.13
Student responsibility/freedom	0.15**	0.07
Uncertain	-0.35**	0.07
Dissatisfied	-0.58**	-0.15*
Admonishing	-0.54**	-0.06
Strict	-0.40**	-0.18**
Multiple Correlation, R	0.71**	

*$p < 0.05$ **$p < 0.01$ n = 405

Whereas the simple correlation *(r)* describes the bivariate association between attitudinal outcome and a QTI scale, the standardized regression weight *(β)* characterises the association between attitudinal outcome and a particular QTI scale when all other QTI dimensions are controlled.

An examination of the simple correlation *(r)* figures in Table 2 indicates that there were eight significant relationships ($p<0.05$), out of eight possible, between student/teacher interactions and student attitudinal outcome; this is 20 times that expected by chance alone. An examination of the beta weights reveals four out of eight significant relationships ($p<0.05$), which is ten times that expected by chance alone.

The simple correlation *(r)* figures indicate statistically significant associations between the students' attitude to class and all QTI scales. The beta weights show that some of these associations retain their significance in a more conservative test with all other QTI scales controlled. In classes where the students perceived greater leadership and helpful/friendly behaviours in their teachers, there was a more favorable attitude towards the class. The converse was true when the teacher was perceived as strict and dissatisfied.

HOW TEACHERS CAN USE THE QTI AS A SEED FOR CHANGE

A number of mathematics teachers have used the QTI as a basis for self-reflection. The process begins with the teacher completing the two teacher versions of the QTI which ask the teacher to rate how they see themselves and how they see their ideal teacher.

A number of mathematics teachers have participated with the authors in research and professional development using the QTI. These teachers are provided with a report that provides the results from using the QTI in their classrooms. The report begins with a brief description of the model for interpersonal teacher behaviour, on which the QTI is based and includes three sets of data representing the three versions of the QTI. Namely, the teacher actual, teacher ideal, and mean student actual perceptions of the classroom teacher-student interpersonal behaviour.

Mathematics teachers using the QTI have reported that they found that the administration of the questionnaire takes little time and that the instructions to participants are quite clear. Some students reported difficulties with understanding words such as "lenient" or "sarcastic", though no problems were encountered when these were explained to participants.

After having completed the questionnaire and having had time to read the QTI report supplied to them, mathematics teachers reported that the results had stimulated them to reflect on their own teaching. The results of the QTI led one teacher to comment on verbal communication in her classroom. Based upon her sector profile diagrams, she concluded that she had become more aware of the students' needs for clear communication. This subsequently became a focus for her in improving her classroom environment and her teaching.

When mathematics teachers were asked if the questionnaire had caused students to work towards a better achievement in their classes, they suggested that students had thought about some of the issues raised by the QTI. However, they felt that more time and further testing would be required to measure any changes in student achievement.

When teachers were asked if they agreed with the results for their classrooms, the findings were revealing. It was reported that, though teachers agreed with the results, they raised further questions relating to their individual teaching practice. For example, the dimension of Helping/Friendly on the QTI produced a surprise for one teacher in that students' perceived a lower level of teacher helpful/ friendly behaviour than did the teacher. This suggested to the teacher that the students either needed more help than the teacher was able to give, or perhaps that the students really "lapped up" the nurturing and wanted more.

Some teachers reported that students often saw them as being more confident and better leaders than they perceived themselves to be. Other teachers suggested that it would be useful to respond to the QTI again after some time had elapsed so that any trends and changes in teacher-student interpersonal behaviour could be monitored.

CONCLUSIONS

This study confirmed the reliability and validity of the QTI when used in mathematics classes. Generally, the dimensions of the QTI were found to be significantly associated with student attitude scores. In particular, students' attitude scores were higher in classrooms in which students perceived greater leadership and helpful/friendly behaviours in their teachers. If mathematics teachers want to promote favorable student attitudes to their class, they should ensure the presence of these interpersonal behaviours.

The three versions of the QTI allow mathematics teachers to obtain their students' perceptions of their interpersonal behaviour, their own perceptions and the behaviour that they consider to be ideal. This valuable information then can be used as a basis for self-reflection by teachers on their teaching performance. Based on this information, teachers might decide to change the way they behave in an attempt to create a more desirable classroom environment.

Sector profiles could be used when considering staff development activities as they provide individual mathematics teachers with information about their actual and preferred classroom environments. This information can be used to identify areas for personal development in specific classroom environments. The sector diagrams also could be used as a basis for discussion of teaching behaviours. For example, mathematics teachers wanting to improve their leadership behaviours could organise professional development activities accordingly.

Mathematics teachers can make use of the QTI to monitor students' views of their classes, investigate the impact that different interpersonal behaviours have on student outcomes, and provide a basis for guiding systematic attempts to improve this aspect of their teaching. Furthermore, the QTI could be used in assessing changes that result from the introduction of new curricula or teaching methods, and in checking whether a mathematics teacher's interpersonal behaviour is seen differently by students of different genders, abilities or ethnic backgrounds.

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BELIEFS AND THEIR WARRANTS IN MATHEMATICS LEARNING

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Abstract

Development of mathematical beliefs, and the warrants for their justification is at the heart of teaching and learning mathematics. How you justify *what you assert or assent to is important in all learning, but it is particularly important in mathematics where justification in the form of 'proof' is part of the substance of the discipline. My investigation employs a philosophical method, (rather than, say, a psychological or sociological one) and mostly draws from a philosophical literature base. One consequence of this orientation is that I use the term 'belief' in its epistemic rather than its attitudinal sense. To exemplify this: 'I believe $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges' is epistemic and 'I believe $\sum_{n=1}^{\infty}\frac{1}{n}$ is difficult to comprehend' is attitudinal. I am interested in different justifications for mathematical statements as students learn a new topic.*

1. Introduction

The question of warrants for mathematical beliefs, that is to say, beliefs about mathematical propositions, is relevant and vital for teachers and students. When a student learns a new topic, he or she does not swallow knowledge as a pill, but, often tentatively, assents to, then, perhaps asserts, propositions about this new topic. Even at the weakest level of just assenting to, for example, '$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges' involves holding a belief . I am interested in the types of justifications, or 'warrants', for such beliefs and how these change.

A further example of the sort of issue in question has been researched by Lee (1994). She found that the same epistemic subjects assented to the mathematical proposition: "0.999...=1" after attending a lecture that included this result, but when interviewed subsequently, while recalling the authoritative answer, indicated their lack of conviction through such phrases as

> " 'it is infinitely close but not equal to', 'there was a page and a half of arguments to show they were exactly equal, but still ..part of me said "no they're not"' and '[yes]...but, I can't picture it'" (Lee 1994, p 131).

This range of type of assent - even from the same person - on the issue of limits will be familiar to any teacher of this topic, and points to the issue of justifications for belief.

For, the warrant for the belief that the epistemic subject employs, (implicitly or explicitly), is crucial in their mathematical knowledge development.

2. Method and methodology.

Methodology

Can academic philosophy provide mathematics educators with insights? I think so. My aim, here, is not to present abstruse parts of academic philosophy with a tenuous connection to education, but to explore concepts that can contribute to student and teacher sensitivity and awareness. With this rationale, then, I have worked through parts of the philosophical literature, looking out for writings that I judge fit this brief. The selection I study closely, is, of course, determined by my interests, here the epistemic notion of 'belief', and by my chance and deliberate encounters. So my 'contribution' is not to the philosophical theories, but to the application of existing theories as they are applied to issues concerning teaching and learning mathematics and to the critque of existing theories by reference to practice.

The particularity of mathematics - beliefs about mathematics, rather than belief in general - is also central to what I am doing, which is why I consider actual topics in the curriculum and the specific issues those topics raise concerning belief.

Method

Using topics in the mathematics curriculum that I have taught, I show that there are philosophical issues about belief embedded therein. I offer a perspective on these issues which is informed by philosophical writings.

3. Knowledge and belief: the move to justifications.

The 'person in the street' would probably say that the job of a mathematics teacher was to teach *knowledge* of mathematics. Why, then, talk about 'beliefs' rather than the real aim: knowledge?

I'll answer this rhetorical question on two levels. Firstly and simply, as exemplified above, there is a practical problem of facilitating our students taking on mathematical beliefs, and transforming these into knowledge. This transformation is rarely within a teacher's ken, so it would be useful to concentrate on 'beliefs'. Secondly, as Plato showed us in the Theaetetus, a useful definition of 'knowledge' is elusive. Knowledge is more than the sum of its constituents, and every component of any putative definition itself requires definition:

> *Socrates* And it is utterly silly, when we are looking for a definition of knowledge, to say it is right opinion with knowledge, whether of difference or of anything else whatsoever. So neither perception, Theaetetus, nor true opinion, nor reason or explanation combined with true opinion could be knowledge. (trans. Fowler, 1921, p255)

The last phrase of the quotation above can be paraphrased as: 'knowledge is justified true belief', which, although it had some promise as a 'definition' of knowledge, under closer inspection prompts us to ask questions like "what is 'justification'?" and "how is 'truth' secured?" In this way, we can get caught, either in circular definings, or in an infinite regression of nested definitions. Nevertheless, this aphorism can help us get our bearings, and, ironically, serve as a good starting place for investigation, for it draws attention to those essential concepts 'justification' and 'truth'. Not only is justification prerequisite for any knowledge claim, but it is through justification that tentative beliefs, like 'I think, the limit is zero' can be strengthened by reasoning, 'the limit is zero, because J', where J stands for some warrant, (about which more below). Knowledge, then, at least requires beliefs that are justified. I am saying that it is the quality of those *justifications* which is the key issue in moving those beliefs towards knowledge. I shall not deal with the meta-epistemological issues of the adequacy of justificatory standards, but refer the reader to Moser, 1993, (esp. pp 60 -105). Here, I focus on learners' justifications for their mathematical beliefs.

In standard Western philosophy texts (e.g. Quine and Ullian, 1970) questions about the nature of knowledge, are quickly brought into more manageable form by declaring that we should focus investigation on propositional knowledge. This means that the content of this knowledge are propositions to which truth values can be associated and thus analysed. So breathing in a certain fragrance might in some sense constitute knowing, but how to appraise this 'knowing olfactory being'? The Western philosopher's trick is to form a proposition from the experience that can considered true or false: 'I am smelling heather honey'. There may be some loss, but the gain is manageability. For the secondary school mathematics teacher, much of the curriculum, about which the students are to have warranted beliefs, can be expressed in propositional form: 'there is a limit to this sequence of numbers', the sum of the angles of a planar triangle is π radians', 'the 4th decimal place of π is 5', etc.

Much of the history of epistemology, from Plato to the present day, is concerned with responses to skepticism. Viewed negatively, skepticism can be characterised as a knee-jerk reaction ('we can't know we know'), but viewed positively, skepticism's challenge forces us to focus on the justifications for our assertions. The contemporary epistemologist Alvin Goldman observes "It may be possible to have rational beliefs even if knowledge is unobtainable" (1986 p 40), which reinforces my rationale for concentrating on belief. His conception of justified belief "depends critically on the use of sufficiently reliable cognitive processes" (ibid. p 39). This notion of a 'reliable cognitive process' is clearly relevant for educational interests. So Goldman provides a theoretical basis for linking the philosophical notion of justification with the practical concerns of those developing mathematical beliefs (either teachers or students). Exactly how these 'reliable processes' are recognised, exploited and communicated is not within his thesis; the specifics for mathematics learning are surely to be located in the domain of mathematics education. Goldman's contribution is significant because, like teachers, he recognises that the justification is entwined with the justifier.

Justification, then, is the key issue in this epistemological analysis of mathematical belief. Next, I want to distinguish various different types of warrant for beliefs that have a relevance for the mathematics classroom.

4. Types of belief.

Beginning a new topic involves forming a belief. Progress towards knowledge is made when those beliefs are justified. Let's take an example to flesh out the meaning of this statement using trigonometry, which is a new topic for secondary school pupils. Of course, the children will have beliefs already formed about triangles and angles and other concepts that are involved in trigonometry, but in this new topic new beliefs are to be formed.

As an exercise, list some of the beliefs that are pertinent to early trigonometry, to see how they might be classified. Here's my list all of which concern right angled triangles:

1 - the side opposite the right angle is called the hypotenuse

2 - the hypotenuse is the longest side

3 - if you mark one of the smaller angles, call it α, then α is formed by the hypotenuse and another edge of the triangle

4 - this other edge is called the adjacent side to α

5 - the side of the triangle that does not help form α is 'opposite' this angle

6 - if the angles of the triangle are fixed, then the ratios of pairs of sides of the triangle are the same no matter what the size of the triangle

7 - these ratios have special names, for example, $\tan \alpha$=opposite/adjacent

8 - $\sin \alpha$ and $\cos \alpha$ are always less than 1 ($0 < \alpha <$ a quarter turn)

9 - if α and the hypotenuse are given as actual numbers, you can work out (inter alia) the opposite side's length using the formula: opposite = hypotenuse $\times \sin \alpha$

10- $\tan \alpha = \sin \alpha / \cos \alpha$

I can discern different types of proposition here, which I shall call:

(a) information (1,4,5,7): the belief comes from an authority, the warrant is authoritative, often social (being the practice of the community).

(b) consequence (2,3,6,8,10): the belief comes from a deduction, the warrant is logical

(c) perceived (2,6,8): the belief comes from sense-data, the warrant is perceptual-empirical

(d) operational (3,9): the belief is related to action, the warrant is procedural (akin to Vergnaud's 'théorème-en-acte' (Vergnaud, 1981))

Clearly to the sophisticated, 2, 6 and 8 can be 'consequential', I have placed them in the 'perceptual' category too as this could be the belief forming mechanism.

There are different sorts of warrants, because there are different ways that sentient beings proceed in their thinking. For example, you may use a logical warrant to justify your assertion, or you may appeal to a perceptual one; in the example above, you may wish to justify 2 by noting that the hypotenuse is opposite the largest angle, a logical warrant, or by measuring many hypotenuses and observing the data (easily done with Cabri or similar dynamic geometry package), a perceptual warrant.

At this stage I want to mark the distinction between the notion of a 'cause of a belief' and that of a 'justification for a belief'. It is my contention, that is compatible with Goldman's ideas about the intimacy of cognition and epistemology, that these notions are not neatly separable. As a trivial example, I might justify a belief by some reference to an authority whence the belief originally came. Nevertheless, I'd say that *belief causes* were principally cognitive and *belief justifications* were principally conceptual. Proposition 6 in my previous list emphasises the distinction: the *cause* of the belief might have been an empirical investigation given several similar right triangles (as the activity in the text book SMP 11-16, Y2 p 22), the *justification* of the belief does not *have* to be given by a corresponding perceptual-empirical warrant - 'I measured them and this is what I got' - but could be justified, for example, 'procedurally' by subsequent enlargements or deductively, by recourse to similarity.

Furthermore, I observe that because of the 'proceptual' (Tall, 1991, "a procept [is] a process which is symbolized by the same name as the product". p 254) nature of mathematical discourse, a conceptual justification at one level can serve as a cognitive cause of the belief at another. To exemplify this, consider the mathematical proposition "1/7, expressed as a decimal, repeats." Suppose that Brian believes this proposition because his calculator displays $\boxed{0.142857142}$ in response to the key sequence $\boxed{7}$ $\boxed{x^{-1}}$ $\boxed{enter}$; the cause of the belief is his trust in the machine (an authoritative warrant) together with what he reads on its display (a perceptual warrant). If Brian were able to do the division $7\overline{)1.00000000}^{\,0.14285714}$, and recognise that the sequence of remainders from the divisions repeated, and would repeat indefinitely because of the very process of executing the division, I would say that this would be a justification of the belief (using a procedural warrant) - and a mathematical one at that! Indeed, once such a mathematical warrant has been used as justification, this entactive competency makes it hard to appreciate the tentative nature of the previously used belief warrants. Furthermore, to elucidate my point above, the explicit division justification feeds (cognitive) causes for beliefs at the level of 'if I work out 1/n as a decimal by dividing I can only get up to n-1 remainders before it starts repeating' etc. So the belief formation process is entwined with the process of justifying beliefs, and hence the justificatory warrants.

What then are the mathematical warrants within the belief formation process? Even for the professional mathematician, Lakatos' analysis (1976) indicates that the deductive does not constitute the only warrant, although it may be the only one the public is privy to. For the learner, the issues are two-fold. First, how are beliefs about mathematical propositions formed? What warrant justifies the tentative belief? Second, as warrants are not of equal value in mathematical justification, what warrants the warrant? How do learners shift to see justification of a mathematical belief in terms of a deductive or procedural warrant rather than an authoritative or perceptual one?

5. Gettier problems.

Following up the aphorism 'knowledge is justified true belief', I want to raise the issue of the applicability of a warrant for justifying a belief. I suggest that as learners we may well form beliefs in certain ways, and, as discussed above, the cause of a belief may be closely tied to its justification. A little more subtle are 'Gettier-problems', (Gettier, 1963). A Gettier-type problem is of the following type: a true belief has a justifiable warrant, but the warrant is misapplied. Examples of Gettier problems in the literature, for some reason, often involve automobile ownership! For example: Fred believes that Jane has a Morris car. This belief is true, Jane does have a Morris car. Fred's belief that Jane has a Morris is justified by several sightings of Jane driving a certain Morris. However, that Morris belongs to Jane's mother. So Fred's belief is true and justified, but the justification is not justification for the actual original statement's truth: 'Jane has a Morris car'. The point is that most people would be uncomfortable asserting that Fred had knowledge of this item, even though he had justified true belief.

I think that awareness of these sorts of problem might be helpful in diagnosing non-mathematical justifications. For example, working analogously: Andy believes that the limit of the sequence '1/n' is zero. This belief is true, the limit of '1/n' is zero. This belief is justified, he thinks, by 'several sightings' of many terms of the sequence. The justification, like the observations above, is not foolproof. Just as the empirical observation of seeing someone drive around in a certain car does not imply ownership, so seeing a sequence get closer and closer to zero does not mean that it does have a limit and that that limit is zero.

So there are two sorts of issue concerned with beliefs and limits of sequences. Firstly, the student's easily taken-on belief that the limit is the number that (approximately) pops out after a large number of terms of the sequence have been calculated, but they still do not believe that the limit actually exists; it is never reached. Secondly, the Gettier type problem: the student might attribute knowledge to his belief that the limit of '1/n' is zero, because of his warrant: 'I have worked it out to thousands of places several times'. This is a case where we might be uncomfortable to attribute knowledge to the student, as his reasonable warrant for empirical propositions, is not the best one with which to *justify* a mathematical proposition.

6. Justification for beliefs

The discussion of beliefs, then, moves to discussion on warrants for belief. It is not the truth or falsity of the proposition that is being investigated, (as we have to do if knowledge claims are being made), but the various ways beliefs are justified, and, hence, I suggest, take hold within us.

What warrants are used in mathematics teaching and learning? I have approached this question, not empirically, but conceptually, drawing on my teaching experience. I have suggested, above, a preliminary sorting of the warrants that come from a set of beliefs that are typical for beginning school trigonometry into 'authoritative', 'logical', 'perceptual', and 'procedural'. I make no claims to be exhaustive, and I have already indicated that different warrants might be employed by different people to believe the same proposition. Although 'authoritative' is but one of the warrants, and an inevitable one if school students are to be inducted into a 'community of practice', I suspect that many kinds of mathematical beliefs held by students are actually only justifiable via this warrant. There is a negative connotation to the word 'only' in the previous sentence because, I assert, that, despite the importance in belief formation, an 'authoritative' warrant is not an 'essentially mathematical' one. Here is a paradox then: we can't do without a 'community of practice' to support mathematical learning, but the warrants for belief in the mathematical propositions held by that community cannot just be the 'authority' of the community itself; to be mathematical belief, the justification must come from other warrants.

In mathematics learning, progress is made when students to shift from using authoritative or perceptual-empirical warrants to using logical or procedural ones. To follow on with the idea of tracking warrants for beliefs: we can envisage the situation where the teacher, having taught the students, might attribute her student's warrant for belief as, say, deductive, yet the student's warrant is authoritative.

The standard epistemological theories about how knowledge is acquired categorise routes to knowledge as 'rationalist' or 'empiricist' (see Grayling, 1996, p 39). This categorisation is exemplified by saying that mathematics and logic are paradigmatic reason-based knowledge, acquired by the former route, and that natural science is paradigmatic empirically-based knowledge, acquired by the latter. How useful are such categorisations for mathematics teaching and learning? Clearly, when any sort of cognitive development is to be considered, there not only cannot be a clean divide between the route to knowledge characterised by 'reason' and that characterised by '(sense)-experience', but also, the dimensions of authority and of automaticity are important too. All of these routes, in some sense, can, in Goldman's terms be 'reliable cognitive processes', but as Moser points out "Notions of justification ...admit of evaluation, at least relative to...certain conceptual purposes" (Moser, 1993, p 13); the purpose in this case is, of course, developing mathematical beliefs. When it comes to practice, epistemic modality - what is believed, known, or taken to be true - in mathematics learning is significant.

Unless awareness of type of warrant is brought to the fore, students' beliefs about mathematical propositions are likely to remain at the level of 'Do I assent to this or not?', rather than, 'If I am to assent to this proposition, what is its warrant?' So for the student the question to be asked is: what is the sort of justification that has helped, or might help, me believe this proposition? For the teacher, the question is, through what warrant(s) am I expecting the students to take on this belief? Clearly, there are important issues concerning which warrants for belief value or promote which qualities in the student.

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Pupils' Strategies And The Cartesian Method For Solving Problems: The Role Of Spreadsheets

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In this paper we focus on methodological aspects of the transition towards the algebraic realm. We specifically discuss issues from an Anglo/Mexican project which are related to the feasibility of switching pupils' informal strategies to algebraic methods of solving word problems by means of a didactic artifact that is a spreadsheet environment. An analytic tool adapted from the mathematical analysis and synthesis process is used to probe the nature of pupils' productions when solving word problems.

INTRODUCTION

Expressing the elements of a problem statement as an equation, and solving the equation to find the numerical value of the unknown is considered an algebraic method. Students' use of algebraic methods has been for many years one of the main goals of secondary school. Nevertheless, a number of studies indicate that pupils at this school level are more likely to use non-algebraic methods when solving word problems (Bednarz *et al.*, 1992; Lins, 1992). This has led in many cases to focus the research work on the analysis of pupils' strategies with the aim of probing the nature of these informal approaches.

Although it is necessary that students experience a detachment from their informal methods in order to acquire algebraic ones, results from one of the studies of the *Anglo/Mexican Spreadsheets Algebra Project* suggest that pupils' informal processes can be used as a basis to build up "more algebraic" methods of solving problems when working in a spreadsheet environment (Sutherland & Rojano, 1993; Rojano & Sutherland, 1992). This collaborative project was developed to help students to bridge the gap between arithmetic and algebraic thinking alongside two evolving lines: 1) basic algebra concepts and 2) problem solving methods .The main aims of this project were to:

- investigate the way in which pupils use a spreadsheet environment to represent and solve algebra problems relating this to their previous arithmetical experiences and their evolving use of a symbolic language.

- characterise pupils' problem-solving processes along the dimension arithmetic/algebraic as they evolve through working in a spreadsheet environment.

The project consisted of two phases. The first phase was carried out with two groups (one in Mexico and one in Britain) of eight pre-algebra pupils (aged 10 to 11 years). Whereas in the second phase we worked with two groups of eight 14-15 year olds (one in Mexico and one in Britain) who had had a history of being unsuccessful with school mathematics. All these pupils were involved in spreadsheet activities which focused on the notion of function and inverse function, equivalent algebraic expressions and the solution of algebra word problems. They used a spreadsheet cell to represent the unknown and then with the mouse or the arrow keys expressed algebraic relationships in terms of this cell. Dealing with the unknown, both in a symbolic and in a numeric way, allowed pupils to make a step in accepting the idea of operating with an unknown quantity, an idea that many secondary students find difficult (Filloy and Rojano, 1989) but which, in turn, constitutes the core of the Cartesian (algebraic) method for solving word problems.

Results of the Anglo/Mexican project emphasising the conceptual development of the pupils have been synthesised in previous papers (Sutherland & Rojano, 1993; Rojano & Sutherland, 1992, 1993 and 1994). In the present paper we focus on methodological aspects of the transition towards the algebraic realm. We specifically discuss issues related to the feasibility of switching pupils' informal strategies to algebraic methods of solving word problems by means of a didactic artifact that is a spreadsheet environment and applying an analytic tool derived from the *mathematical analysis and synthesis process* to interpret children's productions. We use examples from the study with 14-15 year olds.

THEORETICAL ELEMENTS FOR THE RESULTS ANALYSIS

Arithmetic Methods - Algebra Methods

When trying to characterise arithmetic as well as algebraic methods, there always exists the difficulty of referring at the same time to the sorts of problems that are being solved. Puig and Cerdán (1990) retake some of the questions posed by Kieran & Wagner (1989, page 226) related to the nature of problems and solving methods, such as a) *Are there word problems that are intrinsically algebraic rather than arithmetic.* b) *What makes a method of solving a word problem algebraic rather than arithmetic?* In an attempt to answer these questions, Puig and Cerdán develop an analysis of the **translation processes** of the problem statement into an arithmetic or algebraic expression. These authors use as tools of analysis two general methods: **the method of analysis and synthesis** and **the Cartesian method**. The latter is considered the algebraic method *par excellence* and explicitly involves a translation of the problem into the algebraic code, whereas the first one (the method of analysis and synthesis) leads to a translation process of an arithmetical nature, which consists of transforming the initial text of the problem into a new text in which the elements that intervene in more elementary translations are made explicit, in order to make explicit, as well, the way these elements are linked within the arithmetic expression that solves the problem (Puig and Cerdán, 1990, pp. 38-39). The intermediate texts produced in this process involve intermediate variables or unknowns called *the antecedents of the unknown* (Lakatos, 1978) and the idea is to produce only *given* in the final step of this sequence of transformations. This is the *analysis process*, and the inverse one (to perform the operations with the givens to find the unknown value) constitutes the *synthesis process*.

Puig and Cerdán use an *ad hoc* diagram to represent the *analysis process* (Botsmanova, 1972) which is illustrated in Figure 1.

The Problem
Four pieces of cloth of 50 m each will be used to make 20 suits which need 3 m of cloth each. The rest of the cloth will be used to make coats which need 4 m each. How many coats can be made?

The intermediate text

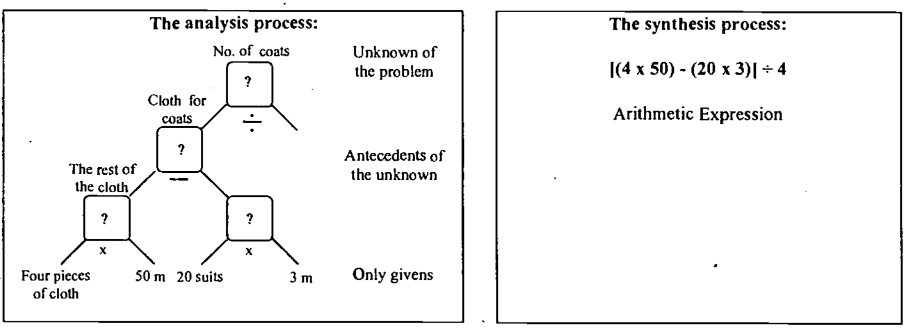
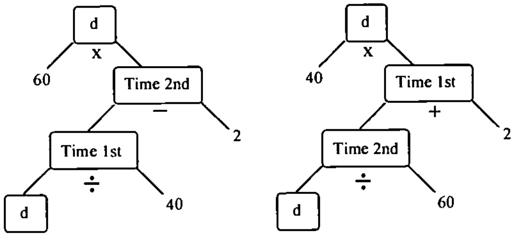


Figure 1

Puig and Cerdán (1990, pp. 40-42) give examples of problems that can not be reduced to an arithmetic expression via the *method of analysis and synthesis* but which when applying *the method of analysis* leads to an equation in terms of unknowns instead of an expression involving only givens. So, in these cases, the *synthesis process* is impossible. One of the examples is:

The Problem

A car departs from a point A bound to a point B with a uniform velocity of 40 Km/h. Two hours later, another car departs from A bound to B with a uniform velocity of 60 Km/h. What is the distance from each car to A?



$$60\left(\frac{d}{40}\right)-2 \qquad d = 40\left(\frac{d}{60}\right)+2$$

Figure 2

According to these authors, *in the limit of the analysis-synthesis the method becomes algebraic when the unknown of the problem is considered as a given, useful to determine the unknown itself, that is,* ***unknown and givens are treated in the same way.***

The former is a way of coping with the problem of trying to characterise word problems and solving methods as either arithmetic or algebraic and can be used to highlight phenomena observed in studies with pupils solving algebra word problems. The adaptation of the *analysis - synthesis* process carried out by these authors provides a tool of analysis that is used to interpret pupils' productions in the *Anglo/Mexican Spreadsheets Algebra Project*.

ARITHMETIC / ALGEBRAIC APPROACHES

Analysis of a problem of the pre-interview.

Children's strategies to solve algebra word problems give account of a solving approach which proceeds from the known to the unknown. This approach is in opposition to that of algebra in which working with unknown quantities is in the core of the method. *The Chocolates Problem* is a word problem that was presented to the students in the pre and in the post-interview. The relationships between the unknowns are explicitly given in this problem and it is considered of a high degree of difficulty because it involves three unknowns. For this reason, this problem was presented only to the 14-15 year old pupils (in Mexico and in the UK).

The Chocolates Problem

100 chocolates are distributed amongst three groups of children. The second group receives four times the number of chocolates as the first group. The third group receives ten chocolates more than the second group. How many chocolates does each group receive?

An algebraic solution to this problem leads to a set of equations such as the following:

$$\begin{aligned} y &= 4x \\ x &= y + 10 \\ x + y + z &= 100 \end{aligned}$$

where x is the number of chocolates of the first group; y the number of chocolates of the second group, and z the number of chocolates of the third group

Which gives the solution: $x = 10$; $y = 40$ and $z = 50$

Table 1 shows solutions and strategies used by **14-15 year olds** in the *Chocolates problem.*

	Pre-interview		Post - interview	
México	**Solution**	**Strat**	**Solution**	**Strat**
Giselle	No Soln.		10, 40, 50	SA (C&S)
Aida	20,33,47	W/P	10, 40, 50	SA (C&S)
Enrique	No soln.		10, 40, 50	SA (C&S)
Zazil-Ha	No soln.		10, 40, 50	SA (C&S)
Pilar	No soln.			
Alejandra	No soln.		10, 40, 50	SA (C)
Edgar	10, 40, 50	T/R	10, 40, 50	SA(M/C)
UK	**Solution**	**Strat.**	**Solution**	**Strat.**
Eloise	No soln.		10, 40, 50	SA
Sally	33, 33, 33	W/P	10, 40, 50	SA (C)
Carla	33, 132, 1	T&R	22, 132, 1320; 10, 40, 50	T&R SA(C)
Lucy	33, 3	WP	10, 40, 50	SA (C)
James	10, 40, 50	T&R	10, 40, 50	SA (C)
Lee	No soln.		10, 40, 50	T&R
Anthony	No soln.		15, 60, 25; 10, 40, 50	T&R-SA(C)
Dennis	No soln.		10, 40, 50	T&R

SA = Spreadshet/Algebraic; C = Computer; T&R = Trial & Refinement; E/P = Whole/Parts; S = Support

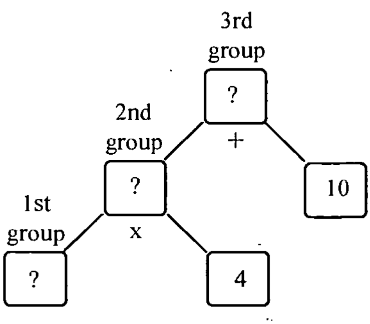
In the pre-interview only 2 out of 7 pupils approached this problem in Mexico and 4 out of eight pupils solved it in the UK. **Whole/parts** strategy (dividing 100 by 3) and **Trial & Refinement** strategies (assuming a numerical value for one of the unknowns and varying this value until conditions of the problem are fulfilled) were used in this problem; the latter led James (in UK) and Edgar (in Mexico) to the correct answer. The presence of the **whole/parts** strategy was most of the times accompanied by leaving aside the constraints of the relationships between the unknowns. The **whole/parts** strategy corresponds to proceeding from the known (the total amount of chocolates) to the unknown (number of chocolates for each group).

From the analysis of pre-interview outcomes, it can be concluded that students' informal strategies in the group of the 14-15 year olds, involved in most of the cases proceeding from the known to the unknown, which can be interpreted as a preference to deal with known quantities. This is a clear manifestation of a non-algebraic way of thinking. Such informal strategies can be described in terms of the **analysis and synthesis method**.

Proceeding from the unknown to the known

Concerning the **trial & refinement** approach, it is noticeable that excepting Carla in the UK, the rest of the children who applied this strategy in the pre-interview could reach a correct answer in the **Chocolates Problem** (James in the UK and Edgar in Mexico). The way in which these children worked out this problem, assigning one of the unknowns a "provisional" numerical value (for example, the number of chocolates of the 1st group) suggests the following **analysis** process diagram:

Trial and Refinement Strategy for solving the Chocolate Problem



Synthetic expression:

((X x 4)+10)

The total amount is incorporated:

((X x 4)+10)) + (X x 4) + X = 100

The latter is clearly an equation whose solving process involves dealing with one of the unknown quantities (**X**) from the very beginning. What some children do is to assign a numerical value (for example, 33) to the 1st group and (in the diagram) this is a way to reduce the antecedents of one of the unknowns to numeric al data, and the corresponding synthetic expression would be: ((33 x 4)+10))+(33+4)+33.

The actual process that the pupils using this approach carried out was to treat the unknown as a given and then find out the second and third unknowns proceeding upwards (according to the diagram) by means of establishing the relationships between the unknowns and using the total amount of chocolates as verifier of their testing out procedure. It is necessary for being successful with this strategy to be aware of and keep in mind the complete set of relationships present in the problem and to explicitly use a verifier of the attempts undertaken. Indeed, most of the pupils using this strategy didn't leave aside any of the problem's constrains. This case recalls the example developed by Puig and Cerdán (1990) which illustrates the idea that some problems can not be reduced to an arithmetical expression applying the

analysis-synthesis method, but instead, this method leads in such cases to an equation and then can be seen as **Cartesian.**

We want to stress the idea that when pupils, who are reluctant or unable to use algebraic tools to approach an "algebraic" word problem, bring into play informal (T&R) strategies, which have one aspect in common with the algebraic (Cartesian) method, that of **considering the unknown as a given, useful to determine the unknown itself**. In other words, **the unknown becomes one of the antecedents in the analysis-synthesis process.**

DEVELOPMENTS IN APPROACH.

Table 1 shows children's solutions and strategies when solving the *Chocolates Problem* in the post-interview. It can be noticed from this table that the whole/parts strategy is still present in the post-interview. A combination of **Whole/parts** with a **spreadsheet-algebraic** (SA) method appears in the *Chocolates problem* in the post-interview. In most of these cases, the whole/parts approach was used to make a "first estimate" of one of the unknowns of the problem before carrying out the variation of the unknown when searching for the value of the other unknowns.

The combined strategy (W/P and T&R) can be attributed to the spreadsheet method that makes it possible to conciliate a non-algebraic approach (W/P) which proceeds from the known to the unknown, with a "more" algebraic approach, which proceeds from the unknown to the known.

PRE-ALGEBRA STRATEGIES, THE SPREADSHEETS METHOD AND THE CARTESIAN METHOD: FINAL REMARKS

It is important to notice that whereas in the cartesian method the part of putting in equation corresponds to the action of finding out two equivalent algebraic expressions for the problem's state of affairs, and then linking these expressions through the equality sign, in the spreadsheets method used in the experimental work, all the partial (or elementary) relationships between givens and unknowns and between unknowns are symbolised in separate but related cells and all these relationships are finally synthesised in one expression which serves as control of the variation of one of the unknowns. From this point of view, besides proceeding from the unknown to the known, these two methods don't seem to have more in common.
Nevertheless, if we analyse pupils' productions when working with the spreadsheet method, in the light of the **analysis-synthesis** procedure diagrams, it is possible to observe that at least in the **analysis** part, both methods are alike. This is because, in the cartesian method, the production of the two equivalent expressions is preceeded by the production of partial (elementary) relationships which incorporate the

unknowns to the procedure at the same level of the givens, in order to get all the necessary antecedents to determine the unknown values. Just what children do when elaborating the spreadsheets formulas involving unknown quantities!

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FALLIBILISM AND THE ZONE OF CONJECTURAL NEUTRALITY

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A fallibilist approach to teaching and learning mathematics depends on classroom dialogue in which students' conjectures are articulated and tested. Whilst the examination and possible refutation of such beliefs is a neutral process from an intellectual point of view, it is difficult for many students to approach it with emotional detachment. The notion that the conjecture is on trial, and not the student, is a subtle one, and students can feel personally threatened when their conjectures are debated and tested. This paper examines a construct named the Zone of Conjectural Neutrality (ZCN), a neutral space in which students' ideas are tested. Proposals are made for some methods whereby fallibilistically oriented teachers might locate students' conjectures in the ZCN.

FALLIBILISM

An absolutist view of mathematics would hold that mathematical truths are sharp and certain, and in some way represent objective knowledge. Indeed, in this view, mathematics stands above and apart from empirical science in its purity and freedom from experimental error. Science can only offer 'theories', whereas the products of mathematical thought are objectively 'true'.

> 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, *because it is so*, because mathematical reality is built that way. (Hardy, 1940, p. 130)

Over the last century, absolutism has been worked out in two major forms, logicism and formalism. The logicism of Frege and Russell attempted to reduce all mathematics to pure logic. Hilbert took the formalist view that mathematics is more than pure logic, but is capable of being axiomatised. Both forms have been questioned from within mathematical logic; the deductive arguments which terminate in mathematical theorems must begin from a baseline of axioms, which are plausible products of observation or intuition. Any claim to absolute truth must then be suspect, since the very foundation is beyond the reach of demonstration.

A different 'fallibilist' critique of absolutism is presented by Imre Lakatos in *Proofs and Refutations* (1976). Central to Lakatos' critique is the failure of formalism to account for the growth of mathematical thought, either in peoples (phylogenesis) or in individuals (ontogenesis). Lakatos offers an alternative view of mathematics as the product of human mathematical activity and inter-personal dialogue.

> [...] informal, quasi-empirical mathematics does not grow through a monotonous increase in the number of indubitably established theorems, but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations. (*op. cit.* p. 5)

> Instead of presenting symbols and rules of combination, [Lakatos] presents human beings, a teacher and his students [...] he presents mathematics growing from a problem and a conjecture [...] doubt giving way to certainty and then to renewed doubt. (Davis and Hersh, 1980, pp. 346-7)

Lakatos' account of mathematical growth is set against the background of Polya's mathematical heuristic and Popper's critical philosophy of science. The term 'quasi-empirical' mathematics refers to the observation that conjectures are the inductive outcome of consideration of 'data' collected in mathematical activity. An asymptotic refinement of definitions, theorems and proofs, argues Lakatos, is the outcome of human dialectic, acted out in the histories of cultures, and again (though not necessarily in the same way) in the classroom. In this fallibilist view, mathematics is a relative and subjective form of knowledge, perpetually open to revision.

Sandy Dawson has explored the profound implications of Lakatos' quasi-empiricist philosophy for the teaching of mathematics.

> It was from ideas contained in Lakatos' articles and book that an alternative way of working in mathematics classrooms developed. [...] Lakatos claimed that the creation of mathematics comes about as the result of a process [...] in which a conjecture is created, tested and proved, or refuted and modified, or rejected outright. A classroom designed for pupils to operate in a fallibilistic fashion would provide pupils with a problem about which they could make conjectures as to its solution. [...] Opportunities to test and examine critically each conjecture must also be provided. (Dawson, 1991, p. 197)

With such an epistemological climate in mind, John Mason has described the qualities of what he calls a 'conjecturing atmosphere', in which

> every utterance is treated as a *modifiable conjecture*! (Mason, 1988, p. 9)

One of the features of a classroom "designed for pupils to operate in a fallibilistic fashion" is the exploration of problems - what have been called 'investigations' in the UK - with the aim of arriving at plausible mathematical insights as a result. It is then essential that such speculations ('guesses' even, according to Polya and Lakatos) be articulated by students, so that they are available for critical examination.

AFFECTIVE CONSIDERATIONS

A fallibilist view of mathematics has implications for classroom conduct. In mathematics talk there is an affective subtext just below the surface of the propositional text. It is there because mathematics is a human activity: the participants care about the mathematics, but they also care about themselves, their feelings and those of their partners in conversation. The possibility of active construction of knowledge from reflection on experience is at the heart of a constructivist view of learning. Such a view puts an onus on the teacher to try to understand the form, content and robustness of that knowledge, as an observer of and participant in pupils' mathematical activity - an "accultured" participant, moreover, who "can legitimise certain aspects of their mathematical activity and sanction others" (Cobb *et al.*, 1992, p.-102). At the same time, a fallibilist view of mathematical knowledge requires that the teacher is not a uniquely privileged arbiter of pupils' conjectures, but rather one who urges the pupils themselves to take a reasoned position in the acceptance or refutation of such conjectures.

Students' self-constructed beliefs may be fragile; in particular, any inductive conjecture would be expected to be tentative. The burden of the affective baggage associated with mathematics in school then necessitates that the pupil articulate the belief whilst distancing her/himself from full commitment to it. That is to say, they must convey their propositional attitude to the substance of their assertion. The rich variety, in some cases the subtlety, of hedges and modal forms deployed by pupils for this purpose is discussed by Rowland (1995, 1996a), and is evidence of this affect-oriented dimension of pupils' communicative competence. These markers of modality are linguistic pointers to uncertainty and attendant cognitive vulnerability. The teacher's subtle task at such moments is to promote the trial and possibly the rejection or modification of such assertions as regards their truth, whilst minimising the personal sense of threat to the students who utter them. In the next section, we see how one teacher tries to achieve this.

THE ZONE OF CONJECTURAL NEUTRALITY

In the following fragment of transcript, a primary [elementary] school teacher, Hazel, is talking with two ten-year-old girls in her class, Faye and Donna. The conversation is, in effect, an exploration of the difference between b^2 and ac, where a, b, c are consecutive terms of an arithmetic sequence. Initially the girls considered the case when the common difference is 1.

Early in the conversation Faye observes a difference of 1 between 10x12 and 11^2.

7 Faye: There's only one, umm ten multiplied by twelve is a hundred and twenty. Eleven multiplied by eleven is a hundred and twenty-one

8 Hazel: Okay

9 Faye: So there's one number difference

Hazel highlights the observation, and asks:

10 Hazel: One number difference ... do you think that will always happen when we do this ... ?

Faye readily agrees, but Hazel seems to want to give the girls more of an option to disagree.

12 Hazel: What makes you think that? Just 'cos I asked it ... or ...?

Donna gives hedged agreement [14].

14 Donna: I think so.

Hazel encourages the children to try out two more examples with three consecutive integers. They obtain a difference of 1 in each case and Faye affirms her belief that, as Hazel puts it [10, 26], "that will always happen".

26 Hazel: Do you think that will always happen then?

27 Faye: Yes.

28 Hazel: How can you say for certain 'cos you've only tried out three examples?

When pressed by Hazel to account for her belief [33], Faye attempts a start, but immediately backs off [34]:

33 Hazel: ... why do you think that for certain?

34 Faye: Because ... well, I don't know for certain but I think ... 'cos the numbers that we've done are quite close to the first ...

Faye's "well" [34] suggests that she had foreseen the inadequacy of her explanation, and cautions that this is not the whole story (Wierzbicka, 1976, p. 362).

Donna offers a brief diversion:

35 Donna: I don't think it will happen if you do like eleven, fourteen, twenty-two.

Faye brings the discussion back on course with a 'crucial experiment' (Balacheff, 1988) with the three consecutive integers 110, 111, 112 [60]:

51 Faye: I still get one number different.

52 Hazel: So that ... so do you ... will it always work d'you think?

53 Faye: Yeah ... I think.

54 Hazel: How can you be sure?

55 Donna: Umm

56 Faye: [laughing] Well ...

57 Hazel: Are you sure?

58 Faye: Well not really, but ...

59 Donna: Quite yeah.

60 Faye: I think so. Yeah quite sure. Because it has worked because we've done ten, eleven ... Well I've done ten, eleven, twelve, nine, ten, eleven which are quite similar and then I've jumped to, um, um ... a hundred and ten, a hundred and eleven, and a hundred and twelve. It's quite a big difference. So yeah?

61 Donna: Yeah so do I.

Hazel probes the extent of the pupils' confidence in the generalisation, and the basis of their belief, and is reluctant [52] to influence their commitment to it on the mere grounds of her own authority. Faye's intellectual honesty is very evident here. Her crucial experiment [60] provides another (presumably weighty) confirming instance of the generalisation [51] yet her assent to it is hedged, reluctant [53, 58]. Perhaps Donna's hedged, but accepting, stance [59] finally prompts her to re-examine the evidence-in-hand [60] and affirm her own conviction.

This transcript is evidence of Hazel's commitment to quasi-empirical enquiry with her pupils, and of her persistence as she presides, as an accultured participant, over the testing of the conjecture that $b^2-ac=1$, probing the girls' conviction that it will hold for *all* consecutive integers a, b, c. I have given the name 'zone of conjectural neutrality' or ZCN (Rowland, 1995, pp. 350) to the space between conviction and assertion. One senses that Hazel has identified and explored a ZCN with Faye, who understands that it is the conjecture ('it always works') which is on trial. She is free to believe or to doubt.

If it can be accepted that truth and falsity may be decided in the ZCN, then a person may articulate a proposition without necessarily being committed to its truth; for the proposition is on trial, not the person. Whilst mathematical conjectures are formed as private, cognitive (perhaps inductive) acts, they are validated in public polemic of some kind. Moreover, the learner ideally participates in the discourse since, as Balacheff submits (1990, p. 259), children must take responsibility for the validity of their own solutions "in order to allow the construction of meaning". At the same time, a conjecture is not fixed and immutable, but modifiable. This is the quasi-empiricist, fallibilistic approach to teaching and learning.

> A teacher who is functioning fallibilistically [...] establishes a classroom climate in which an atmosphere of guessing and testing prevails, where the guesses are subjected to severe testing *on a cognitive rather than an affective level* [...] where knowledge is treated as being provisional. Because of the provisional nature of knowledge, pupils are encouraged to confront the mathematics, their peer group and, where appropriate mathematically, even their teacher. (Dawson, 1991, p. 197, emphasis added)

Not only is uncertainty an intellectually tenable position, but the assertion of uncertainty draws the attention of the teacher to the existence of a ZCN, and thus opens up the possibility that s/he might provide for the student some cognitive 'scaffolding' (Wood *et al.*, 1976) to support, and perhaps transform that state.

INTERACTIONAL STRATEGIES

The ZCN can usefully be viewed as a (metaphysical) space where a conjecture can be located whilst it is tested (and possibly refuted or modified). The issue central to this notion of ZCN is summarised in the question "Where are pupils' conjectures located? Who is responsible for them?" The default position must be that a conjecture belongs to the one who utters it. If the conjecture is asserted with conviction (better still, if it is subsequently validated as true), then this is not an affective problem. But if a conjecture is offered tentatively, then it is better that it be located somewhere neutral before it is tested, in order that there be some real prospect of Dawson's promise (*ibid.*) of "testing on a cognitive rather than an affective level", in defiance of the cultural norm that the pupil is judged to be 'right' or 'wrong' rather than the 'answer' 'true' or 'false'; that it is s/he who is on trial, not her/his beliefs.

This is at the heart of pupils' communicative competence in the use of various kinds of vague language in the assertion of conjectures, especially in the use of hedges such as *maybe*, *I think*, *about*, *basically*, *quite*.

34 Faye: Because ... well, I don't know for certain but I think ... 'cos the numbers that we've done are quite close to the first ...

Such forms of linguistic 'shielding' (Rowland, 1995) have the effect of reifying the ZCN and locating the conjecture in it, thus distancing the speaker from the assertion that he or she makes. A 'plausibility shield' such as *I think*, *maybe*, or *perhaps* does this in a very direct way, because the marker of propositional attitude lies outside the statement that follows it. Epistemic 'approximators' (such as *approximately* or *about*) are more subtle: they do not require the speaker to disown her/his conjecture, but they do make it "almost unfalsifiable" (Sadock, 1977, p. 437). Whilst subtle, this is less than helpful since a consequence of its vagueness is that, strictly speaking, it can neither be validated nor modified. The conventional force, however, is clearly to present the conjecture as fallible, possibly in need of modification.

The teacher who recognises the epistemic force of a hedged conjecture has the option of assisting its placement in the ZCN. One way to do this might be to write it on a chalkboard/flipchart and say something like "OK, let's take a look at this conjecture", possibly without reference, for arbitration or interpretation, to the one who proposed it. Another way is to form small discussion groups which then tend to assume some corporate ownership for the conjecture and their findings about it when reporting back to the class. I sometimes 'return' a conjecture, or an agreed modification of it, from the ZCN back to its originator when the "severe testing" is over; I do this, for example, by marking its changed status with reference to the conjecture as 'theorem' (sometimes 'lemma') and naming it Yuko's Theorem or Tom's Theorem.

A student's conjecture may be the inductive outcome of an extended investigation, or simply the answer to a teacher's question, such as "Is 91 a prime number?", or "How many non-isomorphic groups are there of order 8?" By default, the one who answers the question 'owns' the answer and is subsequently right or wrong. One way of trying to bring the answer into the ZCN before it is spoken is to pose the question as an 'indirect speech act' (Gordon and Lakoff, 1976) such as "Can you tell me if 91 is a prime number". Hazel, the teacher discussed earlier, was notable for her frequent use of this technique (Rowland, 1996b). Another, rather different, technique is to pose questions as statements (with the tacit or explicit "Discuss"). Thus, "91 is not a prime number". Or by attribution: "My friend says that 91 is a prime number". The conjecture then goes straight into the ZCN; at the very worst, only the teacher (or his 'friend') are 'wrong' if the statement turns out to be false. But this technique has limitations, and cannot help with extended enquiries "in which a conjecture is created, tested and proved, or refuted and modified" (Dawson, *ibid.*).

Another affective issue in fallibilistic teaching is raised by Sixth Form College teacher Rachel Williams (1995), in her discussion of a teaching episode with two eighteen-year-old students who had laboured at length on a combinatorial problem. When at last one of them, Di, identifies combinations she had previously overlooked, Rachel can restrain herself no longer:

31	Di:	[puzzled] ten, eleven, twelve, thirteen ...
32	Rachel:	That's all right.
33	Juliette:	[puzzled] That's O.K?

Williams comments:

> I had to confirm that Di was correct [32], I couldn't bear the uncertainty and wanted them to know they had got to the correct number of ways. Looking back, it would have been better to let them sort it out.

The student is required to take risks, but the teacher may have to "bear the uncertainty" when she judges that the student must resolve uncertainty him/herself. That is not to say that the teacher cannot participate in the ZCN, but her/his role may be best restricted to light scaffolding as s/he oversees the debate.

SUMMARY

Whilst constructivism emphasises students' coming-to-know as an outcome of reflection on mathematical activity, it is necessary that knowledge so constructed is somehow affirmed (or otherwise). To quote Dawson (1991, p. 195), "Learning mathematics does not (necessarily) mean constructing the right knowledge". Whilst it may frequently fall to the teacher to legitimate or deny students' constructed beliefs, such a norm presents a somewhat sterile and authoritarian view of mathematical knowledge.

Lakatos' fallibilist philosophy offers a more dynamic paradigm for the authentication of constructed knowledge, which initially may be accorded the status of conjecture. In a conjecturing atmosphere, a pupil may articulate a conjecture without necessarily being committed to its truth. As Mason (1988, p. 9) says, "... let it be the group task to encourage those who are *unsure* to be the ones to speak first". Yet the testing of such a conjecture is a carried out in "a clash of views, arguments and counter-arguments" (Davis and Hersh, 1980, p. 346). A necessary precursor to such testing is the willingness of students to articulate their conjectures, without fear of humiliation on being found to be 'wrong'.

The zone of conjectural neutrality may therefore be viewed as an affectively- neutral space where conjectures are lodged for inspection. Both the pupil and the teacher may adopt strategies to influence the relocation of the conjecture from the pupil to the ZCN.

The conjecture is then tested, modified or rejected in the ZCN. In such a cognitive and affective *milieu*, it is the proposition that is on trial, not the person. The ultimate goal, for the fallibilistically committed teacher, would be for the class to understand that this is the case.

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THE EXPERIENCE OF MATHEMATICS TEACHING
Human awareness as a possible object of enquiry in research in teaching mathematics.

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In this paper I argue that taking an experiential perspective on the teaching process in terms of the different ways in which teachers experience mathematics teaching, can contribute to a better understanding of the teacher and the teaching process as well as what is learned. The point of departure is that this experience is a relation between the subject and the world. When something is experienced it is experienced as something and can be described in terms of in what way the awareness of the subject is structured. By analysing what becomes the fore of the teacher's awareness, i.e. what aspects of mathematics teaching they direct their awareness towards and how this is done, it is possible identify that they open up for different dimensions of variation in their teaching.

Introduction

Why is teaching carried out differently and what can be identified in the teachers' teaching that might lead to a variation in what and how students learn? Although there seems among teachers to be an agreement about for example what is important to learn in mathematics and how mathematics is learned, their teaching can be very different.

The aim of this paper is to explore how research from an experiential perspective can contribute to an understanding of the teaching process in mathematics, particularly in respect of differences between what is taught and how this is done. This paper gives an account for some parts of a study which aims at describing those intentions, explicit as well as implicit, that teachers have for their teaching (Runesson, 1996). In order to reveal and describe the different ways mathematics teaching can be experienced by teachers, a combination of data from interviews with teachers and from classroom observations has been used.

Background

Due to the shift from the process - product to a cognitivistic paradigm in research on teaching in the 1980s, there was an increasing interest in trying to identify and understand teachers' conceptions and beliefs systems. A number of studies in mathematics education have indicated that teachers' beliefs about mathematics and its teaching play a significant role for their practice. Many of these focus on the relation between teachers' beliefs and their practice. The results from these studies look, however, different. This discrepancy has been explained in different ways; for instance due to the research design, but mostly it has been explained by the researcher neglecting the social context (Thompson, 1992). The awareness of the strong influence of the social context in the classroom on teachers' practice, has led to a changed focus in research on mathematics teaching. This implies that the importance of social interactions within the classroom has been taken in consideration. Several studies have been undertaken that give account for the student - teacher interaction in the classroom, either from linguistic (c.f. Pimm, 1987) or sociological perspectives (c.f. Mellin-Olsen, 1987).

Recently there has been a need to broaden the sociological perspective in research on mathematics teaching. Therefore a socio-cultural view on the mathematics classroom has been taken of many researchers. Inspired by constructs from symbolic interactionism or from Lave & Wenger' s social practice theory (Lave & Wenger, 1991), the teacher and the students are seen as participants in a culture of using mathematics. The teacher' s role in the mathematics teaching is seen as a representative of the mathematical community (Yackle & Cobb, 1996) and that mathematical meaning is jointly negotiated in the classroom.

Thus, there has been a shift from seeing the teaching and learning as something mental to regarding this as social. From my understanding however, these different research perspectives have something in common, since both of them are based on the same ontological assumption. This assumption is that the subject and the object are separated; i.e. on a dualistic ontology, since *either* the mind of the teacher *or* the social context has been the object of study.

It has been argued however, that these (the mind and the social) must be seen as complementary (Cobb 1994) . I argue that they must be seen as inseparable; the "mind" and "the social" cannot be divided. Teaching - and learning- is both cognitive and social. A teacher's intentions for example, can neither be reduced to something "inside the head" of the teacher which she tries to realise in the classroom, nor is it only a result of the social interaction in the classroom.

Teaching and awareness

Phenomenography as a research approach offers an alternative perspective for understanding the teacher. Instead of studying teachers' thinking in terms of beliefs, attention is paid to the ways teachers are aware of or experience their professional world (Marton, 1993). Phenomenography studies empirically differing ways in which people experience, conceive of or understand various phenomena in the world (Marton, 1981; Marton & Booth, in press).

Experiencing should not, however, as within psychology be understood as a mental representation or a cognitive structure, i.e. as related to the subject only. Instead experience refers to a *relation* between the experiencing person (subject) and what is being experienced (the object). This implies that phenomenography takes an non-dualistic ontological point of departure since subject and object are regarded as in-divisible. Thus there can not be a conceiving or experiencing without something perceived or experienced. However when we experience something, it is always experienced *as something*, has a meaning. But in order to assign it a meaning, it must be experienced in a certain way; for example, it must be discerned from its context and its parts and how these parts are related must also be discerned. From this follows that the experience has both a structure - how something is experienced - and a meaning - what it refers to.

How something is experienced can be described in terms of the structure of the awareness at a certain moment. The awareness is assumed to have a figure - ground structure, i.e. it has a structurally differentiated character. Gurwitsch (1964) makes a distinction of what is the object of the focal awareness, *the theme*, and those aspects in which these are embedded, *the thematic field*. We are aware of a numerable things, but not at the same time and in the same way. When for instance, a teacher is teaching, she is aware of many different aspects of her teaching (the subject, the students, physical and material frames) but she directs her awareness towards some of these aspects *and* she does this in a certain way. "The aspect of the phenomenon and the relations between them that are discerned and simultaneously present in the individual's focal awareness define the individual's way of experiencing the phenomenon" (Marton & Booth, in press).

Teaching in terms of teacher's awareness

From an awareness perspective, the notion "teaching" is considered as an intentional act aiming to establish some kind of relation between the student and the world. This implies that the teacher wants the student to become aware of or experience the world in a particular way. In order to make this happen, the teacher tries to direct the awareness of the students towards some aspects of a phenomenon. When a specific content is communicated, the learner's and the teacher's thoughts are coming into contact. From that follows that the "teacher's awareness - --- has to be interwoven equally to the threads of the learner and content" (ibid.). In line with the underlying non-dualistic assumption, teaching thus can be seen as a meeting of awareness through a shared object of learning. So what the teacher focuses on and the variation of their awareness in the teaching situation, becomes a central issue.

Research within the phenomenographic research approach has given account for that teachers, when they talk about their teaching, direct their awareness towards different aspects of the teaching process. Some dimensions and aspects come in the fore, become the focus of the awareness and are thematized, whereas others are taken-for-granted. (Alexandersson, 1994). In a study investigating the different ways in which teachers experience and handle the content, Patrick (1992) has shown that the way teachers experience the disciplines (physics and history), affects the way they communicate and handle the subject matter. Patrick describes this in terms of the teacher constructing an object of study towards which the students direct their awareness in their learning. This implies that as the teacher focuses some aspects of the content and the learner, she opens up for variations in some dimensions and invariations in others. Marton & Booth (in press) use the notion "architechture of variation" as one principle for teaching and they point out the importance of a such variation for the learning process .

Methods and procedure

The overall aim of the study is to reveal and to describe the object and aim as regard students' learning in mathematics (Runesson, 1996). The point of departure for the analysis is the structure of teachers' awareness in terms of what is focused in the teaching process, in which dimensions a variation is opened up for and how this variation comes into play.

The study includes five teachers and their pupils in four different schools. Four teachers teach in grade 7, the fifth in grade 6. The selection is done by "purposive sampling" (Cohen & Manion, 1986, p 103). In this case, this means that the group that is investigated consists of teachers with varying degrees of in-service training.

The data consists of audio-recordings of lessons and two interviews with each of the teachers. The teaching has been followed during six consecutive lessons in the respective classes and extends over the first eight week lessons of the teaching section rational numbers. Altogether there are 21 hours of mathematics teaching respectively 8 hours of interviewing documented. This material has been transcribed and typed out word by word. The analysis has consisted of repeated readings of the very extended data set in order to identify those instances that can be considered relevant for the investigation of the phenomenon in question, namely the different ways in how teachers experience teaching mathematics

Some preliminary results

Due to the fact that all five are teaching the same unit of the curriculum, (four of them even using the same textbook), it is possible to identify some aspects of the teaching process that are independent of the content that is taught. Even though the teachers work with the same subject content, use the same tasks or the same type of manipulatives in their teaching, in the interaction with the students they direct the students' awareness towards different aspects of mathematics and mathematical knowledge. Thereby it is possible to show that although there are many similarities in their outer visible methods, there exists something else in addition to this that may be of importance for what is learned. The different acts of teaching and the oral statements from the interviews, indicate that some aspects of the teaching process become the focus of the teacher' s awareness, i.e. they become thematized in a certain way. The teachers thereby constitute their own, personal curriculum for the teaching of mathematics. At present I have been able to identify three such different curricula.

In order to illuminate that the teachers' awareness can be directed towards different aspects and that the awareness can be structured differently, we will take a closer look at two of the teachers in the study.

Their teaching is similar in one respect; in the textbook that is used, there is a unit about the relation between different aspects of rational numbers. It aims at pointing out that for example 6/3 is a fraction, i.e. part of a wholeness, as well as an operation (division).

Case 1, Ms Irvine

Ms Irvine starts the lesson by pointing to a pie-chart on the board. She uses this as a representation, and asks the students:

T: How would you express this?..How would you write what you can see on the black board here?
S1: Six thirds or two.
T: Yes... that equals two, since if you put the parts together, it is exactly two. OK. I'll write that.

$\frac{6}{3}$ or 2

BEST COPY AVAILABLE

T: But if I'd asked you about this a couple of weeks ago, what I had written if I had written like this, six fraction line three, you would not have said six thirds, because then we were talking about something quite different..."divided by three", that's what you would have said...but is it the same?
S (mumbling): yes, no
T: Well, it is said differently,... but is it the same? Yes it is. So six divided by three is the sa-...,gives you the same result as six thirds.

six thirds
$\frac{6}{3}$ =
six divided by three

T: I can show you, if I'd use a picture, I'll do like this: Let's take six pies or what ever, and divide them by three. (Draws six pies on the black board) What am I doing? Well I can divide them into three. Now you can see...if I divide this by three it equals two. So it is the same, but it is said differently. So if I would say: "20 divided by 4" but it doesn't matter what I say 'cause it equals five anyway. But you should know they are actually two different things but the result will be the same.

In this excerpt we find that the teacher directs her awareness towards *one aspect at the time*. She starts out with the part-whole aspect by using a pie chart representation. The students have to give an account for how this can be represented mathematically and expressed orally. Then the teacher concludes: six thirds equals two. Subsequently the teacher focuses upon the division aspect. She reminds the students about their previous experience of division. Previously mathematical expressions with fraction lines have been interpreted as division. Finally she uses a representation for six divided by three. This and the pie chart represent two different aspects of the fraction, but both equals the same integer, " it is the same".

Case 2. Mr Turner

Mr Turner also uses a representation. On the blackboard there is a picture of 12 apples (Example 1).

T: All right. I'm going to give you two questions to think about. Just think! Don't say the answer out loud yet. I think you'll find them easy. OK, here's number one: Twelve apples that you and two of your friends are going to share equally. Now I want you to consider; how can this be written mathematically? And what would that turn out to be?....OK. Do you remember number one now?! All right...number two.

On the black board there is a picture of a bowl with pieces of apples in it (Example 2).

T: This is a bowl...there are pieces of apples in it. Each apple is divided into three parts with the same size. What do we call such parts? You can answer that question now if you want to.
S1: Thirds
T: Right. Well do you remember example number one? How many apples did you get there? OK I'll go on asking. Let's suppose you take pieces of apples from the bowl, just as many so you get exactly as much as you had ...there (example 1). How many pieces of apples do you have to take? In order to get just as much? ...OK. I want you to write that down,...how could this be written? Well, let's go back to question number one again, I asked you how you would write that mathematically and what that turned out to be. How would you do that Erica?
S2: Twelve divided by three.
T: Twelve divided by three. $\frac{12}{3}$

That's what you do, isn't it? You've got twelve apples and you divide them by three, that equa-
...four. So you'll get four apples. OK, the next one. I'd like to know, how many thirds did you have to take in order to get as much apple as you got there? David?
S3: Twelve.
T: Twelve, what?
S3: Thirds.
T: Twelve thirds. How do we write twelve thirds mathematically ? Caroline?
S4: Twelve and then such a... line
T: Such a line, yes a fraction line. Do you follow? Well how many pieces am I going to draw now? Twelve, yes! Now...I want you to look at number one and on number two. Can you find any similarities?...Mm...If so, what is similar?...What is similar...?
S5: You have divided by three.
T: Yes. In the same way, Here you can read 12 fraction line 3. Here you can read 12 fraction line 3. But what would you say in this case (points to number one) would you say twelve thirds there?...Susan?
S6: No, twelve divided by three.
T: Twelve divided by three... and what about this (number two)...But if you'd take these 12 thirds and stick them together. How many apples would you get then. Dan?
S7: Four.
T: Why four?... This is hard. Nobody is raising his hand now!?...Here it comes, Andrew:
S8: 'cause there are four in a whole.
T: so in order to get a whole....so 12 thirds, that's also four. Thus, you can say...
(writes on the blackboard) $\frac{12}{3} = \frac{12}{3}$

that's the same as 12 thirds.....you get the same result. Listen. 12 divided by 3 and 12 thirds, the result is the same. So no matter if you take 12 apples and divide them or if you take 12 thirds, you'll get the same result. OK.

In this case, the teacher directs the awareness of the students towards the two aspects of the fraction *simultaneously*. Just as in case 1, the awareness is directed first towards the division aspect and the to the part-whole aspect, *but* without interuption of letting the students give the answer to example 1. Instead the teacher continues directly with example 2. By not changing the focus of the students' awareness, this is kept directed towards something whole, i.e. a relation. The way in which the teacher acts, indicates that his awareness has a character of something whole, since several aspects of the fraction is focused simultaneously.

Conclusions

By taking the teacher awareness as an object for analysing my data, it has been possible to identify the differences of the focal awareness of the teachers. For Ms Irvine the mathematical problem and its solution becomes the fore in her awareness. Her intention is to teach the students how to solve different kinds of mathematical problems. As regards Mr Turner, it is not the problem *as such* that becomes the focus of his awareness but the structure and the meaning of mathematics. His intention is that the students by understanding the structure of mathematics will be able to solve mathematical problems. Thus, the teachers have different goals as regards the students' learning in mathematics. It is also possible to identify the structure of their awareness. Whereas Ms Irvine's awareness has a sequential structure - the aspects are focused step by step - Mr Turner focuses these aspects simultaneously.

It has also been found that what becomes the focus of the teacher's awareness, becomes the focus of the students' too. For example; the students in Ms Irvine 's class are very eager to come up with the right answers, whereas in Mr Turner 's class they are eager to give arguments for their solution.

By focusing different aspects of the teaching process, the teacher opens up for different dimensions of variation in the classroom. This has implications for what is learned. The importance of exposing students to a variation when learning to solve mathematical problems, has been reported by Ahlberg (1992). So there are reasons to believe that which dimensions of variation that are opened up for is of importance for students' learning in mathematics.

Final reflections

In what way can human awareness as an object of study contribute to our understanding of mathematics teaching? From my point of view the main contribution is to make variations within the social context visible; for instance variation among teachers in respect to those aspects that are less obvious than the outer methods for instance whether group work, manipulatives etc. are used or not. This implies that the variations in respect to how the subject matter is handled can be revealed. Which aspects of the teaching process are varied, which become invariant? In which aspects of the teaching process are dimensions of variations opened up for? What is the nature of that variation and what importance has this for students' learning? These are some central questions, important for understanding both the teacher, the student and the teaching process, that can be answered by taking the human awareness as an object of study.

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CALCULATOR USE BY UPPER-PRIMARY PUPILS TACKLING A REALISTIC NUMBER PROBLEM

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This paper analyses how a structured sample of pupils in the last year of English primary education tackled a realistic number problem, focusing on their use of a calculator. Use of the machine allowed a small number of pupils to work with the representations and approaches underlying the solution strategies of Multiplication Trial, Division Trial and Repeated Subtraction, and the sense-making strategy of Clarificatory Division. It also enabled pupils using the predominant solution strategy of Direct Division to execute a computation for which they had no other method. But here, some pupils chose not to use the calculator, and others failed at the stage either of formulating, executing or interpreting the division. Response patterns were no different in schools long offering a 'calculator aware' curriculum.

The introduction of information technology into education has been accompanied by suggestions that computational tools need not simply be assimilated to traditional patterns of thought, but are capable of provoking fundamental changes. Pea (1985), for example, has made the theoretical distinction, further developed by Dörfler (1993), between a tool as 'cognitive amplifier' on the one hand and 'cognitive reorganiser' on the other. The calculator has a relatively long history of school use, and remains the computational tool most generally and readily available to school pupils. Moreover, in many countries, school mathematics curricula are being reformed in ways which cast the calculator as both amplifier and reorganiser.

In England, the influence of the Calculator Aware Number (CAN) project (Shuard et al., 1991) can be seen in National Curriculum requirements that upper primary pupils should be "given opportunities to use calculators..as tools for exploring number structure and to enable work with realistic data" and that "pupils should be taught to understand and use the features of a basic calculator, interpreting the display in the context of a problem, including rounding and remainders". Equally, the curriculum reflects the CAN emphasis on informal mental calculation, requiring that "pupils should be taught to.. extend mental methods [of computation] to develop a range of non-calculator methods..progressing to methods for multiplication and division of up to three-digit by two-digit whole numbers"; to "extend methods of computation to include..all four operations with decimals..using a calculator where appropriate"; and to "gain a sense of the size of a solution, and estimate and approximate solutions to problems" (Department for Education, 1995, 7-8)[1].

With the notable exception of the work of Groves (1993, 1994), there has been little systematic research into the ways in which pupils make use of calculators in the process of tackling problems within a 'calculator aware' number curriculum.

This paper reports some work in progress as part of ongoing research[2] into the part played by the calculator in children's numerical learning and thinking. The

[1] Reference is to this version as it provides a more succinct synopsis of material also in earlier ones.

[2] The financial support provided for the *Calculator as a Cognitive Tool* project by the Economic and Social Research Council of the United Kingdom under research award R000221465 is gratefully acknowledged, as is the contribution of my project collaborators, Di Chaplin and Laurie Rousham.

research has two components. A macro-study is examining the long-term impact of approaches to number in the primary school on pupils' attainment and attitude. A micro-study is analysing how pupils tackled realistic number problems, focusing on their use (and non-use) of a calculator in doing so.

The research focuses on the mathematical experiences, attitudes and achievements of a cohort of pupils aged 10/11 who entered reception class during the 1989/90 school year and had progressed to Y6, the final year of primary education, in the 1995/96 year. Data was gathered in six neighbouring primary schools, all covering the full primary phase from R to Y6. Three of these schools had participated in the original CAN project between 1986 and 1989. More recently, the teaching of mathematics in all six schools had been influenced by the requirements of the National Curriculum and its assessment, introduced from 1989 onwards.

The situation studied here differs, then, in two important respects from that researched by Groves and her colleagues. First, the pupils are rather older. Perhaps more significant, their mathematics learning has taken place under less propitious - and more typical- conditions than those of a special development project, although many of the teachers involved had previously participated in such a project.

This paper reports one element of the micro-study involving a structured subsample of pupils. The sampling frame incorporated 80 cells: defined by school class (5 in post-CAN schools, 5 in non-CAN); sex; number concept attainment, dichotomised as Above- or Below-average on a 30-item scale (α=0.92) from a written test; and attitude to making use of the calculator, dichotomised as Amenable or Reluctant relative to the neutral position on a 4-item scale (α=0.71) from a written questionnaire (with three-quarters of pupils falling to the Reluctant side). Sampling was restricted to pupils who had remained in the same school throughout their primary education; and to those between the tenth and ninetieth percentiles of attainment, so as to exclude pupils of exceptionally high or low attainment. One eligible pupil was chosen from each non-empty cell, by random selection where necessary, to produce an achieved sample (after 5 recording failures) of 56 pupils.

Each pupil took part in an individual interview during which they were asked to tackle a range of number problems. The videotaped interviews were conducted by an experienced advisory teacher for primary mathematics, and held in whatever private space was available in each school. Interviewer and pupil sat together at a table on which pen, paper and calculator were available. At the start of the interview, pupils were told that they could work out the problems however they liked; using their head, pen and paper, or calculator, or a mixture of them. They were asked to tell the interviewer when they thought they had an answer and how they had worked it out. At the start of each item, a large-type version of the problem, printed on a raised card, was placed on the table and read to the pupil. The interview protocol specified how pupils should be probed when a solution was given (to elicit evidence of method), or after a long period of silence (to offer encouragement and elicit evidence of thinking about method).

This problem was the fifth to be tackled: *313 people are going on a coach trip. Each coach can carry up to 42 passengers. How many coaches will be needed? How many spare places will be left on the coaches?* It is a variant of an item used by

Foxman et al. (1991, 3-23) and resembles one set in a recent national test at this level (School Curriculum and Assessment Authority, 1995, item 13a). It matches the curriculum goal that "pupils should be taught to understand multiplication as repeated addition, and division as sharing and repeated subtraction..and recognise situations to which the operations apply" (Department for Education, 1995, 8).

From the videotape of the interview and any records made by the pupil during it, pupils actions and accounts were analysed in terms of the succession of strategic moves as they related to finding the number of coaches (with only one pupil formulating a spare places conjecture first). For example, the response recorded in Figure 1, was analysed into the 4 moves shown as starting at (0-01), (1-10), (1-56) and (2-40); the first 3 coded in a way which will be explained, and the last move uncoded because it does not relate to finding the number of coaches. This may, of course, be an incomplete record of the pupil's strategic thinking. However, it comprises all the moves which reached the point of being implemented through written recording or use of the calculator, and it also plausibly indicates that she engaged in no other sustained strategic move. Moves conducted wholly mentally are potentially most problematic. Some may have gone unrecorded, although a solution, abandonment or lengthy pause was always probed. The records may, then, under present mental moves, particularly where they proved fruitless or did not involve recording intermediate results. Across the sample, however, the records are likely to provide an indication of the types of mental strategic move used by pupils.

The strategic moves were classified in terms of the scheme shown in Table 1. Each move is characterised in terms of a computational procedure, its referents and intent. The coding of each move had to be inferred from the way that the pupil reported, computed and interpreted it. Direct Division, for example, might be reported as "313 divided by 42", or simply evidenced by the keying or writing of that computation. Often, the form of the computation was critical in classifying a move, since a report such as "You need to know how many 42s there are in 313" could accompany Direct division, Direct Multiplication, Repeated Addition and Accelerated Addition. This shared form of words brings out a common underlying representation of the problem across these different strategic moves. Equally, however, computations of superficially similar form, could have different referents and intent, as in the distinction between Direct, Trial and Clarificatory Division.

The categories from Direct Division through to Repeated Subtraction represent the well-founded strategic moves which were observed. The next two categories cover the misconceived moves displayed by pupils. Then, Clarificatory Division is linked to making sense of the result of calculator division. Finally, a move in which a pupil repeatedly multiplied in search of a factor of 313 is classed as Other Move.

Each pupil's record was coded as a sequence of strategic moves, and each move in terms of the primary mode of computation employed. Direct Division produced the greatest range. When pupils reported a mental strategy, this was classed as Mental with Recording if they recorded intermediate results; Mental with Calculator if they used the machine for subsidiary calculations; otherwise, Wholly Mental. If pupils wrote the division out in column format this was Written Column; if they keyed it, Calculator. In other moves, however, the Calculator coding could subsume a degree of subsidiary mental computation (such as maintaining a tally or assessing a difference).

Figure 1: Record of one pupil's work on the coach problem

(0-01) DD/W/-
Writes 42)313
(0-27)
Scores out 4
(0-36)
Writes in 15
then crosses out

~~15~~
~~42)313~~

(0-40)
Writes

~~42~~ √313
~~/8~~

(0-56)
Crosses out

(0-58)
Inter: "How are you thinking about it?"
Pupil: "I'm trying to work out how many forty-twos in three hundred and thirteen but I don't know how many there are."

(1-10) DD/C/SC
Pupil: "Can I use the calculator?"
Inter: "Mmm."

Keys 313÷42=7.452380952

(1-32)
Inter: "What have you got?"
Pupil: "Seven point four five two three eight oh nine five two."
Inter: " Any good?"
Pupil: "No"
Interviewer: "No?"
Pupil: " Well you'd probably get seven coaches but you'd have some left."
Inter: "How are you going to sort it out?"

(1-56) DM/W/SE
Writes 7
(2-16)
Writes 42
Scores out 7
Writes new 7
Writes 294

~~7~~
42
7
294

(2-40) -/ -/-
Writes:

210
313
\- 294
019

(3-02)
Pupil: "Nineteen spare seats."
Inter: "So there would be what.. seven.."
Pupil: "Seven coaches and nineteen spare seats."

Table 1: Types of strategic move and their characterisation

Strategic move	Characterisation
Direct Division	Division of size-of-group (313) by capacity-of-coach (42) to find number-of-coaches
Direct Multiplication	Single multiplication of capacity-of-coach (42) by conjectured number-of-coaches
Trial Multiplication	Repeated multiplication of capacity-of-coach (42) by conjectured number-of-coaches to match size-of-group (313)
Trial Division	Repeated division of size-of-group (313) by conjectured number-of-coaches to match capacity-of-coach (42)
Repeated Addition	Repeated addition of capacity-of-coach (42) to match size-of-group (313), enumerating to find number-of-coaches
Accelerated Addition	Addition process accelerated through use of doubling
Rounded Addition	Addition process using rounded capacity-of-coach (40)
Repeated Subtraction	Subtracting capacity-of-coach (42) repeatedly from size-of-group (313), enumerating to find number-of-coaches
Misconceived Addition	Adding capacity-of-coach (42) to size-of-group (313)
Misconceived Multiplication	Multiplying capacity-of-coach (42) by size-of-group (313)
Clarificatory Division	Division of 313 to clarify decimal answer
Other Move	Other strategic move

Table 2: Number of pupils indicating use of specific strategic moves, and the mode(s) of calculation employed

Strategic move	Pupils using	-----Mode(s) of computation employed----- Calculator	Written column	Mental + recording	Mental + calculator	Wholly mental
Direct Division	36*	27	11	1	2	3
Direct Multiplication	4	3	1			
Trial Multiplication	3	3				
Trial Division	2	2				
Repeated Addition	15	4	1	7		3
Accelerated Addition	9	1	2	2		4
Rounded Addition	3			1		2
Repeated Subtraction	2	2				
Misconceived Addition	2	2				
Misconceived Multiplication	7*	6	1			1
Clarificatory Division	2	2				
Other	1	1				
None	2					
All pupils	56*	34	14	11	2	12

*Row total of mode frequencies is greater because pupils made several moves using different modes

Table 3: Number of pupils successfully computing and interpreting specific strategic moves

Strategic move	Pupils using	Of whom successfully computing	Of whom successfully interpreting
Direct Division (C)	27	19	0
Direct Division (NC)	15	0	/
Direct Multiplication (C)	3	3	0
Direct Multiplication (NC)	1	1	0
Trial Multiplication (C)	3	3	1
Trial Division (C)	2	2	1
Repeated Addition (C)	4	0	/
Repeated Addition (NC)	11	1	1
Accelerated Addition (C)	1	0	/
Accelerated Addition (NC)	8	3	2
Rounded Addition (NC)	3	2	0
Repeated Subtraction (C)	2	0	/
No well-founded move	3	/	/

Of the 56 pupils studied, 53 displayed at least one well-founded strategic move; 1 produced only misconceived moves; and 2 were unable to suggest any move at all. Table 2 indicates the number of pupils reporting or displaying each type of strategic move, and the modes of computation employed. From this it is clear that the most common moves were based on direct division (used by 36 pupils) or one of the forms of cumulative addition (used by 24 pupils, some trying more than one). Much rarer were moves based on trial multiplication (3) or trial division (2) which appear to depend on calculator use. These issues will be developed later.

Each well-founded strategic move was also classified in relation to two levels of success. Where the relevant computation was successfully executed the move was classed as Successfully Computed; if the result was successfully interpreted in terms of the problem, this became Successfully Interpreted. In the case of Direct Division by Calculator, for example: keying 42÷313= (misformulation) or 3313÷42= (miskeying) achieved no level; correctly keying to produce 7.452380952 but reporting "It doesn't really tell you the answer" (non-interpretation) or inferring that 7 coaches were needed (mis-interpretation) achieved the first level; and keying a correct result and interpreting it as indicating 8 coaches, reached the second level.

Table 3 indicates the highest level of success achieved by the pupils employing each strategy, differentiated by calculator and non-calculator mode. Overall, 5 pupils completed the problem successfully. On Direct Division, no pupil using a non-calculator mode computed successfully; and while the majority of those using a calculator computed successfully, none interpreted successfully. All pupils adopting either form of trialling computed successfully by calculator, some interpreting successfully. Whether executed by calculator or not, the various forms of addition were rarely computed successfully; but there was then some successful interpretation. The different move types will now be examined in detail, starting with Direct Division, reported by 36 pupils, and the first move for 32 of them.

Of the pupils who tackled Direct Division in a mental or written mode (15 pupils), the majority (11/15) got little further than stating the calculation or writing it down (in one case reversing 42 and 313) and carrying out some tentative calculations, often concluding their deliberations with a comment such as "I'm not sure how to do it" or "I've forgotten how to do it". One of these pupils had already used the calculator to tackle this computation. Not all the remaining pupils proceeded to use the calculator (only 5 did so); others abandoned the problem at this stage (2) or switched to another strategy (3). The minority (4/15) who persisted with Direct Division in a non-calculator mode all conjured an answer out of some variation on column methods. One pupil, for example, started, in effect, by dividing 300 by 2 to give an answer of 150 coaches, commenting "It's probably wrong"; she then proceeded to improvise a more elaborate malgorithm, based on adding the results of 300÷40 and 13÷2 to give 14, with these computations executed on the calculator.

Amongst the pupils who tackled Direct Division with the calculator (27 pupils), the majority (19/27) computed successfully at the first attempt. Many seemed to find the resulting 7.452380952 surprising: comments ranged from "Too big a number" through "It's too long" to "That don't share equally". This breakdown of 'exactness' evoked reactions apparently intended to restore its order or affirm its loss: rekeying the calculation (1); adopting a written method (1); varying the calculation by reversing it to 42÷313 (1), or changing operation to produce a Misconceived Multiplication (3) or Misconceived Addition (1); proceeding to Trial Division in the hope of finding an exact relationship involving 313 and 42 (1); or carrying out Clarificatory Division to see if dividing 313 by another number looking like 42 would produce a similarly 'inexact' result (2). Another type of response -sometimes following the previous one- was to seek to use the result: by (mis)interpreting it as implying a solution of 7 coaches (6), often taking the next digit to signal 4 spare places; or by carrying an estimate forward into a Direct Multiplication (3) or Trial Multiplication (2) move.

Amongst those who were unsuccessful in executing Direct Division with the calculator at their first attempt (8/27), one miskeyed, but most reversed the computation (7). Again, responses included varying the calculation by reversing it to a (well-formulated) Direct Division (1) or a Misconceived Multiplication (1); or interpreting 0.134185303 as 13 coaches (2) or 134 coaches (2).

All the pupils employing Direct Multiplication (4 pupils) had already used Direct Division. Most carried forward 7, then misinterpreted it in the standard way (3).

Equally, 4 of the 6 pupils using Misconceived Multiplication did so in response to the decimal result of Direct Division. Nonetheless, to adopt this variation of the division calculation there clearly had to be some degree of misconception. One pupil volunteered: "We were doing yesterday that times and divide is the same thing. If you do it the other way round you get the same answer".

Trialling, too, usually followed on from Direct Division, and was consistently associated with use of the calculator. The two pupils who came to Trial Multiplication from Direct Division both computed successfully but one succumbed to the standard misinterpretation, while the other switched strategy. The last pupil, however, had adopted Trial Multiplication from the outset, following a clearly formulated trial strategy which involved testing 12, 10 and 8 on the calculator and then mentally calculating the difference between 336 and 313 to find the number of spare places and confirm that it was sufficiently small. Similarly, the fully successful use of Trial Division was by a pupil who followed a trial strategy from the start, testing 5, 7 and 8, to see which number of coaches would give just fewer than 42 passengers per coach -a rather unusual problem representation. The second user came from Direct Division with a rather different conception and intention, testing 8 and 9 in the hope that they would divide 313 to give exactly 42.

Now to the various forms of cumulative addition, used by 24 pupils, and the first reported move for 17. In mental mode, the difficulty of Repeated Addition lies in computing and retaining both running total and count. Of those computing mentally (10 pupils) some abandoned before completing (3) and others miscomputed (6). The only wholly successful pupil recorded running totals in a column, and then counted them. She was observed to tap her cheek rhythmically, and reported that at each stage she first added 2 then added four 10s. The one pupil using the written mode calculated the additions correctly but then miscounted them. Most of those who used a calculator (4 pupils) keyed successive +42 operations, either lapsing in the mental count (2), or miskeying (1); the other pupil had the idea of setting the calculator constant, but misformulated this as multiplication by 42.

Of those pupils who used Accelerated Addition in written or mental mode (8 pupils), the majority either abandoned (1) or miscomputed (4). Of the minority who successfully computed, one made the standard misinterpretation, but two succeeded. The pupil who used the calculator, keyed 42x2=84, 84x2=168 and 168x2=336, then mentally (mis)calculated 33 spare places, and was unsure about the number of coaches.

Rounded Addition was used by 3 pupils, all calculating mentally. One miscalculated and then abandoned; the others reached a total of 320 and a count of 8, interpreting this as 8 coaches and 7 spare places, without compensating. A pragmatic decision was made to code this response as successfully computed but misinterpreted. It is worth noting that an alternative to compensation would have been to carry the estimate forward to another strategy, but no pupil reported this. On a related point, nor was there any evidence of pupils estimating from the approximate proportions 40:320, or 40:300, or from the corresponding divisions.

Both cases of Repeated Subtraction were built on an unusual problem representation focusing on the number of people still to be placed. As a mental strategy, the use of subtraction rather than addition, and the resulting combination of a falling total with a rising count, make it challenging. Both attempts made use of the calculator: one failed through miscounting, the other through miskeying.

In what ways, then, could the calculator be said to be acting as a cognitive reorganiser or amplifier for these pupils? As a reorganiser, use of the machine enabled pupils to work with the unusual problem representations underlying Trial Division and Repeated Subtraction. Equally it made the iterative solution strategies of Multiplication Trial and Division Trial available, and the sense-making strategy of Clarificatory Division. But we should also note their limited incidence. Finally, although some pupils used the calculator to form a distributed system combining machine and mental computation for Repeated Addition and Subtraction, they did not do so effectively. As an amplifier, the calculator offered pupils tackling Direct Division a means of executing a computation for which they had no other method. But this was to little effect for those choosing not to use it, for those misformulating the division or misexecuting it, or for those thrown by the decimal result. This brings us to the nub of the issue. Effective use of the calculator requires development both of confidence and competence in its operation, and of mathematical concepts underpinning its use.

Finally, closer examination revealed very similar patterns of response in post-CAN and non-CAN schools, probably reflecting the convergence in their approaches to number in response to the introduction of the National Curriculum.

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SPATIAL ABILITIES, VAN HIELE LEVELS, & LANGUAGE USE IN THREE DIMENSIONAL GEOMETRY

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We investigate the van Hiele levels in three-dimensional geometry, and the spatial abilities, of a group of pre-service secondary teachers. A small subgroup of these teachers was studied for the use of language in the identification of the properties of polyhedra. Whilst the language generally reflects the van Hiele levels there are examples of discussion at a higher level than the written tests indicate.

Introduction

How do students of geometry come to understand the properties of 3-dimensional objects such as polyhedra? The van Hieles developed a theory of levels of understanding in Euclidean (2-dimensional, flat) geometry (van Hiele 1986; Hoffer, A. 1983, Fuys et al 1988, Senk. S. 1989; Burger and Shaughnessy; Mayberry 1983; Treffers, 1978). Gutierrez and colleagues have applied this theory to the understanding of three-dimensional geometry (Gutierrez et al, 1991, 1996). However, also associated with geometric understanding is a developing sense of general spatial perception (Del Grande, 1987; Bishop, 1980, 1983). Del Grande (1987) considered activities that might enhance spatial abilities, based on work of Frostig and Horne (1964) and Hoffer (1977). He proposed seven spatial abilities that seemed to be of greatest relevance in academic development in geometry. Bishop (1980, 1983) also pointed out the importance of spatial abilities in mathematics education. The object of this study was two-fold. The first was to develop an instrument to assess van Hiele levels and spatial abilities. The second was to relate students' questioning and the use of language to their observed van Hiele levels and Del Grande's spatial perception abilities.

Method

We gave 25 students enrolled in secondary initial teacher training (PGCE) a written test designed to estimate their van Hiele levels and spatial perception. 12 of these students were from the year 1995-1996 and the other 13 were in the year 1996-1997. From the group of 25 PGCE students, 7 from the 1996-1997 group volunteered to participate in group discussions on identification of 3-dimensional shapes (all, but one, were polyhedra). In the group discussions one student had a 3-dimensional shape out of sight of the others, whose task was to identify it by asking questions. We videotaped each group session for analysis.

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Test questions

We designed the test to determine van Hiele levels of achievement in three-dimensional geometry understanding, and to access Del Grande's spatial perception categories. Each of the seven questions contained various sub-questions. The arrangement of the questions in relation to the van Hiele levels and the Del Grande categories was as follows:

	Level 1	Level 2	Level 3	Level 4
Perceptual Constancy	Question 1		Question 7	
Figure Ground Perception		Question 2		Question 6
Position in Space Perception		Question 3		
Visual Discrimination				Question 4
Spatial Relationships			Question 5	

Table 1 - Questions corresponding to van Hiele levels and Del Grande's categories

Sample questions

Table 1 above shows, for example, that Question 3 is linked with van Hiele level two and Del Grande position-in-space perception. In this question we requested students to distinguish figures of different three-dimensional shapes and recognise the equivalence of different views to successfully group them. The question also asked for a list of properties for each group.

Question 7 is linked with van Hiele level three and Del Grande perceptual constancy of shape and size. The objects drawn consisted of representations of three dimensional shapes formed from cubical blocks. These blocks were attached face to face forming rigid three dimensional structures. The representation of each pair of identical three dimensional objects differed only by rotation. Students had to shade the right hand faces of objects and to match them in pairs.

Reliability of the test

Table 2 below shows the Kuder-Richardson inter-term reliability for each question. Question 1 had zero variance of scores, and we removed question 5 from the test scores, and the overall calculation of reliability, due to its very low reliability coefficient (considerably less than 0.5).

Question:	1	2	3	4	5	6	7	overall
Reliability:	-	0.64	0.87	0.66	<< 0.5	0.63	0.91	0.76

Table 2 - Kuder-Richardson inter-term reliability coefficients

General results from the test

Gutierrez et al (1991) proposed a coding system to assign students to a specific degree of acquisition within each van Hiele level. We used a modification of that coding system to construct Table 3, below. For three of the students in the study we were not able to assign a clear van Hiele level. The frequency distribution of students according to the degree of acquisition and van Hiele levels shows the hierarchical structure of the latter.

	Not attained		Intermediate	Attained	
Codes:	Blank	Inappropriate	Insufficient	Adequate	Precise
Level 1				5	20
Level 2			3	11	11
Level 3	2		5	10	8
Level 4		15	8		2
Level 5					

Table 3 - Number of students according to degree of acquisition of van Hiele levels

Codes:	Blank	Inappropriate	Insufficient	Adequate	Precise
Perceptual Constancy	2	1	3	3	16
Figure Ground		4	2	17	2
Position in Space			1		24
Visual Discrimination		8	4	7	6
Spatial Relationships		3	2	9	11

Table 5 - Number of students according to degree of acquisition of Del Grande perceptual abilities. Note that the Del Grande Spatial Perception categories are not hierarchical.

Sample results from the discussion groups

This part of the experiment was carried out using concrete three dimensional shapes. Pen and paper were available to the students who were encouraged to draw the shapes during their description. After considering each object, the students stopped only when they believed they had successfully completed their oral description. Note that we presented no constraints to the students regarding which words or technical terms they could use. Throughout the discussion the student with the shape provided information, on request, to the rest of the group. The table below shows the van Hiele levels and degrees of acquisition of spatial abilities of the 7 students who participated in the discussion groups. Note that although student B is assigned level 2 in transition to level 3, in fact in was not possible for us to assign a clear van Hiele level since there was not evidence of a clear attainment of level 2. Similarly, student F is assigned level 1 in transition to level 2, but this student exhibited evidence of level 3 thought.

	Van Hiele level	Spatial Abilities: Constancy of shape and size	Figure ground	Position in space	Visual discrimination	Spatial relationship
Student A	3⇨4	Precise	Adequate	Precise	Precise	Precise
Student B	2⇨3	Precise	Adequate	Insufficient	Insufficient	Inappropriate
Student C	4	Precise	Adequate	Precise	Precise	Precise
Student D	3	Adequate	Insufficient	Precise	Adequate	Precise
Student F	1⇨2	Precise	Inappropriate	Precise	Precise	Precise
Student K	2	Precise	Adequate	Precise	Inappropriate	Inappropriate
Student R	3	Precise	Adequate	Precise	Adequate	Adequate

Table 6: the van Hiele levels and degree of acquisition of spatial abilities on the written test, of the seven students engaged in oral discussion

Transcript of discussion group 1

Student K (level 2) manipulated a shape behind a screen, out of sight of students A (level 3⇨4), D (level 3) and C (level 4). Student A begins with a quantitative and relevant question, but uses the word "sides" rather than "faces":

A: How many sides has the shape got?
K: Six.

Student D checks that "sides" means "faces" (2-dimensional polygonal boundaries):

D: Is that six faces?
K: Yes, six faces, sorry.
C: Are they all the same?
K: Yeah.

Now student A uses "sides" to mean "1-dimensional simplicial boundary"

A: How many sides does each face have?
K: Four
A: So is each face a square?
K: No

Student A then asks for metric information:

A: Are all the angles of each of the faces right angles?
K: No

and student C, apparently not aware, or forgetting, that the faces have 4 edges, asks:

C: Is each face a triangle?
K: No
A: What, with four sides?
General laughter.

Again student A asks for pertinent information, leading her to identify the shape in her own mind:

A: Are opposite sides of the face, ahh .. parallel?
K: Yes, and they are all equal. I would say.
A: So, it's a rhombus?
K: Yes.
D: (Inaudible)
C: (To D) Each face is a rhombus.

However, further questioning by student A reveals that the identification of the shape as a "rhombus" is still somewhat confused:

A: Would the shape be as if you had a cube, and then you tilted it one way?
K: Mmm, yeah.. if I think about it I know what you mean.
Teacher: ... he said to elaborate on something like "tilt" for example, ... if she wanted to.
K: What do you mean by tilt? Do you mean sort of almost pushed to one side?

Student A explains that if one puts a cube on the ground, such that it keeps its face on the ground in a fixed position, and in addition pushes the top, the required shape would be formed. This student did not take into account that just the lateral faces of a cube would change from a square to a rhombus, with the top and bottom remaining the same shape: a square. This was clear to student C, who was functioning at level 4. She demonstrated an ability to understand the logic of the situation:

A: If you keep .. if you put it on the ground, if you put a sq.. a cube on the ground, and you kept the face that's on the ground absolutely in position, and then pushed from the top .... so that the whole thing sort of went out shape. Just skewed over. Does that make any sense?
C: No,... the faces at the bottom will still be a square, won't they?
D: Kind of like you had a cube but just sheared to one side.
K: Yes. That's what I was trying to say.

Student C initiates a discussion on parallelism of opposite faces, which leads the group to agree that they have identified the object:

C: Are, ... are opposite faces parallel?
A: Does it have 3 pairs of opposite parallel faces?
A: Top and bottom, are they parallel?
K: Yes, If you were to look at it umm.. head on you could almost have a , well you have a ...... diamond shape. ....
Others: yeah, yeah.
K: That's right to say diamond shape, isn't it?
A: It's a kite, yeah? Like a kite.
K: Yes, ... 8 vertices.
A: Don't get technical on me!

Transcript of discussion group 2

Student F (no assigned level: level 1⇨2 generally, but shows evidence of level 3 thought) manipulated a shape behind a screen, out of sight of students B (no assigned level: level 2⇨3, but lacks some aspects of level 2 thought) and R (level 3).

Student B begins with a question he asked several times in group discussions. This question, which is more or less subjective, does not focus on properties of the object, and is in keeping with B's lacking some aspects of level 2 thought:

B: Is it a complex object?
F: No.

Nowhere had we told the students about the flatness or otherwise of the faces of the objects, so R's next question is quite reasonable.

R: Is it made up of flat surfaces or curved ones?
F: Flat.

Student B asks a relevant question about the nature of the faces. Notice, however, that B uses "sides" in two senses. Then B asks two questions which on the basis of the known information, are merely guesses. Finally, B asks a pertinent question about the number of faces.

B: Does it have the ... are some sides 4 sided?
F: Ah They all are.
B: Is it a cube?
F: No.
B: Is it a cuboid?
F: No.
B: How many sides does it have?
F: Six.

Student R asks for metric information about angles - in keeping with the assigned van Hiele level 3:

R: What are the angles between the faces?
Teacher: (After a puzzled pause from student F) We haven't got a protractor!
R: An approximation will do.
F: Umm ...
R: 30, 45, 60, 90?
F: Uh, in one face, there are two sets of two sides which meet at 45 ... and uhm, the other two corners uhm .. the angles are greater than 90 ... and the, The sides are parallel.

Student R then asks about the number of faces despite B having already ascertained that there were 6.

R: How many faces are there?
F: Six.

Student B again asks a series of questions which indicate a functioning van Hiele level 2, in that the questions are more analytical. They also involve visual discrimination in which this student tested poorly.

B: Are all the faces identical?
F: Yes.
B: And are they rectangles or cubes... ahh, squares?
F: Neither.
B: But they are all four sided?
F: Yes.
B: Are opposite sides , (inaudible) are they parallel?
F: Yeah.

Student R makes an attempt at identification, and after being unable to draw the object, asks a question involving symmetry, which one would expect at level 4. Note that this student has good spatial abilities in all areas and very good position-in-space abilities:

R: So it's a sort of squashed cube really?
F: Yeah.
R: I can't draw it ... (inaudible)
R: If you, er, if you pick it up, and turn it around in 90 degrees and put it down again, would it, would it fit back into its previous position?
F: Yes. And there is a lot of symmetry in, in each face as well. Between, if you uhm, put a line between them ... bisecting angles.

We found that the interchange of knowledge that took place during the discussions contributed to the development of the students. This led some of them to a higher van Hiele level than that observed in the written test. For example, in the written test student B presented very poor responses related to properties of shapes. This student also showed problems associated with position in space perception. On question three the student regarded tetrahedrons and square based pyramids as identical. The discussion above indicates a relatively higher achievement for this student in relation to the written test.

Comparing the above dialogues it becomes clear that group 1 used a better mathematical language than group 2. They were able to make clear when the discussion was about two-dimensional shapes (a face of the solid) or three dimensional objects. In the beginning of the discussion, without consistent information, group 2 tried to determine the shape. Also this group used the word "side" without explaining carefully, or negotiating, what they meant.

Conclusion

The students' oral description of a certain shape depend on a combination of the students' general geometric level, their spatial ability, and their ability to express the properties of the shape using language. The results from the discussion groups show the importance of both spatial ability and language use in the on-going development of geometric thought. The assessment of van Hiele levels of thinking and Del Grande spatial perception applied to three dimensional geometry confirmed the hierarchical nature of the van Hiele levels (Usiskin 1982, Mayberry 1983, Gutierrez et al 1991). However the assessment of Del Grande spatial perception is not hierarchical. This research shows that it is possible for a student to acquire the abilities pointed out by Del Grande without any regular sequence.

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ALGEBRAIC EXPRESSIONS AND EQUATIONS: AN EXAMPLE OF THE EVOLUTION OF THE NOTIONS

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The work focuses on a 15th year old student and on her knowledge about algebraic expressions and equations. Three interviews with her allow us to study her progression in constructing a personal concept of equations which appears to be, at the end, very different from what it was at the beginning and much more efficient. For this purpose the interviewer leads the interviews in a way which is based on two theoretical tools, one concerning the structure of knowledge, the other one the different possibilities of evolution of the mathematical knowledge.

In this work we shall concern ourselves with a girl student, Leslie, aged 15, through three interviews. We shall see how she builds up gradually some local bits of knowledge on algebraic expressions, starting with two which are at the beginning in a violent opposition. As we proceed we shall point out some aspects of our work which validate the theoretical tools we use:

- it is possible to make a diagnosis of the way a student works in mathematics
- it is possible to change this way of working when it appears to be inappropriate, and to lead the student to have a reflexive way of thinking about his work, so as to induce a remediation if necessary.

These interviews have been collected as part of a research project. They had of course an influence on Leslie work at least during the interviews, but there was no specific aim to help her perform better at school. Thus we did not work on many of her local bits of knowledge although we identified them.

THEORETICAL TOOLS

Local bits of knowledge

The notion of local bit of knowledge (Léonard & Sackur 1991) is quite close to those of representation or misconception but it includes two ideas which have, for us, the greatest importance, and make the former word preferable to us.

- they are bits of knowledge, which means that they are true
- they are locally true.

We consider them as actual knowledge, which implies a positive connotation whereas "misconceptions" is more negative.

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To explain what we really mean by locally true we need to consider three areas which are useful to interpret the student's activity: the psychological area in which the local bit of knowledge is coherent for the student, that is has no internal contradiction; the social area in which the local bit of knowledge is valid, accepted by a certain community (the social group of mathematicians); the area of reality in which the local bit of knowledge is efficient, thus yields a correct answer if used to solve a problem. A local bit of knowledge possesses these three properties within certain limits which the student ignores (that is why s/he makes mistakes and so local bit of knowledge bear some resemblance to misconceptions). These limits can be identified by someone whose knowledge is less partial than the student's, an expert.

A local bit of knowledge in algebra is for instance: "when one multiplies x by a the result is greater than x". It is easy to identify the limits of such a bit of knowledge: the set of numbers bigger than 1. One can imagine where this bit of knowledge comes from since many students have difficulties dealing with numbers which are not positive integers. It has the three properties just described: if one considers only numbers greater than 1 this bit of knowledge is coherent, it is recognised as true by mathematicians and produces correct results on these numbers. Of course as the student ignores the limits of this bit of knowledge s/he may use it outside these limits (with other numbers) and this will lead to some incorrect results. We shall discover some other local bits of knowledge when studying the work of Leslie.

The polarization of the activity

We use three types of polarization to describe the activity of a person working in algebra, each of these type corresponds to one of the three areas presented above: if the student is mostly concerned with the internal coherence of his own activity, if s/he assesses his results by his own standards, we call his polarization of work understanding; if his standards of assessment are borrowed from those of mathematicians, if they are rules which are mathematically correct, we call it conformity; if s/he takes his standards in the reality, we call it performance. In mathematics it is quite difficult to define the reality; for the moment we are still working on it, and we consider the mathematical reality as an exterior standard which permits the student to point out contradictions. We shall see how Leslie deals with it.

Conformity

Conformity plays a very important role in mathematics and especially in algebra. It should not be taken in a pejorative sense, quite the contrary. From the very beginning of algebra and Cartesian geometry, mathematicians have developed rules which permit them to make computations without referring to the meaning of the computations. Historically, this particularity of algebra has been pointed out by Leibnitz who wrote about "Blind Calculation". This led us to refer to students who make computations without any control as "blind calculators" (Drouhard, Léonard, Maurel, Pécal & Sackur, 1994).

The problem we found monitoring the progress of students working in algebra is that they learn a catalogue of rules without links and sometimes they can't even imagine that there is a rational presiding over these rules, and that they could know it and use it to control their knowledge.

Purpose of the research

Our research tends to build up tools to help students learn algebra and to cope with their difficulties; we have observed several students and collected interviews. The use of our work for remediation purpose has just started and has yield some results which we can't discuss here.

We observed that most of the time, a student who has difficulties in algebra uses rules in relation with the type of polarization which we call <u>conformity</u> and his rules are local bits of knowledge with a very small validity domain, so that they are only formal:

"when there is a sign "-" before brackets, I change all signs..."

"when there is a 2 before x I put it on the other side, changing signs..."

We believe it is necessary for the student as a first step to shift to a different type of polarization in his work, if possible to the <u>understanding</u> type, to be able to coordinate the different local bits of knowledge and build up more comprehensive ones. For this we try to confront her/him with the mathematical reality, using <u>performance.</u> To explain this, very shortly, let us say that we don't tell her/him "look your rule is false, the right one is...", neither do we explain to her/him again the reasons why s/he makes such a mistake. Instead, we try to put her/him in a situation where his own local bits of knowledge produce a result which is contradictory from a mathematical point of view (such as $1 = 2$) so that s/he has to change something to resolve this contradiction.

This has led us to develop a particular way for interviewing students which we call "Write False Interviews", as we always start by asking the student to write something s/he knows is false (Sackur 1995).

II- THE INTERVIEWS WITH LESLIE

First interview

Leslie is asked to write a false equality with $\frac{7x}{7+x} =$. She writes $\frac{7x}{7+x} = \frac{7x}{7x}$ and she says that it is false. She gives two reasons for this equality to be false and these are:

1) a "+" sign has been changed into a "×" sign so this is not the same and the equality has to be false,

2) $\frac{7x}{7x}$ equals 1 and $\frac{7x}{7+x}$ cannot be equal to 1, so the equality is false.

She tries it with some values of x. ($x=1$: $7/8 \neq 1$; $x=2$: $14/9 \neq 1$). Then Leslie is asked whether this equality is always false. To answer this question she places herself in a contradictory situation. There is a conflict between two local bits of knowledge:

LBK 1. if one changes a computing sign in an algebraic expression, then the expression changes, which means that its numerical value changes according to numerical values of x. (The formal aspect of the expression changes and so does the expression.)

LBK 2. if there is a sign "=" between two algebraic expressions using the letter x, this is an equation and some computations should "logically" give a value for "x".

The first of these two local bits of knowledge is an answer to the paradoxical injunction (Watzlawick, Beavin & Jackson 1967) "write false" using conformity: the student knows a rule; to produce something false s/he modifies the rule; using conformity the change is only formal.

The interview is led in such a way (Vermersch 1995) that it does not permit the student to go on working using conformity. She has to shift to another type of polarization, performance or understanding. Thus other local bits of knowledge are used. For Leslie, in this situation, there is a conflict which appears clearly in what she says:

S1. one should have 7x = 7+x and one knows that this is false

S2. logically if we go on we should be able to find x.

Those two sentences appear frequently during the first interview. Leslie keeps talking about equations and about finding x, but she cannot do it. What is interesting for us is that she is no longer able to solve an equation as simple as $7x=7+x$ (this she can perfectly do in the classroom). She has been brought out of her type of work, which is conformity and she seems to have no longer any adequate knowledge to solve this problem; then it will be possible for her to build something new and more appropriate. As the interview goes on she tries several computations to solve this equation but she always fails and she keeps repeating sentence S1. At the end of the interview she seems to have convinced herself that $\frac{7x}{7+x}$ is always different from $\frac{7x}{7x}$.

With this first interview one can make a diagnosis about the way Leslie works in algebra and so one can think of a remedial activity. An effect of the interview is that Leslie left aside the conformity type, she has confronted mathematical reality and it will be possible for us to act on her local bits of knowledge.

Second interview

The second interview takes place twelve days later and Leslie recalls easily what the problem was:

Le: one had to know if $\frac{7x}{7+x}$ could really be equal to 1

In: so what do you think about it?

Le: no, it is different

In: when you say it is different, could you explain what is different?

Le: here we have a sign "+" and here we have a sign "×", and that makes it different.

She is still working on the formal aspect of the expression. The difference of sign produces a difference of values as she shows again with some values of x. So the interviewer asks her whether it is possible for an addition and a multiplication to yield the same result. Leslie produces $3+x$ and $4x$ which are equal if x equals 1, but she very clearly states that if the number and the x are the same ($4+x$ and $4x$ or $7+x$ and $7x$) then the result is different and she says that she is demonstrating it using values of x. At this time Leslie is strongly trapped in the first local bit of knowledge:

LBK 1 "changing a sign into another changes the expression".

At that moment the interviewer has to make a decision. It doesn't seem very interesting to create the same conflict as in the first interview, just in the same way as Leslie didn't manage to resolve it. So the interviewer asks:

In: what is the use of the values of x you test?

Le: well one gives us an equality, and we don't know whether it is false or not... and so we shall try to find the x in $\frac{3+x}{4x}$..., so that if we place it into $\frac{3x}{4x}$ we can see whether it is equal or not

In: and we could try to do it

Leslie is now constructing a new bit of knowledge about equations:

LBK' 2 "we have an equality and we don't know whether it is false or not..."

This local bit of knowledge has nothing to do with the very formal one that we saw at the beginning:

LBK 2 "an equation is made of two algebraic expressions linked by an "=" sign"

Leslie solves the equation $\frac{3+x}{4x} = 1$ not very quickly but she solves it,

and then with $\frac{7+x}{7x} = 1$ there is no problem. At the end she claims:

LBK 3 - "an equation always have one solution and one only; $\frac{7x}{7+x}$ equals 1 if x equals 7/6 otherwise it is different from 1".

With the second interview we can see that when she is confronted with a specific situation, Leslie recalls quickly the tools she possesses to solve a problem. When she has solved one equation she has no difficulty to solve another one. On the other end she has a tendency to construct rules about things: "all equations have a solution...". This is her way to learn mathematics. Nevertheless it has been possible to make her elaborate by herself on her own knowledge: she constructed her personal way to talk about an equation: confronted with the reality "multiplication and addition can yield the same result", which she didn't believe at first, she had to construct a system which could take this into account.

The third interview

This interview takes place after the summer holidays that is two and a half months later. Leslie is first asked to solve an equation: $x+2 = 5x$ which she does without difficulty; she checks that the solution is right although she has not been asked to do it, and she comments on what she is doing:

"one doesn't know what is the x which permits to find $x+2 = 5x$ so one has to isolate x...

...so one cannot say that $x+2 = 5x$ is false as long as one has not found the x which goes with this equation".

We find here that after a very long period when she had no opportunity to discuss the problem the local bit of knowledge LBK'2 about equations that she developed in the second interview is still efficient. This is a characteristic of Leslie's work in mathematics, bits of knowledge acquire very quickly a strong stability; this can explain why she is quite good in school and why she has some difficulties which resist all explanations. The other local bit of knowledge LBK 3 we found at the end of the second interview is also still valid: "all equations have solutions". This we can observe in what she says:

"one must first find if there is one x or several which could go in this equation"

and also in what she does:

- confronted with the equation $24\ a^2 = -8$, she solves $24\ a^2 = 8$ "because a square number can never be negative". She cannot imagine that this equation has no solution.

Again she will shift from the <u>conformity</u> type to another one by the end of the interview; this time also the way to make her progress in her activity is to make her confront reality: how is it that one solves the equation $24x = -8$ and has to change equation $24\ a^2 = -8$ into $24\ a^2 = 8$? She then recalls an equation she worked on in the classroom that very morning $0x = 9$ and she concludes:

LBK' 3 "some equations have no solution".

By this time it seems that she starts being able to take responsibility for her own thinking in mathematics, which means, for us, that she is able to leave the conformity type aside in order to switch to her own standards.

Interpretation

We shall now briefly sum up the main observations one can make about these interviews concerning the way Leslie works in algebra.

• What is typical of Leslie's work in algebra is that she needs to have very stable local bits of knowledge. She has no flexibility at all. When she leaves aside a bit of knowledge because she realises that it doesn't fit the problem she constructs immediately another one almost as stable as the one before. There are no links between them and she doesn't seem aware that there should be links. Even when she gives spontaneous explanations of some property, one can see, observing the words she uses, that there is no personal investment but a mere repetition of things that have to be done; an example of this is: "One cannot divide by zero because in the math class when we have a denominator we always look for the value of x which makes it equal to zero". As she has a good memory and works hard she identifies easily the clues specific to one situation and she performs quite well in algebra, although, as we have seen, her knowledge is poorly structured.

• We would like now to examine briefly, neither the evolution of Leslie's bits of knowledge, nor the way she shifts from one type of activity to another but her behaviour during the interviews. In the first interview she was always hesitating, she had great difficulties to find the correct word and kept on making slips of the tongue (writing "$7 + x$" and saying "seven x plus one"). We interpret this great confusion as being created by the conflict mentioned at the beginning and by the fact that she had no way to resolve it. From the point of view of her elocution the second interview was much better and in the third we saw a complete change. She then spoke clearly, the sentences were almost correct with few blanks and appropriate words in the right place.

In the first interview the interviewer spoke a lot, repeating what she had just said to help her go on thinking. In the two others she was really leading the work. Several times she decided what to do, which equation to solve, when give an example and so on. This doesn't mean that there were no mistakes, quite the contrary, there were plenty which we did not work on. We can assume that these mistakes are an evidence that she was doing "her" mathematics, solving "her" problems, using "her" methods. She was not solving an exercise given by the teacher with an appropriate well known method. She was wandering inside mathematics, in a way which seemed to us rather erratic but was certainly not for her. She had begun to learn that mathematics are not only a list of similar exercises to be solved in an unique given way. We think that this perception of mathematics, which resembles what we call the understanding type can be a great help to a student who has difficulties and takes refuge in misunderstood conformity.

CONCLUSION

The interviews, and the theoretical tools that we use to interpret them, permit us:

- to identify the local bits of knowledge of a student, their limits within mathematics and to examine how stable they are,
- to make a diagnosis on the dominant type of polarization used by the student to work (conformity, performance or understanding),
- to make the student shift from one type to another so as to acquire some autonomy in her mathematical activity, to make her have reflexive thinking and lead her to construct the necessary links between her local bits of knowledge,
- to try a remediation on the bits of knowledge. This means working on three things:
    - breaking their inner coherence,
    - making them fail in solving problems,
    - making them contradict accepted mathematical rules.

It is necessary to prepare carefully the interviews; while the first interview can begin with almost any question, the followings must permit the interviewer to confirm what s/he first found and to proceed as well in his knowledge of the student's thinking as in the opportunities s/he gives her to develop her autonomy. This is mainly the job of a mathematician as long as s/he can easily deal with our theoretical tools. So in this work mathematics and psychology must go hand in hand.

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INFERENTIAL PROCESSES IN MICHAEL'S MATHEMATICAL THINKING

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This paper analyzes the zigzagging inferential processes (abductions, inductions deductions) that characterized the mathematical activity of a third grader as he faced the challenge of solving a novel mathematical task. The task was to find all the possible pairs, triples, and quadruples from two, three, and four groups of cards.

Types of inference

The mathematician and semiotician Charles Sanders Peirce (1878a, 1878b, 1903) stressed three forms of inferential reasoning in our inquiries: abduction, induction, and deduction (see Figure 1). These three types of inferences do not work in isolation although they have different characteristics. According to him, a deductive inference is an analytical process by which particulars follow from general premises; thus, no new knowledge is produced in the process since there is nothing in the particular cases that is not implied by the premises. In contrast, he argued that new knowledge is gained "through the mechanisms of induction, which itself functions with the logic of probability....Induction works by taking a random sampling of a class of things so that some classwide conclusion can be drawn about respects held in common by all members of the class" (Peirce, as quoted in Corrington, 1993, p. 43). That is, through inductive inference new general principles or laws are generated from particular instances.

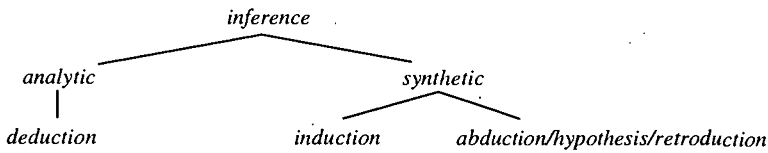


Figure 1. Types of inferential reasoning.
(Adapted from Peirce, 1878 a)

Peirce (1903) clarified that "the first starting of a hypothesis and the entertaining of it, whether as a single interrogation or with any degree of confidence, is an inferential step which I propose to call *abduction* [or *retroduction*]...I call such inference by the peculiar name, *abduction*, because its legitimacy depends upon altogether different principles from those of other kinds of inference" (p. 151). For Peirce (1878a, 1878b, 1903), deduction and induction alone cannot account for the introduction of new knowledge without abduction. He argued that abduction provides the reasoner with a hypothesis that accounts for the observed facts. "Unlike inductive inference, which moves from random samples toward a conclusion about a class of objects, hypothesis goes in the opposite direction, proceeding from a general rule or theory toward a given particular case" (Peirce, as quoted in Corrington, 1993, p. 46). The generation of a hypothesis is a creative act that may come to us "in a flash" or through an extended period of fantasy or work. According to Merrell (1995), an abduction or an abductive conjecture or hypothesis is often a guess but an educated guess "ideally following from much living, much study, and much contemplation of life in general and of a particular problem one has at hand" (p. 56).

The generation of a hypothesis is only one aspect of the abductive process; besides creation, abduction also involves application. A hypothesis must be applied to the case at hand and compete with other hypotheses that could possibly explain the observed facts (Corrington, 1993). Peirce also emphasized the transitory, slippery and temporary nature of the process of generating hypotheses as well as its relative lack of certainty, since more than one hypothesis could be entitled to account for observed facts (Anderson, 1995). Peirce (1903) acknowledged that the selection of a hypothesis is subjected to certain conditions which were already recognized by logicians long before he classified it as an inference. "The hypothesis cannot be admitted even as a hypothesis, unless it be supposed that it would account for the facts or some of them. The form of the inference, therefore, is this: The surprising fact, C, is observed; but if A were true, C would be a matter of course; hence there is a reason to suspect that A is true. Thus, A cannot be abductively inferred, or if you prefer the expression, abductively conjectured until its entire content is already present in the premis(e), 'if A were true, C would be a matter of course'" (pp. 151-152).

Mathematicians and mathematics educators have recognized the influence of abductive processes in mathematical thinking, although under different names. Polya (1945), for example, cast heuristic reasoning under the light of a plausible guess and different from the deductive type of reasoning that furnishes a proof or the attainment of a complete solution of a problem. He says, "We must often be satisfied with a more or less plausible guess. We need the provisional before we attain the final" (p. 113). Lakatos (1976) acknowledged the nonlinearity of inferential reasoning as he says, "Discovery does not go up or down, but it follows a zig-zag path; prodded by counterexamples, it moves from the naive conjecture to the premises and then turns back again to delete the naive conjecture and replace it by a theorem. Naive conjectures and counterexamples do not appear in the fully fledged deductive structure: the zig-zag of discovery cannot be discerned in the end product" (p. 46). Hence, new mathematics knowledge is not only dependent on abductions but follows the zigzagging inferential process of abductions, inductions and deductions. Consequently, the zigzagging nature of inferential processes in mathematics inquiry does not appear written in textbooks and will not be foregrounded in teaching. Mason (1995) points out that in trying to avoid difficulties, "the curriculum turns everything into behavior, avoids awareness, assumes deduction, tolerates induction, and ignores abduction" (p. 4).

Methodology

Teaching experiment

The constructivist teaching experiment methodology consists of long-term interactions between the researcher/teacher and students. This methodology focuses primarily on the students' conceptual constructions. The main goal is to infer the students' constructions of mathematical concepts and operations (Steffe, 1983). Interactions between the researcher/teacher and the students are intended to bring forth the students' mathematical activity and the constructive evolution of his/her mathematical concepts. A constructivist teaching experiment was conducted with six third graders who were interviewed 17 times throughout the school year to inquire about their constructive efforts to conceptualize fractions.

About Michael

Michael was a nine-year old boy who participated in the teaching experiment with five other children. He was perceived by his school teacher as a good student, but not doing as well in mathematics as the others. Michael was a quiet and reflective child, always willing to accept intellectual challenges and able to recapture his mental activity and express it in words. He had participated on eight interviews on whole numbers and fractions before he participated in the interview reported here. This interview served as a warm-up after the Christmas break. In an earlier paper (Cifarelli and Sáenz-Ludlow, 1996), Michael's mathematical activity in solving this task was partially analyzed to illustrate hypothetical reasoning. The present paper extends this analysis to specify the mediating role that abduction had in his thinking.

About the task

The task posed to Michael in this interview was to find the total number of pairs, triples, and quadruples from different groups of objects. He was presented with four groups of cards in the following sequence: (a) a group of letters and a group of numbers; (b) a group of letters, a group of numbers, and a group of figures; (c) a group of letters, a group of numbers, a group of figures, and a group of colors. The cards were displayed over different sheets of construction paper as a means to define a boundary for each group of cards. The number of cards in each set was changed to vary the degrees of difficulty of the task. Michael was asked for the number of pairs, triples, and quadruples that might be possible to make if he were to choose one card from each set. This task was used by Leslie P. Steffe in his teaching experiments with second and third graders and he encouraged the author to use it.

Analysis

Making pairs

The following dialogue took place when the teacher/researcher (T/R) asked Michael for all the possible pairs that could be made from the letter-set with the letters A and B and the number-set with the number 1.

1 T/R: *Do you know what a pair is?*

2 M: *Yes, it is two of a kind.*

3 T/R: *(displays cards with the letters A and B over the first sheet of paper; and a card with the number 1 over the second sheet of paper)*[1] *How many pairs can you make here?*

4 M: *None.*

5 T/R: *Why?*

6 M: *Because they are not alike.*

7 T/R: *OK. Suppose that even if they are not alike you can still pair them by taking first a letter and then a number.*

8 M: *(After some seconds) OK. One, A1... Wait A1, B1.*

9 T/R: *(displays cards with the letters A and B, over the first sheet of paper; and cards with the numbers 1 and 2 over the second sheet of paper) How many pairs can you make now?*

10 M: *(moves the cards A and B close to 1) A1, B1; (moves the cards A and B close to 2) A2, B2.*

Upon the agreement of the constitution of a pair from two different sets, Michael seems to have found a pattern or strategy to form such pairs. This pattern is indicated by his explanations and gesturing (Lines 8 and 10). A moment of reflection on the above dialogue brings to mind several questions: Where did the pattern come from? Will Michael apply the pattern again? Will he be able to find the number of pairs before finding the actual pairs? Is he aware of the pattern and will he be able to describe it? The following dialogue (Lines 11-18) indicates some answers to these questions.

11 T/R: *(displays cards with the letters A and B over the first sheet of paper; and cards with the numbers 1, 2, and 3 over the second sheet of paper) How many pairs can you make now?*

12 M: *Six. A1, B1; A2, B2; A3, B3.*

13 T/R: *(adds a card with the letter C to the cards with the letters A and B and the card with the number 4 to the cards with the numbers 1, 2, and 3) How many pairs can you make here?*

14 M: *This is going to be hard (after some seconds he moves his fingers from each of the letters A, B, C toward the number 1, some seconds later he gives the results) 3, 6, 9, 12.*

15 T/R: *Why?*

16 M: *Because there are three letters here (showing the letters A, B, and C). A, B, C can be paired with 1, that's 3; A, B, C can be paired with 2, that's 6; A, B, C can be paired with 3, that's 9; A, B, C can be paired with 4, that's 12. (touching each number-card he says) 3, 6, 9, 12.*

17 T/R: *(adds a card with the letter D to the letter-set. Now the letters over the first sheet of paper are A, B, C, and D and the numbers over the second sheet of paper are 1, 2, 3, and 4) How many pairs can you make here?*

18 M: *Sixteen either way. These (showing the letters) can go over here (showing the numbers) or these (showing the numbers) can go over here (showing the letters). All four letters can go to 1, that's 4; all four letters can go to 2, that's 4; all four letters can go to 3, that's 4; all four letters can go to 4, that's 4. So there are sixteen in all.*

19 T/R: *Very good!*

Michael is not only able to apply the pattern again but finds the total number of pairs before he describes them (Line 12). This is an indication that Michael is well aware of the pattern he generated. Michael applied his conjecture for making pairs when he was asked for the number of pairs from the three letter-set (A, B, and C) and the four number-set (1, 2, 3, and 4). After some seconds of thinking, Michael came up with the total number of pairs by saying "3, 6, 9, 12" (Line 14). The four times that he counted by 3's indicates that he was pairing each time the three letters with

[1]Comments in parenthesis are descriptions of Michael's and teacher/researcher's nonverbal actions.

each one of the four numbers. In Line 16, he verified his conjecture describing such pairing in detail indicating an awareness of the pattern. In Line 18, he described the pattern demonstrating again the application of his initial abduction (hypothesis or conjecture) of pairing the letters with each of the numbers.

Up to this particular point in the interview, Michael had proceeded in an inductive manner supported by his first *organizing abduction* (coupling all the letters in the letter-set with each of the numbers in the number-set). The word *organization* comes from the Greek term *organon* which means *instrument*; in this sense, the organizing abduction makes reference to the instrumental nature of Michael's insight as to the constitution of the relational order which could lead to the completion of all the possible pairs from the sets of letters and numbers. Thus the answer to the first question "Where did the pattern come from?" is that it came from a creative act which was the result of an abduction that I have called an "organizing" abduction. Here, there is an instance of a creation and application of an abductive hypothesis. If he had not made that abduction, it would have been impossible for him to solve the tasks because there was nothing in the teacher's questioning that would have directed him to do so.

Making triples

In the following dialogues the teacher posed the problem of finding all possible triples from three different sets. The teacher systematically increased the number of objects in each set making the task more difficult each time.

20 T/R: *(places two cards with the letters A and B on the first sheet of paper; one card with the number 1 on the second sheet of paper; and one card with a ▲ on the third sheet of paper) How many triples can you make here?*

21 M: *Four. AB1, AB▲, A1▲, B1▲.*

22 T/R: *That is perfect if I wanted to have more than one element from each group. But if I want to have a letter once, a number once and a figure once, how many triples would you be able to make?*

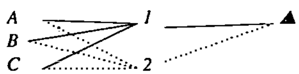
23 M: *Then you have two, A1▲, B1▲.*

24 T/R: *(displays the letters A, B and C over the first sheet of paper; the numbers 1 and 2 over the second; and the figure ▲ over the third) Now how many triples will you be able to make?*

25 M: *(after some seconds) Three.*

26 T/R: *OK, which ones?*

27 M: *(with his fingers makes a movement from the set of letters to number 1 and to the ▲, then from the set of letters to the number 2 and to the▲) A1▲, B1▲, C1▲; A2▲, B2▲, C2▲. (The figure below shows the trajectories Michael made with his fingers). Six.*



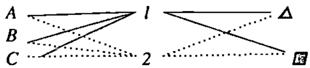
In Line 21, Michael described all possible *combinations* with three elements from the four elements in the three groups (A, B; 1; and ▲). Once an agreement was reached about the conditions for making the triples, he found two triplets (Line 23). As the teacher changed the conditions of the task adding one letter and one number (A, B, C; 1, 2; and ▲), he quickly answered "three". However, when he described them he found six (Line 27). His description of the triples indicated that he was pairing each of the possible couples from the first two groups with the element in the third group. The question that arises here is whether Michael would apply this pattern again or generate all possible triples from three sets in some random manner.

28 T/R: *(displays the letters A, B and C over the first sheet of paper; the numbers 1 and 2 over the second; and the figures Δ and □ over the third) Now how many triples will you be able to make?*

29 M: *(silently moves his fingers from left to right like tracing trajectories) Nine...; wait I can make ten.*

30 T/R: *Can you describe them for me?*

31 M: *(describes the triples while moving his fingers from left to right over the cards; the lines in the diagram represent the sweeping action of his right hand).*



A1▲, B1▲, C1▲, three; A2▲, B2▲, C2▲, six; A1■, B1■, C1■, nine; A2□, B2■, C2■. Twelve.

In line 28 the teacher varied the problem by adding a card with a square to the third group. Now the cards are (A, B and C; 1 and 2; and ▲ and ■). After giving the answer "nine...; wait I can make ten", Michael systematically described the triples while inscribing trajectories with his fingers to join the cards. A careful observation of the pattern of the triples indicates that the pairs from the first two sets are systematically coupled with each of the cards with the figures. Hence, Michael applied his pattern for making triples again. That is, all the elements from the first set (in this case couples) are paired with each of the elements of the second set (in this case ▲ and ■). If Michael had not considered each pair as an entity in itself, he would have not been able to generate this systematic strategy. Instead, he might have generated them in some random order making it difficult to keep track of them. The question that comes to mind here is whether or not Michael was aware of this pattern and in what degree. The following dialogue gives an answer to this question.

32 T/R: *(displays the letters A, B, C and D over the first sheet of paper; the numbers 1 and 2 over the second, and the figure ▲ over the third) In this case how many triples can you make?*

33 M: *(after some seconds) It's 8 because the number of triples is the same as the number of pairs if I take this triangle away.*

34 T/R: *(displays the letters A, B, C and D over the first sheet of paper; the numbers 1 and 2 over the second; and the figures ▲ and ■ over the third) So now how many triples would you be able to make?*

35 M: *(takes the triangle in his hands and says) like that I can make 8 triples; (puts the triangle back) that's 8. 8 and 8 is 16.*

36 T/R: *(puts four cards in each of the sets: four letters A, B, C and D; four numbers 1, 2, 3 and 4; and four figures ▲, ■, ● and ✳) How many triples would you be able to make now?*

37 M: *(after some seconds) 62.*

38 T/R: *Why?*

39 M: *Sixteen couples with numbers (Showing the set of letters and numbers). 16 and 16 is 32 (showing the ▲ and the ■) and 32 and 32 (showing the ● and the ✳) is 64.*

40 T/R: *(makes three piles of cards instead of leaving them displayed on the sheets of paper) How many triples can you make with these three piles of cards?*

41 M: *(Looks at the piles for some seconds and then counts the number of cards in each pile) Three cards here (cards with letters), and four cards here (cards with numbers), that is 12 pairs; 8 cards here (cards with figures), that makes 96 triples.*

42 T/R: *Why?*

43 M: *Because 12 and 12 is 24; two twelves are gone. 12 and 12 is 24; and 24 and 24 is 48, four twelves are gone. 48 and 48 is 96.*

In Line 33, Michael's solution was completely novel. His expression "the number of triples is the same as the number of pairs if I take this triangle away" indicated the use of each pair as an entity in itself to generate all the possible triples. In Line 35, he applied his abduction by coupling each pair from the first two groups with each figure in the third group. When the cards displayed were A, B, C, and D; 1, 2, 3, and 4; and ▲, ■, ●, and ✲, he found immediately 62 triples and made no attempt to describe them. As he described the process, he found 64 because "sixteen couples with the numbers (showing the set of letters and numbers). 16 and 16 is 32 (showing the triangle and the square) and 32 and 32 (showing the circle and the star) is 64." In Line 41, he also used this strategy in a deductive manner since, after several trials, he assumed its viability and applied it to the case at hand.

The novelty in his solution is accounted for by an abduction that is more complex than the *organizing abduction* that led him to establish a particular way of relating the objects of two groups to make pairs. What is novel in his new insight is that he implicitly considered the triple as a couple whose first element is a couple itself. I call the above abduction a *structuring abduction* because it is more complex (considering a couple as a single element) and general (considering a triple as a couple) than his organizing abduction. The word *structure* comes from the Latin verb *struere*, which means to construct. Michael's abduction here is called "*structuring*" to emphasize his complex insight (or abstract organization) in taking one pair as a unique entity and each triple as a complex pair.

Making quadruples

Michael's explanations to find pairs and triples in the above tasks indicated abduction-induction-deduction chains of reasoning. Would Michael be able to carry out this chain reasoning to find the number of all possible quadruples with four different groups of cards? The following dialogue indicated that he was able to do so.

44 T/R: *How many quadruples can you make here?*

A	*1*	▲	*blue*
B	*2*	■	*yellow*
C	*3*		
D			

45 M: *(Looks at the cards) This is hard! (After some seconds) Let me see how many couples are over here [takes the triangle and the square in his hands] twelve. Twelve and twelve is twenty-four triples (returning one figure-card at a time). Then twenty-four and twenty-four is forty-eight quadruples (showing each color-card).*

46 T/R: *Why?*

47 M: *I started making pairs, then triples and then quadruples.*

Michael's solution to the above task indicated another abduction-induction chain rooted in his prior structuring abduction. His explanation in Line 45, indicated that he took each couple as a unitary entity to make triples (as complex couples) and then each triple as a unitary entity to make quadruples (as complex triples). He explicitly expressed his strategy in Line 47. That is, his structuring abductions did influence the building up of a generalization as a product of abductions, inductions, and deductions.

Conclusion

Michael's mathematical activity was sustained by his motivation to stand up to the challenge of solving an evolving task that was novel and different to him. His solutions and explanations provide an illustration of the zigzagging nature of the inferential processes in his mathematical activity. Michael's burst of abductions occurred synergistically with inductions and deductions and not as isolated cognitive processes. In questioning him, it was important not to pose leading questions that would hinder his idiosyncratic way of thinking. Due to the novelty of the task, his abductions (in Peirce's sense), or conjectures (in Polya's sense), or insights (Mason, Burton, and Stacey's sense, 1985), or engendering and metamorphic accommodations (in Steffe's sense, 1991) were easy to capture. Michael provides us with an example of a child's inferential processes at his own level of logical reasoning, while solving a mathematical task.

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STUDENTS *APPROPRIATION* OF MATHEMATICAL ARTIFACTS DURING THEIR *PARTICIPATION* IN A PRACTICE: *À PROPOS DE* A. SFARD...

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In this paper we will present for discussion some ideas that arrive to us both from reflecting on the results of our own research during the last three years and from some readings namely from Jill Adler, Jean Lave and Anna Sfard. We will present briefly the research we have conducted, with some detail on the conceptual framework and the results. After that, we will summarize some ideas presented by A. Sfard' in Seville (1996) underlining aspects more closed related with our own research. Finally, we will propose some questions we are now working on after looking back to those results and the Portuguese educational context.

Introduction

We were working, for the last three years, on the Project MARE[1] seeking for a better understanding of how students relate mathematics and reality when they are learning mathematics through problem solving activities. On MARE' conceptual framework we assumed a situated approach to learning, in Lave's sense, and strong connections with activity theory. The elaboration of its final report was a good reason to look back at our own work — process and learning — and, hopefully, be another good learning moment. Some thoughts and questions came to us in this final moment (some of them we are now working on in our new project "Thinking mathematics learning with Cap Vert"). We find some connections between our concerns and J. Adler and A. Sfard ideas presented in 1996 in two conferences on mathematics education — PME20 and ICME8 — recognizing them as good reasons to think on mathematics learning. In this paper we intend to share our reflections on these process and thoughts. Firstly we will present a brief report of main aspects of our research, secondly we will summarize some ideas of A. Sfard's paper, finally we will discuss two questions relating our research to Sfard and Adler positions.

What have we done and how we spoke about it

The main focus of our research was to understand how children's mathematical knowledge is structured in their everyday activities in mathematics classroom.

[1] This project was funded by Junta Nacional de Investigação Científica e Tecnológica under grant #PCSH/585/93.

Theoretical background[2]

The bases for MARE theoretical background came, mainly from three authors and approaches: (i) Vygotsky and the activity theory; (ii) Lave and a situated perspective of cognition and learning; (iii) Schoenfeld and an approach of mathematics learning as a process of developing a mathematical point of view. On the other hand, we had always in mind the recognition of the cultural nature of both human activity (as for instance, the mathematical one) and human cognition.

Methodology

The research problem and the conceptual approach justified a methodological orientation with concerns that, in some aspects, are usual in ethnographic researches: (i) we were attentive not only to what people say, but also to their acting and to the artifacts they use (recognized by Spradley (1979) as some characteristics of a ethnographic study); (ii) our efforts to understand students points of view and to maintain a close relation between the research process and the phenomenon in it natural setting (considered by Ball (1993) as important concerns in ethnography).

With this conceptual and methodological ideas we studied a small group of 8th grade Portuguese students in their mathematics classroom, considering the unit of analysis proposed by Lave (1988): "(...) the activity of persons-acting in setting" (p. 177).

We observed one ordinary (mathematics classroom — in a public secondary school of Lisbon, with a male teacher and 28 students, without special arrangements on the curriculum or the activities — for every mathematics class during one month. We registered on video and on tape-recorder (two groups of students and the teacher discourse). We also interviewed the teacher and the students observed. We worked on the tapes transcription trying to interpret students activity. Saxe's analytical framework (Saxe, 1991) was very helpful in order to make sense of our preliminary interpretations.

Some results of data analysis

The description and analysis of the everyday practice of students in mathematics classroom context enabled us: (i) to identify the structure of the school mathematics practice, as well as the students motives and goals to participate in the activities; (ii) to stress the role of social interactions between the various participants; (iii) to understand how students appropriate school mathematical artifacts in that practice. We analyzed some situations of students' use of the mathematical artifacts (mediatriz, scale drawing, Pytagoras theorem) in order to understand what we called *Students' appropriation process of mathematical*

[2] This theorical framework is fully presented in Santos, M. "*Na aula de Matemática fartamo-nos de trabalhar* — Aprendizagem e contexto da matemática escolar" Master thesis.

artifacts [3]. The concept of structuring resource[4] played an important role on the analysis of students' process.

Then it was possible to identify some mathematical objectives (to the students in observation) that emerged from students mathematical practice, such as:

• *To understand by doing and to do by understanding, talking and collaborating with others* — To understand and to do are two processes that help each other and that are necessary one to the other. In order to understand what is being done we need to look for a sense for that "doing". For all of these it is important to share information with others, to discuss with them our solving processes, solutions and reasons to do such thing in such way.

• *To use the proper mathematical artifacts* — In mathematics we use certain methods, schemes, concepts, rules, materials and there are also proper conventions. We learn mathematics when we are learning to use all those and we are able to justify why we are using a certain thing in a certain situation.

• *To legitimate results and processes* — Mathematical results and processes can be legitimated in various ways: authority (eg. the teacher or the book), consensus with the colleagues (in our group) or with the best students (in the class), the processes (eg. geometrical) and the artifacts we use (eg. scale drawing, theorems, the rigor).

Some of Sfard's ideas

In order to enable the readers to follow our reflection, we will summarize some ideas (those that are more connected with the points we want to discuss in the present research report) presented by Anna Sfard (1996) in her paper "*On two metaphors for learning and on the dangers of choosing just one*".

As A. Sfard says, her aim was "to arrive at a kind of comprehension that would enable a reflection on tacit assumptions which seem to guide our thinking on learning from behind the scene" (p. 2). To do so, she identified metaphors for learning (*conceptual metaphors* in Lakoff' sense) that "underlie both our spontaneous everyday conceptions and scientific theorizing" (p. 2). After making a search on the professional literature she considered two major metaphors for learning — acquisition and participation — in mathematics education research. She notice that in recent texts both these metaphors are present simultaneously, however each of them was more dominant in different periods (the first in older

[3] In Santos, M. and Matos, J. F. (1996) "Mathematics Learning — where cognition and culture meet" it was described one of those processes — the student appropriation of mediatriz notion.

[4] To analyze the articulation between different activities and to understand the process that makes possible that the "same" activity in different occasions could have different meanings, Lave (1988) proposed the concept of *structuring resource* — something (activity, person, objects, etc.) that help the structuring of a process. So, we can see this idea of structuring resource as something that help us seeing how activity and context interrelate.

studies and the other in more recent ones). As the title of the presentation shows, it is not made any claim on exclusivity for one metaphor in Sfard discussion[5]. Sfard summarizes, in the following map, her point of view on both metaphors about some fundamental aspects considering that in mathematics education we are living (thinking and talking) in "the twilight zone in between the two metaphors" (p. 11).

Acquisition metaphor		**Participation metaphor**
individual enrichment	**goal of learning**	community building
acquisition of something	**learning**	becoming a participant
recipient (consumer) (re-)constructor	**student**	peripheral participant, apprentice
provider, facilitator	**teacher**	expert participant, mediator, preserver of practice/discourse
property, possession commodity (individual, public)	**knowledge concept**	aspect of practice/discourse activity
having, possessing	**knowing**	belonging, participating, communicating

Fig. 1: The metaphor mappings (Sfard, 1996, p.11)

Sfard strengths, several times, that she is analyzing how different schools of thought assume the *nature* of learning rather than to their positions on the *mechanism* of learning. Doing this remark she considers that in various frameworks (from the moderate constructivism to the sociocultural theories)

> "in spite of the many differences of 'how', there has been no controversy about the essence [...] focusing on the 'development of concepts' and on 'acquisition of knowledge' they implicitly agreed that this process can be conceptualized in terms of the acquisition metaphor" (p. 7).

During her presentation of the participation metaphor she calls our attention to the changing of words (and the meanings of those changes) used by researchers to talk about learning aspects: instead of "concept" or "knowledge" they use "knowing", they look to *activities* instead of *states*. She considers also important that "the ongoing learning activities are never considered separately from the context within which they are taking place" (p. 8). One fundamental aspect of this perspective stressed by Sfard is that the learner is viewed as someone interested in *participation* (in a certain kind of activity) rather than in accumulating private possessions. In this context, the teachers are the preservers of the community' continuity and the learner "from a lone entrepreneur [...] turns into a integral part of the team" (p. 8) being this team, to A. Sfard, the *mathematizing community*.

[5] In this aspect could be interesting to read S. Lerman's (1996) paper on his position about these two metaphors.

This last point, seems to be, to J. Adler (1996), one of the most problematic question to address when we shift the "Lave and Wenger's theory of social practice" (p. 3) into school learning. For Adler it is difficult to identify what is the community of practice in school mathematics, and related to this, what might constitute participation (legitimate peripheral participation in Lave's sense) in the mathematics classroom.

À propos de Sfard, Adler and Lave... some thoughts on our work

The school mathematics practice observed in our research, can be described as solving problems in groups or pairs of students, with a lot of opportunities for students to discuss and to communicate their ideas to others (colleagues and teacher). Those problems were not "realistic problems" but prepared by the teacher in order to enable students to work with the mathematical artifacts[6] he was interested on (mainly by institutional motives).

During those solving activities, the relations observed between peers frequently generated, recontextualized, or interrupted students patterns of practice. For instance, we saw one boy (F), usually interested in a more scholar approach to problems (to give an answer to the problem even cheating the proposal), becoming actively involved in a more effective work on that problem due to his interest in continuing to be a preferable peer to his colleague (T — someone who needs to understand what he is doing). In the same way, we saw this last boy (T) very confused changing his usual way of dealing with problems and with F, following his peer approach in order to accomplish the pace of work imposed by F in a situation where this partnership was in dangerous (in T's mind) due to an enlargement of the group and a manifest interest of F in working with the new colleague.

It was also observed the students valuing or not their colleagues collaborations and ideas according to the power relations existing on the group. We identified two main sources of this power: (i) social status — the boys popularity among girls, who could maintain a dialogue about certain interests and to show some knowledge about them; (ii) scholar status — better school marks or the ones that seem to be able to maintain a good dialogue with the teacher (longer, or with more intimacy).

All these aspects seemed to be strongly connected to students **motives** to involve in classroom activity. In fact, it was possible to identify in the three boys observed different motives for participation in that practice. For instance, F is a

[6] In Saxe (1991) terms, *artifacts* are "historical products that can be conceptual (for ex., the cientific concepts), symbolic forms (for ex., numerical system) or material (for ex., tools)" (p. 4). So, it seems that not only the rulers, compass, calculators, and so on, could be thought as artifacts but, also, the mathematical notions such as strategies, methods, rules and concepts.

boy very much interested in the <u>social</u> aspect of classroom life. He his very attentive to the impression he causes on others, he tries to mark the pace, to be helpful to others, to be recognized as the one who is right. T said (about F) in the interview:

F thinks that he is the boss [...] sometimes he needs to be a very good person and enjoys to help the others and teaches them everything.

On the other hand, T is a boy who knows what he wants to do in life (to study pharmacy) in order to take care of family business. He is always very concerned with the <u>understanding</u> of what he is doing, trying to be able to say why something should be done in a certain way. He is the one who usually asks "why" instead of "how" (as F usually asks). We can see this idea when T talked about himself:

This year I know mathematics,... sometimes I can't know exactly how to solve a problem but I know what I'm doing and why I'm doing in that way...

The third boy of the group (M) worked with F and T for the first time during the observation period. He was always changing of group and as T and F said in their interviews *He is rejected*. His participation in that group seemed to have one strong motive — <u>to be accepted</u> by the two boys - and, becoming part of this group, to be recognized by the colleagues and the teacher. He was almost all the time a good listener of F (who likes to speak to someone when he is working), he showed to the other groups that they already solved something (not usual to F and T), he tried to made the calculations needed and offered those results to F and T (trying to be useful).

This practice was lived by students, but in the classroom there is also the teacher. He was responsible for choosing and designing the tasks; he decided completely the curriculum and class organization; he gave information when students asked or when he decided they needed; he was a model in terms of discourse about what they were doing. However, his discourse and ideas were followed within the group usually, according also to students motives and power relations in action at each moment in the group. In some moments they were more dependent of teacher orientations than others. For instance, during the first five or ten minutes of the class they were the traditional pupils asking for a more traditional teacher than when they become involved in the mathematical activities (during the rest of the hour). It was as if they needed some time to change from one practice to another, being this changing due to institutional aspects (the bell rings to finish the time break and to begin a different class in another room, with another teacher as well as to another subject). They were obliged to change some aspects of their practice although maintaining others (the scholar ones).

We said before that these students *appropriate mathematical artifacts through their participation in mathematics classroom practice*. In this way we now can see us as someone who uses old words with new meanings or, as Sfard says, one way of living in the "twilight zone" in between the two metaphors. But were we thinking on students as containers of those mathematical artifacts? Not exactly. We used the word "appropriation" to show the difference between: (i) using the artifact as a structuring resource — something that it is there to be used and they use because makes sense for our purposes — (ii) or using the artifact with a mathematical point of view, something that mediates their approach to new problems. The former happens when, for instance, F used Pytagoras theorem because the teacher considers the geometric resolution not rigorous enough (he wants to please the teacher), or when M (he wants to belong to the group) accepted that he needs to have a strategy before to use the calculator. The later was the case, for instance of T proposing to do a scale drawing as a good strategy to solve a particular problem (without any suggestion from teacher or the task text).

However, we understand this changing in students actions possible because they are participating with others in a system of activity which they recognized as having a specific practice and rhetoric. So we think that the students are participating in a kind of community of practice when they are working in group or in pairs, according to what Lave (1991) said about community of practice: "It [does] imply participation in an activity system about which participants share understandings concerning what they are doing and what that means in their lives and for their communities" (p. 98). Although the different students had different motives to involve in class activities, we believe they shared some understanding about what they were doing (in terms of school mathematics activity). However, we feel necessary more research in order to identify what constitutes a community of practice in the mathematics classroom (from students point of view) and to enlighten the teacher's role in that community. At this moment, we can not see the teacher being identified by the students as the "old-timer" of a community where they want to belong. They see him more as a "preserver" of a certain kind of knowledge existing within a community (people using mathematics) or someone who represents that community and have the responsibility to teach them how that community works. But Lave is also enlightening the importance of those activities' meaning to students lives both as individuals and as elements of a certain community. In fact, in 1995, Lave is much more clear in connecting learning to identity-making "learning, taken here to be the first and principally the identity-making life projects of participants in communities of practice, has a crucial implication for the teaching in schools" (p. 157). If she is right about this point, schools (and teachers) need to address the challenge of incorporating class activities into the life projects of students and helping them to accomplish their projects. In order to do so, we need to know much more about the different communities of practice (inside and outside schools) where students belong (or want to belong), what they are learning there and how, and to understand how they relate their participation in school activities with their participation in society.

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DOES TEACHING MATHEMATICS AS A THOUGHTFUL SUBJECT INFLUENCE THE PROBLEM-SOLVING BEHAVIORS OF URBAN STUDENTS?[1]

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This research was conducted to study the impact on students of a five-year teacher development project in mathematics for elementary teachers in an urban school district. Students taught by project teachers performed better in both classroom problem-solving activities and task-based interviews than students taught by non-project teachers. In addition, there were major differences in the problem-solving behaviors of the two groups. Experimental students displayed greater mathematical confidence, and were more likely to see mathematics as a powerful way of thinking about the real world and approach mathematics as such.

In the past several years, professional development projects have been implemented with the goal of helping teachers to create classroom environments where children have the opportunity to build concepts and ideas when they are thoughtfully engaged in meaningful mathematical explorations. The projects intended to help teachers develop a deeper understanding of mathematical concepts and an increased awareness of the ways in which children learn.

The ultimate goal of these interventions is improved student learning of mathematics. An additional intended consequence is that students would develop productive problem-solving perspectives including greater self-reliance and appreciation for the power of mathematics. Documenting mathematical achievement and problem-solving behaviors however, must be done in a manner consistent with the goals of the interventions. An emphasis on obtaining a more accurate picture of children's problem-solving performance challenges us to raise our expectations about student success from improved standardized test score data to an approach which focuses on the way students think about mathematical tasks (Romberg, Wilson, and Khaketla, 1991; Maher, 1991; Esty, Hall, and Fisch, 1990; Peel, Rockwell, Esty, and Gonzer, 1987; Davis, 1984). Assessment information intending to document deeper and higher order understandings

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and productive problem-solving perspectives, should be taken from a variety of contexts which include clinical interviews and classroom observations (Lesh, Lamon, Behr and Lester, 1992).

Research designed to examine the impact on children's mathematical thinking and problem-solving behaviors, with an emphasis on student performance variables, was conducted in an inner city school district designated as having "special needs" with respect to its student population.. Teachers in this district had participated in a five-year professional development intervention in mathematics. A goal was to analyze the process by which this project impacted the children's mathematical thinking and problem-solving behaviors (Schorr, 1996). Limitations of space do not allow a complete description of the teacher development intervention, its design and implementation, and in-depth analysis of the impact on children's mathematical achievement. Since a thorough discussion of the above would far exceed the constraints of this forum, this paper will be limited to describing the framework for studying the differences in mathematical and problem-solving behaviors for children taught by highly assimilated project teachers and those who were not, as well as a brief discussion of the results with an emphasis on a comparison of the problem-solving behaviors for the two groups of students.

Methods and Procedures

Guiding questions for this study included the following: (i) Would students taught by "project" teachers for a period of three years perform better in both classroom problem-solving activities and task-based interviews than students taught by non-project teachers for the same time period? and, ii) Were there differences in the thoughtfulness and problem-solving behaviors of the two groups of students?

The framework for analyzing the children's mathematical behaviors involved a multi-step process which included the selection of teachers and students.

Selection of teachers. For the purposes of this study, two groups of teachers were identified. The assignment of a teacher to a given category (experimental or comparison) was determined by agreement of three Project mathematics educators who were actively involved in the teacher development intervention. The following criteria formed the basis for selection of experimental (project) teachers and comparison (non-project) teachers. Experimental teachers were those who: (i) chose classroom activities that emphasized inquiry and exploration; (ii) recognized that mathematical learning involves the active manipulation of meanings; (iii) understood that individuals learn by building understanding and knowledge through their actions on objects (which may be mental objects) and their interactions in a variety of social contexts; and, (iv) guided classroom instruction based on children's mathematical thinking.

In addition, experimental teachers displayed a high level of interest that was characterized by the frequency of interaction with Project staff for planning, implementation, and evaluation of classroom lessons. These teachers were described by Project staff as displaying a greater tendency to integrate project philosophy into their regular classroom instruction (Schorr, 1996).

Comparison teachers were those who: (i) taught in a manner which emphasized a more procedural approach to the instruction of mathematics; (ii) did not regularly focus on classroom problem-solving tasks that emphasized inquiry and exploration; and, (iii) did not emphasize meaningful connections among mathematical ideas.

It should be noted that while the classroom practices of comparison teachers did not reflect the philosophical perspectives of the teacher-development intervention, these teachers were highly regarded and respected within their own school communities.

Selection of students. In order to look for large differences in treatment variables, it was decided to select a representative group of sixth grade students (experimental group) who had at least three years of teaching from experimental teachers, and to compare them with students (comparison group) who had virtually no experience with experimental teachers. The two groups of students were carefully matched according to the following important variables: (i) gender; (ii) ethnicity; (iii) school community; and (iv) standardized test scores in both reading and mathematics which came from data obtained prior to the intervention (when students were third graders). In addition, the students were selected to evenly represent high, middle, and low achievement levels, as identified by their classroom teachers. Statistical planning and management for student selection was carried out by a University Professor of Statistics using statistical data provided by the school district. These constraints, compounded by the extremely high rate of student mobility within this urban district, limited the number of students who could be studied to twelve. Once the twelve students had been identified, data were obtained from three activities: a classroom problem-solving investigation; a follow-up interview relating to the classroom task; and a second interview relating to the use of fractions. These activities are described below.

The classroom problem-solving activity. Students were videotaped while they worked in small groups, on an authentic open-ended mathematical problem. The classroom teacher was asked to assign the children to their respective groups, thereby allowing students to work comfortably with familiar and/or typical partners. This made it possible to observe how well the students planned a strategy for dealing with the problem, how well they were able to analyze their work, and how well they were able to work with other students. The classroom activity was administered by an outside instructor who was not associated with the district nor with the intervention project.

The problem activity was an investigation that was carried out during the typical classroom mathematics instruction time, and one hour was allotted in all cases. The particular activity was designed to require students to formulate a convincing argument that they had found all possible solutions to a given problem, with given constraints. This activity allowed students to apply exhaustive thinking for the purpose of creating their persuasive, and defensible argument. Prior to the study, this problem task had been field tested in several other project and non-project classrooms and found to work well with both groups after modifications were made.

Interview relating to the classroom problem-solving activity. Students were videotaped during a task-based interview about the problem task that they had completed. This made it possible to see how well the students had understood the mathematics, how well they could carry out the work when they were alone and not aided by other students, and how well they could explain and discuss their work with an interested adult. To ensure consistency, all students were interviewed by the same interviewer, within the same time period of the classroom activity. So that the study would remain blind and unbiased, a third group of students who had not been selected as either experimental or comparison but had participated in, and been videotaped during, the classroom problem-solving activity were also interviewed by the same interviewer.

Interview relating to the use of fractions. Students were videotaped in a second task-based interview on the use of fractions in solving real-world problems. Fractions were chosen because they are a well-recognized part of the typical school curriculum at this level and because they are known to present conceptual difficulties. Again, the interviewer was the same for all three groups of students. A protocol developed and field-tested specifically for this interview served to guide the interviewer in probing and questioning the students.

Each of the activities was videotaped with two cameras, one focusing on the child (or pair of children in the classroom task), and the other on the work that the child was doing. (In the case of the classroom activity, several other groups of children were videotaped so that the targeted students would not feel uniquely involved.) The videotapes were intended to capture the student's work, facial expressions, and overall character of the classroom activity and interviews. Videotapes were then transcribed and checked for accuracy. The data for analysis for each child included all of the videotapes and transcripts for that child, and his or her written work completed during each session.

The following implementation issues were addressed in order to ensure that the study would remain blind and unbiased:

Selection of an unbiased, independent classroom presenter. An outside, independent teacher who had no knowledge of student or teacher assignment was trained to present the classroom activity. This was intended to eliminate teacher variability.

Selection of interviewers. Two mathematics educators not associated with the district nor with the intervention project, conducted the interviews. Each of the interviewers holds a doctorate in mathematics or mathematics education, and has extensive experience in mathematics education research.

Creation of rating rubric. A scoring rubric for evaluating the videotapes was developed. This contained a list of appropriate criteria for analyzing the children's mathematical performance. These included the following:

a. Ability of the student to go beyond the execution of procedural rules;

b. Richness, depth and completeness of solutions;

c. The nature of the representations that were constructed;

d. Ability of the student to be metacognitive in a useful way;

e. The student's ability to investigate the accuracy and goodness of fit of the mathematical descriptions made;

f. The effectiveness of the student's use of language and communication;

g. The student's ability to work cooperatively with others; and

h. The student's expectation that mathematics is a thoughtful endeavor and that solutions to mathematical problems should make sense.

An analysis instrument, focusing on these criteria, was developed and tested on a representative set of the videotapes by a team of mathematics educators. The instrument was divided into four sections. The first section focused on the classroom problem-solving task; the next two sections focused on each of the task-based interviews; and the fourth section focused on the raters' overall impressions of the students.

Selection and preparation of raters. A team of graduate students, not previously involved in the project, were selected and trained to use the analysis instrument[2]. Inter-rater reliability was high (above 90% for each item). When there was disagreement, it was never by more than one rating point.

Results

The analysis of data indicates that experimental students (those taught by highly assimilated project teachers for at least three years) generally outperformed their comparison counterparts. When the data are considered by section, experimental students outscored their comparison counterparts in 87.5 percent of all sectional comparisons, and scored the same in an additional 5.2 percent of all comparisons. Thus, experimental students had equal or better scores than their comparison counterparts in 92.7 percent of all comparisons taken by section.

[2]A rating system was devised by a team of university mathematics educators. Space restrictions do not permit further description, however, the raters had extensive training on the use and interpretation of this system.

Data such as the above can appear to show "significance" even though the actual results may not seem "important" in the everyday sense of the word. In the present study, there appears to be important differences in student performance. Recall that this was a "blind" study in which the interviewers, classroom presenter, and raters did not know which teachers or students were experimental and which were comparison. However, the interviewers, raters and classroom presenter were able to correctly identify the experimental and comparison students on the basis of their mathematical problem-solving behaviors in virtually all cases. More specifically, the interviewers identified the following two areas as those in which experimental students clearly demonstrated superior performance:

- experimental students were comfortable and effective in talking with adults about mathematical matters, making conjectures, testing conjectures, developing lines of argument, criticizing lines of argument, discussing alternative strategies, etc.; and
- experimental students saw mathematics as a powerful way of thinking about the real world, whereas the comparison students saw math as a matter of paper-and-pencil algorithms that they could not easily bring to bear on real world questions.

To shed further light on the significance of the numerical ratings, the following documentation provided by the raters as they analyzed data from the classroom problem-solving activity and subsequent interviews will be presented[3].. Experimental student 1 "seemed to be confident and willing to risk trying something different and something over again, [the counterpart, comparison] student [1] did not." The raters noted that in both interviews, experimental student 2 had a stronger expectation than his comparison counterpart, that solutions should make sense. In fact, it was noted that experimental student 2 "doesn't make mindless guess(es), but is trying to find additional solutions". With regard to experimental student 3, the following rater comments were made: "The student tries multiple approaches to problems, and is willing to consider different approaches when suggested by others or discovered by herself....The student looks for multiple approaches to problems during the problem solving process....[She] demonstrates confidence in her ability to look for solutions....[She] monitors her own work,...communicates well, asking questions of others when she feels it is necessary,...[and] discusses the usefulness of mathematics at points, and expresses interest in solving problems." Of her comparison counterpart, the following comments were made: "[S]he does not look for different approaches...The student does not seem to perform self-checks of her responses....[She] says she enjoys the problems, but this does not bear out in her actions". In the case of comparison student 4 it was noted that she did "not check her solution to make sure that it makes sense. She seems to find a solution solely to have a solution". With regard to this student's counterpart, experimental student 4, the raters noted that "[this student] thinks aloud before writing her explanation. [If this

[3]All quotations are taken directly from documentation which accompanied the rating instrument.

student] is confused about something...[she] tries to ask [her partner] a question". In the case of comparison student 5, the raters noted that the student copied her partner's work "but does not check for validity". They went on to add that comparison student 5 makes errors and does not "even realize the error and that it doesn't make sense". The raters made these comments about this student's counterpart, experimental student 5 "[This student] uses real life examples and [expects] answers that are reasonable". When certain students could not follow their partner's thinking during the classroom activity, they repeatedly probed until a suitable response, or level of understanding, was attained. For example, in the case of experimental student 6 it was noted that, "[he] does an excellent job of getting his partner to evince what his chip pattern means". However, comparison student 6 "never questions his partner's [work], just accepts it". The importance of these comments is that, in general, when experimental students did not understand a solution, they questioned their own or their partner's work and thinking. The comparison students, however, as indicated by rater comments, were more likely to accept a solution without understanding.

In addition to the documentation provided by the raters, the following comments provided by the classroom presenter will highlight some of the differences between the experimental and comparison students:

> A comment should be made about my role in this project. Again to insure impartiality, I was instructed to be dispassionate and yet informative while at no time revealing any hint or clue. My delivery of the task was to be as identical as possible in each class. I was not told prior to my visits, if the class I was seeing was experimental or control in nature, nor which of the students being videotaped was experimental or control. Through observation, I reached conclusions; they were drawn by the responsiveness of the students, by students' ability to communicate with me, and their fellow students, and by the students' ability to speak and write mathematically. These perceptions were so easily confirmed throughout the sessions by the witnessing of attitudes such as the way the majority of each class would initially view the problem, the ease with which students felt they could discuss their feelings and findings, etc..
>
> Was it easy to spot the control and experimental classes? Yes, for many reasons: The experimental classes showed a greater proficiency at being able to communicate mathematically....In a few of the control classes students mentioned when we were finishing that they thought this activity was going to be about math, but they hadn't seen any math at all....[I]n the experimental classes there was this common thread of students who were comfortable with each other, not afraid to question, more interested in the substance...of [a] solution, [and] able to admit they had taken a wrong path, and continue [solving the problem]....(Schorr, 1996, pp. 167-168)

Conclusions

These results suggest that this intervention had an important impact on students' achievement, their mathematical confidence, and their understanding of what it means to approach mathematical situations thoughtfully. Based upon these results, students of teachers who were actively involved in the intervention and had a greater tendency to integrate project philosophy into their regular classroom instruction, outperformed their comparison counterparts. These results are encouraging as we continue to work to improve the quality of teacher development interventions to help create an over-all atmosphere and school environment where mathematics can become a more thoughtful study, and one for which all students can feel greater respect. In addition, these results indicate that urban children can and do benefit by having, over an extended period of time, teachers who have developed a deeper understanding of mathematical concepts, an increased awareness of the ways in which children learn, and have revised their models of classroom instruction to reflect that knowledge.

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Framing in mathematical discourse

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Abstract. This paper presents one segment in a series of studies on discursive construction of mathematical objects (Sfard, 1996). The focus here is on the way in which the student may become able to make use of newly introduced mathematical symbols, supposed to signify objects which she could not yet have constructed. It is claimed that through linguistic associations, the new symbol may evoke some old meanings which, in their turn, would suggest certain ways of use rather than some others. The phenomenon is known as *framing* of the discourse. The main claim of this paper is that framing, far from being a deviation from the desirable route toward meaning, is often a necessary first step in the construction of mathematical objects.

This paper presents one segment in a series of studies on discursive construction of mathematical objects (Sfard, 1996). In that series, it is argued that the discourse of mathematics may be viewed as an *autopoietic* system (Maturana and Varela, 1987), which is continually self-producing. According to this conception, the discourse and mathematical objects[1] are mutually constitutive and are in a constant dialectic process of co-emergence.

As it is always the case with autopoietic systems, if we adopt this model, we doom ourselves to the dilemma: How does the ongoing process of co-emergence begin? If, indeed, the language and the meaning constantly produce each other, the symbols of mathematics meant to present mathematical objects cannot become fully meaningful before they are used; on the other hand, how can one use a symbol before the object it is supposed to present has been constructed? As will be explained in this paper, the way out of the autopoietic circle may lead through old habits -- through language games which are already well known and deeply rooted. Indeed, it is only natural that in order to circumvent the dilemma of having to use new words before we are aware of their unique uses, we resort to uses with which we are already familiar. We do it by putting the new words and symbols into slots of well known, well remembered, *propositional templates*.

Framing. The ability to make use of symbols that have never been seen before was observed many times by my colleagues and myself in studies with children who were not yet acquainted with algebra, and nevertheless displayed a certain intuitive understanding of algebraic symbolism (Sfard and Linchevski, 1994). At a further stage,

[1] It is very important to understand that in this expression, *the word "objects" does not stand alone and does not signal an existence of special entities which regulate the discourse*. It only has sense within certain phrases and its use is essentially metaphorical. Within our present discussion, I talk about *object-mediated use of symbols*, which means a certain distinct way of manipulating symbols, of solving problems and of communicating -- a mode which reminds us of what can be observed when people talk about physical objects, whether actually present or only recalled.

when faced for the first time with an expression $2x+3x=$, many students would spontaneously complete it to the proposition $2x+3x=5x$. As reported by Demby (1994), students usually explain their decision by saying "Two apples and three apples make five apples". Clearly, the new symbol, x, was substituted here instead of "apples" and thus x took the role of a label rather than of a number. This tendency to view algebraic variables as fitting label slots in non-mathematical discourse may account for a common error of completing such expressions as $2x+3$ to $2x+3=5x$. This error was observed many times in the study Carolyn Kieran and I have carried out in Montreal. We conjectured that the mistaken completion may be due to the underlying reliance on the template "*Two ____ and three more make five ____* " (e.g., *Two apples and three more make five apples*). The plausibility of this explanation was then reinforced by the fact that when the multiplication sign was explicitly written in the expression ($2{\cdot}x+3$), the error usually disappeared. Indeed, the multiplication sign re-directs the student to a different discourse and associates the expression with a cluster of mathematical rather than everyday templates. In the arithmetical templates, a slot on either side of an operator can only be filled with a number. Thus, once the multiplication sign appeared, it became clear that the x is a replacement for a number and that $2x+3$ should be used according to different rules than initially assumed.

To sum up, the very first use of a new signifier *frames* the discourse (Tannen, 1993), namely has the power of directing us toward certain uses rather than toward some others. Together with old templates come old uses, old meanings. Using another language, we may say that we are dealing here with the issue of *expectations* and *verifications*. At the first appearance of a new signifier, certain metaphors come into play and some expectations are born as to the nature of its signified. From now on, the learner will be testing the expectation, sometimes finding that they were justified and some other times proving them untenable. The expectations may come from the way in which the first use of the new signifier is made, but they can also arise due to the associations evoked by the signifier itself. In any case, the mechanism of metaphorical projection from the familiar to unknown is at work.[2]

Expectations that work -- building uses for a new signifier. Let me now turn to an example which will show the dialectic process of expecting and verifying in action. The episode is taken from my recent study in which a mathematically precocious 14-year old student -- let us call him Dan -- learned a number of new mathematical notions. The aim of the experiment was to try to understand better the discursive

[2] Expectations are known also as *prejudgements, prejudices, intimations* or *intuitions*. The motif of meaning constructed through to-and-fro movement between what we expect and what we find goes back to Heidegger and Gadamer, on the one hand, and to Bartlett, Piaget, and Vygotsky on the other hand. In the immediate context of mathematics it became known as an issue of conjecturing vs. proving/refuting (Lakatos, 1976; Lampert, 1990). In the present paper the focus is on linguistically induced expectations.

construction of math objects and, more specifically, to *expose the linguistic elements of this mechanism.* I designed the experiment in such a way as to see the "wheels of the symbolism" turning on their own. In other words, I strove to see how much can be attained by a formal introduction of symbols, unsupported by a meaningful context. This is why I chose a learning situation which by constructivist standards (in fact, by *any* standards) must appear extremely "unfriendly". I created a teaching material in which new symbols were introduced within a sterile context of formal manipulations. Examples of operations that could be performed on these symbols were the only available source of their meaning. In this experiment, therefore, the new concept was presented to the student as purely *artificial*, in Vygotskian sense, namely as one that grows neither from a network of already constructed concepts, nor from its earlier "spontaneous" version.[3] Thus, while solving the problems that were presented to him following a brief exposure to the examples, Dan could only rely on deductive reasoning and on his linguistic associations.

The first in the series of the new mathematical notions to which Dan was exposed in the course of the experiment is presented in Fig. 1. Since the intention of the study was to observe an intra-mathematical production of meaning, the whole process began with a formal introduction of a new signifier. In the experiment, I acted both as an instructor and a researcher. I observed Dan closely during ten one-hour long meetings while he was paving his way toward meaning.

Fig. 1: Introducing calculus of whole-numbers-pairs

During the present meeting we will define addition and multiplication between *pairs of whole numbers*. Here are a few examples:

a. $(2,3)\cdot(5,4) = (10, 12)$	b. $(11,2)\cdot(5,6) = (55,12)$	
c. $(3,4)+(2,5) = (23, 20)$	d. $(5,2)+(1,7) = (37, 14)$	
1. Complete:	$(a,b)\cdot(c,d) =$	$(a,b)+(c,d) =$
2. Compute:		
a. $(1,3)\cdot(2,5)$	b. $(5,1)+(2,3)$	c. $(3,5)+(7,5)$
d. $(2,15)\cdot(10,3)$	e. $(8,3)+(0,5)$	f. $(7,8)+(3,12)$
g. $(5,4) : (1,2)$	h. $(8,15) : (2,3)$	i. $(11,9)-(5,3)$
3. Complete:	$(a,b):(c,d) =$	$(a,b)-(c,d) =$

[3] Although I prefer, as probably most of the readers do, to see this "clinical" situation as purely theoretical and far removed from the reality of today's classrooms, the sad truth is that many would recognize it as only too familiar; even if it becomes more and more rare in schools, it is still quite frequent in colleges and at universities; I am also sure that the conversation between Dan and myself that resulted from my experimental script is likely be considered by some people as a classical mathematical discourse -- the kind of discourse that is generated by those who use to transmit mathematics to others by lecturing or through professional mathematical texts.

The new signifiers introduced during the first meeting were pairs of whole numbers which could be multiplied and added in certain well defined ways. A careful reader will immediately recognize the pairs as another "representation" of rational numbers. For Dan, however, the isomorphism with rational numbers remained unnoticed until the third meeting. Therefore, for more than two hours he acted as a *tabula rasa* as far as the new signifiers were concerned. It is important to stress that during the first two meetings I refrained from referring to the pairs as "numbers", lest this particular name frame the discourse in ways that might distort the processes I wished to observe (I wanted Dan to arrive at certain conclusions from scratch, and not just because some particular kinds of behavior could be expected from objects called numbers). The dialogue between Dan and myself was held in Hebrew.

Fig. 2: Dan defines division between whole-numbers-pairs

[1] D: ... I have a problem here.
[2] A: ?
[3] D: Am I supposed to try now?
[4] A: Do what you can. Write this down [*points to item 2g*], read it.
[5] D: OK, five coma two divided by... I think I will do the same [*as in the case of multiplication*], only I will change to division instead of multiplication, 'caus the symbol of operation here...
[6] A: What do you mean? What are you going to do?
[7] D: Five divided by one, coma, four divided by two, equals five coma two [*writes:* $(5,4):(1,2) = (5/1, 4/2) = (5, 2)$].
[8] A: OK, now I would like you to explain why you did what you did.
[9] D: My reason... In principle, I wouldn't... It is only because you gave me these operations... In fact, I wanted to remark already in the beginning that you shouldn't have used the symbols of addition and multiplication 'cause it is confusing.. . 'cause here [*points to the inside of brackets*] we use multiplication as one operation, and here in the equality [*points to the multiplication sign appearing outside the brackets, between two pairs*] we used it for a different operation. But since you gave me this in that form, I tried to solve it according to what I know.. that division is an inverse of multiplication. So I just did reverse operations...

After Dan discovered the general formulas for addition and multiplication (problem 1 in Fig. 1) and applied them to a number of concrete cases (items *a* -- *f* in problem 2), he unexpectedly came across the operation of division (*g*) which had not been introduced to him, so far. The conversation that followed is presented in Fig. 2. As can be seen, Dan did not have much difficulty with deciding how division should be performed. Moreover, he was also very eloquent about the reasons for his decisions. In [9], after criticizing the teacher (me) for using the familiar multiplication sign to denote a non-standard kind of operation (between pairs), he stated that this was that very sign which had made him act the way he did (see the last sentence in Fig. 2). In an exchange that took place a few minutes later, after Dan successfully verified his result by multiplying *(1,2)* and *(5, 2)*, he explicitly confirmed the role of sign-induced expectations (Fig. 3, [25] and [27]- [29]).

It is noteworthy that Dan's decision was grounded exclusively in linguistically evoked expectations. The appearance of signs with some previous meaning was the only reason for the way in which he chose to broaden the use of the new signifiers. Rather than stick to deduction (which, in this case, would leave him empty-handed, because of the insufficiency of the information at hand), he decided to rely on intuition and analogy. What he eventually created constituted a consistent whole. It was now up to my reaction as a teacher to confirm his interpretation or try to change it. The role of the social in the process of sign-building stands in full relief again. My instructional interaction with Dan was an interplay of Dan's individual constructions and my regulatory interventions.

Fig. 3: Dan explains why he defined division the way he did

[24] A: Listen, I defined the addition and the multiplication as I wanted. I had my reasons to do it the way I did, but I will keep them to myself for now. The question is... could you do the same when defining the division? Were you free to define it as you wanted?
[25] D: No, I was restricted by my associations.
[26] A: What do you mean? What kind of associations?
[27] D: That this sign is a multiplication...
[28] A: And this one is division? ...
[29] D: Yes, and for all I know, they are related.

Expectations that do not work. After demonstrating the strength of linguistically driven expectations, let me turn to the obvious pitfalls of projecting from old to new. First, by activating old uses, the new signifiers may lead to beliefs that obstruct creation of new meaning and create interdiscursive contradictions. Second, the expectations may be superficial and fuzzy, so that their implications become difficult to implement or to test.

The overprojection of old uses results in the phenomena known as misconceptions. This may be best illustrated by the example of the notion of infinity. One may envision the following scenario. A person first gets used to utterances of the form "*Function* f *grows infinitely*". This may well be the phrase through which "infinity" makes its first appearance. At this point the new signifier has no existence of its own. The basic meaningful units are the expressions "*function* f" and "*grows infinitely*". And then, borrowing the template "*Function* f grows/*tends to* _____" from the discourse on functions and numbers ("*Function* f *grows/tends to a number* y_0 *(when* x *tends to*

x_0)."), the learner would say *"Function* f *grows/tends to infinity"* or even in the symbolic form, *"Function* f *grows/tends to* ∞". Once inserted into a slot originally meant for numbers, the word "infinity" and the symbol "∞" have a tendency to sneak into any place destined for numbers. Thus, since the phrase *"Function* f *tends to a number* y_0" may be translated into *"The limit of function* f *equals* y_0", it seems only natural to say *"The limit of function* f *equals* ∞". Here, because of a clear ontological shift (from operational *"grows infinitely"* to the structural *"equals (is) infinity"*), the name "infinity" and the symbol "∞" get a life of their own and start acting in language as signifiers of an independently existing *object*. This is a perfect example of *hypostasis* -- bringing a new mathematical object into existence just by a change in the rules of language game. Up to now everything seems fine. However, the expectation that ∞ should fit any slot meant for numbers, if not restricted, would soon produce statements creating intra-discursive anomalies and contradictions. The common error $^{\infty}/_{\infty}=1$ is a good example.

The fact that using an old template is a package deal finds its other expression in the common expectation that whatever appears within expressions with arithmetic operators, must also be applicable in utterances about quantities and magnitudes. Thus, the fact that complex numbers cannot be ordered appears counterintuitive. The other weakness of expectations -- the one resulting from their blurred inexact nature -- may be illustrated with yet another episode taken from the study with Dan. I have just shown how the appearance of known signs (operators '·' and ':') enabled Dan to act in a meaningful way in an unknown situation (division of pairs of whole numbers). Dan's expectation that division should be "an inverse of multiplication" proved sufficient as a basis for constructing a working definition of this new operation. As may be seen from the excerpt in Fig. 4, this was not the case with the operation of subtraction. The mention of subtraction (the appearance of the sign '-') invoked the phrase "Subtraction is the inverse of addition" but did not give precise directives about the way the term "inverse" should be applied.

Fig. 4: Dan looks for a definition of subtraction -- first trial

[1] A: Good... now, would you, please, do *i* [*(11,9)-(5,3)=*]?
[2] D: OK, This is already more complicated. Would you mind if I made side notes?
[3] A: On the contrary, suit yourself.
[4] D: *(a,d)* minus *(c,d)* equals... This is a problem... a problem... there are... there are much more possibilities for the inverse operation and I have to check them.
[5] A: Fine. Let's see. Continue.
[6] D: I will try to do [it]. Suppose, *a* equals... equals *a* divided by *d* minus *b* divided by *c*, coma, *b* divided by *d* [*writes* $(^a/_d - {}^b/_c\ ,\ {}^b/_d)$], and I reverse all the operations that are here.

Dan's first impulse was to reverse anything that could be reversed -- all the operations on numbers that appear in the formula for addition of the pairs. After all, reversing the component operations did work in the case of division. The only difficulty in the present case was that there seemed to be more possibilities for combining different reversals. By testing the suggested formula in a concrete case Dan soon realized that the reversing of all the operations did not work -- it did not result in a pair of whole numbers which, when added to the subtrahend would produce the minuend. Thus, he ventured a new conjecture, as presented in Fig. 5.

Fig. 5: Dan looks for a definition of subtraction -- second trial

[56] A: Yes, good. It seems that the conjecture did not prove itself, did it? What next? Where were we?
[57] D: I have to derive the operation of subtraction.

....

[66] D: Now, I have an idea.
[67] A: A brilliant idea, of course...
[68] D: No, I am not sure. Now, when I have seen ... When I tried to perform this operation here, I reversed all the signs and the right-hand side answer was correct, the answer that did not require reversal of the operation of addition. So perhaps I only have to reverse the multiplication sign in order to get...
[69] A: Namely?...
[70] D: Namely, that I will do *(a,b)* minus *(c,d)* equals *a* divided by *d* plus *b* divided by *c*, coma *b* divided by *d*. [*writes* $(^a/_d + ^b/_c\, ,\ ^b/_d)$]
[71] A: So the difference between this and what we had before is that we have now plus instead of minus?
[72] D: Yes.

More often than not, the certain difficulty stemming from an inexact nature of expectations is not an insurmountable obstacle. Substantial progress may be made either in a gradual way, by a succession of trials and errors , or in one big step -- by translating the anticipation into an algorithm for finding a working definition. This is how Dan eventually overcame the present difficulty: He translated the claim about the relation between addition and subtraction into a symbolic statement '$(a,b)-(c,d)=(x,y)$ iff $(x,y)+(c,d)=(a,b)$' and then, after applying the formula for addition, solved the resulting equations for x and y.

Concluding remarks: The power of framing. In the clinically sterile setting, created for the sake of the experiment, I hoped to be able to find out how much can be achieved through "symbol games" alone, unsupported by links to the previous knowledge or by a reference to student's needs. It was Dan's job to build these links for himself while it was my job as a teacher-researcher to provide him with a regulatory feed-back. Dan's later success in forging the missing intra- and interdiscursive links in these extremely unfavorable circumstances was truly impressive[4]. In the course of the following few meetings he discovered the number-like quality of the pairs of numbers and, eventually, identified them as 'another representation of rational numbers'. This meant, among others, an immediate re-connection of the mathematical discourse to the 'real world discourse'. It also gave Dan a clear sense of the objects to which the symbols were supposed to refer.

Thus, the extremely important point to stress here is the issue of context. Whether "real-life" or purely mathematical, it is the context which makes growing ideas meaningful and helps in establishing object mediation. Thus, looking at what has been achieved by Dan in the artificially decontextualized situation, one cannot help wondering at the power of linguistic framing. This centrally important mechanism is certainly something to be remembered by mathematics teachers and investigated further by researchers.

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[4] Not every student could be expected to reach as far as Dan did. Dan's outstanding ability to cope with mathematical problems was evidently very important element of his self-image, and this made him a willing participant of the kind of discourse we led. He seemed to recognize the decontextualized situation as normal, and he readily complied with the rules of the symbolic game, never questioning them or wondering about them.

THEORY OF GLOBAL AND LOCAL COHERENCE AND APPLICATIONS TO GEOMETRY

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Abstract

Consider the following statement: "A square with two parallel sides and two equal sides is a parallelogram". In the study described hereby many students claimed the statement was true. Was it because they didn't know anything about equilateral trapezoid? In fact, they knew everything about it, but on the moment of considering the statement they didn't integrate all the pieces of knowledge they hold in their "knowledge base". Is it possible that people, while relating to statements, will not consider all the knowledge they hold regarding the statement, and not act like "rational thinkers"? And if so. what is the reason to that phenomena?. In a research conducted to test this issue one of the basic assumptions was that there is a connection between the limited capacity of the Short Term Memory and the ability to integrate all the pieces of relevant knowledge. The theory that was applied for this matter was The Theory of Global and Local Coherence, which will be described later in this paper.

1. Background

1.1 Geometry Studies in School

Many researches have been engaged in trying to find out what reasons cause students to fail, especially in geometry. In the research literature there are two main categories of explanations:

A. Cognitive Difficulties - Students have difficulties in: a. arranging their thoughts and building logical arguments (Dreyfus & Hadas, 1987); b. dealing with deduction and aspects concerning formal proves (e.g. Schoenfeld, 1985); c. understanding the necessity of writing a formal proof (e.g. Hanna, 1983).

B. Unsuitable Teaching Methods - a. According to the Van-Hiele Theory, teachers tend to teach in a higher Van-Hiele level than their students are capable to (e.g. Usiskin, 1982); b. Teachers tend not to use discovery methods. They simply let their students prove known statements, and thus prevent them from engaging in discovering theories and statements by themselves (e.g. Schoenfeld, 1985).

Following those two categories we looked for answers to questions like - **Why** do students have cognitive difficulties? **Why** can't they understand the meaning and the necessity of formal proof? **Why** do they have difficulties in building logical arguments? **Why**, after so many years of studying geometry they are not reaching a high Van-Hiele level? Is it more difficult to distinguish between "givens" and "what should be proved" when geometrical objects are involved? Are the objects concerning geometry more difficult to grasp then objects from other fields of mathematics? Are the difficulties rooted in the relationships between the geometrical objects? And so forth. In our research we tried to study those questions and others from a different angle, one that places the structure of memory in it's core - The Theory of Global and Local Coherence.

1.2. The Theory of Information Processing and the memory structure

According to the Theory of Information Processing (e.g. Sprinthall & Sprinthall, 1990) there are three main memory units: Sensory Memory, Short Term Memory (STM) and Long Term Memory (LTM). Remembering things means activating (retrieving) the stored information in the LTM and sending it to the STM. The STM can retain information for few seconds (e.g. Broadbent, 1984) while the LTM can retain it during the whole lifetime. The STM has a limited capacity: it can code and hold 7 ±2 distinguished items (Miller, 1956) simultaneously while there are not known limitations regarding the capacity of the LTM (e.g. Craik, 1990). In this research the information storage in the LTM was represented as an Associative Network (AN) consisting of nodes (concepts) and arches (connections between the nodes) (Anderson, 1985). Transferring information from the LTM to the STM means activating the nodes in the LTM. In the LTM information spreads from node to node in the AN in accordance with the connections between them. The knowledge is representing in connected information units. Their configuration and structure in the AN determine whether or not one is able to use this knowledge efficiently (Chi & Koeske, 1983). A useful structure is a coherent one. The notion of coherence relates to the entire structure (global coherence - GC) and to it's sub-structures (local coherence - LC).

1.3. Theory of Global and Local Coherence (Global Coherence View - GCV, Local Coherence View - LCV)

Given the capacity limitations of the STM, it is assumed that during an execution of a mental operation one is not always able to connect at a given moment all the relevant information that is stored in the LTM. The consequent is a "competition" between the relevant possible elements, that ends with choosing some elements in accordance with the view (a group of elements) that was assumed in that moment (Bar-On, 1993). The theory states that the tendency is to look for a lack of contradiction within the view (LC) rather than for a lack of contradiction between possible views (GC). One will not look for the best alternative but will choose the one that first occurs to him and looks like a coherent one at that time. This assumption stands in contradiction to the assumption that people tend to think in a GC (or "rationally") manner. The basic questions that raise from the theory concerns the number of information units (NIU) which could be considered simultaneously, the way they are chosen, the process of integrating them to a coherent view, etc. Following the AN model, the way concepts are connected to each other determines the rate of GC and LC of a structure. The hierarchy and patterns of relation between sub-structures determines the GC, while the interrelation within the sub-structure and the rate of their sharing attributes determines the LC (Chi et al., 1989).

2. The Research

2.1. Assumptions - Mistakes students make in relating to geometrical statements are not necessarily due to lack of knowledge. Because of the STM limited capacity, the tendency is to connect between elements that produce a LCV. Geometry, inherently, requires a GCV. Theorems, definitions, connections, laws of deduction, etc., have to be taken under consideration simultaneously. The type of connections between the elements and the structures within the AN will determine the ability to perform an integration of the existing knowledge, and will determine which elements shall be taken under consideration.

2.2. Research questions (we shall relate here only to two of them):

1. Using the AN, how can the reactions of students be interpreted concerning: a. Factors connected to the reaction time (RT) toward statements; b. The amount of elements (NIU) considered simultaneously for relating to a statement; c. The types of connections between the element students choose to consider simultaneously: Associative connections (AC); Attribute connections (AtC); The rate of the elements' activation; The rate of coherency of the structures that contain them.

2. What factors differentiate between correct and wrong answers?

2.3. Subjects

In the research participated subjects from 9th, 10th and 11th grade, from all study levels. The subjects came from 4 large schools in northern Israel.

2.4. Phases

Phase A: Preliminary research - 321 students answered a questionnaire: In it's first part they had to reply "yes" or "no" to one geometrical statement concerning squares within 30 seconds (writing the first answer that occur in their minds). There were 10 different statement. In it's second part the students were asked leading questions, and then had to reconsider the original statement. The aim of this stage was primarily to test the effect of leading questions on the coherency of answers.

Phase B: Case studies - 23 subjects were interviewed and videotaped. Each interview was built from 6 phases: 1. Drawing all the known squares; 2. Grouping the squares by various criteria; 3. Putting the squares into families and sub-families; 4. Writing all that is known about each square (referred to as "knowledge base"- KB); 5. Relating to ten statements from various types about squares (five true and five false statements); 6. In cases the subjects gave wrong answer, leading questions were given, until the answer was correct.

Phase C: Analyzing the interviews - a. Phases 1 - 4 were used for building the subject's AN; b. The Analysis of answers given in phase 5 was based on the AN. Other parameters used for the analysis were - RT, NIU considered, the rate of information activation, and the rate of the structure coherency.

Phase D: Summarizing the Findings: Tools and Methods

A. The Model for Building the AN - The model of the AN that was used for the research purposes was based on the AN described by Chi & Koeske (1983). The AN was composed from three components: 1. Object-Object Connections (AC); 2. Object - Attributes Connections; 3. Classification of the relevant squares into families.

B. Activation Measurements - The activation measurements were based on a computer program called "iac" (Rumelhart & McLelland, 1986).

C. Coherency Measurements -The rate of coherency the subjects demonstrated was measured using a computer program called "Convince

Me". The program was developed by Ranny M. & Schunk P. based on a theory called "Theory of Explanatory Coherence" (Thagard, 1990). The theory relates to the way people decide how much they believe in an explanation they give or a statement they relate to.

3. Summary of the results and discussion

One of the main findings was a reassurance of the basic assumptions of the Theory of Global and Local Coherence: answers of students can be characterized as having a LCV and mistakes are a result of a lack of GCV. In every case of giving a wrong answer the subjects actually held the required knowledge in their "KB", but they failed to integrate all the knowledge components. The results of this research show that the existence of structures in the AN that contain all the required elements and connections, especially attribute ones, will enable giving an answer from a GCV, and thus reduce the RT towards a statement. The low RT is a consequence of the information's access rate, and of the fact that there is no need to scan numerous structures in the AN. Thus, a focal answer, one that doesn't force a trail passed through various structures, causes the consideration of fewer NIU.

Considering that all the findings are beyond the scope of this paper, we shall briefly summarize only those who have possible utilization in teaching and learning theories:

A. AtC are more meanings then AC regarding the consideration of geometrical statements. The conclusion is that while teaching geometry the teacher must establish the knowledge relevant to AtC between the various geometrical objects.

B. RT, NIU considered and the number of connections between squares were among the factors that differentiate between correct and wrong answers. It was found that correct answers involved lower RT and NIU considered then wrong answers, and greater number of connections in the AN. The fact that correct answers are characterized by fewer NIU considered can be related to the existence of a "special mechanism" that selects between IU. The conclusion is that in spite of the positive correlation that was found between number of connections in the AN and the chance that the answer will be correct, still in average correct answers carried fewer NIU. That means that not in every condition multiconnections would guarantee right answer. Student should learn how to consider those connections. Many teachers tend to encourage their student to produce long answers assuming that the more there are information units the better the student knows the subject. The results of this research point to the fact that assisting students to built their own "filter mechanism" would lead to better results.

C. Teachers should also assist their students "building" the AN structures. On one hand it was found that LCV helped reducing the RT, and on the other hand lower RT was tied to correct answers. It can be deduced that the reduction in time stemmed from the student's ability to identify clearly and quickly the proper structure, the one that contains all the relevant information. That is to say - the student can consider the structure locally, since this well established structure already contains all the information necessary for relating to the statement. Such a structure is in fact producing a view which is global in its nature, because having all the required connection in it saves the time wasted on searching the required information. Teachers can assist their students built AN well equipped in their hierarchical structures and relevant connections and thus influence the ability to cope with the various objects of geometry.

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GENERATING THEORETICAL ACCOUNTS OF MATHEMATICS TEACHERS' PRACTICES

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Abstract. *This paper articulates a particular theoretical and methodological perspective that can play a key role in studying mathematics teacher development. This perspective focuses on generating theoretical accounts of teachers' mathematics teaching practices as they take part in current mathematics education reform efforts. These accounts provide an alternative to studies that focus on teachers' deficits and to teachers' own accounts of their practice. A description of and rationale for the generation of theoretical accounts of teachers' practice are presented followed by an example drawn from one account that the authors developed.*

Background

This paper articulates how a particular theoretical and methodological perspective might contribute to a research-based understanding of teacher development necessary for advancing the current mathematics education reform agenda. The reform implies that teachers must develop practices that differ greatly from what has characterized their mathematics teaching and learning in the past. We understand a teacher's practice to include not only everything a teacher does that contributes to her teaching (planning, assessing, interacting with students), but also everything the teacher thinks about, knows, and believes about what she does. In addition, the teacher's goals, intuitions, skills, and feelings about what she does are part of the practice. Thus, we see the teacher's practice as a conglomerate that cannot be understood by looking at parts split off from the whole (i.e., looking only at beliefs, or questioning, or mathematical knowledge, etc.).

Educating experienced teachers to transform their teaching practices to be more consistent with current reform principles places significant demands on the mathematics education research community. Mathematics educators need research-based understanding of how teachers develop from traditional teachers toward teachers who contribute to and implement the mathematics education reform agenda in order to design and implement successful learning opportunities for teachers. Our current research,[1] which includes case studies of teachers, is based in part on the notion that understanding teacher development involves generating comprehensive theoretical accounts of teachers' practice. We continually generate and revise our theoretical accounts by inferring from data collected through observing the teacher's practice and interviewing the teacher about specific aspects of what is observed. (A detailed account of our methodology is found in Simon, in press). These accounts represent our commitment to articulate how the teacher organizes her experiential reality with respect to teaching. In other words, we strive to create a coherent story of the teacher's practice by *explaining the teacher's perspective from the researcher's perspective.*

This last phrase is the key because it distinguishes our work from both deficit studies of teachers and from work in which the teacher articulates her own perspective (cf., Schifter, 1996). Thinking about "the teacher's perspective from the researcher's perspective" involves a subtle, but important distinction. The researchers attempt to understand and articulate teachers' approach to the problems of practice: how and what the teachers perceive and how they make sense of, think about, and respond to the situations as they perceive them. However, the result may be very different from what the teachers would say about their own practices. The researchers structure their accounts of the teachers' practices using particular conceptual lenses (often not shared by the teacher) that define their focus and guide their interpretations.

Theoretical accounts of a teacher's practice generated at different points in time contribute to analysis of transformation/development of a teacher's practice. In our research, these accounts provide the context for understanding a teacher's response to our interactions with the teacher that are intended to promote development.

Rationale for Developing Accounts of Teachers' Practice

Several considerations have contributed to our perspective that accounting for teachers' practice can make an important contribution to understanding teacher development. Although the restrictions of language require them to be listed sequentially, it is the collective impact of these considerations that have led us to our current perspective.

1. Several research projects have successfully promoted changes in teachers' knowledge, although expected concomitant changes in teachers' practice did not result (c.f., Wilcox, Schram, Lappan, & Lanier, 1991; Simon & Mazza, 1993). We suggest that not only are multiple areas of teacher knowledge necessary to support new forms of practice (e.g., knowledge of and about mathematics, knowledge of students' learning, knowledge of curriculum design and adaptation), but that growth in particular areas of knowledge depends on growth in other areas. For example, a teacher's growth in understanding how students develop concepts of fractions may be limited by her understanding of the mathematical concepts herself, and by her understanding of what it means to understand and do mathematics. Furthermore, if she views teaching as getting students to respond to particular problems in particular ways, her inquiry into students' thinking may be limited to how students'

thinking differs from what she is attempting to elicit. Developing accounts of the teacher's practice has the potential to enhance understanding of the role of the teacher's knowledge in her practice and of the processes involved in the growth of interrelated areas of teacher knowledge.

2. Deficit views of teachers (articulating what they do not know or understand) are insufficient for understanding teacher development, because such views provide no insight into how teachers assimilate new experience, how they define their roles, or how they view students and mathematics. Rather, accounts that characterize the practice of thinking, feeling, acting teachers are needed in order to understand the impact of particular professional development opportunities on teachers' practice.

3. Studies of the mathematical development of students provide a useful analogy to studies of the professional development of teachers. The studies that have provided the most insight into learners' mathematical thinking have characterized the particular ways that learners at different stages think about particular mathematical ideas (cf., Steffe, 1992; Fischbein, Deri, Nello, & Marino, 1985). In work of this type, researchers often need to postulate new constructs to account for the mathematical activity of the observed students. This is necessary because existing ways of describing the concepts of mathematically sophisticated adults are often not appropriate for describing the students' mathematics. In this analogous context, we see that mathematics educators, who bring rich mathematical understandings to bear upon their research, characterize the mathematics of students in ways that the students themselves could not. The mathematics educators' understandings play a critical role in inferring the mathematics of students, in seeing the mathematics of students as rational and coherent, and in devising ways to promote students' learning through instruction.

We use this analogy to think about teacher thinking, practice, and development. Our goal is to account for the development of mathematics teachers in ways that capture key features of their practice. The sophistication of the perspectives that are brought by the researchers to these analyses are important to the effectiveness of the account that is constructed. However, as in the analogous situation of studying the mathematics of students, researchers may need to develop new constructs to account for the teachers' practice, constructs that are not part of their previous analysis of traditional teaching or of reform teaching.

Developing an Account of Practice: One Example

Our work to date suggests that teachers seem to interpret the reform as discouraging a telling and showing approach to mathematics teaching. Consequently, they face the problem of developing an alternative to this deeply entrenched approach. Looking at how teachers address this problem has proved useful in each of our studies of a teacher in transition. Thus, it has become part of our conceptual framework, focusing our inquiry and providing a level of commensurability that structures our attempts to look across case studies. We discuss our attempts to develop an understanding of Sally's solution to this problem as an example of our work in generating theoretical accounts of teachers' practice.

Based on transcripts of our videotapes of Sally's mathematics teaching in her first-grade class and audiotapes of interviews with Sally regarding the observed lessons, we began a line-by-line analysis. We used our perspectives to understand Sally's mathematics teaching from her perspective. Our goal in discussing this example is to illustrate our theoretical and methodological considerations, *not* to convince the reader of the appropriateness of our interpretation, thus the absence of actual transcript data.

From the outset, it was clear that Sally did not use a traditional "show and tell" approach. Rather, she tended to begin her lessons with open-ended questions or tasks. She seemed to be committed to teaching for understanding. She involved the students in hands-on tasks, encouraged collaboration, and questioned them about their understandings. On the other hand, her focus seemed to be on getting the students to do or say what she was looking for, rather than on evidence of their conceptualization. In short, we thought of her teaching as demonstrating aspects of two different teaching paradigms, traditional and reform.

To characterize Sally's teaching practice this way was *not* particularly helpful for several reasons. First, we found ourselves repeatedly struggling over whether her focus was procedural or conceptual. Neither of these terms seemed to capture what we were observing. Second, all teachers might be characterized as demonstrating aspects of traditional and reform teaching. Thus, this characterization shed no light on Sally's specific solution to the pedagogical problems that she faced as a teacher in transition. Finally, such a characterization fails to incorporate an important principle of research on teaching, that every teacher's approach is rationale and coherent from his or her perspective. Therefore, we struggled to generate an alternative account of Sally's practice that was consistent with this principle.

We began to make progress when we examined the inappropriateness of our terms "conceptual" and "procedural" for characterizing Sally's focus. As we observed and re-observed her interactions with the children, we noticed that she closely monitored what they were doing to solve the tasks and questioned them to evaluate their understanding of what they were doing. By generating a new construct, "doing-with-understanding," we were able to break away from our previous ways of categorizing teaching. We designated this construct *to imply a unified approach*, not a combination of separate foci. The notion of doing-with-understanding (DWU) continued to be explicated as we worked with additional data from Sally's practice.

We came to understand Sally's DWU approach as an emphasis on getting students to successfully perform certain kinds of activities while demonstrating understanding, usually by providing particular types of verbal descriptions of their activity. Sally spent most of her class time monitoring students, individually or in small groups, to see if they were successfully demonstrating the competence that she was looking for and if they could provide the verbalizations that she had identified as being evidence of understanding. When students were not able to perform in these ways, she intervened in an attempt to encourage them to alter their responses. These interventions were often questions that were narrower and more leading and use of examples (often from other students' work) of appropriate responses to the tasks or questions.

We further came to understand Sally's teaching as guided by a set of reform strategies that she expected would result in students DWU. These included beginning with broad open-ended questions or tasks in hope that the students would generate the desired responses with little guidance from her, encouraging or requiring use of manipulatives and graphical representations[2], monitoring students' work regularly and modifying her teaching based on students' responses, and demanding verbal explanations of their solutions.

In summary, the creation of the construct DWU was an attempt to recognize the coherence in Sally's practice. This practice incorporated many of the strategies that characterize the reform yet contrasted with our notion of reform teaching in her focus on student performance as opposed to our focus on including teacher's inferences regarding students' underlying conceptual structures and operations. Our theoretical account of Sally's practice is intended to characterize how she attempted to advance children's learning at one point in time.

Discussion

This glimpse of our attempts to generate a theoretical account of Sally's practice demonstrates an important tension for the researchers. On one hand, our ability to "notice" important aspects of Sally's practice was directly related to the perspectives that we were able to bring to bear on the analysis of data. On the other hand, coming to an understanding of Sally's perspective (through the lenses of our perspectives) required us to go beyond the limitations of our extant concepts, a step that necessitated a conceptual reorganization on our part in order to "learn" Sally's perspective.

We believe that generating theoretical accounts of teachers' practice has the potential to contribute to understanding of mathematics teacher development in several ways:

1. These accounts can shed light on the nature of the pedagogical problems encountered by teachers in transition and of the pedagogical solutions that they develop.

2. These accounts contribute to understanding the meaning that teachers attribute to various aspects of the mathematics education reform and for the professional development experiences in which they participate.
3. These accounts provide a basis for professional development interventions, including those that are part of research on mathematics teacher development.
4. These accounts can lead to hypotheses of landmarks of mathematics teacher development from traditional to reform teaching.

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[2] Sally seemed to assume that "seeing" what was being represented was a result of using the physical and graphic representations.

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TOWARDS A NEW THEORY OF UNDERSTANDING

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Abstract

We discuss an important breakthrough for us in our understanding of the meaning of 'understanding' in mathematics education. We provide a background to this discovery, the catalyst for the breakthrough and a concise explanation of our new definition of the term within our existing theory of natural, conflicting and alien experiences. In discussing the consequences of the definition - both in the theoretical terms of internal characteristics and external manifestation, and some early practical consequences for a teacher attempting to model a learner's understanding - we begin to describe a more comprehensive theory of understanding.

Much of our work during the past five years has been devoted to developing a theory of learning in which we identify learning experiences as either *natural*, *conflicting* and *alien*, to which we postulate learners will respond in different ways (Duffin and Simpson, 1993). We have used this theory to analyse a number of learning incidents encountered in our work. In the course of such analyses we found that the word understanding often entered the discussion and eventually we felt compelled to seek a definition of that word which would fit our theory and enable us to further our work. The quest for such a definition has occupied us now for around three and a half years and we will use this paper to explore our current position in that quest.

The Search for Understanding

Our search has fallen into three phases during each of which something emerged which has been of considerable significance for us and, though subject to modification as the work has progressed, each has in itself contributed to our present position.

The first phase was one we call the theoretic phase because, contrary to our usual practice, we were not looking at critical incidents from our experience as teachers, learners and investigators of learning but were examining the affective aspects of understanding, using our own internal perceptions as well as those of colleagues with whom we talked. These we called the internal characteristics of understanding.

During this phase we did a preliminary literature search from which we discerned two apparently discrete approaches to this important concept amongst researchers in the field. There were those who appeared to associate understanding with the ability to do or having a set of skills, among whom we were somewhat surprised to find Gagné (1970), while others (detailed in Byers (1980) including Skemp (1976), Poincaré (1908)) appeared to demand more than doing and knowing as criteria for the concept.

In line with our own theory, we found ourselves rejecting and ignoring those whose perceptions seemed alien to our own and were happier to align ourselves with those who expected more for postulating the presence of understanding in a learner. It is significant for us that later developments forced us to return to this dichotomy to reassess our position in relation to the literature.

With our theory in mind, and our sense of affinity with some of the literature and the ideas of colleagues with whom we shared our developing perceptions, we arrived at our first definition of understanding:

Understanding is the awareness of internal mental structures

Having achieved this first definition our attention was temporarily diverted from our search until its second phase overtook us when two incidents occurred in quick succession and drew us back to our quest. These incidents caught our attention because it seemed to us that one of them showed a learner with a very good understanding while the other seemed, at least on first sight, to demonstrate total confusion and lack of understanding.

Our attention was re-engaged by these incidents which we then tried to analyse in the light of our definition. In the incidents, one learner tried and failed to carry out a task while the other, who confessed to having forgotten what was wanted, was able to recreate the processes successfully by calling on connections he had available.

It was this recognition that, while both were about learners who had forgotten something, only one was successful in recapturing what had been forgotten, which was the catalyst to further investigation. It was clear to us that the success came from awareness of appropriate connections on the part of one learner, precisely in accordance with our definition. The failure appeared to come because of a sense we got that the other learner was merely trying to recall the steps of a procedure without using any significant connections that we might interpret as an understanding of the procedure.

From this we used the two words *reconstructing* and *reproducing* as depicting, respectively, the actions of those who do or who do not understand. It seemed to us that the learner who did not understand could only attempt to reproduce a forgotten procedure, while the one who did was able, despite lack of recall, to reconstruct what was wanted by using awareness of appropriate connections in order to do so.

After our breakthrough, the notion of reproduction vs. reconstruction remains an important facet of our theory.

Moreover, the case of the learner who did understand confirmed one of the internal characteristics we had identified: that if you understand you do not need to remember all the details of something. At the same time, our attention was drawn back to Poincaré who associated understanding with creativity, an element we felt we had now identified in this process of reconstruction.

It was during this phase also that we began to consider ways in which teachers might be able to recognise understanding in their students: we began to consider what we came to call external manifestations (or indicators) of the internal characteristics we had identified. We saw these two elements as important to our developing work though, as in the case of reproduction and reconstruction, later work was to modify and change our perception of the ways in which these elements are part of the concept of understanding.

It was at this point that we entered what was to be our third phase before we came to our present position. We felt we were now ready to write up our work on understanding, our aim at the beginning of our quest. We had identified internal characteristics of understanding and had some tentative ideas about how a teacher might perceive learners' actions in order to determine something about their state of understanding.

But the article failed to materialise however hard we tried to put it together. Admittedly we were again temporarily diverted to another problem associated with our theory of learning but this was not sufficient to explain our inability, over a period of at least two years, to write about our definition of understanding related to our theory of learning.

We returned to the literature. Earlier our attention had been drawn to three writers of whom Nickerson (1985) had provided us with an extension of our perception of understanding and its consequences; on this occasion we decided to concentrate on the other two: Sierpinska (1990) and Mason (1994).

From the new look at Mason arose a paper on methodology; from the new look at Sierpinska came the breakthrough which brought us to the point at which we now stand.

A Breakthrough

In reading Sierpinska (1990) initially, we had been puzzled by her question "is understanding an act, an emotional experience, an intellectual process or a way of knowing?". In our existing theory we saw understanding as a *state* and became puzzled when Sierpinska decided to work with the notion of understanding as an *act*. As we returned to work on understanding again, we went back to Sierpinska's question and looked for clarification in her more substantial book on the topic (Sierpinska, 1994).

We had also, however, begun to see a problem with our existing theory: there seemed to be a 'hole' between the notion of internal characteristics and external manifestations - a 'hole', though we were unaware of it at the time, that came about because of the tension between seeing characteristics as static and manifestations as dynamic. This, coupled with the confusion about whether understanding was an act, a state or a process, left us ripe for the breakthrough.

This came with the sudden realisation that we *could* interpret Sierpinska's notion of an act of understanding and that it might be a vital idea for our theory. We interpreted her 'act' as the *use* of connections to solve a problem*. This seemed to us to be a vital part of our theory - it is only in 'enacting' understanding that a learner uses their connections in such a way that a teacher can make some judgement about the understanding they might have.

This led us to an important partial analogy with the naive view of potential and kinetic energy. In the case of a ball with potential energy, that energy cannot be directly seen. Indeed, even in the case of a ball which is falling and has kinetic energy, we still cannot see that energy: we can only infer that the ball has energy from the observation that it is moving and, only from that can we infer that the ball began with some potential energy. Similarly, unless understanding is enacted, we have no way of inferring anything about the level of understanding that a learner has and, even when the learner does something (such as gives an explanation, or solves a problem) we cannot see the understanding being enacted, we only interpret our observations as the use of understanding and from that infer that the learner began from some level of understanding.

This analogy, partial as it is, led us to think about an analogue to *giving* a ball potential by lifting it off the ground. That is, what might we mean by building understanding. Thus we were led to a three component theory.

Three Components of Understanding

We named the three components of understanding *building*, *having* and *enacting*. The first of these is the process of the formation of the internal mental structures, the connections that constitute the breadth and depth of understanding that a learner has at any particular time. The mechanisms by which these connections are formed (or broken) belong more properly to the underlying theory of responding to natural, conflicting and alien experiences (detailed in Duffin and Simpson, 1993). It is important to note, however, that our new theory retains a notion of breadth and depth of understanding: in the former case, someone may have a *number* of connections from the notion under consideration, while in the latter the learner may be able to form long *chains* of connections from that single concept. This notion, originally from Nickerson, goes beyond the questions of whether understanding is

* It is important to note here that we now believe our breakthrough came from *misreading* Sierpinska. We now interpret her 'act' of understanding as what we will come to call 'building understanding' and that her notion of overcoming epistemological obstacles is what we have previously called resolving a conflict (Duffin and Simpson, 1993)

an 'all or nothing' concept or whether there is a single dimensional continuum of understanding. Instead we can talk of learners who have breadth - perhaps evidenced by the number of different starting points they have access to in solving a problem - or depth - perhaps shown by the ways in which they can unpack each step in an argument they produce.

The second component we term *having*. This is the state of connections at any particular time. A learner may only be enacting or building single strands of their understanding when they are working, or indeed they may not be enacting or building any part of their understanding if they are passive (or if, as we shall suggest later, they are *reproducing* a response). However, in determining what understanding learners *have* at a particular moment, we are interested in the totality of the connections available to them. As we shall see however, modelling the understanding that a learner has is an indirect and complex thing.

The third component, *enacting*, is one which we will suggest has the most to offer in terms of enabling a teacher to model a learner's understanding. By *enacting* we mean the use of the connections available in the moment to solve a problem or construct a response to a question.

Note that these three components can bring us back to our problem with Sierpinska's original question: whether understanding is an act, a state or a process. Our theory suggests that there are aspects of all three: an ongoing process of the development of connections (building), a state of the available connections at a given time (having) and the act of using the connections in response to a problem (enacting).

Internal and External Aspects of the Three Components

Much of the development of our original theory came in deciding what internal and external characteristics could be discerned when someone has some understanding. In doing this we developed two questions which we asked of ourselves and of colleagues (an approach which fits the methodology we have developed: looking at ourselves and others 'as if from inside' and ourselves and others 'as if from outside' (Mason, 1987), detailed in (Duffin and Simpson, 1996)). These two questions can be modified to determine what we might mean by the internal and external aspects of our new theory.

We can try to determine internal characteristics by asking ourselves and others the questions:

> How do I feel when I [am building]/[have]/[enact] my understanding?

and we can try to determine the external manifestations by asking ourselves and teachers the questions:

> What would I expect to be able to see in my students if they [are building]/[have]/[are enacting] their understanding?

This has led us to begin to build a matrix of these possibilities (figure 1.)

mental structure component	building	having	enacting
internal characteristics	[noticing?]	feeling comfortable, feel able to forget (and reconstruct later)	able to do problems, able to explain, able to see in other contexts, able to derive consequences
external manifestations	eyes lighting up, 'Aha!'	NONE	doing*

figure 1

We suggest that the internal characteristics of building an understanding may be hard to determine - that the focus of the learner may be so firmly on the process of forming connections that they cannot simultaneously notice what is happening to themselves (though perhaps this is at the heart of the discipline of noticing that Mason (1994) has developed). What may be accessible to a teacher, however, is the reflex reactions to making a breakthrough - a lighting up of the eyes or other signs of excitement (or relief!).

The internal characteristics of *having* may be more accessible. We may feel at ease with a topic if we feel we have sufficient understanding to be able to cope with the kinds of situations we expect to encounter. We may even feel that if we forget the details of an algorithm (for example) we can reconstruct it using our available connections without much difficulty. However, we suggest (as our earlier energy analogy indicated) there can be no external manifestations of having understanding: external manifestations can only come when something is being done and the state of having understanding is a passive one. The most important implication of this is that in trying to model the understanding that a learner has, a teacher cannot use direct manifestations: they can only model from the available manifestations - those of building or enacting.

Enacting can have a number of internal characteristics. A learner may see themselves using their connections to explain something to someone else, derive a consequence, or to see an aspect of a concept in a new context. These internal enactments manifest themselves in a number of ways - speech, writing, drawing - all of which we put under the heading of 'doing'. It is in the *interpretation* of the actions of a learner that the crux of the problem of a teacher's modelling of understanding lies.

* It is interesting to note that we now see 'doing' as so important in modelling understanding, when we initially rejected Gagné's use of it - but by 'doing' we mean any form of action which may be interpreted by the teacher, not just the completion of routine tasks; indeed, as we shall suggest, it is routine tasks (which may be easily reproduced) which are the least useful in helping a teacher model understanding.

It is important to note that the analysis of understanding that we give - in three components with internal characteristics and external manifestations - is not meant to indicate that these are discrete elements in learning. We do not suggest that there is a period of building, followed by a period of having, followed by a period of enacting. Indeed, it is likely that in solving a substantial problem we use some recalled facts, enact some understanding, get stuck, find conflicts and resolve them by building new connections, enact those new connections, bring in more recalled ideas and so on; all this possibly taking place in a very short space of time.

Modelling Understanding

In trying to make some judgement about a learner's understanding, a teacher can only work from what they observe and from their interpretation of those observations: that is, teachers model the understanding of pupils by building an understanding of it. The observations that are most readily accessible are the external manifestations of enactment, which has two associated problems: an enactment is a single use of a single pathway of connections (so does not represent the totality of available connections) and a learner's solution may not involve them in *enacting* their understanding if, instead, they can reproduce an answer.

The former problem seems to confirm that the act of modelling understanding is a long term one. A teacher builds and modifies their model of pupils' understanding from repeated interpretations of numerous observations of what the pupils do, and can provide questions and directions to the pupil to encourage them to enact their understanding in as many different ways as possible.

The issue of reproduction vs. reconstruction is a more substantial one for us. We suggest that a person will reproduce a response if they can (even if they are quite capable of reconstructing one, which involves enacting their understanding by using their available connections) since this involves considerably less effort, as implied by Bartlett (1932). Thus in trying to model understanding, a teacher may be misled into thinking that a solution to a problem is one obtained by enacting understanding, but which may instead have been reproduced from memory (whether or not any understanding lies behind it). Even if someone cannot simply reproduce an answer, a further difficulty may come from a learner's awareness in the moment: while a learner may have the necessary connections to solve a problem, at a particular moment, in response to a particular situation, they may not have an awareness of those connections and thus may not be able to enact their understanding.

Both of these ideas lead us to some practical consequences for a teacher trying to model their pupils' understanding:

- The teacher needs to be aware that modelling understanding is a long-term and indirect process in which they need to build a model from a large number of questions which allow a range of broader and deeper understandings to be enacted.

- To avoid the problem of reproduction, the teacher might use questions which make any reproduction more obvious: requests to explain, questions set in unusual contexts or questions which ask learners to derive consequences (this latter came from Nickerson). The experience and sensitivity of the teacher is important in making interpretations of what the learners do
- Questions might be put in many different ways and contexts, and put at different times to ensure that a momentary lack of awareness is not misinterpreted.

These are the beginnings of the implications for us of our developing theory of understanding. It is important to realise that our theory develops by moving between the theoretical search for implications, the discovery of practical consequences for ourselves and others, and the application to incidents from our teaching and learning.

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HOW FAR CAN YOU GO WITH BLOCK TOWERS?

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For this report, we focus on the development of combinatorial reasoning of a fourteen-year old child, Stephanie, who is investigating binomial coefficients and combinations in relationship to the binomial expansion and the mapping of the binomial expansion to Pascal's triangle. This research reports on Stephanie's examination of patterns and symbolic representations of the coefficients in the binomial expansion using ideas from earlier explorations with towers in grades 3-5 to examine recursive processes and to explain the addition rule in Pascal's triangle. This early work enabled her to build particular organization and classification schemes that she draws upon to explain her more abstract ideas.

The teaching experiment reported here is a component of a longitudinal study[1] of the development of mathematical ideas in children. Attention has been given to studying how children build mathematical ideas, create models, invent notation, and justify, reorganize, and extend their ideas. We have been observing Stephanie doing mathematics for nine years. Stephanie's early work in combinatorics began in grade 2, building models to justify her solutions and validating or rejecting her own ideas and the ideas of others on the basis of whether or not they made sense to her. In the earlier studies Stephanie simultaneously referred to and monitored the strategies of other group members and integrated the ideas of her partner into her own representations. This enabled her to keep track of her data and to cycle through constructions, thereby producing more powerful representations (Maher & Martino, 1991; Davis, Maher & Martino, 1992; Maher & Martino, 1992 a and b).

In grade three, Stephanie was introduced to investigations with block towers[2], which enabled her to build visual patterns - such as the local organization within specific cases based on ideas like "together", "separated", "how much separated" - to show us her ideas. She recorded her tower arrangements first by drawing pictures of towers and placing a single letter on each cube to represent its color, and then by inventing a notation of letters to represent the color cubes. Stephanie's working theories about the towers provided striking and effective ways of working with mathematical ideas, They triggered for her the spontaneous use of heuristics (guess and check, looking for patterns,

[1] Earlier work for this study was supported in part by grant number MDR9053597 from the National Science Foundation. The opinions expressed are not necessarily of the sponsoring agency and no endorsement should be inferred.

[2] A tower is an ordered sequence of Unifix cubes, snapped together. Each cube is called a block. Each block, and hence each tower, has a bottom and top. The height of a tower is the number of its blocks. For a positive integer n, an n-high tower is a tower of height exactly n. We say two towers are the same if their colors match, block by block, from top to bottom.

think of a simpler problem, etc.); the development of arguments to support a component of a solution; and the extension of arguments to build more complete solutions.

Theoretical Framework

Guiding our work is the view that children come to their investigations with theories that are subsequently modified and refined in contexts that include a mixture of personal exploration and social interaction. Their theories include criteria for deciding what to investigate, for determining how to go about their investigations, for choosing what to look at, and for establishing more precisely what the discourse is about. We find that children's working theories empower very striking and effective ways of working with mathematical ideas, often using concrete objects, in very particular ways, first as evidence for specific arguments, then as anchors for quite abstract constructions. In turn, children's theories and their ways of working with these theories help us to constitute our own conceptions of children's work and thought, and affect the way we build the discourse that is shared with them.

We view the interview structure as our way of <u>working</u> with the children. In our interactions with them, we, too, are building a theory. The process is reflected in the research-interview structure. Initially, the interviewer engages the child in an exploration attempting to estimate the theory that guides the child's thinking. Later in the same interview or in subsequent follow-up interviews, the ideas are pursued by the child who initiates and takes on the responsibility for the direction of the discourse. In the interview cycle, there is a "folding back"[3] that begins with very concrete discussions centered on local justifications. This is followed by a "teaching phase" intended to investigate deeper connections. In these interview settings, children often make big connections early and surprise us. These become our opportunities to learn from them.

Situations, in which children make connections on their own initiative, stand out from carefully structured interview sequences in which connections have been carefully delayed. One outcome of such a situation is that new hypotheses emerge that we tentatively hold and that now redefine our working text about how children make abstract connections. Once the child's connection breaks the flow, the interviewer invites further explanation. The child is asked about the structural similarity that is visualized or being constructed.

In <u>Mindstorms</u>, Papert reflects on the building of his personal mathematical understanding, which strongly influenced his later work, from his early experience as a child playing with gears. As with Papert, some of Stephanie's early mathematical understandings can be traced to her activities using block towers to investigate counting problems. Speiser's paper (1996) takes block towers as a concrete microworld for exploring several quite abstract arguments, with combinations as a guiding theme. It was triggered by the "Gang of Four" study (Maher and Martino, 1996a and 1996b), which

[3]See Pirie and Kieren (1992).

shows the reasoning and development of ideas which led Stephanie and three other children, at age 9, to discover the idea of mathematical proof, in order to conduct specific arguments about block towers. The central recognition of both papers, directly constituted by the "working theory" of these children, is that block towers, viewed as concrete mathematical objects, can function as essential tools for building much more general ideas. In particular, Stephanie's actions on these blocks have helped to anchor our theoretical descriptions of her new, more abstract work, as much as they have helped to anchor Stephanie's own discourse, with us, about that work. Hence our own views of what to look at, how to go about it, what is out there, and what descriptions should consist of —our theory, in other words— seems to start with towers, seen both mathematically and metaphorically. Building from towers (and from Papert's gears) we develop precise, particular descriptions, first of how Stephanie actually does powerful mathematics based on towers, and second, of what, concretely and precisely, constitutes that power.

Setting

The research was initiated with a class of 18 first-grade children in 1989 with university researchers and classroom teachers to study together how mathematical ideas develop in children (Martino, 1992; Maher and Martino, 1996b). Stephanie was one of six children selected for the study as a more or less representative sample of first-graders, at a public school, in a blue-collar district. Stephanie and her classmates were challenged in their mathematics classrooms to build solutions to problems and construct models of their solutions. This setting, which for Stephanie continued to grade 7, encouraged differences in thinking that were discussed and negotiated. In fall 1995, Stephanie moved to another community and transferred to a parochial girl's school. Her mathematics program for grade eight was a conventional algebra course. Stephanie continued to participate in the longitudinal study through a series of individual task-based interviews. A subset of these interviews provides the data for this study.

Guiding Questions

The following questions guide our analysis in order to consider, systematically the ways in which Stephanie's past experience is drawn upon: How does Stephanie work with towers in building images and understandings for higher mathematical ideas? What is the role of past experience in building new ideas? How are her ideas modified, extended, and refined over time?

Procedures, Data Source, and Task Design

Data come from two of eight individual task-based interviews of Stephanie that were videotaped with two cameras to capture what was said, written and built, as well as less tangible data such as tone of voice, speech tempo, and where people are looking while they talk and do things. Transcripts and analyses of the interviews were made by a team of researchers including the authors and graduate students. Stephanie's written work from outside the interview and observers' notes are also data sources. The teaching experiment was conducted over a six month interval (11/8/95 to 5/1/96). Each interview, approximately one and one-half hours in length takes on a particular format. It typically begins with inquiries about the mathematics that Stephanie is currently studying in eighth-grade algebra, leaving open the opportunity to talk about that mathematics and further to pursue her thinking about fundamental ideas in greater detail.

Results

During the March 13, 1996 interview, Stephanie, unprompted, makes a connection to towers in examining her symbolic representation of the expansion of $(a+b)^2$ and $(a+b)^3$.

S: So there's a cubed [a^3.]
I_1: That's one.
S: And there's three a squared b [$3a^2b$] and there's three a b squared [$3ab^2$] and there's b cubed. [b^3] [Interviewer writes 1 3 3 1 under the 1 2 1 as Stephanie speaks] Isn't that the same thing?
I_1: What do you mean?
S: As the towers.
I_1: Why?
S: It just is. (March 13, 1996; lines 718-724)

Stephanie has asserted (in her own way) that each 3-high tower gives a non commutative monomial of degree 3 in 2 variables, and has indicated that these non commutative monomials, indexed by the towers, collect to give the coefficients for the commutative ones. Our interpretation, therefore, is that Stephanie visualizes the towers (referring to mental models—she does not have plastic cubes) in order to organize her monomials. More precisely, we think that Stephanie is visualizing towers and reasoning from her visualizations.

Stephanie, working at home before the interview, had written down the first six powers of the binomial a + b. Interviewer 1 covered Stephanie's paper, guessed the coefficients for the sixth power expansion, and then wrote down the terms in full. Her coefficients agreed with Stephanie's, but one polynomial term was different, nonetheless. A few minutes later into the conversation, Stephanie gives further evidence, in a little more detail, that she is visualizing towers and reasoning from her visualizations.

I_1: So you have two factors of a. Right?
S Um hm.
I_1 You have one of those. One thing with two factors of a. One thing with two a's in it.
S: Um hm.
I_1: I don't want to think of a's. I want to think of red.

S: Okay. [laughing]
I_1: Can you switch that a minute?
S: Yeah.
I_1: So now I have one thing with two reds. What thing can I be thinking of with two reds?
S: That's a tower that's two high.
I_1: Okay. And here I'm talking about two things.
S: Um hm.
I_1: One is
S: Red and
I_1: one is
S: one is yellow.
I_1: Is that possible in two high?
S: Yeah.
I_1: Having one red and one yellow? There are two of them?
S: Yeah.
I_1: Which two?
S: 'Cause the one is the red could be on the top or the bottom, with the yellow the same thing.
I_1: What about b squared?
S: Um. Two yellow. (March 13, 1996; lines 745-768)

In a March 27, 1996 interview, Stephanie, is invited by Interviewer 1 explain to Interviewer 2 (who was unfamiliar with her recent work), what had happened in the previous (March 13, 1966) interview. Stephanie begins with towers, and then introduces the binomial coefficient notation (this time, following the earlier February interviews) as C(n,r), in a carefully sequenced progression of examples, based on organizing towers. Here Stephanie then explains that "r is a variable" which can range from 0 to n. This observation, which shifts the level of abstraction upward from concrete towers to patterns of formal symbols, also views the index n, the height of a tower, as a variable. This implication seems to trigger an extraordinary, detailed recapitulation, by Stephanie, of the recursive construction of the towers of height n from the towers of height n–1 (introduced by classmates, Milin and Michelle in grade four[4]).

From this unprompted discussion of the recursion on n for building towers, which organizes towers in a very different way, Stephanie returns to the current exploration of combinations (folding-back, which she is clearly leading) and, for increasing n, writes down several further rows of Pascal's triangle. At this point, with the triangle, including its addition rule clearly in view, Stephanie goes on to explain, to Interviewer 2, that she can use Pascal's triangle to predict the terms of $(a+b)^n$ for new, and hence larger, exponents.

On March 13, indeed, Stephanie had also used Pascal's triangle, in particular its addition rule, to make predictions, but she had done so in a conceptually quite different

[4] See (Maher and Martino, 1996b).

domain: to predict, in effect, the numbers of n-high towers in each given case (k reds, say, for $k = 0, \ldots, n$) for new values of the height. Stephanie's choice to center, in her folding back, directly on binomials strongly confirms that Stephanie now grasps the isomorphism between Pascal's triangle, which she had built, at first to summarize her towers cases, and the array of coefficients for her polynomial expansions for the powers $(a+b)^n$, for increasing values of n.

On the one hand, the text cited on March 13 is mainly that of Interviewer 1 with Stephanie an eager, active listener. On the other hand, the discussion here fourteen days later with Interviewer 2, innocent of what had gone before [and with a somewhat different agenda], is just as clearly being led by Stephanie. The levels of abstraction noted, then simply as a guess, in our description of the March 13th interview now seem, at least in part, confirmed by Stephanie's new variation. Again, the towers serve as anchor for a more abstract discussion, but now for polynomials, whose coefficients now encode Pascal's addition rule.

The investigators' choices, made to plan the teaching phase for the March 27th interview, grew from a careful, and quite fundamental, folding back, now by the investigators, who had been reconsidering their conceptual description of the underlying mathematics. This reconsideration built upon a very careful, systematic look, in great detail, at how Stephanie might be imagining combinations, viewed, as Interviewer 1 suggested to her on March 13, more as selections than as towers. Interviewer 2, deliberately kept unaware of the content of the March 13 interview, began this new direction, building on the exposition which Stephanie had just provided, by proposing a new exploration of block towers. The goal of this exploration, which will cover quite familiar ground for Stephanie, but in new ways, is to offer Stephanie precisely the tools to construct a formula, originally due to Fermat (Weil, 1984), which expresses the relationship between two successive binomial coefficients, and hence leads quickly to the standard formula for C(n,k). In this discussion, a central feature is to fix n, the height of a tower, and to vary k, beginning with a known case, either k=0, or, as Interviewer 2 did, with k=1.

The point of the new task, in the context of Interviewer 1's ongoing work with Stephanie, is that the fundamental observations needed to determine any C(n,k)—an important question raised, but not resolved, on March 13—do not, in any way, depend on the order of positions in a tower, and hence seem much more concrete and direct than the usual approach through permutations. This discovery, at the purely mathematical level [for the full story, see Speiser, Block Towers and Binomials], through its fresh look at the fundamental concepts which underly the work with Pascal's triangle, changed drastically the investigators' line of questioning.

The Interviewer's new look at the mathematics depends fundamentally on equal-handed treatment of the two colors used to build the towers. Hence, independently, not knowing how deeply the March 13th interview had explored symmetry, Interviewer 2 wonders about a difference between the way Stephanie has built her towers and the way she subsequently organizes them.

I_2: About when you um I think it was when you built these six towers. Uh. It looked to me like you were making pairs of opposites....at the beginning, when you were constructing them....but that when you were explaining to me how many there were, you organized them differently....um, could you say a little more about that?
S: Oh. Well. Because it's easier for me to look at them as opposites when I'm building them.
I_2: Um hm.
S: Then, 'cause I know, 'cause it's like pairing them up, like if there's one separated on top, there's one, you know.
I_2: Yeah.
S: But, it's easier for you to look at them when they're done if they're like this. So you can see the pattern that they make. That you can't build down any more.
I_2: Um hm.
S: Or you can't build up any more 'cause there's no more to....do it.
I_2: So it was more for your explanation that....you rearranged them.
S: Like you could see it better like this, than if I said....I mean....'cause when we first did the towers problems, we went through....I mean there were tons of Unifix cubes and all it was, was those two are opposites. 'Well, how do you know?' [imitating earlier questioning]
I_2: Um hm.
S: And I don't know. I didn't know how to explain it. So it's easier for you to see that.... there's the, you know, because it goes down 'til it can't go anywhere. That's why.

(March 27, 1996; lines 168-187)

In this discussion, Stephanie distinguishes clearly between her construction technique, in which opposite towers seem to help her connect, and perhaps also cross-check, different cases, and the text she's building, which will verify a major claim.

Conclusions

Images, patterns and relationships have become mathematical objects which Stephanie sees and works with mentally in building abstractions. Our conversations with her elicited both spoken and written texts that, along with our interpretations, have helped narrate the development of certain mathematical ideas. These texts (works in progress) extend over time and serve as records of particular events upon which later texts can comment. Further, they serve as raw material from which new texts can be composed. Analysis of clear cut mathematical representations, such as Pascal's triangle and its addition rule, provides one major strand which interweaves with much more tentative reflective discourse. Most importantly, the discourse, as we have seen, folds back on itself through critical review and purposeful reshaping of specific items of past work, or through more thorough reconsiderations over time of how past text relates to current and future needs. We revise our texts and so does Stephanie as our experiments proceed through detailed interactions with each other. Hence, as Stephanie's developing judgment enters the discussion, she helps to focus and direct the investigations. Our agenda for the interviews continues to be rewritten in response, often, to the direction that Stephanie pursues.

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CHANCE ESTIMATES BY YOUNG CHILDREN: STRATEGIES USED IN AN ORDERING CHANCE TASK

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This study focused on children's representation and reasoning strategies when estimating chance. Children aged 7 and 8 years were asked to order three sets of coloured marbles with different degrees of chance of drawing a marble of a given colour. The role played by the favourable, unfavourable and possible cases was equally emphasised in the instructions so that children were aware of their importance to estimate chance. The results indicated that children correctly estimated and ordered the sets on the basis of part-part comparisons between the favourable and unfavourable cases. The strategies adopted to solve the task showed how young children deal with the relevant variables and crucial relations involved in the chance concept. The importance of estimate in children's reasoning and the educational implications for the teaching of math are discussed.

The concept of chance involves three crucial quantities (two parts – favourable and unfavourable cases, and a whole – all cases) and relations between these quantities. These relations can be represented either in part-whole (comparisons between a part to a whole) or in part-part relationships (comparisons between the parts)[1]. Thus, in order to express the probability of any event (e.g., blue marbles) in a set of three blue and nine pink marbles, one may correctly represent this either in terms of a part-whole relation (3/12) or in terms of a part-part relation (3 : 9).

Studies on the concept of proportion (e.g., Spinillo & Bryant, 1991; Singer & Resnick, 1992) revealed that children tend to apply part-part relations when the task to be solved is opened to the two types of representation (part-part and part-whole). It is possible that these representations could be applied to other relational concepts such as chance. Which type of representation children use and which quantities they consider in a chance problem whose solution involves either part-part or part-whole relationships?

Piaget & Inhelder (1975), for instance, argue that children tend to focus on the favourable cases in absolute terms, ignoring the role played by the possible cases (total number of cases). Analysing the probability tasks the authors used, one may say that the crucial question addressed to the children ('Which of the two containers shows the best

[1] These representations are associated to quantities that are complementary parts of the same whole.

chance of drawing a blue marble: here, there or the chance is the same?') made reference only to the favourable cases; so children credited great importance to the favourable cases and neglected the part played by the total number of marbles in each set. Thus, this task calls their attention to the favourable cases.

Acredolo, O'Connors, Banks & Horobin (1989) tested the idea that children can attend to both favourable and possible cases. They devised a task in which the favourable as well as the possible cases were salient across trials. It was found that even young children take both cases into account when making probability estimations. Thus, this task calls their attention to the favourable and possible cases.

Although these investigations suggest a contradictory picture of children's knowledge about the concept of chance, an alternative interpretation might be given: children perform according to task demands (see Konold, Pollatsek, Well, Lohmeier & Lipson, 1993). The task devised by Piaget and collaborators emphasised the favourable cases; while the task adopted by Acredolo et al. emphasised either favourable and possible cases. In both studies children performed in accordance with the emphasis given to the variables in each task. One might ask whether children would consider the three variables (favourable, unfavourable and possible cases) in a task in which they were equally emphasised. In such a task, children could either establish part-part or part-whole relationships. Even though this possibility cannot be discarded, one may argue that children would tend to represent the relations in part-part terms by comparing the favourable vs. unfavourable cases. Evidence for a part-part representation is supported by the fact that it is easier to deal with two parts of the same whole that can be directly compared (part-part) than to deal with comparisons of a part of the whole it belongs to (part-whole). There is already empirical evidence to support this idea (Spinillo & Bryant, 1991; Singer & Resnick, 1992; Spinillo, 1995).

Thus a study was devised so that: (a) the three variables (favourable, unfavourable and possible cases) were equally emphasised; (b) the task was opened to the two types of representation (part-part and part-whole); (c) children could estimate the chance rather than making calculations for precise responses. The aims of this research were to examine how children deal with, represent and relate the crucial variables involved in the chance concept, and also to explore the solution strategies adopted.

Method

Subjects, materials and procedure

Forty elementary school children aged 7 and 8 years were individually presented with 12 trials each consisting of three sets of blue and pink marbles. They were asked to order the sets according to the degree of chance of drawing a blue marble. The importance of favourable cases (blue marbles), unfavourable ones (pink marbles) and possible cases (total number of marbles) was explicitly stressed by the experimenter. The verbal justifications and the strategies children used were analysed.

The sets of marbles and the trials

There were three types of trials with the following structure[2]:

Type 1: the sets had unequal number of favourable and possible cases (e.g., 4/8 vs. 16/16 vs. 9/12, 2/8 vs. 9/12 vs. 8/16). Even though they had different degrees of chance, correct ordering could be made on the basis of the absolute number of favourable cases.

Type 2: the three sets had unequal number of the possible cases and two of them had an equal number of favourable cases and had different degrees of chance of getting a successful draw (e.g., 6/8 vs. 4/16 vs. 6/12, 8/16 vs. 8/8 vs. 9/12). Correct ordering could not be made on the basis of the absolute number of favourable cases.

Type 3: the three sets had unequal number of favourable and possible cases (e.g., 3/12 vs. 6/6 vs. 8/16, 12/16 vs. 3/12 vs. 8/8). Differently of Type 1 trials, correct ordering could not be made on the basis of the absolute number of favourable cases.

Results

Correct ordering

An Analysis of Variance indicated significant main effects for Age ($\underline{F}$ (1,38)= 8.67, p=.005) and Type of Trial ($\underline{F}$ (2,76)= 11.27, p<.001). There was an increasing number of correct responses from younger to older children, and Type 3 trials were much more difficult than the other two (Table 1).

[2] Each fraction represents a set in which the numerator corresponded to the number of favourable cases (blue marbles) and the denominator corresponded to the number of possible cases (total number of marbles).

Table 1: Means of correct responses (out of four) with standard deviations in parentheses.

	TYPES OF TRIALS			
AGES	**Type 1**	**Type 2**	**Type 3**	**Total**
7	3.85 (.489)	3.50 (.827)	2.75 (1.410)	3.36
8	4.00 (.000)	3.90 (.308)	3.70 (.733)	3.86
Total	3.92 (.350)	3.70 (.648)	3.22 (1.209)	3.61

The interaction Age x Type of Trials was significant ($F(2,76)= 3.70$, $p<.05$), revealing that the effect of the types of trials on performance was age dependent. Type 3 trials were particularly difficult for the 7-year-olds but not for the older children who did equally well with the three types of trials. This result was confirmed by Friedman Test (Two-way ANOVA) ($p<.05$). In fact, at the age of 8 children experienced no difficulty with Type 3 trials. U-Mann-Whitney Test showed that 8-year-old children did significantly better in trials Type 2 ($p<.02$) and Type 3 ($p<.05$) than did the 7-year-olds. In both age groups children had the same good performance in Type 1 trials.

In sum, all children performed successfully in the task. The main difference between ages occurred in relation to the types of trials: for the 8-year-olds the three types of trials were equally easy while the 7-year-olds experienced difficulties with the Type 3 trials. One possible explanation for this result is that Type 3 trials required a relative approach by the part of the children, and the three sets had to be taken into account simultaneously. The strategies described below help to understand this issue better.

The strategies adopted [3]

It was found three major types of strategies based on children's explanations, actions and responses given to the experimenter's questions during the task[4]:

[3] The protocols were analysed by two independent judges whose reliability of coding assessment between them was 83,95%.
[4] E - Experimenter's interventions in parentheses.

Strategy 1: Comparing only favourable cases in absolute terms

Item: 3/12 vs. 6/6 vs. 8/16 (Type 3) Ordering: 3/12 - 6/6 - 8/16

This (3/12) has less chance because it has 3 blue ones only. This (6/6) is the one in-between because it has 6 blue ones, and 6 is more than 3 and it is less than 8 blue marbles. (E: Look, in this (6/6) there aren't pink marbles to mess the game up, while in this one (8/16) there are 2 pink marbles.) Because in this (8/16) there are more blue marbles than in this set (6/6). And in this set (6/6) there are more blue marbles than in the other set (3/12).

Strategy 2: The child isolates one of the sets and compares the other two

2A - the child isolates the set with a different number of favourable cases from that in the other two sets, and compares the remaining sets on the basis of absolute number of unfavourable cases.

Trial: 8/16 vs. 8/8 vs. 9/12 (Type 2) Ordering: 8/16 - 8/8 - 9/12

I take this (9/12) apart. (E: Why?). Because it is the different one. (E: How is it different?). It has 9 blue ones and the others have 8. And this is also the one with more chance than the others. Now, this (8/8) and this (8/16)... Let me see... The chance could be the same in both but... because of 8 and 8 (points to the blue marbles). But... this (8/16) comes first and this (8/8) follows. (E: Why if they have the same number of blue marbles?). Because the chance in here (8/8) is higher than in this (8/16) because it has 8 pink ones, and the pink marbles do not help in the game. They are bad marbles. While in the other (8/8) there are only blue marbles. (E: But here (9/12) there are 3 pink marbles, those that are not good). But there are more blue marbles than in the others, so the chance is high.

2B - the child isolates the set with 100% of chance, and compares the remaining sets on the basis of absolute number of favourable cases.

Trial: 8/16 vs. 8/8/ vs. 9/12 (Type 2) Ordering: 8/16 - 9/12 - 8/8

This (8/8) stays far from the other two. (E: Why?) Because it makes one wins, it has all marbles in blue. So, this (9/12) has medium chance, and this (8/16) has the lowest chance of all because it has few blue marbles and many pink marbles. (E: And how about this (9/12)?) Ah! This has 9 blues (9/12) and this 8 blues (8/16). Thus, low (8/16), medium (9/12) and high (8/8). (E: But how it could be? Both (8/8 and 8/16) have the same number of blue marbles.)The chance is higher here (8/8) because all the marbles are blue.

2C - the child isolates the set with 100% of chance, and compares the remaining sets on the basis of favourable and unfavourable cases.

Trial: 3/12 vs. 6/6 vs. 8/16 (Type 3) Ordering: 3/12 - 8/16 - 6/6

This (6/6) is the highest (moves it to the right side). (E: Why?) Because I am sure that anybody can take a blue marble with this set, all the marbles are blue. Now I take this (8/16) and this (3/12). This (8/16) stays in the middle, and this (3/12) is the one with less *chance of all because it has more pink marbles than blue marbles. The pink ones mess the game up. (E: But why did you think that this one (8/16) is in the middle?) The blue and the pink are equal, 8 and 8 (E: But this one (8/16) has more blue marbles than this one (6/6), isn't it?) I know, but even with less blue than the other this one (6/6) has more chance because all of the marbles are blue, there is no pink at all. When it happens no matter whether the blue marbles are many or few. One will always win.*

Strategy 3: the child compares the three sets in relative terms, considering the favourable and unfavourable cases in each set.

Trial: 8/16 vs. 6/8 vs. 3/12 (Type 3) **Ordering: 3/12 - 8/16 - 6/8**
The highest is this (6/8), because 6 blue marbles against only 2 pink ones. The pink ones are not good, they did not help. Then the next is this one (8/16) and in the last place this (3/12). This (6/8) is the highest because it has more of the lucky ones (blue) than of the unlucky ones (pink). Here (3/12) is different because there are few blues and many of the bad ones. This (8/16) is in the middle because the chance is the same inside: 8 blues and 8 pinks, the lucky ones and the unlucky ones are the same inside. So we never know for sure. (E: You know for sure that in 6/8 you get a blue one?) No. But at least I know that in this game here (points to the all three sets on the table) the chance is bigger with this one (6/8). (E: But why this (8/16) is not the highest? It has more blue marbles than the others.) But it does not help much to have the same inside (points to the blue and pink marbles in 8/16). (E: What does help then?) When there are more of the lucky ones than of the unlucky ones. And to have all of them lucky ones. That's the best.

It is worth noting that the children never made reference to the possible cases, representing the task in part-part terms (favourable vs. unfavourable cases) as observed in Strategy 2C and 3. The main difference between these strategies was that in Strategy 3 the three sets were taken into account simultaneously. The use of these strategies varied according to the types of trials and according to age (Table 2).

Strategy 1 was never used in Type 2 trials, and Strategy 2A was only to be found in trials Type 2. This pattern of result was much the same in both age groups. The fact that Strategy 1 was never adopted in ordering sets in Type 2 trials is due to two of the three sets having the same absolute number of favourable cases. This called children's attention to absolute number of unfavourable cases since the favourable ones were constant across the two sets (Strategy 2A). This explanation suggests that Strategy 2A appears as an alternative solution for Strategy 1 (absolute number of favourable cases) when the latter does not work.

The main difference was that 7-year-olds adopt Strategies 2B more often than the older children did, while Strategy 3 was predominant at age 8 (70%). Strategy 2C and 3 are invariably accompanied by a correct ordering of the sets.

Table 2: Percentage of strategies by Age and Type of Trials

STRATEGY	Type 1*	Type 2*	Type 3*	Total
		7 yrs		
1	9	0	13	7
2A	0	25	0	8
2B	35	14	36	28
2C	22	22	21	22
3	34	39	30	35
		8 yrs		
1	3	0	3	2
2A	0	4	0	1
2B	10	7	11	10
2C	16	15	20	17
3	71	74	66	70

* Percentages extracted from the total of 80 responses in each type of trial per age

Discussion and Conclusions

The following discussion concentrates on the cognitive nature of the strategies and on what they tell us about the way children initially reason about the concept of chance. This is not only of psychological interest, but also of didactical importance. The study showed that 7-8 -year-old children can correctly estimate and order different sets according to the chance of drawing a given element. They did so by representing the relations as ratio (favourable vs. unfavourable cases in each set) rather than in terms of part-whole relations (fractions). This shows that young children have a systematic and appropriate approach to deal with the concept of chance even before they are taught probability in school. The fact that children adopt part-part instead of part-whole relations (according to Singer & Resnick, 1992, children are part-part reasoners) suggests that the concept of chance (and other relational concepts as well) could be initially grasped in terms of ratio before it acquired the form of fraction. Estimating is an important intellectual activity in which children have the opportunity to deal with the principles behind the concept without worrying about calculations that they may not master yet (e.g., Streefland, 1982, 1984, 1985). Instruction should produce a shift toward more effective and powerful forms of reasoning by calling children's attention to (a) the whole and to the parts (see Nunes & Bryant, 1996; Spinillo, 1996), and (b) the fact that the chance could be altered when increasing/decreasing the variables (see Lamon, 1995). This study might be an argument for starting to teach complex concepts while children are still in elementary school. Instruction on such concepts may have a crucial role on children's development of mathematical reasoning.

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MULTIPLE REFERENTS AND SHIFTING MEANINGS OF UNKNOWNS IN STUDENTS' USE OF ALGEBRA

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In this paper we show that an interpretation of a variable as having multiple referents or shifting values is evident in the thinking of a sample of Australian students. We show how their imprecise and varying meanings for "the unknown" affected their reasoning as they worked through solutions to problems. In interviews with students we identified three modes of use of variables: to refer to different quantities in the one equation; to refer to different quantities at different stages of a solution; and as a general label for any unknown quantity or a combination of unknowns.

In the early years of algebra learning, students are taught to use letters to represent specific unknowns or sets of possible values of variables. Some students learn easily and succeed on school algebra tasks whereas others are completely mystified. Like many other researchers, we have explored the variety of ways in which students interpret and use algebraic letters. We have presented evidence (MacGregor & Stacey, 1996, in press) for the following interpretations and beliefs (the first two already well-known) in the context of simple translation tasks :

- the letter is perceived as an abbreviated word.
- the letter is assigned a numerical value that would be reasonable in the context.
- the letter is assigned a numerical value related to its position in the alphabet.
- the letter has the value 1 unless otherwise specified.
- the same letter can represent different quantities.

These interpretations were seen in responses to items that required students to write simple expressions. For example, students were asked what they could write for David's height, given that David is 10 cm taller than Con and Con's height is h cm. Common responses, reflecting the interpretations listed above, were:

- Dh (meaning "David's height")
- 110 cm (10 cm taller than 100 cm)
- r cm (h is the 8th letter of the alphabet, $8 + 10 = 18$, and the 18th letter of the alphabet is r)
- 11 ($h = 1$, therefore $10 + h = 10 + 1$)
- $h = h + 10$ (h can represent two different quantities)

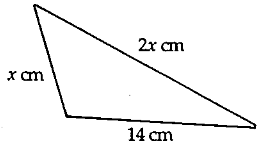
It is possible that the lack of any context or purpose associated with the test items used in studies by ourselves and others affected students' decisions about how to interpret letters For example, in certain contexts DH might mean "David's height" just as CD means "compact disc"; and in codes or puzzles, h might stand for the numeral 8. These interpretations are not unreasonable, and can be associated with the use of certain teaching materials as well as with students' everyday experiences (MacGregor & Stacey, in press).

As part of our subsequent research program, we wanted to find out whether students would be more likely to use letters as unknowns in the conventional way if they were given tasks in which the use of algebraic notation has a clear purpose. Students learn that one important purpose is to solve problems, and they are shown how to use algebra to solve certain types of word problems. We expected that in a familiar problem-solving context most students attempting to use an algebraic method would use letters to represent unknowns. The data show that this was generally the case. However some students used x to mean "anything unknown" and accepted that there could be multiple referents or a series of referents for x as they worked through a problem. Some indications of this interpretation - a letter representing different quantities - had been found in our previous work (see above). Fujii (1993) found that it occurred in a sample of Japanese students, and suggested that it represents students' emerging understanding of the non-specific nature of variables, that is, "x can be any number". In Fujii's study, students accepted that if $x + x + x = 12$, then the first x could be 2 and the other x's could be 5. This belief in multiple referents for x has been rarely mentioned in the literature, although a related misconception- that two different letters cannot have the same value - is widely recognised. In this paper we show that an interpretation of x as having multiple referents or shifting values is evident in the thinking of a sample of Australian students. We show how their imprecise and varying meanings for "the unknown" affected their reasoning as they worked on problems.

Procedure

We prepared a set of word problems to be used by teachers for their own classes. The students involved were aged 13-15. The majority were in their third or fourth year of algebra learning. We collected written problem solutions from approximately 900 students in 10 schools, and we carried out interviews with 30 individual students in three schools. Discussion with these individuals while they worked on problems gave us insights into their reasoning and explained much of the behaviour evident in the written solutions collected from the large sample. Three of the problems, which we refer to in this paper, are shown in Figure 1. They belong to a category of problems frequently used for beginners in algebra. In devising the problems, we have chosen the simplest set of possible relationships (see Bednarz & Janvier, 1966, for variants and their complexity). Janvier (1996, p. 235) has pointed out that there are at least three valid interpretations of letters in school algebra - place-holder, unknown, and variable - producing different patterns of reasoning. In the problems referred to in this paper, only one of these interpretations - letter as specific unknown - is required.

1. The perimeter of this triangle is 44 cm. Write an algebraic equation and work out x.



.... $x =$ .....

2. Some money is shared between Mark and Jan so that Mark gets \$5 more than Jan gets. Jan gets \$$x$. Use algebra to write Mark's amount. .... If the money to be shared is \$47, how much would Jan get? How much would Mark get?

3. A bus took students on a 3-day tour. The distance travelled on Day 2 was 85 Km farther than on Day 1. The distance travelled on Day 3 was 125 Km farther than on Day 1. The total distance was 1410 Km.
Let x stand for the number of Km travelled on Day 1.
Use algebra to work out the distance travelled each day.

Figure 1. Problems, called TRIANGLE, MARK, and BUS in this paper

Success rates for the three problems

A large proportion of students wrote no equations. Others tried to write equations but then switched to non-algebraic methods to solve the problems. Table 1 shows the percentages who wrote a correct equation (whether it was subsequently used or not), and the percentages who obtained a correct answer to the problem by any method. Methods included trial-and-error, logical arithmetic reasoning, and the solving of an algebraic equation. Many students who began using algebra changed to another method to get the answer.

Table 1. *Percentages for equation correct and for answer correct by any method*

		1. TRIANGLE		2. MARK		3. BUS	
Year	*n*	equation	answer	equation	answer	equation	answer
9	249	-	-	15%	76%	24%	70%
10	700	38%	63%	30%	73%	32%	60%

Note. The Year 9 sample used a version of the test that did not include the "Triangle" item.

As Table 1 shows, approximately one-quarter of the students in the sample did not work out correct answers to these very simple problems. Some of these students had tried to use algebra, but without success. Students in two particular schools, who had been taught a problem-solving routine for selecting and naming an unknown, generating expressions, and formulating an equation, were almost all able to write algebraic solutions that were concise and correct. At another school, teachers wanted to find out whether specific instruction in formulating equations would be beneficial. One class was given demonstration at the chalkboard and practice on 12 problems before attempting our test. The results, for both writing equations and solving the problems, were far better than the results for a parallel class who had followed the normal curriculum. Success rates for BUS, for example, were 78% and 27% respectively for the two classes. This outcome suggests that the majority of students should be able to learn to use algebraic methods for solving problems.

Excerpts from interviews with students

Below we present excerpts from interviews with six students (age 14-15) to show how different uses of a letter to symbolise "the unknown" affected their attempts to use an algebraic method. These modes of use were:

- x may refer to different quantities in one equation.
- x refers to different quantities at different stages of the solution.
- x is a general label for any unknown quantity or a combination of them.

1. Marianne, trying to make sense of TRIANGLE, sees x as a label for two different unknown quantities. She writes the correct equation $2x + x + 14 = 44$ but says that she cannot solve it unless she knows the value of x to substitute in $2x$. If she knows this x, then she can work out "the other x". She sees her equation as containing two unknowns, both called x (see line 4), and she knows that an equation with two unknowns cannot be solved.

1 M: *You can't solve that unless you know what x equals. If I knew what x*
2 *meant in there* [indicating the side labelled $2x$] *then I could do it. There's*
3 *no way you can work it out from between those two* [indicating $2x$ and
4 14]. *What we are trying to do is to find out what both these x's mean, and*
5 *we can't do it unless we know what that x* [indicating $2x$] *means.*

2. Justin sees the sole aim of the task as finding the length of the side labelled x. He does not seem to believe that the instruction to find x also applies to the side labelled $2x$. To find x, he assumes that the side labelled $2x$ is 2 cm long. Thus the value of x is 44 - (14 + 2), giving 28. (This answer was relatively common in the main sample.) The interviewer begins by asking Justin how he worked out $x = 28$.

1 J: [pointing to the sides labelled 14 and $2x$] *That's obviously 16.*
2 I: *Why? Did you see that x there?* [indicates $2x$]
4 J: *Yes, I noticed it, but I thought no, it's obvious that you need to find out*
5 *what this x is.* [indicates the single x]
6 I: *Could you write the equation?*

J: *Yes.* [writes $2 \times x + x + 14 = 44$]

The interviewer asks Justin to solve his equation. He says he can't, and looks back to the diagram, again expressing his concern about not knowing the value of x in $2x$ (lines 8,9). He tries to resolve this dilemma by saying that x must have the value 1 (an error referred to earlier in this paper), but later rejects this idea (line 13) because he sees that the triangle could not have sides 1 cm, 2 cm and 14 cm because 1 + 2 + 14 gives 17, not 44.

J: *That's what I don't understand, when you have got 2 times x and you don't know what x is, you know the 14, so seeing that* [points to side x cm] *is just x, it would be 1. And that's 2 times 1* [indicates side $2x$] *so it's still 2.*
I: *Why is x equal to 1?*
J: *Because if it doesn't have a number there* [to the left of x] *then x equals just 1 ... but that's not right.*

3. Dean thinks of x as signifying the total of unknowns. For TRIANGLE, he works out x to be 10 but writes down his solution as $x = 30$. He then says that is wrong, and writes $x = 10 \times 3$ as his final answer to the problem. He explains, "It is three lots of 10". He seems to think that x should represent everything that was not explicitly given in the data, although he knows that the side labelled x cm is 10 cm long and the side labelled $2x$ cm is 20 cm long.

4. Joel has in mind multiple and shifting referents for x in the MARK problem. He writes the correct expression $x + 5$ for Mark's amount. Then he writes $x + 5 = 47$ and the interviewer queries him.

I: *What does it say?*
J: *You've got one starting number and you add 5 and get 47.*
I: *This* [indicating 47] *is the total amount, this* [indicating 5] *is the extra five, so what is this x?* [indicating x in the equation $x + 5 = 47$]
J: *The amount they both get. The amount that Jan gets. I just like to keep the three of them, 47 dollars, x, and 5 dollars more, and make something out of them.*

Although Joel has written the correct expression for Mark's money in terms of x, he then sees the unknown x as "the amount they both get" (line 5). He also sees it as Jan's amount (line 5). When asked to explain his equation, he does not relate it back to the problem situation but interprets what he has written as a narrative about numbers - a sequence of events (line 2). He says that he needs only one x in his equation (lines 6,7). His equation states a relationship between the numbers 5 and 47 given in the problem and some unknown amount. However it is written for the interviewer, and not seen by Joel as useful for obtaining a solution to the problem. When he solves it and obtains $x = 42$, he says that Jan's amount is \$42. He has lost connection with the meaning of the problem which he initially understood. His earlier spontaneous approach to solving the problem (interrupted by the interviewer asking for an equation) was to divide 47 by 2. His thinking seems to move back and forth between x as the amount to be shared equally (\$42) and x as the value of each of the final shares (\$21 and \$26).

5. Les also uses x to refer to different quantities in MARK. He uses mathematical notation informally to try to keep track of his thinking. His first equation for Item 2 has an x that is the first thing he wants to work out ("what is left", lines 1 and 7 below), but then he talks about "sharing two x's" (line 3). Les knows how to work out the answer by keeping in touch with the situation, but does not know how to write down his procedure. He finally writes a description of the steps in his calculation, trying to make it look like algebra (line 11). To begin, Les writes the equation $5 + x = 47$ and explains what it means.

1 L: *x is what is left out of $47 if you take 5 off it.*
2 I: *What might the x be?*
3 L: *Say she gets $22 and he gets $27. They are sharing two x's.*
4 I: *What are the two x's?*
5 L: *Unknowns ... they are two different numbers, 22 and 27.*
6 I: *So what is this x?* [pointing to his equation $5 + x = 47$]
7 L: *I thought that was left over from $47, so it's $42.*

The interviewer points out that he has three meanings for x, which makes the equation hard to solve. Les decides to use x and y for Jan's amount and Mark's amount. He writes $x + 5 + y = 47$, and explains what he is going to do.

8 L: *He's got 5 more than her, so you take 5 off. Would you minus 47 from*
9 *5 plus x plus y?*

It appears that he wants to subtract 47 from both sides of his new equation, remembering a method he has been taught. The interviewer wants to see how he will do this (line 10), but he reverts to arithmetic thinking (lines 11, 12).

10 I: *OK, try that.*
11 L: [writes $47 - 5 + x + y$] *Then minus 5 is out of the way, so you split it in*
12 *half. If you take the 5 off, then you've just got two unknowns, 21 and 26.*

The logic of Les's reasoning is sound, but he does not know how to express it clearly in words or how to write down his solution procedure. Like Joel, he has thought of x as standing for 42, 21 and 26, that is, for any quantity that was unknown and needed to be worked out.

6. Tim writes $x + 5$ for Mark's amount, but extends it to make $x + 5 = x$, saying that the x after the equals sign is "Jan's x". The interviewer queries him about the meaning of the other x.

1 I: *So what is this x?* [points to the first x in $x + 5 = x$]
2 T: *That's Mark's x.*
3 I: *And why do we add 5 to it?*
4 T: *Because Mark has 5 more dollars than Jan. No, that's not right, it should*
5 *be Jan's x plus 5 equals Mark's x.*
6 I: *Could you write an equation to say that Mark and Jan have $47 in total?*

The interviewer explains that to write an equation you don't have to work out a numerical answer first. Tim now thinks he should write what he would do to work out the answer (line 7).

7 T: *x divided by half equals x* [writes $x \div \frac{1}{2} = x$]

Here Tim is writing x to mean "some total amount of money" and then x again to mean "half the money". For him, $x \div \frac{1}{2} = x$ makes sense because he knows, at least momentarily, what each x refers to. He wants to share the $47 equally first.

8 I: *So you take the money and you halve it. It that what you mean?*

9 T: *Yes.*

During the interview, he has used x to mean "Jan's amount" (line 5), "the total" (line 7), and "half the total" (line 7). Although he recognises that if Jan has $\$x$ then Mark has $\$(x + 5)$, he is not sure whether the x in $(x + 5)$ is "Jan's x" or "Mark's x". Since $(x + 5)$ represents Mark's money, he first thinks, not unreasonably, that the x in it should be "Mark's x" (line 2). He later uses x to mean any unknown quantity he is thinking of.

Discussion

The interviews reported here explain some of the solutions written by students in the larger sample. They indicate three different modes of use of a letter to symbolise "the unknown" (or, as we have seen, "unknowns"). Some students believed that x can stand for more than one unknown (including the total of all unknowns) simultaneously; some changed its referent at different stages of the solution; and some saw the unknown as any or every quantity they didn't know in the problem.

Mode 1. Interviews with Marianne and Justin illustrate the first mode. These students used x to refer to two different quantities in one equation. They both explained why they could not solve Item 1: they saw the two x's - in the diagram, and in the equation which they both wrote correctly - as having different values. Justin seemed to think that the item asked for only the side precisely labelled as x to be found.

Mode 2. Dean's interview shows how he thought of x as representing a total of unknown quantities. This thinking may be the reason why some students in the large sample wrote $x + 85 + 125 = 1410$ as their equation for the BUS problem.

Mode 3. Joel, Les and Tim changed the referent of x at different stages of their thinking. Joel and Les each used at least three referents within a minute or so. As the solution process continued, they used the one letter to stand for the many different unknown quantities which are present in even the simple situations portrayed in the problems. In the light of this, the standard instruction given by teachers to let a letter stand for "the unknown" seems particularly inappropriate and highlights the arbitrary nature of the way in which teachers automatically classify problems as having a certain number of unknowns.

Tim's use of variables exhibits all modes. His equations ($x \div \frac{1}{2} = x$ and $x + 5 = x$) are action statements about how to get from one unknown quantity to another. "Equations" such as these, which he and many other students wrote, represented steps in their reasoning but were not useful except as an informal reminder of the most recent idea they had in mind.

Many of the students interviewed were puzzled by our request to write equations, apparently not appreciating how this could help them solve the problems. In fact many students in the large sample wrote, as their equations, formulas for the answers worked out by logical arithmetic reasoning. One student, for example, wrote

$$\text{Jan } x = \frac{47 - 5}{2} \quad \text{Mark } x = \frac{47 - 5}{2} + 5$$

The success of students at the three schools mentioned above indicates that explicit teaching of the logical basis of algebraic problem solving is effective and should be more widely used.

Mathematics educators and psychologists continue to wonder why students pass through three or four years of algebra courses without being able to use algebra to solve problems. It is widely accepted that these students have not learned to "think algebraically". Many of the students in our sample exhibited logical and complex thinking. However when asked to write an equation, or when their thinking was pushed to its limit on harder problems, they did not know how to use algebraic notation as a tool to organise, record and extend their ideas. Their problem-solving strategies are restricted to a series of independent calculations working from what is known towards the answer. They record ideas and calculations using written symbols informally and inconsistently, believing that they are doing algebra.

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Elementary Components of Problem Solving Behaviour

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Abstract
The paper discusses problem solving abilities of children in primary schools. It focuses on gestalt operations and on heuristic techniques used by children when solving problems. We show that even in non-geometric contexts gestalt operations can be found, and we illustrate that even in primary education *combinatorial* as well as *backtracking techniques* are used by children in a natural way without prior instruction. The tasks are non standard tasks in different contexts (e.g., puzzles, arithmetic, money). To stimulate problem solving behavior, all tasks are unsolvable which means that the goal of the problem can not be reached. The unsolvability is of a kind which can be understood and proved by children of this age.

1 Question of research and basic considerations

In the history of mathematics as well as mathematics teaching *problem solving* always has played an important role. The research which is described on the following pages has connections to three fields of research:

1. Gestalt Psychology (see, e.g., WERTHEIMER 1961)
2. Heuristics of problem solving (see, e.g., POLYA 1967)
3. Information processing psychology (see, e.g., NEWELL & SIMON 1972)

A more complete analysis of the literature will be found in my next publication.

Since 1992 I work - together with several groups of student teachers - on problem solving. We focus on such components of problem solving ability which are *not* subject of mathematics lessons. As a consequence, we do not deal with word problems and other "classical problems". The central idea of our research is to use sets of tasks which are all *unsolvable* which means they have a goal which can not be reached. The unsolvability, however, is of a kind which can be understood even by younger children (e.g.: try to find exactly 4 different numbers out of the set {1, 2, 3, 4, 5} which give the sum 9). Subsequently, we shall use the term *impossible task* as well.

The main purpose of our research in its present state is - like a "constructive existence proof" - the identification of elementary components of problem solving behaviour which are actually used by younger children[1]. The search for those components is organized as search for noticeable patterns in the subject's behaviour. We call such patterns *seeds of strategy*. This means that we do *not* assume a conscious use of the strategy in question. It may "grow" and come to consciousness (e.g., if the interviewer

[1] A project in progress which is supported by a grant from *Deutsche Forschungsgemeinschaft* will systematically compare the behaviour of pupils of grades 3/4 and of grade 8.

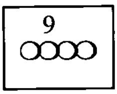
intervenes), or it may "die out".
In this paper, I describe *gestalt operations* and *problem solving techniques* (see also for more complete desciptions and more tasks: STEIN 1995 and STEIN 1996). A paper in preparation will focus on *reasoning abilities*.
In section 2 we shall first describe some of the tasks used, and then make some remarks on the construction principles and reasons for this type of task design. Sections 3 and 4 give some examples for the results which can be obtained by our method.

2 Tasks and Design of the Study

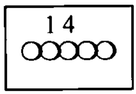
2.1 Arithmetical tasks

Given are round tokens (each only once), bearing the numbers 1, 2, 3, 4 and 5 (task 1) resp. 1, 2, 3, 4, 5, 6 (task 2). The problem solver gets a pattern which informes him/her how many tokens he/she may use. In the first unsolvable task the pattern shows 4 circles. Consequently, the problem solver shall lay down 4 tokens (for each circle one token), *which sum up to 9* (This task will be called sum-9-task). In the second task, the sum 14 shall be built using 5 different tokens out of {1, 2, 3, 4, 5, 6} ("sum-14-task").

sum-9-task



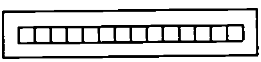
sum-14-task



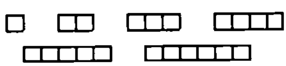
2.2 Mixture of geometrical and arithmetical aspects

The tasks described in the previous section can be embedded in a geometrical context: As an example, for the sum-14-task, a strip with a length of 14 units shall be filled with exactly 5 stripes. For this task, the solver has stripes of legth 1 unit, 2 units, 3 units, 4 units, 5 units, 6 units (each stripe may me used only once).

Pattern:



Stripes:



Having finished one attempt to reach the goal, the solver can start the next attempt in a new row. The same rules apply in this row. There is enough material given; as a consequence, the problem solver has full information about his/her earlier attempts on solving the problem.

2.3 Towers

This is the well known task to build all possible different towers of a given height from yellow resp. red stones. To make it impossible to reach the goal, there are more spaces to place the towers than different towers exist. I.e., in the case of towers of height 3 built from two different colurs, there are *nine* spaces given (though there exist eight different towers only).

2.4 Basic principles for the construction of the tasks

The tasks are constructed following some basic principles.

- They demand no prior knowledge which exceeds basic abilities of mental arithmetic.
- They use implicit rules which can be understood immediately.
- Every task has a certain goal. *This goal can not be reached.* It can be easiliy understood, however, that reaching the goal of the task is impossible.
- The material encourages *enactive* activities.

Using impossible tasks has several reasons:

1. Better understanding of and more information about problem solving behaviour
   The impossibility of reaching the goal inspires pupils to try many different approaches. They try in many different ways to find the solution or a partial solution.
2. Stimulation of Systematic Bahaviour
   Solvable problems - i.e. problems the goal of which can be reached - can be solved by incidence. Our *impossible problems*, however, can be understood only by a systematic analysis of the problem space (in the sense of, e.g., NEWELL/SIMON 1972). The size of the problem space depends on the solver's ability: in case of the sum-9-task (see section 2.1) a problem solver may use the laborous technique of systematically checking all possible combinations of 4 different numbers out of the set $\{1, 2, 3, 4, 5\}$, or he/she can just add up $1 + 2 + 3 + 4$ and argue that this is the smallest sum which can be constructed using four different numbers out of $\{1, 2, 3, 4, 5\}$, and that this sum is > 9.
3. Getting Information about "natural reasoning behaviour"
   Although this is not the central part of this paper, it should be mentioned that our impossible tasks are excellent tools for measuring reasoning behaviour: since the pupil is not able to reach the goal of the problem, he will start reasoning about the "unsolvability" of the problem without enforcement by the experimenter.

2.5 Design of the study

All problems are solved by groups of two pupils. The pupils are informed that some of the tasks can be solved, other tasks not, and that they have to find out what is the case and why.

As has been said, the search for components of problem solving behaviour is a search for noticeable patterns in the actions of the children. We do *not* assume, however, that

such a pattern is guided by a consciously applied "strategical insight". If a teacher - or, in our case, the interviewer - starts asking questions about the behaviour, the pupils may discover the strategical implications of their behaviour and start talking about it as if they had planned their proceedings from the very beginning.
As a consequence, the interviewers *only watch the pupils work*. When the pupils *say* that the task can not be solved, or *show some "suspicion"* the interviewers ask whether there might be other ways to solve the task... This may be repeated once. Only at the end of the interview the interviewers are permitted to talk with the children about their understanding of the situation and about the process of solution.
The interviews are filmed with a video camera. Every action of the children is protocolled. The final transcription has the character of a script for a movie which allows to replay the interview.
The script is interpreted turn by turn (MAIER 1991) by a team of interpreters which includes the interviewer. The behavior of the children is analysed under a broad range of aspects. In many cases there will be more than one interpretation of the same behaviour. The same action may be interpreted, for instance, as a consequence of a social conflict between the two children, or may be seen as influenced by gestalt operations or be understood as guided by explicit use of heuristic strategies.
In the following sections we give only such examples which have a rather unambiguous interpretation.

3. Gestalt-Operations

Many actions of problem solvers can be understood as reactions on the *gestalt* of the situation, or they aim at constructing special shapes or patterns. The analyses of transcripts under this aspect are influenced by WERTHEIMER 1961.
We give some examples of typical *gestalt* procedures.

3.1 Production of Standard Patterns

Arithmetical Tasks

The number 10 plays a special role in mental arithmetic. In that sense this number is a "standard pattern". As a consequence, we rather often find pupils who in the "sum-14-task" fill up to 10 - though it could easiliy be recognised that no way leads from this sum to the demanded sum 14.

Tower tasks

Opposite pairs and staircases

The "opposite pair" is a standard pattern which can be found rather often, e.g.:

g	r
r	g
r	g
g	r

Obviously, this pattern is a reaction to the fact that the towers are built using *two* colours.

We observed pupils who solved the whole task using opposite pairs: (the following example was built by pupils in grade 4).

w	r	w	r	w	r	r	w	r	w	w	r	r	w	r	w
r	w	w	r	r	w	r	w	r	w	r	w	w	r	r	w
w	r	w	r	r	w	w	r	r	w	r	w	r	w	w	r
r	w	w	r	w	r	w	r	w	r	r	w	r	w	r	w

Opposite pairs are of limited use only since there is only *one* tower which is "opposite" to a freshly found tower. As a pattern, the *stairs* are far more useful. Discovering this pattern, the problem solver can construct a whole series of new towers.
The following pupil (he works together with a second pupil, but the following action is done by him alone) gives a good example (grade 4).
The pupil has generated the following towers.

w	w	r	r	w	r	r	r	w
r	r	w	w	r	r	r	r	w
w	r	w	w	r	w	r	r	w
r	r	w	r	w	r	w	r	w
9	8	7	6	5	4	3	2	1

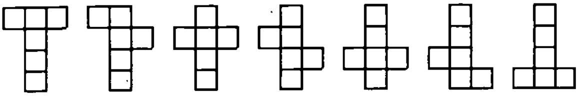
He starts to rebuild this series of towers. We show the last three steps of his work:
The pupil removes towers 5 and replaces it by the next "step". He says: "One higher".
We conclude that he has discovered the staircase-pattern in the towers 2 and 3.
We now have the following situation:

w	w	r	r	r	r	r	r	w
r	r	w	w	w	r	r	r	w
w	r	w	w	r	w	r	r	w
r	r	w	r	r	r	w	r	w
9	8	7	6	5	4	3	2	1

The pupil now removes tower 6, says: "One higher" and finally builds the last "step":

w	w	r	r	w	r	r	r	r	w
r	r	w	w	r	w	r	r	r	w
w	r	w	w	r	r	w	r	r	w
r	r	w	r	r	r	r	w	r	w
10	9	8	7	6	5	4	3	2	1

He finally says: "Now I reached the top".
It is interesting that the "stairs" are a pattern which is independent of context. Hasemann (1985) describes a similar behaviour when pupils have to find all patterns which can be used to fold cubes.



Symmetry

Using opposite pairs, some pupils find patterns of surprising symmetry. In the following case, the final pattern shows a "nonverbal proof" of the fact that it is not possible that there are exactly nine different towers of height three, using 2 colours:

4 Problem solving techniques

POLYA 1967 describes problem solving techniques as *working forward* or *working backward*. Good descriptions of elaborate techniques can be found in ANDERSON 1983 and LAIRD et. al. 1986.

4.1 Systematic combinatoric checking

The sum-9-task in its geometrical representation leads in the case of the following interview (pupils in grade 4) to an impressing result. I show the first 1.5 minutes of the interview.

The stripes which are layed down are marked by their length. The two pupils are referred to by S1 and S2. The stripe of lentgh 5 is named "5er", the other stripes are named respectively.

S1 lays down the 5er. In nearly the same moment S2 takes the 4er in his hand, takes it away again and takes the 3er. S1 says:"We shall need always mostly the small ones". S1 lays down the 3er. We have the following situation:

5	5	5	5	5	3	3	3	

S1 fills up the pattern with the 1er. So the pattern is fully filled, but the condition was to use *exactly **four** stripes*. Immediately after laying down the 1er the pupils start to work on the second row which contains the same pattern. Firstly, the 4er is laid down.

5	5	5	5	5	3	3	3	1
4	4	4	4					

S1 lays down the 2er and says: "The 1er, the 1er!". S2 lays down the 1er.

5	5	5	5	5	3	3	3	1
4	4	4	4	2	2	1		

In the same row, the pupils try to use the 3er (which is too long) and the 2er (which violares the rule that in one row each stripe may be used only once). None of these attempts remains on the pattern.
S1 now lays down the 3er, S2 the 2er, S1 the 4er:

5	5	5	5	5	3	3	3	1
4	4	4	4	2	2	1		
3	3	3	2	2	4	4	4	4

S1 says: "Oh, that does not fit" and laughs. S2 now uses the next row to lay down the 3er. S1 lays down the 2er, S2 the 1er.

5	5	5	5	5	3	3	3	1
4	4	4	4	2	2	1		
3	3	3	2	2	4	4	4	4
3	3	3	2	2	1			

S1 first tries to fill up with the 4er (which is too long). S2 holds the 3er over the gap, but S1 says that this would violate the rule that each stripe must only occur once in a row.
They start to fill up the fifth row with the 3er. S2 takes the 1er in his hand. He first hesitates to use it, but is encouraged by S1: "Yes, use it, yes yes." After a quick check of the 4er (which is not laid down), the 2er is laid down.

5	5	5	5	5	3	3	3	1
4	4	4	4	2	2	1		
3	3	3	2	2	1			
3	3	3	2	2	4	4	4	4
3	3	3	1	2	2			

This last picture shows the relevant stations of a rather systematical checking. If we additionaly bear in mind the stripes which were checked but not finally laid down, we have a remarkably complete combinatoric analysis of the situation.
It is especially remarkeable, that from an adult point of view, the last picture is sufficient to give good reasons for the impossibility of reaching the goal of using exactly four different stripes to fill up one row. Our pupils, however, start with new attempts and work for another five minutes without getting better insight of the situation. This is exactly what we mean by using the term *seed of strategy*: There is a

strong impression, that the behaviour of the pupils is guided by the combinatoric strategy of systematic checking - but this guidance appears to be sub-conscious, and dies out if there is no external stimulation by the interviewer.

4.2 Backtracking

Backtracking is a problem solving technique which is used in computer science. Its special way of "going forward and then backward in one's own tracks" leads to efficient computer programs. The term *depth-first search* by NEWELL/SIMON describes the same phenomenon. With some of our problems, it can be an efficient - but highly non-trivial - technique of analysis. If we find a "backtracking pattern" in our interviews, we shall usually realise that- since our interviewers do not intervene - it dies out after some time: the seed is planted, but it does not grow to reach consciousness.

The following example shows to pupils (grade 2) working on the sum-14-task. In this presentation of the task, the pattern in which the tokens are to be layed is given only once. So here each row of the transcript means a new action. The sequence is laid by S2. Nearly from the beginning, S1 holds token 2 in his hand. Considering this, the analysis is complete, if we accept that the pupils do not try actions which are obviously "ridiculous" (trying token 1 when the sum is reached or exceeded).

- 5 - - -	token 5 is laid down in the second space
6 5 - - -	token 6 is put in space 1
6 5 4 - -	the 4 is laid down (sum is 15, too high)
6 5 - -	token 4 is removed
6 5 3 - -	token 3 laid down (sum correct, but too many free spaces)
6 5 - -	token 3 is removed
5 - - -	analysis complete: token 6 is removed

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CHANGING TEACHING AND TEACHER CHANGE

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Although reform efforts have some times managed to change teaching, the changes implemented are many times superficial and represent just the paste-on of a few new adjustments over the old practice. This paper argues that for true changes to occur in mathematics education it is necessary to go beyond changing teaching and work towards changing teachers. To change mathematics instruction in an effective way, however, it would be necessary to change teachers ideologies, given that ideas and beliefs about mathematics and its teaching and learning are part of a broader vision of the world and cannot be separated from this wider web of ideas. Teachers teach according to their value-rich visions of the world and, to deeply change mathematics lessons, reformers need to challenge these visions, forcing teachers to accommodate reform ideas instead of simply assimilating them.

Embora tentativas de reforma às vezes consigam modificar a forma como professores ensinam, as mudanças implementadas são, normalmente, superficiais, e representam uma colagem de novos ajústes sobre a velha prática. Este artigo coloca que para a ocorrência de uma reforma verdadeira em educação matemática é necessário ir além da mudança da prática e trabalhar no sentido de mudar os professores. Para efetivamente modificar o ensino de matemática, entretanto, seria necessário mudar as ideologias dos professores, dados que idéias e crenças sobre a matemática, seu ensino e sua aprendizagem, são parte de uma visão mais ampla do mundo e não podem ser separadas desta rede maior de idéias. Professores ensinam de acordo com suas visões e valores, e para modificar profundamente as aulas de matemática é preciso desequilibrar estas visões, forçando professores a acomodarem as idéias de reforma ao invés de simplesmente as assimilarem.

Discussing efficacy and the current reform in mathematics education, Smith (1996) illustrates a range of different reactions teachers have when facing the challenge of change. He classifies one type of teacher's response to calls for reform as "paste-on adjustment", which he defines:

> The reform combines a theory of mathematics content, learning, and teaching with some more specific prescriptions for teaching (e.g. using manipulative materials or small-group problem-solving). In response to strong suggestions that they "implement" the reform, some teachers add these specific elements to their practice without addressing the more fundamental issues that underlie and inspire them. (p.396)

These paste-on adjustments allow teachers to implement new teaching ideas in their classrooms without having to deeply change their practices or beliefs. As Smith (1996) explains, "small-group work, student projects, and manipulatives can be easily assimilated to views of content that

[1] This paper is based on the author's doctoral dissertation—directed by Frank K. Lester, Jr. During the doctoral program and the research project, the author was supported by a scholarship from CNPq, Brazilian government.

emphasize the standard rules and algorithms, the teacher's role of knowledge telling, and students' roles of listening and practicing, leaving the pedagogy of telling fundamentally intact" (p. 396).

In a study that analyzed the implementation of the California Mathematics Framework (California State Department of Education, 1985), Cohen and Ball (1990) noticed that some teachers added bits and pieces of the reform document to their teaching scheme, developing what the authors considered contradictory practices that combined elements and theories incompatible in essence. For them, the document consisted of many different (important) ideas which could be "picked up in random bits and then enacted in variously interpreted permutations of each bit" (p. 332).

Teachers, as is true of all people, "adapt recommendations instead of adopting them" (Darling-Hammond, 1990, p. 341), and we have long learned that they interpret reform documents, translating reform rhetoric into their own language and practice (e.g., Olson, 1981; Saranson, 1982). However, as I see it, translations, interpretations, adaptations, contradictory practices or paste-on adjustments exist because reform attempts tend to concentrate their efforts on changing teaching without deeply addressing the more difficult issue of changing teachers. Therefore, teachers can assimilate reform ideas without having to accommodate them in a new understanding of what mathematics education should be and what its functions in schools and societies are.

This paper addresses the tension between changing teaching and teacher change in the current wave of reform in mathematics education. Based on the research I describe and discuss, I argue that reform in mathematics education fails to look at teachers from a holistic perspective, taking into consideration their ideologies, that is, their "value-rich philosophy or world view, a broad inter-locking system of ideas and beliefs (Ernest, 1991, p.111). Consequently, reformers discuss teachers' knowledge and beliefs about mathematics and its teaching and learning, neglecting to consider that these beliefs are part of a wider web of ideas which forms the teachers' ideological visions of society, of education and of their students, among other elements.

The Research

The Model of Educational Ideology for Mathematics (Ernest, 1991) served as the theoretical framework for the in-depth case studies of three elementary school teachers who were trying to "implement" the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989).[2] [3] Through intense classroom observation and five unstructured interviews with each teacher, one of the goals of the study was to understand how the primary elements of this ideological model existed and influenced the classroom practices of the teachers, shaping the way they adapted the Standards. These elements are: epistemology, philosophy of mathematics, set of moral values, theory of child, theory of society, educational aims.

Ernest (1991) says the first three primary elements of his ideological model are very abstract and "ideologies must relate them to the experience of being a person living in society" (p. 131). Therefore, he introduces the next two elements by explaining that they relate to the first three but are connected to reality in more practical ways, and that many philosophers and educators have given the child, society, or both, a central place in their theories. Finally, Ernest clarifies that educational aims, the last of his primary elements, "represents the intentional aspect of the ideology with respect to education, drawing together elements of the underlying epistemology, system of values, theory of the child and theory of society" (p. 132).

Teresa M. Walker, Betty J. Finkel, and Julie Farnsworth[4], the teachers who kindly agreed to participated in this study, did not talk about their theories of child, society or education. The elements fo their ideological visions emerged in this work through the study of their practices. After coding and analyzing the data, using the Constant Comparison Method (Glaser & Strauss, 1967; Strauss & Corbin, 1990), two themes seemed to most deeply influence what the teachers did in their mathematics teaching: their perceptions of what their *students need* from school, as well as their views of their own *roles as educators*. These themes where then related to Ernest's primary ideological elements.

These teachers did not have a clear epistemological position or a well-defined philosophy of mathematics—elements that Ernest (1991) classifies as more abstract. However, the three teachers based their teaching on their moral values. On the more practical side, the teachers had well-grounded working models for their theories of the child and of society. Although they did not

[2] In a previous paper we discribed the research project and presented initial data on one of the teachers that participated in the study (Sztajn & Lester, 1994)
[3] The Standards should be understood here as an example of reform documents. The issues discussed in this paper can shed some light in the reform discussion in general given that they do not specifically refer to this particular document.
[4] These are all pseudonyms.

explicitly talk about such theories, their working models were present in the comments they made about their students and what these children need from school and education in order to function in society. Each teacher, therefore, combined her working models of children and society to form what she called *students' needs.* Concerning educational aim, each teacher's perceptions of her *role as educator* constituted the practical representation of Ernest's last theoretical element.

Although there is no space to present the data that serve as evidence for the claims I make in this paper[5], in what follows I introduce a brief description of the teachers and their classrooms, analyzing them from an ideological perspective. Finally, considering these three cases, I discuss the idea of why reform is failing to change teachers—although it might succeed in superficially changing teaching.

The teachers and their teaching

When they were observed, Teresa, Betty and Julie worked in public schools in small midwestern American towns and all the students in their classrooms were white. Teresa's classroom had 19 students, Betty's had 24, and Julie's 25, in schools that had about 300, 1000, and 500 students enrolled, respectively. The school where Teresa teaches includes grades K through 5, and there are two other third grade rooms. Betty's class is one out of six third grade rooms in a school that also includes grades K through 5. Julie teaches in a 3-6 school where there are five fourth-grade classrooms. These three schools, however, widely varied with respect to the socioeconomic background of the children, measured by the percent of children on free or reduced lunches (40%, 21%, and 10%, respectively) and the educational credential of parents, based on the principals' estimations and the teachers' opinions.

All three teachers had heard about the Standards before the beginning of the project, and all claimed that they were trying to align their teaching to what they perceived as the new trends in mathematics education. With different frequencies and styles, the teachers in this study used manipulative materials, did some group work, tried to do problem solving and attempted to implement some alternative assessment methods in their classrooms. Except for Julie who said that, whenever she could, she had always tried to incorporate these elements into her mathematics classes, the other two teachers believed their classrooms were different because they had heard about these

[5]See Sztajn (1995) for the complete study.

"new" ideas for mathematics teaching. All three teachers, nonetheless, believed they were somehow in the process of changing the way they teach mathematics.

These three classrooms, however, looked very different and the mathematics taught to the students was definitely not the same. In Teresa's room, the most traditional of the three, children had no voice or choice and all students always worked on the same assignment at the same time. Despite some group work, Teresa's students mainly worked alone so that she could evaluate them and compare them to the "norm" she expected them to conform to. Students were not free to move around the room and all desks faced the blackboard. Teresa talked most of the time. She mainly relied on the textbook to guide her teaching and problem solving was a fun way of "resting" from drill—the real important thing to do in mathematics.

Betty's and Julie's rooms were very different from each other, but they were both less traditional than Teresa's. In these rooms, children had some options and were allowed to participate in decision-making processes. They had a few choices concerning the work they would do and their group work counted towards their assessment. The desks were either arranged in groups or in different positions everyday. Students could move around as they pleased and they talked almost as much as their teachers. Both Betty and Julie tried not to rely exclusively on textbooks for instruction, creating and adapting activities to give to their classes.

Translating the many elements of her ideological position into action in her classroom, Teresa wants to teach to form responsible citizens. According to what she says, her poor students come from disrupted lives, know very little, have learning problems, cannot behave, and their families do not care about their education. These children, Teresa concludes, *need* to learn discipline, organization, basic facts, and other skills that might help them become better integrated into society and the workplace. As a teacher, Teresa believes it is *her role* to teach her students the social values and norms their families are failing to provide them (a typical example of the educational deficit perspective).

By contrast, Betty teaches to develop thinking people. She has a wider vision of society as a place where people need to work collaboratively and be problem solvers. She believes all children can learn, independent of their socioeconomic background. For Betty, her students' main *need* is to become thinkers, capable of facing problems and working collaboratively to solve them. Therefore, *her role* is to help these "future thinkers" to become self-motivated learners who are able to gather and critically analyze information.

Finally, Julie teaches to promote the growth of happy and caring individuals. She says her students come from families with professional backgrounds where parents are willing to put a lot of effort and money into their education. Concerning both knowledge and behavior, these children come to school having already learned the "basics" at home. What these children *need*, according to Julie, is to experience long-term, sustained, and challenging efforts. Julie believes *her role* is to make school a place where her students have an enjoyable time and learn to be caring people, thereby enhancing the future of the human race on the planet.

For Teresa, for Betty, and for Julie mathematics instruction has to help them accomplish their broader educational goals. Their aims, however, vary according to their vision of society and to the socioeconomic background of the students they deal with. Teaching poor children some minimal skills to "properly" behave in society, or teaching children to cooperate in order to solve problems, or teaching children from professional background how to endure long-term challenge and be happy, Teresa, Betty and Julie have to teach very different mathematics. Therefore, despite reform ideas such as manipulative materials, group-work, problem solving and alternative assessment, Teresa students,who were mainly from low socioeconomic income families, learned to follow instructions; Betty's children came from middle-class families and learned to think; and students in Julie's classroom, who were from families in the middle- to upper-class range, learned to enjoy their learning experiences.

Discussion

Although teachers might paste-on some adjustments to teach according to what they see as the new reform ideas, it is in their ideological visions that lies the core of what they do in their classrooms. Therefore, to change mathematics education in schools, reform needs to go beyond changing teaching and toward changing teachers. The interesting point raised by presenting these three elementary teachers is the justification each one of them gives for working the way she does. It is in the teachers' reasonings that their ideological, value-rich perceptions of the world play an important role, and it is toward the understanding of these reasonings that Ernest's (1991) ideological elements are helpful.

Concerning the Standards, this research shows three teachers who knew very little about it and interpreted and implemented what they knew in very different ways. This three teachers made the document fit their own, previous perceptions of education. They assimilated the document without having to accommodate their philosophies of teaching. Thus, what these teachers learned about the Standards did not challenge their ideological visions. Quite the contrary, their ideological visions shaped the possibilities of interpreting and implementing the Standards, allowing the teachers to use a few of the teaching elements the document supports, without having to consider the more deep ideas and beliefs which the reform movement espouses.

For each teacher, factors such as the school where she teaches, the socioeconomic backgrounds of the children she works with, and her perceptions of what society expects from these children influenced what she did when "adopting" the Standards. More specifically, these teachers' perceptions of their *students' needs* and of their own *roles as educators* determined the way they adapted the mathematics reform rhetoric to fit their previously existing visions of education and of the world.

As researchers, teacher educators, reformers, and so forth, the mathematics education community needs to realize that teachers do not, and cannot, turn their ideologies off once their mathematics lessons begin. Teachers also do not leave the world outside the door once they walk into the classroom to begin their day. Members of the mathematics education community cannot continue researching, educating teachers, or attempting to reform mathematics education without considering ideological and social factors that affect mathematics teaching. Not only what one teaches and how one teaches matter; where one teaches, why one teaches, and to whom one teaches are also of the utmost importance.

These factors are ideological in nature and, unfortunately, they are not among the traditional ones mathematics educators discuss when talking about change in mathematics classrooms. To date, these factors have not been part of the debate on reform in mathematics education, most probably because they go beyond the traditional border of what we consider mathematics, its teaching and its learning to be. These issues, however, need to be addressed if we are to change teachers instead of superficially changing teaching.

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IS THE LENGTH OF THE SUM OF THREE SIDES OF A PENTAGON LONGER THAN THE SUM OF THE OTHER TWO SIDES?

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Abstract

This article reports on the responses of 289 students, K to 11th grade, to comparisons dealing with the sum of lengths of several sides of given polygons with the sum of lengths of their remaining sides. Most participants answered correctly when comparing the length of one side with the sum of lengths of the other two sides in a triangle, or the other three sides of a quadrilateral. However, the percentages of correct responses decreased when students had to compare, for instance, the length of the sum of two sides in a pentagon with the sum of the remaining three sides. In these cases, students tended to claim that the sum of more sides (three sides in a pentagon) was larger than the sum of the smaller number of the remaining sides (the other two sides of the pentagon). These results are in line with the intuitive rule: 'More of A- more of B'.

Introduction

People continually undertake comparison tasks- trying to determine whether two entities are equal or not equal with regard to a certain characteristic. While we feel that our judgment takes into account only relevant information, in many cases there are factors which unconsciously bias our judgment. For instance, when comparing two points, one the intersection of two lines and the other the intersection of six lines, students tend to include the irrelevant factor "number of lines" when performing the comparison.

It was found that these interfering factors follow certain "intuitive rules", for example, the rule "*More of A- more of B*" (e.g., Stavy & Tirosh, 1994). Hence, in many cases we hear such claims as: "the point of intersection of six lines is bigger or is heavier than the point of intersection of two lines". This appears to reflect an idea that *more* lines create *more* weight and size.

The notion crops up surprisingly often in everyday situations- *the more* you learn *the more* you know; *the more* you eat *the fatter* you get. What is even more amazing is that this same factor influences the way scientific and mathematical comparisons are conducted.

In mathematics and science, comparison tasks are used both as a means and an end in themselves. As a means they serve to investigate students' alternative conceptions of scientific and mathematical notions. When asked to compare quantities which are equal, by including an irrelevant characteristic and applying the rule of *more of A - more of B* erroneous conclusions are reached (e.g., studies on the development of the concepts of *temperature:* Strauss, Stavy & Orpaz, 1977;

actual infinity: Tsamir, & Tirosh 1992; 1994; *time:* Levin 1982; *point:* Fischbein, 1993; Tsamir, 1997; *angle*: Foxman & Ruddock, 1984; Noss, 1987 Stavy, Tirosh & Tsamir 1997; *gravity:* Gunstone & White, 1981 etc.).

Thus teachers should be aware of how this rule may interfere with students' mathematical and scientific conceptions. Mathematics and science educators' familiarity with this intuitive rule and with its influence can serve as a means for both *understanding* and *predicting* students' alternative conceptions.

The aim of the present study was to examine how the intuitive rule *More of A - more of B* affects students' performance of mathematical comparison tasks. Students were asked to examine four polygons - a triangle, a quadrangle, a pentagon, and a septagon- and to compare the sum of the lengths of several sides, of each polygon to the sum of the lengths of the remaining sides of that polygon. Our prediction assuming the intuitive rule *More of A- more of B,* was that *in all cases*, students would tend to view the sum of the lengths of *more sides* as being *longer*, even though this was true only in the cases of triangles and quadrilaterals.

Method

Subjects

Two hundred and eighty-nine students between age 5 and 17 sampled from kindergartens, regular and advanced math classes in the Israeli public school system participated in this study (Table 1). All 9th and 11th graders were from a high school for generally average achievers. The "regular" 9th and 11th graders were working on the second level, out of three levels of mathematics classes, while the "high" 9th and 11th graders were math majoring.

Table 1: Number of participants from each grade level

Class:	K	2nd	4th	6th	9th-R	9th-H	11th-R	11th-H	TOTAL
No. Students	37	36	46	34	34	36	35	31	289

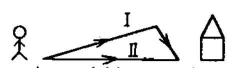
Materials

All participants were presented with the following four problems:

Questionnaire

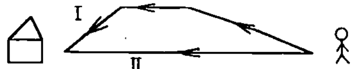
Danny goes home. The drawing in each problem represents two possible routes both on a horizontal plane. In your opinion, which of the two routes is shorter?

Problem 1



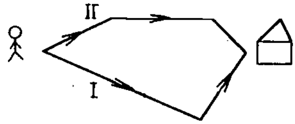
In your opinion, route I is shorter / equal / longer than route II.
Explain your answer.

Problem 2



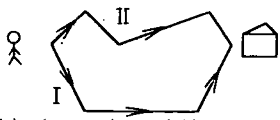
In your opinion, route I is shorter / equal / longer than route II.
Explain your answer.

Problem 3



In your opinion, route I is shorter / equal / longer than route II.
Explain your answer.

Problem 4



In your opinion, route I is shorter / equal / longer than route II.
Explain your answer.

Procedure

One of the researchers conducted 37 individual interviews in a kindergarten. Each interview lasted between 15-55 minutes. The other participants answered the same questions but in writing. The written assignment took about 15 - 30 minutes.

Results

I. The Triangle and the Quadrilateral - The Length of One Side vs. the Sun of Lengths of the Remaining Sides

As expected, almost all students from all grade levels argued in the cases of the triangular and quadrilateral drawings, that the one-sided route is shorter than either the two-sided or the three-sided routes (Table 2).

Many students offered no explanation to their judgments. When relating to both the triangular and quadrilateral drawings, quite a number of participants of all ages, who did justify their answer, correctly explained that *"a straight line is the shortest line between two given points"* [2nd grader]; or that *"In any triangle the sum of lengths of two sides is always bigger than the length of the third side"* [9th grader]. When referring to the quadrilateral, some 9th and 11th grade math majors drew a diagonal and justified their claim by twice using the above mentioned argument, while others just ambiguously claimed that *"this is a generalization of the theorem relating to triangles"*. A number of (mainly) 6th graders based their answers on measurements they performed, and others only counted the segments in each route.

Table 2: Students' responses (%) to the triangular and quadrilateral routes

The Longer Route

Triangle	K	2nd	4th	6th	9th-R	9th-H	11th-R	11th-H
*** TWO SIDED ROUTE**	**100**	**100**	**80.4**	**85.3**	**100**	**91.7**	**82.9**	**96.8**
ONE SIDE	-	-	6.5	5.9	-	5.6	8.6	3.2
EQUAL	-	-	2.2	8.8	-	-	5.7	-
NO ANSWER	-	-	10.9	-	-	2.8	2.9	-

Quadrilateral	K	2nd	4th	6th	9th-R	9th-H	11th-R	11th-H
*** 3 SIDED ROUTE**	**81.1**	**97.2**	**71.7**	**82.4**	**97.1**	**88.9**	**85.7**	**96.8**
ONE SIDE	10.8	2.8	8.7	5.9	-	5.6	2.9	3.2
EQUAL	8.1	-	6.5	11.8	2.9	2.8	8.6	-
NO ANSWER	-	-	13.0	-	-	2.8	2.9	-

* correct answer

The typical justification of the K and 2nd graders participants, and quite a frequent justification of 4th and 6th graders was that *"more lines create a longer route"*. Several other students used the same line of reasoning to claim that *"the bigger the number of corners (turning points) - the longer the route"*. The latter rationale was explicitly based on the intuitive rule: *the more - the more.*

Both formal reasoning as well as intuitive justifications almost always accompanied valid mathematical judgments. Yet, quite surprisingly, there was a small number of cases in which students incorrectly viewed **the two routes as being equal** or even considered the single side as being longer than the sum of the other sides. Most of these students gave no explanation. An unexpected aspect of the *more- more* idea was found in younger children's justification of their answer that **the two-sided route (or the three sided route) was as long as the one-sided route**. They argued that *"both routes are equally long since they have the same starting point as well as the same ending point"*. Insinuating the idea of--The same edge points implies the same length of lines connecting them. *"Same of A - same of B"*-- as a version of the intuitive rule: *"The more- the more"*.

II. The Pentagon and the Septagon - The Sum of the Lengths of three Side vs. The Sum of the Lengths of the Remaining Sides

In the cases which of drawings representing the pentagon and septagon only about 25% of the students in each grade level correctly responded that **the three sided route was shorter than the two sided route** in the pentagon-shaped drawing; and that **the three sided route was longer than the four sided route** in the septagon-shaped drawing (see, Table 3). As before, most students did not justify their responses. Yet those who did provide explanations either said that *"it seems shorter"* (usually K and 2nd graders), or performed (accurate) measurements considerations *"I have measured both routes and found this one to be longer"* (4th

to 11th graders).
Quite a number of the participants from grade 4 on, used their rulers in doing the comparisons. However, some of these students performed hasty inaccurate measurements, concluding that **the routes were equal**. Others just vaguely claimed that *"It seems that the two sided route is equal to the three sided one"* [6th grader], or *"Both start and end at the same point, therefore these must be equal routes"* [4th grader].

A response which was typical to mathematics majors was that the situation described in these problems was **inconclusive**. About 10% of 9th high-level students and about 35% of the 11th mathematics majors, analyzed the given *specific* case from an (over)generalized perspective, claiming that it was impossible to decide which sum was larger: *"One cannot determine whether two sides are larger than the other three or not. It depends on the given pentagon"* [9th grader, high level].

Table 3: Students' responses (%) to the pentagon and septagon shaped routes

The Longer Route

Pentagon	**K**	**2nd**	**4th**	**6th**	**9th-R**	**9th-H**	**11th-R**	**11th-H**
* TWO SIDED ROUTE	18.9	5.6	15.2	11.8	23.5	13.9	14.3	25.8
EQUAL	5.4	-	4.3	32.4	8.8	25.0	45.7	-
INDECISIVE	-	-	-	-	5.9	11.1	-	35.5
THREE SIDED ROUTE	75.7	94.4	67.4	52.9	52.9	47.2	31.4	**32.3**
NO ANSWER	-	-	13.0	2.9	8.8	2.8	8.6	6.5

Septagon	**K**	**2nd**	**4th**	**6th**	**9th-R**	**9th-H**	**11th-R**	**11th-H**
* THREE SIDED ROUTE	13.5	11.1	13.0	29.4	14.7	19.4	8.6	22.6
EQUAL	5.4	-	17.4	23.5	8.8	16.7	34.3	16.1
INDECISIVE	-	-	-	-	2.9	11.1	-	35.5
FOUR SIDED ROUTE	81.1	83.3	58.7	44.1	52.9	44.4	45.7	19.4
NO ANSWER	-	5.6	10.9	2.9	20.6	8.3	11.4	6.5

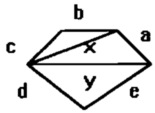
* correct answer

As predicted, a substantial number of students argued, in the case of the pentagon and septagon, respectively, that **the three-sided route was longer than the two-sided one;** and that **the four sided route was longer than the three sided one**. The most frequent justification presented at all grade levels, was that *"the <u>bigger</u> the number of sides - the <u>longer</u> the route created"* [11th-R grader], and *"the <u>bigger</u> the number of turning points - the <u>longer</u> the route"* [9th-R grader]. Both explanations were in line with the intuitive rule: *more - more*. Many participants also said: *"It seems shorter/larger"*.

A number of 4th to 11th graders from regular mathematics classes presented a combination of the rule *"More of A - more of B"* and an (over)generalization of the previous cases of the triangular and quadrilateral routes: *"All the problems are basically similar. If one understands that 'the two sides are larger than the third',* *then three sides are larger than the other two' is also true"* [6th-R grader]. Some used a more formal yet invalid phrasing: *"It is actually an extension of the theorem regarding triangles that: 'In any triangle the sum of any two sides is larger than the third side'; Accordingly 'In any pentagon the sum of three sides is larger than the sum of the other two sides'. As in algebra, adding an equal increment (1) to both sides of a given inequation (2>1), preserves the inequation (3>2)* [9th-R grader].

Mathematics majors who justified the claim that "the three sided route is longer than the two sided route" usually presented similar ideas in a more sophisticated manner. An 11th grader claimed, for instance, that: *"The pentagon is a generalized case of the triangle. One should try to prove this mathematical characteristic by using 'induction'. The theorem is true for the triangle and this should be grasped as a substitution of n=1 (one side is smaller than the sum of the others). The pentagon is actually the case of n=2 (the sum of two sides is smaller than the sum of the others); the septagon n=3 etc. Validity should be assumed for n=k and proved for n=k+1".*

Other 9th and 11th graders fabricated invalid "proof", such as: *"In order to prove that $c+b<a+d+e$, one should draw two diagonals (x and y) from one vertex. Three triangles are created and the following occurs: $d+e>y$, $y+a>x$*



hence $d+e+a>x+y$ but $c+b$ is only bigger than $x \Rightarrow d+a+e>c+b$" [9th grader].

Discussion and Conclusions

Our findings strongly support our prediction. A substantial number of students from all grade levels (including those majoring in math) tended to view all routes consisting of fewer sides to be the shorter ones.

The questionnaire essentially presented two types of problems. The first relating to a mathematically unconditional situation. Here, not only was the answer determined by the specific drawing, a triangle and a quadrilateral, but also by the generalized rule that the length of one side of a polygon is *always* shorter than the sum of lengths of its other sides. This correct answer was consistent with the intuitive rule *More sides - longer.*

Accordingly, almost all the participants who did provide an answer, correctly claimed that, in both the triangle and quadrilateral drawing, the one sided route was the shorter one. They justified their claims in a formal, general mathematical manner; in a concrete way or by visual, intuitive means. It seems as if all reasoning methods led to a single conclusion-- the correct answer. Yet, a few students contrary to all previous reasoning, still viewed the routes as being equal or viewed the shorter route as being longer. Their explanations were either irrelevant, involving false topographical conditions, or surprisingly, and in line with the intuitive rule "*Same of A - same of B*", claiming that equal routes are the result of equal end points.

The second kind of problems dealt with pentagon and septagon shaped drawings. Unlike the first set of problems, there is no mathematical rule on which to base the answers. The comparison must be conducted for the specific case presented. The drawing which accompanied each problem was meant to serve as a concrete, visual clue to the correct answer, since in both the pentagon and the septagon the route consisting of "more sides" was depicted as the shorter one. However, the visual clue was ignored - the students usually did not measure the lines but relied upon their intuitive grasp of the situation.

Naturally, by applying the intuitive rule of *the more - the more* (*the less - the less)*, in the pentagon and septagon drawings, the route consisting of fewer lines was incorrectly declared by a substantial number of students to be the shorter one. Even though this tendency to rely upon intuitive rules declined with age, it was so powerful that still almost 20% of the 11th graders majoring in math made this same error regarding the septagon, and over 30% regarding the pentagon.

To conclude, the intuitive rule, "*More of A - more of B*" seems remarkably influential in directing students' line of reasoning. It was more powerful than the provided relevant drawing, which graphically presented the correct result, and also more powerful than students' geometrical formal knowledge. By using our experience with this rule we can propose a reliable prediction of students' problem-dependent, correct as well as erroneous responses. Such an ability to foresee possible intuitive triggers and obstacles, should serve as a tool for meaningful instruction.

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INVESTIGATING CHANGE IN A PRIMARY MATHEMATICS CLASSROOM: VALUING THE STUDENTS' PERSPECTIVE

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ABSTRACT

Data referred to in this paper are from a case study of one teacher's self-initiated attempt to progressively change her approach to teaching mathematics in her year seven primary classroom. While there have been many investigations of change in mathematics education, this case study is unique because its focus is on interpreting change from the students' perspective, rather than that of the teacher. In this paper the importance of giving the students a voice in classroom research is first discussed and then demonstrated by reference to data from the study. A particular instance is described which illustrates the subtlè way in which the students' covert actions powerfully influenced their teacher's approach to change, a process which would have been obscured and misrepresented if the researcher had not had access to the students' subcultures.

INTRODUCTION

Current thinking about the nature of mathematical understanding and learning requires a methodology for classroom study which is open to alternative interpretations and which also takes account of the complexity of the teaching/learning situation (Adler 1996; Bauersfeld 1992; Bishop et al. 1996; Clarke 1996; Cobb et al. 1992; Doyle 1990). From the teachers' perspective, a mathematics lesson may be about the students developing a new mathematical idea and from the researcher's perspective it may be about observing a particular form of interaction between teacher and students. However, from each student's perspective it may be about some distinctly different activity, which may or may not have anything to do with engaging in learning the mathematical idea, or entering into any particular form of interaction with the teacher. It seems then that if the researcher wishes to investigate any part of the classroom milieu, there are dangers in limiting the viewpoint to the agenda of the teacher, or of the researcher, and ignoring the multiplicity of agendas assembled by the students (Erickson and Shultz 1992; Hoyles 1982; Kouba and McDonald 1991; Woods 1990).

Changing the way the learning of mathematics is undertaken in schools has been of central interest to mathematics educators for a very long time, and the difficulties of implementing curriculum change at the classroom level have been well documented (Delaney 1996; Ellerton and Clements 1988; Levenberg and Sfard 1996; Swinson 1995; Weissglass 1992). For the most part, however, the perspective taken in such studies is that of the teacher, not the students (Erickson and Shultz 1992). For example, in his research, Clarke (1995) drew attention to the "struggles, challenges, and highlights" (p.183) teachers encountered as they attempted to use innovative mathematics materials in their classrooms and listed twelve factors which he suggested influenced their changing roles — yet none of factors listed related to students. Was that because the students exerted no influence, or was it because their voices were excluded from the study?

Certainly there are studies outside of the mathematics research community which do suggest that students' responses need to be considered when trying to implement or understand classroom change processes (Hunt 1990, pp. 45-66). McQuillan and

Muncey (1994) describe a large scale United States school reform project and reports that students,

> *...have not been passive recipients of Coalition reform efforts at their schools. ... [There were] numerous instances of students resistance (both active and passive). ... Some were sceptical, distrustful, or outright rejecting of efforts to empower them ... [and sometimes] were quite cynical about why a new opportunity was provided (p. 273).*

In Britain, Woods (1990) describes how the teachers in his studies provided both explicit and implicit information about their teaching style and expectations early in the year. Students combined this information with their own preconceptions about teachers' roles and responsibilities in order to construct "rules" which the teachers tacitly agreed to follow.

> *The emphasis varies, of course, from classroom to classroom, but in all such instances the agreement is implicit, and the teacher's behaviour is constrained whether this is realized or not (p. 5).*

The methods by which students constrain a teacher's behaviour are many, from outright non-compliance to more subtle subversive techniques. While the overt balance of power in classrooms clearly favours the teachers and the school system, and most students are careful to avoid any overt challenge to this, Wood provides evidence of covert power structures which students utilise to at least partially shape their own learning environment.

In her research in Australian schools, Zevenbergen (1995) has revealed student behaviour in mathematics lessons which parallels Woods' findings. She found that while students learned, and appeared to conform to, the implicit cultural values of the mathematics classroom, they had nevertheless "developed many elaborate practices for appearing to comply ... but in actuality, resisting the values which are integral to the classroom culture" (p. 562). Zevenbergen also found that teachers in her study had no idea that students had developed such complex strategies for subverting many of the mathematical teaching goals, while appearing to overtly comply with the classroom behavioural norms and expectations.

These studies suggest that changes in teaching approach, as well as research into such change, need to take more account of the students' subcultures, which may incorporate particular views about mathematics and schooling, the teacher's role and purpose, and their own roles and purposes. Hence the case study, from which this paper arose, focuses specifically on the students' viewpoints in order to explore the ways in which their actions and reactions may impinge on their teacher's efforts to change their mathematics learning environment. The purpose of this paper is to demonstrate, using just two anecdotes from the case study, how the inclusion of students' voices provides important insights into classroom learning situations, and how their exclusion could result in research findings which are essentially spurious.

THE CONTEXT

The case study involved a year seven class (the final year of primary education in Western Australia) in a government school. The teacher volunteered her class of eleven to twelve year old students for three consecutive school terms of research. While highly competent in the language area, she willingly admitted to a lack of confidence in teaching mathematics, a profile which fits many primary school teachers in Australia. Nevertheless, she claimed to have moved away from a "boring workbook" emphasis and improved her skills, but was still dissatisfied:

In language I've a very clear view of where I'm going, I know the purpose for what I'm doing and I can think about what is the best way of doing it, but with maths it's not like that. I know what I was doing before was no good, and I'm doing things differently now, but I still don't really know what is the best way of teaching it, so I just try to do a lot of different types of maths, using different resources, filling in the gaps, but I don't have any real sense of where I'm going, I haven't got a 'big picture'. (26/4/96)

The teacher intended to begin using in maths lessons the kinds of collaborative group learning processes she found to be successful in her language teaching. For the first two terms of the cases study, the researcher did not directly have input into the content or approach taken by the teacher. However, at the teacher's request, the researcher provided the teacher with advice and resources to support a more substantial change of approach during the final term, namely, a series of chance and data investigations based on Lovitt and Lowe (1993). (The anecdotes included in this paper relate to that period.) Data collection methods included lesson observations, audio-taped group and individual interviews with teacher and students, student journals, questionnaires and surveys. Students were informed of the researcher's obligation to maintain confidentiality, both in talking to their teacher and in reporting to the research community

The case study methodology involved elements of the 'bricoleurs' approach described by Reid (1996) in his paper presented at PME20 on enactivist research, in which it was recognised that the researcher also becomes an integral and changing part of the complex system under investigation:

Just as an individual's structure changes in changing the context, so our expectations change even as we observe, interview, and analyze according to our expectations (p.208).

Mousley (1996) also draws attention to the integral place of the researcher in terms of his or her hermeneutic situation, ie the particular exegeses brought to an event which "evolve as we re-develop our social contexts" (p. 36). Thus, the researcher's evolving interpretations are not seen as aiming towards the discovery of some definitive 'truth' in the data. Rather, they are attempts to represent the phenomena under investigation in ways which enable others to enter into the experiential space of the participants (in this case students, teacher and researcher) in order to more authentically reflect upon the participants' circumstances, and so better inform future actions. The interpretations presented in this paper are offered in that spirit.

THE STUDENTS' PERSPECTIVE

During the interviews, students were given the opportunity to voice their opinions about a wide range of issues associated with their mathematics learning environment. For example, the following excerpt comes from a discussion session with four students towards the end of the study. They were asked if their teacher's decisions about mathematics lessons were ever influenced by students' actions:

Kelly: *Maybe sometimes, like Mrs B, she likes to make our maths really fun for us.*

Angela: *Well, she tries!*

Kelly: *Yes, but at least she tries for us, so I think maybe that a little bit she probably gets of it.*

Researcher: *What makes her decide to make it fun, do you think?*

Kelly: *Because she's probably learnt over the years of her teaching that kids don't like it when you just give them a page of maths, and like they groan and all that.*

Therese: *But no teacher is perfect, and they're never going to learn everything that kids like.*

Megan: *And I think you coming also has sort of changed the way she thinks a bit.*

Angela: *And she probably feels threatened too, because I mean, we talk behind her back anyway, and we've got names for her and that, that I'm not going to say on tape, and but, anyway I think she feels even more threatened by another adult.*

Megan: *I think she realises that we do talk about her and her ways of maths in these sessions.*

While the question related to the impact of students' actions, the children went on to consider other influences on their teacher and jointly constructed interpretations of her motivation and feelings. Note how the students have taken account of the researcher and attributed perceived changes in their teacher's thinking to the researcher's presence, and also theorised about their teacher's affective reactions to the research process. However, the mathematics itself seems not to be the issue here — maths is maths, you could present it in a boring way, or a fun way, the choice was the teacher's, and the implication is that teachers ought to provide, but couldn't be expected to know, what students like. These students recognised that, unlike some other teachers, theirs at least tried to make it 'fun'.

These responses were typical of the kinds of insights the students offered during the study, and it was surprising to the researcher that, at only eleven and twelve years of age, children were able to reflect with such clarity upon their teacher's thought processes. From a methodological viewpoint it is most important to note that this depth of reflection only became apparent during the later stages of the study. Clearly it took a long time for the student/researcher relationship to develop to the stage where students were willing to provide access to their personal world. What is particularly important to report is that the earlier interviews *appeared* to have provided this access at the time. It was not until after the later interviews that the researcher realised the students were initially providing only limited and tightly controlled entry into their subculture.

In planning the study, the researcher had anticipated that aspects of the classroom culture other than those directly related to mathematics lessons would form an important part of the study. However, the researcher had not expected that the students would continually redirect the discussions towards such a wide range of personally relevant topics. Gender issues, power relationships between teachers and students and among students, subversive classroom behaviours, comparative teaching styles and parental influences were all discussed by the students at various times in the later part of the study, and the relevance of these issues to the students' mathematics learning has yet to explored. Nevertheless, the researcher has been challenged to extend the boundaries of her research paradigm to encompass a much broader than anticipated student agenda.

A SNAPSHOT OF STUDENT ACTION AND REACTION

A chance and data investigation, undertaken during the final term of the case study, provides a simple illustration of the subtle way students' subversive actions (of the kind reported in Zevenbergen 1995) may satisfied their own purposes, yet also influence their teacher's subsequent action, without either being consciously aware of the nature of that interactive process.

Students had begun an investigation into whether the different colours of small chocolate sweets called 'Smarties' and 'M & Ms' (trade names) were produced in equal or unequal proportions. During the first lesson, students in pairs were given two or three small boxes of Smarties and asked to consider the likelihood of any box containing equal numbers of sweets of each colour. The initial part of the lesson progressed smoothly. Students were clearly interested, they put forward a variety of opinions and questions of their own, then proceeded to count and record the contents of their boxes in order to compare the frequencies for each colour. The teacher indicated to the researcher that she was very impressed with their response.

After each pair recorded their data, the next step was to collate the full class set. Rather than choosing to quickly organise this herself (so the data could be compared before the close of the lesson) the teacher asked the students to each collect data for three of the colours from all the other students, having first demonstrated a simple plot for one colour on the blackboard. At this point, what could be described as a 'divergence of purpose' between teacher and students began to occur.

Teacher: *You have to go and collect this data from each group or each pair. Yes, Aaron?*

Aaron: *How are we going to do that if everyone's asking each other?*

In keeping with the earlier mode of working where student opinions were valued, it seemed reasonable that Aaron should draw attention to what, perhaps, should have been seen as a possible area of difficulty. The teacher, however, did not see this.

Teacher: *You're just going to be in a controlled manner and you're going to just go up and ask someone. That actually is not really the issue at all, and if there are fifteen people talking to you, Aaron, obviously then others are not using their common sense. I actually don't want to get into that, that's not maths.*

Aaron: *I was going to say, we could maybe take, if you got it, put each one on pieces of paper, you could just put them around the classroom and people could go to them.*

Still under the apparent impression that there was room to negotiate, Aaron had offered an alternative. Incidentally, note the teacher's statement, that's not maths!

Teacher: *Well, I'm happy for you to just go to each group and speak with each other, this is quite an acceptable way of doing it, but I need for you to have three little diagrams that indicate how frequent these colours are occurring in each person's box.*

Here the teacher did not reject the offer explicitly, but repeated her vision of how the data collating should proceed, then cut further conversation by specifying the expected end result of the activity. This signalled a noticeable change in the students' responses, compared to those in the first part of the lesson. Procedural questions followed, to which the teacher responded with authority.

Sandra: *Is that rough or good? (Meaning 'rough draft' or 'good copy'.)*

Teacher: *Oh, rough at this stage.*

Troy: *Can we use textas*

Teacher: *No, that is exactly how it must be set out (indicating the blackboard), so I would suggest perhaps to take down this one quickly, just to get an idea of what your other colours are going to look like, then move off and start collecting the data. Off you go, please.* *(14/10/96)*

Finally, the teacher gave very explicit instructions to direct the students' actions.

Following this directive, students began to move around the room in an orderly way, happily talking, writing and apparently completing the task, with a high, but clearly acceptable noise level. The teacher remarked to the researcher that she was pleased to see evidence of this ability to work cooperatively and sensibly and suggested that it would not have occurred earlier in the year; it was something she had consciously built up with the students. She reasoned that Aaron's question was probably the outcome of his absence on an overseas trip for most of third term. She felt he had not had sufficient opportunity to practise the collaborative learning skills she had focused upon during that time, and pointed out that the rest of the class had quickly got themselves organised for the activity. The teacher was, however, somewhat perplexed that after ten minutes, none had completed the task (which she had viewed as very simple) and she expressed concern that it would have to be continued into the next lesson.

In her reflections following the lesson, she said she felt sure the students enjoyed it, but, in view of the fact that none had completed the data collation by the end of the lesson, she thought perhaps they'd not understood the mathematical idea in the task.

It was not until students were interviewed that some insight into those ten minutes emerged. Students were asked if there were any changes the teacher could have made to the lesson. Several groups referred to the data collation as the point where they either lost interest or felt it was too difficult logistically to complete.

Aaron: *Probably the only really boring thing was the Smarties one when you had to go around the whole class and just ask everyone. It was, you know, noisy and rowdy, and that.*

Carl *Yeah, that was probably the worst part of it.* *(15/10/96)*

A group of girls had initially attempted to collect the data but found they could not keep track of the pairs, and (along with many others in the class) resorted to subversive action. They feigned compliance to the teacher's instruction by overtly 'working' cooperatively and sensibly, but covertly, they falsified their data and exploited the social opportunities made available by the situation.

Karen *Yeah. In the end we didn't, we just wrote down any old thing and just did nothing really, just talked about, like, other stuff and that, when she wasn't around.*

Researcher: *Not about the Smarties?*

Sandra: *No, anyway, everyone was doing that, faking it, not just us.* *(15/10/96)*

It is important to reiterate that neither the teacher nor the researcher had realised at any time during those ten minutes that most of the students had actually disengaged from the teacher's purpose and begun to follow their own interests. Nevertheless, even though the subversive actions themselves went unnoticed, the teacher was influenced by the outcome of those actions. The teacher modified the next lesson according to her interpretation of the students' failure to complete the task, which she felt was related to aspects of the mathematics, rather than any procedural difficulties. She had decided they needed more structure to the investigation, in order to better focus on the mathematics, so constructed an instruction sheet for the students. Her introductory comments revealed her change of approach:

Teacher: *OK as you know yesterday we started a little bit of an investigation and we had a little bit of fun with a couple of boxes of Smarties, well the fun will continue, but there is a purpose for all of this and you'll see by looking at the sheet that we are trying to achieve something, so just follow as I read the first part. The actual task is split up into five parts and we are going to work our way through those five parts, with the culmination of this* *particular aspect by Monday. So it really is a time for really focusing, listening and getting on with it, all right?* *(15/10/96)*

The teacher proceeded to modify the initial intention of the investigation and, although the students still saw the sequence of work that followed as being distinctly different than previous mathematics lessons, the initial momentum had been lost. For the most part, the divergence of purpose between the teacher and students widened during the following lessons. In the weeks that followed, the students expressed concern that they were learning little or no mathematics, and many said the activities were becoming boring and repetitive. Nonetheless, they found ways to convey enjoyment of the activities to the teacher, while confessing to the researcher that their enjoyment was in having the freedom to pursue their own social interests in their pairs and groups, while feigning attention to the tasks. Throughout this period the teacher struggled to adjust her methods in order to better develop the mathematics. The students, though, had effectively disengaged from this purpose and continued to operate in ways which were difficult for the teacher to interpret.

CONCLUDING DISCUSSION

Undoubtably the establishment of a sound theoretical framework for such research greatly assists the development of understanding (Clarke 1996). Contructivist theories of learning (Bauersfeld 1992; von Glasersfeld 1991) have powerfully influenced the way research in mathematics classrooms have been conceptualised in recent years, requiring as it has a re-focusing on and valuing of the child's way of doing and thinking mathematics, as well as the social influences on this within the classroom culture (Cobb et al. 1992).

However, it seems that constructivism has become a limiting theory for some types of classroom investigation (Confrey 1995a). Although the child's thinking processes may be seen as central in constructivist research, usually what is of interest is only those thinking processes which relate to the learning of the particular mathematics under investigation, ie the focus is on the agenda of teacher and researcher. The assumption seems always to be that the students are willing and active participants in that agenda. But, as Zevenbergen (1995) and Woods (1990) have suggested and this paper illustrates, it is often the case that students disengage themselves from their teacher's purposes and proceed to follow their own diverging goals. The remarkable feature of this is that the process can be invisible to teachers and researchers alike. If these subtle complexities are to be exposed and understood however, new theoretical frameworks which are able to account for the students' alternative agendas are needed.

Enactivism, as discussed by Davis (1995) and Reid (1996), seems to provide a promising theoretical starting point. There is insufficient space here to describe the theory, but essentially it focuses on the complex and fluid interrelationships between organisms and their contexts, and does not privilege any one perspective over another. Reid (1996) also points out that, unlike 'theories *of*' which implicitly claim to represent models of some existing reality, enactivism is a 'theory *for*', ie it is a theory developed *for a purpose*, and its "usefulness in terms of that purpose ... determines ... [its] value" (p. 208). For these reasons, it appears to be a theoretical approach particularly appropriate for the purpose of incorporating students' voices into the growing body of educational research into the learning of mathematics.

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STUDYING CHILDREN'S ARGUMENTATION BY INCORPORATING DIFFERENT REPRESENTATIONAL MEDIA

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In this paper, we investigate children's argumentation while assigning meaning in different learning situations. The activities used in the study were initiated by the haptic exploration of 3-D objects and integrated verbal, written, and 3-D forms of representation in the field of geometry. The initial results suggest that argumentation is an ongoing process of developing intuitions and revising interpretations.

In the seventeenth century, Vico identified language as the human institution that makes possible the formation of societies. Language, according to his view, was not simply a tool for human communication but a constitutive element of human reality. The recent "linguistic turn" in the human sciences has led to the reconsideration of Vico's position and to a broader questioning of the distinction between language and reality as well as between mental and physical activity, consciousness and the material world (Foucault, 1993; Bakhtin, 1981, 1984; Williams, 1977). As a number of recent commentators have pointed out (Rotman, 1993; Walkerdine, 1988), the resistance to thinking of mathematics as bound up with other linguistic practices might be attributed to both disciplinary and social prejudgments.

With data from a classroom experiment, this paper argues that the development of geometrical thinking can be better evaluated and enhanced if the place of linguistic representations and metaphors in mathematical education is studied. We concentrate our attention on the potential value that different modes of communicating mathematical understanding might hold for children. Specifically, we examine children's verbal and written descriptions arising during the haptic exploration of three-dimensional objects and the construction of physical models based on these written descriptions.

In teaching geometry, vision is frequently considered to be dominant over other senses. A number of studies have indicated the limitations of this prioritization (Clements and Battista, 1992; Fischbein, 1993; Hershkowitz, 1989; Vinner, 1983). Overcoming these limitations may be achieved through the development of ways of thinking to complement perception (Fischbein, 1987, 1993; Michotte, 1991; Parzysz, 1988; Wittgenstein, 1967). This 'fusion' of conceptual qualities and spatial characteristics has been attempted in various ways. For instance, haptic exploration of geometric shapes, by imposing an interruption to visual immediacy, joins visualization with action (Piaget and Inhelder, 1956; Triadafillidis, 1995). Constructing three dimensional models of objects is another approach which requires a comprehensive mental representation of the object along with the analysis of the single components (Bishop, 1988; Marrioti, 1989; Potari and Spiliotopoulou, 1992).

Talking to oneself and others is also crucial in achieving control over geometric images (Pimm, 1995). Verbal representations may relate to instrumental or relational understanding (Byers and Herscovics, 1977; Skemp, 1979) or to an intuitive comprehension of a situation. Writing about mathematical concepts demands commitment both from the writer's side in order to communicate his/her understandings and from the reader's in order to assign meaning to other pupils' mathematical records (Pimm, 1987; Rotman, 1994). Writing then might be proposed as a means to heighten awareness of thought processes and conceptual relationships, thus facilitating reflection and ownership of mathematical knowledge (Connolly, 1989; Shepard, 1993).

The importance of integrating different media has been acknowledged in mathematics learning. This has been especially exemplified by the use of multiple representational genres in computer environments (Kaput, 1992; Schwartz and Yerushalmy, 1995). Other researchers have studied pupils' mathematical performance and strategies when they integrate visual and written forms of representations (Ben-Haim, Lappan and Houang, 1985; 1989; Gaulin, 1985). In the present study, we combine written and verbal forms of representations in learning situations that include haptic exploration and the construction of models of three dimensional objects. We approach this aim by exploring children's 'argumentation' while trying to produce and interpret texts carrying mathematical meaning.

Methodology

The study is a classroom teaching experiment which took place in a primary school in Patras, Greece. It falls within the ethnographic research tradition (Wolcott, 1988) as it attempts to address the classroom environment in which children worked on the activities. We consider not only the ways in which pupils themselves assigned meaning to the activities, but also the ways in which these understandings and intuitions were communicated among pupils, in group- and in whole-classroom arrangements, and between pupils and the teacher. We encouraged teachers to be actively involved in the study, introducing the activities to their class and assisting in the organization of the work throughout all the phases. This was not part of an attempt to situate ourselves as external observers. Leaving room for teachers' initiatives provided an opportunity for us to investigate the meaning that they themselves assigned to the activities.

We argue that the construction of robust geometric concepts may be enabled by learning environments that encourage assigning meaning to, and conveying meaning through, various forms of textual representation. Texts designate any configuration of signs that is coherent and comprehensible to a group of users, irrespective of the form these signs may have (Hanks, 1989). Texts then can be written, verbal, or material constructions (a painting, a sculpture, a photograph, a written composition, a poem read aloud), according to the medium chosen by the user in order to communicate his/her emotions, ideas, perceptions, intentions. Assigning meaning to a text requires the active engagement of the interpreter. Equally important is the involvement of the person who creates the text, in achieving a coherent representation of his/her ideas, intuitions, or understandings according to the conventions determined by the medium chosen for communication and the context in which communication takes place (Hanks, 1989; Ingarden, 1973).

Two classes, a fourth grade of 20 pupils (8-9 years of age) and a fifth grade of 22 pupils (9-10 years of age), participated in the experiment. The classes were informally visited for two weeks before the main study to get to know the classroom environment and familiarize the teachers with the materials we planned to use. The children worked mainly in groups of four, which consisted of two pairs that worked separately to produce the textual artifacts for each activity. All sessions were video-taped and the discussion in all groups was tape-recorded. In addition to the authors, three postgraduate students from the University of Patras participated as group observers.

The study consisted of four phases each lasting for ninety minutes. To prevent pupils from seeing the objects while exploring them, 'feely boxes' were used (see Triadafillidis 1995 for a description). Phase one acted as an introduction to the 'rules' of the game. An object was drawn randomly from a bag full of geometrical solids made out of cardboard and other objects from everyday contexts. A volunteer was assigned with the task of haptically exploring the object and answering to various questions asked by the rest of the class about its features. The rest of the class could see the explored object. In a variation of the game, the object was not visible to the class. Pupils then had to ask the volunteer questions in order to guess the type of object that was in the feely box.

In the second phase, each pair within a group had to explore haptically a plastic vinegar bottle or a glass bottle of soda and produce a written description of it. The objects consisted of cylindrical and conical surfaces. The written reports were exchanged and used by the other pair of the group to construct the object out of cardboard. Therefore, we cautioned pupils to include in their descriptions as many features of the object as possible. Disclosing the name of the object, for those who had recognized it, was not an appropriate clue. In the third phase, pupils worked in the same manner, only on different objects made out of multilink cubes in different arrangements. These constructions formed cubes made of three layers but with some pieces missing. In this phase, the children were given multilink cubes to build the object described in the written report. At the end, in both phases, the initial shapes were shown to the children. Each pair then had to evaluate their own constructions and comment verbally and in writing on the other pair's written description. In the fourth phase, we discussed with the whole class two written descriptions, one for the vinegar bottle and one for a cubic arrangement, employing clues that pupils had used in the two preceding phases. This encouraged children to reflect on their own work and voice personal opinions concerning their choices.

Analysis

The whole process outlined in the methodology section aimed to turn children's attention in the communication of their mathematical understanding. The written reports served as a way for the children to reflect on their verbal communication and become conscious of actions performed during the experiment. By requesting that the students construct objects only using information given in the written reports, we emphasized the importance of producing a coherent, concise and detailed description. Considering the objects used in the study as symbols (Pimm, 1995), we would characterize the haptic exploration by the pupils as a process of signification in which meanings emerged through the use of formal, mathematical, or informal, metaphorical, signs. In our analysis, we attempted to study these meanings through children's argumentation as developed in the different phases. As defined by Krummheuer (1995: p. 229) argumentation can be considered "as a social phenomenon, when cooperating individuals [try] to adjust their intentions and interpretations by verbally presenting the rationale of their actions".

In the present study, we encountered several different types of argumentation. In the first and fourth phases, argumentation occurred in the setting of the whole class discussion. In the first phase, this was expressed through children's *questioning* while in the fourth phase through children's *reflections* and *justifications* concerning their work. In the other two phases when they worked in pairs or in groups, children's activities were more goal-directed and their argumentation was marked by the *negotiation* of opinions within pairs and by the *defense* of these positions in groups especially during the construction and evaluation process. The result of these various types of argumentation was expressed in verbal and written forms and in the constructed physical models.

For the purposes of this paper, we will demonstrate the types of argumentation used in the classroom discussions and in the work of one group from fifth grade. We will proceed with a description of aspects of children's geometrical thinking about solids evident when in their production and interpretation of texts.

Initial whole class discussion:

In the children's questions during the familiarization phase, we identified a number of "good reasons" (Alro and Skovsmose, 1996) they developed in their classroom interaction. In the fifth grade, children initially asked questions about the physical characteristics of the objects and their possible function: "What is it made of?", "What is its color?", "What is it used for?" "What does it look like?" are some examples. These were followed by questions concerning geometrical aspects such as shape and size: "How many surfaces does it have?", "How many angles?", "How many equal parts does it have?", "How big is it?". This transition from contextual to mathematical questions was encouraged by the teacher's intervention at a stage where the children appeared to have exhausted their repertoire of questions. In the fourth grade, there was no such transition point as these two types of questions were frequently alternated. A small number of children in both classes utilized their personal everyday experience to describe the objects : "It is like a gift box", "It looks like a glass object which has a blue liquid inside". One reason that the children did not use everyday metaphors might be that they knew that they worked in a school, or particularly a mathematical, environment.

The questioning process seemed to be greatly influenced by the way that the teachers themselves interpreted the activity. In encouraging the students to produce correct and swift responses, the fifth grade teacher demonstrated a traditional attitude. When he intervened, he said things such as: "We'll see in the end in how many of your questions he managed to answer correctly", "Just say a number (angles of a tetrahedron), any, to check it later", "Allow him two minutes to reply". This response to children's questions and answers demonstrated a rather superficial consideration of the activity. On the contrary, the teacher of the fourth grade adopted a less directed teaching approach which allowed children to express their personal intuitions and strategies. For example, a child who haptically explored the number of surfaces of a metal mould for baking cakes, which had the shape of a cone's frustum without bases, replied that it had two surfaces. From the discussion that followed, it appeared that the children who agreed with this position had only considered the physical characteristics of this object. The teacher furthered the pupils' thinking by encouraging them to construct this shape with a piece of A4 paper in order also to consider its geometrical aspects.

Working in groups:

The second and third phases were characterized by the revision of arguments. This was partly due to the fact that different representational media were used. First, the children had to negotiate their opinions of haptic exploration of the objects to their partners. In one pair, the written report that was produced was a collection of individual opinions. On the other hand, the written report of the other pair reflected the children's agreement which was the result of verifying verbally and/or haptically their opinions. In one pair the written report emphasized the physical characteristics of the objects while in the other more emphasis was placed on geometrical aspects, especially of the cubic arrangement. Points that reflect their geometrical thinking concerned the size of the objects, the relation between parts and whole, and the way that the object could be constructed. For example, children used qualitative ways of appreciating the size of the objects. This started with a rough estimation such as: "small", "medium" or "large", combined with a comparison the a familiar equivalent unit: "It has the same size as a washing-powder container". Finally, the feely-box itself, was used as a measure for comparison: "It is much smaller than the feely-box". Both written reports appeared as a transcription of pairs' verbal interactions and they did not give a coherent description of the object.

The processes of interpreting the written text, building the object and evaluating the report were all interrelated. Children's actions were shaped by, and reshaped each of these processes. In particular, children had to agree on the information on which they would base their construction. By continually evaluating the results of their actions they were assigning new meaning to the written text. The criteria for judging the other pair's report reflected the difficulties they had in building the object but also revealed a tendency to look for information that they themselves had included in their reports. The comments of one pair were of the "right-wrong" kind, consonant with teacher's expectations, while the other pair's comments were more specific and explanatory. The comparison of their constructions with the initial objects provided an opportunity for reflection. This was at first expressed as a defense of their decisions which later on developed into a recognition of some of the inadequacies of their description. By studying the solids which the children produced, we identified differences arising from the nature of the initial objects. For example, the construction of the two bottles passed from two- to three-dimensions by the use of nets for each surface of the object, while the cubic arrangements required working in 3-D as they were build with multilink cubes.

Final whole class discussion:

In this phase, the children were more critical of their judgments. They tended to seek for a coherence in the written text which we asked form them to evaluate. They often singled out redundant statements: "Sir...sir a cylinder always has curves.... I mean if we relate it with the previous (statements)". The fifth grade children rejected statements that described the physical characteristics of the objects and looked for more formal, mathematical expressions. The fourth grade children, on the other hand, developed geometrical reasons based on the metaphorical aspects of the descriptions. For example, the statement "It has a whole in the middle, like a skylight (a cubic arrangement)" led to a concretization of an intuitive appreciation of the shape which was "a cube from which we have taken out a few small cubes".

Concluding Remarks

Children's argumentation was an *ongoing process* with different expressions in each phase. These were shaped by the classroom climate that the teacher promoted, by the forms of representations and the nature of the activities. It appears from the study that children attributed a complexity of meanings to the objects and their properties. This complexity was demonstrated by children's tendency to consider not only geometrical elements of physical models normally used in the mathematics classrooms but also their physical or functional characteristics. The study also revealed children's personal intuitions about the properties of three-dimensional objects and how these intuitions developed during the experiment. Haptic exploration seemed to encourage the formation of children's geometrical understanding and intuitions thus complementing perception. Natural speech, writing, signification, sense, meaning, interpretation, and different textual representations, were all considered as aspects of children's argumentation. They may all be welcomed as linguistic practices that could enhance our understanding of the development of geometrical thinking. By further analyzing our data we hope to extend our research to the role that age and children's metaphors may play in the development of children's geometrical thinking and argumentation.

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PERCEPTIONS OF UNFAMILIAR RANDOM GENERATORS - LINKS BETWEEN RESEARCH AND TEACHING

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This paper reports some findings from two independent studies, one in Northern Ireland, the other in South Australia, into young children's understanding of the behaviour of unfamiliar Random Generators (RGs). Our findings indicate that probability understanding is often influenced by the physical properties or appearance of RGs. Sometimes it is their physical arrangement in a container that influences responses. Teaching of the topic depends on teachers being aware of these misconceptions are before planning for teaching.

FOCUS OF THIS PAPER

The introduction of probability as a mathematics topic in primary classes has been encouraged by new curriculum documents in many countries in the Western world (NCTM, 1989; DES, 1991; AEC, 1991; NZ. Ministry of Education, 1992), and this has meant that for the first time children's understanding of chance events is seen as part of the whole mathematics curriculum. The writers of the Australian *National Statement* :

> ... recognise that misconceptions about chance processes are widespread, and that many become established while children are still quite young and are then difficult to overcome. (AEC, 1991, p.163)

However, this movement towards the inclusion of chance events has meant that

> many ... teachers find that they are not adequately prepared by their own education to teach these topics. As well, mathematics educators ... have not ventured far in studying how children can best handle the topics in these new areas. (Watson, 1995, pp. 120-121)

The majority of curriculum documents lack reference to children's learning in the area of probability and generally consist of 'outcome' statements, which for many teachers provide one of the few resources for planning teaching programmes, e.g.:

> Analyse simple experiments (e.g. those involving single coins, dice and simple spinners); make a systematic list of possible outcomes and assign simple numerical probabilities based on reasoning about symmetry.(AEC, 1991, p. 170)

This activity is for Years 5-7 students but there is no indication of the complexity of this task nor the level of understanding of the behaviour of RGs necessary for a child to be able to tackle such problems in a systematic and developmental way.

In Australia one other support document for teachers, published by the Curriculum Corporation is *Work Samples* which outlines lesson plans, and suggestions for each teaching strand. The primary Chance & Data segment focuses only on sorting activities and some games based on dice, spinners and coloured blocks. No lesson plan suggests comparisons of outcomes of these games or comparisons of related RGs.

Yet during Ritson's study some children aged 10-11 years were specifically taught, in preparation for an examination, that there is an equal chance of getting any number on a die. But shortly after these examinations Ritson questioned these children about the outcome of rolling one die. About one third of them had reverted to their original perceptions about dice.

This is a very important finding which is supported by (Bramald,1994, p.85) who made the observation that;

> ... one of the root difficulties associated with probability concepts appears to be the lack of transferability of pupils' curriculum based knowledge and understanding. Could this be that this is caused by the urge to get children to work too quickly with estimates of probabilities which assume an underlying symmetry of outcomes.

Ritson's is one example of the absence of the few linkS that are being developed between research and practice. Green (1982) and Fischbein (1975) both consider that even with a carefully structured teaching programme, the understanding of probability is often poorly understood, and the view that structure holds the key to understanding may not be valid.

FOCUS OF ANALYSIS

We have both found that many children as old as twelve years, when asked, for example, 'when an ordinary 6 sided die is thrown which number or numbers is hardest to throw, or are they all the same?' will respond confidently, 'all numbers have the same chance'. The same children, however, when shown a 12 sided die and asked the same question, frequently specify that one number is easier to throw than others. Clearly knowledge of one RG does not transfer simply to another.

So part of our research has been to investigate young children's interpretations of the behaviour of RGs in order to understand the salient focus for them. Ritson's study was devised partly to investigate the changes in student perceptions of different RGs over a period of time; while Truran's study has investigated students' perceptions of five sets of common, related RGs with different embodiments and has produced and trailed in-service material for primary teachers.

We shall show how our findings have made clear three important factors about children's interpretations and perceptions of RGs. The physical properties of RGs, and the physical arrangement of RGs both have significant influence on children's perceptions. Preferences about the objects to which the questions referred also influenced the children's responses. We believe that the first two points present new findings, and that all points must be considered in future curriculum planning and in-service teaching.

We suggest that there should be more informal discussion about the features of RGs used in the classroom, and that the children are exposed to experience with different and unusual RGs as a matter of course.

SUMMARY OF SOME PREVIOUS RESEARCH

What is seen as the classic investigation into children's understanding of probability is that of Piaget and Inhelder (1951) who analysed children's thinking about probability into stages of development, which they argued followed a developmental sequence, as a continuum, culminating in a level that they describe as formal understanding. Green (1982), in a survey of 3000 secondary school pupils aged 11-16, showed how their development indicated a hierarchy that was modelled on Piaget's. But Green also concluded that most English students finish secondary school without achieving the level of formal understanding.

Kahneman, Slovic and Tversky (1982) believed that even adults reasoning is intuitive and characterised by what they described as heuristics; intuitions, the most common of which are which recency and representativeness. Fischbein (1975) showed that some intuitions and biases in young children's thinking are important in helping their pre-formal probabilistic thinking. He believed that these biases, which are the result of personal experience, can be overcome by an appropriate teaching programme

Lecoutre (1992), on the other hand, suggested that children used what she called an 'equiprobability bias'. She claims that children view as chance 'naturally' equiprobable; and that this bias is as important in understanding their thinking as those biases described by Kahneman et al.

Random Generators play a fundamental role in the study of probability and it is often taken for granted that children see these devices as we do. However, Amir and Williams (1993) claimed that the way children see RGs depends on their culture, and their previous experiences and perception of the physical properties of the RG and the way that one handles these devices. Zaleska (1974) suggests that the perception of independence of RGs is influenced by the concreteness of the generator and their ability to operate it themselves ... [and] by their familiarity with the generator. Amir and Williams illustrated this claim in their own study by describing perceptions of coins by children whom they interviewed: some saw coins as linked with cheating; others with experience and although they could not explain why, some suggested that some people are luckier than others.

In our studies we have heard many of the explanations similar to those discussed above. Our concern is that these be recognised in some formal way, leading to their eventual acknowledgment by curriculum planners and teachers, so that appropriate modifications of curriculum documents can be made. We believe that the things that children say to us in interviews are not just 'engaging and cute', but real indicators of the children coming to an understanding of probability thinking as well as indicators for curriculum planning.

DETAILS OF BOTH STUDIES

The studies described below involve children from a range of schools, ages, and socio-economic background. While different approaches and random generators were used, both studies are similar in structure and indicate similarities in the children's responses as well as evidence of returns to basic beliefs when confronted with different or unusual RGs.

Ritson has conducted a study of 46 children aged initially from 5 to 9 years, over a period of almost three years during which each child has been interviewed individually on twelve separate occasions at two month intervals during each academic year. These interviews, which were recorded on video, all began with a game involving the random generator which was to be the subject of the interview. The main threads followed throughout the study were children's perceptions of the likely outcome of:

- a variety of rolling dice - standard, or marked with a variety of numbers, colours, letters and shapes: dice with different embodiments and in some questions more than one die was used.
- spinners yielding equal and unequal probabilities;
- sampling with replacement from bags containing equal and unequal numbers of coloured beads;
- sampling with replacement from cards marked with 'blobs' of different colours and sizes.

Very few of the children had had any previous experience with either spinners or sampling so it is believed that their perceptions in these areas have not been influenced by any specific teaching. All of the children have had experience of playing board games with a single 1-6 sided die, though some have had more experience than others.

Truran has conducted group tests administered to whole classes in a normal classroom situation, to approximately 300 children from Years 5 and 7. Individual interviews were conducted with six children randomly chosen from each of the Years 5 and 7 classes previously tested. Year 3 children were considered too young to cope with a group test, so six children were randomly selected from five Year 3 classes and interviewed individually. The study investigated children's responses to the behaviour of a range of RGs:

- coins;
- raffle tickets;
- 6 and 12 sided dice;
- spinners with 6 and 12 segments;
- urns with 6 or 12 numbered table tennis balls.

Both studies rely on a significant number of individual interviews which were recorded. This method was used because in both studies young children were involved and it is believed that they respond best to a situation where the interviewer:

... asks the child to verbalise his thought, to give reasons for his actions and generally to reflect on what he has done. ... two types of reflection are often involved. One involves process,...a second involves the rationale for a solution. (Ginsburg, 1981, p. 6)

Ritson's additional use of a video made not only the verbal response clear, but also indicated the child's physical demeanour during the interview and helped to clarify situations when the child's explanation was based on pointing to an object or demonstrating a strategy.

COMMON ELEMENTS IN BOTH STUDIES

It is not surprising that in both studies young children, trying to make sense of their world, should exhibit similar beliefs regarding RGs over which they have no control. It has been observed by both researchers that the children's answers often do not relate to mathematics, but rather to impressions, prior beliefs and illusions about the physical attributes of the RG and its behaviour.

The children involved in both studies have had limited experience with some of the RGs used, and none with others, for example Ritson's 'blob' cards and Truran's urns. The urns are similar in context to the 'Lotto' urns that are shown on television but none of the children interviewed indicated that they could see any connection between the two.

Three examples of cues and reasoning as a basis for their responses, taken from both studies, will be presented as evidence of the similar responses being used by children. The following are examples of different, but similar interpretations. The physical 'separateness' of the RGs was seen as being important in making a decision.

A common element observed in both studies was that physical properties or appearance of RGs were significant in their outcomes

- One of Ritson's questions, asked the children to consider whether there was a better chance of getting a number that they wanted with a standard die or with six disks numbered 1-6

  KMc(9:6)

  > The disks are better because they are separate and can all jump about in the box, but the numbers on the dice can't move.

  Truran replicated a question used by Fischbein et al (1991) which asked children to consider the best method of tossing three dice so that the outcome of each die would be five. The majority of responses to this question indicated children's belief that it would be better to toss each of the dice separately. If they were all tossed together the dice would 'rub against each other, and the numbers would move around' so the probability of getting three fives would be less likely.

Children were shown an urn with six numbered table-tennis balls and a six sided die and asked whether if they chose a table-tennis ball and tossed a die the outcome would be the same or different. The majority of responses to this question also indicated that because of the different shape and composition of the RGs involved the numbers would be different in each case. Very few children suggested that all numbers have the same possible outcome, although that was not always the case as the following example shows.

TC (F 9:8)

I Would the game be the same or would it be different if I used the table-tennis balls in the urn instead of the dice?

TC [hesitates] It's still the same numbers as the dice.

I So, what do you think. would the game be the same or different?

TC The same it's still got the same numbers.

Urns or bags of counters were seen as being particularly unreliable, because of the probability that the objects in them would roll about and cause certain elements to be more likely to be chosen. In the case of the urn it was pointed out that the table-tennis balls would roll 'into the corners' thereby making it unlikely that all elements had the same probability of being chosen.

The physical arrangement of RGs with a different embodiment were frequently seen as being liable to different outcomes because of this.

DD (F 9:11)

The dice is hard, kind of - and the raffle tickets are paper, so you won't get the same numbers.

WD (F 7:9)

I When I toss this die [twelve sided] is there a number or numbers that is easiest to get or are they all the same?

WD It has more sides than the other one, a lot of angles, I think six sides are better, I don't like this one.

Perhaps it is not just the question that we ask but the objects about which the question refer that influences children's responses.Yost et al.(1962) discussed the need to eliminate preferred and least-liked colour of RGs before the formal interview. The following indicates how colour-preference might influence a response.

JM (M 8:2)

I Holding a six faced die. Have you seen one of these before?

JM No

I Not at all?

JM Well I haven't seen one like that, that's red.

In the teaching of probability there are so many implications that influence children's responses. Our teaching will be more effective of we know what the misconceptions are so that they may be addressed directly. Madsen (1995, p. 90).

CONCLUSION AND FURTHER QUESTIONS

This paper emphasises the need for a link between research and practice. The pedagogy of probability teaching is not well defined. The psychology that explains issues of how children learn probability is increasingly well studied, so research findings like these described here are necessary if we are to create links between the two areas. Teachers and curriculum planners need to be provided with information about the influence that physical properties and physical arrangements of RGs have on children's probability thinking, as well as childrens' preferences about RGs used during lessons.

It needs to be ensured that documents in the future reflect children's real thinking about probability. While not specifically reported elsewhere it has been both researchers' finding that primary school teachers attending in-service courses are especially concerned about their own lack of knowledge of stochastics; and lack confidence in planning the continuation of a stochastics topic without support from mathematics associations and in-service courses. They are eager to discover what children think about stochastics and ways in which to present children with appropriate, wider-ranging and more challenging activities than they frequently are at present.

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REPRESENTATIONS OF POINTS

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Abstract

This article offers an analysis of the data concerning a comparison task, where two points, the intersection point of two lines and the intersection point of six lines were to be compared. This study of kindergarten to 9th graders, suggests that (a) students tend to grasp the two points as unequal. Moreover, the intersection point of six lines was frequently viewed as bigger and heavier than the intersection point of two lines; (b) this tendency declines with age, yet about 25% of the 9th graders still held this belief. The analysis considers the findings from a large perspective using the intuitive rule: "More of A - More of B": The larger the number of lines - The larger and heavier the point.

Introduction

'Point' is a basic notion in geometry. Rather than asking what a point is, it seems much more appropriate to ask what it is not, since 'point' has no definition and no characteristics by which it can be described. A point has neither width nor length - no dimensions at all. It just marks a place on a plane. This alone is quite paradoxical: How can something possibly mark a place when it has no dimensions?

To a naive student, it seems that the point, in spite of its being dimensionless, is the origin of segments with finite length, lines with infinite length, planes of width and length, in fact, all the possible geometrical figures. In other words, a certain accumulation of 'zero length' points seems to produce a segment of 'n' length. However, the geometrical continuum involves the emergence of positive length segments from zero length points. This, too, is an non-intuitive facet of the mathematical concept of 'point'.

In spite of the above mentioned even very young children are familiar with points, because the point is not just a geometrical notions; it has various meanings in other mathematical and naturalistic frameworks too. Children first encounter points early on in their lives. Even in kindergarten, children draw dots and points, talk about big points and small ones, round ones and colored ones - very perceptive and visual ones... Moreover, one uses a 'point' to separate whole numbers from decimals (e.g., 1.3 is one point three); a pen has a 'point'; there is a 'point' to my listing all these examples; and this drawing '•', also concretely designates a 'point'.

Using Tall and Vinner's terminology (1981), in spite of the fact that 'point' has no <u>concept definition</u>, students have, from very early on, a fairly clear <u>concept image</u> of points. This concept image is rooted in their real life experience and in their geometrical experience. Yet, points, in an everyday context, are frequently physical in their nature, therefore they actually contradict the pure, dimensionless, geometrical notion of 'point'. This geometrical notion, much like the whole of geometry, is antagonistic to reality-- "[geometry] deals with points which are not points but vague unspecified items; lines which are not lines but classes of items; and planes that are not planes but classes of classes" (Goetz, 1956/ 1962 p. 187).

Moreover, even the schematic drawing of points, in geometry lessons, might trigger an erroneous perception of the notion of point in the students' mind. According to Fischbein (1993), when referring to geometrical figures there are three categories of mental entities: the definition, the image and the figural concept. Fischbein differentiates between the drawing, which is a concrete entity, and the corresponding abstract idea, strictly determined by its definition. However, students tend to confuse the drawing with the geometrical idea.

This contradiction between the mathematical meaning of the notion of 'point' and the irrelevant, interfering interpretations which are attached to it, may lead to strange and problematic ideas that occasionally contradict the mathematical ones. Teachers should be aware of these ideas when preparing instructional materials.

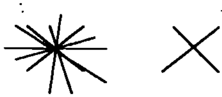
It is surprising to find how little research has done to investigate students' conceptions of the notion of 'point'. Hence, one of the main aims of this study was *to look further into students' concept of 'point'*. Another aim of this study was *to look for general rules, which apply beyond the specific issue of 'points' and intuitively guide students when comparing any two given entities*.

Method

Three hundred and one Israeli state-school students aged 5 - 15 participated in this study (57, 45, 64, 70, 65 students from grades K, 2, 4, 6, and 9, respectively). It is noteworthy that the 9th graders were not math majors. One of the researchers conducted 57 individual interviews in a kindergarten. All other participants (2nd - 9th grade) answered the same questions, but in writing.

All students were asked to compare the intersection point of six lines (A) with the intersection point of two lines (B). This part was based on Fischbein's study (1993) of children's figural perception of geometrical figures. The questionnaire is given below:

The Questions:



A B

Point A is the intersection of 6 lines   Point B is the intersection of 2 lines

1. Are points A and B equal? Yes/No
2. Is one of the points bigger than the other? Yes/No
3. If the answer to question 1 is yes - which point is bigger?
4. Is one of the points heavier? Yes/No
5. If the answer to question 3 is yes - which point is heavier?
6. Do the points have the same shape? Yes/No

   The shape of point A is: .... The shape of point B is: ....

Results

The results will be presented in the sequence of the questionnaire.

Are Points A and B Equal?

Most of the young participants (75-80% of kindergartners and 2nd graders) claimed that points A and B are *not equal,* whereas most 4th and 6th graders (about 65%) and most 9th graders (about 75%) viewed points A and B as equal (Table 1).

Table 1: Students' tendency (%) to view points A and B as being equal

Class:	**K** (n=56)	**2nd** (n=43)	**4th** (n=62)	**6th** (n=68)	**9th** (n=64)
A equal B	19.3	24.5	64.1	67.7	75.8
A not equal B	80.7	75.5	35.9	32.3	24.2

An important question that naturally arises is, "What factors facilitate students' perception of the equality or inequality of points A and B?" Three aspects of participants' grasp of "equality" were considered: (1) size, (2) weight, and (3) shape.

Students' Perception of the Sizes of Points A and B

A. Points A and B have the Same Size

Most 6th and 9th graders (65% - 75%), about half of the 4th graders and some of the young participants (about 10% of kindergartners and of 2nd graders) claimed that the points have the same size. Some explained: *"Because both are points"*. Regretfully, most participants did not explain their claims at all (Table 2).

B. Points A and B are of Different Size

Most kindergartners and 2nd graders (about 75-80%), about half of the 4th graders, but also about 30% of the 6th graders and 20% of the 9th graders claimed that **point A is larger**. The most common explanation was *"Because there are more lines passing through it"*; or *"it looks bigger"*. But again, usually no explanation was given.

Surprisingly, about 13% of the kindergarten participants and of the 2nd graders, and a few other students claimed that **point B is larger than point A**. The reason given was that point B had more place to expand as it was not limited by so many lines.

The differences between the answers of students from various class levels, were significant ($X^2 = 88.63$ $df = 8$ $p < .01$).

Table 2: Which point is larger?

Class:	K (n=54)	2nd (n=33)	4th (n=50)	6th (n=69)	9th (n=64)
A=B	11.3	6.1	48.0	65.2	73.4
A>B	75.7	81.3	47.9	30.4	21.9
A<B	13.0	12.6	4.1	4.4	4.7

Students' Perceptions of the Weight of Points A and B

About 10% of the 9th and 6th graders found this question too strange to be answered, arguing that weight is entirely irrelevant to points. A few added that they never thought about this aspect in relevance to points, and a number of others said that if there is such a thing as 'the weight of points' then it should be equal for all points.

A. Points A and B have the Same Weight

About 65% of the 9th and 6th graders, and about 40% of the 4th graders claimed that **both points have the same weight**, *"because both are points"* or *"because they are equal"* (Table 3).

B. Points A and B Differ in Weight

Almost all younger participants (kindergartners and 2nd graders), about 60% of the 4th graders, and about 35% of the 6th and 9th graders claimed that the points differ in weights and almost all of them specified that **point A is heavier**, usually *"because it consists of more lines"*. There were significant differences between the answers of students from various class levels ($X^2 = 82.21$ $df = 8$ $p < .01$).

Table 3: Which point is heavier?

Class:	K (n=53)	2nd (n=29)	4th (n=47)	6th (n=67)	9th (n=61)
A=B	3.8	-	38.3	64.2	61.7
A>B	94.3	96.2	61.7	35.8	36.7
A<B	1.0	3.8	-	-	1.7

Students' Visualization of the Shape of Points A and B

The graphic representation of points A and B elicited the following three main explanations of the shape of these points:

A. Pictorial Description

Points A and B were visualized as being similar to everyday objects. Point A, for instance, was visualized as a flower, a star, the sun, or a hedgehog. Point B was visualized as being similar to different objects, such as, the letter X, a butterfly, scissors, etc. About 35-50% of the K, 4th, 6th and 9th graders, and about 80% of the 2nd graders gave this type of description (Table 4).

B. Dimensional Description

Points A and B were described in terms of size. When this manner of description was used (about 50% of the kindergarten, and about 10% of the 2nd, 4th and 6th graders), it was used to describe both points. Students claimed that point A is big, thick or wide, while point B is tiny, small, thin or slender. Although the same style of description was used for both points, the specific words used to describe each point indicated that these students believed the points were not equal.

C. Geometrical Description

Both points were described as circular figures in general, or as small circles. This was quite common in 4th to 9th class levels (about 30-50%).

Table 4: Students' Visualization of Points A and B

	K (n=50)		2nd (n=32)		4th (n=42)		6th (n=68)		9th (n=52)	
POINTS:	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**
Pictorial description *sun, hedgehog* *vs. butterfly, scissors*	**32.0**	**30.0**	**81.8**	**81.8**	**52.4**	**54.8**	**31.6**	**35.1**	**48.9**	**48.9**
Dimensional description *big, thick, wide* *vs. tiny, thin, small*	**46.0**	**46.0**	**12.1**	**12.1**	**9.5**	**9.5**	**8.8**	**8.8**	**2.1**	**2.1**
Geometrical description *a (small) circle, circular*	**12.0**	**11.0**	-	-	**28.5**	**26.2**	**49.2**	**45.7**	**32.0**	**32.0**
Looks like a point	**10.0**	**14.0**	**6.1**	**6.1**	**9.5**	**9.5**	**10.5**	**10.5**	**14.9**	**14.9**
No definite shape	-	-	-	-	-	-	-	-	**2.1**	**2.1**

There were significant differences between students' descriptions of point A in different class levels ($X^2 = 51.12$ $df = 20$ $p < .01$), as well as differences in students' descriptions of point B ($X^2 = 42.32$ $df = 20$ $p < .01$).

A connection was found between the students' manners of description and their tendency to relate an identical shape to both points. Students who described the two points in a pictorial way and those who described the points in terms of size, usually differentiated between the shapes of point A and point B. On the other hand, participants who described the points as a geometrical figures (e.g., round or circular) and those who merely said that a point looks like a point, tended to attribute the same shape to both points.

Two additional ways of describing points A and B were: (i) *"They just look like points"*, and (ii) *"They have no definite shape"*. These indefinite responses did not, in fact, reflect how the students visualized points. Still, about 10-15% of the kindergarten, 4th, 6th and 9th graders used this description. The response that *"points* *have no definite shape"* was actually presented by one 9th grader who consistently responded to all questions by saying that *"a point is a basic notion representing a geometric figure with no dimensions. Hence, it cannot be defined, drawn or referred to by its shape".*

Conclusions

The findings of this study agree with the findings of previous research showing that students relate to mathematical notions attributes which are not necessarily in line with the mathematical theory (e.g., Tsamir, 1996). These findings strongly indicate that students' responses are often influenced by the way in which the problem is presented (e.g., Tirosh & Tsamir, 1996). Moreover, geometrical drawings play a special role in students' conception of geometrical notions (e.g., Fischbein, 1993).

Geometrical drawings usually assist in the solving of geometrical problems. Yet, they are also a source of confusion. Students do not distinguish between *necessary* and *accidental features*. When presented with a drawing, they connect pure geometrical notions with *accidental* features. Such features are existent only in the specific figures drawn or visualized for the case in question. The confusion arises when these irrelevant features are mistakenly considered to be *necessary* features of the geometrical figure in question. However, the latter are formally derived from notions and postulates of the theory (see, Black, 1959). Our findings show that students of various ages attribute to geometrical points various accidental features, such as shape, size and even weight, features which are part of the visual image of the geometrical drawings.

Under such circumstances the notion of 'point' should be formally presented and constantly repeated. First of all, due to the impact which students' misconceptions about 'points' have on their performance within various mathematical domains. For instance, when asked to compare the number of points in two different line segments (in set theory), some students claimed that the answer depends on the size of the points in each segment (Tsamir, 1994). Or when asked to compare two angles, 4th graders claimed that the angle with the bigger point in its vertex is bigger (Tsamir, 1995). Secondly, the notion of 'point' is very important in itself, but it is even more important since it represents the whole idea of 'basic notion'. It can and should be used as an illustration for the structure of the hierarchy of notions within a particular mathematical theory, which can only be done within the framework of a mathematics lesson.

In the upper grades of high school and in college courses for pre-service teachers, it would also be very appropriate to discuss "the usefulness of the impossible" (see, Goetz, 1956/1962). Schaaf explained Goetz's expression as follows: *What professor Goetz here calls the "impossible" in mathematics refers, of course, to the abstract nature of mathematical concepts and relations. They are impossible because they are ideal or "perfect". But by virtue of their perfection, they are amenable to the laws of logic and internal consistency, and thus we can think about them in rigorous terms*

(p. 186). This "perfect-pure" nature should be differentiated from the concrete physical application of the mathematical terms.

What has *not* been explained so far, is the pattern noticed in most students' answers when asked to compare the point of intersection of six lines (A) with the intersection point of two lines (B): Why did most students, who claimed that points A and B are of different size, also consider point A to be larger than B, and similarly, why did those who claimed that points A and B differ in weight also grasped point A as being heavier?

It seems that these patterns are not content-specific, and do not relate to 'points' or any other specific concept in particular. Comprehensive research is currently being conducted, which considers students' similar reactions to a wide variety of scientifically unrelated situations in both mathematics and science education, by Stavy and Tirosh (e.g., 1994). They suggest that many erroneous responses to comparison tasks, which the literature calls alternative conceptions, could be interpreted as evolving from a small number of basic intuitive rules. One of the rules they identified is: ***The more of A, the more of B.***

The simplest way to demonstrate the widespread cue of these intuitive rules is through their manifestation in common phrases in daily speech, such as "*the more you study, the more you know*" etc. Comparing quantities and determining whether they are equal or not is something people do frequently. Tirosh and Stavy have specified two procedures for the comparison of quantities: the direct method, and the indirect one. In some cases, judgment is based on *direct*, visual information. That is, one can directly perceive that, in respect to the quantity in question, one object is equal or bigger than the other. Take for instance, two identical sticks, one of which, however, is longer than the other. It is obvious to children, from a relatively early age on, which one is longer.

However, often, such direct perceptual cues are not available, thus requiring an indirect approach. In these cases the target comparison is often based on another quantity. That is, there are many cases in everyday life in which a perceptual quantity (A) can serve as a criterion for comparing another type of quantity (B). In these cases "the more of A (the perceptual quantity) implies the more of B" (the quantity in question). Unfortunately, frequently quantity A is irrelevant to the required comparison, or cannot, by itself, serve as a criterion for comparison. For example, when young children compare two cups containing equal amounts of water but where one cup is narrower and taller than the other, they often claim that "the taller - contains the more".

This intuitive rule affects students' responses to comparison tasks, regardless of the specific nature of the content domain. Thus misconceptions apparently related to different domains are actually only specific instances of the use of this general rule.

In the case of points A and B, where A was the point of intersection of more lines, it was also usually grasped as being larger and heavier, reflecting the rule "the more lines-- the larger the point" or "the more lines-- the heavier the point". Interestingly, even students' explanations of the view that point B, the point of intersection of only two lines, was larger than point A, the intersection point of six lines, could be interpreted by means of this very same intuitive law. Point B was considered larger, because students argued that "the point which still has more place to expand, is bigger". An unexpected instance of "the more A- the more B".

Last but not least, the findings of this work also strongly indicate the need for further research to bring to light students' conceptions of 'points'. It would be interesting to widen our data base by means of a larger sample of students, including mathematics majors. This work should also refer to additional aspects of basic notions in general and of points in particular.

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UNDERSTANDING OF DIFFERENT USES OF VARIABLE: A STUDY WITH STARTING COLLEGE STUDENTS

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University mathematics courses require a solid and flexible understanding of the concept of variable. In this paper we present a detailed analysis of the responses given to a series of open-ended items involving different uses of variable (unknown, general number, variables in functional relationship) by 164 starting college students. The results show the persistence of many misconceptions and problem solution strategies characteristic of lower school levels. There is evidence showing that the majority of the students are still restricted to an action concept of variable. This hinders most students from attaining a level of abstraction which will enable them to consider variables as objects whose role can be analysed.

The mathematics taught at university level requires a solid and flexible understanding of the concept of variable. Even though much attention has been brought to secondary students' conceptions of variable [*2,3,7,9*], a considerable less effort has been dedicated to the study of the ways in which newly undergraduates work with this concepts. Our study aims to contribute in filling this gap.

In previous articles [*5,6,8*] we have presented the first results of a project that intends to investigate college students' conceptions of variable. We presented a decomposition of the concept of variable that was used to design a questionnaire of 65 open-ended items. A quantitative and a qualitative analysis of 164 starting college students' answers suggested that they have great difficulties to deal with different uses of variable, namely, unknown, general number and variables in a functional relationship. Moreover, this analysis allowed us to refine our decomposition of variable. In this article we present a detailed analysis of the responses they gave to some of the items involving these different uses of variable. This analysis aims to highlight the ways in which they work with them and to explore up to which level they have developed a capability to cope with them adequately in simple school problem.

THEORETICAL FRAMEWORK

The understanding of the concept of variable implies, from our point of view, the possibility to overcome simple calculations and operations with literal symbols, to develop a comprehension of why these resources work; to foresee the consequences of using them; to distinguish between the different uses of variable and shift from one to another in a flexible way, integrating them as components of the same mathematical object. Coping with each one of the different uses of variable implies the capability to recognise its role in a given situation; to operate on and with it when required by the task; and to use it in order to symbolise a problem situation. It could be expected that after several elementary algebra courses undergraduate students will be able to cope with the different aspects of variable, and that the understanding of each one of them will be equally developed.

A starting point for understanding the way in which students conceive and work with variables can be a careful analysis of what is meant by understanding this concept, through isolating its components and explicitly describing the relations between them [*1*]. In a first approach this analysis or decomposition of the concept of variable was based on our own understanding of this concept, on our experience as teachers and on what we considered the necessary mental constructions for developing it. This first decomposition [*8*]has been refined after analysing students' responses to the questionnaire. In the following paragraphs we present a new decomposition of the concept of variable highlighting the aspects that, from our point of view, are basic for understanding its different uses.

We consider that an understanding of the variable as unknown implies: to recognise and identify in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem; to recognise the symbol that appears in an equation as an object that represents specific values that can be determined by considering the given restrictions; to be able to substitute to the variable the value or values that make the equation a true statement; to determine the unknown quantity that appears in equations or problems by performing the required algebraic and/or arithmetic operations; and to identify the unknown quantity in a specific situation and to symbolise it posing an equation.

It could be argued that an unknown is not a manifestation of the variable because it represents a fixed value; nevertheless, we consider that the first perception of the literal symbol when working in algebra is, or should be, that of a symbol representing any value, and that only in a second moment its role in the expression in which it appears can be defined. So, when presented with an equation we recognise that the variable represents a specific value only after having actually or mentally performed the necessary manipulations that allow us to recognise it as an equation and not, for example, as a tautology.

The understanding of the variable as general number implies: to recognise patterns in numeric sequences or in families of problems; to recognise a symbol as representing a general, indeterminate object; to develop the idea of general pattern/method by distinguishing the invariant aspects from the variable ones in a problem situation and to symbolise them; to simplify or to develop expressions.

The understanding of variables in a functional relation (related variables) implies: to recognise the correspondence between quantities independently of the representation used (tables, graphs, verbal problems or analytic expressions); to determine the values of one variable (dependent or independent) given the value of the other one (independent or dependent); to recognise the joint variation of the variables involved in a relation independently of the representation used (tables, graphs, analytic expressions); to determine the range of variation of one variables given the domain of the other one; to symbolise a relation based on the analysis of the data of a problem.

For each one of the three aspects of variable, this decomposition stresses different levels of abstraction at which it can be handled. College students should be able to cope with all of them, moreover, in order to handle the variable as a mathematical object they should be able to integrate its different uses in one concept and shift between them depending on the requirement of the task.

METHODOLOGY

Our first decomposition of variable was used to classify and interpret the answers given by 164 starting college students [*8*]. As already reported the score for each one of the uses of variable was very low. Therefor, in order to obtain a better comprehension of students' understanding of different uses of variable we decided to focus our analysis on the incorrect answers. For this analysis we selected a series of items that involved each one of the different uses of variable. Moreover, some students were briefly interviewed in order to obtain a deeper comprehension of their understanding of the concept of variable. This information was further complemented by observing the way in which algebra is taught in high school.

QUALITATIVE ANALYSIS OF THE RESPONSES

Table 1 presents a selection of the items involving variable as unknown (items 14, 16, 50 and 52) and variable as general number (items 4, 5, 8, 22). For each item the percentage of students giving correct, incorrect or no answer are shown. Additionally, examples of the most frequent incorrect answers are presented.

The analysis of students' responses suggests a difficulty in discriminating between variable as unknown and variable as general number (items 5, 13, 14) that implies a feeble conceptualisation of these uses of variable. Instead of manipulating the expressions in order to obtain a form that might help them recognise if the variable involved represents an unknown or a general number, students seem to look for signs allowing them to give a quick answer. Their answers seem to be an automatic response to external *stimuli*, suggesting an understanding of variable at an action level. Evidence for this is provided by the answers given to item 14. The incorrect answers to this item show two big tendencies: some students answered by writing the number 2; a second group considered that the letter could assume an infinity of values. The first answer seems to be induced by the presence of the quadratic exponent on the left side of the equation. This sign seems to act as a *stimulus* inducing students to consider that they are facing a second degree equation and, using a memorised fact, they automatically answer that the letter can take two values. The second group, less numerous but considerable, seems to deepen a little more their analysis and get aware of the quadratic term on the right side of the expression. This seems to lead them to consider that they are facing a tautology and, therefor, the letter is viewed as representing an infinity of values.

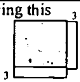
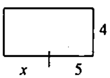
Item	No answer	Correct	Incorrect	Examples of incorrect answers
How many values can the letter assume in the following expressions? 5. $x + 2 = 2 + x$	4%	52%	46%	1
13. $(x + 1)^2 = x^2 + 2x + 1$	23%	28%	49%	1; 2
14. $4 + x^2 = x(x + 1)$	31%	23%	46%	2; ∞; several; R;N
16. Write the values the letter can assume. $(x + 3)^2 = 36$	17%	10%	73%	$x = 3$
50. Write a formula for solving this problem. The total area of the figure is 27. Calculate the side of the shadowed square. 	38%	7%	55%	$a^2 + b^2 = 27$; $L = \sqrt{27}$; $3(x) = 27$; $(1 - 3)(1 - 3)$; $27 - (\sqrt{27} \cdot 3 + \sqrt{27} \cdot 3)$; $x - 3$; $27 - 3^2$
52. To rent a car costs $25 daily plus $.12 per kilometre. How many kilometres can Diego drive if he only has $40?	13%	46%	41%	125
4. Write a formula to express an unknown number divided by 5 and the result plus 7.	2%	29%	682%	$\frac{x}{5} = y + 7$
22. To calculate the perimeter of a figure we add up the length of each side. Write a formula to express the perimeter of the figure. 	5%	54%	41%	$2(b + h)$; $(5 + 4)\,2$; $8 + 5 \times 4$; $2(5) + 2(4)$; $x + 5 + 8 + 9 = P$; $x = 5$ $L + L + x + x$; $8 + 5x \cdot 2$

Table 1

A superficial conceptualisation of variable as unknown and as general number is confirmed by the answers given to item 13. Once more their answers seem to be influenced by stimuli perceived as external and not by an analysis of the expression. We suggest that the presence of the equal sign led 49% of the students to conceive the expression as an equation and the variable involved as an unknown. The presence of the quadratic exponent led half of them to consider that the letter may assume two different values, while the other half considered that it can take only one single value. That the presence of the equal sign induced a perception of the variable involved as an unknown is suggested as well by the answers given to item 5. However, if compared to items 13 and 14 it can be observed that the number of correct answers considerably increased for item 5. We would suggest that the reason for this is that item 5 can be solved by simple inspection, without any manipulations, while to answer items 13 and 14 it is necessary to manipulate the expression, actually or mentally, in order to recognise the role of the variable.

A tendency to avoid manipulation and to determine the value of the variable by inspection is suggested by the answers given to item 16. Although they correctly perceive the variable as unknown, they avoid manipulation and by simple inspection they consider that the variable can assume only one positive value. This behaviour suggests the persistence of arithmetic methods in the solution of equations and an elemental action conception of the variable as unknown. The way in which the problem is posed seems to influence students' answers (compare items 14 and 16 that ask respectively "how many values" and "the values" the letter can have).

Difficulty in conceptualising variables as unknowns is confirmed by students responses to verbal problems. Face to a simple problem (item 52) half of them answer by giving a correct number without posing an equation. Although they conceptualise the unknown of the problem, they determine its value by arithmetic methods. When the complexity of the problem does not allow them to use an arithmetica approach (item 50), they give an arbitrary solution showing a lack of ability to symbolise the problem. It can also be observed that when they use a symbol this is not denoting the unknown but another quantity involved in the problem. All these results indicate that most of the students handle variable as an unknown in an elementary level in which they barely recognise it as an unknown number, and their possibilities to process all the information are very limited.

Responses to item 4 show that although students have some capability of symbolisation and manipulation of the general number, they seem to feel a certain discomfort to consider x/5 as an object to which another operation can be applied. Students seem to need to write explicitly a new general object to perform the next operation and they do not consider necessary to separate the expressions x/5=y and y-7. The combination of both expressions in a single one leads them to an incorrect result. The use of the equal sign as a connection between the solution steps can be considered as a reflection of their abbreviated thinking process and as an evidence of their insecurity to assign a particular role to the variable in the problem. The capability to construct an expression which involves a general number slightly increases when the symbol has a clear reference and the expression obtained represents a result. Evidence is given by the response to item 22, in which the literal indicates the indeterminate length of a segment, and the result is a perimeter. For items like this one a strong tendency to use memorised formulae was also observed, suggesting once more an action conception of variable.

The use of general number implies a process of generalisation. Less than a half of the students were able to interpret patterns and deduce a general rule. Most of them were able to recognise a pattern at an action level, finding the relation for specific numbers, but they had serious difficulties in generalising the process. Students show a very low comprehension of variable as a general number. They can recognise a symbol as the representation of something indeterminate in simple cases, but show great difficulties when the level of abstraction is a bit higher.

When students deal with related variables (Table 2), we find that no matter the representation used to express the relationship, students can adequately handle the notion of correspondence between specific numbers, but they have difficulties with the idea of related variation. For example, the answers given to questions 44 and 58 can be classified into two groups. Students who list only the whole numbers in the interval and those who answer in terms of intervals. The first group demonstrates a weak notion of the continuos property of the real numbers and a discrete conception of the relation. Students in the second group obtain the interval by evaluating the function at the end points of the given interval, without taking into account the specific characteristics of the function (item 58). This shows an inappropriate generalisation of the monotonical property independently of the relationship.

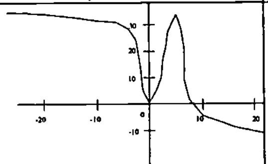
Item	No answer	Correct	Incorrect	Examples of incorrect answers
39. Write a general rule if n stands for the number of copies. Copy | Price 5 | \$6.25 10 | \$12.50 15 |	12%	65%	23%	n price; $6.25\,n = x$; $5n = 6.25$
43. Write a general rule to link the numbers on the left side of the table to those on the right side. Sec | Velocity 0 | 0 m/s 10 | 30 m/s 15 | 20 | 60 m/s	29%	32%	38%	$10s = 30v$; $x + 1 = y + 1$; $1s = 3m/sec$; $x + m = y$; $d = \frac{v}{t}$
44. Observe the following expression $y = 3 + x$. If we want the values of y to be greater than 3 but smaller than 10, which values can x assume?	3%	18%	79%	1, 2, 3, 4, 5, 6, 7 ; {1, 2, 3, 4, 5, 6, 7}
For each kilogram the tray of the balance moves 4 centimetres. 48. Express the relation between the weight of a merchandise and the movement of the tray.	12%	28%	60%	1 kg = 4 cm
49. If the tray moves 10.5 cm., how many kg are bought?	5%	68%	26%	1 kg = 4 cm 2.62 ? = 10.5 cm
From the data of the table 53. What happens to the value of y when the value of x increases? x | y 0 | 0 10 | 100 -15 | 225 25 | 625 20 | 400 -10 | 100 15 | 225 20 | 400	10%	11%	79%	increases ; decreases
58. If x assumes values between -2 and 26, which will be the range of values for y?	37%	2%	61%	4 and 676
59. $40 - 15x - 3y = 17y - 5x$. Which is the value of y for $x = 16$?	34%	21%	45%	$y = 10$; $y = -60$; $y = 6$; $y = 32$; $y = 13$; $y = 100$
 62. Between which values of x he values of y increases?	16%	41%	43%	0 and 5 and 0 and -5; 0 and 5 and 0 and -2; [(0, 10) ; (0, 5) ∪ (0, -∞); -20 to 0 ∪ 0 to 5
63. Between which values of x, the values of y decreases?	16%	27%	57%	5 and 0 and -5 and -10; (5, ∞); (4, ∞); (∞ and -5) and (5, ∞); (10, 20)

Table 2

The way students deal with tables and graphs suggests the use of intuitive methods to interpret variation. Student's difficulties with the notions of order and density of the real numbers appear again when they try to find intervals in which a function increases or decreases. In the case of item 62, some students take a particular value of the independent variable as a reference and analyse the behaviour of the dependent variable to its left and right, concluding that the values of the dependent variable increase when the independent variable is between 0 and -5, and between 0 and 5. Another example of the use of intuitive methods is found in the way students complete tables or solve verbal problems (item 49) using proportions as a tool to find particular values.

The symbolisation of a functional relationship also presents serious difficulties to students. Only when the symbols for the variables are given explicitly, as in item 39, students are able to symbolise the relation. This is an indication that students require external support to recognise and express in symbols the relation between two variables. When faced with word problems, as in question 48, they try to synthesise part of the information of the problem by using symbols. A large number of students write: 1 kg = 4cm, showing that they interpret the problem in terms of the relation between quantities. Here the equal sign is once more used as support in the analysis of the problem and as a reflection of the mental process of the students. They are not able to generalise the rule and express it in an analytical form. When several operations are needed in the solution of a problem: substitution, transposition, and grouping alike terms, students demonstrate a weakness in their capability to manipulate and to handle related variables. In item 59 we found several inadequate manipulations and, again, a need to perform actions one by one following a specific order and writing down all the steps, without analysing if they are correct.

In general, the answers to those problems that involve relations show a great inconsistency and inadequate generalisations, even for very simple problems. Most students do not seem to conceive the relation as a transformation process or as a dynamical process of variation. It seems clear that their conception is limited to a static idea of one to one correspondence.

The class observation at high school level showed that teachers' attitudes seem to reinforce student's conceptions about the role of memorisation and automatic skills in algebra: When teachers approach algebraic problems, they handle the different uses of the variable, and go from one to the other without any explanation that would enable the students to understand the role the variable plays in a specific moment in the process of solution. The class time assigned to mechanisation of procedures is always larger than that used to application or conceptualisation exercises. When students make a mistake teachers put more emphasis on the correction of the algorithm steps than on the conceptual aspects involved [*4*].

CONCLUSIONS

The results of this study confirm that learning the concept of variable is a difficult and slow process. Students' conceptualisation of this important concept remains at an action level after several years of study. The attention of the students is centred on superficial characteristics of the expressions, and they tend to use arithmetic methods for solving problems. The exposition to previous courses does not develop in a meaningful way the possibility to generalise procedures and patterns. Even though the majority of students are able to recognise the role a variable plays in very simple expressions and problems, a slight increase in the complexity of the problems leads to inadequate generalisations and to the search for memorised or trial and error solutions. Students' strategies are dominated by procedures which have not been interiorised, this is shown in their need to make explicit the steps they follow in the solution process and their incapability to analyse them and detect possible mistakes. Many of their actions seem to be caused by *stimuli* that they perceive as external and which induce them to answer in a certain specific way, avoinding manipulation, suggesting that they are anchored to an action level.

The results of class observation suggest that the type of problems found in this study are strongly related to the way in which algebra is taught in previous courses. It seems that there is a necessity to re-think the way in which this concept is approached at school.

Finally we want to stress that our decomposition of the concept of variable allowed a detailed analysis of students' responses. It would be desirable to prove it in more complex situations and in the design of strategies and activities for a more meaningful teaching of the concept of variable.

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GENDER DIFFERENCES IN COGNITIVE AND AFFECTIVE VARIABLES DURING TWO TYPES OF MATHEMATICS TASKS

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Abstract

In this study gender differences in cognitive (performance, solution strategies) and affective variables (confidence, persistence following failure) were studied. Subjects were sixth-grade students (ages 11-12) who were observed when working on two types of mathematics tasks: algorithms versus applied problems. Results demonstrated gender differences in cognitive and affective variables with respect to applied problems only. Girls perceived lower confidence compared to boys, and also compared to their own confidence while solving algorithms. Our data suggest that performance alone does not account for differences in confidence. Analyses revealed that boys were inclined to be overconfident, whereas girls were inclined to be underconfident.

Introduction

Theoretical framework

The importance of including motivational variables when studying individual differences in mathematics learning has been stressed by many authors (e.g. Pintrich & De Groot, 1990). In our study, elements from research within cognitive psychology on the one hand (e.g. Schoenfeld, 1985) and elements from research that is directed at a task-specific approach of motivation and learning on the other hand (e.g. Boekaerts, 1991) are integrated.

The reasons for setting up this research were twofold. First of all, this research was aimed at gaining better insights to different aspects of motivational variables and achievement in mathematics, especially at the task-specific level. The study that is described here draws on research that has been directed at students' motivation in concrete learning situations (Seegers and Boekaerts, 1993). Secondly, the research was set up to further explore gender differences in mathematics. In many studies it has been reported that already in primary school, girls perform worse than boys in mathematics, especially when it comes to applied problem solving. In two studies that were executed five years apart by the Institute for Educational Measurement in The Netherlands, clear differences in performance between boys and girls in the final year of primary school were reported (Wijnstra, 1988; Bokhove et al, 1996). Over the years a wide range of explanations about the causes of gender differences in mathematics achievement have been offered. Different models, which are complex and multivariate, have been presented (e.g. Ethington, 1992). Our starting point is that gender differences in mathematics performance are the outcome of complex interaction effects, in which both cognitive and motivational variables play a role.

Research perspective

We investigated gender differences in both cognitive (performance, solution strategies) and affective variables (Vermeer, 1997). With respect to the latter, emphasis was put on students' expectancies concerning successful goal attainment while they were working on a mathematics task (perceived confidence), and their persistence following failure. Drawing on studies in which gender differences have been reported involving content-specific areas of performance (e.g. Marshall and Smith, 1987), a distinction was made between two types of mathematics problems: algorithms and applied problems. Algorithms are characterized by the fact that a precise, systematic and detailed plan should be executed. An applied problem may also include a sequence of steps, but this sequence is less complete and less systematic than within an algorith. Especially with context problems, which are considered to be an important part of the mathematics curriculum within realistic mathematics education (Treffers and De Moor, 1990), an extra difficulty is involved because students should evaluate their solution within the context of the problem.

We hypothesized that, in general, students would show more confidence while solving algorithms, because this appeals more to the execution of a precise and systematic plan than solving applied problems. We also expected students to give up earlier after failure when it came to solving applied problems compared to algorithms. With respect to gender related differences, we did not expect to find differences in boys' and girls' confidence while solving algorithms, nor in their reactions to failure. However, we expected girls to have lower confidence than boys while solving applied problems, and hence to give up more easily after failure experiences.

Our research questions were:

Do gender differences exist with respect to the cognitive and affective variables used in our research?

- How are (in)correctness of the solution, gender, type of task, and students' perceived confidence related?
- What are the influences of perceived confidence, type of task, and gender on persistence following failure?

Method

Subjects

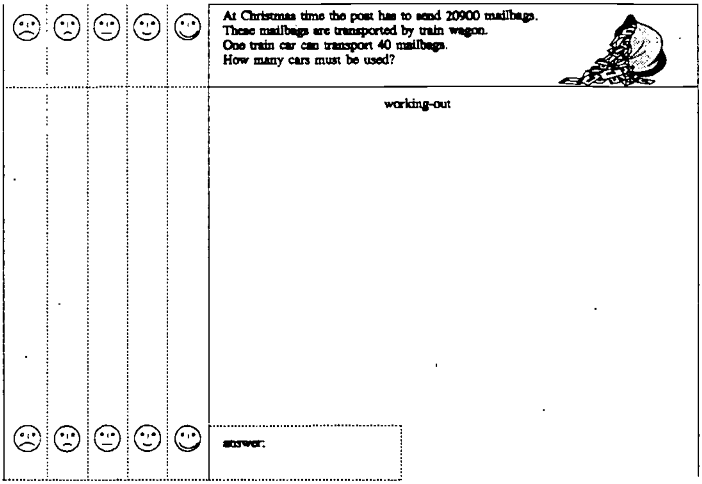
Subjects were 158 sixth-grade students (ages 11-12), 79 boys and 79 girls. These students came from 12 different schools in The Netherlands, at which all teachers work with the same realistic mathematics method. At first, all students made a shortened version of a test for non-verbal intelligence. Those students who scored within the highest or within the lowest decile were excluded from our research. From the pool of remaining students 158 were randomly selected.

Measures

Two mathematics tasks were constructed for this research. Task A consists of six algorithms, whereas Task B consists of six applied problems. The contents of the tasks were chosen in such a way, that Task B could be solved by applying the same algorithms as in Task A, only using different numbers. In this way, the students' required domain-specific knowledge with respect to procedures and computations were the same in both tasks, and therefore comparisons between behaviors during the two tasks could be made more easily.

The Confidence and Doubt Questionnaire (CDQ) was developed for this project (Boekaerts, Seegers, & Vermeer, 1995). This questionnaire is an instrument for registering confidence during mathematical problem solving. In order to investigate these processes on-line, a special notation system was developed. In the left margin of every working paper on which a problem was written, five faces were drawn ranging from very sad to very happy in their expression. They symbolized the degree of doubt or confidence a student had while working on the problem (see Figure 1). While working on the task, students were asked to indicate to what extent they thought that their strategy would lead to the right solution. The students were asked to put a mark under one of the faces (1) after having read the problem (the orientation phase), (2) at 40-seconds intervals during the solution process (the execution phase), and (3) after having found an answer (the verification phase). Marked faces were translated to scores ranging from 1 (not confident ar all) to 5 (very confident).

Figure 1: Example of the Confidence and Doubt Questionnaire



Procedure of the individual sessions

Students were tested individually while they worked on the mathematics task. The procedures for both sessions were the same. First, Task A was given to the students in combination with the CDQ. After about three months, Task B was administered to the students in combination with the CDQ. At the beginning of every session, the students received detailed instruction about the procedure. Students were instructed to write down their solution process and calculations in as detailed a manner as possible. After they had understood the instruction, they completed two pretest problems. While doing the tasks, the students filled out the CDQ while they worked on the problems. After having given a solution, the students were told whether it was correct or not. If their answer was incorrect, they were asked whether they wanted to try the problem again. After having solved the problem for the second time, no feedback was given. If students had given up, they were instructed to go on to the next problem. Working time was recorded for each problem.

Variables

We calculated a *mean confidence* score across the three phases of the solution process for every student on every problem. Mean confidence ratings ranged from 1 (not confident at all) to 5 (very confident).

Students' solutions were scored as either *correct* or *incorrect*. In addition, *solution strategies* for each problem were listed, based on the students' written work and on notes that were made during the individual sessions. This resulted in different lists of solution strategies for each problem. In order to compare solution strategies across problems, they were further categorized into three general solution strategies: (1) ineffective, (2) conventional, and (3) unconventional. First, a solution strategy was called ineffective, when students either, (1) showed no attempt to solve the problem at all, or (2) used the wrong computations or combination of computations, or (3) did not complete all the necessary steps in order to solve the problem. Secondly, from the solution strategies that were not ineffective, a solution strategy was called conventional if one or more standard computational strategies were applied, such as the execution of long division. Thirdly, a solution method was labeled unconventional if it was mainly non-routine, for example primarily based on students' insight and/or logic, such as estimation or mental computation.

We dichitomisized the variable *persistence* into the categories high and low, by taking into account the total number of incorrect solutions students had on the whole task, and the number of times students decided to try to solve a problem again. Students who retried less than half of the problems they had solved incorrectly, were considered to be low in persistence, whereas students who retried at least half of the problems they had solved incorrectly were considered to be high in persistence.

Results

Consistent with expectations, boys performed better than girls on Task B. In addition, there were more girls than boys who performed better on Task A than on Task B, and there were more boys than girls who performed better on Task B than on Task A. The data also revealed gender differences in the use of solution strategies: Boys used more unconventional solution strategies than girls for the problems in Task B. An unexpected finding was that girls were more inclined to persist on Task B after failure experiences.

Intra-individual differences in students' confidence while working on the mathematics tasks were analysed across tasks using a multivariate analyses design with repeated-measures. As was expected, boys perceived higher confidence than girls while working on the problems of Task B. We found that only for girls, the mean confidence during Task B was lower than the mean confidence during Task A.

In addition, we analysed to what extent there was correspondence between students' confidence and the (in)correctness of a solution. Figures 2 and 3 display the mean confidence scores on each problem of both tasks for four categories of students: (1) boys with a correct solution, (2) boys with an incorrect solution, (3) girls with a correct solution, and (4) girls with an incorrect solution.

Figure 2: Students' mean confidence scores on the problems of Task A

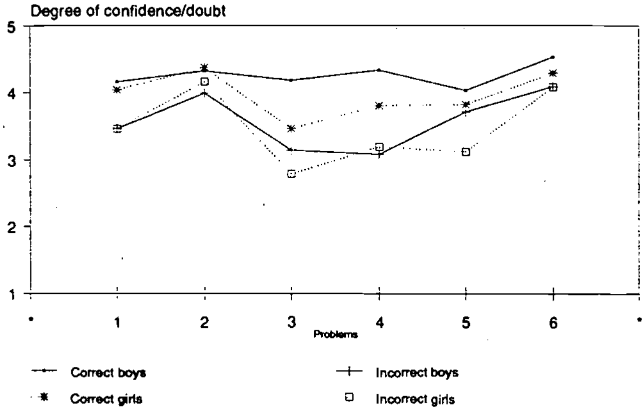
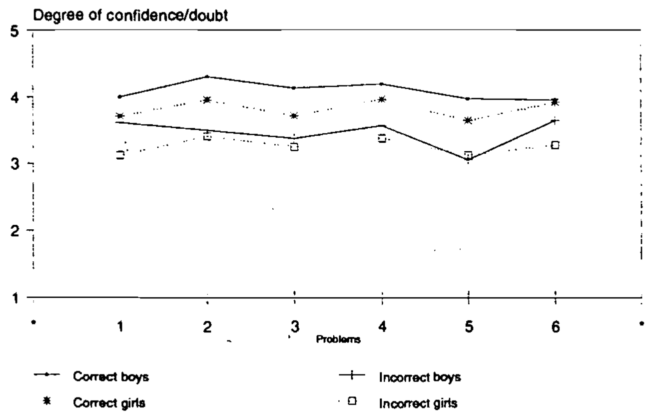


Figure 3: Students' mean confidence scores on the problems of Task B



Regression analyses revealed that students with correct solutions for all problems of both tasks displayed more confidence than students with incorrect solutions. However, after accounting for the (in)correctness of the solution, a gender effect on perceived confidence was found in relation to some problems. During two of the problems of Task A, boys showed higher confidence when the solution was incorrect than girls did, whereas during two of the problems of Task B boys showed higher confidence than girls when the solution was correct. One of these problems also elicited higher confidence in boys than in girls, when the solution was incorrect. This pattern suggests that boys were inclined to be overconfident, while girls were inclined to be underconfident.

Our last research question addressed relations between perceived confidence, gender, and persistence following failure. Logistic regression analyses were applied to the data with persistence (low versus high) as the dependent variable. For Task A, we found positive relations for both boys and girls between confidence and persistence. The gender effect appeared to be stronger than the confidence effect with respect to Task B. Although boys' perceived confidence was higher than girls', girls persisted longer than did boys.

Conclusions

The results of the present study demonstrate that both the cognitive and affective variables measured during mathematics tasks revealed gender differences. These differences in problem-solving behavior were also dependent on the contents of the mathematics tasks. Consistent with our expectations, we found that girls perceived lower confidence than boys, but only while working on the applied problems. Intra-individual analyses revealed that girls not only rated their confidence lower than boys during the applied problems, but also compared to their own confidence ratings while solving algorithms. Our data suggest that performance alone does not account for differences in confidence. We found that boys were more inclined to be overconfident, whereas girls were more inclined to be underconfident.

Relations between perceived confidence and persistence following failure were only partly confirmed in our study. Although we did find that girls displayed lower confidence than boys during applied problem solving, this did not result in lower persistence for girls.

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COMING TO KNOW PUPILS: A STUDY OF INFORMAL TEACHER ASSESSMENT OF MATHEMATICS

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Abstract

The author visited a mathematics classroom regularly throughout a term, recording the observable behaviour and written work of a small number of pupils. The teacher was consulted frequently about her developing knowledge of the pupils as learners of mathematics. Information obtained through the research was shared formally at two points during the study, and informally. In this paper, the teacher's emerging picture of a pupil is described and the relevant processes of informal assessment are analysed and critiqued. The author finds that while behavioural information dominates the observable data, it is other kinds of action that convey most about mathematics.

Introduction

This paper describes part of an investigation into how, in their usual classroom practice, teachers recognise and know what their pupils know and can do in mathematics. The project addresses comments such as "problems stem from a system which arranges for evaluation on behalf of others to be made" [Wheeler,1968], by discovering more about the processes of evaluation, and "there appears to exist a great need for a study of teacher judgements of pupils' mathematical potential" [Bishop and Nickson,1983] by directly studying how those judgements are made.

It was prompted by moves towards incorporating teachers' judgements into statutory assessment requirements in the UK from 1988 onwards, reinforced by Dearing [1994]: "Ongoing teacher assessment is central to the assessment of the performance of the individual child"(para3.38) and "Assessment is the judgement teachers make about a child's attainment based on knowledge gained through techniques such as observation, questioning, marking pieces of work and testing" (p.100)

In the early part of the project a description of the types of evidence teachers reported using informally in practice was developed [Watson,1995]. The types were all problematic in terms of meaning, interpretation and potential for bias and have varying levels of acceptability as evidence, according to different audiences and purposes of assessment. Defining *evidence* as *the raw material used to make judgements*, the following were found to be used, to a greater or lesser degree, consciously or unconsciously, by teachers: oral evidence from teacher-pupil talk or overheard pupil-pupil talk; written evidence, including tests, rough work, extended writing, exercises; actions while working, particularly during practical work; unprompted use of mathematics, occurring through any of the three modes above;

knowledge of the pupil, leading the teacher to infer what has not been directly observed; the teacher's own view of mathematics, which frames the work and expectations of the class; behaviour, body language and facial expression. Oral communication and unprompted use of mathematics were deemed to be the strongest types of evidence. Written work was generally considered problematic as many pupils have difficulty expressing their understanding on paper. Tests, particularly, were thought to give an incomplete picture of knowledge.

Subsequent interviews gave detailed information about issues of fairness, interpretation, analysis, understanding and teachers' attitudes, some of which has been published elsewhere [Watson,ibid;1996a;1996b]. However, results were largely descriptive and did not offer any information about how evidence was moulded into pictures by teachers. There is a dearth of literature describing how informal judgements are made in mathematics. Literature on teacher expectations [Nash,1976] describes how teachers can be influenced by what has been said already, and by whom it was said, so that first impressions are founded on received impressions. Nisbett and Ross [1980] describe how interpersonal judgements are made based on outstanding external features in the early stages of knowing people and a reluctance to change those views, even in the light of contradictory evidence. Symbolic interactionists [Blumer,1978] suggest that we develop our views of others through interaction. This places a burden on the participants to give observable, interpretable signs in interactions [Goffman,1959]. In a classroom, the younger, less socially-skilled participants have to communicate their thinking through interpretable signs to the teacher.

In order to find out how views are formed we must know more about the signs of pupils' knowledge which are available to be observed by teachers in classrooms.

The study

I observed a small number of pupils in each of two mathematics classrooms once a week for the first term with a teacher. It was agreed that I should write down everything I could observe and hear from the focus pupils and have access to written work and other information about their mathematics. This way I would have the same kinds of observable information as the teacher had, although I would have it in short, intensive chunks rather than thinly spread over all lessons. I would not tape-record or video pupils as this might alert them to the fact that they were the focus of my work. I would share information with the teachers as soon as possible if it was aberrant, immediately relevant, or in conflict with the teacher's current expressed views; I would also give them copies of my notes at certain points during the term.

This way they would have access to all the information I had and hence be better able to form well-founded views of the focus pupils[1].

Bauersfeld [1988] warns against decomposing classroom events into pupils' actions and teachers' actions, commenting that it can blind the researcher to the interactive joint construction of classroom reality [p29]. However, I shall illustrate that the outcomes of a small part of the study suggest that more can be learnt about judgements arising in the interactive classroom by such decomposition. I shall describe some aspects of the data relating to one pupil, G, in one of the classes. It was a mixed-ability year 6 middle school class (10 and11 year-olds). The teacher had talked to me of her beliefs about mathematics and learning; briefly, that children learn better if they are doing something *practical*, that *communication* is very important, that giving children the chance to *choose and explore* is important in mathematics, that she has to ask them to do *too much writing* because of the demands of inspectors, that children had *different strengths* in different aspects of mathematics, different subjects and different learning situations.

G[2]

A sample of the observations, with contextual comments

i) G put her hand up to answer questions about shapes. She is not chosen to answer. (G put her hand up for nearly every question in all plenary sessions; when chosen to answer she was usually correct.)

ii) She tries to attract my attention with irrelevant chat. (This happened several times during the first few lessons.)

iii) She draws a square as asked but it is the wrong size. She notices this, rubs it out and says "this is so easy". (She frequently rubbed out spatial work many times; always with noticeable energy. The decision to rub out always appeared to be her own.)

iv) She says that she cannot make a triangle with three pieces she has been given, as required, and makes a rectangle instead.

v) She volunteers to demonstrate angles by standing up and turning. She turns immediately when words like "whole turn, half-turn" are used but hesitates for "right-angle" or quantified angles, turning after others have called out instructions to her. (Volunteering to "show" or "demonstrate" was a feature of the classroom and G always volunteered.)

[1]This latter decision allowed me to come to terms with ethical concerns about covert observation.

[2]The following comments are chronological, extremely abbreviated and merely indicative of the actual data.

vi) She has her hand up for a long time before starting work. (G frequently needed some personal interaction before working from textbooks, and when making choices.)

vii) She chooses to work outside the room, but returns for various reasons 11 times during the lesson.

viii) She does not correctly carry out instructions which involve angle AND direction simultaneously.

ix) After rubbing out some of her own diagrams, G helps the girl next door measure some angles, although her own work is about reflective symmetry. She goes from one to the other a few times. The symmetry work is largely correct, with some errors in counting squares. (I saw her make errors in physical counting several times during the term.)

x) She does some subtraction with decomposition. One of them she redoes as soon as she has finished it. She tells the classroom assistant: "because I took 1 off 7 and made it 8; but it should go down". At the end of twelve examples she checks them on a calculator. (There were other examples where she self-corrected her work, in both spatial and numerical contexts.)

Sample evidence from written work

In her book her number work is correct, including subtraction with decomposition. Rounding and estimating is correct. Work on sorting shapes is correct. Several pieces of spatial and investigative work are unfinished. Work on arranging and ordering colours and digits is unsystematic and incomplete. The symmetry work is not in her book. In a test on shapes G has successfully identified most of the shapes with equal sides. She has answered correctly several tricky questions about shapes with curved sides and no angles. She identifies shapes with right angles when they are in usual orientations, but does not pick out those in unusual positions. She correctly selects several shapes, including the hexagon, which tessellate, but lists octagon as well. She has given no evidence of any symmetry (but see ix). In another test G gives "10" as the answer to "Which is the odd one out: 2,4,6,8,11,12,14?"[3]

Some teacher's comments about G

a) She is new to the school, although others joined it a year before, having been moved there to avoid being bullied elsewhere. She is immature and very needy socially and emotionally (see ii and vii). Based on a number test, has been put in the third of four groups within the class.

b) On seeing my data, the teacher is surprised that G could keep two things in her head (ix, but see vii also); she was surprised that G was even aware of what others are doing.

[3]This was marked as incorrect

c) G had special language help at her previous school (vi).

d) When rank-ordering the focus pupils she places G lowest for each aspect of mathematics apart from data-handling where others who are more untidy are placed below her.

e) G is better at maths than at language-based work. She has a short memory. She is unfocussed and has a "muddling mind"

f) After seeing my data, particularly the evidence of self-checking (iii and ix), the teacher revises her opinion of G "upwards" in the rank-ordering. G does not self-check in other subjects, because they are all language-based so she has no mechanism for checking.

g) G has a desperate need to be accepted (i,ii and v). In number work she has some understanding, but has a poor memory so cannot remember rules.

Analysis of the data

The excerpts above do little more than give a flavour of the whole of the data. I analysed the data by identifying incidents of the seven types of evidence mentioned above in G's observable actions, and in the teacher's comments. Similar work by Goldin and others [1993] depends on the researchers *interpreting* observed data as being indicative of verbal, imagistic, notational systems etc., but these were generated in clinical interview situations and classroom data did not yield easily to these interpretations.

I had decided to include as *behaviour* only actions which might be interpreted to relate to mathematics. For instance, "hands up" may mean the pupil knows the answer, but might instead demonstrate only a naive understanding of the classroom culture; repeated rubbing out of work may mean G knew it was wrong, but might instead demonstrate a desire to take a long time over the work in order to avoid doing the next task. I excluded apparently gratuitous social behaviour. Even so, behavioural observations dominated the data. Apart from "rubbing out" the teacher and I had similar evidence of behaviour, and the teacher was highly aware of the dominant forms of G's behaviour: seeking reassurance and keenness to take part in whole class sessions. All behaviour has to be interpreted.

As *actions* I included observable involvement in practical work, and other physical actions, such as using rulers, moving forwards and backwards through work, looking in several places for information etc. I also included movement from one task to the next, such as choosing, which gives direct relevant evidence of doing maths. The teacher had not been aware of all actions, or repetitions, or the effect actions might have on G's work, or vice versa. I had much more detailed information than the teacher did for this aspect. For instance, G's habit of self-checking appeared through my data[4].

As *oral* evidence I included public contributions to whole class discussion and question-and-answer sessions and anything I overheard between the pupil and teacher or between pupils which was about mathematics, the lesson, the teacher or maths lessons in general. In Gs case she rarely talked to other pupils, and her teacher-pupil interactions were usually about reassurance or understanding text. In this aspect I had the same evidence as the teacher.

For *written* evidence I saw G's book, tests, and other artifacts. In this aspect I had the same evidence as the teacher, although I often knew more about what had been achieved but not written down.

All observable evidence was identifiable as one of these kinds.

Teacher's and researcher's responses to the study

When discussing the data the teacher talked of assessment as an organic process, not resulting in definite judgements but giving a snapshot view of what a pupil can do. However, her comments do not give a snapshot view; they show the creation of a holistic picture of G as a learner of mathematics. Individual incidents of success and failure were incorporated into her existing picture so that certain parts of the picture intensify or fade. G was described as needy and weak at first. Although the teacher moved slightly from that description, "need" still dominated the teacher's descriptions at the end of the term.

The separation of G's story from the rest of the data allowed me to see patterns within her observable behaviour. I have a coherent record based on close observation for several whole lessons, and looking at written work. The teacher has a coherent record based on her knowledge of G's work within the whole of her school life, relative to the teacher's pedagogic aims and the classroom culture. Although we agree on the evidence, and much of my data confirmed the teacher's impressions, we have constructed stories with different emphases.

My collation of action, oral and written evidence gives a picture of a pupil who can do and describe several kinds of calculation, can imagine and describe some spatial and positional properties better than she can draw or make them, can do everything expected of her in textbooks so long as someone helps her interpret what is meant first, works effectively in structured situations, has some difficulty with accuracy in counting, and self-corrects automatically while working. But the teacher's

[4]I am accepting the teacher's interpretation of self-checking as a strength here. It is also possible that it is a manifestation of low confidence or obsessive fear.

judgements about her mathematical capabilities come, not from absolute data like mine, but from comparisons which necessarily involve consideration of what a pupil cannot do, as well as what they can.

By concentrating on a few pupils, rather than an overview of the interactive classroom, I miss the ability to compare G's work to:

- what other teachers have said (G's reasons for coming to the school were supported with records from elsewhere);
- how others adapt to the mathematics classroom culture (G could not choose, explore, or do practical work well);
- expectations in the National Curriculum (G's observable successes are relatively low in NC terms);
- teacher's expectations of the class (G did well at what she was clearly asked to do);
- different outcomes from other pupils in similar learning situations (G was at her weakest in situations which resulted in most differentiation; I had no access to the whole class outcomes).

The teacher missed repeated learning actions, like self-checking, which indicate more about mathematical processes and potential than answers alone can do. Actions which show useful learning habits are not, in general, obvious in a busy classroom unless the pupil has a way to bring them to the teacher's attention. G's way was to show her book to the teacher and ask for frequent reassurance, which, in the context of already-diagnosed emotional need, is easily seen as a manifestation of that need, rather than an attempt to show what G has done, or inform the teacher about what she has thought. G, by being socially inept, and adopting only a naive view of classroom culture (put your hand up all the time) is spending energy on reinforcing negative judgements rather than informing the teacher about her strengths. She is also unable to demonstrate mathematical behaviour valued by the teacher: choosing, exploring, practical work, use of unprompted mathematical knowledge, clear communication about mathematics. In fact, the only evidence of *mathematical learning* which was available to both of us was in written form, with its associated inadequacies. Other mathematics-rich data, G's actions, were not available to the teacher.

Conclusion

Development of criteria for informal assessment [Grouws and Meier,1992] and attention to documentation of unplanned observations [Clarke,1992] are important but not in themselves enough to ensure teacher assessment is valid and reliable.

G's initial dominant features were emotional and social need. Subsequent behaviour, even excluding non-maths-related behaviour, confirms this for the teacher, although other interpretations are possible. Careful observation revealed that G *did* give signals orally, in writing and physically about her mathematical attainment and learning habits, but not all of these would normally be available to the busy teacher.

Tentative analysis of data relating to other pupils gives similar results. This has implications for the fairness of incorporating informal teacher assessment into statutory assessment, and is especially worrying in view of the amount of training and documentary material supporting teacher assessment which has been available in the UK during the last eight years.

Teachers' informal assessments, even with training, interest and good intentions, are still liable to be biassed by a dominance of behaviour which can be interpreted to confirm initial impressions.

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Supporting Elementary Teachers' Exploration of Children's Arithmetical Understanding: A Case for CD-ROM Technology

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This paper reports preliminary findings from a year-long teacher enhancement project conducted with 27 elementary teachers. The aim of the project was to support teachers' reflection about the nature and quality of young children's arithmetical thinking. A secondary aim of the project was to explore the potential ways the teachers might use an interactive CD-ROM package to analyze quicktime movie excerpts of children's clinical interview sessions. The findings reported suggest that the teachers' understanding of children's conceptual development and the opportunities they had to explore children's arithmetical activity via interactive technologies contributed in part to the teachers' ongoing professional development.

Introduction

This paper is a preliminary report of a year-long project conducted with 27 practicing elementary teachers (grades K-3) of mathematics. The primary aim of this project was to provide opportunities for the teachers to explore the nature and quality of young children's arithmetical thinking. A secondary aim of the project was to explore the ways CD-ROM technology might potentially augment the teachers' reflections about children's arithmetical thinking. As part of the project, during a one-week summer seminar, the project teachers viewed and analyzed various quicktime movie excerpts of children's interview sessions contained on the CD-ROM. In the discussion that follows we report initial findings from our ongoing analysis of the data that was collected during the summer institute when the CD-ROM package was implemented.

For many of the teachers, exploring children's arithmetical thinking via interactive technology was a novel experience. Further, many of the teachers had had minimal experiences analyzing children's solution methods. As such, exploring the CD-ROM package provided the teachers opportunities to consider issues related to teaching and learning of mathematics in ways that they had not previously experienced. In this paper, we elaborate the various issues that were raised by the teachers as they reported their analysis during panel discussions at the close of the summer institute. In so doing, we clarify the role interactive technology might play in supporting the teachers' ongoing professional development. To accomplish this we first provide background information about the project. We then address the underlying theoretical positions that informed instructional design decisions when developing the CD-ROM package. Finally, we report preliminary analysis of the teachers' reflections as they explored the CD-ROM package.

Background

The project teachers participated in 4 mini-conference sessions, a one-week summer institute and 8 three-hour class meetings, respectively. These three components comprised a university course for which the teachers received graduate credit. The project team members co-taught the summer

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sessions and the 8 three-hour class sessions.

One of the primary aims of the summer institute was to provide opportunities for the teachers to clarify the nature and quality of young children's arithmetical thinking. As such, the project team led discussions and designed activities that allowed the teachers to consider various ways children conceptualize number for quantities up to 20 and place value. To supplement these activities, the teachers, working in pairs, explored the CD-ROM package to develop mini-case studies of one child or several children. The teachers presented their findings as a culminating experience during the final day of the summer institute.

The CD-ROM package included movie excerpts of six elementary children solving various problem situations adapted from interview tasks developed by Steffe & Cobb (1988) and Cobb (1995). Of the six children included on the CD-ROM, two children were kindergartners (ages 5-6 years old), three children were first-graders (ages 6-7 years old), and one child was a third-grader (9 years old). Further, the movie excerpts depicted a wide range in the nature and quality of the children's current ways of knowing. For instance, one of the children had not yet constructed a concept of number. Another first-grade child was extremely sophisticated and could mentally manipulate two and three-digit numbers flexibility to solve a range of tasks.

Theoretical Considerations

With regard to instructional design, the goal was to develop a CD-ROM package so that the quicktime movie excerpts were easily accessible and the interface itself was user-friendly. Further, the developers hoped to provide the teachers opportunities to view the movie excerpts in many ways. For instance, the teachers might explore several different children's solutions for a particular type of task, say addition of single-digit number sentences. This would require the teacher to first select a particular child to view (see Figure 1) and then select various movie clips related to single-digit addition (see Figure 2). By way of contrast, the teachers might choose to view excerpts of one child's solution methods across a variety of tasks. These are two of the numerous ways the teachers could view and analyze the movie excerpts.

In addition, we anticipated that the teachers would use the information in various ways as they considered the content contained on the CD-ROM. Whereas the content embedded was carefully selected so that certain issues might arise, it was thought that the teachers would actively construct knowledge based on their current understandings of the content as they explored the movie excerpts. Following (Duffy & Jonassen, 1992), on the one hand we realized that the different ways the user (teacher) might explore the software were constrained, in part, by the information embedded in the software. That is, certain issues might emerge more readily for the teachers if certain examples (e.g., instances of children using counting methods to solve tasks) or related issues were included in the CD-ROM package. Further, how the content was formatted, that is the actual interface, whether or not the interface was openended, etc., would constrain how and in what ways the teachers could explore the information. On the other hand, following a Neo-Piagetian position (Cobb, Yackel & Wood, 1992; von Glasersfeld, 1995), the developers took the position that the teachers would construct their own meanings as they explored the CD-ROM package. That is, as the teachers explored the information embedded in the CD-ROM package, they would construct and reflect on issues that might later become objects for further reflection. In this sense, the CD-ROM environment was constrained by the individuals' current interests, experiences, and understandings of the content. In sum, we postulated that there was an interdependence between the information embedded in the CD-ROM package as it was designed by the developers and the actual ways in which the teachers used the package. The apparent contradiction of imposing "structure" on how the teachers would move through the CD-ROM environment was transcended by acknowledging that the various ways they explored issues were constrained and enabled by their previous experiences. With regard to our project goals, we hoped that the project teachers would access and consider information that was specific and/or relevant to their own teaching practice.

Methodological Issues

Data collected during the project included pre- and posttests, field notes of observations conducted during each of the 17 sessions, the teachers' portfolios (containing weekly reflections during the course of the project, written assignments and other artifacts the teachers chose to include), and videotapes of each of the sessions. Two cameras were used to videotape whole-class and small group discussions of each session. Videotape segments were then transcribed for further analysis.

Qualitative procedures that fit with the constant comparison method (Glaser & Strauss, 1967) were used to analyze the transcriptions of the teachers' presentations. The transcripts were coded to identify issues that emerged during the presentations. From this process, more general categories were identified. Using these generalized categories, the researchers then developed working hypotheses related to the teachers' interpretations and the potential ways they used CD-ROM package to clarify their interpretations. Each of these is summarized below.

Preliminary Findings

Interpreting Children's Arithmetical Understanding. During the presentations, the teachers provided psychological analyses to support claims about the children's arithmetical thinking. These claims varied from pair to pair and were both descriptive and analytical with regard to the quality of the children's arithmetical interpretations. Two of the prevailing constructs the teachers attempted to tease out related to whether or not a child had developed a concept of number and/or some understanding of "tenness." Further, many of the teachers were intrigued by one of the first-grade children's interview segments. This child randomly applied doubles strategies incorrectly to several of the tasks that were posed during the interview session. As such, several of the teachers offered conjectures about the child's beliefs about what it meant to know and do mathematics. In addition, the teachers speculated as to the kind of experiences this child encountered in her mathematics classroom. In particular, some of the teachers gave social explanations for the difficulties that the child experienced (e.g., the child attempted to comply with the norms for doing mathematics in her classroom) during the interview session. In a crude sense, the teachers attempted to coordinate the child's individual interpretations with the social situation in which those interpretations evolved during classroom instruction. In retrospect, from the project team's viewpoint, including this first-grade child's interview was critical and provided unforseen opportunities for the teachers to explore a range of issues related to establishing norms for knowing and doing mathematics.

As the teachers presented their findings, a second related issue emerged as to the viability of their analyses. The teachers struggled with whether or not they had sufficient information to make certain claims about the children. That is, they did not feel qualified to make inferences about the children's interpretations because they were not the child's teacher and/or they did not have enough background information about the child. These concerns were raised by several of the groups as they presented their findings to the class. This issue had not been anticipated by the project team and yet, as it surfaced, it was interesting how the different groups handled the conflicting goals of analyzing the children's arithmetical activity and remaining true to their own beliefs about what constituted viable inferences.

At least three different means of handling this dichotomy surfaced as they teachers presented their analyses. One of the groups stated that, initially, they did not "know how they would know what [the children] were thinking." They then explained that as they attempted to clarify the quality of the children's thinking, their conjectures became claims that they could make with a great deal of confidence. By way of contrast, another pair of teachers did not develop the same confidence about the claims they made. They explicitly stated that they struggled with making assumptions. "It was uncomfortable." They argued that they needed more information about the child to support the claims that they made. At the same time, this pair was able to use the child's interpretations across tasks to *adequately* clarify the quality of the child's concept of number. Further, in response to this pair's difficulty with making claims, another pair of teachers indicated that not knowing the child's background freed them to consider a range of possible reasons for the child's interpretations.

These varied ways of dealing with the apparent contradictions the teachers encountered as they analyzed the movie excerpts were quite informative. The teachers brought their beliefs and experiences to bear as they generated hypotheses that were unique and distinctive for the purposes at hand. Yet in many instances, they developed claims that were quite compatible with those offered by their fellow classmates. Further, the claims they made fit with constructs found in the current literature on children's development of number.

As a further note, as the teachers presented their findings, the most poignant moment came when one of the teachers recast this struggle anew at the close of the summer institute. It so happened that the third-grade child interviewed was a student in one of the project teachers' classrooms. This teacher and her partner chose to analyze the third-grade child's mathematical activity. Referring back to the analysis given by the teacher of the third-grade child and her partner, a teacher commented that she was amazed that the analysis given by the child's teacher who had the background information on the child fit with the analysis she and her partners generated. The teacher then proceeded to salute the project team to acknowledge her support of the project team's decision to engage the teachers in analyzing the children's interview sessions. This instance further pointed to the significance of the issues that the teachers addressed as they explored the CD-ROM package.

<u>The Role of Technology</u>. We were particularly interested in the ways the teachers would use the CD-ROM package to analyze the children's mathematical activity. We hoped that the teachers

would have opportunities to revisit issues multiple times to clarify their own thinking as they formulated conjectures about the children's interpretations. Interestingly, when the teachers presented their analyses, some of the teachers explicitly referred to the process that they went through to develop claims about the children's interpretations. This process included developing conjectures and counter conjectures to refine claims that they made. Further, this process was ongoing and resulted from revisiting the movie excerpts several times over a four day period. For instance, one pair explained that the initial conjectures that they made as they analyzed two kindergarten children's understanding of number changed during the course of the week. From the outset, they considered the two children's understanding of number to be similar. The day prior to presenting their findings, they realized that the two children's interpretations were quite different in both the nature and the quality of their thinking. The teachers indicated, "As we got farther into looking at the both [children], we saw that they were quite different, even though they may have answered the question the same, the thinking behind it was so different..."

As a further point, the teachers continued to refine their conjectures during the presentations. For instance, as a subsequent group of teachers were presenting their analysis to the class, one of the teachers indicated that she changed her initial claims about a particular child's interpretations as a consequence of seeing one of the movie clips during the subsequent group's presentation. We suspect that this process would continue if the teachers had other opportunities to explore the CD-ROM package over time.

Discussion

These findings have both pragmatic and theoretical implications for incorporating interactive technologies to support teacher professional development. Pragmatically, with regard to instructional design, the teachers could easily access the movie excerpts in several ways. This provided some flexibility in what the teachers decided to explore as they developed their mini-case studies. The software also provided the teachers the opportunity to revisit the various clips to develop and refine conjectures. This was a critical component that contributed, in part, to the quality of the teachers' analyses. In addition to being able to randomly access the movie clips, we suggest that it was absolutely crucial for the teachers to view excerpts of children engaged in arithmetical problem solving in order to conceptualize the potential ways the children could solve arithmetical problems. Whereas the CD-ROM package afforded the teachers random access to children's interview clips, as the teachers explored the various clips, they had opportunities to conceptualize and possibly reconceptualize their understanding what it means for children to know and do mathematics. As such, developing legitimate activities for teachers to construct a deeper understanding of what it means to know and do mathematics is not unlike developing curriculum materials for children.

Theoretically, these findings suggest there was a interdependence between how the teachers used the interactive technology package and the teachers' conceptions about teaching and learning. On the one hand, as the teachers explored the CD-ROM they developed conjectures about the nature and quality of the children's current ways of knowing. The relationships that the teachers made were also the result of their previous experiences and understanding of what meanings children construct.

In this sense, the claims the teachers made were the results of interpreting the contents against the background of their own experiences. Engaging in this kind of activity thus provided the teachers opportunities to conceptualize or reconceptualize their understanding of children's arithmetical development. For some of the teachers, these refinements were made possible as they continued to visit and revisit the movie clips over and over. As such the teachers' conceptualizations were dependent on the ways they explored the CD-ROM. On the other hand, as a consequence of the teachers' understanding of children's arithmetical thinking, they could explore the CD-ROM differently to develop generalities about the children's understanding. The teachers could access excerpts to clarify the claims that they made. As such, the ways in which the CD-ROM could be explored were dependent, in part, on the teachers' current ways of knowing. More generally, how the teachers explored the CD-ROM and the interactive technological environment constructed by the teachers were interrelated; neither existed without the other.

The results reported here are preliminary and point to one of many ways technology might be integrated in inservice teacher development. The researchers hope to further clarify these and possibly other issues as they continue their ongoing analysis. Upon further analysis, attempts will be made to triangulate these findings with the other data collected during the project. In so doing, the researchers hope to determine the processes by which the teachers developed the claims that they made about the children's interpretations.

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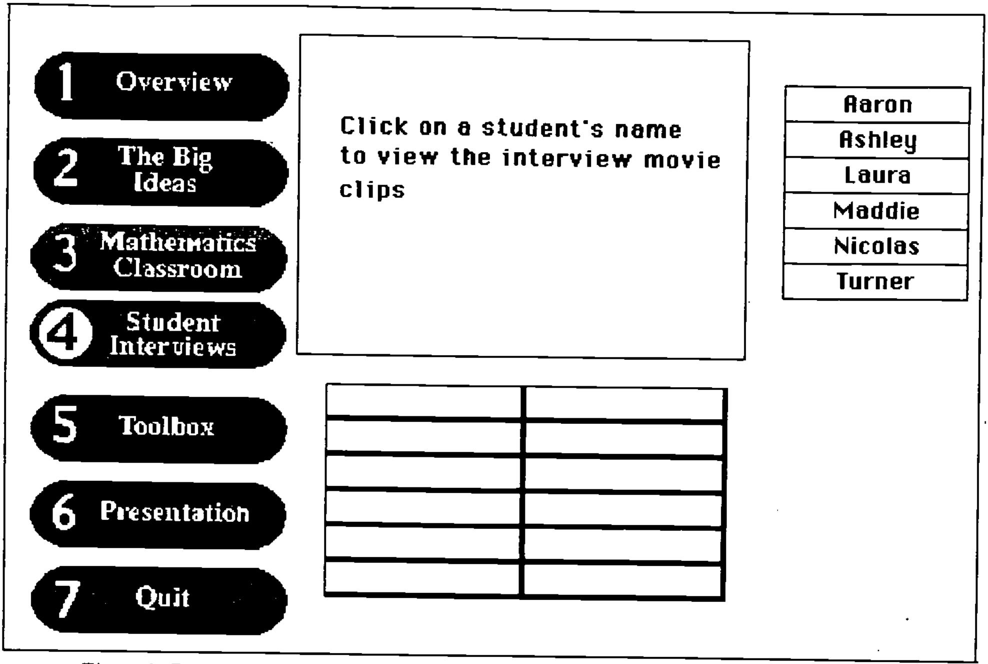

Figure 1. By selecting on Student Interviews, a prompt appears in the movie window for the user to select a particular child to view.

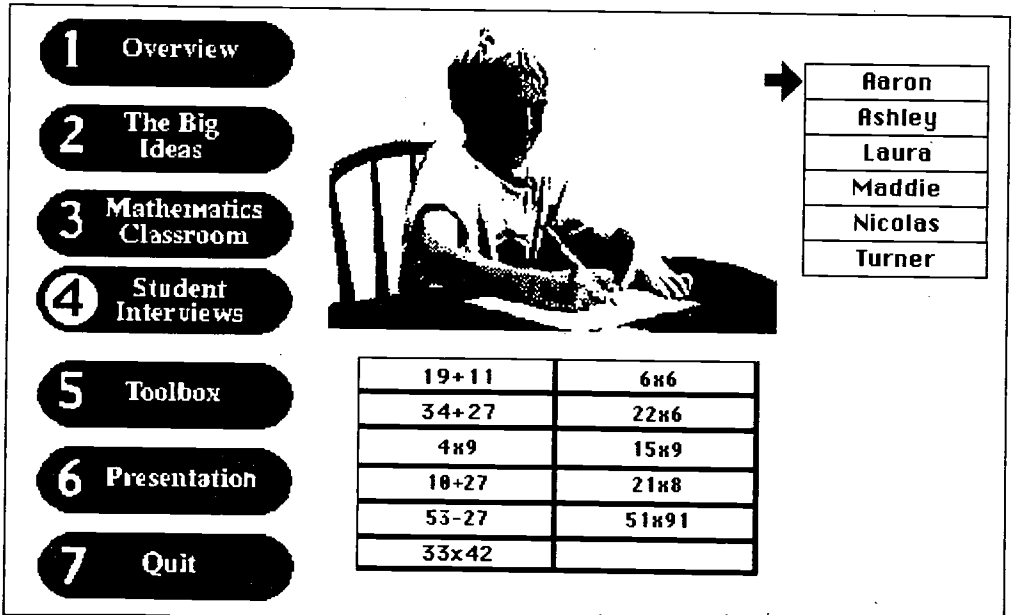

Figure 2. By selecting on a particular child's name, the subsequent window displays the various tasks that are available to view.

Numbers versus letters in algebraic manipulation : which is more difficult?

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ABSTRACT. *The purpose of this paper is to describe some of the situations where students have greater difficulty in applying the same procedural knowledge to manipulate similar algebraic expressions involving numbers, and try to explain why there are such differences. Data were collected through testing, follow-up interviews and case studies in an average standard secondary in Hong Kong.*

Introduction

Elementary algebra could be regarded as generalized arithmetic with the use of letters to represent numbers its principal characteristic. Inadequate arithmetic knowledge will lead to difficulty in learning algebra. On the other hand, the discontinuity between arithmetic and algebra, such as the difference between the meaning of concatenation, could be one of the major sources of difficulties (Kieran, 1990). Thus, in discussing the difficulties in learning algebra, students' knowledge of operations on letters cannot be isolated from their knowledge of operations on numbers, and numbers and letters might be perceived as mathematical objects in different but related reference fields (Kaput, 1987).

Letters are more abstract entities than numbers. It was found in the CSMS study that many children were not able to interpret letters as generalized numbers or variables, and some students are unable to treat letters as numbers, but as some concrete objects (Booth, 1988). Collis (1975) shows that in tests about operations, the nature of the elements, namely small numbers, big numbers and letters, in an item can have a marked effect on facility. It is found that younger children can only deal with items with small numbers. Collis argues that the difficulties stem from the extent to which the elements lack meaning for the child.

Although letters are not used in arithmetic, numbers are very common in algebraic expressions, for instances, as the constant term, as the coefficients of the variables, or as one of the data input elements. Sometimes the procedural knowledge involved is different due to the presence of numbers versus letters. For instance, the brackets in the following expressions are removed by different procedures :

$$a(b+c) = ab + ac; \quad a(3+4) = a(7) = 7a$$

But in some cases, the procedure is identical, for instance :

$$a^2 \cdot a^3 = a^{2+3} = a^5; \quad 4^2 \cdot 4^3 = 4^{2+3} = 4^5$$

If we accept the view that information is stored in our long-term memory in some orderly way, it will be interesting to study how students' knowledge about letters is connected with their knowledge about numbers. Hiebert and Carpenter (1991) have proposed to use the metaphors of vertical hierarchies as well as a spider's web for the structure of internal networks of mathematical knowledge. Using these metaphors, we can imagine that the connections among information about letters are at a higher level of the vertical hierarchy of the internal networks of mathematical knowledge than those about numbers. Successful learning of algebra involves building connection within as well as across these two layers of networks.

In some educational systems such as that of Hong Kong, students learn to transform algebraic expressions according to some standard procedures in early age[1]. They may fail to do the transformation correctly when the familiar letters are replaced by numbers. It is the assumption of this study that sometimes students have greater difficulty in dealing with numbers than with letters in manipulating algebraic expressions.

The purpose of this paper is to describe some of the situations where students have greater difficulty in applying the same procedural knowledge to manipulate similar algebraic expressions involving numbers, and try to explain why there are such differences. It is hoped that the data in this report will lead to better understanding of the relationship between students' knowledge of numbers and letters.

Methods of enquiry

The information of this report came from two sets of data[2]. *The first* is from the testing and selected follow-up interviews of four classes of Hong Kong secondary school students. The test is to examine whether students execute the same procedures in items involving numbers and parallel items with letters instead. The items are restricted to the multiplication items (e.g. a·a·a·ab·a) and the manipulation of exponents. The procedures for manipulating these types of expressions are identical or similar, no matter there are numbers or letters instead. The correct percentages are compared and the errors are categorized. Why students executed different procedures for the parallel items are then explored in the follow-up interviews. *The second* is two case studies based on clinical interviews to observe how well the subjects apply the rules they learn in school to manipulate algebraic expressions with different elements. The clinical interview procedure allows the researcher to get in touch with the learner's thinking processes in doing mathematics.

[1] Hong Kong students start learning solving equations in Grade 4 and simplifying expressions in Grade 7.

[2] The two sets of data were collected by the writer for another study about students' cognitive obstacles in learning the laws of indices. Part of the data are about students' performance in manipulating numbers versus letters and are reported here separately.

Subjects and procedures

The study was conducted in an average standard secondary school in Hong Kong. The 156 Secondary Four[3] students were tested at the beginning of the academic year and 7 students were selected for follow-up interviews. Besides, 2 Secondary Three students, Patrick and Fiona, were involved in the case studies. Each of them was interviewed individually for six times. Four of the interviews were conducted at the period they were learning how to manipulate exponents in school.

Results and analysis

In the set of test items that could be simplified by employing the rule $a^m \times a^n = a^{m+n}$, there are significant differences[4] in the item facilities, that is, correct percentages, for the pairs of items :

Pairs of items for comparison			
Exponents with numbers versus letters		Bases with numbers versus letters	
Item	Item facility	Item	Item facility
$2^5 \cdot 2^7 =$	78.8%	$2^a \cdot 2^b =$	60.9%
$2^a \cdot 2^b =$	60.9%	$m^a \cdot m^b =$	79.5%
$x^5 \cdot x^2 =$	94.9%	$2^5 \cdot 2^7 =$	78.8%
$m^a \cdot m^b =$	79.5%	$x^5 \cdot x^2 =$	94.9%
$53^4 \cdot 53^3 =$	88.5%	$53^4 \cdot 53^3 =$	88.5%
$2^a \cdot 2^b =$	60.9%	$x^5 \cdot x^2 =$	94.9%

The students have poorer results when the exponents are letters in the parallel items. On the other hand, they have better results when the bases are letters in the parallel items. It is found from the error analysis that many students multiply the indices together in expressions with algebraic exponents [for instance, 18% of them got $m^a \cdot m^b = m^{ab}$] but multiply the bases together in expressions when the bases are small numbers [for instance, 10% of them got $2^5 \cdot 2^7 = 4^{12}$]. The following episode from one of the follow-up interviews may help explain the results in these item :

[3] Secondary Four in Hong Kong is equivalent to Grade 10 in some countries.

[4] The differences are significant to 0.01 level, except for the last pair of items in the second column.

[The student was asked to simplify $3^x \cdot 3^y$ and he wrote $3^x \cdot 3^y = 9^{x+y}$.]

I : Why?

Student$_3$: 3 times 3 gives 9. x and y could not be multiplied together. y should be added to x.

I : Why couldn't we multiply x and y?

S$_3$: The book says so.

I : Do you mean this is what you had learnt from your textbook?

S$_3$: Yes.

[He was then asked to simplify $x^5 \cdot x^3$ and he wrote $x^5 \cdot x^3 = x^8$.]

I : Why?

S$_3$: Both are x. They are equal and need not multiply. Then 3 plus 5 gives 8.

I : You say that both are equal and need not multiply. How about that one [*pointing at* $3^x \cdot 3^y$]? Why do you multiply them ?

S$_3$: [*Silent for almost 10 seconds*] .... I don't know.

I : Is it correct ?

S$_3$: No. It should be 3^{x+y}.

I : Why ?

S$_3$: Because the numbers need not be multiplied.

The above student knew the correct procedure for this type of problems. But he executed a different procedure when the bases are small numbers. Another student [student$_4$] gave similar responses in the interview : "$3^a \cdot 3^b = 9^{a+b}$" and "$53^4 \cdot 53^3 = 53^7$". He said that when he saw the symbols "$3 \cdot 3$", he wrote 9 almost automatically.

In the above pairs of items, the results are less consistent :

Items with letters only	Item facility	Parallel items involving numbers	Item facility
m·n·n·m·m =	90.4%	y·y·8·y =	94.9%
a·a·ab·a =	74.4%	k·k·k·3k·k =	59.6%
$x^n + x^n$ =	52.6%	$4^7 + 4^7$ =	21.1%

It is difficult to explain the differences. It is found that some students got the correct answer $x^n + x^n = 2x^n$ by the faulty procedure of adding x + x without adding the bracket as they should have. It seems that the difficulty caused by the presence of numbers depends on the context of the item.

In the case studies, there are also some interesting findings. After Patrick had learnt the law $(ab)^n = a^nb^n$ in school, he was asked to simplify the following expressions and he made very consistent mistakes : $(2a^m)^n = 2a^{mn}$, $(2x^3)^4 = 2x^{12}$, $(2x)^4 = 2x^4$ and $(7k^2)^3 = 7k^6$. However, when it came to the question below, he answered correctly without hesitation :

$$(hk)^n = h^n \cdot k^n$$

After that, he was able to do those previous questions correctly. He explained why he made the mistake , $(2x)^4 = 2x^4$: "*I think that 2x is a number. So I do not use bracket. It is just like 3 to the power four [He writes : $(3)^4 = 3^4$]*". It seems that the presence of "pure letters" in $(hk)^n$ reminded him of the formula $(ab)^n = a^nb^n$ he learnt in school.

Another student Fiona made different mistakes in related to the above formula. She was able to transform $(2x^3)^4$, $(2x)^5$ and $(my)^2$, but made the mistakes $(3x)^2 = 9x$, $(6x)^2 = 36x$, and $(7t)^2 = 49t$. Later it was found that she was able to simplify $ax \cdot bx$ and $ma \cdot na$ but made the mistakes $3a \cdot 3a = 9a$ *and* $4k \cdot 3k = 12k$. Since she interpreted expressions in the form of $(ab)^2$ as "multiplying itself", it was natural that she got $(3x)^2$ $[=3x \cdot 3x] = 9x$. That is why she made mistakes only when the exponent is 2 as well as when there is a number inside the bracket. Her mistake of $3a \cdot 3a = 9a$ was probably related to her interpretation of 3a as a+a+a and such interpretation is quite unlikely in expressions such as $ax \cdot bx$ and $ma \cdot na$.

Conclusion

Data in this report suggest that in some situations students are more likely to make mistakes in expressions involving numbers due to different reasons.

When students are able to manipulate algebraic expressions with letters but not with numbers, we may say that they fail to activate the appropriate information items in their mind. Some cognitive scientists have pointed out the importance of the degree of strength between the connections of information items in learning situations. They argue that learning is not simply "adding nodes to the network", but is also the adjustment of link weights between the nodes so that "activity flows into the appropriate nodes at appropriate times " (Partridge and Paap, 1988, P.140).

In conclusion, whether algebraic expressions involving numbers are more difficult depends on the types of expressions as well as the algebraic knowledge of individual students. Although it is not yet possible to draw conclusions about the relationship between students' knowledge of numbers and letters, it is clear that the relationship is too complicated to be described by any simple models.

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"YOU HAVE TO PROVE US WRONG": PROOF AT THE ELEMENTARY SCHOOL LEVEL

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*In solving a variant of the 'chessboard' task, a team of fifth grade elementary students are convinced **that** their pattern works. They use what they know of the pattern to refute an argument by peers. There is evidence of conjecture, refutation and generalization, and aspects of proving. It is the teacher's contention that in order for an argument to be considered a proof, the students need not only **convince**, but also to **explain**. Thus teacher involvement and personal inquiry are in this instance necessary to provoke thought concerning **why** the pattern works as it does .*

In my current research I am endeavoring to see how the learning of mathematics is interactively accomplished within my fifth grade classroom. I consider what individual children and the teacher contribute to this collective activity. In this paper I will be concerned specifically with proof in this context. I will show how in work with one task which was not at first assigned with any intention of attending to proof, I found a number of the elements identified by Hanna (1995), namely "assumption, conjecture, example, counterexample, refutation and generalization" (1995, p. 48). I will focus on three of these aspects -- conjecture, refutation, and generalization -- and show that aspects of proving arose spontaneously during the activities.

There is little in the research literature on proving in relation to young children, with the exception of the seminal work being done by the Maher team at Rutgers (for example, Maher & Martino, 1996) and Lampert (eg., 1990), and preliminary work by Jones (1994). Maher, and Lampert propose that involvement in inductive and deductive reasoning which leads to the construction of proofs should begin at the elementary school level. Gardiner suggests that "the 'form' and 'language' of the reasoning changes as learners grow older, but <u>not</u> the requirement that mathematical reasoning be: (a) general (that is, valid for all possible examples in the universe under consideration), and (b) completely convincing" (1992, p. 4). Gardiner also highlights the essential ingredient, that of the notion of infinity, saying: "To tame infinity we need proof" (1992, p. 10). De Villiers has spoken of the "reasoning which young children exhibit in situations which are real and **meaningful** to them" (1991, p. 254, boldface in the original.). I take as my starting point the definition of 'proof as a convincing argument' (Hanna, Balacheff, & Pimm, 1991, p. xxxiii). I will show how the children at times exhibit careful reasoning, as they build their arguments and attempt to convince. With reference to Mason's (1982) statement that when you prove, first you convince yourself, then convince a friend, and then convince an enemy, I will show instances within the children's interaction of 'convince yourself', and 'convince a friend' that you are correct. My goal as a teacher is nurturing a higher level of agency and autonomy for learners, and the ways in which the students ask their own questions, direct their own inquiry and engage in sustained conversation about generalizations and proving is the prime focus of the paper. The children were convinced ***that*** their pattern worked. It is my contention that in order for an argument to be considered a proof, the students have to not only *convince*, but also to *explain*. I will indicate, briefly, ways in which my involvement and personal inquiry were in this instance necessary to provoke thought concerning ***why*** the pattern works as it does .

The school community and classroom setting, and assigned tasks

St. George's is a private, non-denominational school, with a middle class population of mixed ethnic, religious, and linguistic backgrounds; the population is predominantly English-speaking. The homeroom class size in the 1995-1996 year was 26; the work, however, is always done in half-groups (13 children in each group) of heterogeneous ability. Problem-solving is at the core of the mathematics curriculum in this classroom. The school and classroom learning site is a community of practice which Richards (1991) has called *inquiry math*; it is one in which the children are expected to publicly express their thinking, and engage in mathematical practice characterized by conjecture, argument, and justification (Cobb, Wood, & Yackel, 1993, p. 98). Of interest here is the intersection between the last-mentioned items, and proof.

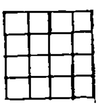
Mathematics class periods are 45 minutes, and twice a week are extended to 90 minutes. In addition to the in-class problem-solving sessions, each week the children also work on one challenging problem at home. They are expected to record their work and reflect on their strategies in a Math Log which serves as the initial basis of their group discussions in class. In class much of the session is conducted by the children as they discuss the problem first with a partner, then in a group of four or five, and finally with the entire group of thirteen students.

The children are videotaped throughout the school year on a rotating basis as they work in their groups. In addition to the videotape records, data sources include focused observations, student artifacts (math logs), teacher-composed questions eliciting opinions (written responses), and retrospective interviews.

The mathematical context of the problem/discussion

The COUNT THE SQUARES task is a variant of the 'Chessboard' problem (see Mason et al, 1982; Anderson, 1996). The work was assigned as follows:

Task #1 (April 29, 1996) :



Find all the squares .

Can you prove that you have found them all?

After the conclusion of the discussion on Task #1 (discussion was held May 1), **Task #2** was given (**May 1**):

> **What if . . . this were a 5 by 5 square? How many squares would you have?**

The students wrote their explanation for their answer to Task #2 in their Math Logs at home (as with Task #1), and then in class I asked them discuss their answers to the 5 by 5 grid, to think about and discuss the questions below, and to go as far in their exploration as they were comfortable. The following extensions were posed:

> **What if this were a 10 by 10 square? How many squares would there be?**
>
> **What if this were a 60 by 60 square? How many squares would there be?**

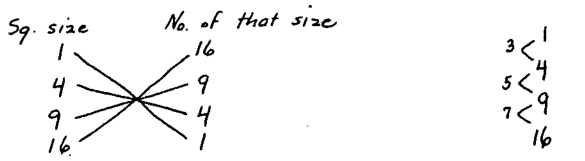
I did **not** assign the tasks with a view to provoking discussion on proof. However, the children's interaction in that 1995-1996 year, and the focus on proof at a number of conferences (Canadian Math Ed Study Group, May 1996; PME 20 and ICME 8, July 1996) aroused my interest in proof, which in turn led to this analysis.

Children's perceptions of the term 'prove'

The word 'proof' has a wide range of meanings, from everyday usage to the idea of formal rigorous mathematical proof. Amongst the children within the mathematics classroom there are as well subtle and important differences in how they interpret the term 'prove'. You will note in Task #1 the request: '**Prove that you have found all the squares.**' In conversation with me about the ways in which the children responded, Tommy Dreyfus suggested that there were two kinds of *assertions* in the data (personal communication, Dec. 10, 1996). The first is in regard to a single case; a number of students prove by *checking* that their answer for the specific question is correct. For Task #1, the correct answer is 30 squares. *Point finale*--no need for proof. Others are seen to refer to a pattern which has been discerned and which a number of the students contend will continue forever. This second kind of assertion is a general statement, and thus in principle requires proof.

'Convince yourself'

Will's pattern, his conjectures and his testing of them drove much of the work in his team of three and in the group of five. At the very outset, when the task was first assigned, Will seemed to have an intuition of a pattern, as he is seen to look and ponder and suddenly say: "I know what to do, I know exactly what to do"; he then proceeds to write the 'criss-cross' pattern on the Math Log page (Figure 1). When meeting with his partners two days later, Will says to Lew and to Ross, his partners that day: "I was pretty sure there would be a pattern, so I was keeping my eyes open and I found one." He then says he hasn't tested it on a different size square yet, but "the chances are if it works for those it works for others." Lew, in response to seeing Will's pattern on the page, is very impressed. Will checks out the answer for the next size square (5×5), and finds that his conjecture is correct, and says: "So, it's basically the same. I never realized it [i.e., the pattern] would be so helpful." Will checks his work up to the 5×5, and then assumes that the pattern which has worked for up to 5 will continue to work in the same way; he is subsequently seen to act in accordance with this assumption.



criss-cross **differences**

Figure 1: **Will's two patterns**

At one point in the first day's discussion, Will notices a 'pattern of differences' (Figure 1), hence a second pattern. There were other approaches voiced by the group members such as for example Gord's sum of squares, and Ross's sum of the little squares.

It is important to emphasize that the 'criss-cross' pattern is one I rarely see; over the past 3 years, only two children (of the 75 students) have discerned it, realized that it might be significant, and then pursued it (Will, and Alan). Will's criss-cross pattern is the one I assumed would be focal. It lends itself to the summing of the squares. However, Will himself is seen to spend much time using the pattern of differences to arrive at the numbers which will be the addends for the answer to the 60×60 task. He is adding to get the next number, and adding to get the total. Will does not at that point seem to be attending to the 'squares' component. It is Gord who from time to time refers to the squares. It is Gord and Lew who see that the solution for the 60 by 60 could be arrived at by multiplying (deriving the squares) and then adding: 60×60 + 59×59 + 58×58 and so on. They declare with excitement: "We're a genius!" The "we're a genius" speaks both to thrill of discovery and to their acknowledgement of the fruitfulness of the collaborative endeavor.

'Convince a friend': An argument and three counter-arguments

In order to 'convince a friend', Will, Lew and Gord proceed by saying, and showing, that they have a pattern and that it works. They use what they know of the pattern to refute the proposal presented by another pair, Ross and Ted. My caution to the reader is that the summary which follows makes it all sound far too straightforward; the unfolding of the argumentation, the challenges posed therein by the 5 individuals, the discoveries and reconfirmations which occur during the 15-minute interaction are not represented here.

We take up the May 3rd discussion at the point at which Will, Lew, and Gord (the group of 3) meet with Ross and Ted (group of 2), and form the group of 5.

Although their approaches to arriving at the answers have varied, the five peers have up to this point all been in agreement with the answers for the tasks up to the 10 by10. The answer for the number of squares in a 4 by 4 is 30, in a 5 by 5 is 55, and in a 10 by 10 is 385. It is during the discussion of the answer to the 60 by 60 question that there is a disagreement, and one group is challenged by the other to refute: "You have to prove us wrong."

As they begin their group of five discussion they agree that the answer for the 10 by 10 is 385 squares. The group of three states that they-- Will, Lew and Gord -- had not yet completed getting the answer for the 60 by 60. Ross and Ted feel that they have the solution for the 60 by 60, which is to take the 385 (the answer for the 10 by 10 square) and multiply it by 6 to get the number of squares for a 60 by 60; the resulting answer is 2310. [Of interest is to note that 10 of the 26 children in the class used this strategy.] Will and Lew are very sure that Ross and Ted are wrong:

L: I'll make you a bet.

W: I'll make you a bet.

L: I'll bet you anything in the world.

R: I'm not betting. You have to prove us wrong.

Will and Lew and Gord then proceed to use three counter-arguments, all based upon their generalizations, to refute what Ross and Ted have said. The first argument (Will, Lew) is that 3600 (the result of multiplying 60 by 60) is already bigger than 2310. Lew adds: "And that's just the little squares." One needs only one counter-example to disprove, but Ross and Ted are not seen at that point to concede. For the second argument Lew and Will create a generalization. They propose that if what Ross and Ted are saying is true, then it should work in general; and they then proceed to give a counter-example. Lew and Will consider the answers for the 4 by 4, and then the 8 by 8 square. They use the information to show that the answer for the 8 by 8 -- 204 squares-- is not simply double of the number of squares in the 4 by 4 -- 30 squares. The point they make is that just as one cannot multiply by 2 to get the answer, one cannot multiply 385 by 6 to get the answer for a 60 by 60. The third and last argument of the three arguments put forward is given by Will, and supported by Gord; he points out that there is a pattern at work, and that doing a move such as taking 385 and multiplying that number times 6 means that one is not allowing the pattern to continue to grow, but that rather one is 'restarting' the whole pattern.

In their counter-arguments to Ross and Ted, Will and Lew (with Gord supporting Will and Lew) are in the end successful in refuting, and in convincing. Ted is heard to say: "Yeah, they're-, you guys are right, I go along with you guys." It occurs after the second argument has been presented, but it is the idea of the 3600 (the first argument) to which Ted directs his attention, and which he now assets is correct. Ross does not voice his agreement explicitly, as did Ted, but is later seen to support

Lew and Will when, in the larger group, Lew and Will work to refute another team's presentation of the 385 times 6 strategy.

The findings seem to suggest that the students who succeed in convincing their peers (Will, Lew and Gord, in this instance, and others) are those whose justifications are based upon the generalizations. Will, Lew and Gord assert that the pattern must be adhered to. Will insists that it will continue forever: "You can use the pattern to calculate any number, even a googol times a googol." What Will, Lew and Gord do and say reflects their certainty that the pattern is correct in all cases. There is evidence of conviction prior to proving; their arguments are based upon their conviction *that* their pattern works in all instances. The pattern of summing the squares, i. e., $1^2 + 2^2 + 3^2 + \ldots$, is indeed correct. To be taken up next with the children, then, is proving in the sense of *explaining* the mathematical basis of the generalizations.

The teacher's role in pushing to explain *why it works*

One element which was not pursued by the students was that of explaining why the pattern works as it does. In investigating other patterns such an interest was present. (as in Zack, 1995; Graves & Zack, 1996). The absence might in part be due to the challenge inherent in this task. I myself was absorbed by the questions the task evoked. In follow-up interview and presentation sessions conducted seven months after the assignment was done, in December 1996, ten of the students re-immersed themselves in the task. What I was able to do then was to suggest that questions of *why* had to be addressed, and that I myself had many queries. I shared with them my own questions and my own search for illumination (Note 1). Thus, I provided a model of inquirer, of teacher as student (Freire). I built upon the elements which they had discovered and introduced to me, and showed how it related to some of what I had learned from other mathematics educators who had served as intermediaries for me: Bill Nevin, Sept. 24, 1996, and David Reid, Oct. 13, 1996 (personal communications) and John Mason (1982, pp. 18-21). I presented demonstrations of the explanations I had encountered, and also provided them with a "non-obvious expression" which I had found in a work by Anderson (1996, p. 35). I told the children that I myself did not understand how Anderson had arrived at that expression, nor why it worked. I did indicate to the children that there was a way to derive the formula, but that neither I nor they had the tools to do so. The discussion which ensued allowed a preliminary investigation of the children's criteria for proof.

All agreed that the pattern of summing the squares was intensely time-consuming. During the May 3 class time, one child had spent much time with his partner seeking a formula (Alan with Keiichi), with no success. (Please note that in the past he had had great success deriving algebraic expressions, as had others in the group as well). When shown the Anderson formula, $n(n+1)(2n + 1)/6$, the ten students interviewed felt that the Anderson formula would be useful, and economical. However, perhaps due to the emphasis in our classroom work that we have put upon explaining oneself, in their emergent definitions of what they felt proof ought to be, the students emphasized that their criteria for proof included: (a) a need for evidence, (b) that the proof must make sense, and (c) that the person presenting must say why it works. It was in written responses to the prompt "What do you think of Johnston Anderson's rule?" that the children expressed their positions. Ross, for example, stated that Johnston Anderson's rule was "brilliant, but he should explain why it works." Lew commented: "I think that if the Johnston rule had evidence, if Johnston himself explained why it worked it would be more convincing." Rina felt that Anderson's expression was "a great way to figure out the problem but it doesn't make sense . . . I think a mathematical proof is when you say why it works and if it works for everything show why." Only one child, Sanjay, did not voice a need for further elaboration, saying of the rule: "It's sort of like pi, it just works." Thus, despite finding Anderson's formula expedient, the majority of the students stressed that one ought to know why it worked as it did. In revisiting the problem, one of my objectives was to make the students aware of the importance of seeking to explain the mathematical structure. Hanna has asserted that in education proofs that explain should be favoured over those that merely prove (1995, p.48); the children as well are seen to seek proofs which explain.The criteria the children stated represent a healthy 'habit of mind' in our push to have learners think meaningfully about proof.

Note 1: My questions were: Why does it go from 9 2by2's in the 4×4 to 16 2by2's in the 5×5, etc.? Are there only 25 little squares added on when one moves from the 4×4 to the 5×5? Nevin showed me that it was 25 squares of different sizes, while Reid showed me that one could consider that only 25 1by1 squares appear which have never been there before; all the rest are expansions. Mason (1982, p.18-21) explained to me the derivation of the general formula $1^2 + 2^2 + 3^2 + \ldots$, relating it to the number of lines touched by the squares top to bottom, and side to side.

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CONSTRUCTING KNOWLEDGE BY CONSTRUCTING EXAMPLES FOR MATHEMATICAL CONCEPTS

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ABSTRACT

In this paper we discuss tasks in which students are asked to give an example of a mathematical entity with certain properties. We analyze students approaches and difficulties when facing this kind of tasks. We suggest that problems of constructing examples may help students in meaningful mental construction of mathematical objects.

Consider the following tasks:

(1) Give an example for a 6-digit number divisible (a) by 9, (b) by 17.

(2) Give an example for a function which value at x=3 is -2.

(3) Give an example for a sample space and an event that has a probability of 2/7 in that sample space.

These tasks address different mathematical topics and require different knowledge for solving them. Nevertheless, they have a lot in common.

1. They all ask to provide an example of a mathematical entity. The required example has to be a mathematical object (a number, a function, an event) with certain properties. The importance of addressing mathematical concepts through an examination of their attributes is discussed in Hazzan (in press). One of the arguments for this approach is that such a discussion invites treating mathematical notions as other objects in our life, which have properties.

2. These tasks appear to be "inverses" of standard and more familiar tasks, because usually learners are asked to determine divisibility of a given number, the value of a given function at a given *x* or the probability of a given event. In this sense the usual roles of what is "given" and what is "to be found" are reversed here.

3. Usually there is no prelearned algorithms on how to create examples and the solution for these tasks is not unique. The importance of students' exposure to and encounters with problems that have many solutions has been discussed by Zaslavsky (1996).

These factors contradict some rooted beliefs about mathematical subject matter, such as existence of the "correct way" and existence of the "correct solution" (Schoenfeld, 1985). We found that one of the interesting common features of these tasks is learners' approaches when facing them. If the example is to be "given" by a learner, where from is it to be "taken"? Learners' ways to provide examples are of our interest in this paper.

We start by presenting several "expert" approaches to the tasks above. Then we discuss students' approaches, obstacles they face and ways to overcome these obstacles. We conclude with a discussion of a potential contribution of including "give an example" tasks in students' mathematical experiences.

EXPERT APPROACHES

(1) Give an example for a 6-digit number divisible by 9.

The first examples that come to mind are 900000 or 999999. In general, considering the divisibility rule for 9, when 5 digits of a number are chosen at random, the sixth one is determined. A "lazy" expert would avoid calculating the total sum of the digits by choosing pairs or groups of digits with a sum of 9, like 362718, 450054 or 333009.

(2) Give an example for a function which value at x=3 is -2.

Probably f(x) = -2 is the most trivial example. Another possible way to generate examples is to choose an arbitrary function and "adjust" it at by a constant to get a value of -2 at x=3. Adjusting y=x, y=2x or $y=x^2$ we get y=x-5, y=2x-8 and $y=x^2-11$ respectively.

(3) Give an example for a sample space and an event that has a probability of 2/7 in that sample space.

Choosing a sample space with 7 possible outcomes, two of which are included in the event provides a variety of examples satisfying the request.

METHOD

The first task was administered as a part of interviews with 20 preservice elementary school teachers. This was a part of a larger ongoing study on students' learning introductory Number Theory (Zazkis and Campbell, 1996). The second task was administered in a written questionnaire with a different group of 22 preservice elementary school teachers. The third task was posed to a two groups of preservice elementary school teachers during a classroom discussion.

In each task, students were asked to explain and articulate their strategy for generating an example and to describe their ways of thinking while addressing the tasks. In addition, after providing one example for the desired object, students were asked to provide five more examples, or, alternatively, to describe a general strategy to be used if they were asked to provide five more examples. Here is for example the full formulation of task 2:

a) Give an example for a function which passes through the point (2,13).
b) What is the value of the function y=2x+183 at x=1?
c) Give an example for a function which value at x=3 is -2.
d) Explain how did you find the function in (c).
e) How will you guide a friend who encounters difficulties in the solution of the task?
f) If you were asked to give 5 additional examples to such functions (which are described in (c), who would you do that?

STUDENTS' APPROACHES

In what follows we will present strategies students used to provide examples. We will not bring a summary of students successful or partly successful solutions, rather, we will focus on common themes in their approaches.

Random trial and error: This strategy was a common one, at least in the beginning, among participants working on task 1. Trial and error was observed on several levels. There were students that picked numbers at random and checked with a calculator their divisibility by 9. There were others that picked numbers at random and checked whether the sum of their digits was divisible by 9. If we compare random trial and error with other approaches, it seems like all the following strategies require a construction of an object. It is probably a more demanding cognitive task to construct an object than to pick an object at random and check whether is satisfies a given property. The latter allows one to reduce a new task to a familiar carrying out of a algorithm and to avoid any decision making. This tendency is an example of "reducing abstraction level" (Hazzan, 1995).

Informed trial and error: This was a strategy that "bridged" a random trial and error and a construction. For instance, if a randomly picked number N gave a reminder of 2 in division by 17, the next "trial" for a student utilizing this strategy was N-2. This can be seen as a mid-way between "finding" as searching in a dark and "constructing" a mathematical object using the known properties.

Constructing an object: This approach presented a challenge of choice, because for the tasks presented in our study a wide variety of objects could be considered as a solution. It was our assumption that such a situation did not make the task harder. However, the wide openness of tasks left many students wondering whether their approach was "right" and the existence of a variety of solutions did not allow the satisfaction of getting "the right one". In the third task, for example, the students' first quiry was about the objects forming the sample space: Should they be numbers? Letters? Different objects? Here we observed emotional or cognitive difficulty, not a mathematical one, in making choices. It is related to the certainty that students are looking for and that in mathematics they are used to have one path to the solution. (Carpenter, Lindquist, Matthews and Silver, 1983).

Designing an algorithm: This was a common approach in generating examples. When dealing with the second task, a majority of students used this approach. They found a way to get a variety of objects using an algorithm which they created. As stated above, in task 2 the students were asked to describe how they found the function, how they would guide a friend who encounters difficulties in the solution of the task and finally, to describe how they would find 5 additional examples for such functions.

Elena wrote: *The generation of the function which gets the value -2 at x=3 is carried out in the following way: It is given that x=3 and y=-2. We will look for a function for which if we plug in x=3 its value will be -2. We will write the formula of a line y=ax+b. Two parameters are known from what is given. We will plug them in the line formula: -2=a*3+b. As we said, a and b are not known to us [...]. We will select some a (which determines the lines slope). In my example I chose a=2. Let's plug it in the line's formula, and now we will calculate the point in which the function cuts the y axis. -2=2*3+b ; b=-2-6=-8.*
My function is a=2 , b=-8 , y=2x-8.

This shows that students have a strong desire to follow an algorithm. When such an algorithm is not given, they create it (Hazzan, in preparation). Tendency to stick to known algorithms was observed also with task 1. While a 6-digit number divisible by 9 was easily obtained by considering the sum of the digits, finding a number divisible by 17 provided a greater challenge. The reason for this discrepancy could be that for checking divisibility by 9 students had a recently reviewed algorithm (process) which had to be inverted. For checking divisibility by 17 there was no learned algorithm. Therefore the algorithm had to be created first and then inverted.

Trivial examples: By this we call a number like 170000 in question 1(b) and a function like y=-2 in task 2. Although it was agreed in advance to honor trivial examples but also to request additional ones, we were surprised by a relatively small number of "trivial" examples that were given. Possible explanations to this phenomenon is that our "trivial" examples are not "prototypes" of these objects: A function is expected to be an expression involving a variable x; a 6-digit number is usually combined from a wider variety of digits. Another possible reason for avoiding trivial examples is the students' assumption that a teacher or an interviewer expects something different.

Validation: There was a strong tendency among students who have systematically constructed their examples, rather than found them by trial and error, to check the correctness of their answers. For example, students that "created" their divisible by 17 numbers by choosing a multiplier and performing multiplication, wanted to verify (by division) the correctness of their example as a solution.

Coping with degrees of freedom: This strategy presented a challenge that we learned to appreciate. Interpreting task 1 as a request for a sequence of digits whose sum is divisible by 9, students still had a difficulty to choose the digits. Lucy explained her strategy:

Lucy: *I need a sum to be something divisible by 9. So we can choose 18. OK, 8 and 2 give 10, and now I break the-8 into 4, 3 and 1. So 82431 should be divisible by 9.*
Interviewer: *Good, can you think of another strategy to find such a number?*
Lucy: *I could make the sum something else, like 27 or something...*

We note here Lucy's resistance to think of a "number divisible by 9" without a specific number in mind. This difficulty, however, is simply resolved by focusing on a specific number, rather than on its property.

DISCUSSION

In this paper we discuss "give an example" tasks which are different from the more standard tasks (which require an execution of a procedure) in several ways: First, they inverse the traditional roles of "given" and "asked for"; Second, this kind of problem invites an exploration of properties of mathematical notions; and Third, they have many, at times infinitely many, solutions.

Inverting traditional tasks is a useful research technique and it was utilized by several researchers. For example Ball (1990) asked students to give an example of a problem situation that can be solved using a given division exercise. Furthermore, Simon (1993) asked students to create a problem that can be solved by a division exercise including a division by fraction. An appreciation of problems created by students and of an activity of problem posing side by side to problem solving is becoming more and more popular in mathematics education research literature (Brown & Walter, 1983). Here we suggest to consider this kind of tasks not as research tools but as an integral part of learning.

We feel that one of the most significant results of our study is the acknowledgment of the relative difficulty of solving "give an example" tasks. Even though majority of students came up with correct answers, we conclude the relative difficulty as a result of the amount of time it took a student to respond in an interview, the lack of ease with which additional examples were generated, the amount of calculations it took to produce an example and finally, students' confidence in the examples they generated. Talking about "relative" difficulty we mean relative in comparison to tasks which require the same mathematical sophistication, but are "standard execution" tasks, rather than "construction" tasks. In other words, checking divisibility of a number, calculating the value of a function or the probability of an event, seemed to be easier than providing examples of or constructing objects with given properties. In the next paragraphs we debate on what could be perceived by a learner as difficult in solving a "construction" problem.

Usually, when we hear the term construction problems, the immediate association is with a compass and a straight edge tasks in geometry, like construct a bisector of a given angle or inscribe an equilateral triangle in a circle. In our study students were asked to create a mathematical object, which satisfied some properties. It turns out that such a construction goes through a sequence of stages, where at some points there are several ways to proceed. Thus, for example, in a more advanced context, while constructing an isomorphism between two groups of order 6, students have the freedom to choose the match between elements of the same order in the two groups (Leron, Hazzan & Zazkis, 1995). In this paper we showed that a similar phenomenon occurred in a more elementary context. Students exhibit and acknowledge emotional difficulty to deal with degrees of freedom. They feel uncertainty when put in a decision making situation, and sometimes prefer to quit and avoid making choices when there is no one dictated way to proceed. When facing this difficulty, we may draw students' attention that in their lives they are making decisions, some easy some harder, all the time. Decision making in mathematics should be practiced and not avoided. We believe that such a practice could influence students' beliefs about mathematics, their habits, their search for certainty and their need for feedback.

Another activity that is practiced a lot outside of mathematical classrooms is a discussion of properties of "things" and classification of objects according to their properties. We wish to bring this discussion into the mathematics classroom, by showing the analogies between "regular" activities and mathematical activities. We believe that a construction of a specific mathematical object described by its properties, may help students in the mental construction of the relevant mathematical notions on an higher level of abstraction. This is because in the process of constructing an object, students have to deal with the concept through its properties, and not by carrying out several calculations, which could be executed without an understanding of the essence of the concept under the discussion. Therefore, we suggest that "give an example" tasks should be implemented at different levels and with different mathematical contents. Several examples are below:

- *Give an example of two linear (non-linear) functions which graphs intersect at (3,-1).*
- *Give an example of a continuos function that is not differentiable at x=4.*
- *Give an example of a sequence with a limit of 3.*

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CHANGES THAT COMPUTER ALGEBRA SYSTEMS BRING TO TEACHER PROFESSIONAL DEVELOPMENT

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Abstract

Computer Algebra Systems (CAS), with their powerful combination of numeric and symbolic computation, 2D & 3D graphic and programming facilities, are a natural and logical continuation of the scientific and graphical calculators that are becoming popular in school mathematics. Mathematics teachers have mathematical backgrounds that enable them to understand what a CAS is doing and to appreciate the facilities offered by this new technology. Most teachers, however, are not aware of the existence of such software. Here we discuss the features of a professional development course using a CAS as a technological tool in the mathematical environment. From a test given in the course we present an example that became a turning point for the teachers. Toward the end of the course the teachers developed projects that we analyzed with regard to the goals of the course.

The idea of using computers to perform symbolic, rather than numerical calculations led to the development of Computer Algebra Systems (CAS) in the early sixties (Harper, Wooff & Hodgkinson, 1991). The availability of CAS for microcomputers in the mid-eighties attracted mathematics educators to the possibility of using them in the classroom. Recently there has been an increase in research and development work regarding symbolic manipulators. Hillel (1993) investigated CAS as a cognitive technology. He analyzed the strengths of CAS as a learning tool and its opposition in instruction, and he concluded that because of its extensive mathematical coverage a CAS can be used as a long-term mathematical learning and solving tool. First attempts to set up a theoretical framework for using CAS in education have emerged. Kutzler (1994) introduced the "scaffolding didactics" model. According to his approach CAS can serve as scaffolding that supports students in moving to a higher mathematical level. Kutzler's idea is based on a didactical model for the use of computer algebra in mathematics education, called White-Box/Black-Box and Black-Box/White-Box. The principle of this model is that when students learn a new mathematical skill or concept, they should first do the operation "by hand"; only after mastering it somewhat the Black-Box is issued to perform the operations. In another learning situation the sequence can be reversed – CAS are used to generate examples for exploration (the Black-Box phase), leading to concept formation in the White-Box phase. Drijvers (1995) investigated several educational examples regarding this model. He pointed out the advantages and limitations of the model; its main difficulty is that it fails to deal with the role of CAS in open-ended investigations for which a CAS is very useful. Drijvers recommended searching for an appropriate model for different examples by applying and adjusting general didactic theories, or by using specific theories on the didactic use of technology. David Tall (1993, 1996) developed a theory of using a computer environment for learning mathematics. One of his ideas, "the principle of selective construction", refers to the flexibility of the learner in focusing on one aspect of cognitive learning while the computer carries out the others. Educators must provide appropriate activities to enable students to focus on selected mathematical concepts and processes. An application of this principle in teaching the limit concept with CAS was discussed by Monaghan, Sun and Tall (1994).

Although it is generally acknowledged that teaching with computers should be adapted to individual preferences of students in processing information (Corno & Snow, 1986; Ford & Ford, 1992), little is known about individual differences in the professional development of teachers. Bottino and Furinghetti (1994) examined teachers' views on the role of computers in mathematics teaching and found that they were mainly a projection of their own views on mathematics teaching. Computer Algebra Systems have presented a great challenge to mathematics teaching today. Thus it is important to adequately prepare teachers for the future in this age of technology.

A professional development course

Many of the mathematics teachers active today previously had solved problems by pencil and paper. The technological tool available then was a book of logarithm tables, and later on the slide rule. The scientific calculator was a big step forward technology-wise, and the graphic calculator even more so. CAS technology, however, with its power in symbolic computation is significant in changing the traditional mathematical environment. Freed from performing manual techniques, the problem solver can now focus on mathematical meaning, methods and explanations. By combining various representations of mathematical problems, teachers can invent new ones. In considering the future needs of teacher preparation and involvement, we designed and implemented a 90-hour professional development course with the following goals:

- By making use of the new possibilities offered by CAS teachers will be provided with opportunities to refresh and extend their mathematical knowledge.
- The teachers themselves will experience learning with this new technology as a motivation to acquire further knowledge, to experiment and integrate CAS in their teaching.
- CAS will become a familiar and integral part of the mathematical environment of the teachers.

To achieve these goals we identified four characteristics of the course:

(1) Choice of mathematical topics

The software can be used in those problem-solving situations that address important mathematical ideas. The solutions obtained using CAS have pedagogical benefits and demonstrate the advantages of using the software.

(2) Introducing the various components of CAS

The problems are presented in a sequence that gradually introduces the system components. We start with the infinity of prime numbers problem using CAS's powerful numerical capacity, we add and combine symbolic and graphical representations, and conclude by programming utility files for computing circular functions (e.g., computation of π) that also combine a visual animation of the iteration process (Mann & Zehavi, 1996).

(3) The technological aspects of CAS

In any integration of software it is important to acquire basic skills in its technological aspects. *Derive* (Soft Warehouse, 1990) is a menu-driven computer algebra system that is extremely easy to use. We found it ideal for use in the teacher course. Another issue that also needs to be addressed is that when teachers are first exposed to CAS they usually like its "magic", and then some react, "but it just solves without explaining the way" or "how does computer algebra work?" Teachers should be aware that a computer algebra system solves problems in a different way than they ordinarily would. They should benefit from learning about the system to obtain some theoretical understanding of its symbolic computation, data structures and algorithms. This knowledge will help them, for example, to answer a curious student with programming experience who wonders how the software represents and manipulates very long numbers.

(4) Implication for the classroom

In order to attract teachers to use this new technology they must be involved in developing CAS teaching methods. Such involvement includes learning about cognitive technologies, discussing current experiments, and creating problems for students (Zehavi, 1996).

A test problem

The first three chapters in the course include examples from number theory, a wide range of symbolic manipulations and the advantages of the graphical representation. In the test that was given to the teachers at this stage we used a problem taken from Alan Schoenfeld's mathematical problem-solving class, cited by Arcavi (1994).

"Explore the relationships between the values of 'a' and

the number of solutions of the pair of equations:

$$x^2 - y^2 = 0$$

$$(x-a)^2 + y^2 = 1"$$

The problem is stated in algebraic terms. Thus the teachers used *Derive* to perform the laborious algebraic manipulation. The solution obtained, $x = 0.5(a \pm \sqrt{(2 - a^2)})$, led to the interpretation that "any solution of x represents two points in the plane", ignoring when $x = 0$, which occurs for $a = 1$. Some teachers also tried to add conviction by changing the representation to graphics. *Derive* enables implicit plotting of equations. Figure 1 shows the graph of the first equation, which is represented as two lines, y = ±x, and several circles whose centers lie on the x-axis for specific values of 'a'. When a = 1 there are three rather than four intersection points. After observing this some teachers performed a multi-representation analysis of the problem and gave the correct answer of 0, 2, 3, or 4 solutions.

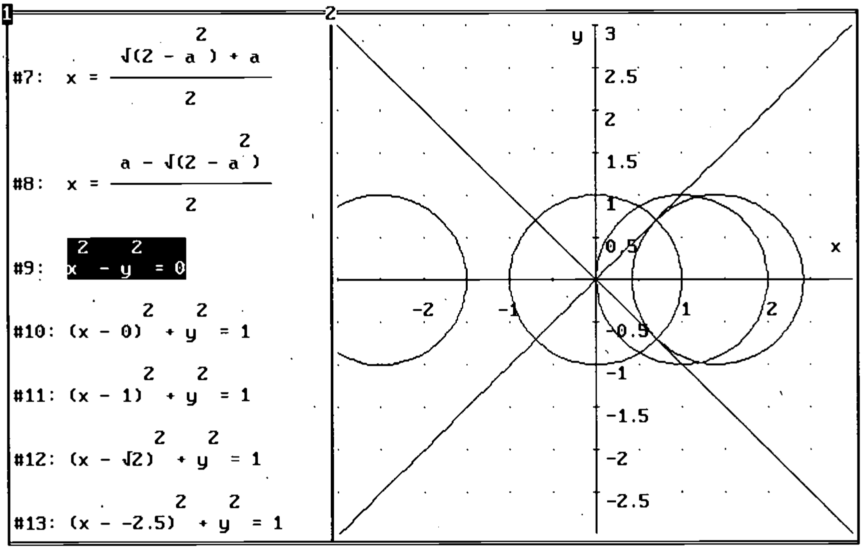

Figure 1: Symbolic and graphical solutions

The distribution of the teachers' responses (n = 25), presented in the table, reveals their solution strategies:

Method	*Correct solution*	*Wrong solution*
symbolic	4	7
symbolic and graphic	10	1
graphic and symbolic	1	2

The following anecdote describes a change of attitudes. After the test, one of the participants, a quite experienced high-school teacher, compared notes with another teacher. She would not agree about the case of three solutions. He suggested plotting. She said, "I am pragmatic, the graph does not add information, it just illustrates what we know. I like solving the algebra with *Derive* and that should be enough." He explained how the implicit plotting, when a = 1, helped him to complete the answer and to understand the algebraic explanation. When she found her mistake she became enthusiastic and changed her approach, "I must educate myself to use the graph more seriously, especially when implicit plotting is available." Here the step forward of implicit plotting integrated with the symbolic mechanism of the software enhanced the users' cognitive mathematical integration.

A problem created by a teacher

Toward the end of the course the teachers were challenged to create problems that revealed pedagogical advantages of the software. We present here a problem, designed by a teacher, which combines analytic geometry with a minima problem:

"Given the lines x = -2 and y = 3

find a line y = mx to complete a right angle triangle

(a) with a minimal length of the hypotenuse,

(b) with a minimal area."

The teacher was very familiar with the commands and options of the algebraic and graphical windows. She used the vector notation to denote the vertices of the triangle, A:= [-2, -2m] and B:= [3/m, 3] (see Figure 2). When she tackled the problem she discovered that she did not have to explicitly display the length AB as a function of m to indicate that she was interested in the derivative. She liked the idea that when she issued the commands "Calculus Differentiate" for expression #4 the program showed that the derivation was for m, and she explained the possible advantage of this feature for conceptual understanding of calculus. Furthermore, when she issued the command "soLve #5 to zero" (see annotation at the bottom of Figure 2) she got the value of m for the minimal length of AB. The teacher emphasized this cognitive technology of a black-box with some peeping holes. She discussed with the other teachers its potential use in class where students would have to interpret the steps shown. The technology enables an innovative pedagogical approach in which the student can focus on key concepts regarding the variable of the differentiation without bothering with the complicated expressions of the function and of its derivative. This is not the only approach; it is the problem solvers' responsibility to select the direction they want to pursue.

The teacher repeated the procedure for area S of the triangle. The surprising aspect of the problem is that the minimal area *is not* obtained for the minimal length of AB. This is nicely illustrated by the graph in the picture that the teacher prepared using the option of "tracing" a graph (see Figure 3). On the left-hand side are shown the symbolic function AB(m) and its graph. The minima, AB = 7.02, was obtained for m = 1.11471 (in decimal approximation). On the right-hand side two minima were obtained for m = 1.5 and for m = -1.5. In the positive case the area is minimal, but AB = 7.21. The teacher began, at this stage, to look closer at the format of the symbolic expressions obtained for AB and S. Relating their structures to their graphs leads to asking additional questions regarding the initial geometrical

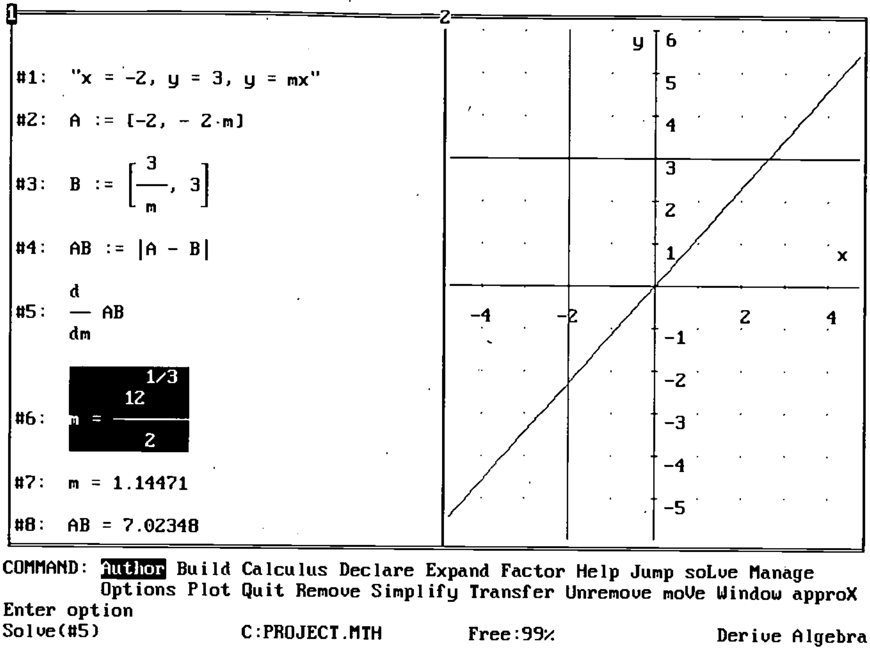

Figure 2: Geometrical representation of a minima problem

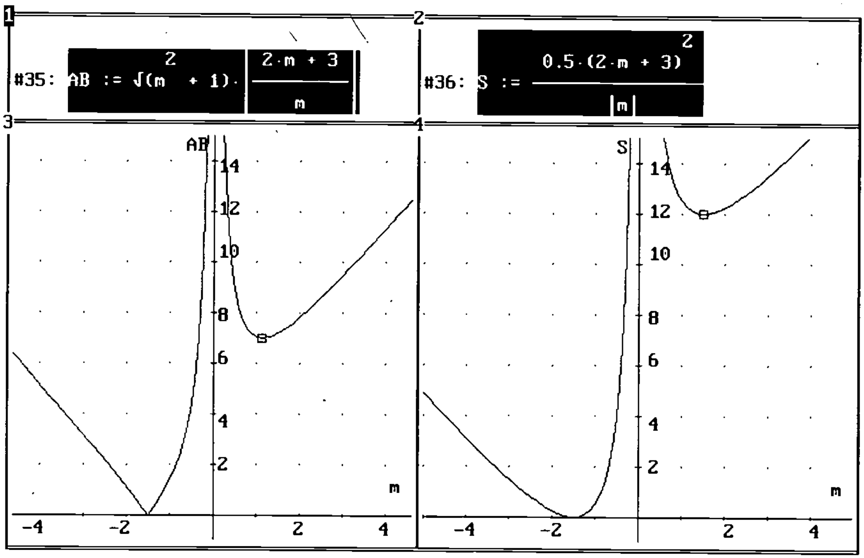

Figure 3: Graphical representation of a minima problem

problem. For example, what happens for negative values of m? Where m = -1.5 it is clear – there is no triangle and the values of AB and S are both equal to zero. Then another teacher raised the question whether there is another case where the numerical values of AB and S are the same. The teachers selected and combined various representations, symbolic, graphical, and geometrical to answer this question. The most popular way was to formally solve the equation AB = S, which yields m = -5/12, and then to interpret this result by drawing the triangle (as in Figure 2), and later to look for the intersection points of the two graphs that appear in Figure 3.

The teachers suggested further ideas for teaching as well as technology tips. Clearly, CAS support different approaches of coping with mathematical problems. An added value of the coursework was that teachers became aware of the variety of possible ways of carrying out a learning activity.

Conclusion

By inventing and solving problems using a CAS, teachers became more familiar with the software. The problems that teachers invented reflect a variety of backgrounds, views and teaching styles. In a feedback questionnaire most of the teachers in the course indicated that the teaching method used provided them with greater understanding and insight regarding how to use *Derive*, as well as refreshed their mathematical knowledge. Half of the teachers, however, did not agree that the course extended their mathematical knowledge explaining that they could not understand all the problems. These findings indicate the importance of teachers doing and learning mathematics themselves with the software, as an essential part of their professional development. When teachers were asked about their readiness to meet the challenge of incorporating *Derive* in their teaching, their first reaction was that it would require a fair amount of preparation time. Nevertheless, they suggested identifying segments of the problems they felt comfortable with, and adapting them for use with students, implementing "the principle of selective construction". They all expressed their desire to participate in a follow-up course for designing curricular materials. Even if, today, there are no clear answers to the major research questions on integrating CAS in teaching, teachers should be motivated to learn and to be involved in developing teaching methods for this new technological age.

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