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ABSTRACT

The third volume of the proceedings of 21st annual meeting of the International Group for the Psychology of Mathematics Education contains the following papers: (1) "Graphics Calculators Use in Precalculus and Achievement in Calculus" (P. Gomez and F. Femandez); (2) "Tapping into Algebraic Variables through the Graphic Calculator" (A. Graham and M. Thomas); (3) "Collaborative Mathematical Reasoning in an Inquiry Classroom" (B. Graves and V. Zack); (4) "Working from the Inside with Theory from the Outside" (U. Hanley and T. Hardy); (5) "Gender Difference and Their Relation To Mathematics Classroom Context" (M. Hannula and M.-L. Malmivuori); (6) "Are Lawyers-Prey To Probability Misconceptions Irrespective of Mathematical Education?" (A. Hawkins and P. Hawkins); (7) "An Expression of the Idea of Successive Refinement in Dynamic Geometry Environments" (O. Hazzan and E. P. Goldenberg); (8) "Effectiveness of a Strategy as a Sociomathematical Norm" (P.G. Herbst); (9) "How Equally Suited is Realistic Mathematics Education for Boys and Girls? A First Exploration" (M. van der Heuvel-Panhuizen); (10) "Teacher as Amplifier, Teacher as Editor: A Metaphor Based on Some Dynamics in Communication" (D. Hewitt); (11) "Using the Computer To Improve Conceptual Thinking in Integration" (Y.Y. Hong and M. Thomas); (12) "Investigating Children's Collaborative Discourse and Verbal Interaction in Solving Mathematical Problems" (H.-M.E. Huang); (13) "An Analysis of Student Talk in 'Re-learning' Algebra: From Individual Cognition To Social Practice" (B. Hudson, S. Elliott, and S. Johnson); (14) "The Cognitive and Symbolic Analysis of the Generalization Process: The Comparison of Algebraic Signs with Geometric Figures" (H. Iwasaki and T. Yamaguchi); (15) "Making Sense of Mathematical Meaning-Making: The Poetic Function of Language" (M. James, P. Kent and R. Noss); (16) "Children Learning To Specify Geometrical Relationships using a Dynamic Geometry Package" (K. Jones); (17) "Change in Mathematics Education: Rethinking Systemic Practice" (L.L. Khisty); (18) "Area Integration Rules for Grades 4, 6 and 8 Students" (G. Kidman and T.J. Cooper); (19) "Teachers' Pedagogical Content Knowledge of Multiplication and Division of Rational Numbers" (R. Klein and D. Tirosh); (20) "Gender Differences in Algebraic Problem Solving: The Role of Affective Factors" (S. Kota and M. Thomas); (21) "Students' Representations of Fractions in a

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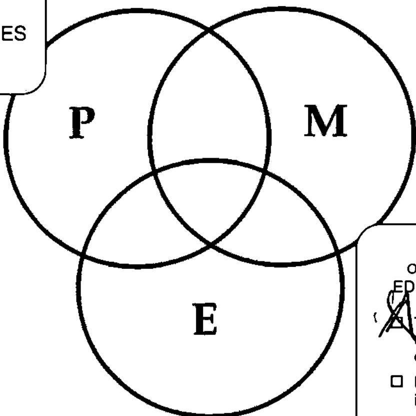

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RESEARCH REPORTS

(continuation)

Graphics Calculators Use in Precalculus and Achievement in Calculus

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Most studies on the effects of graphics calculators on students' achievement in precalculus use specially designed tests that are implemented immediately after the introduction of the technology. In many cases, the way the new technology is integrated into the curriculum is not taken into account. This study analyzed the achievement on calculus of students who took a curriculum innovation in a precalculus course that involved graphics calculators use. Even though no differences were found between the non–calculators and calculators groups at the adaptation phase, significant differences were found between these two groups at the consolidation phase, and between the calculators groups of the adaptation and consolidation phases.

Introduction

Current research on the use and effects of graphics calculators suggests mixed results (Penglase and Arnold, 1996). Some studies show that graphics calculators can enhance the learning of functions and graphing concepts and the development of spatial visualization skills. They can also promote a shift from symbolic manipulation to the graphical investigation and examination of the connections among the several representation systems associated to a given concept. Nevertheless, other studies show that graphics calculators use might not promote the development of some necessary skills and in some cases may result in some "de–skilling". Most studies use specially designed tests to assess the effects of graphics calculators use. These tests are administered immediately after the experience and no follow-up is presented. Furthermore, in many cases it is difficult to distinguish between the effects of the graphics calculator as a tool and the effects of the instructional process in which its use was involved. There has been little attention paid to the effects of graphics calculators use depending on the level of integration of the tool into the curriculum.

In this study we were concerned about the effects on students' achievement on calculus of graphics calculators use in a precalculus course. We wanted to see whether students that had taken a precalculus course involving a curriculum innovation that included graphics calculators use could obtain better results in the second calculus course of the mathematics cycle (in which there was no graphics calculators use), when compared with other students who took the standard precalculus course. Additionally, we were interested in seeing whether the effects of graphics calculators use depended upon the phase at which the technology was integrated into the curriculum.

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Graphics calculator and students' "mathematical future"

In this study we were not concerned about the effects on students' understanding of graphics calculators use. Research has shown that in most cases graphics calculators use can have enhancing effects on students' understanding of precalculus concepts. Even though this understanding is clearly important, it is meaningful if, for instance, it can help students succeed in their performance in the calculus courses that follow the precalculus one at the university level. Whether graphics calculators use has relevant "de-skilling" effects depends upon whether students will need those skills in the future. If students are allowed to freely use graphics calculators in all their mathematics activities through their career, then it might be possible that, even if this "de-skilling" takes place, it does not affect the students' "mathematical future". However, graphics calculators use is not generalized in all educational institutions and at all levels. This was the case of the university in which this study took place. Graphics calculators use in some precalculus courses was seen as an "experiment", and graphics calculators were (and are) not used in any other mathematics course. This meant that students taking the course that involved the curriculum innovation with graphics calculators were not going to be able to use graphics calculators in the two calculus courses that followed. This posed the question of whether, if there has been some "de-skilling" due to the curriculum innovation involving the graphics calculator, this "de-skilling" had any effects on the students' "mathematical future".

Graphics calculators integration into the curriculum

Graphics calculators cannot be simply introduced into curriculum. They can be used at different levels and they can have different roles in curriculum design and implementation. The effects of graphics calculators use can depend upon how they are integrated into curriculum. Following the ideas suggested by Kissane, Kemp and Bradley (1996) for assessment and graphics calculators use, we introduce four phases concerning graphics calculators use in curriculum design and implementation: non–existent, introduction, adaptation and consolidation. We consider five elements of curriculum: students' use, teachers' use, tasks proposed, textbook, and assessment. Each of these elements can be in any of the four phases. The first phase is evident: graphics calculators are not used or mentioned at all. The table in the following page shows how each of the three other phases is defined on the basis of the curriculum elements considered.

The main difference between the adaptation and the consolidation phases concerns whether advantage is taken of the graphics calculator possibilities. This means whether graphics calculators are used in order to create new learning opportunities through promoting mathematical investigation and exploration and emphasizing relationships among representation systems. The above categories do not take into account the way graphics calculators are used by teacher and students when presenting an explanation or solving a problem.

These categories are proposed in order to classify curriculum innovations that involve graphics calculators use. It is clearly possible for a given curriculum innovation implementation to be located in different phases for different elements of the curriculum. This can be the case, for example, when teacher's use of the graphics

	Phases		
	Introduction	**Adaptation**	**Consolidation**
Students	Students have restricted access during some classes. They do not have access outside the classroom.	Students have unrestricted access to graphics calculators.	
Teacher	The teacher has a basic knowledge about the graphics calculator operation. He/she does not use it during classroom activities at all, except for explaining how to use it.	The teacher uses the graphics calculator when it is necessary or when asked to do so by the students. His/her explanations do not take advantage of graphics calculator possibilities.	The teacher takes advantage of the graphics calculator possibilities for explanations and problem posing.
Tasks	The only tasks that involve the graphics calculator are those used to learn how to use it.	Few tasks take advantage of the graphics calculator possibilities.	Most tasks take advantage of the graphics calculator possibilities.
Textbook	Reference is made to graphics calculator as far as how to use it. Problems and exercises do not take advantage of the graphics calculator possibilities.	Some problems and exercises are specially designed for graphics calculators use. The way content is presented and learning is promoted do not take advantage of the graphics calculator possibilities.	Problems proposed and the way content is presented and learning is promoted take advantage of the graphics calculator possibilities.
Assessment	Graphics calculators are not allowed in tests.	Questions are "calculator–neutral". There is no advantage to students with a graphics calculator.	Unrestricted calculator access. Students decide when and how to use the graphics calculator.

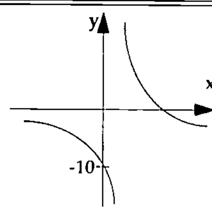
calculator remains at the introduction phase, whereas curriculum design imposes conditions for graphics calculators use at the adaptation phase on the other elements. In this sense, the teacher plays an important role in the process. This can also be the case concerning assessment. If assessment remains at the introduction phase, the effects of graphics calculators use might be curtailed even if other elements are at the adaptation

phase. Nevertheless, even if no curriculum innovation can be accurately classified in one level, it seems reasonable to think that most elements will adjust themselves so that they are approximately at the same phase.

Context

In the university this study was done, first semester students of Engineering, Business Administration, Economics and Biological Sciences are classified according to their results in the mathematics section of the State Examination. Those students with best results enter directly to the first calculus course. The rest, approximately half of them, start their mathematics cycle with the precalculus course. The students who succeed in the precalculus course are supposed to take the first calculus course during the following semester. If they succeed in this course, they should take the second calculus course immediately thereafter. Students are allowed to drop any course before the mid–semester without getting a grade. Those who fail a course have to take the course again the following semester or during the summer holidays. The study considered only those students starting the mathematics cycle with the precalculus course who were able to succeed in the three courses comprising the cycle during the three consecutive semesters.

The established precalculus course is an introductory course to the study of functions in which some emphasis is given to the graphical representation and to problem–solving. Usually the teacher presents some theory at the beginning of the lecture, and the rest of it is spent solving exercises with some students at the blackboard. The curriculum innovation involving graphics calculators use introduced some changes to this precalculus curriculum. A stronger emphasis was given to the connections between the symbolic and the graphical representations and the concept of family of functions was introduced. Lectures were mainly developed around problem solving activities (Gómez et al., 1996) that followed the ideas of higher–order mathematical thinking (Resnick,1987). As an example of some of the differences between the two courses, the table shows a question of the final exam from each course.

No calculators	**With calculators**
Solve: $\frac{x}{x-1} \leq x\lvert x\rvert$	$x^3 - k \geq g(x)$ has $[0, \frac{1}{4}]$ U $[c, \infty)$ as solution set. It is known that k and c are positive numbers; g(x) cuts the X–axis at $\frac{5}{4}$; if h(x) = x + 2, the function h(g(x)) has y=0 as asymptote. The figure shows the graph of g(x). Find k, c and g(x). 

Graphic calculators are not allowed in the two calculus courses that follow the precalculus course. In these courses students are expected to develop operational skills for

symbolic manipulation. Lectures are taught in a similar way to the standard precalculus course.

The curriculum innovation involving graphics calculators use underwent the three phases (introduction, adaptation and consolidation) described previously. The three phases were developed during three consecutive semesters. Some results are already known concerning this curriculum innovation. Mesa and Gómez (1996) found no differences in some aspects of understanding between the students who took the traditional course and those who took the curriculum innovation at the adaptation phase. Gómez (1995) and Gómez and Rico (1995) found that the students of this group participated more actively in social interaction and in the construction of the mathematical discourse, changes that can partially be attributed to a different behavior of the teacher. Even though she changed her behavior, Valero and Gómez (1996) found that the teacher could not change completely her beliefs system. Finally, Carulla and Gómez (1996) found that the teachers and researchers who participated in the curriculum innovation (at the adaptation and consolidation phases) underwent significant changes on their visions about mathematics, its learning and teaching.

Problem

We wanted to answer two questions:

- ▲ Were there any differences in the students' final grades in the second calculus course between those who took the traditional precalculus course and those who took the curriculum innovation involving graphics calculators use?
- ▲ Were there any differences in the students' final grades in the second calculus course between those who took the curriculum innovation involving graphics calculators use at the adaptation phase and those who took it at the consolidation phase?

Design

Two groups of students starting the precalculus course during two consecutive semesters were taken into account. The first group was divided into two subgroups. The first one (**G1C**, with 134 students and five different teachers) took a precalculus course in which the curriculum innovation was implemented. The second sub–group (**G1NC**, with 111 students and five different teachers) took the established precalculus course without calculators. A different group of students starting the precalculus course the following semester were divided in the same way: those taking the precalculus course in which graphics calculators were used (**G2C**, 58 students and two teachers), and those who took the traditional precalculus course (**G2NC**, 125 students and four teachers). The graphics calculators sub–group of the first semester (**G1C**) followed a curriculum innovation that was at the adaptation phase. The curriculum innovation for the graphics calculators sub–group of the second semester (**G2C**) was at the consolidation phase. Students were randomly assigned to each teacher.

This was a longitudinal comparative study. Students' achievement was measured on the basis of the students' final grades in the second calculus course of the mathe-

matics cycle. The comparisons between groups and between graphics calculators adaptation and consolidation phases were established on the basis of the difference of sampling means of the final grades of the second calculus course. The parameter analyzed was of the form $\mu_A - \mu_B$. The statistical significance of the difference of sampling means was measured with a two tails p–value associated to the t–test of comparison of two independent populations. In order to analyze the possibility of confusing factors, the teacher's effect at the adaptation phase was taken into account. In the first group there were ten teachers. Five of them implemented the curriculum innovation. Only two of these five teachers implemented the curriculum innovation at the consolidation phase.

Three comparisons were made: between the calculators and non–calculators groups corresponding to the adaptation phase (G1C and G1NC); between the calculators and non–calculators groups corresponding to the consolidation phase (G2C and G2NC); and between the students of the two teachers that implemented the curriculum innovation at the consolidation phase and the students from these two teachers at the adaptation phase (G2C and G1C(2T)). Since the proportion of students who succeed the precalculus course differs from one teacher to another, in order to establish appropriate comparisons, we considered the 25% of students who obtained the best grades in the second calculus course from each group.

Results

The table presents the grades' mean and standard deviation and the percentage of students considered for each of the five groups mentioned above, together with the results for the three comparisons proposed. We observe that, for the first comparison (G1C and G1NC), even though the difference was negative, it was not significative (p=0.14). Nevertheless, when we look at the other two comparisons, we observe that there were very significant differences. In the case of the two groups corresponding to the consolidation phase (G2NC and G2C), the difference favors the calculators group (p=0.0034). In the case of the comparison for the same two teachers (G1C(2T) and G2C), the difference favors the group corresponding to the consolidation phase (p=0.00057).

	G1NC	G1C	G2NC	G2C	G1C(2T)	G2C
$\bar{X}$	3.43	3.22	3.56	4.04	3.42	4.04
s	0.6	0.44	0.5	0.4	0.385	0.4
%	24.3%	24.6%	25%	24.1%	26%	24.1%
Dif	-1.53		3.09		3.94	
p	0.14		0.0034		0.00057	

Discussion

We cannot assure that the results obtained in this study are valid for other circumstances except for a hypothetical situation in which similar students take the same courses with the same teachers and curriculum implementation. The statistical analy-

sis refers to that hypothetical population.

The results show that the effects of graphics calculators use in this study depended directly upon the phase at which graphics calculators were integrated into the curriculum. While no significant difference was observed between the calculators and non–calculators groups when the curriculum innovation was at the adaptation phase, significant differences were found between these two groups at the consolidation phase, and between the calculators groups of the adaptation and consolidation phases. This might be due to the fact that during the consolidation phase, graphics calculators were used to create new learning opportunities through the promotion of mathematical investigation and exploration and the emphasis given to the relationships among representation systems. Furthermore, these differences (specially those concerning the two teachers that participated at the two phases) might also be explained by the change that teachers and researchers had of their visions about mathematics, its teaching and learning as a consequence of the way graphics calculators were integrated into the curriculum (Carulla and Gómez, 1996). These results show that, at least as far as achievement is concerned, graphics calculators effects cannot and should not be studied independently of the way the new technology is integrated into the curriculum. Furthermore, it might be possible, as it was the case for the experience reported here, that the use of graphics calculators needs to go through an "integration process" in which in order to attain a given phase, the previous phases have to be completed. It remains to be seen whether a successful consolidation phase (as far as achievement is concerned) can be attained without a change in teachers' visions.

The results obtained in this study do not support the "de–skilling" argument that is sometimes presented against graphics calculators use. The two calculus courses that follow the precalculus course considered in this study do not allow graphics calculators use and follow a traditional curriculum in which students are expected to develop operational skills that emphasize symbolic manipulation. If, in fact, some "de–skilling" took place, then either it was not relevant, or its negative effect was overcome by other skills and knowledge developed by the students who used graphics calculators.

Even though this study did not analyze the new skills and knowledge developed by the students who used graphics calculators, it showed that graphics calculators use had positive effects on their "mathematical future".

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TAPPING INTO ALGEBRAIC VARIABLES THROUGH THE GRAPHIC CALCULATOR

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There has been much discussion about how best to introduce algebra into the student's mathematical experience. However this is attempted it is our belief that an understanding of the concept of variable is fundamental to progress in algebra. This paper describes a study in which we used a module of work based on a graphic calculator to provide an environment in which students could begin to build an understanding of variable. The graphic calculator proved to be a motivating instrument for successfully achieving a significant improvement in student understanding, something which has often proved difficult.

Introduction

The experience of teachers and a wide range of empirical research inform us that children find great difficulty in understanding the algebra of generalised arithmetic (e.g. Küchemann, 1981; Wagner, Rachlin & Jensen, 1984; Thomas, 1988). One of the most important obstacles to progress involves a concept that is too rarely discussed in most classrooms where algebra is presented and yet one which underpins all that students learn. This is the concept of variable. It is important that students gain some understanding of variable if they are to progress beyond basic processes. Küchemann (1981) showed clearly that extremely few students reach a working knowledge of variable, with only 9% of 15 year-old students in his study having gained an appreciation of variable beyond that of specific unknown. This paper addresses how the graphic calculator may be effectively utilised in the classroom to improve student understanding in this area.

Background

A *procept* has been described (Gray & Tall, 1994) as a combination of mathematical symbols, a process (which they may invoke) and the concept (which they may represent). For example, $x + 1$, is a symbolisation which simultaneously represents an expression (or function) and the process of adding one to an unknown value. It is, however, important to encapsulate the generalised process of adding one as the object $x + 1$, because the process cannot be carried out directly unless x is given a value. However, many students only see the symbol $x + 1$ as a process and not as a mental object in its own right, capable of manipulation in an abstract form. A theory of *procepts* helps us see that whilst arithmetic expressions may be successfully interpreted as signalling a process to calculate the answer, algebraic expressions are different and require *proceptual thinking*. Prior to the introduction of algebra, children become accustomed to working in an arithmetic environment where they solve problems by producing a numerical "answer" (Kieran, 1981), leading to the expectation that the same will be true for algebra. To cope with the difficult

transition from arithmetic process-oriented thinking to proceptual algebraic thinking, teaching has tended to emphasise the process side of algebra; the calculation and manipulation of algebraic expressions. Students have been taught the rules of algebra so that they could develop the necessary manipulative ability, but there has been little addressing of the concepts. Kieran (1994) presents three different views of algebra: operational; fixed-value; and functional and suggests that these have often been introduced to the learner in this order. She proposes a different approach which would start with functional algebra and the use of letter as variable. Sfard (1995) agrees that this might help to reduce the difficulties of students. Examining algebra beyond the introduction of symbols leads to an examination of combinations of letters and numbers in strings of symbols and Sfard and Linchevski (1994) have described four different views of these which different contexts may evoke: computational process; specific unknown; function; and mere string of symbols. Whichever starting point is used for algebra, the student, to be successful, has to grow to an understanding of the use of symbolic expressions which will encompass these four strands and take into account the current understanding of the initial learner of algebra. We believe that the success rate can be significantly improved by giving a coherent meaning to the letters used. We have had considerable success in the past in doing exactly this using the computer (Thomas, 1988; Thomas & Tall, 1988; Tall & Thomas, 1991), demonstrating that it was possible to improve students' understanding of variable by giving them environments in which they could manipulate examples, predict and test and gain experiences on which higher-level abstractions could be built. However, as with much research, the beneficial effects are often slow to permeate into the mathematics classroom, if indeed they ever do. Whilst there are a number of possible reasons for this (see, e.g. Thomas *et al.*, 1996; Thomas, 1996), one often mentioned by teachers is the lack of resources, both in terms of computers and relevant, tried and tested software. The graphic calculator is now a portable, affordable alternative option to the computer for many schools and it has two very useful qualities. Firstly, like the computer it intrinsically employs variables in its operation. Secondly, the multi-line display enables one to see, reflect on and interact with, several previous input/output rounds. It is important to appreciate that the calculator is a tool with these important attributes which can be integrated into a teaching module (Penglase & Arnold, 1996). This present research study attempted to combine these advantages with the principles and techniques we had learned from using the computer and put them into practice on the graphic calculator.

Method

Teachers from five United Kingdom schools volunteered to take part in the research project. Each of them agreed to teach a module of work in algebra to one of their classes, based on the TI-80 graphic calculator. In addition they chose a control group of pupils, similar in ability and background to the experimental group, against which to make a comparison. The control group received algebra work to parallel the experimental group, but were taught using usual teaching methods.

Students were from years 8 to 10 (age 12-14 years) top and middle ability groups. The module was taught during early 1996 by the classroom teachers, each of whom had attended a weekend course (run by one of the researchers) which was designed to help them gain proficiency in the use of the calculator. The researchers were not present in any of the classrooms while the students were working on the project. The classroom groups were all given a pre-test and a post-test which comprised questions based on, and extending, the Küchemann (1981) research, since these still provide a normed measure of understanding. The two tests used were different, with the pre-test having 28 questions and the post-test 68. This latter test was more difficult, containing 63.2% of level 3 and 4 questions (specific unknown and generalised number) compared with 53.6% for the pre-test.

The algebra module

The module of work was designed to last about three weeks. The first section comprised an introduction to using the graphic calculator, since it was assumed that almost no students would have had experience of using them. In the previous research study we had used simple programming in BASIC, such as:

A=3 followed by PRINT A+2

so that the computer responded with the number 5. Students could then conjecture what would happen if they typed

PRINT A+3 or B=A+2 PRINT B

and so on, in order to begin to formulate theories about the consistency with which the language handles the symbols and to build an understanding of their purpose. On the graphic calculator (we used the TI-80) the above sequence became:

3—›A (using the STO› key) A+2 [Enter] A+3 [Enter] (or A+2—›B) B[Enter]

but the essential elements remain the same. Figure 1 gives an idea of the layout used in the module, illustrating the 'Press', 'See' and 'Explanation' features which were universally used.

You can use letters as stores for numbers. Try the following:

Press	See	Explanation
4 [STO] [ALPHA] **A** [ENTER]	4 → A 4	The value 4 is stored in A.
[[CLEAR]		This clears the display.
[ALPHA] **A** [ENTER]	A 4	This confirms that the number stored in A *is* 4.

Figure 1. An example of the layout of the work in the algebra module

Building on this introduction, a typical early exercise was:

Store the value 2.5 in A and 0.1 in B.
Now predict the results of the ten sequences listed below.
Then press the sequences to check your predictions.
A+B, B+A, A – 5B, 2A + 10B, A/B
AB, BA, 2A + 2B, 2(A+B), 4(A+5B).

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One of the novel aspects of the module was the use of *screensnaps*, where the student was given a screen view and required to reproduce it on their calculator screen. Examples of these which were given include those in figure 2. These have the advantage of encouraging beginning algebra students to engage in reflective thinking using variables. This is beneficial since, unlike experienced mathematicians, they do not reproduce them by using algebraic procedures but by predicting and testing. Other topics covered included squares and square roots, sequences, formulas, random numbers and function tables of values. In all of these activities the student is actively involved in a *cybernetic* process where the technology reacts to the individual's actions according to pre-programmed and predictable rules. This environment provides consistent feedback in which students may predict and test, enabling them to construct an understanding of letters in algebra as stores with labels and changeable contents.

A+B	A+B	A+B	A+B
0	11	11	11
A/B	A–B	A–B	A–B
-1	1	5	-3

Figure 2: Examples of *screensnaps* from the algebra module

Whilst this is not the full story of the mathematician's perception of a variable, the attainment of such an understanding represents a considerable advancement on that which many students currently reach.

Results

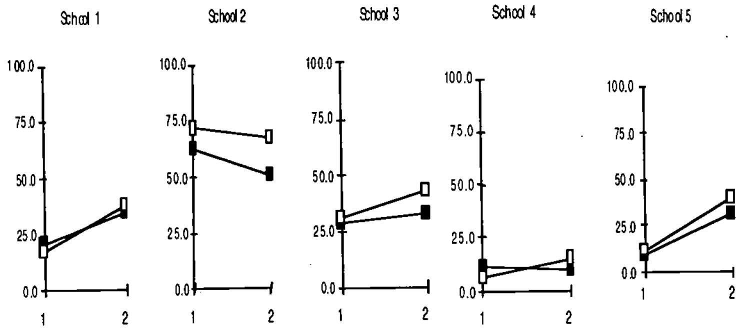
A summary of the results of the schools in both of the tests is given in table 1. These results may be easily compared by examining figure 3, which gives the mean percentage scores in the pre- and post-tests. There were fewer questions in the first test and that included more of the relatively easy questions, which may account for the apparent drop in performance in school 2. However, in each case, the relative improvement of the experimental students over the control students is clearly seen. Examining these results we see that, whilst the groups do not differ at the pre-test, the post-test results of the experimental groups are significantly better than that of the controls for 4 of the 5 schools. Since the tests were constructed so that they were a direct measure of the students' level of understanding of letter as specific unknown, generalised number and variable in algebra, we conclude that the graphic calculator module has improved the students' conceptual understanding of this concept.

In order to see the extent of this improvement we analysed the performance of the two groups on those questions at levels 3 and 4 only (understanding letter as specific unknown and generalised number respectively), as described by Küchemann (1981).

Table 1: A statistical analysis using t-tests of the post-test results for each of the five schools

	Experimental means (SD)		Control means (SD)					
	Pre-test (max=28)	Post-test (max=68)	Pre-test (max=28)	Post-test (max=68)	N_e (Pre/Post)	N_C (Pre/Post)	t	p
School 1	5.07 (2.78)	26.17 (7.30)	5.79 (3.23)	24.16 (5.95)	27/24	24/25	1.05	n.s.
School 2	19.26 (3.35)	45.71 (8.46)	17.56 (4.12)	34.45 (7.96)	27/28	27/29	5.17	<.0001
School 3	8.75 (2.71)	28.96 (7.86)	8.0 (6.39)	22.24 (15.06)	29/29	24/24	2.08	<.05
School 4	2.04 (1.43)	10.17 (5.91)	3.25 (2.05)	6.88 (3.40)	23/23	20/17	2.23	<.05
School 5	3.32 (1.96)	27.93 (11.35)	2.6 (1.92)	21.13 (6.50)	31/31	29/30	2.79	<.005

At these levels the experimental group students were again getting significantly more questions correct.



Control group (black) Experimental group (white)

Figure 3: The mean percentage scores in the pre- and post-tests for each of the five schools

Table 2: A comparison of questions examining conceptual understanding

Question	Experimental proportion correct (N=130)	Control proportion correct (N=129)	χ^2	p
$x-y=z-y$ — always, never, sometimes ... when ?	0.30	0.16	7.73	<0.01
$a+b=b$ — always, never, sometimes ... when ?	0.35	0.12	17.7	<0.001
$L+M+N=L+P+N$, — always, never, sometimes ... when?	0.31	0.19	5.15	<0.05
$3h=c+3$ and $h=2$, then $c=?$	0.58	0.44	2.39	n.s.
$r=s+t$ and $r+s+t=30$, then $r?$	0.34	0.14	14.1	<0.001
Area of rectangle 5 by $e+2$	0.15	0.08	3.68	n.s.
Which is larger: $3q$ or $q+3$?	0.08	0.02	4.23	<0.05

In Table 2 we give some examples of specific questions (abbreviated) at levels 3 (specific unknown) and 4 (generalised number) where the understanding of the students who had used the calculators was better. Of the five questions shown where they did significantly better, four of them are at a level requiring an understanding

of letter as generalised number. This seems to represent a considerable advance in understanding. In the previous study (Tall & Thomas, 1991) we had noticed that, initially, the computer students had performed less well on the traditional skill type questions. What was pleasing to see in this study was that the students who used the graphic calculators did at least as well on these questions in virtually every case and

Table 3: A comparison of questions examining procedural skills

Question	Experimental proportion correct (N=130)	Control proportion correct (N=129)	c^2	p
Simplify $(a+b)+a$	0.62	0.39	14.3	<0.001
Simplify $2a + 5b + a$	0.61	0.50	2.83	n.s.
Simplify $3a - b + a$	0.17	0.09	3.30	n.s.
Simplify $(a-b)+b$	0.18	0.08	6.51	<0.05
Simplify $3a - (b + a)$	0.44	0.37	1.18	n.s.
Simplify $(a+b)-(a-b)$	0.03	0.02	—	n.s.
$2a+2b=2(a+b)$ — always, never, sometimes ... when ?	0.44	0.46	—	n.s.

significantly better in two (see table 3). Since there had been no attempt to teach explicitly these skills these results are very encouraging.

Student comments

Both the teachers and their students were asked to comment on their experiences with the graphic calculator teaching module. The majority of students felt that the experience of using the graphics calculator was of benefit in improving understanding and making the learning of algebra more palatable by providing a useful diversion, with typical students remarking:

> The work we did using the graphics calculator was very interesting and it made algebra seem a little more fun. Algebra was a lot easier on the graphics calculators that it was doing it the ordinary way. . .The graphics calculators have also given me a better understanding of algebra.
>
> I think I understand algebra more after this course and if it worked for me it should work for almost anybody.

It was clear that the majority of the participating pupils enjoyed the experience. However, a small minority found it hard or unsatisfying. This demonstrates what we had expected, namely that a few students do not need such a prolonged introduction to variables, while others seem to find algebra difficult however it is approached.

Teacher comments

Each participating teacher submitted an invaluable commentary on their own impressions of the project, including ways in which it could have been improved. The project was not designed to provide a set of comprehensive, polished or coherent classroom materials. Nevertheless the teachers were clearly interested in using these materials again in the future and made a number of useful suggestions on how they could be organised more effectively, for example commenting on the practicality of the worksheets:

> I thought the worksheets were extremely well presented and the pupils were able to follow them easily. I gave the exercises involving predictions as homework to check their understanding.

The module contained no work on manipulating expressions etc. and our aim was to ensure that it did not 'teach the test'. This certainly appears to have worked, since several of the teachers commented on the apparent lack of a relationship between the calculator work and the tests:

> At times I was not entirely sure what the purpose of some of the exercises was. I felt that I might be emphasising the wrong thing.
>
> The project work provides a lot of very useful practice in algebraic ideas but there is very little practice of work resulting in non-numerical answers. By this I mean answers like $25m$ or $8t + 3$, i.e. answers like those required in the post-test.

Overall the teachers' comments were very positive and they felt that their pupils' algebra did benefit as a result of working on the calculator. One particular area mentioned centred around the primary purpose of the module, namely the idea of a letter as a store for a number and the value of a physical metaphor for this concept:

> I think it was useful to use the calculators for the idea of 'storing' a number. This was a concept that the children found easy to grasp. It was much easier to get this idea across with the calculators because the number was physically stored.

Further, the opportunities created for discussion were seen as valuable, if a novel experience for some:

> My pupils are not good at discussing mathematics! This may be partly my fault, of course, and the TI-80 work was good for encouraging discussion but, with little previous practice, I don't think the pupils were able to get as much from the discussions as they might have. Having said that, I do think the idea of discussing the work is excellent.

All the teachers felt that the pupils enjoyed the work on the project. Most were unqualified in their enthusiasm, although one or two noted that pupil interest started to wane a little at the latter stages and this is a fair indication that the work may be a little longer than is necessary.

> The work took around three weeks and at no time did they seem to get bored – normally, three weeks on any topic results in at least some pupils becoming disillusioned. The project work itself was varied and easily kept pupils interested and motivated throughout. They particularly enjoyed the screensnaps.
>
> Their enthusiasm is undiminished.
>
> The kids really enjoyed the work. It made the algebra much more interesting and obviously the novelty of the graphics was a hit!

The comments from the teachers are most encouraging since we realise (Thomas *et al.*, 1996) that the most important element in the successful introduction of technology into the classroom is the attitude and support of the teachers.

Conclusion

The evidence that we have presented from our study shows that students can obtain an improved understanding of the use of letters as specific unknown and generalised number from a module of work based on the graphic calculator. Approaching algebra by gaining an appreciation of the use of letters as labelled stores will, we believe, help students construct an understanding which will improve assimilation of later concepts. Certainly they enjoyed learning about algebra in this way, with the technology providing strong motivation in the short term. Their teachers too

appreciated the value of the experience and were keen to use the method again. With the assistance and support of classroom teachers, innovative strategies such as that we propose here can make a difference. To try to show the universal value and appeal of this approach to learning about variables we have also used the module in a parallel study in New Zealand. The results of this study are also extremely positive and we will be reporting these in the near future.

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COLLABORATIVE MATHEMATICAL REASONING IN AN INQUIRY CLASSROOM

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This paper investigates the informal reasoning of elementary school children as evidenced in their language and discursive practices, while they engage in collaborative problem solving in an inquiry mathematics classroom. The reasoning evolves in the context of the activity as thirteen fifth grade students discuss a variant of the "chessboard problem." We present transcriptions of video data which illustrate how the interpretive and argumentative strategies are applied in conjunction with the domain-specific knowledge of mathematics. We describe how a collective voice emerges from the coordinated and distributed reasoning among the children by providing instances where they complete each other's ideas, paraphrase each other's expression, repeat each other's language and articulate claims based on what another child says.

Theoretical framework. This paper investigates the informal reasoning of fifth grade children as evidenced in their language and discursive practices, while they engage in collaborative problem solving in an inquiry mathematics classroom. Currently, there is increasing interest in the study of discourse as an interaction of individual, social and cultural processes. This research which has examined the social and functional uses of language (cf., Bernstein, 1996; Freedman & Medway, 1994; Halliday, 1975; Lemke, 1995; Vygotsky, 1978), has also included examinations of the specific discursive practices which occur in mathematics and science classrooms (Ball, 1991; Halliday & Martin, 1993; Lemke, 1991; Roth, 1995; Walkerdine, 1991). In addition, much research into mathematics education has lent support for the view that students develop the understanding of what it means to do mathematics from the practices into which they are socialized (Lampert, 1990; Lave, 1988; Schoenfeld, 1992; Steffe, Nesher, Cobb, Golden, & Greer, 1996).

What then is the relationship between the individual and the broader social and cultural contexts? Our framework for understanding this relationship draws on the work of emergent theorists (Kieren, Calvert, Reid, & Simmt, 1995; Maturana & Varela, 1980; Varela, Thompson & Rosch, 1993). This position maintains that the individual and the social/cultural are equally privileged and mutually constituted in their interactions and should always be viewed in conjunction with one another. From this perspective language functions as a system of orienting behavior which permits individuals to construct a consensual domain of behavior in interaction with one another (Maturana, 1972/1980, p. 30). This can occur, however, only if the domains of interactions share a comparable framework. In addition while the linguistic interactions serve to orient the individual they do not control the subsequent performance.

Taken together these different research perspectives support the view that not only the classroom and school community but also the wider social and cultural contexts, model attitudes and practices which result in the generation of specific types of discourse and reasoning. It is in accordance with these multiple contexts that reasoning individuals make inferences, defend their choices, and provide explanations. We use the expression *informal reasoning* to identify an argument practice in which a valid

conclusion requires the presentation of supporting evidence, and the quality of the argument rests on the strength of the evidence gathered in support of the conclusion (Voss, Perkins, & Segal, 1991). In this way informal reasoning shares much with the view of explanatory proof as put forth by Hanna (1995). With respect to informal reasoning in mathematics, the specific problem-solving activities in which the children are engaged create an arena for argument discourse. The goal is "understanding; the coin of the realm [is] argumentation" (Schoenfeld, 1991, p. 338).

Method. This paper examines a large group discussion of thirteen fifth grade students who are discussing a variant of the "chessboard problem" and describes how they apply strategies, see patterns, identify mathematical structures and connect this information to support their formulations. In our data we are interested in evidence of a collective voice (Smithson & Díaz, 1996) as it emerges from the coordinated and distributed reasoning among the children.

The problem activity and its context. The specific problem activity along with many others in this inquiry mathematics classroom is structured in the following way: For each problem assignment the students first work individually, then collectively in groups of 2/3, and then in groups of 4/5. Finally they meet in a large group to discuss the problem. In practical terms this means they have been engaged with the problem on four separate occasions. The group discussions typically begin with a comparison of the students' answers and then proceed to a comparison of solutions and strategies. At each step of the way students are encouraged to reflect on what they did, justify their formulations with evidence, understand how someone else went about solving the problem and assess the value of different strategic approaches.

During the weeks prior to this particular group discussion, the students had calculated the squares for both a 4×4 and a 5×5 figure. In general, their initial strategy was count, check, and double check. Over many discussions this was eventually replaced by adding the squared values of the different types of squares, that is, $5^2 + 4^2 + 3^2 + 2^2 + 1^2 = 55$ (for a 5×5 figure). This performance does not mean that the students were always explicitly aware of the squared nature of the values since often what they added were the values, $1 + 4 + 9 + 16 + 25$. The students were then asked to identify the procedure for finding the number of squares in a 10×10 and in a 60×60 sided figure. The fact that they were not required to work out the solution but rather just describe the procedure for arriving at a solution was a less familiar task and almost all calculated the number of squares in the 10×10 (385). In most cases they transferred their successful strategies from solving the simpler 4×4 and 5×5 problems and calculated the sum of the squared values to arrive at 385. While this approach is applied by a number of students to the 60×60 figure, there is a divergent strategy adopted by 6 of the children which was to multiply the number of squares in a 10×10, that is 385, by 6 to get 2310 squares in a 60×60. It is at this point that we enter the large group discussion in which several children argue against this strategy.

The data. The data for this presentation consist of video-tape recordings of the children as they discuss in a large group. It begins with Will at the board while the other children are seated in a semi-circle on the floor. The entire argument takes twelve

minutes and is sustained by the students themselves. There are only a few conversational turns contributed by the teacher and these for points of clarification, and discourse management. Five excerpts from the transcript have been chosen and presented in their original sequence. The first sets up the problem; the second, third and fourth present the argument data, the claim and the warrant/explanation respectively (Toulmin, 1995); the fifth excerpt is included to illustrate collective understanding and argument. The excerpts were chosen so that the reader can focus on the coherence of the argument itself and the evolving understanding of the students. In addition the amount of idea completion, repetition and agreement contributed by various students as well as the large amount of overlap in their talk as signaled by / / and the immediate uptake or continuation of ideas as signaled by =, are presented as evidence for the collaborative and distributed nature of the argument.

Name	Transcription	Description and comments
1) Setting up the problem		
Will:	I can prove that that doesn't work. Well, in this pattern uh-	[Will goes to the board. He writes the following: 25, 16, 9, 4, 1 in a vertical column, and in a second vertical column shows how the values increase by adding two each time to the preceding number]
	(Lew seated in the circle, provides an "easier proof" and explains that since there are thirty-six hundred small squares in a 60×60 figure, the answer must be greater than thirty-six hundred. He says conclusively: "That's more than the answer you got. That in itself proves it wrong.")	
	Here. I can prove it wrong in another way.	
Will:	In these numbers here	[Points to the column of increasing values]
Ruby:	/What is that?/	[Request for clarification]
Mary:	/Which are?/	
Will:	Well for example, in one two three four,	[Draws a grid counting the lines as he draws.]
	one two three four five	[Draws vertical lines]
	one two three /four/	[Draws horizontal lines]
Gord:	/four, five/=	[Completes idea].
Will:	=five. So this is a five times five. So wait	[Writes the values for a 5×5 resulting from adding two to the preceding number]
Mary:	What are you doing?	[Request for clarification]
Gord:	You'll see	[Sets up expectation]
Ross:	That's a good question.	[Validates request]

In this first excerpt, Will explicitly states his goal which is to disprove the multiplication strategy used by a number of students. In preparation for his argument, he sets out on the board the values and diagram which he feels are necessary for

understanding the argument. Two of the students are asking for clarification as he proceeds, "What are you doing?" and Gord replies, "You'll see," which sets up the expectation for what will follow. Gord knows what Will is about to do since Will, Lew and Gord have already argued against this strategy in their group of five. In the meantime, Ross' response, "That's a good question" validates the request for clarification and underlines the fact that Will must consider the audience as he proceeds.

2) The data: Identification and definition of a pattern

Will:	Well it's to-, these numbers are a pattern that keeps on going on forever. /And you can-,/	[Points to pattern emphasizing forever with a downward gesture of his hand]
Gord:	/one two three four/ five six seven eight nine ten=	[Spoken very rapidly]
Will:	=you can use this to calculate the amount /of-, of/=	
Nora:	/It's not one two three four five six seven/	[Spoken very rapidly to G:]
Will:	=squares in any size of square. Well there's-, like even a googol by a googol, as long as you keep on going long enough.	
	(This is followed by four turns which revolve around what a googol is.)	
Will:	Anyway, so to get this you can either-, since it's five times five, you can either just go five down which is twenty-five? uh here or you can just go uh five times five which is twenty-five so you get to there, right? there and if you add the one, the four, the nine, the sixteen and the twenty-five you'll get the total amount of squares in this.	[Points to grid] [Points to values on board] [Points to board and underlines the last value]
Jane:	Right	
Will:	Well-, so you agree that this is a working pattern?	[Checks audience]
Ruby:	Hm hmm	
Will:	Well these-, what makes these a pattern and not just random numbers, one, four, nine, sixteen, twenty-five is it-, the number=	[Points to the sequence of numbers he just read out]
Gord:	=/increases/	[Idea completion]
Will:	/the difference/ in them increases by two each time. So the difference between one and four is three. Then the difference, so in other words=	[Elaborates]
Nora:	=oh okay, I understand.	
Will:	it's always a difference in two between the one-, here look, to get the four you /add three/	[Repeats]
Nora:	/I understand/	

Will:	and to then get the nine you add five to that and it always increases by two.	[Elaborates and repeats]
Ruby:	We understand how you get it.	

In terms of the mathematics, Will argues the numbers constitute a pattern which "keep[s] on going on forever." In addition these numbers are not random since they systematically increase by two. This is an important principle for his argument and the means by which he calculates the number of squares in a square of any size. He gives specific examples to back this and continues to do so until Nora has said, "I understand" on two occasions, and Ruby says, "We understand how you get it." Once this has been established he continues:

3) The claim

Will:	Now to prove-, so you need to know this to understand my proof. So what happens uh, if this restarted itself, say when-, like after maybe ten of them or something then, then /that would work./	[Pointing to the continuing pattern. Here he introduces the notion of "restarting" as the necessary condition for being able to multiply.]
Lew:	/It would work/ but it doesn't.=	[Overlaps and expands this idea.]
Nora:	=Wait Will, what are you proving that-, that-that the answer is=	[At the same time N requests clarification]
Will:	=that you cannot just take like a certain figure and find out-, like a figure five times bigger, that you can't just multiply the number of squares in there by five=	[States the claim]
Ross:	=take a number (()) and multiply it times six	[Extends the explanation]
Will:	For this five times five, to get a ten times ten which is two times bigger you cannot just multiply the answer here by two=	[Re-states the claim]
Lew:	=by two=	[Echoes W's claim]

At this point the claim has been put forth with respect to multiplying by 5, 6, and 2 by Will, Ross, and Lew.

4) The warrant or explanation for the claim

Will:	Okay well that goes on forever=	[The pattern as data]
Gord:	=that always increases by two more than the /last one/	[Re-iterates and elaborates]
Will:	/and it/ always is increasing this by a lot. Like-, (4 sec. pause) like=	[Points to board] [Students are speaking to each other]
Gord:	=Can I just say /something?/	

Will:	/Wait./ Now if-, now this keeps on increasing. Now what you're doing in multiplying, is just taking it and stopping it here and then re-starting it=	[Maintains the floor and connects the notion of restarting and the multiplication strategy]
Gord:	=You already said this=	
Will:	=and keep restarting it six times instead of having it keep increasing like it's supposed to. You're just restarting it and then restarting it again and then restarting it, and you're restarting it six times=	[Explanation continued]
Gord:	=Here, /Will/	[Hand raised for permission]
Will:	/instead/ of letting it go until you get sixty of these numbers=	[Explanation continued]
Gord:	=Will=	
Nora:	=/Oh okay/	
Gord:	/Will,/ Will=	
Will:	=Got it?=	[Checks audience]
Ruby:	= Okay but why=	
Gord:	=can I just say something?	
Will:	Yes=	
Gord:	=See if you're-, if you're doing-, say if you want to do a six-sided figure and you have a three-sided figure and you know the answer already for a three sided figure okay?	[Goes to the board] [Points to values on the board]
Will:	/so a six-sided figure/	[Repeats]
Gord:	/what you're doing/ is you're just making it stop wherever you want and then it's not a pattern=	[Reiterates W's notion of restarting.]
Will:	=Yeah so like here=	
Ruby:	=Okay	

The discussion continues and the same arguments are repeated. Lew then comes to the board to present his argument which he has stated at the outset of the discussion while Will was writing on the board.

5) Collective understanding

Lew:	Okay so what was the answer that you got?	[Requests the counter position]
Mary:	Two thousand three hundred and ten.	
Lew:	Okay two thousand three hundred and ten.	[Repeats]
Ross:	/Yes two thousand- me and Terry-, Terry and I got the exact same thing./	
Lew:	/Sixty times sixty You get thirty six thousand/=	[Writes on board]
Gord:	=hundred	
Lew:	Yes thirty-six hundred, sorry, and that's more than the answer you got and that's only the little squares /the little individual units/	[Makes a hand motion to signify "little-size."]

Will:	/That's the amount of the little squares/ but then there are-, but then there are	
Nora:	No, you can't prove that	
	/yes you can/ *(chorus of voices)*	
Lew:	/yes I can/=	
Mary:	= cause you go sixty times sixty equals=	
Gord:	=the area of the whole square	
Lew:	Look, for this one-, for this one you go one two three four five, five times five is twenty-five . If you count these squares there's going to be twenty-five of them.	[Points emphatically moving up the vertical right edge] [He counts softly one two three four five six seven eight nine ten]
	That's the exact same thing= *(chorus of voices)*	
Lew:	=it's just bigger numbers	
Will:	In sixty by sixty you go sixty times sixty you'll get the total amount of little squares	
Nora:	Right	
Jane:	So already your ((answer is)) wrong.	[Addressing the counter position]

Collective argument. If you take the preceding discussion removing the names of the speakers, it coheres into the voice of a single speaker. Just after Nora: says, "No, you can't prove that," the rest follows in one voice.

> "Yes you can. Yes I can cause you go sixty times sixty equals the area of the whole square. Look, for this one-, for this one you go one two three four five, five times five is twenty-five. If you count these squares there's going to be twenty-five of them. That's the exact same thing. It's just bigger numbers. In sixty by sixty you go sixty times sixty you'll get the total amount of little squares. Right. So already your ((answer is)) wrong."

While Will has the floor and often exhibits explicit strategies to maintain that position, what we see is an inter-active discussion of the ideas involving eight students. Specifically, we see instances where they complete each other's ideas, paraphrase each other's expression, repeat each other's language and articulate claims based on what another child says. The reasoning evolves in the context of the activity. While claims, justification, and counter-argument are viewed as rhetorical activities (Billig, 1996), these interpretive and argumentative strategies are applied in conjunction with the domain-specific knowledge of mathematics. The strategies both emerge from and depend on the mathematical activity in which they occur.

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WORKING FROM THE INSIDE WITH THEORY FROM THE OUTSIDE

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Working from a long term interest in supporting teachers'development of their practice, our recent efforts in the the area have led us to examine more closely the process involved in researching and working on practice. It seemed to us that reflection and analysis used alone as techniques in this process were limiting and this has led us to look more closely at the role that 'theory' and 'theorising' might play.This paper discusses our construction of the bringing together of theory and practice and describes teachers' responses to strategies we used in attempting to promote 'theorising practice' and 'practising theory'.

Introduction

Practitioner research as part of a teacher's professional development has gained in prominence in the UK in higher education courses. Whilst in these courses there has been a centralisation of teachers' own practices and experience there are obvious difficulties in linking school-based and institutional parts of these courses.

There has been discomfort with the false binary that could be seen underpinning previous academic courses and research and indeed teachers' perception of their own practices and professional knowledge; that is, a false binary in the form of the polarisation of 'theory' and 'practice'; of educational theory from teachers' day to day professional activities.

That this is of international concern is indicated by the formation of the PME 'Teacher as Researcher' working group and its extensive work. Indeed, there have been many moves in maths education to variously rehabilitate 'theory' or 'practice' as valid sources of professional knowledge.(eg Carr &Kemmis 1986)

Attention to practitioner research has generated a dissatisfaction with the methodologies available to structure teachers' enquiries. Alternative frameworks have been developed that offer a more authentic basis for such enquiries. When John Mason gave a plenary address to PME17 (1994), where he presented his development of a research methodology 'Noticing' for practitioner research, he emphasised that he was speaking to our experience.

Our own researches have been in the development of methodological frameworks on which to base courses for teachers that articulate the bringing together of theory and practice, of crashing the binary.

In giving here our construction of a dialogic approach to theory and practice in teacher development we draw on our work over the last 2 years with practitioners from both primary and secondary classrooms who have taken part in a term long module as part of a modular Master's course in Mathematics Education at the Manchester Metropolitan University. This course acted as a vehicle for us to devise strategies that might promote 'theorising practice' and 'practising theory'.

For many of the teachers attending this course, development of practice has been closely associated in the past with initiatives and ideas developed by 'experts' elsewhere which are thought to be replicable in a variety of classroom contexts. Whilst our privileging of teachers' own knowing offered no credence to this transference model of professional development our experience indicates that for many teachers reflection and analysis on existing views of practice served to create teachers skilled exclusively in this reflection rather than in conceptualising different forms of practice.

This led us to search for strategies that offer a means of looking again at over familiar classroom situations. This also raised questions about the professional knowledge per se and what it means to 'come to know' in our profession.

A view of knowledge for teacher development.

Knowledge about teaching comes in various forms and from many sources. Much of our personal knowledge is in the form of generalisations that are derived in part from our interaction with the world. More specifically for teachers this means knowledge derived from experience, amassed through practical work in the classroom. There is a 'taken for grantedness' about this knowledge; much of it remains unexamined and unarticulated (Elbaz 1990).

In order to examine this knowledge, we must become more aware of our professional acts, our professional decisions, the justifications we offer for these, and, importantly reflect on these in order to develop a critical sense of how our professional knowledge is formed. Through a deeper awareness and clearer articulations of our professional acts, we can hold up and acknowledge these as a source of professional knowledge and theory. Experience is not automatically theoretical; however it is open to generalisations, to theorising. By being able to form valid generalisations from instances of experience, it is possible to create an overall sense of current beliefs and preferred practices; and to imagine the possibility for refinement or change. It is very difficult to envisage changes to practice when experiences feel singular and unrelated.

To break down this sense of singularity we work from the belief that the development of practice requires the closer examination of the things that we currently do, the examination of the personal knowledge and the theories to which we attribute aspects of our practice and the broadening these horizons through consideration of the theoretical frameworks of others.

Our work with 'Noticing' (Mason 1992) has been significant in this articulation of our professional 'coming to know' and in our subsequent design of this teacher research module.

So to address our concern that the techniques of reflection and analysis alone do not assist teachers' conceptualising of different practices we sought strategies to use the 'theory of others' in the re-examinaton and reformation of our practices, professional knowledge and our own theoretical frameworks.

A framework for bringing research into practice and vice versa

For many teachers, there has long been a distinction between the knowledge offered by 'theory' or research and that which can be derived from classroom experience. There is also the sense that 'experience' can only be described in practical terms and not, for example, as reading or reflection. A question for us was how to work with 'theory' in such a way as to dissolve this distinction. In order for theory to appear relevant, there is a sense in which it needs to be recognised. Before we can recognise and understand something, we already need to have a pre-conception of it. 'We drive at an insightful and explicit understanding of something only on the basis of "something we have in advance"'(Gallagher 1992, p.61, citing Heidegger). There needs to be a resonance or jarring with something already existing in my cognition. In this there are possibilities for re-cognition, for while I still have my existing understanding, my attention has been shifted towards alternatives and the possibility of other ways of thinking.

In this sense, we are working towards creating a dialogue again, this time between our experiences, our generalisations and those of others.

We are not advocating here the straightforward acceptance of the propositional truths which others' theories appear to carry, but to consider our reactions to our reading. We need to be able to articulate our own response to these readings, clarifying our interpretations, and to consider the way we position ourselves in relation to the propositions on offer. Forms of language and concepts made available to us can (if they acknowledge the complexity of our practice) assist in our examination of our existing beliefs and understanding and perception those aspects of practice which are open to generalisation.

Linking our theoretical framework (for teacher development and research) to our practice (as tutors on a teacher research course)

When coming to work with teachers on our Master's level practitioner research unit, we employed strategies that were sympathetic to a form of research that gives explicit recognition of our professional knowledge and its role in theorising our practice. This is described in detail elsewhere (Mason 1992; Hardy, Wilson, 1996). In particular the process of anecdoting was used. Briefly here, the roles of teller of and listener to an anecdote are identified. The teller reviews the anecdote, the listener seeks resonance with her own experience. The listener also assists the teller in identifying where the significance of the moment lies. The teller may then consider systematically other incidents from her practice, to consult literature from her field of enquiries, to tell and re-tell these anecdotes to colleagues, discussing similar experiences and seeking recognition. This systematic reflection leads to a search for strands within her own experience that will throw up relevant questions to ask, areas on which to focus enquiries, interpretations that might be constructed, generalities which might be made.

Within this Master's course we employed particular strategies for engaging with theory alongside the process of anecdoting. Such a strategy was our use of readings from mathematics education literature, and it is this strategy we are choosing to describe in detail as it illustrates well the necessary (for us) position 'theory' holds in practitioner research. The illustrations we give are drawn from work which took place in university sessions in summer 1996 and from the writing which that group of teachers regularly produced. Both tutors acted as participant observers in tutor led and student led sessions keeping notes of student interactions and taking copies of written work.

Our choice of literature was inevitably influenced by readings that as tutors, we had found particularly powerful or useful. Some texts were chosen because they offer a model that teachers can easily recognise from their own practice, they offer reassurance; for example, Barbara Jaworski's article "'Is' versus 'seeing as': Constructivism in the Mathematics Classroom" (Pimm 1988). Others, for example, Brousseau's notion of the didactic contract and Bateson's of the double bind have a jarring effect, making (over) familiar practice seem less familiar and so open to re-examination and led to lively and fruitful discussion of teaching-learning interactions.

In the 1996 course we gave students two writings related to these, one John Mason's chapter 'Tensions' (Pimm 1988) , the other a section from Stieg Melin Olsen's book 'The Politics of Mathematics Education' (1987).

The students' first task was to read through both articles and highlight a section that resonated strongly or jarred in some way with their sense of classroom dynamics, and also to identify a section that they found inaccessible or unclear (the response 'all of it' was not allowed!) and bring these to the next session. The task was not one of gaining a 'full' understanding of the theoretical framework being offered but of finding some personal response to the writing.

We then spent some time as a group discussing their highlighted sections, mapping the notions and language used in the texts onto our experience, developing our sense of recognition or dissonance with our own stories about our practice, in a process of anecdoting as we described above.

John Mason (Pimm 1988) describes the didactic contract as

> '...between teacher and pupil although it may never be explicit. The teacher's task is to foster learning, but it is the pupil who must do the learning. The pupil's task is to learn, or at least to get through the system. They may wish to be told what they need to know, and often they wish to invest the minimum of energy in order to succeed. ... it contains a paradoxical dilemma. Acceding to the pupil's perspective reduces the potential for the pupil to learn, yet the teacher's task is to establish conditions to help the student to learn. .....The dilemma is then that everything the teacher does to make the pupil produce the behaviour the teacher expects, tends to deprive the pupil of the conditions necessary for producing the behaviour as a byproduct of learning; the behaviour sought and the behaviour produced become the focus of attention.
> Put another way, the more the teacher is explicit about what behaviour is wanted,

the less the opportunity the pupils have to come to it for themselves and make the underlying knowledge or understanding their own.'

The notion that these are inescapable classroom phenomena with no simple resolution, that you cannot satisfy both sides of the contract, seems uncomfortable for teachers who are striving to improve their practice. The need to resolve these tensions is clear, their inevitability is disabling.

The ways forward offered by Mason and Melin Olsen may be seen as circumspect and unclear. They certainly offer no slick solutions.They both discuss the power of awareness of (or sensitivity to) the 'bind' in unblocking the energy wasted in these tensions.

'To stay alive as a teacher, it is necessary to be aware of the variety of perspectives (...that students and teachers have as to the nature of learning and the role of teachers) and the fact that they are very deeply rooted.
In the midst of a lesson we respond to the pressures of the moment. But I have also caught myself locking up energy in resentment or guilt or ' if onlys'. .....
I believe that it is important to be open to these dilemmas, to take opportunities to talk about them with colleagues, to try to become precise in our articulations, because then it is possible to unlock the blocked energy and exploit it positively'
in Pimm 1988

Melin Olsen talks about working on a metalevel

The method of avoiding its (the double bind) damaging effects is to loosen it by communicating at the metalevel as often as possible, thus releasing the contradictions which determine it.'

He also talks of '...Bateson's conception of metalearning and double bind are all useful for a full understanding of learning behaviour. ... what are being offered are thinking tools which help to understand the pupil's predispositions for learning'

We are asking students to approach such theoretical notions as 'double binds' as tools, to apply to their classroom experience and see what awarenesses might be thrown up for them, and not as theories to be analysed for their truth in an absolute or external way. The task is not about identifying 'what I should have done'.

In the next paragraphs we discuss teachers' response to working with these theories and the difficulties they encounter. When faced with making meaning of a theoretical framework they often revert to a technical solution. In this sense students find working with 'theory as a tool' an unfamiliar notion. It requires effort and practice and in that sense is not easy. Using 'theory as a tool' is rarely teachers' initial response to the reading tasks we give them.

We followed on the discussion of highlighted sections from readings on double binds and the didactic contract by giving students the task of jotting down an incident from their teaching over the coming week that contained within it some kind of double bind and bringing it along for anecdoting in the next session.

The following anecdote was offered by Judith:

A lively, enthusiastic year 7 class 'bounce' into the classroom, buzzing with questions and answers for challenges from previous lessons. Kevin comes in two minutes later looking at the floor and walks up and down from the front to back of the classroom. Eventually he picks up a chair and drags it to the back of the classroom and sits on his own. I set a few questions for the class to occupy them and avoid too much attention on Kevin.

'Kevin, what's happened ? I can see you are upset, what's the problem?'
'Nothing, nothing!' came the forceful reply.
'Kevin, let me help if I can - who do you want to sit next to ? '
'Nobody wants to sit next to me'
'Why Kevin, why is that?'
'I don't know but I can't do maths, french, anything'

We continued in this vein for a few minutes where I tried to be positive. Kevin had produced some excellent work in the last few lessons in percentages. He agreed to start afresh after a lot of praise.

The group then worked on the incident against the theoretical notions of double bind and didactic contract with a sense of t working with theory as a tool.They were able to identify a range of interpretations in terms of the double binds that actors in the interaction were in:

- A bind for the teacher working with a mixed ability class:

I believe that confidence is important for all children to work successfully at mathematics. I am especially concerned that Special Needs kids develop this confidence. I want to help them break out of the demotivating failure cycle and break their self image as failures and of maths as 'too hard'. This would lead me to give separate (not too hard) work that they can succeed at (In practice such tasks prove unengaging and being given this sort of work rarely boosts children's confidence – rather it stigmatises.). At the same time I believe challenge is important and so think that they should 'hang on' with the rest of the class and that I should offer them the support they need to stay with the group and engage in this work.

This form of double bind is expressed by John Mason:

> The confidence - challenge tension leads educators to simplify the tasks given to low-attaining pupils, ...Any intellectual challenge is removed on the grounds that they cannot handle it and all edge, all interest is gone. in Pimm 1988

Thinking of Kevin's position in the incident the group developed the following possible scenario

- A bind for Kevin:

I want my own work. I want to feel special and to work on something that I can do. At the same time I want to be with the group.When I work with the group they know I'm slow and I don't like that. They expect me to struggle and don't listen to what I say or ask. They think I slow them down. I don't expect to have anything useful to offer to group work. I want to be able to be part of the group, to be accepted. It's important not to be different. I want to be able to show my worth.

In the group reworking we were not looking for the 'I should have s'. The group was not concerned with finding the 'truth' of Kevin's acts or the best

teaching approach. The concern for us was to use theory creatively as a 'thinking tool' (after Vygotsky's dialectical notion of tool-and-result, see Newnam & Holtzman 1993).

Teachers then had some time to capture in writing the reworked incident and their discussion, recording what had struck them, what resonated, jarred. What they are doing here can be seen as producing data for their own enquiries.

Another scenario that had been identified was a 'bind' for the teacher: Should my attention be with the whole class or the individual child?

Another teacher, Andrew has offered an anecdote earlier about the tension of breaking off a whole class discussion or exposition to follow up some query or misunderstanding voiced by one student.

He expressed this tension:

> 'Do I tell them to shelve that concern for now and just follow the class input (I'll pick it up on an individual basis later) or do I take the whole class's time dealing with the query in the belief it will prove beneficial for all of them ? If one child is having a problem then others will be too. What about those who have followed so far and whose time I am wasting?'

Andrew recognised this aspect in the retelling of Judith's anecdote:

> 'It's the same as in my anecdote. Should my primary concern be with the whole class, the pace of work they need, the support they need or with the individual who has come to my attention ?'

He quizzed Judith about her response of setting the whole class 'work to get on with' so that she could engage with Kevin individually.

> Was this quality work? Were you just occupying them? How did you justify it ? Should we pick up all queries as they come up? or work with a belief that children will make connections in their own time if they keep listening? that they won't/don't need to sort out every detail before they move on.'

Andrew recorded these as questions for his line of enquiry. Judith's anecdote acted as data for her enquiries and Andrew's response to her anecdote acts as second-layer data for him. The exercise is about creating data, getting a sense of what there is to be studied and reflected upon. In this way, his sense of recognition give validity to the focus of his enquiry.

Conclusion

We have described a process where theoretical tools are used to shed new light on over-familiar classroom interactions and illustrated with their responses the effectivity of teachers working this way. However our concern with the relationship between theory and practice endures. It continues as problematic. The issue of making theorising useful and valid remains for us. Different theoretical tools shed light on different aspects of classroom experience, offering a range of interpretations, indicating different possible responses decisions or choices and promoting different questions and enquiries. We are

not advocating an orgy of interpretations for their own sake but are still aware of the need for some way to discern the usefulness of theories generated.

Recognition and resonance or jarring with existing cognition perhaps determines the usefulness of any one tool. The theory must speak to my own experience. In this way also, validity may be added to the generalisation we make from our practice.

We see our role as teacher educators as instrumental in enabling teachers to open up their practice for examination. This involves becoming involved in group activities which support a process of coming to recognise and articulate personal theories which in their various guises underpin professional practice,that is becoming actively involved with 'theorising' practice and identifying pertinent questions. We may have to find different ways of thinking and talking about practice in order to begin to answer such a question other than superficially, and this can involve working with others' theoretical frameworks. We believe that theory and practice are symbiotic, that the articulation and clarification of one illuminates the other and that both aspects of this relationship need to be reflected upon in order for meaningful professional development to take place. We continue to work on our articulation of this dialogue and continue our work with teachers in investigating the implications for all involved in practitioner research.

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GENDER DIFFERENCES AND THEIR RELATION TO MATHEMATICS CLASSROOM CONTEXT

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In this article some relationships between gender and mathematics are examined. Quantitative analysis is based on a sample of 739 Finnish ninth-graders from 50 different lower secondary schools. Gender is found to act as an important mediator between success, self-confidence and classroom environment in mathematics. The effects of classroom environment are stronger for girls and they are seen most markedly at the classroom level. Teaching variables explains 60 percent of the variation of self-confidence between girls from different classes.

Previous studies on mathematics education show some clear gender differences in pupils´ mathematical attitudes and performances. One of the most consistent findings of these relates to girls´ lower confidence in learning mathematics than that of boys´ (e.g. Fennema, 1989; Leder, 1995). Also boys generally tend to score better than girls in mathematics tests (e.g. Friedman, 1989; Pehkonen, 1992). Various instructional, environmental or social factors has been suggested as determinants for these differences. Maybe the most often referred are the variables attached to characteristics or activities of the teacher. The way, that teacher "creates" these differences, however, is not well known. Shaugnessy et al. (1983) in their study of the relations between attitude toward mathematics and some environmental factors reported that teacher variables correlated with mathematics attitude more strongly for females than for males. Again in Forgasz´s (1995) examination of the relations between pupils affective variables and classroom environment, the pattern of relations was not the same for males and females, and the gender differences in affective factors were more marked on the group level than for individuals. Some relations has been found also between teacher-student interactions and students´ gender or their levels of confidence in learning mathematics (Hart, 1989).

This study was designed to consider the significance of environmental factors for the gender differences in pupils´ mathematical performances and some of their attitudes or beliefs. A special focus was in pupils´ levels of self-confidence in mathematics. These results are viewed to guide the efforts to find means for affecting the perceived central gender differences in mathematics within classroom context of which pupils self-confidence seems particularly influential. The results

of the study derive from a Finnish research project considering mathematical beliefs and performances with 739 (363 girls, 376 boys) ninth-grade Finnish pupils from 50 mixed classes and the mathematics teachers of these classes. The data of mathematical beliefs was based on pupils´ responses to a structured questionnaire measuring their views about mathematics, mathematics learning and teaching, and about self as mathematics learners (Malmivuori, 1996; Malmivuori & Pehkonen, 1996). Their mathematical performances were measured through the national grade 9 examination concentrating on mathematics at everyday situations (Pehkonen, 1996). The teacher factors of the study were again constructed on the basis of teachers´ responses to a questionnaire with 28 (open and closed) items covering teachers´ backround information, teaching practices, mathematical beliefs, and evaluation methods (Pehkonen, 1996).

Differences between boys and girls

The obtained results from the study were consistent with the previous findings of gender-related differences in mathematics. Boys scored on average 25.7 points compared with girls´ 23.8 points in the mathematics test. The perceived gender difference in favor of boys was statistically significant at 0.05 level, but still only minimal compared with the related standard deviation (12 points). To consider gender-differences in pupils´ beliefs, nine factors were constructed from their responses to the questionnaire on the basis of the performed factor analyses (Hannula, 1996). Statistically very significant ($p<0.001$) differences were found in three of those factors, but only two of these factors with largest gender-differences were selected for further analyses.

The first factor represented the constructed self-confidence measure on the questionnaire, based mainly on the items used in Fennema & Sherman´s (1976) Mathematics Attitudes Scales and partly on items constructed for Finnish research projects. It involved statements as "I am not the type to do well in mathematics." or "I think I could learn more difficult mathematics.", with positive loadings referring to high self-confidence. The other considered factor was named Co-operation with positive factor scores referring to pupils´ preference for active learning and interactions with other pupils and the teacher. The included items in the factor of Co-operation together with the related factor loadings are given below .

"Co-operation as a way to learn mathematics":

You can learn mathematics by asking help from other pupils (0.73)

You can learn mathematics by thinking together with other pupils (0.66)

You can learn mathematics by making mistakes (0.66)

You can learn mathematics by asking as much as possible from your teacher during the lessons (0.43)

Factor scores for these two factors were calculated for all pupils. Statistically significant (at the 0.1% confidence level) t-test values for the differences between girls´ and boys´ scores were found for both of these factors. Boys were more confident than girls on their abilities in mathematics ($t = -6.54$), whereas girls reflected more often than boys a tendency for co-operation in their learning of mathematics ($t = 5.00$). These results on mathematical beliefs were contrasted against pupils´ mathematics successes.

Interrelations within classrooms

Correlations between pupils´ self-confidence or co-operation, and their success in mathematics were calculated for boys and girls separately. Statistically significant correlations were found only between pupils´ success and their self-confidence in mathematics ($p < 0.001$), where the related positive correlation was slightly stronger among boys than among girls. Other correlations were very small. In order to examine these interrelations and gender-related differences at classroom level, mean scores for the three variables were computed for boy-groups and girl-groups within each class of the study. Below are presented the obtained correlations both at individual level and at classroom level (i.e. with the means of the scores) between these variables (Figure 1).

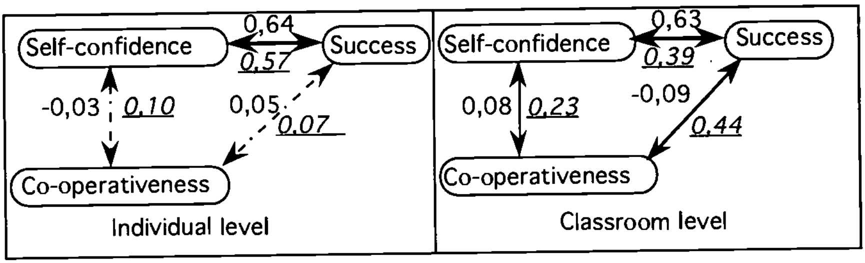

Figure 1. Correlations (separately for boys and *girls*) between self-confidence, co-operation and success at individual and classroom levels.

Some interesting correlations emerged at classroom level, that could not be found at individual level. At individual level there was no significant correlation between pupils´ mathematics success and their co-operation, but correlations at classroom level displayed fairly strong positive connection between girls´ success and their co-operation (with $p < 0.01$). This correlation was even slightly stronger than the positive correlation between self-confidence and success or between self-confidence and co-operation within girl-groups. The strongest correlation within boy-groups could be found between means of self-confidence levels and

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mathematics successes. Similarly as at individual level, the correlations between boy-groups´ self-confidence and co-operation, and between their co-operation and success were very small, indicating rather low significance of co-operation for boys´ levels of success in mathematics. Instead, there was again a significant positive relation between boys´ means of self-confidence and their means of successes.

These results are consistent with the obtained previous results of the central and rather independent role of self-confidence levels for boys´ mathematical performances compared to that of girls´, and again the significance of co-operation for girls´ learning of mathematics (e.g. Malmivuori, 1996; Pehkonen, 1992). These findings further indicate that significant environmental effects on pupils´ mathematical views and behaviours may operate within classrooms, that would not appear if the classroom context is omitted. Moreover, the environmental factors seem to have separate impacts for girls´ and again boys´ learning of mathematics. How important in this would be mathematics teacher-variables will be considered in the further analyses below. But first is presented a figure (Figure 2) that illustrates the distribution of girl- and boy-groups´ self-confidence levels and the correlations between these groups´ self-confidence and success in mathematics.

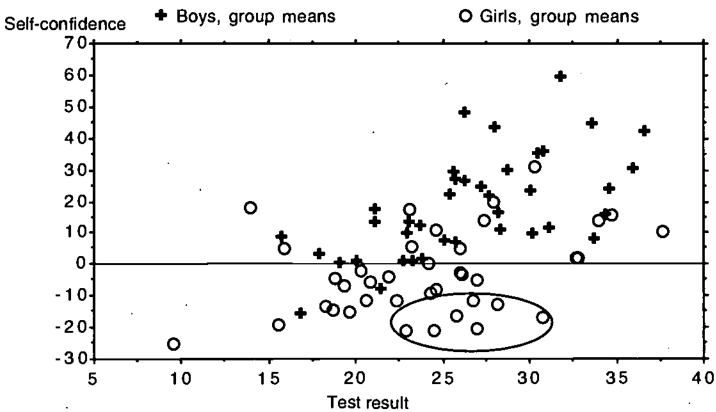

Figure 2. Self-confidence of girls and boys in different classes as a function of groups´ success (the classroom level). Seven low-confidence groups of girls are encircled.

The Figure 2 shows that most of the self-confidence levels of girl-groups´ display a negative attitude toward self, whereas the boy-groups reflect basicly positive attitude. Further, the positive relation between boy-groups´ self-confidence

and their success is more apparent than that of girl-groups´ - the result shown already in the differences in the correlations given above. Extremely clearly can be discerned the appearance of self-confidence levels of middle achieving girl-groups, that express very low self-confidence regardless of their average or above average success in mathematics. Seven of this kind of girl-groups are encircled in Figure 2. It seems that some very significant environmental or social features may affect girls´ attitudes toward self in mathematics, especially within the groups with large amount of middle achievening girls. They represent a group that may lie particularly open to these kind of influences. Examples of these possible influences are considered below with some teacher variables.

Connections between pupils´ self-confidence and teacher factors

In order to consider the effects of some contextual factors in mathematics learning, correlations were calculated between the variables obtained from teachers´ responses to the teacher questionnaire, and pupils´ self-confidence, co-operation and success respectively. In the Table 3 are presented correlations between the self-confidence levels of boy- or girl-groups´ and the teacher variables for these groups. All the statistically significant ($p < 0.01$) differences between girls´ and boys´ correlations (with teacher variables) are represented in the table.

Teacher variable	Correlation with self-confidence boys	girls	Difference (stat.sign.)
Use of textbook's teacher manual for planning	-0.14 (-)	0.30 (-)	**
Use of school-made material for planning	-0.15 (-)	0.52 (***)	***
Use of school-made material for teaching	-0.06 (-)	0.52 (***)	***
Teacher feels need for research problems that enlighten the structure of mathematics	0.22 (-)	-0.22 (-)	**
How often one uses following working methods:			
Exercises in small groups	0.23 (-)	0.63 (***)	**
Solving problems in pairs or small groups	-0.17 (-)	0.40 (*)	**
Co-operative learning	0.07 (-)	0.46 (**)	**
Changes in teaching in recent years	-0.14 (-)	0.43 (**)	***
Mathematics tests have changed	0.16 (-)	-0.26 (-)	**

Table 3. Correlations between some teacher variables and the self-confidence levels of girl-groups and boy-groups. 'Difference' refers to the difference of correlations between sexes.

Many of the teacher variables correlated only with the views of one of the two groups (boy- or girl-groups). In the presented table this concerns only girl-groups. Considering all teacher variables, correlations were usually stronger for girls, but there were also variables which correlated significantly only for boy-groups. The most (statistically) significant correlations were however the positive correlation

between girl-groups´ self-confidence and their teachers´ use of school-made material for planning and teaching, or of exercises in small groups as working methods. Positive correlations were found also between girl-groups´ self-confidence and their teachers´ emphasis for co-operative learning and for use of pair or small group problem solving in teaching. Recent changes in teachers´ teaching was also positively related with girl-groups´ self-confidence levels, but instead the number of working years of the teachers´ or the sex of the teachers´ did not have any significant correlation with the self-confidence levels of either of the groups´.

A stepwise regression analysis was further performed in order to find some examples of possible causal effects between teacher variables and girl-groups´ levels of self-confidence in mathematics. The results of the performed regression analysis are given in the Table 4 below.

Variable:	Coeff.:	Std. Err.:	Std. Coeff. :	F to Remove:
Teacher values also the prosess-nature of mathematics	0.384	0.194	0.196	3.926
Use of school-made material for planning	9.123	2.058	0.428	19.652
Use of working methods: Exercises in small groups	6.435	2.233	0.335	8.307
Changes in teaching in recent years	3.879	2.022	0.214	3.68
Test results of girls	0.65	0.224	0.284	8.408
		$R^2 = 0.64$		

Table 4. A regression analysis for girl-groups´ self-confidence in mathematics and some teacher variables.

Girl-groups´ means of test scores together with four teacher variables (the best predictors) explained over 60 % of the variation in self-confidence levels between girl-groups, from which the four teacher variables explained the most variance (almost 60 %). The best two predictors were teachers´ use of school-made material for planning and their use of exercises in small groups. The third predictor - girl-groups´ means of mathematics test scores - explained alone about 16 % of the variance in the self-confidence means of girl-groups´. The last two chosen predictors represented teachers´ recent changes in their teaching and their views of mathematics as a process.

Epilogue

In the presented results for studying ninth-grade girls´ and boys´ mathematical beliefs, clear gender-differences were found in pupils´ mathematical performances, in their confidence in learning and doing well in mathematics, and in their views of co-operation in learning mathematics. Also there was evidence that the influential aspects included in girls´ learning of mathematics in classroom context may differ from those features operating in boys´ learning, and that these aspects may importantly affect the perceived gender-differences in mathematics. Consideration of the variables of this study indicated further that mathematics teacher may represent an important factor in constituting these aspects. These factors were related to mathematics teachers´ activity and especially to things as teachers´ emphasis for co-operation in learning groups. The characteristics involved in the co-operative type of work in classrooms seemed to play an important role in girls´ successes and confidence in mathematics, but not in boys´ learning. As with girls, also boys´ mathematics performances were highly positively related to their self-confidence in mathematics, but not their self-confidence nor their successes could be directly connected to their teachers´ actions nor to their own preference for co-operation, as was with girls at classroom level. This result was different from Forgasz's (1995), in whose study the connection between self-confidence and learning environment was found for boys but not for girls on classroom level. However, in both cases the learning environment is related with gender diffrences.

Behind the studied variables and relations may be found a key to the explanations for girls´ generally lower confidence in their mathematical abilities than that of boys´, as well as to the possible ways of increasing girls´ levels of confidence in mathematics classrooms. These factors could be traced back to the learning processes and environmental features operating in mathematics learning situations, that constitute the framework for the appearance of the different experiences and lives of girls´ and boys´ in and outside classrooms (see e.g. Bem, 1993; Leder, 1995). As the results above show, much responsibility for these features may be assigned to mathematics teachers and their actions, at least in the case of girls´ learning of mathematics. Moreover, the kind of teacher variables considered here can be directly connected to the prevalent characteristics and processes of schools (e.g. factors reflected in the amounts of teachers´ use of school-made material). Thus teachers actions may not arise only from their personal views, characteristics or experiences as mathematics teachers, but also from the features and lives of schools.

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ARE LAWYERS PREY TO PROBABILITY MISCONCEPTIONS IRRESPECTIVE OF MATHEMATICAL EDUCATION?

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Recently there has been a trend towards admitting expert statistical evidence in UK court cases. There have been a number of cases, however, in which outcomes have been distorted by statistical or probabilistic misconceptions and by faulty inference. Typically, lawyers receive no training in these areas apart from their compulsory school mathematical education. The data from five groups of trainee lawyers demonstrate that their errors in assessing likelihoods persist irrespective of the level and type of mathematical education that they have received. The typical approaches and content of mathematical education at school or college need re-thinking. Data from two other groups of subjects (one of statistical educators) with different types of mathematical backgrounds were available for comparison purposes.

Background

The project was a collaborative initiative between the RSS Centre for Statistical Education in Nottingham and staff at the College of Law, the UK's largest law school (recruiting over 3,500 students a year at its four main branches in London, Guilford, York, and Chester).

In the past twenty years, there has been a steady growth of empirical research findings which identify the problems and misconceptions that people have in the areas of statistics and probability. (See reviews by Garfield, 1995; Shaughnessy, 1992; Kapadia & Borovcnik, 1991; Garfield & Ahlgren, 1988; and Hawkins & Kapadia, 1984.) It is clear that many of these misconceptions are peculiarly resistant to conventional scientific and mathematical education, and indeed that some may worsen with such schooling (Fischbein & Schnark, 1996). During the same period of time, there have been curriculum changes and developments of teaching methods and materials. The content and style of these developments have, however, tended to be belief-driven, rather than based on research findings. In any event, there is evidence to demonstrate that the misconceptions still persist, even after encounters with these newer teaching approaches.

If lawyers are also prey to these misconceptions, their ability to evaluate cases adequately will be affected. They will not easily be able to use, and may therefore avoid using, quantitatively-based arguments when these are appropriate. Nor will they, as advocates, be able to give correct guidance to members of the public who serve as jurors. They may also, therefore, be responsible for promulgating more widespread scientific misconceptions. The present study focused specifically on lawyers' facilities with assessing likelihoods, a skill that is fundamental to the execution of their professional duties.

Typically, lawyers in the UK receive no training in these areas apart from their compulsory school mathematical education. If they were to conform to earlier research findings with other (non-lawyer) respondents, they would exhibit a number of predictable misconceptions. Indeed, the outcomes of some recent court cases have suggested that this is a very likely state of affairs. This study is the first to focus on the precise nature of *lawyers'* misconceptions. It is a precursor to developing and implementing teaching materials and approaches specifically aimed at combating the difficulties they have with probabilistic information and evidence. The effectiveness of lawyers' previous mathematical training in preparing them reliably to assess likelihoods was also evaluated.

Methodology

A *Likelihood Schedule* was administered to seven groups of people, there being 217 respondents in all. Five of the groups were trainee lawyers. The schedule was designed to evaluate respondents' abilities to assess likelihoods in a number of different contexts relevant to what lawyers might encounter in their work.

All the groups were advised that it was usual for the schedule to take about six minutes to complete. No respondent was prevented from spending longer if necessary, but only a few did require any extra time. The groups were asked not to confer, and the researchers were available to ensure that the respondents did indeed work independently.

Information was also collected about the mathematical backgrounds of the respondents. A list of qualifications, ordered by level, was provided and the respondents recorded the appropriate code number on their schedules which were then handed in anonymously. A similar exercise was conducted for the statistical backgrounds of the statistical educators. The legal background of the first five groups was already known, and was also recorded for the final group.

Subjects

Five of the groups consisted of 168 trainee lawyers (all graduates) who had had a minimum of one year's legal training. A further group was made up of 23 mature postgraduate students, who were all following distance-learning courses in statistical

education, and who were also all teachers of statistics. The final group comprised 26 retired or semi-retired professionals or businessmen (PROBUS members). Three had received some legal training, including one who was a magistrate.

Most (91%) of the statistical educators had studied mathematics to degree level and beyond. Of these, at least 39% had qualifications in statistics that were lower than their mathematics qualifications. 30% had no formal qualification in statistics, and only 57% had a college level qualification. Only four (17%) had studied statistics in conjunction with education. [Note that this does not necessarily mean that they had studied *statistical* pedagogy.] Most of the trainee lawyers (94%) and PROBUS members (81%) had no formal qualification in mathematics beyond school level, although this includes 32% of the lawyers and 24% of the PROBUS members who had secured the equivalent of 'A'-levels in a mathematical subject.

Research Instrument

The research instrument was developed to reflect probabilistic misconceptions and heuristics which have been described in the literature (e.g. Kahneman *et al*, 1982). This included two items on the *Availability* or *Simulation Heuristic*, whereby people give an incorrect answer because they find it easier to imagine than the objectively correct response (Tversky & Kahneman, 1973). Like all the questions, the first item was couched in a longer, more wordy, form but it essentially required respondents to assess which was more likely to produce distinct panels of judges from a pool of ten judges - panels of 3, panels of 7, or neither. The second item referred to assessing the likelihood that one person would be more upset than another by what were objectively the same outcomes in terms of penalties experienced. One person, however, was seen to incur the penalty by the elapse of a long time period whereas the other person 'only just missed' arriving in time. A 'neither' response was available.

One item demonstrated possible over-reliance on the *Representativeness Heuristic*, under which people respond according to the degree to which they believe that a sample of observations matches up to their expectations about a population of outcomes (Kahneman & Tversky, 1972; Shaughnessy, 1992). This was a version of the fairly classic item related to dichotomous events that *appear* to occur in too systematic an order to be random (see also Green, 1982). Again the objectively correct answer 'neither' was given as one of the options.

One item was based on the possible existence of the *Conjunction Fallacy*, whereby people tend to overestimate the likelihood of two or more things that occur in conjunction with one another (Tversky & Kahneman, 1982). Respondents were given a potted description about Roger and then asked to rank the likelihoods of his being a law student, a student, someone who likes listening to jazz, and a law student who likes listening to jazz.

Problems with *Inferential Asymmetries* (described as *interpretation of conditionality as causality* by Falk, 1988) were also explored. The research literature suggests that respondents find it easier to reason the forward influence of events than their backward influence. They were told that two barristers and two solicitors were all going independently to a meeting, and were asked what was the likelihood of (i) one of the solicitors arriving second, given that the other one arrived first, and (ii) one of the solicitors arriving first, given that the other one arrived second.

Finally, the research instrument tested respondents' ability to assess likelihood in the face of two (conflicting) items of evidence. The problem requires the application of Bayes theorem to arrive at an objectively correct response. The research literature suggests that here, in the absence of recourse to Bayes, subjects will tend to ignore base-rate data which they perceive to be incidental, but not base-rate information which they see as causal (Bar-Hillel, 1980; Hawkins *et al*, 1992; Hawkins & Hawkins, 1992). A version of the taxi-cab problem was used, asking respondents to assess whether a blue cab company was liable in an accident given a certain percentage of blue (as opposed to green) cabs in the district, and a witness who identified the colour to have been blue but who was shown to be unreliable a certain percentage of the time.

All of the questions were couched within legal situations in order to remove the possibility that the trainee lawyers would be confused by an unfamiliarity with the context. Answers required only a tick (or in one case the entry of rank orders) in answer boxes printed beneath the questions. Respondents were also asked to give reasons for their answer to the base-rate question.

Outcomes

It was found that mathematical background only appeared to be a significant factor influencing performance on the first three items. Higher levels of mathematical qualification seemed to combat *to a certain extent* misconceptions associated with availability and representativeness. It must be remembered, however, that those more highly qualified in mathematics tended to be concentrated in the group of statistical educators, and they also tended to have received more *statistical* training. The performance of the different groups has been analysed for the present study, and also in comparison with research findings reported by others. The following overview gives a general idea of how the different groups performed. Many of the differences reported were significant at the 0.01 level. More detailed analysis is available for presentation at PME-21.

On the first *Availability Heuristic* item, the findings supported those reported in the research literature with most of the lawyers and PROBUS members (83-84%) giving an incorrect answer. However, the PROBUS members appeared to be less inclined to conform to the predicted bias (answering 'Panels of 3') than were the lawyers. The statistical educators were relatively immune to errors.

The second *Availability* or *Simulation Heuristic* item posed difficulty for *all* the groups, particularly the lawyers. Of the lawyers and statistical educators *who were incorrect*, 95% and 90% respectively chose the answer predicted in the literature (the 'near miss' person), but only 67% of the errors made by PROBUS members were biased in this way.

The statistical educators did not have any problems with the representativeness item. The PROBUS group did particularly badly, however, and they seemed to be more susceptible to give the answer that accorded with the *Representativeness Fallacy* than the lawyers (i.e. they chose the unsystematic order to be the most likely).

The *Conjunction* item was certainly the question that appeared to cause most difficulties for the respondents, especially the PROBUS group where the missing response rate is higher overall than for the other questions. There were significant differences between the groups with respect to the ability to correctly rank the likelihoods. Overall, the PROBUS members did much worse than the trainee lawyers and statistical educators. However, on ranking 'likes jazz' and 'law student who likes jazz', the lawyers and statistical educators did less well, and their performance dropped towards that of the PROBUS group, even though they might have coped correctly with ranking 'student' and 'law student who likes jazz'.

The statistical educators were relatively impervious to the potential problems in the *Inference Asymmetry* item. The PROBUS group did much worse, with only about 55% getting the right answer to each part (and not the same 55% either). Most of the lawyers (84%) correctly managed the likelihood of a second specified event given the first, but their performance was reduced to the level of the PROBUS group for the likelihood of a first event given the second.

On the *base-rate* item, one that specifically asked for a judgment of whether a company was liable for an accident, it was particularly disappointing to find that the lawyers performed worse than either of the other two groups. Once again, the statistical educators did best, but they were by no means infallible.

An analysis of the reasons given by respondents for their answers to the base-rate item revealed great confusion and eccentric reasoning. In particular, the lawyers often resorted to statements that had merit neither in legal terms nor in probabilistic terms. There was very little evidence of any real ability to bring these two separate strands of their understanding together in a coherent and constructive way, and several respondents admitted to resorting to pure guesswork. Guesswork in the absence of sound probabilistic intuitions, however, does not make for correct inferences.

Discussion

The research literature has typically classified errors according to certain labels. In some cases, the researchers have then started from these labels and devised means of

demonstrating that the phenomena do indeed exist, using essentially 'tricky' probability questions designed to trip up the respondents in predictable ways. With respect to identifying the real nature of the misconceptions, such an approach *can* become somewhat circular. However, it is clear that the outcomes of studies of probabilistic understanding are extremely sensitive to small changes in the wording of the questions. The present researchers therefore chose to adopt versions of tried and tested research questions, rather than embarking with a new, and therefore potentially unreliable, test instrument.

Nevertheless, a new framework of explanation was also adopted, derived from Glickman (see Hawkins *et al*, 1992). This was based on the identification of more general types of error - *failure to formulate and/or failure to enumerate possible outcomes of* - the required probability model. The analysis of the results was indeed also conducted with a view to validating, or otherwise, the more specific categories of error reported elsewhere in the research literature. However, it was felt that the Glickman framework was more likely to yield insights that could be applied to enhancing statistical education strategies, and that these insights would relate better to pedagogic reforms observed in many mathematical curricula. Greater emphasis on representation and modelling is entirely in keeping with moves towards transferable mathematical skills. Too much emphasis on specific error types tends to obscure our understanding of general cognitive skills and strategies. Ways must be found whereby the teaching/learning process will succeed in inclining students to use the 'right' approaches to probability questions. It is not clear that their encounters with statistics and probability in the conventional mathematics curriculum is succeeding in this respect.

The *Prosecutor's* and the *Defendant's Fallacies* have received much media attention recently because they have been associated with a number of miscarriages of justice (Thompson & Schumann, 1987; Finkelstein & Levin, 1990; Donelly, 1994; People v Collins, 1968; R v Deen, 1994). The relationship of these misconceptions with the lawyers' performance on the *Conjunction Fallacy* and *Base-rate* items in the present study, as well as with the Glickman framework of explanation, is promising as an area for research.

Conclusions

The conventional forms of mathematics training that these groups of respondents had received were not sufficient to instil the necessary understanding of chance, probability and likelihood reflected in these test items. Even the statistical educators who were generally more highly qualified in mathematics had difficulties with some of the items. Those items on which they did better were the ones that were more directly related to their own statistical training, and to the content matter that they were now teaching. It was certain, however, that their grasp of the relevant concepts

was sufficiently fragile to cause considerable concern over their competence to teach statistics.

If it was the *statistical*, rather than *mathematical*, backgrounds of the statistical educators that were contributing to their enhanced performance on some of the items, then we must find out how to provide other groups with equivalent preparation in appropriate strategies with which to attack probability questions. In fact, it is clear that there is room for improvement in the training of *all* groups in this respect.

The lawyers had considerable difficulty with most of the questions, confirming the inadequacy of their preparation for the increasingly quantitative decision-making now facing them in their work. If the mathematical education now found in the National Curriculum (which is all that most lawyers receive) is not a satisfactory preparation, it is certainly necessary to find and implement remedies. It might be that some form of quantitative training should be introduced within all legal training courses. An innovation such as this will be more effective if the methods and materials relate directly to what research can show us about the nature of lawyers' existing misconceptions.

The PROBUS group were a less homogeneous group in many respects, and accordingly they seemed to produce less predictable answers. Their years of experience in business and the professions had not been sufficient for sound intuitions of likelihood to emerge. This is not something that we want of tomorrow's business managers and professionals.

The Glickman framework provides a useful alternative or adjunct to the conventional error-based classification schemes in the literature. It is indeed an interesting starting point for developing more generalised and effective teaching strategies. Future research is needed that will develop and evaluate such strategies in the classroom.

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AN EXPRESSION OF THE IDEA OF SUCCESSIVE REFINEMENT IN DYNAMIC GEOMETRY ENVIRONMENTS

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Abstract

In this paper we deal with students' work in Dynamic Geometry Environments (DGEs). More specifically, we analyze students' thinking from the perspective of successive refinement: a problem that is non-trivial for its solver tends not to be solved in one shot, but rather through a sequence of steps, in which each step is an attempt to "improve" on earlier ones. In some cases, steps lead to deadends, and one must backtrack and look for a new path to the solution. We will show how this idea is expressed as students construct a square in a DGE and as they determine the extent of generality under which a theorem is true.

Introduction

Dynamic Geometry Environments (DGEs) enable one to construct geometrical figures by specifying certain relationships among their components. A distinguishing feature of such environments is their "dragging mode," which allows one to manipulate geometric constructions by dragging various parts (like points, segments, etc.), while preserving the specified relationships. One can then study the unanticipated invariants—properties and relationships that were not explicitly specified, but are consequent to those that defined the construction.

Most of the research about learning in DGEs focuses on its geometrical aspects, e.g., understanding of geometrical concepts (Capponi, 1992; Laborde and Laborde, 1995), and an analysis of such environments for learning geometry (Laborde and Straßer, 1990; Holzl, in press). Other research shows how a DGE may help students understand related mathematical concepts, like the notion of function (Hazzan and Goldenberg, in press). In this paper we attempt to show how DGEs enable students to approach problems through a process of successive refinement.

The idea of successive refinement is widely used in computer science, where people look for ways to overcome the complexity of certain very large programming tasks. Leron (1994) describes how learning and teaching with this idea in mind differs from the traditional approach:

> The standard model of dealing with complexity advocates decomposing a topic into a linear sequence of tiny "atoms," then proceeding along the sequence, mastering the atoms one piece at a time. The model of successive refinement offers a viable alternative for dealing with complexity. One starts with a simplified version of the phenomenon under study, and refines it successively to include more and more details, subtleties and precision. Through the entire process, the student constantly deals with the whole picture, though it may be vague or imprecise in the intermediate stages.

In this paper we present our ideas around two tasks which we asked 9th and 10th grade students to deal with. In the first task the students were asked to draw a square in a DGE. It turns out that this task, which might at first appear quite simple, involves a number of non-trivial steps for beginners. The students refined their construction in stages, so that in each step they improved the square, taking into account additional essential conditions of this geometric shape. The second task was to check the scope of the hypothesis under which a theorem is true. The discussion was based on the theorem: The midpoints of a quadrilateral form a parallelogram. When one tests this theorem in a DGE and drags a vertex of a quadrilateral (as ABCD in the first and second drawing, left to right, in Figure 1), he or she is likely to encounter four-sided shapes that were not anticipated at the outset of the experiment (the other two drawings in Figure 1).

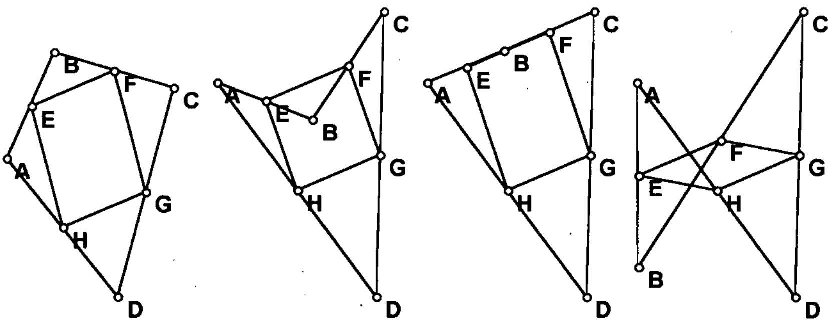

Figure 1: Is ABCD a quadrilateral?

This phenomenon raises questions, including whether these new figures are to be considered quadrilaterals and, independently of that definitional question, whether the

theorem holds for these other shapes. Such a task invites an initial formulation of what is a quadrilateral, an examination of the theorem and of the quadrilateral definition, a verification of the essential conditions, and perhaps even a *refinement* of the definition or theorem.

Our observations confirm Leron's observation that students do not reach complex targets in one shot, and help to describe the process by which they do reach the target. In other words, students approached these tasks not with a "top down" plan, but through an iterative process—a sequence of stages in which they reexamine and make successive refinements to their mathematical ideas.

We believe that the dragging feature that makes DGEs dynamic, strongly influences students' learning approach in the direction of successive refinement. In both tasks, we will emphasize how this property of DGEs enables students to rethink, reexamine and reflect both on their own thinking and both on the mathematics involved. In the first case the object under the discussion was a real mathematical object - a square; in the second case it was a meta mathematical object - a definition.

Research background and analysis

The analysis described here is taken from interviews with high school students. Six of them were not familiar with DGEs before the interviews and dealt during the interview with the construction of a square; seven have been familiar with such environments and dealt during the interview with the quadrilaterals definition. For reasons of space limitation we include here only a few short excerpts. More excerpts will be presented in the talk and in a broader paper of the research described here.

Task 1 - What is a square?

On a paper, a square is a square. The situation is different in DGEs, giving vivid meaning to the idea of what Healy, Hoelzl, Hoyles and Noss (1994) call an "unmessupable figure." In DGEs, one can create malleable figures. If all configurations into which the figure can be adjusted retain the originally intended character, we call the figure "unmessupable." In other words, a sketch that can be dilated or translated or rotated by dragging the vertices, but that always remains square (or rectangular or

rhombic or whatever) is an unmessupable square (or rectangle or rhombus or whatever).

There is more than one way to think about successive refinement in relation to this task. Students might, for example, sketch a square as four segments attached at their vertices and then "successively refine" the sketch by adjusting each vertex carefully, staring hard at the drawing, and then making further adjustments until the picture looked perfect. It is certainly *possible*, especially if there is a requirement that the square not be "level," that students may think at other than a perceptual level and somehow articulate more clearly for themselves some of the mathematical properties that are inherent in squareness.

We made a game out of the task of drawing an unmessupable square. If we succeeded in messing up a construction, students could start a new sketch. Thus, they'd always eventually "win" through a process of successive refinement. Step by step, they'd analyze the conditions for squareness, and add one (or more) additional feature to ensure that squareness. Each step improved the sketch, and also improved the students' understanding of squareness.

Jill started by drawing a square "by eye"—four segments adjusted to look right. Of course, they could readily be *un*adjusted, messed up. Then she added the condition that opposite sides must be parallel. Even though Jill's construction looked like a square, it behaved like a parallelogram when point A was dragged (Figure 2):

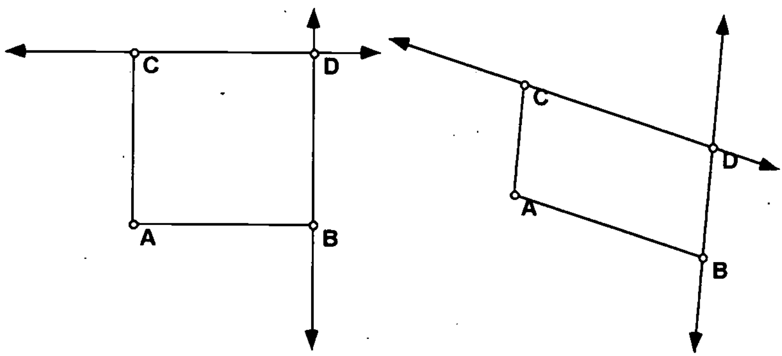

Figure 2: An unmessupable parallelogram

She recognized the problem and added the constraint that the angles remain fixed at 90°. The resulting invariance was rectangularity (as point C is dragged in Figure 3), but still not squareness. Finally, she saw that she had to ensure that the four sides remained equal.

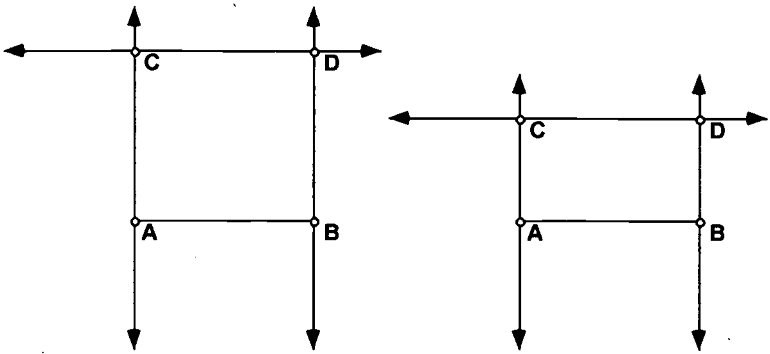

Figure 3: An unmessupable rectangle

Here are some of the mathematical, epistemological and educational thoughts about this kind of tasks, the spirit of DGEs and the idea of successive refinements:

- Such an experiment may lead to a discussion about the idea of invariance in general, and also provides an opportunity to discuss shapes as an invariant. This kind of thought experiment would likely seem arbitrary, not to mention difficult in a static environment.
- Such an experiment makes quite salient the hierarchical relationships among geometrical objects, showing which shape is more general and which is more specific. As a result of the successive refinements in the process of constructing the square, it is easy to see that a parallelogram "requires" less conditions then a square, and hence the squares are a subset of the parallelograms.
- The "bugs" in students' initial solutions have a real positive contribution. Just as Papert (1980) describes in relation to Logo, our students' errors help focus their attention on what remains to be added to refine their solutions (and mental constructions, of course).

Task 2 - What is a quadrilateral?

We asked the students to construct a quadrilateral and then connect the midpoints of its sides. They would assert that the resulting "inner" shape is a parallelogram. We then asked them to check, by dragging various points or segments, to see if that inner

parallelogram results for *all* quadrilaterals, and also to see if it results for anything *other* than quadrilaterals. That is, are there monster[1] shapes whose midpoints, when connected, still produce a parallelogram? (See examples in figure 1.) This question is raised because these shapes are deformations of the original quadrilateral and because even for strange shapes, which do not look like a quadrilateral, the inscribed shape is still a parallelogram.

In this case we see how DGEs enable to reexamine and to restate the formulation of the quadrilateral definition. It may lead to a new definition of a quadrilateral, to a refinement of its first definition or to a creation of a new concept, which captures all the shapes for them the theorem is true. This kind of task is quite difficult to do in a static environment.

The idea of redefining a concept is similar to the case described in Lakatos (1976) 'Proofs and Refutations' story. In that story a teacher with his class discuss the question of what is a polyhedron. This question is approached using the process of finding monsters and barring them by a refinement of the polyhedron definition. Goldenberg and Cuoco (1996) raise some questions in the same spirit in relation to the quadrilateral definition discussed here:

> How do [students] handle the fact that moving a single point can...create "monster quadrilaterals," such as the triangular configuration or the self-intersecting bowtie? [See Figure 1] Do they seem to fail to notice these cases altogether, or ignore them as if they do not exist or are a kind of irrelevancy? Do they think of these as interesting but separate byproducts of a set of observations about quadrilaterals? Or do students experience this as conflicting with their previous notions of "quadrilateral"? If so, do they extend the definition? Define exceptional cases? Do they resolve the conflict by what Lakatos (1976) calls "monster-barring"—deliberate and careful reworking of all relevant definitions (in this case, of quadrilateral) for the purpose of rejecting aberrant or troublesome cases?

In the presentation we will address questions like the above. For example, most of the students described the self-intersecting bowtie quadrilaterals (See Figure 1) as two

[1] The term "monster" comes of course from Lakatos' book *Proofs and refutations*. This is a metaphor for instants of mathematical examples which lead to reexamine definitions or theorems.

triangles. Two questions are relevant here: What prevents the students from accepting this shape as a quadrilateral? Why did they prefer to conceive it as two triangles? Here are some arguments:

Tom: They're not like -- you can't say that -- they're not a theoretical thing because they're triangles. [...] They're not -- they're like two of them. It's not one object. [...] It's two. [...] But it still makes a parallelogram.

Arvin: Here it's not [a quadrilateral]. [...] Because it has more than four sides. It's more -- it's like two triangles.

Such instances of the quadrilateral, in addition to cases where we do not get a parallelogram, led the students to examine, redefine and refine the quadrilateral definition.

Conclusion

The ability to solve a problem in the process of successive refinement depends largely on the environment in which one works. Sfard and Leron (in press) describe the contribution of the computer and its relationships to problem solving in the process of successive refinement:

> Unlike paper, the computer speaks to you. It responds -- sometimes angrily! -- to anything you might be doing. [...] Mistakes (or "bugs" as they are now called) become part of the deal -- they turn into stepping stones for improved proposals, for more promising conjectures, for further progress. Half-baked ideas gain legitimacy, since they are now seen as a necessary first step toward a fully satisfactory solution. Here it is taken for granted that an answer to a problem can only be obtained by a spiral process of partial solutions and their successive refinements.

This paper is one in a series of papers in which we are trying to present an epistemological framework for dealing, working and thinking in DGEs. In relation to the topic of this paper, the question of whether the successive refinements we see in students work are learning about the geometry or learning about the software, is still bothering us. The fact that the software tool does not recognize the attempt to draw a square the way any human would, and take that to be the intent, plays a role here.

Other questions which we will address in the talk are: Why successive refinement? Why would it not be better to think everything out logically in advance? Why is it not enough just to analyze the definition of squareness, which includes all the

properties that must ultimately be built in? Partial answer comes from our interviewees' performances which were evidence that having the definitions was not enough. It allowed them to articulate the *nature* of the failures -- more than a mere perceptual task -- when they saw them, but apparently did not allow them to predict the failures in advance.

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EFFECTIVENESS OF A STRATEGY AS A SOCIOMATHEMATICAL NORM

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This paper presents 'effectiveness of a strategy' as a plausible example of a sociomathematical norm emerging from a problem solving session with preservice mathematics teachers. Effectiveness is used to illustrate the claim that the notion of sociomathematical norms needs to be further specified in order that it can be clearly differentiated from social norms. Such specification needs to account for the systemic intentions regarding the mathematics at stake.

The notion of *sociomathematical norms* has been characterized as "criteria of values with regard to mathematical activities" (Voigt, 1995, p. 196) and has been differentiated from "general classroom social norms that apply to any subject matter area and are not unique to mathematics [because they focus] on normative aspects of mathematics discussions specific to students' mathematical activity" (Yackel and Cobb, 1996, p. 460-461). I contend that the difference between the notions of social and sociomathematical norms needs to be further specified if they are going to be used differentially. I suggest that this shortcoming is concomitant with the absence (in the theory) of mathematics as cultural knowledge: Although *mathematical activity* is present as the basis of the differentiation, such presence is token in the analysis done by the 'emergent theory' team (Cobb and Bauersfeld, 1995) and accounted to the "subtle influences" (Voigt, 1995, p. 199) of the teacher in the classroom.

This paper intends to contribute to the theoretical debate concerning the use of this construct by analyzing the development of effectiveness as a socio-mathematical norm in a problem solving session that was part of a case study with preservice teachers—hereafter Jack and Jill (Herbst, Mesa, and Gober, 1996). Jack and Jill were engaged in a problem that required them to work together in comparing areas under four curves given by their graphs. Graphing calculators were among the available tools.

Lack of consensus and difference: Which is sociomathematical and which is social?

Yackel, Cobb, and Wood (1991) defined *social norms* and gave as an example "that partners should reach *consensus* as they work on the activities" (p. 397). Yackel and Cobb (1996, p. 461ff) identified *mathematically different* as a *sociomathematical* norm. The following ex-

[1] I am grateful to Jeremy Kilpatrick for many helpful comments on the original manuscript.

amples drawn from Jack and Jill's problem solving session suggest that the attribution of those labels is problematic.

Example 1: Despite Jack's attempt to solve the problem by modeling the equations, Jill's strategy of drawing blocks of 1 square inch each has been successful to keep them working together and making progress—unlike Jack's approach that had failed in Graph 2. They have used Jill's strategy for the first three graphs and they are now facing the fourth:

> *Jill*: This is very hard to do [in my way], Jack. There, one whole block, two whole blocks. This is gonna be 3, 4,... 5, 6 [counting whole squares first and then compensating 2 pieces for 1 square and three pieces for one square], don't you think?
> *Jack*: 7 [compensating some pieces].
> *Jill*: About 7 and a half maybe [teasing him].... Jack, I'm gonna guess it's B!...There must be some mathematical way to do this Jack! [teasing, very secure]
> *Jack*: I was trying to do that in the last one, and you didn't let me! [frustrated]
> *Jill*: [laughs] Cause, I don't remember how to do that.

Jill's teasing is a multifunctional statement. Beside its propositional content to which Jack reacts, it also conveys the message that an answer has been achieved nonetheless. Her use of *mathematical* could well have been replaced by *elegant* or *precise*.[2] The *mathematical difference* of the strategies seems to be acknowledged by both participants. Still, the overall social norm which indicates that they should work together is the source of Jill's justification for insisting on her not-so-precise, although efficient, strategy. They both agree that the strategies bear different relations to mathematics on the basis of a *sociomathematical* norm (i.e. difference). However, they disregard it in working the problem on the basis of a *social* norm: that they would work together and produce one agreed-upon solution.

Example 2: Jill's approach had successfully answered the question posed. The pair had offered a final answer based on Jill's solution. Then they engaged in a conversation with the interviewer (N); the conversation was smoothly leaving Jill out and focusing on what Jack had been missing. Then comes the following excerpt:

> *Jack*: I just, I would set the range up on [the graphing calculator] to be the exact range of [the interval] and so I'd have this side of the picture here.... I'd use key points....It's still not exact... but ... it's closer than [Jill's approach]
> *Jill*: No-o-o! [long as in a lament] But my way worked!
> *Jack*: But, I mean, that's...the same idea. I mean,...if you get down to it that's where [hers] would go to. *Jill*: What you call that?

[2] In an interview immediately after the session Jill was asked about the status of her strategy. She qualified it as mathematical but less precise than Jack's.

Jack: If you can make that smaller and smaller, that's the limit ... that ... started what I was trying to do.

Jill: Riemmann sums [sad, as in a lament. Jack and N are talking in the background.]

By focusing on Jack's approach, the interviewer was pointing at the mathematical difference between Jack's and Jill's approaches. That context allowed Jack to downplay the agreed-upon solution. As Jill's reaction suggests, this action went against their consensus. Jack acknowledged her by implying that for all present purposes both solutions were equivalent. *Consensus* acted as the sociomathematical norm.

Mathematical and social for who?

The notion of sociomathematical norms is a very useful one. Indeed, the whole problematique of the development of constituted sociomathematical norms is crucial when studying classroom mathematics. Still, one needs to point out that when one attributes the emergence of those norms from mathematical activity such attribution is made by an observer (not by the teacher or the students); an observer who is able to distinguish the mathematics at stake from the problem at hand. The participants cannot make such distinctions until they are required to do so by the logic of the practice in which they are involved. In the 'emergent theory' classroom,

> [The teacher addressed] an issue which is not often distinguished by young children[:] the difference between the meaning of a disagreement that is personal and one which arises from differences in ways of thinking mathematically (Wood, 1996, p. 431).

The example that Yackel and Cobb (1996) provide to illustrate how the teacher intervened to establish "the expectation that rationales should be mathematical" (p. 469) shows that the differentiation between the mathematical and the social was not clear for the participants: The teacher intended to persuade Donna that in the same way as she would not accept that her name is Mary because her name is Donna, she should not accept that the solution is eight if she believes it to be six. The episode is not an example of Yackel and Cobb's contention: From Donna's perspective, the rationale is not mathematical at all; it seems more an occasion of direct moral instruction.

I don't intend to blame the teacher. To the contrary, the point is that even with such a collaborative person, there are sociocultural issues that have to be accounted for, as they matter for the issue of how norms are constituted. In particular, the existence of mathematics as culturally shared knowledge (which finds the teacher as an agent in this case) and the exis-

tence of expectations and ways of functioning associated to the teacher by her socialization into the school as a system. The teacher is justified (moreover, coerced) in providing the norm to Donna on the basis of her social responsibility and of her knowledge that it is true in mathematics. From Donna's perspective, even if in the future she does not agree with what she does not believe, what a mathematical way of thinking is may become even more problematic. As a consequence, if the students did not have ways to differentiate personal from mathematical disagreements, the teacher's action of pointing this difference out can only provoke an effect of the opposite direction to the intended.

In fact, if not all that can be attributed to knowledge from the observers' perspective is within what the participants regard as knowledge, it is likely that at least when accounting for social interaction one cannot isolate cognition. The observer needs a broader systemic model that accounts for the relations between power, knowledge, and discursive practices. (See Wertsch and Rupert, 1993, p. 228ff; Rouse, 1994). If this argument seems plausible, we can conclude that the quoted distinction between disagreements that are personal and those that are mathematical is artificial. In fact, taking the sociocultural perspective advocated by Lerman (1996), one would expect that the unfolding of mathematics as a discourse differentiated from the ordinary discourse of ordinary life would initiate the participants' awareness of the different nature of those ways of disagreeing.

If a classroom as a system where shared meanings are produced is going to be an *inquiry* mathematics classroom, this decision will have to act on the conditions and constraints for the differentiation of mathematics as knowledge (that in Vygotskian terms are dual to the differentiation of the self and the production of consciousness). It looks like these intentions are among the *subtle* influences that Voigt invokes: But, actually, this issue seems to point at a theoretical black hole that has eliminated from sight the social project of education—not quite a subtlety.

It becomes a technical problem for the educator to find the ways in which the activities at hand and the mathematics at stake can be coordinated so that what the observer recognizes as sociomathematical norms are seen by the participants as mathematical norms. The theory needs to acknowledge this basic intentionality of education: "In mathematics education we are concerned with students acquiring the concepts and language of the community of mathematicians" (Lerman, 1996, p. 145-146).

The Jack and Jill Project cannot stand close comparison with the 'emergent theory' team project, primarily because the former was not a classroom research project. However, some observations are appropriate for a discussion of sociomathematical norms. The problem involving Jack and Jill was an intentional environment with respect to the kind of knowledge that they would invest. They were not conditioned, but some of their actions could be predicted by the characteristics of the situation. The problem was very open but it created conditions for Jack's *modeling strategy* to be plausible but not easy, and for Jill's *block strategy* to be plausible, yet questionable. Solving the problem within the norms of a collaborative work demanded the negotiation of a common strategy. During that process emerged a norm that I would call *effectiveness* and can be traced as follows. After Jill's *block strategy* produced a result for the first question, Jack proceeded to involve her in his strategy. Working together (but on different tasks), they achieved an equation for the first graph. Then comes this excerpt:

Jack: Okay, this is 1/2 x +1 cubed plus 4.... Easy?
Jill: But look at the second one *Jack*: All right
Jill: I'm telling you Jack, I've got it figured out. [*Jack* smiles at her paper. Jill laughs.]
Jill: I already know which one.... Don't you think?... [*Jack* complains inaudibly]
Jill: But that's right!... Don't you think? Or am I wrong? *Jack*: I don't know
Jill: I figured this whole thing is about 7, but see there's seven whole blocks right there [referring to Graphs 1 and 2 alternatively. She is saying that Graph 1 has already 7 whole blocks inside plus the ones that are not completely contained, but Graph 2 even compensating incomplete blocks only adds up to 7.]

That is, the apparent difficulty of applying Jack's strategy to the following graph is contrasted with the apparent effectiveness of Jill's strategy to answer all questions. After the above mentioned exchange, the pair follows Jill's lead with Jack's acceptance that they can discriminate between the first two graphs:

Jack: I'm sure it's right *Jill*: Do you think? [Tone as in 'Let's talk about it'].
Jack: All right... so.... Which one of these? [He points at Graphs 3 and 4.]
Jill: Oh, man! ...This is gonna be x square... This is gonna be a parabola ... quadratic.
Jack: [Drops pencil] Do it with the ruler thing! See if that will work on this!

I take this as an application of the same norm used in the previous excerpt, but reversing the direction—that is, questioning whether Jill's strategy is correct by testing whether it is effective in a particular case. After both work on the proposed pair of graphs using Jill's strategy, the dialogue continues:

Jack: All right [Jack goes back to Graphs 1 and 2]
Jack: So which one is bigger? This one or that one [points to Graphs 2 and 4]
Jill: [After looking at the graphs for about 8 seconds,] I don't know.
Jack: All right.... Do you know ... do you have any idea of a [formula] whose graph would look like...?

The answer achieved by Jill's strategy (comparing Graphs 3 and 4) cannot be integrated with the previous question (comparing Graphs 1 and 2) so as to sort the four graphs.[3] Jack's implicit claim that Jill's strategy was not effective and his explicit invitation to return to his modeling strategy were based on that impossibility. Jack led the work trying to model Graph 2 which he had correctly understood as a quartic. After several attempts that gave him graphs that were close but not close enough to satisfy his standards he stared at the problem for a couple of minutes and said,

Jack: Okay let's do—I'm pretty sure this is the biggest one, right? [Pointing at her work].
Jill: Well, that one's the biggest, period.... I think [Jill points at one of the graphs].
Jack: Which one comes next?

As a consequence of the lack of success in achieving an equation within what was for him a reasonable amount of time, Jack allowed himself to abandon his own strategy and favor a new look at Jill's. The norm of *effectiveness* was molded within the constraints of correct application of a strategy to a problem and reasonable allocation of time for work. As I noted earlier, Jill was positive in saying that her way had worked. From an observer's perspective, she was absolutely right: Her strategy satisfied a norm of effectiveness related to the problem at hand. My question is, What kind of norm is effectiveness and how does it interact with the mathematics at stake (not just with the problem at hand)? This question is particularly important if one admits (as I do), that from a cultural perspective, Jack's strategy is indeed better than Jill's.[4]

A source of a possible answer is found in my interview with Jack alone immediately after the problem session. These first exchanges confirm Jack's use of a norm of effectiveness:

I: I noticed that when she did this approach with the squares, you kept looking for a formula.

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[3] The problem had required them to compare first Graphs 1 and 2, and then Graphs 3 and 4. Eventually they were required to sort all four graphs.

[4] This claim has to be supported by a more specific description of the strategies that is beyond the scope of this paper. In particular, Jill never showed that she would reduce the error by refining the partitions; she relied, instead, on compensating incomplete blocks.

Jack: Yeah, I don't know about that.... I was just trying to get more, I guess, precise.... [Hers] had worked a lot better than what I thought ...cause I didn't think ... they all looked so close I thought we would come up with ... you know like seven blocks in each one....
I: Those blocks were too big for your [taste]?
Jack: Yeah.... Well, I mean, I thought that once she finished it this would come up with seven and this would come up with seven ... I didn't think she would get eight [or] bigger differences

Jack expressed a preference for his own strategy, but had submitted to the choice of a strategy for the group solution that fulfilled a norm of effectiveness, which he knew how to apply. When I tried to get more of how he valued both strategies, he expressed these values in terms that were less conflated with personal ownership: "You know, mine was the more—... I mean,... I used more math." Later he said: "If I was doing something for a job,... You know what I'm saying? If I were doing something extremely important ... I'd do it my way."

Effectiveness was a sociomathematical norm. Still its validity was regulated by some features of the context that go beyond the social interaction between Jack and Jill. Jack (a college student who has had three courses in calculus) saw the need to manipulate effectiveness in favor of what (he thought) was more mathematical (although what was at stake was sorting those particular areas). He also saw the need to insist on that preference depending on the social importance of the task at hand. Within the social importance of the situation in which Jack was operating, effectiveness had been modified by an allocation of time for work. More on the characteristics of this interaction was implied by Jack when I asked him how he would change those problems if he was going to give them to his students:

Jack: I'd probably make it easier, ... you know.... A lot of times you were multiplying by one half.... I don't think I'd change it much just maybe the equations of the graphs... to make them so—I mean this one [points at Graph 2] was just so hard to find... I would try to make that one a little easier to find.... Kids like the critical point at one and a half ... two and a half... and just move it a little bit... would make it a little easier.

In other words, the conditions of the problem had violated Jack's expectations as to what was legal in that didactical contract: When one poses those kinds of problems one does not use strange numbers to deceive students. He did not seem to realize that such issue could compromise some effectiveness-related features of the areas.

The session with Jack and Jill was an environment whose (systemic) intentions as far as mathematics education is concerned were loosely specified. The researchers' purpose was never to make it otherwise. My comments above intend to show that such loose specification

is likely to be connected with the impossibility to decide whether *effectiveness* as a sociomathematical norm refines the social norms or depends on the social norms. They suggest that a further differentiation of sociomathematical from social norms would depend on systemic specifications of the cultural knowledge that is intended to be used.

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HOW EQUALLY SUITED IS REALISTIC MATHEMATICS EDUCATION FOR BOYS AND GIRLS? — A FIRST EXPLORATION

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The results of two successive national assessments of educational achievements in primary school in The Netherlands placed the gender issue on the research agenda. These results suggested that girls and boys did not profit equally from the assets of the new Dutch approach to mathematics education. Therefore, the MOOJ Project, a study of the gender aspect of Realistic Mathematics Education, was started. This paper will address the first part of this study. The focus of this part is to provide an overview of the gender differences in mathematics achievements in The Netherlands, and to investigate whether there are schools in which the achievements of girls are at least equal to the achievements of boys. The survey involves data from approximately 70% of the Dutch primary schools.

1. Introduction

Twenty five years ago, in The Netherlands the first steps were taken towards the reform of mathematics education, which later became known as "Realistic Mathematics Education" (RME). Freudenthal and his colleagues of the former IOWO[1] were the founders of RME. Although still under development, and not yet entirely implemented in the classroom practice, this reform has left its mark upon today's primary school mathematics education. More than three-quarters of the Dutch primary schools now use a mathematics textbook that was inspired to a greater or lesser degree by this reform movement.

Characteristic of this new approach to mathematics education is the rejection of the mechanistic, procedure-focused way of teaching in which the learning content is automatized in meaningless small parts and where the students are offered fixed solving procedures to be trained by exercises, often to be done individually. RME, on the contrary, has a more complex and meaningful conceptualization of teaching. The students, instead of being the receivers of ready-made mathematics, are considered as active participants in the teaching-learning process, in which they themselves develop mathematical tools and insights. The basis for this new approach to mathematics education emerged from Freudenthal's (1971, 1973) idea of mathematics as a human activity which he connected with the principle of guided reinvention. This means that, in RME, the own constructions and productions of students play a central role. As Treffers (1987) indicated in his description of the

theoretical framework of RME, the contributions of the students are one of the five elements that constitute RME curricula. The other elements are the major place of contextual problems and real-life situations by means of which the students can both constitute and apply mathematical concepts, the use of models by which the gap between the informal, context-connected mathematics and the formal mathematics can be bridged, the interactive character of the teaching process, and the intertwinement of various learning strands. Aside from the more general idea that in RME, problem situations should always be imaginable for the students, and as a consequence, should always fit within the child's world, the aspect of gender has never been a special topic of investigation within RME.

2. The MOOJ Project

Since 1987, the National Institute of Educational Measurement (CITO) has held National Assessments of the Educational Achievements (PPON) in Dutch primary schools. For mathematics this was done in 1987 and in 1992 (see Wijnstra et al., 1988; Bokhove, Van der Schoot, & Eggen, 1996[2]). One of the surprising results of these PPON gauges was that they suggested that girls and boys did not profit equally from the assets of the new Dutch approach to mathematics education. Both studies showed namely that there were significant differences between the mathematics scores of girls and boys, which were almost all in favor of the boys. Moreover, it turned out that these differences were very stable. No interaction effect was found between sex and the year of the study on the achievement scores.

These results led at the end of 1995 to the start of the MOOJ Project, a more thorough investigation into these differences, funded by the Ministry of Education. The research is conducted by the Freudenthal Institute of Utrecht University and the Center for the Study of Education and Instruction of the State University of Leiden, in collaboration with the CITO.

The general goal of the complete research project is to find out which factors are causing these gender differences, and, eventually, to trace teaching methods which are especially suitable for girls (Boekaerts, Bokhove, Gravemeijer, & Treffers, 1995). The proposed research project contains three parts. Part I is mainly focused on the further exploration of the gender differences in mathematics achievements in Dutch primary schools and on the identification of schools in which the mathematics achievements of the girls were at least equal to the achievements of the boys. In the

MOOJ Project these schools are called "girls schools". Schools in which the boys perform better are called "boys schools". In part II, a deeper exploration will take place of the influence of school, teacher and classroom practice factors, in connection with student characteristics, on these differences in mathematics achievements. This further research will take place in a small selection of the identified girls schools and boys schools. Part III, finally, is meant for investigating how the findings of this gender study can be conveyed to the field of education. In this last part of the study an in-service training project for teachers will be set up. The first part of the study has been carried out in 1996. The present paper will report on it.

3. The research questions and the set up of part I of the MOOJ Project

The main question of this first part of the study is whether there are schools in The Netherlands in which the girls have at least the same level of performance in mathematics as the boys have. Without such schools the subsequent part of the research would be senseless, because without such schools it will not be possible to compare girls school and boys schools. Another goal of this initial part of the study is to get a better understanding of the gender differences in the Netherlands. How large are these differences? And, in what respect do the mathematics achievements differ between the sexes?

To answer these questions, data collected with the CITO final test for primary school has been used. This CITO test is meant for providing an individual student score for making a decision about the enrollment at a school for secondary education. The test is not compulsory and is administered in approximately 70% of the Dutch schools. This means that yearly, a little bit more than 100,000 grade-six students take this test. Along with items on mother language and on what is called "information processing", the test contains 60 items on mathematics, divided in three parts. The items are presented in a multiple choice format.

For the MOOJ Project, the 1993 through 1995 mathematics data of the CITO final test for primary school were analyzed on three different levels: the individual student level, the level of the discrete test items, and the aggregated school level. The first two analyses were meant for gaining insight into how the mathematics achievements differed between girls and boys. The third analysis was aimed at the selection of schools for the second part of the MOOJ Project.

Cito final test for primary school	X̄ (%)	sd (%)	X̄ (%)	sd (%)	X̄ (%)	sd (%)	X̄ (%)	sd (%)	X̄ (%)	sd (%)	X̄ (%)	sd (%)
	1993 girls n=50111*		1993 boys n=49411		1994 girls n=52600		1994 boys n=52133		1995 girls n=52835		1995 boys n=52024	
Total score mathematics (60 test items)	65	19	71	18	67	19	73	19	65	20	72	20
Basic knowledge numbers	67	24	74	22	63	28	71	26	60	28	68	27
Mental arithmetic	66	22	73	21	67	21	73	20	66	22	73	21
Operations	67	24	74	22	63	28	71	26	60	28	68	27
Fractions, percentages, ratios	62	22	69	21	72	22	78	20	64	23	70	22
Measurement, time, money	59	23	67	22	60	22	69	22	65	24	73	23

*The scores in this table only involve the students of which the sex was filled in on the test page. In each year this information was lacking for about 2% of the students.

Table 1 Mathematics achievements of boys and girls on the CITO final test for primary school

PPON 2 (1992) Final-assessment (grade 6) Differences in corrected scores~ between boys and girls (b-g) (* p<.05)		
1	Basic operations	11*
2	Whole numbers: basic knowledge and understanding	20*
3	Decimal numbers:basic knowledge and understanding	19*
4	Mental arithmeric: addition and subtraction	18*
5	Mental arithmeric: multiplication and division	9*
6	Estimation	24*
7	Written algorithms: addition	-2
8	Written algorithms: subtraction	-3
9	Written algorithms: multiplication	-3
10	Written algorithms: division	-1
11	Written algorithms: applications	7*
12	Fractions: addition and subtraction	13*
13	Fractions: basic knowledge and understanding	6*
14	Fractions: multiplication and division	19*
15	Fractions: applications	19*
16	Percentage: basic knowledge and understanding	24*
17	Percentage: smart calculation	17*
18	Percentage: applications	18*
19	Ratio: basic knowledge and understanding	22*
20	Ratio: applications	23*
21	Measurement: basic knowledge and understanding	24*
22	Measurement: counting the numbers of units	14*
23	Measurement: caculating the numbers of units	19*
24	Measurement: measurement systems	23*
25	Measurement: applications	25*
26	Calender sand time: applications	17*
27	Money: applications	4
28	Calculator: applications	3
29	Geometry: applications	16*

~ The scores have been corrected for social background of the students, the age of the students, the kind of textbook that is used and the attainment of the kernel curriculum.

Table 2 Gender differences in mathematics achievements found in the second PPON study

4. The results of the analysis on the student level

The analysis of individual student scores on the CITO final test for primary school showed that the grade-six boys surpassed the grade-six girls in their mathematics performance in each year under investigation (see Table 1). In 1993, the boys answered 71% of the sixty problems correctly, while the girls came up with 65% correct answers. This 6% difference in the total score for mathematics is between one-third and a quarter of the standard deviation of the scores. The same results were found for the total scores in the other years and for the five subscores that are distinguished within the domain of mathematics. These include: basic knowledge of numbers; mental arithmetic; operations; fractions, percentages, and ratios; measurement, time, and money. The gender differences between the subscores were rather small. This was probably caused by the way in which the problems for the different mathematics domains have been operationalized in the test.

In this respect the PPON studies[3] were more revealing. In the second PPON study, Bokhove, Van der Schoot, & Eggen (1996) found (see Table 2) that the differences were the highest, in favor of the boys, on the subscales on measurement, ratios, percentages, and estimation. On the subscales on column arithmetic (written algorithms) the differences were the less. Here the girls even surpassed the boys, but not significantly.

5. The results of the analysis on the level of the test items

In order to get a better idea of how the mathematics scores on the CITO final test for primary school differ between the sexes, an analysis on the level of the discrete test items was carried out. For each of the test items, the percentage of correct answers (p-value) was calculated for the boys and the girls separately. This was done for all the three years under investigation. The differences in p-value (boys minus girls) ran from 26% through -4%. Thus, there were almost no items on which the girls performed far better than the boys. The next step in the analysis was to select the "most extreme" items for the boys and for the girls in each part of the 1993 through 1995 tests. The "most extreme" items for the girls were the items on which the girls had approximately the same performance level as the boys or scored a little bit higher than the boys. In each part of the three tests, for both categories three or four extreme items were selected. This led to a collection of 34 "extreme boys items" and 36 "extreme girls items". Then followed a qualitative analysis of

these items. The goal of this analysis was to find differences between these items. No particular criteria were defined in advance for this analysis. The analysis procedure can be characterized as repeated reading until certain characteristics could be identified, followed by a check on the tenability of these characteristics in other test items.

Eventually, this analysis led to the following gender-specific characteristics of test items:

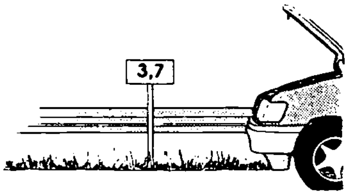
The boys perform better than the girls on

- problems which ask for daily-life knowledge on numbers and measures
- problems in which large numbers with many zeros are used
- problems in which different numbers or different units of measurement are used
- problems which have possibilities for "tinkering" with numbers
- problems which ask for reasoning backwards

The girls perform equally well as the boys or a little bit better than the boys on

- problems which ask for accuracy
- problems of which the text is complex
- problems which ask for (reflection on) strategies and not for calculations
- well-known problems which refer to standard procedures
- straight forward problems
- problems which refer to shopping situations.

Figures 1 and 2 illustrate these gender-specific characteristics of the test items.

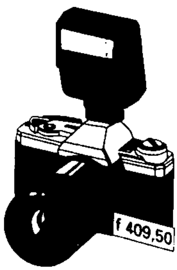


The telephone to call the car service is at the 3.4 km peg.
How many meters is that from the 3.7 peg?

A 0.3 m C 30 m

B 3 m D 300 m

Figure 1 "Extreme boys item" from the CITO final test for primary school (difference in p-value +26%)



Jelle likes to buy this camera. He saved
in January f 40.75
in February f 39.20
in March f 75.15 and
in April f 80.95
His father is paying the shortage.
How much has he to pay?

A f 173.45 C f 233.55

B f 173.55 D f 273.45

Figure 2 "Extreme girls item" from the CITO final test for primary school (difference in p-value -4%)

6. The results of the analysis on the aggregated school level

By means of the analysis on the aggregated school level girls school and boys schools could be identified for part II of the MOOJ Project. This is done by calculating for each school the average score and the standard deviation for both sexes separately. The schools with missing data and the schools with less than 10 students, however, were previously removed from the data collection. After the differences between the average scores were tested by means of the t-test, the schools were categorized according to their t-value. The extreme categories "girls score clearly better" and "boys score clearly better" both belong to a t-value that indicates a difference that is significant at the 5% level. The next categories belong to less significant differences. It turned out that the different categories are not equally distributed over the schools. Globally spoken, the schools are split up in two groups (see Table 3). In nearly one half of the schools the scores of the girls and the boys are approximately the same. In the other half of the schools the boys surpass the girls. There is only a very small minority of schools where the opposite is the case.

	Number of schools					
Total number of schools who did the Cito final test	**1993** 5282		**1994** 5434		**1995** 5388	
Total number of schools (of which the scores of the boys and the girls were compared)	3458 (65%)		3689 (68%)		3649 (68%)	
girls score clearly better	23	1%	28	1%	35	1%
girls score rather better	24	1%	17	0%	18	0%
girls score somewhat better	125	3%	128	3%	137	4%
girls boys	1745	50%	1894	51%	1918	53%
boys score somewhat better	734	21%	754	20%	735	20%
boys score rather better	241	7%	249	7%	244	7%
boys score clearly better	566	16%	619	17%	562	15%

Table 3 Number of Dutch schools in 1993 through 1995
in which the girls perform better, equally or less than the boys

		Number of schools
Total number of schools (involved in this analysis)	girls 3 years better *	0
	girls 2 years better and 1 year equal	9
	girls 1 year better and 2 years equal	97
n = 2134	girls boys	326
* better = "somewhat better", "rather better" or "clearly better"	boys 1 year better and 2 years equal	679
	boys 2 years better and 1 year equal	609
	boys 3 year better	227
	others	187

Table 4 Number of Dutch schools with a particular pattern in the mathematics scores of girls and boys

A further analysis learned that only a small part of the schools belonged to the same category in three successive years (see Table 4). These schools, however, form the source from which the schools for part II will be selected.

7. To conclude

The most important findings of the first part of the MOOJ Project are:

- In the Netherlands, boys surpass girls on standardized mathematics tests.
- The girls do not score lower than the boys in all mathematics domains.
- The test items have gender-specific characteristics.
- The achievement pattern is not the same for all the schools.

In the next part of the study it will be investigated why in some schools the girls have the same performance level as the boys, and what can be learned from these schools for the further development of RME.

Notes

1. Later, this research group is called OW&OC. After Freudenthal's death, in 1990, it is re-named Freudenthal Institute.
2. Many of the results of this second PPON study were already presented at the 1994 "Panama-najaarsconferentie" in Noordwijkerhout, a yearly Dutch conference on mathematics education.
3. The PPON studies have been set up for a detailed evaluation of the output of the educational system. For this purpose an assessment tool is used in which each subscale contains a large number of open-ended items. Also different from the CITO final test for primary school, a sampling method was used in which a sampling of students is combined with a sampling of test items. Because of the lack of complete classes who did all the test items, the PPON data was not suited for the selection of schools.

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TEACHER AS AMPLIFIER, TEACHER AS EDITOR: A METAPHOR BASED ON SOME DYNAMICS IN COMMUNICATION

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In this paper, I discuss some dynamics involved in communication and introduce the image of a Neutral Zone through which all communication takes place. I discuss some of the difficulties involved with learning due to the need for a learner to translate instructions into actions, or transferring from one of the five senses to another. To reduce the need for such translation, I develop the metaphor of teacher as amplifier, teacher as editor.

Communication

I gave a demonstration lesson to some students with a number of teachers watching. In my attempt to explain what I wanted the students to do, I became quite flustered. The teachers observing reported later that they were confused. However, to my surprise, the students had begun work. As I wandered round, I found that each group knew exactly what they were doing, yet no group was doing the same as any other, and none was doing what I had intended. As Donaldson (1986) commented *...the questions the children were answering were frequently not the questions the experimenter had asked.* (p49).

I couldn't directly give the students what I wanted, in the sense of opening up the top of their heads and placing that information inside. I had to go through the media of words, writings, drawings and actions. The students could not open the top of my head either, and take out the information they needed in order to start work. The only things available from which to gain information were the words, writings, drawings and actions I offered. These were on offer for all the students, so why did they not all do the same? There must be some other dynamic involved in order for different students to make different decisions about what they were to do. Cobb, Yackel and Wood (1992) said that ... *we contend that students must necessarily construct their mathematical ways of knowing in any instructional setting whatsoever, including that of traditional direct instruction.* (p28). No matter how clearly, or otherwise, a teacher may say something, it does not mean that any of it is 'received' by students. In fact the notion of something being 'received' in this way does not seem appropriate. Von Glasersfeld (1987) commented:

> Educators share the goal of generating knowledge in their students. However, from the epistemological perspective I have outlined, it appears that knowledge is not a transferable commodity and communication not a conveyance. (p16)

The words, writings, drawings and actions in themselves are hollow. There are no meanings that come with them. Gattegno has described words as hollow in a number of his seminars and von Glasersfeld (1987) states that the idea of words as containing meaning is misguided: *This notion of words as containers in which the writer or speaker "conveys" meaning to readers or listeners is extraordinarily strong and seems so natural that we are reluctant to question it. Yet, it is a misguided notion.* (p6). Also, St. Augustine (1950) was clear about the fact that words do not convey any meaning, and wrote the following in the 4th Century:

> ... we do not learn anything by means of the signs called words. For, as I have said, we learn the meaning of the word - that is, the signification that is hidden in the sound - only after the reality itself which is signified has been recognized, rather than perceive that reality by means of such signification. (p174)

In order to develop some meaning with what I offered in the lesson, each student will have to be active with the material which is on offer. I describe this situation in terms of a *Neutral Zone* - a zone in which I have placed a number of sounds and images that each student can then choose to attend. It is a zone in which offerings are placed. The material with which each person can potentially work is a subset of the offerings in the zone, and is dependent upon the attention of that person in time. Some offerings may be available <u>over</u> a period of time, such as the visual sentences and drawings on a blackboard. Others are available <u>in</u> time, such as speech.

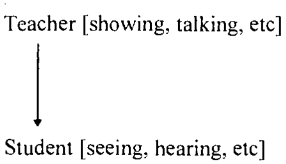

Figure 1: Traditional image of student receiving information.

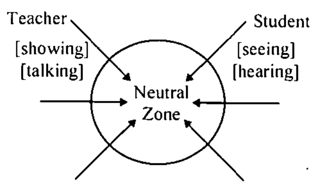

Figure 2: Dynamics involved in communication through the Neutral Zone.

The dynamics of communication are indicated by arrows from each person to the Neutral Zone (see Figure 2). All arrows have a direction away from each person, whether that person be a listener or a speaker; a watcher or a demonstrator. The arrows represent human attention rather than physical entities such as photons of light or sound waves.

Many expressions within the English language imply that arrows go towards a person. For example: *This quote brings to mind... It brought about... It summoned in me... This caused me to think that... This is what prompted me to...* Papert (1994, p83) considered the sentence *The teacher teaches the child,* pointing out that grammatically the teacher is the active subject and the child is the passive object of the sentence, whereas in fact it is learning which is the active process. I am proposing that all arrows go *away* from each person. The Neutral Zone offers an image where both teacher and learner are active in their respective roles.

Translation

Supposing a teacher makes the statement: *a parallelogram is a four-sided shape which has two pairs of sides parallel.* A student may have meanings for words or phrases, such as *parallel, four-sided,* or *two-pairs*, which are similar to the meanings held by the teacher. However, there is still some work for the student to *see* examples of parallelograms. A parallelogram is a geometric, visual image whereas sentences are auditory and consist of a series of words which are said over time. I make this distinction to indicate that although a student may hear, and attach meaning to words, there is still a translation required to turn those meaningful words into a visual image. Janvier (1987) referred to the notion of translation: *By a translation process, we mean the psychological processes involved in going from one mode of representation to another, for example, from an equation to a graph.* (p27). I will extend this notion by considering translation to be any transfer from one of the five senses to another, whether it be within the same mode of representation or not. For example, a teacher may draw an example of a landscape and ask a student to copy it. Even though the student is attempting to draw the same picture, the visual impact of the picture still has to be changed into physical movements of muscles to produce a successful drawing. Thus, I would describe such a demonstration or instruction as requiring a translation from the visual sense to the kinesthetic domain of activating muscles, even though the same mode of representation - both drawings - is involved.

My usage of *translation* involves going from one sense to another. Figure 3 gives a representation of what is involved in a student having to translate from listening to an explanation, to making an attempt at doing what was explained.

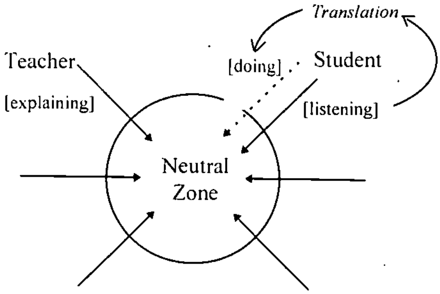

Figure 3: Diagram representing the need for translation with a traditional style of teaching. (The dotted line happens at a later time than the solid lines).

The student has to actively pay attention to what the teacher is saying, but then has to translate those instructions before they can be in a position to inform their own actions. This work is, by its nature, private work. As a consequence it is difficult for the teacher to become aware of what the student is doing in this translation process.

Teacher as amplifier, teacher as editor

I will now consider ways in which a teacher can act in order to try to reduce the need for translation. Instead of a teacher describing or explaining something they want a student to learn, use can be made of what a student has already demonstrated, or already said. Then the student can reproduce what they have already done or said, and this does not require translation. Several years ago, I listened to a teacher of physical education, Jean Lyttle, talk about how she used to help children who could not hit a ball with a bat in the game of rounders. She used to put her arms round the student and hold the student's hands as they held the bat. As the ball was thrown towards them, she would ensure that the bat hit the ball. Initially, this required her taking control and moving the student's hands so that the ball was hit. As time went by, she found she could gradually reduce her own input, until she was able to withdraw her hands and the student was successful on their own. She talked about the importance of the student physically experiencing the way their own arms moved when the bat did hit the ball. This became an experience which the student could begin to call upon in their next attempts to hit the ball. Successful experience could be called upon, whereas if the student had been left to try on their own, there were likely to be only experiences of unsuccessful movements to call upon. Jean's method is quite different to one based on the notion of either demonstrating how it 'should' be done, or describing in words what to do. Both of the latter would require translation, but this is reduced by Jean ensuring that the student gained a personal

experience of how arms were moved with a successful hit. This experience can be called upon without the need to translate what someone else is either saying or doing, since it is already part of the student's own experience.

I will offer two examples within mathematics. The first occurred in a low ability class of 12-13 year olds which I was teaching. The digits '427051' were written on the blackboard. We were engaged in an activity of putting a decimal point in a particular position in order to meet certain requirements: just less than fifty; a bit more than three hundred; etc. I asked that the decimal point should be placed so that the digit '1' should be worth one. A girl, Clare, thought that the point should go before the '1' and wrote '42705.1'. I asked her to say the number in words. Finding that she was able to do this, I asked her to say it again:

Clare:	*Forty two thousand, seven hundred and five point one.*
DH:	Say the words again but don't say anything before the 'five'.
Clare:	*Five point one.*
DH:	Now don't say the five.
Clare:	*Point one.*
DH:	Is that the same as one?
Clare:	*No, it's less than one.*

Once I knew that Clare was able to say the number name, I wanted her to attend to the fact that she had said *point one* and not *one*. Thus I wanted her to become aware of what she had just said. A possible obstacle was that there were a number of other words said in the number-name which were not relevant to this awareness. Thus, I invited her not to say certain parts, which in turn left the part I wanted her attention to be with.

Acting as an editor is one way in which I can attempt to affect someone's attention, and by its nature, the act of editing also amplifies that which is left. Amplifying and editing are techniques for attempting to shift attention and are thus tools for a teacher to use. Mason (1989) has talked of a teacher's role in terms of helping students to shift their attention and the use of split attention - where a student does something and also observes that doing. Here, by acting as editor, I help to shift Clare's attention onto a part of what she said which offers the opportunity for her to become aware of her error. This is an example of a linguistic strategy which Pimm (1991) has discussed: *Teachers, in order to teach, need to acquire linguistic strategies... in order to direct pupil attention to salient aspects of the discourse - or indeed the nature of that discourse - while still remaining in 'normal' communication with the pupil.* (p167). Von Glasersfeld (1995) also talks of affecting attention: *They [moments when students realise for themselves that what they are doing makes no sense] are moments in which the teacher may become a most effective helper, not by*

showing the 'right' way, but by drawing attention to a neglected or counter-productive factor in the student's procedure. (p189).

My second example involves the developing awareness of a whole class. It comes from an extract of a video (Open University, 1991) of a lesson I taught on algebra with a group of 13-14 year olds. Shona was repeating what I had said I did to my unknown number, and Naome had offered a way in which we could find out what my number was.

41 Shona: Think of a number, add three, times by two, equals 14.
42 Naome: 14 divided by two, take three.
...
47 DH *... What is different about what I said and what Naome is saying about how to work it out?*
48 Ben: Turning it around the other way.
49 DH: *Turning what around?*
50 Ben: The numbers.
51 DH: *OK. Can you just say the numbers? Shona.*
52 Shona: Think of a number, add three.
53 DH: *Right, just the numbers.*
54 Shona: Three... two... 14.
55 Naome: 14... two... three.
56 DH: *So that's right is it? OK. Right. Is that right? Uha. And what else is different? The whole lot again* (to Shona).
57 Shona: Think of a number, add three, times two, equals 14.
58 Naome: 14 divided by two, take three.
59 DH: *What else is different? What else is different? Jo.*
60 Jo: Instead of... you got divided and take away instead of add and times.
61 DH: *Right, so can you just say the... which one did Shona say?*
62 Jo: Add three, times two.
63 DH: *OK. So just say the three bit.*
64 Shona: Add three.
65 DH: *Just say the three bit* (to Naome).
66 Naome: Take three.
67 Jo: So it is the opposite.
68 DH: *Uha. And what other number... say the other* (to Shona).
69 Shona: Times two.
70 Naome: Divide by two.
71 Jo: It's the opposite again.

I helped amplify some aspects of what Naome and Shona were saying, by editing what they said. This increased the likelihood of the students shifting their attention to

these parts of what was said. This helped all the students in the class to check the conjectures which were being suggested, and it also helped them to focus on these aspects when I continued with different linear equations of this type. By the end of this lesson, most students in the class were able to solve relatively complex linear equations without having had any explanation from myself. My role was mainly to shift their attention to the relevant aspects of what someone had already said.

Summary

Through considering the dynamics involved with communication, and being aware of the demands and difficulties that the act of translating can put on a learner, I have developed the notion of a teacher acting as an amplifier/editor, on the material that a learner has already demonstrated is within their recent experience, thus reducing the need for the student to translate (see Figure 4). I have also introduced the image of the Neutral Zone, where all offerings are placed and became material to which someone may or may not attend. Considering the particular activity of a teacher, the material can be made use of in an attempt to draw a student's attention to certain aspects of what has already been said or done.

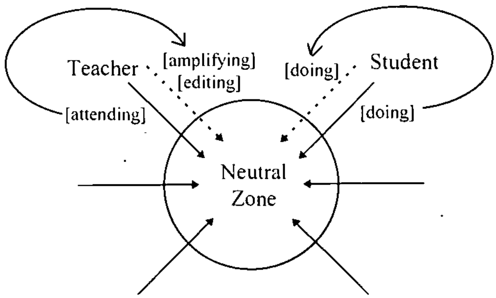

Figure 4: Avoiding the need for translation through the use of editing and amplifying. (The dotted lines happen at a later time to the solid lines, and the student's dotted line happens after the teacher's dotted line).

The role of teacher as amplifier/editor can also help focus the attention of a whole class on certain aspects of what has already been said, written, or drawn by someone in the class. A new awareness can be gained through shifting attention onto aspects of those things which are already known.

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USING THE COMPUTER TO IMPROVE CONCEPTUAL THINKING IN INTEGRATION

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It appears that many students try to learn the calculus as a set of discrete processes. In our research we developed questions to probe understanding of the concepts of integration, and sought to improve it using computer-based modules of work. We describe how tertiary students using spreadsheets and symbolic manipulators exhibited a significant improvement in proceptual understanding with a tendency to understand in a, concept-oriented manner rather than as rote processes. In contrast, the control group students, with their traditional learning of calculus, often experienced no change, showing the same misconceptions in both pre-test and post-test.

Introduction

Our belief is that many students have experienced difficulties with calculus because they have relied on memorising rules and procedures and ignored the conceptual aspects of its objects. This has led many novice calculus students to develop an instrumental rather than a relational understanding (Skemp, 1976), concentrating on calculus algorithms and learning 'how to' rather than why. Introductions to the calculus often depend on students understanding the idea of a limit, but this concept causes conflict between students' intuitive ideas and the formal definition. There are a number of conceptual problems related to infinite processes, and logical and manipulative difficulties which can occur when one is confronted with a complex definition. Problems such as this lie at the heart of learning calculus concepts. We have previously described (Thomas & Hong, 1996) the sort of misconceptions which many students have in calculus. This paper describes the value of computer-based modules of work for supplementing traditional approaches to integration in a way which may help to surmount these difficulties.

Background

Processes and Concepts

Since Piaget (1985, p.49) described how "actions or operations become thematised objects of thought or assimilation" much has been written about the relationship between processes and concepts in mathematics. Research has emphasised that there is a conceptual change involved in the conversion of a dynamic process into a static object. Gray & Tall (1994) have defined the notion of *procept* as an amalgam of three things - process, symbol and concept. Thus an integral symbol may evoke both the process of integration and the concept of integral, with the cognitive combination of all three being a procept. Much of the symbolism used in mathematics carries the dual role of process and concept and distinguishing between each usage is clearly important mathematically.

Computers in calculus

Recently in calculus teaching there have been attempts to move away from a process-oriented style of teaching and learning which may have prevented student understanding of important concepts. Much of the research has sought to use computer software, such as symbolic manipulators to improve this situation (e.g. Small & Horsack, 1986; Palmiter, 1991; Barnes, 1994; Hubbard, 1995). Software has been used to improve understanding of concepts such as limits, in differentiation and integration (e.g. Tall, 1986; Li & Tall, 1993; Thompson, 1994). The limit is an important example of a procept in the calculus. For example, the symbol $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ may represent either the process of getting close to a specific value, or the value of the limit itself. Furina (1994) studied the methods that students used to calculate limits and suggested that more than one technique should be used to promote understanding. Encapsulating both the differentiation and integration processes (which involve limits) seems to be an essential prerequisite for understanding the fundamental theorem of calculus. Someone who has the ability to switch his/her focus between the dual roles of the symbols may be described as a *versatile* mathematician (Tall & Thomas, 1991). It is our contention that using computer software in mathematics courses can encourage the students' understanding of processes, thus facilitating versatility. Evidence for this was provided by Monaghan (1993) who studied the growth of 16/17 year old students' conceptualisation of real number, limit and infinity over one year. Students using the symbolic manipulator software *Derive* were better able to seeing the limits as objects. The major aim of the research described here was to investigate student thinking and misconceptions when dealing with integration. A definition of definite integral (called a Riemann integral) requires an understanding of taking the limit as $n \to \infty$. But the fundamental theorem of the calculus linking areas and antiderivatives is usually introduced before students have encapsulated the concepts and so a definition of the definite integral based on area and limit concepts is often quickly discarded and forgotten. So students' may fail to see that estimating areas by upper or lower Riemann rectangles and letting the number of these rectangles tend to infinity is a process which leads to an object, the integral. Their desire to leave these Riemann sums behind is understandable, because they are tedious to calculate by hand and require the difficult idea of a limit, whereas the antiderivatives involve easier algorithmic processes. Symbolic and numeric methods can be carried out by symbolic manipulators, easily evaluating any number of limits. Hence the possibility arises for an early focus on the limit as process and concept, but with the computer carrying out the calculations internally. We agree with Tall (1993), who suggests that the computer relieves the learner of the tyranny of having to encapsulate the process before obtaining a sense of the properties of the object. By using software which carries out the process internally, it may become possible for the learner to explore the properties of the object produced by the process before, at the same time, or after studying the process itself. The graphic approach to the calculus using the computer is designed to

give students an environment in which to construct a network of related ideas. A computer's symbolic manipulator or spreadsheet software enables students to experience many possibilities with respect to the relationships between numerical, graphical and symbolic representations. Working on this basis, Monaghan's (1993), students began by making hand calculations to compute upper Riemann sums over a small number of intervals, which were then transferred to a Computer Algebra System (CAS) and the number of rectangles extended, first to a large finite number and then to the limit $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{x}{n}\cdot\frac{r^3}{n}$. He found that the CAS students were better able than traditional students to assign meaning to the terms and to describe how to apply the concepts and processes.

Method

A questionnaire was designed comprising two types of questions on integration. The first addressed the standard algorithms which any student of integration could be expected to know, for example, they were asked to integrate using antidifferentiation techniques, $\int f(x)dx = F(x) + C$. The second introduced novel types of questions which we developed to assess deeper understanding of concepts. The questionnaire was given to 161 first year Auckland university students who had already completed and passed the introductory calculus paper and were enrolled in the second calculus paper, which covers Riemann integration. The students were asked to volunteer to take part in the computer work, which was to be given in addition to the standard lectures. In the event, only seven students did so and these became an experimental computer group. Once we knew that we were in the position of having such a small sample we formed 7 matched pairs using students from the control group, matching them on the basis of the number of correct answers in section I of the pre-test. Section II was not used in the matching due to the very low correct response rate from all students The seven experimental group students investigated the processes of integration (using modules of work) with the Excel spreadsheet for four one hour sessions, followed by the Maple symbolic manipulator for two one hour sessions, supplementing the normal lectures. None of them had had ever used Excel or Maple for mathematics previously. The aim of these computer tutorials was to try to improve the aspect of the understanding of the concepts associated with integration by giving the students direct experience of experimentation with the processes which lead to them. The students were all given a post-test comprising the same questionnaire, however the control group had decreased to 100 students in the lecture streams, those leaving being primarily the weaker students who could no longer cope with the paper. In addition, following the post-test, all the experimental students and seven matched students from the control group were individually interviewed by one of the researchers to investigate further their understanding. During the interviews they were questioned about their view of integration and their experiences in the tests. The interviews were recorded and later transcribed by the researcher.

Results

The ten questions of section I were based on standard text book questions, and concentrated on a process-oriented approach which students would be familiar with. The 13 questions of section II were aimed at concepts rather than processes. However, we linked some questions in sections I & II to see if some students had developed techniques of algorithms but did not have the corresponding concepts, or the ability to apply the techniques when solving problems. To accomplish this we generalised some section I questions to functions which were not explicitly stated but left as *f(x)*. We also used some other means which precluded any process being carried out. For example we asked both:

Given that $\int_0^3 x^2 dx = 9$, *find* $\int_1^4 (x-1)^2 dx$ *Given that* $\int_1^3 f(t)dt = 8.6$, *find* $\int_2^4 f(t-1)dt$

Section I   Section II

Here the section I question may be answered either by an understanding of the concept that the translation of the graph leaves the area unchanged or by simply recalculating the second integral. However, the section II question cannot be answered by calculation but only through conceptual understanding. The categories of questions used were as follows: conservation of integral; the maximum values of an integral function; the definite integral and area; integration and transformations; relationship between the definite integral of a function that crosses the x-axis and area; summation using sigma (Σ) & Riemann sum; Riemann integral; and sketching the integral function.

Statistical Comparison

We performed a statistical analysis to see if there was any improvement in the aspects of process-oriented skills (section I) and conceptual understanding (section II) displayed by the students, and hence in their proceptual thinking. Table 1 gives a comparison of the proportions of correct answers obtained in sections I and II by the computer tutorial and non-computer students (using Yates' correction where appropriate).

Table 1: The pre- and post-test proportions of correct answers for the computer and non-computer groups

	Computer (n=70)	Non-computer (n_I=1540; n_{II}=2002)	χ^2	p
Pre-test				
Section I	0.57	0.44	4.47	<0.05
Section II	0.26	0.20	2.24	n.s.
Post-test				
Section I	0.89	0.51	35.1	<0.001
Section II	0.70	0.34	47.8	<0.001

We see from the pre-test results that there was a significant difference between the two groups in section I but not in section II, enabling us to infer that the starting points of the two groups were the same with regard to conceptual understanding. At the post-test, while both groups have made gains, the experimental group's understanding of the conceptual questions in section II is considerably better than the control group. These figures, of course need to be carefully considered due to the relatively small

number of students in the computer group. Table 2 gives the corresponding results of the two matched groups (using Yates' correction where appropriate). It can also be seen that these two groups performed at the same level on the section II questions on the pre-test even though they were not matched on these.

Table 2: The pre- and post-test proportions of correct answers for the computer and non-computer matched groups

	Computer	Non-computer	χ^2	p
Pre-test				
Section I (n=70)	0.57	0.59	0.03	n.s.
Section II (n=91)	0.26	0.27	0.03	n.s.
Post-test				
Section I (n=70)	0.89	0.60	13.5	<0.001
Section II (n=91)	0.70	0.51	7.45	<0.01

These results appear to confirm the significantly better overall post-test performance of the computer group on both the procedural and the conceptual questions and we conclude that the addition of the computer work to the lectures had improved both the process ability and the conceptual understanding of the students.

Riemann sums

Analysing the individual question results in more detail it was pleasing to see that on an understanding of the concepts of Riemann integral the students who had used the computer were outperforming those who had not. Table 3 gives the statistical analysis of the proportions of correct responses for the questions involving Riemann sums (including Yates' correction).

Table 3. Proportions of students giving correct responses on Riemann sum questions

Section and Question	Computer (n=7) Non- Computer (n=154; 100)	Pre-test correct	Post-test correct	χ^2 for post-test	p
I 7.Calculate $\sum_{i=0}^{5}\left\{(i^2-1)\frac{1}{10}\right\}$	Computer	0.71	0.86	2.07	n.s.
	Non- Computer	0.53	0.50		
I 8. a) leftsum calculation	Computer	0	0.57	8.04	<0.01
given function values	Non- Computer	0.14	11		
II 5. Limit of rightsum	Computer	0	0.57	10.05	<0.01
- leftsum is 0	Non-computer	0.08	0.09		
II 10. Riemann sum of $f(x)$	Computer	0	0.71	3.93	<0.05
increase or decrease	Non-computer	0.19	0.28		
with more intervals?					
II 12. Match diagram to	Computer	0.14	0.71	8.77	<0.01
$\sum_{i=0}^{9} f(x_{i+1})\cdot\frac{3}{10}$	Non-computer	0.05	0.17		

We see that the students who had used the computer performed significantly better on every question but one. Since these section II questions in particular require an appreciation of the following ideas:

- Upper and lower Riemann sums both approach the same value in the limit
- For a strictly decreasing function the value of the left (upper) sum decreases as the number of strips increases
- Identifying the diagram for left sums when $f(x)$ is negative in an interval

without being able to verify them by calculation, this is very pleasing. Table 4 records the same data for the matched pairs of students, and while the same statistical improvement is not quite shown here there is weak evidence of the improvement present.

Table 4. Proportions of students giving correct responses on Riemann sum questions for the matched pairs

Section and Question	Computer (n=7) Non- Computer (n=154; 100)	Pre-test correct	Post-test correct	$\chi 2$ for post-test	p
I 7.Calculate $\sum_{i=0}^{5}\left\{(i^2-1)\frac{1}{10}\right\}$	Computer	0.71	0.86	1.24	n.s.
	Non- Computer	0.57	0.43		
I 8. a) leftsum calculation given function values	Computer	0	0.57	3.15	<0.1 n.s.
	Non- Computer	0	0		
II 5. Limit of rightsum - leftsum is 0	Computer	0	0.57	3.15	<0.1 n.s.
	Non-computer	0	0		
II 10. Riemann sum of *f(x)* increase or decrease with more strips?	Computer	0	0.71	0.29	n.s.
	Non-computer	0.14	0.43		
II 12. Match diagram to $\sum_{i=0}^{9} f(x_{i+1})\cdot\frac{3}{10}$	Computer	0	0.71	4.98	<0.05
	Non-computer	0	0		

While we recognise the limitations of having such a small number of students in the experimental group, and the difficulty of generalising from this, we have recently completed two more experiments and now have a much larger group whose progress we will be able to report on soon.

Answers and interview comments

Some of the answers given by computer students in the post-test showed excellent understanding compared with those of the matched students. For example, two students wrote, for question 10 of section II:

> b because the actual value of *f(x)* is between rightsum<*f(x)*<leftsum
>
> b on a strictly decreasing function left endpoints are greater than the actual value. If accuracy increases then leftpoint will move towards actual value i.e. get smaller

When asked in the interview about the same question, one of the computer students responded as follows:

> Int: In this question, 10 equal sub-intervals were given, if, 50 equal sub-intervals were given, then, what's your expectation for the approximation to the integral? Will the value be larger or smaller than the case of 10 equal sub-intervals?
>
> Student: For leftsum, it will be greater, and rightsum could be smaller
>
> Int: For equal sub-intervals, what's the difference between the rightsum and leftsum as $n \to \infty$?
>
> Student: zero, zero

Asked in their interviews about the difference between the rightsum and leftsum as $n \to \infty$, where n is the number of intervals, all 7 of the experimental group said that it was zero. They seem to have grasped the concept of the limit of the Riemann sums being equal, for an integrable function.

In contrast the matched control students were still unclear on some of the concepts. For question 5 of section II, one control student answered 'very small and positive', ignoring the given limit. In his interview he also showed a misconception of Riemann sum, answering 'smaller' for the question which the value of the width, and 'close to zero but not zero' for its limit. When asked about the difference between the rightsum and leftsum, he said 'I don't know'. Another student mistakenly thought that Σ was the same as $\int$, his misconception about Σ becoming clear in the interview:

> What's the difference between the Σ and $\int$? 'No difference'

In addition several control students attempted the calculation of the leftsum using the 'Trapezium rule'. This may be evidence that they are looking to use processes with which they are familiar when confronted by concepts they do not understand.

Comments on the study's value

The students commented in the interviews that the Maple and Excel computer modules had helped them to understand integration better, six of them mentioning Riemann sum specifically. The students (4 had English as a second language), when asked how their understanding had been affected, said:

> Excel & Maple are made easier to understand concept of integration to show process of calculation of integration.
>
> I can see clearer than the between the rightsum and leftsum and the conservation of integral.
>
> Could see the effects of transformations on leftsum etc. values and the graph. When you shift the interval there was relation to area remained the same.

It appears that the opportunity to investigate integration using the computer had given valuable insight into the processes underlying integration and this in turn assisted the students' conceptual understanding.

Discussion

We believe that the evidence presented here confirms when students learned calculus using the computer in a manner where they could investigate its processes they are able to tackle successfully the more demanding section II questions, which required varying degrees of conceptual awareness. In contrast the experiences of the control students to date have left important gaps in their conceptual understanding. Their solutions show that they have a tendency to see the integral calculus as a series of procedures and associated algorithms and have not developed a grasp of some concepts which would give them versatility of thought. In the light of what we have seen it appears valuable to design and use curriculum materials, such as those we have based on the computer, which give an improved cognitive base for a flexible proceptual understanding of the concepts associated with integration, making it possible for the student to develop a perception in terms of both process and concept.

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Investigating children's collaborative discourse and verbal interaction in solving mathematical problems

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Abstract

This research investigated how the verbal interaction in collaborative small groups affected children's analogical problem solving for mathematical word problems, as well as compared children's verbal interaction behavior and their confidence in problem solving. In study 1, children who were in the collaborative small group learning condition outperformed those in individual learning condition on isomorphic problem solving. In study 2, children who had been engaged in the collaborative instruction condition performed significantly better than those in conventional instruction condition on nonisomophic problem solving. Children who had been engaged in the collaborative instruction condition performed more active verbal interaction behavior, asked more questions and answers.

Theoretical Framework and Objectives of the Research

Current mathematics instruction centers on developing children's problem-solving abilities and collaborative discourse practices as well as an emphasis on better questioning skills (NCTM, 1989). As students engage in collaborative discourse during problem solving, they are able to express their opinions, articulate their reasoning process, defend the validity of their solutions in the face of questions and question peers' ideas. During this process they may clarify, elaborate, revise and reorganize their own thinking on the basis of mathematical evidence (Ball, 1993; NCTM, 1991; Webb & Farivar, 1994), the building of community and reasoning occur. This knowledge constructive activity facilitate students to understand the knowledge of the problem domain (Hicks, 1994; Hiebert & Wearne, 1993; King, 1994).

Analogical problem solving involved transfer of a relational structure from a better undrestand problem domain (the source) to another fundamentally similar but less known problem domain (the target) (Novick & Hmelo, 1994; Vosniadou, 1989). The target problem is the new problem that is yet to be solved in analogical problem solving tasks. Problem solvers must understand and notice the correspondence

between the known problem (source problem) and the target problem, then analogical transfer. The isomorphism means that the source problem and target problem share the same structure similarity with identical goal structure, constraints, and problem space. If, however, the source problem and the target problem are slightly structurally dissimilar; i.e., the target problem is not isomorphic to the source problem. Research reveled that the complete mapping occurs when the target problem is isomorphic to the source problem. Children performed better on isomophic problem solving than nonisomophic problems solving (e.g.: Reed, 1987). We believed that students understand the source domain knowledge deeply, then they would analogical transfer well in solving the nonisomophic problems.

There is considerable theoretical support for the idea that collaborative discourse enhance children's ability to make sense of mathematical ideas. But does collaborative small group interaction affect elementary school children's analogical reasoning in solving mathematical problems? The empirical data is less compelling. Students are given more challenges to explain and construct their understanding through collaborative inquiry. In the conventional instruction environment, students learn and model the solution from teachers directly, and work individually (Huang, 1996). The verbal interaction and the questioning skills of students may differ between these two instruction conditions. Many questions remain unanswered on how these differences relate to children's self-confidence in problem solving and analogical reasoning performance. The goal of this research was to fill this gap.

This research comprised two related studies. The first study examined how the verbal interaction in collaborative small groups affected children's mathematical problem solving. The purpose of the second study was to compare children's verbal interaction behavior, their confidence in problem solving , and analogical reasoning problem solving performance. The second study also identified the types of questions and answers generated during peer interaction from two different mathematics instruction conditions. One class was conventional instruction and the other one was a collaborative learning condition with a stronger problem solving orientation.

Methods, Techniques and Data Source

In the first study, two general mathematics classes with conventional instruction of 4th grade students (N=83) (about age 10) were randomly selected from a local public primary school. This primary school implemented two instruction conditions, one was general mathematics classes with conventional instruction, and the other was experimental mathematics classes with collaborative instruction. In the experiment, subjects were first presented a mathematical word problem as a source problem with spedific procedures for its solution. Then they were requested to solve another two tasks (target problems). One task contained an Isomorphic Problem and the other one was a Nonisomorphic Problem. One class of subjects (N=39) was assigned to small groups. They were encouraged to have verbal interaction with peers to discuss the solutions of the source problem. Then they solved the two target problems individually. Students from the other class were assigned to individual learning condition, subjects (N=44), were asked to read and comprehend the solution of the source problem , and then they solved the two target problems individually. At the same time, all subjects were requested to rate their confidence level when solving the problems. Subjects from the two learning conditions were used their previous academic performance (verbal and math acheivement) as the control factor. The result of *t*-test indicated that subjects' previous academic performances were not significantly different before the experiment, with $t(81)=-.83$, $p>.05$.

Subjects for the second study included 39 4th grade students were chosen randomly from a collaborative instruction condition. And 41 4th grade students from a conventional instruction condition. Subjects were from the same primary school as study 1, but exclusive of the subjects used for Study 1. The result of *t*-test indicated that subjects' previous academic performances in the two instruction conditions were not significantly different before the experiment, with $t(78)=1.86$, $p>.05$. Subjects were assigned to small groups then were requested to complete a set of analogical tasks. The source problem and target problems were the same as those used in Study 1. Subjects were encouraged to have verbal interaction with peers and discuss how to

solve the source problem. Then every subject solved the two target problems individually. In the verbal interaction period, six research assisstants observed the small groups. They wrote and recorded the children's verbal protocols, and completed an evaluation of the children's Verbal Interaction Behavior during discussion. Students' verbal interaction was coded by three raters. The reliabilities were .77, $p<.001$.

Results and Discussion

For the first study, see Table 1, results showed that children who were in the collaborative small group condition outperformed those in the individual condition when solving the Isomorphic Problem, $t(81)=2.13$, $p<.05$. The difference between the two conditions was not significant in solving the Nonisomorphic Problem, $t(81)=.79$, $p>.05$. Problem solving performance and children's confidence level when problem solving was significantly correlated, $r=.31$, $p<.01$.

Table 1. Children's problem solving performance for the two learning conditions.

Learning conditions	Collaborative learning (N=39)		Individual learning (N=44)	
Problem condition	IP	NP	IP	NP
MEAN	10.54 (SD= 7.16)	6.87 (SD= 7.85)	7.09 (SD= 7.56)	5.45 (SD= 8.55)

For the second study, results indicated that different instruction conditions significantly affected children's verbal interaction behavior, questioning and answering, and analogical reasoning performance, as well as their confidence level for problem solving. As Table 2 shows, the contents of the verbal interactions were classified into six types of questions which contained thirteen question categories. The first type was Comprehension questions, including: What does... mean? ; Descriptions in the solvers' own words.; Why is this the reason? ; How do you solve this problem? The second type was Connection questions, which included: Relating new material to prior knowledge. The third type was High-level explanations, including: Analyze the

reasons; Clarify the relations; Propose problem solutions; Propose solution and ask for support. The fourth type was Low-level explanations, including: Rehearsal of the answer; Short answer. The fifth type was Critical thinking questions: Noting strengths and weaknesses, justifying and evaluating ideas. The sixth type was Irrelevant dialogue. Children in the collaborative instruction condition asked significantly more questions (total verbal interaction), more high-level explanations type questions and more low-level explanations type questions than those in the conventional instruction condition. The results of Chi-square tests were $\chi 2 = 37.02$, $p<.001$; $\chi 2 = 9.48$, $p<.05$; $\chi 2 = 7.04$, $p<.01$, respectively. There were no significant differences in Comprehension questions and Connection and Critical thinking questions as well as the Irrelevant dialogue. The results of Chi-square tests were $\chi 2 = 4.27$, $p>.05$; $\chi 2 = .32$, $p>.05$; $\chi 2 = .56$, $p>.05$; $\chi 2 = .88$, $p>.05$, respectively. As Table 3 shows, the difference in Verbal Interaction Behavior between these two instruction conditions was significant, $t(5)= 4.52$, $p<.01$. Children in the collaborative instruction condition performed more active behaviors in discussion with peers, than those from the conventional instruction condition. Furthermore, children from the collaborative instruction condition performed better in solving the Nonisomorphic Problem, and had a higher confidence level of problem solving than those from the conventional instruction condition, $t(78)=2.04$, $p<.05$, and $t(78)=2.85$, $p<.01$, respectively. There was no significant difference between the two conditions in solving the Isomorphic Problem, $t(78)= -.53$, $p>.05$. Children's Verbal Interaction Behavior was highly correlated with the analogical reasoning when problem solving, $r=.25$, $p<.01$.

Table 2. Question types in children's verbal interacton in problem solving

	Frequency Collaborative condition		Individual condition	
Comprehension Questions		*(62)*		*(10)*
What does... mean?	1		0	
Descriptions in solvers' own words	36		9	
Why is this the reason ?	9		1	
How do you solve this problem?	16		0	
Connetion Questions		*(1)*		*(1)*
Relating new material to prior knowledge	1		1	
High-level Explanations		*(151)*		*(92)*
Analyze the reasons	16		10	
Clarify the relations	46		13	
Propose problem solutions	77		63	
Propose solution and ask for support	12		6	
Low-level Explanations		*(56)*		*(18)*
Rehearsal of the answer	41		7	
Short answer	15		11	
Critical Thinking Questions		*(26)*		*(9)*
Noting strengths and weaknesses, justifying and evaluating ideas	26		9	
Irrelevant Dialogue	52	*(52)*	29	*(29)*
Total verbal interaction	*348*		*159*	

Table 3. Children's problem solving , verbal interaction behavior performance, and confidence of problem solving in the two mathematical instruction conditions.

Instruction conditions	Collaborative instruction (N=39)				Conventional instruction (N=41)			
Performance	IP	NP	VIB	Confidence	IP	NP	VIB	Confidence
MEAN	9.87 (SD=1 0.03)	11.64 (SD=8.27)	27.67 (SD=3.72)	8.69 (SD=2.25)	10.98 (SD=8.38)	7.56 (SD=9.62)	20.67 (SD=2.16)	7.24 (SD=2.29)

Findings of the first study indicated that children who were in the collaborative small groups, demonstrated better analogical reasoning than those who were in the individual condition on isomorphic problem solving but not on nonisomorphic problem solving. Children's analogical reasoning was also highly related to their confidence level for problem solving. From the results, it seems that the collaborative learning

did help children to learn the source problem and to analogical transfer on the one-to-one correspondent problem solving. But the children who were from the conventional instruction condition, they were not used to discuss with peers in general mathematics class. It is probably that children were unable to perform better questioning and answering skills, then the knowledge construction might not well enough when learning the source problem to be activated for solving the nonisomorphic problem. It can be found from the results of the second study. The beneifits from collaborative learning was limited if the learners were unable to have a better discourse with peers.

Findings of the second study reveled that children who were in the collaborative instruction condition, posed more thought-provoking questions (e.g.: the comprehension questions ; high-level explanations, and critical thinking questions) to each other and answered each other's questions on the material being studied. In constrast, children who were in conventional instruction condition performed less verbal interaction behavior. However, more complex knowledge construction and reasoning is indicated by explanations, justifications and inferences, and the like. When children engage in the thought-provoking questions during learning, their understanding on the learning tasks is enhanced, which in turn activate the knowledge to analogical transfer in nonisomophic problem solving. Furthermore, the better questioning and answering skills which direct to problem-solving enhance solvers' confidence level for problem solving. It seems clear that elementary age children can be cultivated to pose these kinds of questions from collabative instruction condition.

Implications for Instruction

From the present experiments, children's analogical reasoning when problem solving was significantly influenced by the collaborative learning condition. Children who learned in collaborative small groups were able to proposed more questions and answers, and were more actively involved in group discussion. Such interaction may facilitate knowledge construction, which in turn promotes positive analogical reasoning performance (Graesser & Person, 1994). Children from conventional instruction have fewer opportunities to discuss with peers Their verbal interaction with peers and

their questioning skills were poor, and they transferred knowledge poorly on solving nonisomorphic problems. In the mathematics classroom, certain kinds of instruction and discourse produce specific learning outcomes. When students participate in collaborative discourse, they have to concentrate on a deeper level of understanding and reasoning than they would when working individually (Schoenfeld, 1989). It is worth to encourage children to engeged in collaborative discourse, that facilitate them to propose more questions and answers as well as promotes positive analogical reasoning performance.

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AN ANALYSIS OF STUDENT TALK IN 'RE-LEARNING' ALGEBRA: FROM INDIVIDUAL COGNITION TO SOCIAL PRACTICE

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Abstract

In this paper we report on a study with the aim of investigating how a focus on language and meaning can assist students in reconstructing algebraic knowledge. The project is set in the context of ongoing work with students in Higher Education who need to develop their understanding of algebra if they are to make substantial progress within their undergraduate studies. The project is based upon a belief that students' difficulties with algebra are language-related. We have collected extensive data by means of videotaped sessions involving the students talking about their own understandings of algebra. The students involved were drawn from courses in initial teacher education and engineering. This paper presents a detailed analysis of the responses of one student and discusses the ways in which this shifted our attention as researchers from looking at our data from the perspective of individual cognition towards one informed by social practice theory.

Introduction

The Re-Learning Algebra project grew out of the difficulties many students have with algebra which have been observed in the course of working in the arena of Academic Maths Support at Sheffield Hallam University. These students have considerable prior experience with algebra and many have undergone years of drill and practice. They have encountered algebra as both an abstract topic in its own right and also within various contexts. Therefore any additional help offered to such students clearly needed to take account of previous experience but also needed to have a different emphasis. An approach which was seen to be successful in practice involved encouraging interaction using group activities in which the students could share their understanding and experience. The activities also addressed the use and development of algebraic language and have previously been reported on in Elliott and Johnson (1995).

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Related Literature

The literature cited in this section of the paper helped to formulate our thinking and informed our discussions during the course of this study. Amongst the most relevant literature was the work of Lesley Booth (1984) and David Pimm (1995). Also of particular influence has been the work of Anna Sfard and Liora Linchevski (1994) in relation to their *theory of reification*. Of particular interest is the way they consider two 'especially crucial transitions': that from the purely operational algebra to the structural algebra 'of a fixed value' i.e. an unknown and then from there to the functional algebra of a variable. Carolyn Kieran's (1989) emphasis on the recognition and use of structure as a major area of difficulty in algebra was also found to very resonant and in particular the way she highlights the equality relationship between left- and right-hand expressions of equations as a 'cornerstone' of much work in teaching algebra. She observes that for students who tend to view the right hand side as the answer 'the equation is simply not seen as a balance between right and left sides nor as a structure that is operated on symmetrically'. The notion of a *cognitive gap* between arithmetic and algebra as proposed by Carolyn Kieran and Nicholas Herscovics (1994) was also found to be relevant. This can be characterised as 'the student's inability to operate spontaneously with or on the unknown'.

We found a different emphasis in Abraham Arcavi's (1994) ideas about symbol sense which seemed closer to our initial starting point for this study. His work is about describing and discussing behaviours and not about defining and describing research on students' cognition and ways of learning and there is an emphasis on sense-making and on recognising meaning. A similar emphasis on meaning is to be found in the work of Romulo Lins (1994) and Arzarello, Bazzini, Chiappini (1995) who describe their work as moving towards a socio-cultural theory and practice. The analysis of algebraic thinking offered by the latter authors is underpinned by Vygotsky's (1962) ideas of algebraic thought and language as two intertwined, mutually dependent aspects of the same process and his stress on the fact that the word's meaning is a linguistic and intellectual phenomena that evolves in time. The authors identify the importance of the Conceptual Frame as an organised set of knowledge and possible behaviours and this notion is seen to be closely linked to that of the Semantic Fields of Romulo Lins.

Theoretical framework

Given the initial aim of this study, which was to investigate how a focus on language and meaning can assist students in reconstructing algebraic knowledge, we have sought to develop a theoretical framework which takes account of this emphasis on language and meaning.

A key influence has been the work of Lev Vygotsky (1962), underpinning which is a central assumption that socio-cultural factors are essential in the development of mind. In discussing the influence of such a perspective, Stephen Lerman (1996) argues that language is not seen as giving structure to the already conscious cognising mind; rather the mind is constituted in discursive practices. Thus the semiotic function becomes the focus of study, rather than the mental structures (for Piaget).

In exploring the notions of sense and meaning further some useful ideas were drawn from the field of activity theory. In particular Erik Schultz (1994) offers some interpretations of sense and meaning when writing about the hermeneutical aspects of activity theory. He proposes that the purpose or intention of a cultural product is the meaning and further that meaning is a kind of 'cultural intention' in a supra-individual fashion. In all cultural products there is an intention to be found, and in finding it, we interpret the meaning of the product. Sense is the interpretation one makes of the meaning. We also found the work on activity theory of Kathryn Crawford (1996) to be relevant. She highlights how activity denotes personal (or group) involvement, intent and commitment that is not reflected in the usual meanings of the word in English. She draws attention to the fact that Vygotsky wrote about activity in general terms to describe the personal and voluntary engagement of people in context - the ways in which they subjectively perceive their needs and the possibilities of a situation and choose actions to reach personally meaningful goals. In building upon Vygotsky's work, Leont'ev, Davydov and others made clear distinctions between conscious actions and relatively unconscious and automated operations. Operations are seen as habits and automated procedures that are carried without conscious intellectual effort.

Methodology

Data was collected by means of the video recording of a series of one-hour sessions with four groups of students during March 1995. The students involved were from courses in Education and Engineering. The groups had two or three sessions each.

A series of tasks was devised which were designed to get the students talking together about their understanding of algebra. For example the first activity involved 'Algebraic Pairs'. In this activity each group of two students is given a set of cards

with a pair of algebraic expressions on each. The task is to decide which of the two expressions are always, sometimes or never equal. Another activity was to ask them to explain what they understood by mathematical words such as expression, equation, function, variable etc. The sessions were carried out in a small TV studio.

The initial data analysis involved the three researchers simply viewing the video tapes and discussing reactions and questions arising. Following the tape transcription this process was repeated with the transcripts. Our discussions were further informed by our ongoing reading. We also held two internal university research seminars during this period.

In this paper we have chosen one particular section of the transcript which we found to be particularly rich but also very challenging to us to make sense of in terms of the starting point of our study i.e. how an focus on language and meaning can assist students in reconstructing their algebraic knowledge.

Data Analysis and Discussion

This particular section of the interaction took place at the end of the first session with the 2 Year BEd students. They had been working on the Algebraic Pairs activities for the first part of the session and then had spent the latter part in a discussion of mathematical terms such as expression, equation, function, variable etc. As the session was almost complete, the researcher provided the opportunity for any questions, reactions or general discussion. The result was an extensive and articulate series of responses from one student in particular - Anthony (AG). Anthony is a mature student who had previously worked in industry as an engineer.

BH OK, I was going to think about further activity but seeing as we've only five minutes left, I think we'll end. Unless, are there any particular things that struck you as we've been talking, that you want to return to, words which conjure up...

AG It's obvious as we start talking about maths, we start talking about functions, some people have got a clearer view; that my image I realise now, when I'm teaching, I tend to opt for, I like to see it as, that y = some function, it could be a=3b plus something. I keep returning to y = some function of x and if I saw it in a textbook for example that 2(x+3) my automatic reaction would be to write y=(2x+3) before I give it to the children to do. y=2(x+3).

BH What would you be thinking of asking them to do next?

AG I'd be asking them to multiply the brackets out to give me a y=2x+6 or asking them to substitute a value of x and tell me what y is because that on its own as a function - 2x+3. I

suppose to me it is just floating about in mid-air with no relationship to anything. It's totally intangible, what is it? what's it for? So if I ask them to multiply that bracket out I got 2x+3 before, now I've got 2x+6, still doesn't lead to anything, doesn't mean anything. doesn't tell me what it's from or where it's from - so my automatic reaction is to put the y= in. Otherwise you've got that floating about and that is a function, then you've got function. To me, what is a function, where does it come from, where does it come from?

BH You'd be happy to relate it to y. What would that mean then for you?

AG There's a missing number y and a missing number x and if we put any value in for y, or any value for x ...If we can find a value for y then we can find a value for x and if you get into a quadratic there'd be two answers for y, so actually you're using something to solve a problem.

BH Just taking that, say it was y=2x squared plus 6 times something ...You said two values.

AG Again, as soon as you get an x squared, I tend to think that it's probably going to be two answers. Depends on ...

AG I'm coming from a realistic point of view in that I've got a specific problem of trying to find out what this value of y is and in doing that I've made an equation in order to solve my problem and in trying to solve my problem I might find that there are two values of the x.

BH Say we had that? What about if I said y was minus 10?

AG Minus 10? Then there might not be a solution to it ... no real solution. No solution to my real world. This idea of no real solutions you've gone into a hypothetical world. You've gone out of a real life situation. From my experience, in my situation, you've gone out of a real life situation, you're going back to a hypothetical situation. You're going right full back in circles to functions, that's something hypothetical - it's floating about, not related to anything or solving anything. It's not come from any real life situation, it's just a function, it's not related to anything else. I think that's why I have difficulty in seeing where it's coming from.

An initial analysis of this section suggested a number of links with the background literature and theoretical framework previously outlined. In order to help the reader make sense of the transcript, it is worth emphasising at the outset that Anthony does not distinguish between the terms function and expression. In fact he refers to 2(x+3) as a function rather than as an expression at line 7. In relation to activity theory there are a number of references to a lack of purpose when dealing with functions. For example at line 13, Anthony asks 'what is it? what's it for?' and at line 14 says that it 'still doesn't lead to anything' and goes further to say that it 'doesn't <u>mean</u> anything'. This statement fits with Erik Schultz's interpretation of meaning as the 'purpose or

intention of the cultural product' which in this case is the word 'function'. Anthony's description also suggests that he is working operationally for much of the time e.g. at lines 7/8, he says that 'my automatic reaction would be to write y=(2x+3)' and also at line 15 he says that 'my automatic reaction would be to put the y= in'. His comments also suggest a lack of appreciation of the structural properties of equations e.g. at lines 10/11 he would 'be asking them (the children) to ... substitute a value of x and tell me what y is' This suggests a view, consistent with the work of Carolyn Kieran, of 'the right hand side as the answer'. His comments at line 19 'There's a missing number y and a missing number x' suggest that he has not made the transition, in Anna Sfard's terms, from the 'structural' algebra of 'a fixed value' to the 'functional' value of a 'variable'. It seems from Anthony's comments that he sees the purpose of an equation as being to find a missing number and not to express a relationship. In Carolyn Kieran's terms, the equality relationship is not fully recognised i.e. the equation as a balance between right and left hand sides and as a structure to be operated on symmetrically.

To an extent these observations are typical of many students although they were surprising to the researchers, as Anthony was seen to be a mathematically capable, though not strong, student. However much of what Anthony had to say was left untouched by this analysis and we were left with a sense of the inadequacy of the various theoretical frames, through which we had viewed our data, to account for what Anthony had to say. It seemed that there was evidence of resistance to 're-learn' algebra on Anthony's part and much that was being said was about his sense of identity and also his view of the nature of mathematics. None of this seemed to have been addressed in our first readings of the data. As a result of wider discussions with colleagues we decided to look to social practice theory for a 'wide(r) angle lens' (Robert Dengate and Stephen Lerman, 1995) through which to view our data. In particular we turned to the work of Jean Lave and Etienne Wenger (1991) and that of Jean Lave (1996).

Jean Lave and Etienne Wenger stress the essentially social character of learning and propose learning to be an aspect of a process of participation in socially situated communities of practice. They discuss the notion of Legitimate Peripheral Participation (LPP) which describes the particular mode of engagement of a learner in a new community of practice, whose level of participation is at first legitimately peripheral in the practice of the expert. The move from peripheral participation to full participation is seen as a dynamic process, characterised by changing levels of participation. Writing in 1996, Jean Lave describes the direction of movement as a

telos and gives the example of 'becoming a respected, practising participant among other tailors or lawyers, becoming so imbued with the practice that masters become part of the everyday life of the Alley or the mosque for other participants and others in their turn become part of their practice'. She proposes that this might form the basis of 'a reasonable definition of what it means to construct identities in practice .

Returning to the analysis of the transcript, it seems that there is considerable resistance on Anthony's part to re-construct his view of algebra. His view of a function is that 'it is totally intangible' (l 13) and 'with no relation to anything' (l 13). It is 'floating about in mid-air' (l 12), without meaning e.g. 'what is it?' or purpose 'what's it for?' (l 13). It seems that Anthony's view of mathematics is only meaningful if 'you're using something to solve a problem' (l 21). Having a problem to solve is real e.g. 'I'm coming from a realistic point of view' (l 25) and equations are simply tools to solve 'my problem' (l 27) e.g. 'I've made an equation to solve my problem' (l 25). In formulating his views on the nature of mathematics, Anthony also seems to be saying significant things about his own sense of identity. His background is that of an engineer working in industry over many years and his path into Higher Education and teacher training would have been via vocational routes. Anthony seems to be calling on his previous experience (as expert) in this particular community of practice and also on his developing expertise in the practice of 'school teacher' to emphasise his identity as a part of the 'real world' e.g. 'my experience, my situation' (l 31). This contrasts with his view of the community of practice of mathematicians, as exemplified by the researcher, who inhabits 'a hypothetical world' (l 30) and who has departed from the real world e.g. 'you've gone out of a real life situation' (l 31). He stresses his view that the researcher/mathematician is going nowhere e.g. 'You're going right full back in circles to functions, that's something hypothetical - that's floating about. not related to anything or solving anything. It's not come from any real life situation, it's just a function, it's not related to anything else.' (l 31-35). However he does seem to express some sympathy and desire for a greater level of participation in the practice of being a mathematician when he says 'I think that's why I have difficulty in seeing where it's coming from.' (l 35) This also seems to reflect his peripheral participation in this particular community of practice.

It seems that our interest at the outset of this study, in language and meaning, has given us a picture of some of the ways in which our students are working on re-learning algebra. However it has also revealed much more - a complex set of phenomena and questions with which to revisit both our data analysis and also the ongoing development of our own practice.

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The Cognitive and Symbolic Analysis of the Generalization Process: The Comparison of Algebraic Signs with Geometric Figures

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ABSTRACT

Mathematical symbols have significant part in generalization. What seems to be lacking,however,is the consideration on the difference of role in generalization which algebraic signs and geometric figures play although Thom(1973,p.207) said "One sees from this comparison how Euclidean geometry is a natural(and perhaps irreplaceable) intermediate stage between common language and algebraic language." For this purpose, we first set up a theoretical framework for the analysis of cognitive activities in generalization in terms of critical consideration on Dörfler's generalization model from the metacognitive viewpoint and Skemp's director system. We secondly designed two mathematics class which are both problem–solving oriented, and normal. One is "Numbers on the Calender" to examine the change of forms in algebraic signs in generalization. The other is "The Sum of Five Angles in Pentagram (hereafter cited as "The Pentagram") to research the change of meanings in geometric figures in generalization. A close observation, comparison, and analysis of these two teaching practice based on the above framework has shown that there are two types of generalization: one is the generalization of object, the other is that of method.

1 . BACKGROUND FOR RESEARCH

Generalization is so crucial to mathematical thinking, therefore we should pay much attention to generalization process in mathematics learning. But both process and significance of generalization, setting products of it aside, seem to have room for consideration and is worthwhile figuring it out although a lot of effort has been made on this area. It will be useful, to begin with, to sketch out the work of Dörfler because he devotes his research into the generalization process from a epistemological viewpoint and proposes his generalization model as shown in Fig.1(1991,p.74).

His generalization model has two main features. One is to involve the *constructive abstraction* which extends from *system of actions in the starting situation* to *symbols as objects*. This process is based on actions and the reflections of them. The other is the adequate allocation of *symbols as objects* in the generalization process. According to him, symbols in Fig.1 can be of a verbal, iconic, geometric or algebraic nature (Dörfler,1991,p.71). Needless to say, various representations which Dörfler calls *symbols* in Fig.1 play vital roles in mathematical thinking and learning.

Although Dörfler's model provides us with fruitful suggestions, we think there remain still two main issues to be answered. In other words, his generalization model shows some salient key stages which form the generalization process. However cognitive activities which promote generalization are not mentioned sufficiently and those activities are still embedded in lines which connect among these stages. That goes to the heart of the problem in generalization as *process*. Consequently we need to set up the adequate framework for analysis and to examine these cognitive activities on it. This is the first issue.

The second issue is on variableness in mathematical symbols. This is deeply concerned with symbols as objects in Fig.1. That goes to the core of problem in generalization as *product*. We should classify mathematical symbols into two categories, that is, algebraic signs and geometric figures

although Dörfler has another way of classification. This categorizing reflects the quality of generalization in each. Regarding to categorizing, but not regarding generalization directly, Skemp (1987) suggested that the features of visual system make a excellent contrast with those of verbal–algebraic system (p.79).

2 . PURPOSES

Our teaching practice was designed based on Dörfler's generalization model in Fig.1. Our main purpose of this attempt was to examine the generalization process through a teaching practice, and then to complement and elaborate it from cognitive and symbolic perspective. This study would contribute to the cognitive significance of the teaching unit as well which Wittmann (1984,1995) thinks great deal of. But this is not our present concern.

To sum up, the purposes of this paper are the following:

(1) To set up the theoretical framework for the analysis of cognitive activities which promote the generalization process.

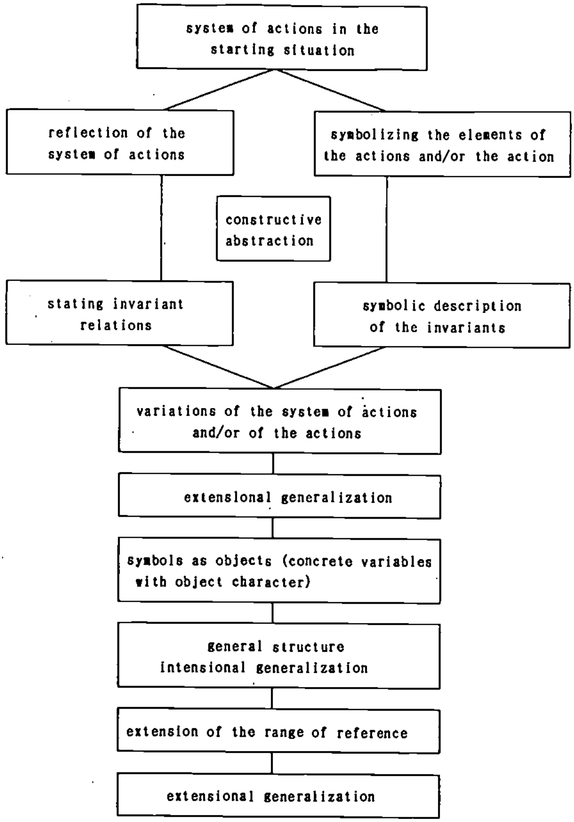


Fig.1 Dörfler's generalization model (1991,p.74)

(2) To examine the quality of generalization process by means of comparison between role of algebraic signs and that of geometric figures in it.

3 . THEORETICAL FRAMEWORK

Dörfler used no more than two words from cognitive terms to explain the generalization process, that is *schema* and *reflection*. He might try to prevent his generalization from psychological discussion because he might think metacognition was not useful enough to describe the process of generalization. We think all other cognitive words than above two words to talk about it are concealed in the line connecting each stage in his model Fig.1. Therefore these two words easily bring us to expand the notion of metacognition which refers to one's own knowledge concerning one's own cognitive processes and products or anything related to them(Flavell,1976,p.232). But some explanations will be needed on metacognition before going there.

Over the past decade a considerable number of studies such as Garofalo & Lester(1985), Schoenfeld(1987), Silver(1985) have been made on metacognition in mathematical problem solving, which exclusively focus on the functions of metacognition as the driving force in *pre*− solving problem. But the generalization would begin substantially *after* solving problem if consciousness of students are attracted to the following activities: extracting mathematical relations among phenomena, reflecting on one's own existing schemata, searching of connections between them, reorganizing or creating those relations as one new kind of schema, and so on.

Many researchers of metacognition investigated the driving force promoting problem solving under the condition of *pre*− solving problem. Therefore they have examined nothing about *post*− solving problem, that is, generalization from the metacognitive perspective. But the generalization could be said to be one of the most metacognitive problem, which has great deal of significance in both mathematics and education. Then we should think about the concept of metacognition which integrates *pre* and *post*− solving problem consistently. We found out its hint in the *director system* of Skemp (cf.1979a,1979b,1987).

We proposed the expanded theoretical framework for metacognition shown in Fig.2(Iwasaki et al.,1995), which was built by combining of the present theory of metacognition in mathematical problem solving and Skemp's director system. This framework works to explain the intellectual development such as *post*− solving problem which includes the generalization.

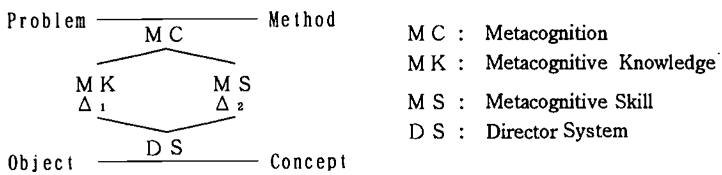


Fig.2 The transformation from metacognition to director system

Van Hiele shows the discontinuous development of mathematical recognition. A method should be re− cognized as an object in new cognitive stage. If so, metacognition should be transferred into the director system of Skemp at the same time.

According to van Hiele and Skemp, a certain method could become an object of thinking in *post*− solving problem. It is to share the new situation of thinking with students in a class that deserves to much attention of mathematics teaching− learning. Even if metacognition has a firm place in the context of *pre*− solving problem, we should extend it to reflective intelligence to describe the cognitive process in *post*− solving problem.

If we consider Fig.2 adequate when examining cognitive activities embedded in the line of Fig.1 where Dörfler is silent, we could cite the following functions of delta− two to reveal them in the intellectual development metacognitively (Skemp,1979b, pp.218−219). That is:

(a) Formulating our concepts and schemas.
(b) Devising experiments by which to test the productive powers of our schemas.
(c) Revising our schemas as necessary in the light of these (and other) events.
(d) Mental experiments, by which we try to optimise our plans before putting them into action.
(e) Examining our schemas for inconsistencies and false inferences.
(f) Generalizing our concepts and schemas.

(g) Looking for connections between events and our existing schemas. This is a reflective activity in which a person is explaining to himself.
(h) Increasing the number of conceptual connections within a schema, as a special case of
(i) Improving and systematizing the knowledge we already have.

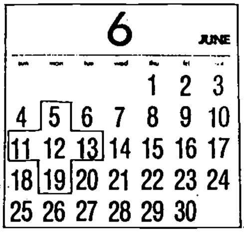
The term *our* in above functions (a) to (i) is worth paying attention because it implies that metacognitive activities inside ourselves could be shared with students. Therefore we think generalization could be realized as cognitive activities in classroom situation.

4 . CASE STUDY(1) : The Analysis of the Generalization Process in "Numbers on the Calendar"

In this section, we design a class *Numbers on the Calendar* for eighth graders in two class hours periods. The aim of this class is to enable students to get interested in a calendar, find the relations among numbers on it, and express them in numerical expressions by use of letters.

[Numbers on the calendar]

This is a calendar of June in 1995.
Let's consider about it.
(1)We enclose five numbers on this calendar with the frame ✙ . What relations can you find among these numbers ?
(2)Move this frame freely. How is the relations you find in problem (1) ?
(3)Changing the shape or location of the frame, find various relations among numbers on a calendar.



In problem (1), the frame ✙ in a calendar is fixed. Students find some relations among five numbers 5,11,12,13,19 by adding, subtracting, and comparing the results of computations through trial and error. Here we pick up and consider one of the relations, namely, "*The sum of three numbers on the vertical direction in the frame is equal to that on the horizontal direction*".

Secondary, problem (2) takes in advance students' following question: "*How is the relation in other location of the frame ?*". Students spontaneously move the frame ✙ freely and consider whether the relations they find in problem (1) are held or not. These activities forms a part of the *extensional generalization* in Fig.1. The issue here encourage the actual state of mental activities arising at the *post*– solving problem (1). In this stage, student's schema about the numerical relations in a calendar is transferred into more adequate one by these mental activities. Since these are closely concerned with the schema construction or reconstruction, we think it is a better way to explain these mental activities by means of the functions of delta–two. In the following sketch, each alphabet in the parenthesis [ ] stands for the function of delta–two mentioned previously.

After the extensional generalization, students who prompt by full of curiosity have offered a question *why* this relation among numbers is held in any location. This question is quite natural because this mathematical relation is tentative and personal one. We can explain the mental operations which support this *why* as following. The students try to connect logically between one's own existing schemas or ideas and several facts obtained by moving the frame [(g),(h)] , and to formulate the logical connections as a new kind of schema [(a)] . The utterance of *why* seems to be raised by these mental activities.

Taking this opportunity, the actual state of activities is transferred into the stage of *verification or testing mathematical relation*, in other words, *schema testing in a public level*. As a result of this thinking

why, students come to pay much attention to the relations between a central number and other four numbers in the frame. At that time, students recognize that four numbers except for a central one are expressed by adding or subtracting 1 or 7 to a central one in the frame. The invariants in the numerical relation are represented for the number " 1 " or " 7 " .

Until this time, a central number has been the invariant in the following form:

$$(12-1)+12+(12+1)=(12-7)+12+(12+7)$$

However, it becomes the variant by means of letter *n* as follows:

$$(n-1)+n+(n+1)=(n-7)+n+(n+7)$$

The essence of this mathematical relation is visualized by expressing the invariant for a number. And it becomes objects of the subsequent operations by symbolizing the variant.

Though the symbolization enables students to express the mathematical relation in a general form, students offer the second question *why*. This second *why* is the question which raise the tentative mathematical relation to the status of relation proofed logically. To put it another way, as students make sure that both the sum of numbers on the vertical direction and that on the horizontal are equal to 3n in any time, they recognize that the previous equation stands for the intention of the mathematical relation. In this sense, this is the *intensional generalization*. From the semiotical viewpoint, though the thinking in earlier stage is in the semantical level, the thinking in this stage is detached from the original context of a calendar by the introduction of symbol n and is in the syntactical level.

The invariants like " 1 " and " 7 " are based on the invariability of actions of arranging seven numbers on each line in a calendar. It seems that this second question *why* is supported by the following mental operations. Those are to formulate one's own schema more exactly by means of symbols [(a)] , to connect one's existing schema with the facts or relations held in any location [(g)] , to generalize the tentative relation as the logically proofed one [(f)] .

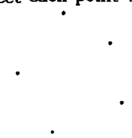
Finally, problem (3) aims at the *extensional generalization* which continues to the intensional generalization. As students change the shape of a frame or the arrangement of numbers on a calendar, they find new relations in new situations. For example, it is one of those activities to rotate the frame ✣ in a 45−degree arc. In this stage, the invariants in the earlier stages are transferred into the variants with the help of *what if not ?* strategy such as changing the initial conditions.

5 . CASE STUDY(2) : The Analysis of the Generalization Process in "The Pentagram"

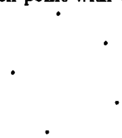
In this section, we design a class *The Pentagram* for eighth graders in 2 class hours period. The aim of this class is to enable students to find the geometrical features of the pentagram, prove those features deductively. In problem(1), students connect with five points in two ways. When they construct the figure like a star in problem (1−b), they name it the *pentagram*. Since students note on some similarities between the figure in (1−b) and the star, we can regard the figure as the *icon*. In problem (2), students find some features of the pentagram which they construct in (1−b). Although they find various features, in this paper we focus on the following feature, that is, $\angle A+\angle B+\angle C+\angle D+\angle E=180°$.

Most students measure the sum of five vertical angles by a protractor at first. As a result of the measuring, they propose the tentative assumption that the sum equals to 180 ° . Next, students draw several concrete pentagrams and confirm that their assumption would be true by summing up 5 angles. However, some of them stick to another measuring results. Students who realize the limitation of this measurement try to prove the anticipation, that is, the equation deductively. This activity is the process

[The Pentagram]
(1) Connect the following five points.
(1−a) Connect each point with the next one.



(1−b) Connect each point with every other point.



(2) Find the features of the pentagram in the above (1−b) .
(3) Explain the reason why your anticipation is true. [Note: ∠ A, ∠ B, ∠ C, ∠ D, ∠ E are vertical angles of the pentagram in (1−b) respectively.]

$$\angle A + \angle B + \angle C + \angle D + \angle E = 180^\circ$$

which extends from *system of actions* to *extensional generalization*.

We can explain the mental process which promote these activities described above by functions of delta− two. Firstly, the activity of drawing various concrete pentagrams is raised in order to improve and organize their assumption [(g)] . In fact, they try to draw several pentagrams [(b)] , to measure the sum of angles, and to modify their ideas if necessary [(c),(e)] . As a result of this, they feel sure that their tentative assumption may be true and try to verify and generalize it [(f)] .

In problem (3), students logically prove that their anticipation is true. The solution process of this problem involves *symbols as objects* in Fig.1. To put it another way, *symbols as objects* is the boundary where students the inductive reasoning is transferred into the deductive reasoning. The importance of *symbols as objects* is the same as that in *Numbers on the Calendar.* In this class, we identify four types of students as follows;

(a)Students insisting that summing up 5 angles which are measured by a protractor is 180° .

(b)Students explaining their result by using the demonstrative noun such as *this* and *that*.

(c)Students first explaining the equation in the special case such as the regular pentagram, and after that they try to prove in the general pentagram.

(d)Students explaining the equation by using alphabetic symbols like *A,B,C* etc.

In this case, it is essential that students must regard the concrete pentagram as the general one in order to reason deductively. From this point of view, students in type (a) have not reasoned deductively yet because they exclusively treat the specific concrete pentagram. On the other hand, students in type (d) have already reason deductively. In this sense, the pentagram which students in type (d) treat as the object of thinking is the general one and the *symbolic sign* because they try to explain by means of alphabetic sign like A,B,C, etc. The terms which students in the type (b) treat are restricted to the demonstrative noun such as *this* and *that.* We think that they have not reasoned deductively yet because of their context boundness. In the case of type (c), for example, students say as the following: *As this angle is bigger, that one is smaller. So I think that my anticipation is true even if it is not a regular pentagram.*

6 . DISCUSSION

In section 4 and 5, we design two mathematics classes and give an outline of them. In our sketch of two classes, we can see cognitive activities embedded in lines in Fig.1 by the functions of delta− two from the expanded matcognitive point of view. We think this attempt contributes to the elaboration of

the model of generalization process. At the same time, we come to recognize the significant difference between the roles of symbolization at the generalization in the algebraic situation and that in the geometric one. As our sketch shows, we can see this difference at *symbols as objects* in Fig.1. The prominent phenomenal difference is the following, that is, in *Numbers on the Calendar,* we can videotape the symbolization process that students replace the central number in the frame ✙ by the letter n and reason deductively by means of it when students go over the boundary *symbols as objects*. On the other hand, it is difficult for us to observe or record the corresponding process in *The Pentagram.* In other words, we can not identify the general and ideal pentagram N in the visual form because the quality of *symbols as objects* in the geometric situation is not the same as that in the algebraic one.

If students go over the boundary *symbols as objects*, some indication of changes might reveal in students' utterances and drawing. As we mentioned in the section 5, four types of students are identified by the analysis of these students' utterances and drawing. In *The Pentagram,* students must regard the concrete pentagram as the general and ideal one when they reason deductively. However, we should not overlook that the object of student's thinking is externally same in both pre and post–symbols as objects. And we should note that the difference among four types depends on the way of viewing the pentagram. Therefore, based on the way of viewing the pentagram, we must set up the criterion of this categorizing.

Regarding this point, Dörfler(1996) notes on the *object* and the *product* of one's own thinking and calls them the *prototype* and the *protocol* respectively. For instance, the concrete pentagram, in the blackboard or in the paper, as the object of student's thinking is one typical example of the prototype. On the other hand, the protocol is a record of one's own activities to the prototype. Cognitive process and its representations such as students' utterances and drawing to prototype are typical examples of the protocol. The above issue on the way of viewing the pentagram is closely concerned with the cognitive activity which supports the transformation from the prototype to the protocol. And this cognitive activity is the expanded metacognition mentioned in the section 3. In this sense, although the term protocol is the integrated notion of process and product, we should distinguish between process and product. Consequently, we come to adopt the notion of the expanded metacognition as the criterion for the analysis of process–aspect of the protocol, and Peirce's classification of symbols, namely, *icon*, *index*, and *symbolic sign* (Yonemori,1981) as the criterion for the analysis of product–aspect of the protocol. From this perspective, the comparison of *symbols as objects* both in *Numbers on the Calendar* and *The Pentagram* seems to be summarized in Table.1.

7 . BY WAY OF CONCLUSION

Main findings in this paper are the following:

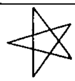
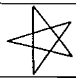
(1) In this paper, we extracted the mental activities, especially metacognitive activities, embedded in lines which connect with stages of the generalization process as shown in Fig.1. Metacognition here constructs or reconstructs one's own existing schemas under the condition of *post*– solving problem and promotes the generalization. This is new roles of metacognition which has not been mentioned yet.

(2)The close observation, comparison, and analysis of two teaching practice based on the theoretical framework in this paper leads to the following conclusion that there are two types of generalization: one is *the generalization of object* in the algebraic situation, the other is *that of method* in the geometrical situation. In other words, in *Numbers on the Calendar,* the object of one's thinking such as the concrete number is generalized by use of letter n. On the other hand, in *The Pentagram,* the way of viewing

itself is generalized. It realizes the change of inference form, that is, from inductive to deductive. This makes the excellent contrast each other.

Table 1. The Comparison of *Symbols as Objects* in *Numbers on the Calendar* and *The Pentagram*

	Prototype	Protocol	
		Process (the utterance raised by the expanded metacognition)	Product (numerical expressions)
pre-Symbols as Objects	the central number in ✚	Let's sum up three numbers in both the horizontal and vertical direction in ✚.	$5+12+19=36$, $11+12+13=36$ (Index)
post-Symbols as Objects	letter n	Let's replace the central number in ✚ by letter n.	$(n-1)+n+(n+1)=(n-7)+n+(n+7)$ (Symbolic sign)

	Prototype	Protocol	
		Process (the utterance raised by the expanded metacognition)	Product (figures)
pre-Symbols as Objects		The sum of five vertical angles equals to 176° in this pentagram.	The concrete and special pentagram as an index
post-Symbols as Objects		The sum of five vertical angles equals to 180° in every pentagram.	The general and ideal pentagram as a symbolic sign

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Making sense of mathematical meaning-making: the poetic function of language

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ABSTRACT. In trying to make sense of mathematical meaning-making, sections of the mathematics education community have increasingly turned to linguistics as a basis for theorising mathematical discourse. In this paper, we critique the standard interpretation of (Jakobson's) structural linguistic theory which has been used by mathematics educators. From the theoretical perspective we outline, based on the work of Jakobson and Barthes, we re-interpret some examples of mathematical meaning-making.

Introduction

Recent attempts to make sense of mathematical meaning-making have drawn freely on the ideas of metaphor and metonymy. Broadly, there are two main strands of research: one drawing its theoretical basis from the work of Lakoff and Johnson (1980) on conceptual metaphors and metonymies (e.g. Lakoff and Núñez 1996, Sfard 1994), and the other from the work of the linguist Jakobson (1956, 1960) on metaphoric and metonymic relations in 'texts' (e.g. Pimm 1990, Walkerdine 1988). Here, we concern ourselves with the latter strand of research and critique the 'standard' interpretation of Jakobson's linguistic theory, in particular the interpretation of 'metaphor' and 'metonymy' which considers the two as dichotomous.

In the standard interpretation, metonymical relations operate *within* a discourse (intra-domain) while metaphorical relations refer to things outside it (inter-domain). Although this interpretation adequately allows meaning to be thought of as developed through the interplay of metaphoric and metonymic relations, it is a partial interpretation of Jakobson. Two crucial features are missing: (1) how the relations may operate at *any* level in a text, not just at the high level of inter- and intra-domain relations; (2) how the relations exist in a dialectic, informing each other as well as informing together.

We mention here a few theoretical constructs whose meanings we will elaborate in the paper. In (mathematical) texts there are multiple systems of signification whose pairwise dependencies we analyse in terms of *denotation* and *connotation*. However, these multiple systems are not a property of the text itself, but of the text and the reader. Thus we look at how a reader (learner) may come to build a *connoted reading* out of the signs of the text. We suggest that a key mechanism for

this is the *poetic function*; and the key to the poetic function is the dialectic of *metaphoric* and *metonymic* relations.

Denotation and connotation

Language can signify something other than 'what it says'. For example, if you are presented with a poem, it does not say anywhere in the text 'this is a poem'. Nevertheless, you attend to the layout of the text, perhaps the regularity of metre, perhaps rhyme; the text therefore signifies 'this is a poem'. This is the traditional understanding of denotation ('what it says') and connotation ('what it does not say').

Barthes (1967) formulates a general semiotic theory of denotative and connotative *systems*. In a semiotic system, signifiers and signifieds are united in the act of signification into signs. His definition of denoted and connoted rests on a relation between the two systems, independent of the nature of the signifieds: the signifiers of the connoted system comprise signs, or collections of signs, in the denoted system. Thus he reformulates 'saying' and 'not saying' in terms of signifying in different, but related, systems:

> I am a pupil in the second form in a French *lycée*. I open my Latin grammar, and I read a sentence, borrowed from Aesop or Phaedrus: *quia ego nominor leo*. I stop and think. There is something ambiguous about this statement: on the one hand, the words in it do have a simple meaning: *because my name is lion*. And on the other hand, the sentence is evidently there in order to signify something else to me. Inasmuch as it is addressed to me, a pupil in the second form, it tells me clearly: I am a grammatical example meant to illustrate the rule about the agreement of the predicate (Barthes 1972, pp. 115-6).

The connoted signified ('I am a grammatical example') has here for its signifier a collection of signs ('because my name is lion') in the denoted system. Barthes names the 'collection of signs' the *meaning*, and the signifier the *form* (*ibid*, p. 117). In order to create a meaning within the connoted system (the grammar lesson), the reader has to do two things. First attention must shift away from the meaning deriving from this sentence about a lion and on to the form. Second, the reader must seek the signified of the form.

We will take as a first mathematical example of denotation and connotation the case of some study materials which form part of a common mathematics curriculum in the UK, the School Mathematics Project (11-16). The writing, and computation, of products involving decimals is initially motivated as a representation of repeated addition (SMP 1983a). This becomes problematic when both quantities are non-integer. SMP introduces this latter case in the context of computing costs where the

number of items and the cost per unit item are given (SMP 1983b). Before asking the child to work out the cost of 3.7m of gold braid the text says

> When you work out the cost of 3m, you do £2.60 **x 3**
> When you work out the cost of 4m, you do £2.60 **x 4** (*ibid*, p.2).

Students are being asked to return to previous texts and by attending to the form, construct the connoted sign: 'this is about a multiplicative structure'. A shift in the site of potential meaning is demanded. In this teaching sequence, the implied role of the text has shifted from representing multiplication, to the object of attention itself. The way such a shifting occurs has been theorised by Jakobson.

Jakobson proposes that language has six functions, each set towards a specific element of the act of communication (Jakobson 1960, p. 357). For example, the referential function of language relates to its capacity to refer to some extra-textual reality. A more than trivial text will rarely fulfil just one function though a particular function may be dominant. The shift of role of the text to 'object of attention' is a result of the dominance of the poetic function. Before we discuss the poetic function in detail, we need to set out Jakobson's theory of metaphoric and metonymic relations. His presentation is characteristically condensed and we have drawn on the elucidations of Lodge (1977) and Hawkes (1977).

Metaphoric and metonymic relations

The workings of metaphoric and metonymic relations are set down by Jakobson in the following terse paragraph:

> The development of a discourse may take place along two semantic lines: one topic may lead to another through either their similarity or through their contiguity. The metaphorical way would be more appropriate for the first case and the metonymic for the second, since they find their most condensed expression in metaphor and metonymy respectively. (Jakobson 1956, p. 76).

Here, 'topic' and 'development' are to be understood extremely broadly: Jakobson is proposing that metaphoric and metonymic semantic development can exist at *all levels* in the text (*ibid*, p. 77). 'Topic' may be the text, a sentence, a word, a combination of words: any discernible 'unit'. Jakobson uses 'metaphoric' for a relation at any level which is based on similarity, and 'metonymic' where the relation is based on contiguity; he reserves 'metaphor' and 'metonymy' for the figures of speech which are the most condensed expressions of such relationships.

A linguistic example of this development 'along two semantic lines', which lies at the heart of structuralist linguistics, is the syntagmatic/paradigmatic polarity. In the syntagm (a technical word meaning 'combination of signs')

'the girl sat on the chair'

the meaning of each word is developed as the sentence is carved out (the *syntagmatic axis*). Thus syntagmatic relations hold between the constituent signs and between the signs and the syntagm, and are therefore relations of *contiguity*. Further, each word's meaning is affected by its relation to other words that could have been chosen (the *paradigmatic axis*) but were not. Thus paradigmatic relations are relations of *similarity* (or dissimilarity, a negation of similarity). Note that paradigmatic and syntagmatic relations hold between signs in the discourse and *not* between signs and some version of a reality 'out in the world'. The meaning of a sign is developed both by its *reference* to some version of reality, and by its *value*: that is, its paradigmatic and syntagmatic relations to other signs in the discourse. For example, 'sat' draws meaning from its contiguity with 'on the chair': a particular way of sitting. It also draws meaning (paradigmatically) from *not* being 'perched', 'lounged', 'crouched', or even 'spat'.

A mathematical illustration. Pimm has written extensively on 'metaphor' and 'metonymy' at the inter/intra-domain level. For example, he has said that activities which develop 'symbolic fluency', such as when children chant a times table, are metonymic; because they focus a child's attention on the "movement 'along the chain of signifiers'" (Pimm 1990, p. 135). But this ignores the fact that, on a different level of topics, there are metaphoric relations present, formed by similarities between the lines of a chant—e.g. '1 times 2 is 2', '2 times 2 is 4', etc.—in the repetition of signs ('times 2') and the regularity of metre. It is these metaphoric relations, generating a sense of movement and rhythm, which, at least in part, cause the text to be metonymic at the level of topics considered by Pimm.

The poetic function in language

The poetic function, whose set is towards the message itself, operates via transgressions of the language system: transgressions that make the text 'strange'. We shall give one example in some detail: the breaking of the syntagmatic/paradigmatic polarity. As remarked before, this is a fundamental feature of language in structural linguistic terms and hence can be expected to be a particularly fruitful site. As we shall outline, the disruption of the polarity shifts attention to form as signifier and to value as potential signified.

One mode of effecting this transgression is by imposing similarity on the syntagmatic axis, where ordinarily (in referential texts) contiguity is expected—this is the principle constitutive device of poetry according to Jakobson (1960, p.358). Rhyme is perhaps the most obvious kind of 'strange' similarity (sounding alike but semantically unlike). In Barthes' phrase (1967, p. 87), rhyming 'corresponds to a deliberately created tension between the congenial and the dissimilar, to a kind of structural scandal.' Jakobson (1960, p. 358) lists other possible strange similarities including, for example, the equalising of word stress

with word stress. An alternative mode of breaking the polarity is to impose contiguity on the paradigmatic axis. In the example given by Lodge (1977, p.77), the syntagm 'ships crossed the sea' can be transformed into 'keels crossed the deep' producing two metonymies. The non-logical deletions, e.g. deleting 'ships' instead of 'keels' from the notional syntagm 'the keels of the ships', render the text strange.

Referential reading becomes interrupted and attention is shifted from the (extra-textual) referent of the sign on to the signs themselves. In poetic texts denoted signs become connoted signifiers. In the case that we have discussed, this occurred through disruption of value and thus value may be brought to the reader's attention: value becomes a potential site of meaning. The poetic text ceases to be solely a window onto something else, but invites the reader to attend to its own form. *But*, it does not cease being a 'window onto': it depends on the reader's focus. Sites of potential meanings are multiplied, not exchanged one in favour of another. In Jakobson's words: 'The supremacy of poetic function over referential function does not obliterate the reference but makes it ambiguous' (Jakobson 1960, p. 371).

Connoted signs that arose out of transgressions, out of breaking the rules of the denoted system, become themselves 'institutionalised' for the reader as he or she develops the connoted system as a site of meaning. In this sense, the new system may become as familiar, and its signifieds as 'concrete', as the signifieds of the original denoted system.

Two mathematical examples

Our examples of the SMP text on multiplication, and the chanting of times tables, have already offered two illustrations of the re-interpretation of mathematical texts. Those, and the two further examples here, show the potential of our theoretical ideas for the analysis of mathematical texts. We should emphasise that we are not claiming to be able to offer a semiotic system of mathematical discourse. We are proposing interpretations by analogy with examples within literary theory and linguistics; thus our interpretations can only be pointers towards a more systematic mathematical analysis.

<u>Example 1</u>

Consider the two mathematical texts

'2 + 3 ='

'2 + 3 = 1 + 4'

It is well known that often, long after a learner is capable of reacting to a text of the first kind by performing the sum, the second produces bewilderment: the learner finds it 'wrong' or 'meaningless'. Several authors (e.g. Kieran 1981) have pointed out two related reasons why children react as they do. Firstly, children interpret the equals sign as meaning 'do the sum'. That is, they have a procedural interpretation

of the equals sign. Even supposing that the children's interpretation can be shifted to some notion of equivalence, a second reason remains: they may have a procedural interpretation of '2 + 3', or any syntagm whose template is 'number-operation-number'. In this case, '2 + 3' will not be seen as the result '5' but as a sum which, if performed, would give the result '5'. So, '2 + 3' cannot be the same as '1 + 4': they are different sums.

In relation to the child's system, the interpretation "'2 + 3' and '1 + 4' both signify 5" is a connoted reading. '2 + 3', a syntagm (recall, a *combination* of signs) in the child's system, is a form, a signifier in the connoted system. Gray and Tall (1994) have written about the 'process-product' ambiguity in mathematical notation, which for them expresses a cognitive 'process-concept' duality, or 'procept'. They posit that a learner's grasp of this 'notational ambiguity' is central to her success or failure in mathematics. From our perspective, the question is: how might the child's entry into the connoted system be facilitated?

One approach may be to tell the child the rules of the connoted system: '1 + 4' is another name for the number 5. But this ignores that, for the child, '1 + 4' is *not* an empty form, it is a syntagm full of meaning. Viewed in this light, the problem is one of denotation/connotation rather than the more general 'ambiguity'; and this highlights an asymmetry of the two systems for the learner. We cannot hope to obliterate the child's denoted sign, 'sums'; and there is evidence that the 'name for a number' approach is not successful (Kieran 1981). The problem is much more difficult: we would need to find ways of building on the child's system, so that she can appreciate a poetic reading of the text: '2 + 3' is equivalent to '1 + 4' because the result of the sum '2 + 3' is the same as the result of the sum '1 + 4'. This reading is a metaphoric relation resting on a metonymy: a sum is like another sum (metaphoric) because their result (metonymy) is the same. Such a reading is not self-evident: the metonymy is non-logical. To comprehend the syntagm as signifier, as formal mathematical discourse would have it, is a matter of enculturation into this discourse. This will not occur through attention to a single text. Enculturation requires that the connoted system be built up by the learner through numerous and diverse activities, with significant attention to poetic readings of texts.

Example 2

Consider the simultaneous equations

$$x+5(y+1)=0$$
$$5y=-(5+x)$$

If the equivalence of these equations is not noticed and a solution is attempted then an ambiguity concerning equality arises which is very different from the 'process-product' ambiguity of Example 1: the calculation will end up with something like '0 = 0'. If attention is focussed on this as a syntagm composed of '0', '=' and '0'

then the statement is tautologous. Clearly, it is a transgression of the 'rules' of the denoted system to supply no information. A student could attempt a poetic reading, focussing on '0 = 0' as a form: as a signifier in the connoted system. But as signifier, its signified is 'the equations are dependent', a far from obvious connection. Perhaps the form is recognised, perhaps the signified is known, but it does not necessarily follow that the sign will be constructed.

Conclusion

In this paper we have briefly outlined a theoretical basis for analysing mathematical meaning-making which calls into question the dichotomous relationship between 'metaphor' and 'metonymy'. The implications are far from esoteric: the theory suggests a need to promote connoted reading of texts by learners, and we are beginning to understand how it may help us to elaborate mathematical meaning-making in terms of webbing—an attempt to explain how a learner struggling with a new mathematical idea can draw on supportive knowledge from a range of sites, rather than simply erecting a hierarchy of abstractions (see Noss and Hoyles, 1996).

We are beginning to make sense of the ways in which carefully-designed computer software can offer learners the means to find more direct entry points into the 'connoted' system, by providing a means for expressing meaning in computational action. Conceived in this way, the computer is a rather special kind of tool in which action involves the formal use of language, and where the usual polarities—meaning and precision, informal and formal—do not hold.

We may speculate that there is a link between this work and our current research on mathematics curriculum design for undergraduate science students. To what extent must the structure of mathematics be understood in order for it to be used effectively as a tool in the sciences? What can we say about the changing relationships between mathematical and scientific epistemologies, and the roles of new technologies in mediating these relationships? In the area of 'service' mathematics teaching there is a standard dichotomy that concerns the ways in which mathematics may be learnt. It can be characterised as 'formal' versus 'informal': one either learns the formal mathematics itself and then 'applies' it to scientific situations, what we might call a 'metonymic' approach (mathematical meaning develops within the discourse), or one simply learns to 'use' mathematics informally in science without attempting a 'formal understanding' of it, what we might call a 'metaphoric' approach (mathematical meaning develops with reference to science). We are questioning this dichotomy—a dichotomy which we speculate is an applied consequence of the metaphoric/metonymic dichotomy with which we began.

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Children Learning to Specify Geometrical Relationships using a Dynamic Geometry Package

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In order to understand the learning taking place when students use a dynamic geometry package such as Cabri-Géomètre, *a particular focus for study needs to be on the learning* mediated *through employing such a resource. In this paper I describe how one pair of 12 year old students begin learning how to specify geometrical relationships in* Cabri. *I argue that, while* Cabri *provides certain elements of the mathematical language necessary for the articulation of relevant mathematical ideas, significant aspects must be provided by the teacher.*

Introduction

The use of concrete materials such as manipulatives, and tools such as calculators and computers, to support mathematics learning is reasonably well-established and widely encouraged. In trying to understand the mathematics learning taking place when students use such devices, the work of Wertsch (1991), amongst others, suggests that we need to consider carefully what stands between the learners and the 'knowledge' that they are intended to learn; that is, we need to focus on the learning *mediated* through employing such resources. Ohtani (1994), for example, presents this in the usual triangular form (adapted slightly as Figure 1).

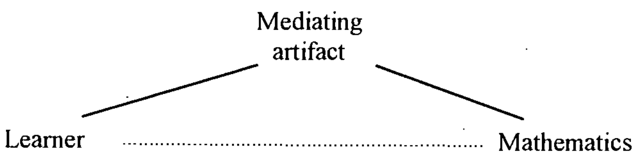

Figure 1

Dynamic geometry environments (DGEs), such as *Cabri-Géomètre,* are one example of such mediating artifacts. Such a package allows the user to experience the *direct* manipulation of geometrical objects (or, at least, the *appearance* of such direct manipulation). Within the computer environment, geometrical objects created on the screen can be manipulated by means of the mouse (a facility generally referred to as 'dragging'; for further details see Hölzl in press). What is particular to DGEs is that when elements of a construction are dragged, all the geometric properties employed in constructing the figure are preserved. This is because one of the significant features of a dynamic geometry package is the ability to specify relationships between geometrical objects (Laborde and Laborde 1995 p 240). In this way, the

software provides the learner with a means of expressing mathematical ideas. As Noss and Hoyles (1996 p 54) argue: "It is this articulation which offers some purchase on what the learner is thinking, and it is in the process of articulation that a learner can create mathematics and simultaneously reveal this act of creation to an observer." Hence when students are using a DGE such as *Cabri* to tackle mathematical problems they are involved in both perceiving and specifying relationships between geometrical objects.

In this paper I focus on the transition from perceiving and specifying geometrical relationships when students are using *Cabri* and how this is mediated by the computer environment. In what follows I describe how one pair of 12 year old students begin learning how to specify geometrical relationships in *Cabri*. I argue that, while *Cabri* provides certain elements of the mathematical language necessary for the articulation of relevant mathematical ideas, significant aspects must be provided by the teacher. The data comes from a longitudinal research project designed to trace the transition of student conceptions of some chosen geometrical objects from informal notions towards formal mathematical definitions. I begin with a brief outline of the theoretical framework with which I will interpret the data.

The Mediation of Learning

One of the central concepts underlining the approach I adopt in this paper is Wertsch's notion of "individual(s)-acting-with mediated means" (Wertsch 1991 p 12) which is itself based on aspects of the work of Vygotsky and Bakhtin. From such a perspective there is an intimate relationship between psychological processes and the sociocultural setting such that all mental processes are considered to be mediated by communication that is inherently and complexedly situated. In this model, when we describe human action we can only do so in terms of the mediating artifact because "action and mediating means are mutually determined" (p 119).

A second central concept is the idea that the move from *perceiving* to *specifying* is at the heart of mathematics learning. In this context, *specifying* requires the use of elements of conventional mathematical language. With certain computer applications, such as Logo, spreadsheets and perhaps DGEs, the computer can become a special tool for mathematics learning because the actions of learners using such applications necessarily involves some formal use of mathematical language. Noss and Hoyles call such a computer environment "autoexpressive" when it contains elements of mathematical language "to talk about itself" (1996 p 69). For a DGE such as *Cabri*, some of the relevant elements of mathematical language (such as mid-point, bisector, perpendicular, and so on) can be considered to be explicitly available via the various menu items. Further elements are implicitly contained within the figure as it is constructed. I will return to this point later in this paper. Given these considerations, the central question here is how we can describe the learning of aspects of plane geometry when mediated by a computer application such as *Cabri*.

With the above in mind, particular foci for the presentation and analysis of the qualitative data from this study are:

- how particular geometric figures presented on paper are interpreted by the students when the aim is to construct them using *Cabri*
- how the figures are constructed; that is how they are specified in terms of the *Cabri* menu items
- what the response is to the feedback presented by the resulting image on the screen
- how the specification is checked
- what form of assistance is sought from the teacher/researcher and what the response is to interventions

I follow the example of Meira by focusing on how "instructional artifacts and representational systems are actually used and transformed by students *in activity*" (1995 p 103, emphasis in original) rather than simply asking whether the students learn particular aspects of geometry better by using a tool such as *Cabri*. This is because what I am interested in is both what the students learn *and* how they learn it.

Description of an Episode

This data comes from a research study in which pairs of students in their regular mathematics classroom tackle a series of tasks focusing on the geometrical properties of quadrilaterals. The pair of students in this extract are 12 year olds who have used *Cabri* on four previous occasions, each one lasting almost an hour, the last time being about four weeks earlier. The class is of above-average attainment in mathematics and from a UK city comprehensive school whose results in mathematics at age 16 are at the national average. The mathematics teacher employs a problem-based approach to teaching mathematics and the students usually work in pairs or small groups. The class has three 50-minute mathematics lessons per week. The version of *Cabri* in use was *Cabri I* for the PC.

The task the pair of students are undertaking is to construct the following diagram, Figure 2, using *Cabri* and hence obtain Figure 3.

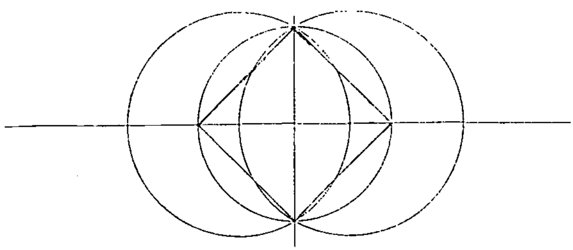

Figure 2

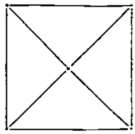

Figure 3

The task then asks the students to "explain why the shape is a square". The students know that they need to construct the figure in such a way that the figure is invariant when any basic object used in its construction is dragged. In the words of Healy *et al* (1994), the figure must be impossible to "mess up".

After a short discussion the pair begin by constructing two interlocking circles, as shown in Figure 4.

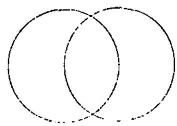

Figure 4

In order to draw the third circle they need to construct its centre. They realise that it has to be midway in between the centres of the two larger circles. In the extracts that follow, *R* and *H* are the students, *I* is myself as teacher/researcher.

R	You want to get that thing in between them, I can't remember what it's called now.
H	Construction is it? No ..
R	Yes, on Construction, and it is ...
H & R (together)	Intersection!

The students attempt to use *intersection*, but, of course, it is not the correct choice. I decide to intervene.

I	What are you trying to do?
R	Make a point in between there.
I	An intersection will only give you the point where two lines cross. But there is something else which will give you something which is halfway between.
R	Go under Construction.
I	Yes, have a look under Construction again.
H & R (together)	Yeah, Midpoint!

They create the third circle and check that their construction is correct by dragging one of the points on their figure.

R	Yeah, that's it Then we want like a diamond shape inside it.
H	So we need to ....
R	Just see if they all stay together first.
H	OK.

69 R Pick up by one of the edge point.

70 H & R (together) Yeah, it stays together!

The next step the students make is to draw two lines, see Figure 5, and again check, by dragging, that their construction is correct.

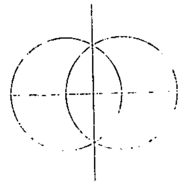

Figure 5

They complete their construction by drawing the four line segments forming the square and once more check, by dragging, that their figure cannot be "messed up". To construct the figure shown in Figure 2 they "erase" (or, more accurately, hide) the requisite lines and finish by constructing line segments as diagonals of the square.

One of the students comments:

167 R A square. Four triangles in it.
168 Or is it a rectangle? Those bits look longer.
169 H They do slightly.
170 R Should I get a ruler?

I intervene by asking them what they can say about the diagonals of the shape.

174 R They are all diagonals.
175 I No, in geometry diagonals are the lines that go from a vertex, from a corner, to
176 another vertex.
177 R Yeah, but so's that, from there to there.
178 I That's a side.
179 R Yeah, but if we were to pick it up like that .... like that. Then they're diagonals
180 I In mathematics, in geometry, a line that goes like that is called an oblique line.
181 It's not vertical, it's not horizontal. It's oblique.

Following this I prompt them into beginning to explain why the quadrilateral is a square. For example, I ask them to compare the lengths of the diagonals and how they intersect.

195 I and what can you say about that line and this line [referring to the diagonals]?
196 H They're the same distance.
197 I They're the same length?
198 H Length, yeah.
199 I OK, so the diagonals are the same length. And what can you say about the way in
200 which they cross?
201 H They cross exactly in the middle.
202 I So you're saying that from there to there is the same as from there to there.
203 H Yeah.

214 I At what angle do they cross?
215 H A right angle.
216 I (to R) Is it a right angle?
217 R No .. yeah.
218 I Yes? So this is a right angle here?
219 R Yeah.

The session finishes with my asking them:

271 I So what sort of shape has got diagonals that are the same, that cross in the middle,
272 so they bisect each other, that cross at 90 degrees, and has got 90 degree corners?
273 What sort of shape is it?
274 H A square.
275 I No other shape is like that?
276 H No.

Analysis and Discussion

The students successfully complete the task, but with particular input from myself as teacher/researcher. This is not altogether unexpected as, in every attempt to reveal the mathematical thinking of learners, the balance between exploration and guidance is always problematic. As Noss and Hoyles explain "This tension is not completely resolvable. We might be able to engineer situations in which a mathematical way of thinking is encouraged. But mathematics *per se* is not discovered by accident" (1996 p 71). What becomes of interest here is the *nature* of the interventions that were necessary.

The students begin confidently enough, although it soon transpires that they have forgotten the term *midpoint*. They know what they want to specify (the centre for the third, smaller, circle) but attempt to locate it using *intersection*, as the drop-down menu calls it (actually the item locates *points of intersection*). An intervention is sufficient to put the students on the right track again.

Lines 69 through 74 shows student R firstly referring to the square to be constructed as a *diamond* (presumably due to its orientation; see Hershkowitz 1990 p 82- 86) and later calling a point on the circumference of a circle an *edge point*. This latter choice of terminology is especially interesting as this particular form of point (and there are several forms of point in *Cabri I*) is referred to in *three different ways* on the screen in this version of *Cabri* (*Cabri I* for the PC). From the creation menu, one can construct a circle using the menu item *circle by centre and rad. pt.* (the user needs to know, presumably, that *rad. pt.* is a shortened version of *radius point*). The pop-up help offers the advice "select or create the centre of the circle, then *a point on the circle*", while the screen pointer uses the terms "this centre" and "this *circle point*" when creating a such a circle. This particular student then invents their own term.

At this point, the students use the drag facility to check that their construction so far specifies the appropriate geometrical relationships. It does. By lines 167-170 in the

transcript, student R is referring to the quadrilateral as a square, but queries the screen image. As I do not think measuring, particularly with a ruler, will resolve the matter I intervene by asking them to reflect on what they have done (transcript lines 174-276). In so doing, I have to introduce terminology that does not occur in any menu item in this version of *Cabri*. At various times I employ terms such as diagonal, vertex, oblique, bisect (note that bisector is a *Cabri* menu item), and right angle.

Finally, the students complete their construction, again checking by dragging that the construction can not be "messed up". They are convinced that the quadrilateral they have constructed is a square and they can articulate some of its geometrical properties.

Concluding Remarks

Overall, the episode portrayed here demonstrates that this particular pair of students had, at their disposal, sufficient technical fluency with *Cabri* to successfully complete the required task (albeit with some timely intervention). It was they who devised the strategy for the construction and consequently it was they who were able to specify their construction using the facilities offered by this particular dynamic geometry environment. They did not merely line up relevant objects by eye nor did they start guessing by randomly opening menus and trying out all the items in some false hope of hitting on the right one (phenomena observed by Noss *et al* 1994 and by Jones 1995).

Yet, at the same time, the computer environment alone was insufficient to allow the students to fully articulate their specification in conventional mathematical language. For one thing, the menu items can not hope to provide the range of terms required (nor could they be expected to do so). For another, a full articulation of why the quadrilateral is a square requires some of those delicate chains of reasoning characteristic of the finer elements of mathematical proof. The explanation of why the shape is a square is not simply and freely available within the computer environment. It needs to be sought out and, as such, it is *mediated* by the computer environment.

On the other hand, the essence of the explanation *is* contained implicitly within the construction. The students' construction of the square is a *general* representation and not a copy of a particular concrete object. What is more, the properties of the figure are derived from definitions within the realm of the Euclidean axiomatic system. The UK mathematics curriculum expects students at this level (above average 12 year olds) to *begin* giving mathematical justification for their generalisations. An objective of the curriculum then is to develop their ability to use mathematical language effectively in presenting a convincing reasoned argument. As currently specified, it is only the more able 14 to 16 year old who are taught to "extend their mathematical reasoning into understanding and using more rigorous argument, leading to notions of proof" (DFE 1995 p 20). It may be that experiences with a DGE such as *Cabri*, and tackling suitable tasks, will help to allow this objective to be realised.

The example provided in this paper shows some aspects of how it is interaction with more knowledgeable others that ensures that at least some of the explanation available with the DGE can become accessible to the student learners of mathematics. Hence,

while *Cabri* provides certain elements of the mathematical language necessary for the articulation of relevant mathematical ideas, significant aspects must be provided by the teacher. This paper has attempted to document at least some of these aspects.

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http://www.soton.ac.uk/~gary/crime.html

CHANGE IN MATHEMATICS EDUCATION: RETHINKING SYSTEMIC PRACTICE

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Interest in the use of systemic thinking and practice to effect educational change has coincided with efforts to reform mathematics education consistent with new visions of what it means to do and know mathematics. The current application of systemic theory as an approach to change has raised questions about its effectiveness. This paper reports on findings of a teacher development project and the issues raised regarding teacher's understanding of new practices. In light of these findings, other key ideas of systems thinking and practice emerge as being more relevant and deserving of consideration.

Since the mid-eighties, there has been much interest in applying systemic theory to educational change (Vinovskis, 1996). Some of the key systems ideas that have been used include *coherence, alignment, and holism.* These ideas have been interpreted and operationalized, for example, as curriculum-driven reform based on frameworks such as the NCTM *Standards* (NCTM, 1989) and as coordinated state policies that provide a coherent restructuring of such elements as teacher development, assessment, and even governance (O'Day and Smith, 1993). For the most part the systemic reforms have been structural in nature. However, there is growing doubt in systemic theory as a useful tool to foster wide-spread change (Scheurich and Fuller, 1995). There is increasing evidence that teaching practice remains untouched by the current efforts (Grant, Peterson, and Shojgreen-Downer, 1996). By inference, since practice and outcomes are so closely related, this also suggests that student achievement remains unchanged.

Systemic has had many interpretations (Holzman, 1993). It has been used to refer to the whole educational system from top to bottom and to the whole system across local schools. Such variations in definitions of "whole" are reminiscent of confusions children have in understanding the nature of a "whole" and its fractional parts, where a "whole" can be anything according to how one specifically defines it. Nevertheless, systems thinking does offer some key concepts that are highly relevant to effective change. These will be presented later in this paper. The discussion will begin with some of the results of a teacher development project. These results form the context for discussion of systemic. It is from these data that critical issues in educational change emerged and from which the need to rethink systemic emanates.

The SYSTEM'S Project

As part of a federally-funded three year project (*SYSTEM'S, Say Yes to Students and Teachers in Multilingual Multicultural Mathematics and Science)* to enhance teachers' instruction of mathematics with Latino students in the United States, approximately forty teachers from

about twelve schools participated in a set of three specially designed courses. The staff development is described elsewhere (Khisty and Adams, 1996) and only the relevant aspects are presented here to provide some context to the discussion. The courses were repeated and staggered so that no more than twenty teachers at a time were in the project with some overlap among groups of participants. The "courses" were designed to be like laboratories that emphasized collaboration, investigation, instructional problem-solving, and dialogue. The project was based on the assumption that by the nature of the target issue (i.e., improvement of mathematics achievement among Latino students), teachers needed opportunities to thoroughly integrate knowledge bases from three areas: innovations in pedagogy reflecting constructivist and sociocultural perspectives, mathematics education, and bilingual/ESL education. The implied objective was to enhance teachers' ability to handle high complexity and to translate this complexity into concrete practices. Also, the teachers were extensively engaged in experiences that stressed meaning-making.

Each year as part of the project's external evaluation, participants were interviewed regarding, among other things, key issues revolving around their understandings of pedagogical and content concepts, their sense of their own change, and their own assessments of their instruction. The teachers also were asked to engage in informal dialogue journal writing as part of each course. These interviews and journal writings form part of the data for this discussion. The data were analyzed yearly for patterns that emerged among the teachers' thinking and understandings.

In addition, the teachers were informally and intermittently interviewed about reform activities, if any, at their school. Occasional visits were made to schools to observe these activities and the project staff were sometimes invited to participate in workshops that were part of these same reform efforts. These observations also form the basis for identifying issues.

Issues of Change

Three issues related to educational change have been selected for discussion. These emerged from the teachers' interviews and writings and from the observations of school reform efforts. The first issue has to do with teachers' lack of deeper understandings of key issues, concepts, and skills found in their own repertoire on teaching and learning. This lack of substantive understanding did not seem to come from teachers' inability to understand but rather from their own previous learning experiences that did not emphasize and ensure meaning-making. As a matter of routine throughout the project and particularly in courses or any other discussions, if a teacher used a term related to teaching and learning such as "constructivism" or "using

students' prior knowledge", the teacher was asked in a positive and supportive manner to extend his/her thinking on the matter by explaining further what the term meant. Teachers were continuously asked to differentiate the term or idea from others (compare and contrast), to elaborate with concrete examples, and to explain what it had to do with students' learning. They also were repeatedly asked to identify and explain a mathematics concept that came up in teaching episodes. Unfortunately, they always responded with too short, incomplete. or rhetorical definitions such as "fractions", "adding fractions", "hands-on". The teachers acknowledged how difficult it was to respond with whole sentences that established relationships (for example, "The students were putting together quantities that were not whole but which would give answers that were greater than a whole...."): Consistently, among all the teachers and in every situation (group discussions, one-on-one talks, journal writings), extended and detailed elaboration was difficult to produce. One teacher seemed to capture the general situation by noting that "...no one ever taught me about math concepts....or asked me to explain what I mean much less...what I think...."

Superficial understanding did not seem to result only from never having to elaborate. It also seemed to stem from experiences where other's discussions were heavily rhetorical and where there was little opportunity for clarifying dialogue as what happened in many inservice workshops. The best and most frequent example of this concerns the idea of parent involvement. After a district-sponsored staff development workshop on parent involvement, the teachers who attended it were asked to discuss it with the rest of the project group. Consistently, the five "workshop" teachers talked about parent involvement as if they truly understood it and as if everyone shared the same conception. However, after some probing, it became apparent that their conception conflicted with that of other teachers. More importantly, their conception was not the same as what is frequently found in the literature. The teachers' assumed conception of parent involvement was "having parents spend time in classrooms". Their conception did not include encouraging parents to ask about the child's school work or to ensure the child has a place to study, nor as taking those steps necessary, such as writing notes telling about upcoming math units, to keep parents informed.

The second issue to emerge has to do with teachers' difficulty in making connections in general, and particularly between concepts and procedures, between mathematical concepts and other subject matter, and between instruction and assessment. In essence, the teachers were not in the habit of thinking holistically. The difficulty in making cognitive connections also seemed to get in the way of understanding how children could learn in a holistic manner and how

multiple learning objectives could be simultaneously developed. In one of the laboratory courses, the teachers were asked to extensively analyze and discuss a curriculum package that has had success with second language learners, *Finding Out/Descubrimiento, (FO/D)* (DeAvila, Duncan, and Navarrete, 1986). The purpose was for the teachers to have a concrete example of a carefully crafted model of integrated science, mathematics, and literacy which develops students' higher order thinking skills, basic skills, and biliteracy. In essence, *FO/D* is designed so that as students independently read and comprehend linguistically sensitive task cards, they are engaging in problem solving and literacy development; and as they carry out the science activity described on the card, they are using mathematical concepts and skills even though the mathematics is not obvious. While the teachers could review and discuss *FO/D* and even try it out in their classrooms, they found it difficult to replicate the integration of subjects and skills in materials or learning experiences they developed on their own.

Difficulty in making connections became most obvious as teachers discussed issues of assessment particularly via standardized tests. They appeared to find little connection between this "new teaching" and students' ability to do well on the district's yearly tests. The idea that students' rich and highly frequent experiences with mathematics in varied contexts could produce positive outcomes on standardized tests, strained their comprehension even though there was evidence to demonstrate it. It seemed to be very difficult for the teachers to relinquish their belief that when testing time drew near, it was time to drill on isolated skills. This is a critical issue since a cursory analysis of sample test items indicate that drilling on skills would not prepare students to deal with the complex thinking needed to solve the problems.

The third issue has to do with how the teachers are themselves taught. The informal observations of the reform-oriented inservice activities provided to the teachers reinforces other work (Lieberman, 1995) that suggests that staff development is too short-term, fragmented, and didactic to really move teachers in a different direction. These staff development activities and others that the author has experienced seem to perpetuate the issues noted above. Teachers were still passive learners even in situations where they were actively engaged in a demonstration. What was lacking were opportunities for the teachers to engage in purposeful dialogue and meaning-making with the workshop presenters. In addition, it seemed to be a common practice among the schools to carry on separate simultaneous workshops for various clusters of teachers (e.g., the mathematics teachers, the specialists, the bilingual education teachers, etc.). Consequently, teachers were cut off from engaging with colleagues in collaborative work that might have encouraged cognitive connections among content areas. The compartmentalization

of staff development activities was intended to bring teachers closer together; however, this one act had very different and severe consequences. It, in fact, fostered disconnected "camps": the mathematics camp, the reading camp, the bilingual camp, etc.

Systemic Reconsidered

The issues described above relate to assumptions about communication and what is meant and how things are connected and related to one another. While prevailing perceptions of systemic focus on structural wholes, systemic also refers to a way of thinking (Hutchins, 1996). Systemic thinking represents a major paradigm shift in how we view the world. It is a shift away from the traditional view of reductionism or thinking about isolated parts that fit a mechanistic model. The mathematics learning that we are attempting to reform is based on thinking about parts and not wholes or connections and relationships. Systemic, therefore, is a philosophy, a way of thinking, that once adopted permeates all thinking regardless of situations or context. As a way of thinking, systemic theory offers us a way of understanding and dealing with the issues noted in the earlier section.

Change requires the change agent to think systemically or holistically. However, since systems are naturally complex there must be *bounded rationality* which means narrowing the scope and setting priorities as to what to address first, but never forgetting that a change in one part affects all others. In other words, systemic thinking suggests thinking globally but acting locally. To change learning outcomes we must select to change teaching practice since there is a clear connection between how and what the teacher instructs and how well and what the student learns.

Consistent with this perspective, changing teaching practice requires teachers to think holistically with a keen awareness of relationships. This can not be accomplished through approaches that reflect thinking in parts as via isolated activities (Cohen and Barnes, 1993). Recent research in mathematics education has been concerned with what teachers believe about the teaching and learning of mathematics (e.g., Boufi, 1994; Becker and Pence, 1996), and it suggests that there is a strong relationship between teachers' beliefs about mathematics and their practice (Raymond, 1993). However, beliefs are not the same as understanding; nor is understanding the same as content knowledge. Understanding entails knowing the meaning and the nature of something, having a mastery of it and being able to discern it wherever it occurs. Moreover, it is possible to believe something and yet not fully understand it. Consequently, beliefs, content knowledge, and understandings must go hand-in-hand.

Therefore, changing teaching practice should include preparing teachers to think in terms

of conceptual meanings and relationships. Another key idea from systems is the notion of *communicative action* (Ellis, 1995) which is a process of talking through conflicting perceptions and agreeing upon a common vision. This process of negotiating and making-meaning, requires persons to be conscious of their own and other's perceptions and meanings and to not assume common understandings. It requires that the actors differentiate, analyze, provide examples, explain graphically and orally, make explicit relationships, and assess new relationships. Such a process can be used with teachers to develop shared meanings of pedagogical and mathematical concepts and to instill a mode of looking for relationships (e.g. how theory relates to practice or concepts to procedures). The process also instills habits of the mind that carry over into everything from how to instruct students to how to collaborate with colleagues.

Concluding Thoughts

The experiences from a multi-year project working with teachers have raised some critical questions about the nature of effecting educational change and questions regarding the appropriateness of prevailing emphases and activities. Evidence from the teachers suggest issues of thinking at a deeper level, thinking holistically, and of the role of communication to foster such thinking, The type of mathematics teaching and learning that we have set forth is complex (e.g., non-routine problems, integrated curriculum, multiple strategies, and varying classroom organizations), and consequently, requires both students and teachers to think with greater complexity. All of this is no small matter. I have suggested that other aspects of systems theory can be used to guide our thinking about how to change teachers holistically which could affect change in the whole system.

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AREA INTEGRATION RULES FOR GRADES 4, 6 AND 8 STUDENTS

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This paper investigates grades 4, 6 and 8 students' use of area integration rules by administering area judgement tasks, using rectangles of varying areas and perimeters. Information Integration Theory and functional measurement procedures revealed that students' responses were determined by both additive (perimeter) and multiplicative rules. It was found that judgement rules change intra-individually. There does not appear to be a relation between judgement rule and Grade level.

Area is one of the essential concepts of mathematics instruction because it is the most commonly used domain of measurement and the basis for many models used by teachers and textbook authors to explain computational strategies (Hirstein, Lamb & Osborn, 1978; Woodward & Byrd, 1983; and Baturo & Nason, 1996). However, it is a concept that textbooks make little attempt to define, and many (for example, Blane & Booth, 1989) discuss with the apparent assumption that students already understand it. In particular, textbooks aimed at the Year 8 level of schooling in Queensland appear to limit their approach to the concept of area to two ways: a combination of basic pre-formula exercises, statements of formulae and exercises using the formulae (e.g., Blane & Booth, 1989; Duffy & Murty, 1988); and directly into statements of formulae and exercises using the formulae (e.g., Clark, Clark, Burza & Conway, 1988; Priddle & Davies, 1989).

This apparent lack of definition and emphasis on formulae seems to be attributing, in part, to well documented misconceptions of the concepts of area in both primary and secondary school aged students (Kidman & Cooper, 1996a; Outhred & Mitchelmore, 1996; Clements & Ellerton, 1995; Bell, Costello & Kuchemann, 1983; Bell, Hughes, & Rogers, 1975). As the research of Hirstein (1981) and Hirstein, Lamb and Osborne (1978), for example, has shown, one of the major misconceptions is confusion between area and perimeter. In particular, as Kidman and Cooper (1996b) found, students have difficulty with the process of obtaining a shapes' measurements, determining which dimensions to consider and how to integrate these dimensions when calculating either area or perimeter. Attributing to this misconception is the possible inadequate knowledge of area by teachers. Research on student teachers (Baturo & Nason, 1996; Simon & Blume, 1994) has revealed similar inadequacies. The student teachers show very little conceptual understanding of the

relationship between area and side length. They are able to apply procedural formulae, but they confuse area and perimeter and use linear rather than square units.

The Information Integration Theory (IIT) method and functional measurement technique of Anderson and Cuneo (1978) has been widely used to identify the rules applied by students to integrate dimension information. As argued by Kidman & Cooper (1996a) and Wolf (1995), IIT and cognitive algebra offer an excellent opportunity to explain the process of area concept development in children. IIT places importance on problems of stimulus integration and multiple causation. According to IIT, "all behaviours reflect a blend of stimuli, and a response is the consolidated resultant of multiple causal forces" (Kidman and Cooper, 1996b, p. 340). The methodological counterpart of IIT, called functional measurement, allows diagnosis in simple algebraic terms, "... of the rules which govern integration of information about perceived stimuli." (Wolf, 1995, p. 49-50).

Recent studies have employed IIT to investigate students' perceptual judgement of area (Wolf, 1995; Schlottman & Anderson, 1994; Lautrey, Mullet & Paques, 1989; and Silverman & Paskewitz, 1988). In these studies, students have been provided with different rectangular shapes and asked to place their area on a linear scale. The general consensus of these studies is that students' judgements of area obeyed two-dimensional rules. At some stage between the age of 5 and 12, a child is expected to make the transition from an additive integration rule to the normative multiplicative integration rule. It appears from these studies that this stage is about 8 years old. At this age, the students were in a transitional stage between the additive and multiplicative rules.

This paper describes an investigation to determine the judgement rules used by students in Grades 4, 6 and 8 in a private college in Queensland and reports on student responses to experiments to explore how the students integrated length and width dimensions to judge area of rectangular or near rectangular shapes. The purposes of the investigation were to:

(1) identify the way in which students integrate stimuli to determine area; and

(2) determine if integration rules change intra-individually.

The experiments was based on the body of literature and the functional measurement methodology stemming from the work of Anderson and Cuneo (1978).

The study

The study used a multi-method design where the quantitative methodology of functional measurement was combined with the qualitative methodology of semi-structured clinical interview.

Participants. The sample consisted of 36 children, 12 students from each of the three grade levels, with equal numbers of boys and girls, and a range of mathematical abilities, one third each of below average, average, and above average, from each grade.

Instruments. The instruments used were three experiments and an interview. The first experiment contained 16 rectangular wooden pieces painted to represent chocolate and with dimensions corresponding to the factorial combinations of 3, 6, 9, and 12 cm. The pieces were to be presented to students who would be asked to judge the area of the rectangular pieces in relation to two end anchors and the previous pieces they had judged. To obtain measures of the students' area judgements, the students were provided with a 19 point scale with two end points. Two special pieces of dimensions 2.7 x 2.7 cm, and 15.8 x 15.8 cm were used as end anchors.

The second experiment used 16 rectangular pieces identical in dimensions to the first experiment, but with a rectangular corner 'bitten' off producing a figure of equal perimeter, but less area. The dimensions of the 'bitten' off corner were all one third of the width and one third of the height of the rectangular stimulus. The third experiment again used 16 rectangular pieces identical to the first experiment, but this time they had a semi-circular 'bite' out of one side producing a figure with less area but greater perimeter. The 'bite' was centred along one dimension with the radius of the 'bite' one third of the length of the shortest dimension.

The interview was short and semi-structured and asked each student to describe the method they used to rate each piece. They were quized as to whether they were aware of any changes they had made to their method over the course of the three interviews. Diagrams of identified methods were sought, from the children. At the conclusion of the interview, the students understanding of both area and perimeter was discussed, and the student was then asked to identify if he/she had employed either or both of these concepts to rate the chocolate pieces.

Procedure. Each experiment was completed with each student. The students were withdrawn from their class and the three experiments and the interview were administered in a separate room. The experiments and the interview were videotaped. The interview followed the third experiment. It took no longer than 30 minutes.

For each experiment, the students were first familiarised with the end anchors which were presented as corresponding to the end points of the scale. The scale had a smiling face at one end and a frowning face at the other. The small end anchor was presented as a piece of chocolate that people would be unhappy to receive while the large end anchor was presented as a piece people would be happy to receive. The students were then asked to judge how happy someone would be to receive each of the 16 pieces if they were chocolate to eat. The pieces, each of equal thickness, were presented individually, and judgement was expressed on a 19-point response scale (see Anderson & Cuneo, 1978, for more details). The presentation of the chocolates

was randomised, and a practice phase preceded the test phase. The students judged three replications of the chocolate stimuli in each experiment.

Analysis. The experiments were analysed using Anderson and Cuneo's (1978) functional measurement methodology. The idea behind functional measurement is to use algebraic rules as the base and frame for psychological scaling. These rules provide the breakdown of the observed response into its functional components, as represented by the scale values and weights of the various pieces of information (Anderson & Cuneo, 1978). This is used to identify the kind of rules underlying the judgements students make when provided with different rectangular shapes and asked to judge their areas on a rating scale. The scale positions for the rectangles (which are specific combinations of height and width) are represented graphically and then subjected to an analysis of variance. Conclusions regarding the kind of rule underlying the judgements are determined from the shape of the graphical plot, and the significance or nonsignificance of the main and interaction effects (Anderson, 1981).

The graphical plot of the responses is against the length of one of the dimensions of the rectangles. Thus, if the plot is an arrangement of parallel lines or parallel curves, the students' judgements are considered to be additively based, that is, they are tending to perceive area of a rectangle in terms of the *sum* of its dimensions. If the plot is fan shaped (expanding lines or curves), the students' judgements are considered to be multiplicative, that is, the students are tending to see area of a rectangle in terms of the *product* of its dimensions. Figure 1 presents hypothetical curves for these rules. If the plot lines or curves intersect, then an inference with regard to additivity or multiplicativity may not be possible.

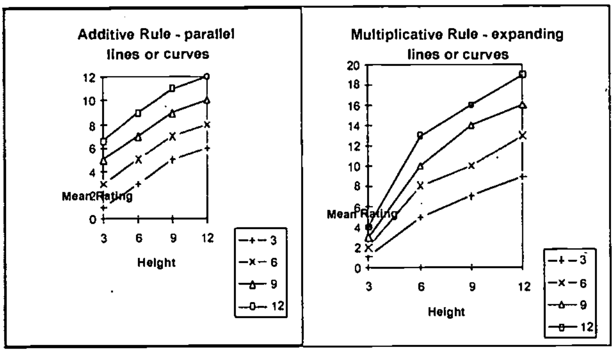

Figure 1

Hypothetical plots for additive and multiplicative based judgements

Factorial plots were drawn for each student for each experiment, as well as a group plot for each of the three grade levels. The plots were then compared to the hypothetical plots shown in Figure 1. On the basis of this comparison, additive or multiplicative integration rules were assigned to that student for that experiment.

The interviews were transcribed into protocols and the students' statements compared with their experiment results in an endeavour to provide a second option for explaining students' responses. The results of this part of the study are not provided in this paper.

Student responses to the experiments

Understanding the instrument. All students appeared to understand the judgement they were required to make in terms of being happy or unhappy with pieces of chocolate in relation to the end anchors and previous pieces of chocolate. They appeared able to express their judgement unambiguously using the 19 point scale. The understanding of the response scale was checked by having the student point to specific sections of the scale (for example, a section depicting a little bit of sadness), as well as making a verbal statement about the section being pointed to (for example, "I would be a little bit sad").

The most important procedural detail concerned the establishment of the frame of reference, "The rating of any one stimulus is always relative to what other stimuli are being rated" (Anderson, 1980, p. 9). The students appeared to understand the end anchors in terms of their being a standard device for setting up the frame of reference. For the students, they were noticeably more extreme, higher or lower, than the rectangles used in the experiments. The students were able to see the end anchors as tieing down the end responses. All the students' responses to the chocolate stimuli came from the interior of the scale. There were no end effects.

Scale positions. Each child's scale positions were analysed with the functional measurement methodology. Table 1 shows the integration rules used by the children, as well as the grouped rules for the three grade levels (Gr4, Gr6 and Gr8) and the three experiments (E1, E2 and E3). The symbol **?** is used to denote an intersection of the plot curves or lines, **X** the multiplicative rule, and **+** the additive rule. **TOT** gives the total number of multiplicative and additive students in each grade.

In the majority of cases the resulting plots were obviously additive with clear plots of parallel lines or curves, or multiplicative with clear plots of expanding curves or lines. In cases where the curves intersected (for example, the curve for a width of 6 cm crossed the curve for a width of 9 cm), the general shape of the plot was recorded, but a '?' was also recorded indicating a 'questionable' rule usage. It was not possible to determine a judgement rule for Ben, a Grade 4 student, doing Experiment 1. This

particular plot had four intersecting locations and no obvious parallel curves or diverging lines.

Table 1

Grade 4, 6 and 8 students' integration rules for area

Gr 4	E1	E2	E3	Gr 6	E1	E2	E3	Gr 8	E1	E2	E3
Jem	+	x	x	Di	+	+	x	Kev	x	x	x
John	x	+	+	Mark	x	x	x	Carol	+	+	+
Cass	+	+	+	Rhea	+	+	+	Brook	+	+	+
Alice	x	x	x	Jenny	x	x	+	Elle	+	+	+
Cathy	x	x	x	Peter	x	x	+?	Ellen	x	x?	+
Kim	x	x	x?	Wade	x	x	x	Kelly	+?	x	+
Jack	+?	+	x	Jodi	+?	x	x	Jay	x	x	+?
Ben	?	+?	+?	Sam	+?	x	+	Matt	x	x	+
Stacy	+	+	+?	Angie	+?	x	x	Mick	x	x	x
Tony	+	+	+	Sean	+	+	+	Sue	+	+	x
May	+	x	x?	Anne	x	+?	+	Phil	x	x	+
Tom	+	x	x	Jose	x	x	+	Marg	+	+	x
Ave	x	+	x	Ave	x	x	x	Ave	+	x	x
TOT	x=4 +=7	x=6 +=6	x=7 +=5	TOT	x=6 +=6	x=8 +=4	x=5 +=7	TOT	x=6 +=6	x=7 +=5	x=5 +=7

The differences between the grades was not as obvious as could be expected. The perception of area of rectangle being related to the sum of the rectangles' dimensions is fairly constant across the grades. The group of Grade 8 students tested do not seem to have progressed much beyond the Grade 4 or Grade 6 level. However, there were two interesting small changes. The first was the increase in multiplicativity in the Grade 4 results from experiment 1 to experiment 3. The second was the increase in multiplicativity from experiment 1 to experiment 2 across all Grades; and, except for the Grade 4 students, the decrease in multiplicativity from experiments 1 and 2 to experiment 3.

It is evident that judgement rules do change intra-individually. Ten students used an additive rule initially in experiment 1, but had altered this to a multiplicative rule by the conclusion of experiment 3. Surprisingly, 9 students did the reverse. They started using a multiplicative rule but changed to an additive rule in either the second or third experiment. Similar to the distribution of additivity and multiplicativity across the Grades, the changes in integration rule intra-individually (within students) was also fairly constant across the grades with the number of students constant in their rule in each Grade remaining between 4 and 6 across the three Grades.

Discussion and conclusions

If the plot of the students' scale positions approximates parallel lines or curves, this reflects a perception of area where doubling the lengths of the sides of the rectangle is seen as doubling the area. This in turn reflects a perception that the relation between area of a rectangle and the dimensions of the rectangle is additive. Thus, this perception can be considered as a confusion between area and perimeter for rectangles. It has been denoted as the Area = Height + Width integration rule. In contrast, a plot which approximates a fan shape reflects a perception that doubling the sides more than doubles the area of the rectangle. This is seen as a correct perception and denoted as the Area = Height X Width integration rule. Plots with lines crossing indicate a very poor conception of area as this means that the student has judged a rectangle with smaller dimensions as having a larger area than a rectangle with larger dimensions.

This study has, therefore, supported the findings of Hirstein (1981) and Hirstein, Lamb and Osborne (1978) that there is confusion between area and perimeter. Around 50% of students from all Grades and in all experiments exhibited judgements that showed they were using the perimeter rule to determine area.

Experiments 2 and 3 were performed to see if modifications to the rectangle would effect integration rules used by students. The removal of a rectangular corner was found to reduce students use of the perimeter rule while the removal of a semicircular piece from an edge did not.

The question is why? It could be argued that a rectangular piece out of a corner of a rectangle gives the effect of adding two more sides and thus the student tends to look at the amount of surface rather than the length and width. It could also be argued that the removal of the semicircular piece from a side has a lesser effect on how the rectangle is perceived than the removal of a corner, and that the addition of a semicircle to the factors that have to be taken into account in making area judgements adds weight to an additive focus on length. However, the reasons for students use of perimeter in the three experiments will have to wait until the interviews are analysed in relation to these experiments.

Over 50% of the students (22 out of 36) changed their integration rules across experiments. Once again the question is why? There appears to be no pattern in the changes: from additive to multiplicative, from multiplicative to additive, and sometimes in both directions. There appears to be no relation to Grade level. Hence, once again, the best hope for reasons is the relation between the experiment responses and the interview statements.

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TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE OF MULTIPLICATION AND DIVISION OF RATIONAL NUMBERS[1]

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ABSTRACT: *The main aims of this paper were to evaluate prospective and inservice teachers' knowledge of common difficulties that children experience with multiplication and division word problems involving rational numbers, and their possible sources. Most prospective teachers exhibited dull knowledge of these two aspects of pedagogical content knowledge. Most inservice teachers were aware of students' common incorrect responses, but not of their possible sources. We suggest that direct instruction related to students' common ways of thinking could enhance both prospective and inservice teachers' pedagogical content knowledge.*

Pedagogical Content Knowledge (PCK) is widely recognized as one of the most significant aspects of teachers' professional knowledge. A major component of PCK is "the understanding of how particular topics, principles, strategies and the like in specific subject areas are comprehended or typically misconstrued" (Shulman, 1986). This component of PCK depends on research on conceptions and misconceptions of students in specific domains.

In mathematics education, one area that has received attention is that of students' conceptual development of multiplication and division. Studies have consistently shown that students have difficulties in selecting the operations needed to solve multiplication and division word problems involving rational numbers (for an extensive review see Greer, 1992). Many children, adolescents, and even adults make systematic mistakes such as changing the role of the divisor and the dividend when solving a division word problem in which the correct solution should have had a divisor greater than the dividend. Researchers have theorized about the sources of these difficulties (e.g., Fischbein, Deri, Nello, & Marino, 1985).

A related question that comes to mind is the extent to which prospective and inservice teachers are aware of students' most common incorrect responses. This issue is of great theoretical and practical importance as teachers' knowledge of students' conceptions and misconceptions can seriously influence the nature of their instruction (Fenemma, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). This paper describes a part of a project aimed at evaluating prospective and inservice teachers' PCK of rational numbers. Here we shall relate only to one aspect of this knowledge, namely prospective and in-service teachers' knowledge of children's

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difficulties with multiplication and division word problems involving rational numbers and their possible sources. We are interested in two main issues:

(1) Are prospective and inservice teachers aware of common difficulties that children experience with multiplication and division word problems involving rational numbers? (knowing that)

(2) To what do they attribute them? (knowing why)

Our evaluation is based on the extensive body of knowledge on children's understanding of multiplication and division word problems involving rational numbers.

Methodology

Sample. Sixty-seven prospective teachers and 46 inservice teachers participated in this study. Thirty-seven of the prospective teachers were in their first year in a four-year teacher education program at an Israeli State Teachers' College, and 30 were in their third year in the same program. The inservice teachers participated in a special two-year program: "Expert Teachers Program" (ETP) aimed at creating a community of leading elementary school mathematics teachers in Israel. Thirty of these inservice teachers were in their first year and 16 were in their second year of this program.

In Israel, the topic of rational numbers is mainly taught in grades 5 and 6. Most of the inservice teachers were practicing teachers in these grade levels (20 out of the 30 teachers in the first of the program, and 11 out of the 16 in the second year). The rest were teachers who taught in other grades, mostly in grades 1,2, and 3.

Instruments. Two types of research instruments were used: A Diagnostic Questionnaire (DQ) and semi-structured interviews. In this paper we shall report only on the following item from the DQ:

For each of the following word problems: (a) write an expression that will solve the problem (do not compute the expression); (b) write common, incorrect responses, and (c) describe possible sources of these incorrect responses

1. One kilogram of tomatoes costs $3\frac{1}{2}$ shekels. What is the cost of $\frac{3}{4}$ kilogram of tomatoes?
2. A car uses $\frac{3}{10}$ liters of gasoline per each kilometer. How much gasoline is needed for 9 kilometers?
3. There are 320 calories in one kilogram of cucumbers. How many calories are there in $\frac{1}{3}$ kilogram?
4. Four friends bought altogether $\frac{1}{4}$ kilogram of chocolate and shared it equally. How much chocolate did each person get?
5. A five meter long stick was divided into 15 equal sticks. What is the length of each stick?

6. Four kilograms of cheese were packed in packages of $\frac{1}{4}$ kilogram each. How many packages were filled with this amount of cheese?
7. A bottle can hold $\frac{3}{5}$ liters of water. $\frac{1}{4}$ liter of water was poured into this bottle. What part of the bottle is filled with water?
8. A group of girl-scouts walked 15 kilometers in 5 hours. How many kilometers, on average, they passed in an hour?
9. $\frac{4}{5}$ kilograms of meat costs 30 shekels. What is the cost of one kilogram of meat

Although our main concern was to explore prospective and in-service teachers' PCK of rational numbers, the participant teachers' Subject Matter Knowledge (SMK) of this topic was assessed as well, as teachers' PCK should be examined in light of their personal mathematical knowledge of the task under discussion.

Procedure. The DQ was administered to the subjects in two sessions of 90 minutes each, during mathematics method courses. As mentioned previously, all inservice teachers who participated in this study were enrolled in the ETP. It is worth noting that this program included the specific course: "Students' Conceptions of Rational Numbers". This course deals with students' ways of thinking about rational numbers. The course was given in the second semester of the first year and thus the inservice teachers enrolled in the second year of the program had taken this course before answering the DQ while those in the first year of this program had not.

Results

Prospective and Inservice Teachers' SMK. Table 1 shows that most inservice teachers provided correct expressions for the multiplication and division word problems (93% of correct responses, on average). The percentages of correct responses given by the prospective teachers were lower (69% on average). Differences between prospective and inservice teachers were especially observed on three word problems. The most difficult one for both populations was problem 7 (77% and 19% correct responses among inservice and prospective teachers, on average, respectively). Two problems, 3 and 9, were relatively easy for inservice teachers but difficult for prospective teachers.

Much like in other studies that examined prospective and inservice teachers' incorrect responses to multiplication and division word problems, our data reveal that prospective and inservice teachers' incorrect responses were similar to those mistakes reported in the literature as made by students (e.g., Ball, 1990; Tirosh, Graber, & Glover, 1989).

Prospective and Inservice Teachers' PCK. As shown in Table 1, some prospective and inservice teachers incorrectly respond to several word problems. Obviously, the analysis of teachers' PCK should take account of teachers' own solutions to each word problem. Since most prospective and inservice teachers' correctly answered

most word problems, our analysis here will mainly relate, for each problem, only to those who provided correct answers to this problem.

Table 1 - Distribution (in %) of correct teachers' responses to word problems (by class)

Prob. No.	inservice teachers: 2nd year N=16	inservice teachers: 1st year N=30	prospect teachers: 3rd year N=30	prospect teachers: 1st year N=37
1	93	94	80	84
2	93	97	83	86
3	93	97	50	70
4	93	97	60	81
5	93	97	97	94
6	86	97	77	81
7	81	73	3	35
8	93	94	77	89
9	93	94	30	57
Average	94	93	62	76
General Average	93		69	

For most word problems, each teacher came up with mostly one (or no) typical incorrect student responses. An exception was word problem 4. Eighteen participants listed two different, common incorrect students' responses.

Table 2 describes prospective and inservice teachers' responses to the request to write common incorrect responses to each word problem. The second column of this table describes students' common incorrect responses to seven out of the nine word problems, as reported in the mathematics education literature on children's ways of thinking about multiplication and division word problems involving rational numbers. According to the literature, word problems 2 and 8 are usually solved correctly by students and adults.

Table 2 shows substantial differences between prospective teachers and inservice teachers in their knowledge of common incorrect responses to multiplication and division word problems (31% and 80% , on average, respectively). The most prominent differences were observed on word problem 7. Seventy-seven percent and 61% of the inservice teachers in the first and second year, respectively, listed common incorrect student responses to this word problem while only 15% of prospective teachers on the first year and no prospective teachers on the third year did so.

Table 2 - Distribution (in %) of teachers' knowledge of common students' incorrect responses (by class)

Prob. No.	mistakes mentioned in literature	inservice teachers: 2nd year N=16	inservice teachers: 1st year N=30	prospect teachers: 3rd year N=30	prospect teachers: 1st year N=37
1	subtraction, DIM*	53	53	37	42
3	DIM	93	93	33	11
4	subtraction, MID** DID***, 1	93	89	67	57
5	subtraction, DID impossible	94	89	21	54
6	subtraction, MID, DID	92	79	56	40
7	subtraction, MID	61	77	--	15
9	MID	67	90	--	5
	Average	79	81	31	32
	General Average	80		31	

* DIM = division instead of multiplication

** MID = multiplication instead of division

*** DID = changed the roles of the dividend and the divisor

As stated previously, the two classes of inservice teachers included teachers who taught in grades 5 and 6 (the grades in which the topic of rational numbers is taught in Israel) and teachers who taught in other classes. We assumed that the actual teaching of this topic provides teachers with many opportunities to observe students' common difficulties, making such teachers more knowledgeable about such mistakes. Table 3 shows that, indeed, 87% of the inservice teachers who taught in grades 5 and 6, on average, suggested students' common mistakes, while 70%, on average, of those who did not teach these grades, mentioned such mistakes. No substantial differences in awareness of common mistakes were observed between first and second year inservice teachers who taught in grades 5 and 6. Yet, the differences between first and second year inservice teachers who did not teach in these grades were substantial.

A closer scrutiny of Table 3, which relates separately to first and second year inservice teachers, reveals that among the first year teachers, the differences in awareness of common students' mistakes between those who teach in grades 5 and 6 and those who do not, are substantial. However, no differences in awareness of common mistakes were observed between second year inservice teachers who teach grades 5 and 6 and those who do not.

Table 3 - Distribution (in %) of teachers' knowledge of common students' incorrect responses (by experience)

Prob. No.	mistakes mentioned in research	Inservice teachers teaching grades 5,6		Inservice teachers not teaching grades 5,6	
		1st year N=20	2nd year N=11	1st year N=10	2nd year N=5
1	subtraction, DIM	73	50	11	60
3	DIM	100	100	60	100
4	MID, DID, 1	80	100	68	100
5	subtraction, impossible, DID	100	100	67	80
6	MID, DID	89	90	56	100
7	subtraction, MID	75	70	84	70
9	MID	93	75	60	67
	Average	89	84	58	82
	General Average	87		70	

Teachers' Knowledge of Possible Sources of Students' Incorrect Responses. In part c of this item, the participants were asked to describe possible sources of each of the mistakes they listed in response to part b of this same item (see Table 4). Four types of possible sources of common students' incorrect responses were mentioned: Intuitively-based sources (e.g., "Children believe that division makes smaller. In the word problem that deals with buying tomatoes (Problem 1), they know that three quarters of a kilogram costs less than one kilogram, and therefore they incorrectly divide"); algorithmically-based mistakes (e.g., "It is easier to multiply by a fraction than to divide by it"); The nature of fractions ("Children have difficulties with fractions. They know how to cope with integers but not with fractions"); and general reasons for lack of success in solving word problems (e.g., "Problems in reading comprehension").

Table 4 shows that only few prospective teachers listed possible sources of students' common incorrect responses to the various word problems. Among the inservice teachers, more sources of students' common incorrect responses were mentioned by the teachers enrolled in the second year of the ETP program than by those enrolled in the first year. Most of the prospective and inservice teachers who mentioned possible sources of students' incorrect responses, related to intuitively-based sources or to the specific nature of fractions. It is noteworthy that all the possible sources mentioned by the teachers indeed affect students' mathematical performance in solving multiplication and division word problems. Yet no teacher directly related to the possible effects of the primitive models of these operations on students' ways of thinking.

Table 4 - Distribution (in numbers) of teachers' responses to sources for students' common incorrect responses (by class)

correct response	incorrect response	sources	inservice teachers: 2nd year N=16	inservice teachers: 1st year N=30	prospect teachers: 3rd year N=30	prospect teachers: 1st year N=37
$\frac{3}{4}\cdot\frac{1}{2}$	DIM	intuitive	7	4	3	1
		fractions	1	5	–	4
		general	–	1	–	1
$\frac{1}{4}:4$	MID [DID]	intuitive	2 [10]	4 [8]	– [6]	2 [5]
		algorithmic	1	4 [1]	–	–
		fractions	–	1 [1]	–	–
		general	–	1 [1]	–	2 [1]
5:15	subtract.	intuitive	–	1	–	–
	impossible	intuitive	1	1	–	–
	DID	intuitive	11	13	–	4
		fractions	5	6	–	2
		general	–	1	–	–
$\frac{1}{4}:\frac{3}{5}$	MID	intuitive	–	–	–	–
		fractions	2	8	–	–
		general	–	1	–	–
$320\cdot\frac{1}{3}$	DIM	intuitive	6	6	–	–
		fractions	1	5	1	–
		general	–	3	–	1
$30:\frac{4}{5}$	MID	intuitive	1	1	–	–
		fractions	2	3	–	–
		general	3	4	–	–

Final Comments

In this paper, we describe some initial findings regarding prospective and inservice teachers' knowledge of children's ways of thinking about multiplication and division word problems involving rational numbers. Our data showed that the majority of prospective and inservice teachers provided correct expressions for the multiplication and division word problems. Some teachers who gave incorrect responses to the word problems also listed the correct expression as incorrect students responses. It is reasonable to assume that prospective and inservice teachers who incorrectly solved the word problems and offered correct responses as examples of incorrect students' reactions would consider such responses as incorrect in a classroom setting.

In respect to teachers' PCK, we assumed that actual teaching of multiplication and division word problems involving rational numbers strengthens teachers'

knowledge of students' common incorrect responses in this topic and their possible sources. Our data reveal that most experienced teachers were familar with students' incorrect responses to multiplication and division word problems involving rational numbers. Teachers who had no experience teaching in these grade levels but participated in the course: "Students' Conceptions of Rational Numbers", also exhibited profound knowledge of such students' ways of thinking, while those who did not participate in this course did not. Thus, it seems that knowledge about common ways of thinking among students could be acquired not only through teaching experience but also by participation in specific courses on that subject.

Our paper also discussed prospective and inservice teachers' understanding of the possible sources of specific students' reactions; i.e., knowing why (Even & Tirosh, 1995). The data indicate that this knowledge was insufficiently developed among both prospective and inservice teachers. Most subjects, in both populations, were unable to provide any sources for the common incorrect responses they themselves listed. By and large, this paper shows that prospective and inservice teachers do not by themselves develop a solid PCK of students' conceptions of multiplication and division involving rational numbers. In light of these findings it seems that teacher education should devote more efforts to develop ways of increasing elementary school prospective and inservice teachers' knowledge concerning possible sources of students' incorrect responses to multiplication and division word problems and possible ways of taking them into account in instruction.

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GENDER DIFFERENCES IN ALGEBRAIC PROBLEM SOLVING: THE ROLE OF AFFECTIVE FACTORS

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There has been much discussion in the literature about the existence and nature of gender differences in mathematical performance. In this paper we describe a research study in which we measured seven affective variables simultaneously with students' algebraic problem solving ability. Evidence is presented to show that there is a gender based gradient effect in the way the affective factors change in early adolescence which corresponds to a change in the problem solving performance of boys and girls. The effect seems to result in a lowering of mathematical outcomes for girls but an improvement for boys.

Introduction

The adolescent years constitute a period of great change for secondary school students. A number of studies have attempted to understand gender differences for the learning of, and achievement in, mathematics during this period. The importance of gaining this knowledge has been described by Tartre and Fennema (1995, pp. 199-200):

> Gender differences related to mathematics pose complex but urgent questions. Important among them are - what factors are related to mathematics achievement for boys and girls and do these relationships change during the critical period of adolescence?

Numerous studies, covering many countries, cited by Brandon *et al.* (1987) and lately by Sayers (1994) have noted gender differences favouring either boys or girls. Research during the last decade revealed that by the end of high school boys outperform girls on mathematics achievement as noted by Skaalvik and Rankin (1994) who hypothesised that either these differences were occurring in late adolescence or were diminishing.

Affective factors

A number of studies have examined the reasons for these differences. Possible explanations provided include biologic, sociocultural and sex stereotyping (Brandon et al, 1987; Fennema *et al.*, 1985; Meece *et al.*, 1982 and Leder, 1992). A growing body of literature is establishing the importance of affective variables in students' learning and achievement in mathematics. For example, Reyes (1984) has shown self-concept, anxiety and perceived usefulness as some of the affective factors that influence learning of mathematics. Self-concept, which changes with age and gender (see Marsh, 1989; Meece *et al.*, 1982; Hattie, 1992), is a reflection of how one sees oneself, and findings consistently show that mathematical self-concept is related to mathematics performance (Marsh *et al.*, 1991). Self-perception, the belief system that includes judgements about one's ability has also been shown to affect

motivation and anxiety (Skaalvik and Rankin, 1995, p. 165), based on the dynamic equilibrium model of Marsh, they predicted:

> ...student domain specific self-perceptions (math self-concept, self- perceived aptitude and self-perceived ability to learn), which are assumed to be based on external and internal comparisons affect intrinsic motivation, effort and anxiety and are affected by academic achievement.

Affective factors and problem solving

Mathematical problem solving can be seen as a result of the interaction of several closely related, independent categories of factors, such as knowledge acquisition, utilisation, beliefs, affects and socio-cultural contexts (Lester, 1994; Boekaerts *et al.*, 1995). Some researchers have hypothesised that attitudes towards mathematics contribute to gender differences in mathematics problem solving (Fennema and Sherman, 1976; Brush, 1985). If such differences between genders are primarily environmentally induced, then it is important to investigate the environmental influences on the intellect and their operational procedure. Since the affective domain is the likely interface for the environment and intellect of the human brain it becomes necessary to monitor changes in these factors and their influence on learning. The aim of this present study was to consider changes in affective factors concurrently with an evaluation of student problem solving ability in algebra, analysing any gender differences and the possible reasons for them.

Method

Subjects

The data for the present study was collected from 345 form 3 and 4 students of ages about between 13 - 15 years, from 8 secondary schools in the Auckland region of New Zealand. Both single-sex and co-educational schools were represented in the study. Each school reported that they were following the 1992 New Zealand curriculum published by the Ministry of Education. One class of average ability students from each school was randomly selected to form the subject group. The students were taught by specialised mathematics teachers.

Instruments

Each individual student in the study was given two questionnaires and these were given during regular mathematics classes in the first half of the academic year 1996, before the algebra syllabus was taught for the year. These questionnaires are currently being repeated at the end of the academic year. The first measured seven affective factors using self-descriptive questions in a 5-point Likert format. The second measured algebraic problem solving ability. The details of the tests used are:

Self-concept: The mathematical self-concept scale of 27 items developed by Gourgey (1982) was used to measure mathematical self-concept. The internal consistency reliability of the scale is 0.96 and Gourgey stated that the analysis of the scale provided support for its validity and reliability.

Interest: The scale developed by Mitchell (1993) and used in his subsequent research was used to measure the level of interest in mathematics. The internal consistency coefficient for the independent subscales used ranges from 0.77 to 0.93.

Anxiety: The anxiety component of the Skaalvik and Rankin (1995) scale was used to measure the students' mathematical anxiety. There are 8 items in this scale and the Cronbach alpha for this scale was reported as 0.90 for a similar subject group.

Self-perceptions: The self-perception constructs were measured by using Skaalvik and Rankin's (1995) three-item scale for each of self-perceived ability and self-perceived aptitude. The internal consistency reliability coefficients are 0.8 for 12/13 years and 0.83 for 15/16 years.

Usefulness of mathematics: The usefulness of mathematics scale of Fennema & Sherman (1976) served to measure the individual student attribute about the usefulness of mathematics.

Mathematics Intrinsic motivation: Of the two types of motivation, intrinsic and extrinsic, we considered that the intrinsic factor was likely to be more consistent and reliable since it originates from the subject. This factor was measured using the English translation of the instrument developed by Skaalvik and Rankin (1995).

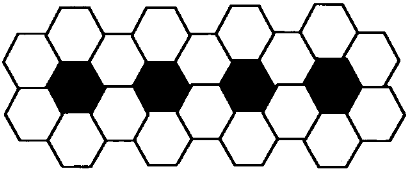
Enjoyment of Mathematics: Students enjoyment levels were measured using the instrument from Aiken's attitude scale as modified by Watson (1983). The internal consistency reliability coefficient was found by Aiken to be 0.95, using secondary school students.

Problem Solving ability: An algebra test consisting of five basic word problems was constructed from the problems used in the three algebra tests which we had piloted (Thomas and Kota, 1996). The problems were constructed based on the objectives for levels 2 to 5 of the New Zealand mathematics curriculum guidelines, published by the Ministry of Education. It was intended that factors like word order, situation, language and difficulty in carrying out numerical operations would not be obstacles in the process of solving the problems successfully. The type of problems used in the test were:

1. The number of girls in a school is 41 less than the number of boys. The total number of students in the school is 1539. How many girls are there in the school?
2. To hire the Pizza House for birthday party costs a basic rate of $70.00 plus $3.50 per person. If the total bill is $346.50, how many people attended the party?
3. Tickets for the school play cost $2 for children and $4 for adults. 500 tickets were sold for $1,640. How many children's tickets were sold and how many adult tickets were sold?

4. The Auckland city council wishes to create flower beds, surrounding them with hexagonal paving slabs according to the pattern shown below.

Complete the table below to find the number of paving slabs needed for 8 and 100 flower beds. Write an algebraic equation to find the number of paving slabs, N, needed to surround F flower beds.



Number of flower beds	1	2	3	4	8	100	F
Number of paving stones	6	10	14	18			

5. A wooden fence is made by placing 3 planks between 2 posts as shown in the figure. (A picture was given). Complete the table to find the number of planks needed for fences of 5 posts; 100 posts and S posts. Write an algebraic equation to find the number of planks L in a fence S posts long.

Number of posts	2	3	4	5	100	S
Number of planks	3	6	9			

Results

Table 1 shows the statistical analysis of the mean scores on each of the affective factor scales and the algebra test for all the form 3 and form 4 students.

Table 1: Means and standard deviations for all form 3 and form 4 students

	Form 3 (n=177)		Form 4 (n=168)		
	Mean	SD	Mean	SD	F
Self Concept	91.70	15.56	87.71	15.56	5.66
Interest	68.74	10.77	66.22	11.74	4.29
Anxiety	26.86	4.17	25.77	4.19	5.79
Enjoyment	37.47	8.55	35.34	7.81	5.78
Self Perception	21.13	3.56	20.85	4.41	0.40
Usefulness	45.96	8.22	47.11	7.83	1.63
Motivation	44.91	10.27	44.59	10.04	0.10
Algebra	12.04	5.47	16.65	5.48	57.37

Whilst these means are for different groups of students (we are currently following the students through to get longitudinal data) they are from the same population in each case, and so it is still of interest that the mean scores of all the affective factors decreased from form 3 to form 4, with the exception of the usefulness of

mathematics, and this decrease was significant (F>3.92, p<0.05) for self-concept, interest, anxiety and enjoyment. There was also a significant increase in the algebra problem solving abilities, as one would hope. However when the data were analysed taking gender into account, the results obtained were those shown in table 2. We see that in form 3 the mean scores of the girls were higher than the boys in almost all the affective factors, with the exception of mathematics self-perception and enjoyment. However, by form 4 all the mean scores of affective factors for the boys' are higher than those of the girls. These form 4 differences are significant (p<0.05) for anxiety, self-perception, and enjoyment and nearly so for interest, but none of the differences between girls and boys was significant in form 3. Looking at the algebra problem solving scores, the girls scored significantly higher than the boys in form 3 and although higher in form 4, the difference is no longer significant.

Table 2: Means and standard deviations of form 3 and form 4 boys' and girls' affective factors and algebra score

	Form 3					Form 4				
	Girls(n=101)		Boys (n=76)			Girls (115)		Boys (n=53)		
	mean	SD	mean	SD	F	Mean	SD	Mean	SD	F
Self Concept	92.57	14.51	90.36	16.83	0.88	87.03	14.83	87.51	20.91	0.03
Interest	68.89	10.57	68.26	11.27	0.14	64.55	13.70	68.60	10.02	3.72
Anxiety	27.04	4.33	26.53	4.00	0.65	25.04	4.81	26.86	3.95	5.82
Enjoyment	36.67	8.61	37.93	9.38	0.86	33.63	7.47	39.23	7.23	20.79
Self Perception	21.02	3.57	21.21	3.61	0.11	20.22	5.34	21.83	2.49	4.39
Usefulness	46.09	8.01	45.80	8.34	0.05	46.61	8.38	48.40	6.45	1.89
Motivation	45.04	10.25	44.78	10.34	0.04	44.23	10.47	45.70	9.02	0.76
Algebra	13.56	5.73	10.07	5.17	17.57	17.02	5.64	16.03	5.19	1.15

However it was when we analysed the changes in the affective factors and problem solving abilities from form 3 to 4 (see table 3) to obtain some insight into the nature of changes for each gender, both in magnitude and direction, that an interesting pattern emerged.

Table 3: A gender based comparison of change from form 3 to form 4

	Girls					Boys				
	Form 3 (n=101)		Form 4 (n=115)			Form 3 (n=76)		Form 4 (n=53)		
	Mean	SD	Mean	SD	F	Mean	SD	Mean	SD	F
Self Concept	92.57	14.51	87.04	14.83	7.65	90.36	16.83	87.51	20.91	0.73
Interest	68.89	10.57	64.55	13.70	6.66	6826	11.27	68.60	10.02	0.03
Anxiety	27.04	4.33	25.04	4.81	10.16	26.53	4.00	26.87	3.95	0.23
Enjoyment	36.67	8.61	33.63	7.47	7.75	37.93	9.38	39.23	7.23	0.71
Self Perception	21.02	3.57	20.22	5.34	1.63	21.21	3.61	21.83	2.49	1.22
Usefulness	46.09	8.01	46.61	8.38	0.21	45.80	8.34	48.40	6.45	3.61
Motivation	45.04	10.25	44.24	10.47	0.32	44.78	10.34	45.70	9.02	0.30
Algebra	13.56	5.73	17.02	5.64	19.86	10.07	5.17	16.04	5.19	41.61

The mean scores for the boys increased in almost all the affective factors, except mathematics self-concept, whereas the mean scores of the girls show a decline in all the affective factors. The increases for the boys were not significant, but the girls declined significantly ($p<0.05$) on four of the factors: self concept; interest; anxiety; and enjoyment. At first sight the significant ($p<0.05$) decrease in anxiety levels for girls might seem desirable, but we have previously found a positive correlation, for girls, (Thomas & Kota, 1996) between anxiety and problem solving performance. The effect of the changes in the factors is shown, we believe, by the way that although the mean problem solving score of the girls increased from form 3 to 4, it did so by significantly less than that of the boys.

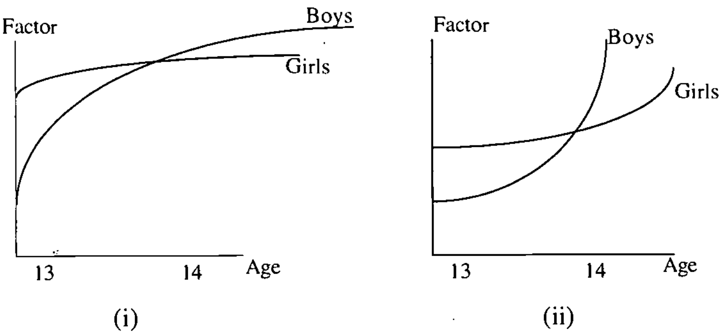


Figure 1: The gradient effect of differing rates of change of affective factors for boys and girls

It appears that the rate of increase in the girls' performance is decreasing, while that of the boys is increasing. One possible reason for this is the effect of the greater rate of negative changes in girls' attitudes to mathematics and their view of themselves relative to it, compared with the boys. This *gradient effect* in the affective factors may be seen in figure 1, where we have pictured what is apparently happening to the changing affective factors of girls and boys from about age 14 years to 15 years.

Where the factors are decreasing (part (i)) the boys tend to decrease at a slower rate, and where they are increasing (part (ii)) the boys do so at a faster rate. These greater rates of change are then exactly mirrored by algebraic problem solving performance and we postulate that there is a causative link between them. The result is that even though girls often seem to start ahead of boys in both the levels of their attitudes to themselves and mathematics, and indeed their actual performance level, they are eventually overtaken by the boys. In order to improve this situation it seems imperative to discover what is behind these changes in the way girls view both themselves and mathematics, and seek to put in place strategies

for preventing any relative decline. It will be interesting to see if the changes we have described here are also presented when we analyse the longitudinal data we have recently obtained from following the same students from age 14 to age 15 years.

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STUDENTS' REPRESENTATIONS OF FRACTIONS IN A REGULAR ELEMENTARY SCHOOL MATHEMATICS CLASSROOM

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The study reported in this paper investigates students' representations of fractions in a regular elementary school mathematics classroom where students' construction of mathematical knowledge is emphasized in the process of teaching and learning mathematics based on a constructive approach. This paper focuses on an analysis of students' representations of fractions as they work on the fraction comparison tasks and justify their solutions in a collective classroom activity. The importance of setting a problematic situation and encouraging students to make various representations for their meaningful learning mathematics is exemplified. Some implications for teacher's activity and school mathematics curriculum are also suggested.

Theoretical Background of the Study

The study reported in this paper makes a part of our research project on establishing a theory for planning and practicing mathematics class that enables students to actively construct mathematical knowledge. Nakahara (1993) has proposed a so-called "constructive approach" and established the lesson process model in the constructive approach that consists of such five steps of teaching and learning activities as being conscious, being operational, being mediate, being reflective, and making agreement. From a different perspective, Koyama (1996) has analyzed an elementary school mathematics class in Japan and showed that the process of teaching and learning mathematics in the classroom actually developed in the line with the horizontal axis, i.e. three learning stages of the intuitive, reflective, and analytical that are set up in the "two-axes process model" of understanding mathematics (Koyama, 1992).

Purpose and Method of the Study

As a result of the Rational Number Projects (Carpenter, Fennema, and Romberg, 1993), it is shown that representations, translations among them, and transformations within them play several important roles in mathematical learning and problem solving. Lesh, Post, and Behr (1987) notes that "the term representations here is interpreted in a naive and restricted sense as external (and therefore observable) embodiments of students' internal conceptualizations —although this external/internal dichotomy is artificial" (p. 33). Moreover, as Goldin and Passantino (1996) notes, students' external representations of mathematical ideas permit us to conjecture or infer their internal representations and conceptual understanding of the ideas concerned.

On the other hand, it has been difficult for students to understand fractions as mathematical ideas and construct meanings of fractions (cf. Lesh, Behr, and Post, 1987; Post, Cramer, Behr, Lesh, and Harel, 1993; Watanabe, Reynolds, and Lo, 1995; Goldin and Passantino, 1996). Post, Cramer, Behr, Lesh, and Harel (1993) especially criticizes the instructional emphasis on developing procedural skill for fraction and the divorce of operations from their meanings, and suggests us as follows: "Fraction order and equivalence ideas are fundamentally important concepts. They form the framework for understanding fractions and decimals as quantities that can be operated on in meaningful ways" (p. 340).

We have the accumulated important information on students' representations and conceptions of fractions by means of performance tests, task-based interviews, or a combination of them (cf. Carpenter, Fennema, and Romberg, 1993; Goldin and Passantino, 1996). We, however, do not have enough information on students' representations of fractions that they make and use to understand fractions and construct meanings of fractions in a regular school mathematics classroom. Therefore, the study reported in this paper focuses on an analysis of students' representations of fractions as they work on the fraction comparison tasks and justify their solutions in a collective classroom activity (cf. McClain and Cobb, 1996).

The sample episode discussed and data of students' representations analyzed in this paper are taken from a fifth-grade classroom in which the teacher, Mr. Miyamoto, has participated as a collaborating member of our research project on the constructive approach. The study reported in this paper is not such an experiment study that enables us to make valid generalizations for neither a wider population nor students in other countries, but should be regarded as one of our investigative case studies in Japan. It, however, may contribute to gain more information on students' representations of fractions in a collective classroom activity, and exemplify the importance of setting a problematic situation and encouraging students to make various representations for their meaningful learning and construction of mathematics concerned.

A Regular Elementary School Mathematics Classroom

The classroom focused on in this paper is a fifth-grade (11 years old) classroom at the national elementary school attached to Hiroshima University in Hiroshima City, Japan. The 37 students (19 boys and 18 girls) in the classroom are heterogeneous in the same way as a typical classroom organization in Japanese elementary schools, but their average mathematical ability is higher than that of other students in the local and public schools. The teacher in the classroom, Mr. Miyamoto, has participated as a collaborating member of our research project on the constructive approach. He is an experienced and highly motivated teacher, and has a relatively deep understanding of both elementary school mathematics and his students.

In Japan the Course of Study as a national curriculum identifies the objectives and typical sequence of topics in elementary school mathematics, and teachers teach their students mathematics usually with a series of mathematics textbooks approved by the Ministry of Education as suitable textbooks. Therefore we should see the outline of the typical sequence of topics related to fraction. According to the current Course of Study (Ministry of Education, 1989), the typical sequence of topics related to fraction begins with an introduction of fractions as quantities, a basic relationship between fraction and decimal (1/10=0.1), and addition and subtraction with two simple fractions with a common denominator at the third grade, and then moves on as follows: fraction equivalence (e.g. 1/2=2/4), fraction order (e.g. 1/5<3/5, 5/7>2/7) with a number line, and addition and subtraction with two fractions with a common denominator at the fourth grade; more general fraction equivalence (e.g. 2/3=4/6, 12/16=3/4), the meaning and procedure of both reduction of a fraction to the lowest terms and reduction of fractions to a common denominator, fraction order (e.g. 2/3>5/9, 4/9<5/6), addition and subtraction with two fractions with different denominators, fractions as operations involving two quantities (e.g. 2÷3=2/3), relationships between fraction and decimal (0.1=1/10, 0.01=1/100), and fractions as ratios at the fifth grade; multiplication and division with fractions at the sixth grade (the last grade in elementary school).

Setting a Problematic Situation

Against this curricular background, I and the teacher, Mr. Miyamoto, elaborated the lesson plan for his students in a fifth-grade classroom. The intention of the plan was to modify the sequence of topics related to fraction, before introducing formal procedures of reduction, by carefully setting a problematic situation in which students might be conscious of and actively work on the fraction comparison tasks. In order to see students' ideas and internal representations, we also decided to ask students justify their solutions by encouraging them to make and use various (external) representations of fractions and any mathematical knowledge that they had constructed.

The classroom episode described in this paper is taken from the first two lessons of successive five lessons on fractions in the Mr. Miyamoto's fifth-grade classroom in November, 1996. We decided to use three different fractions in written mathematical symbols, 4/5, 3/5, and 3/4 for setting a problematic situation at the beginning of the first lesson. These fractions were carefully chosen and might be presented to students not at the same time but one by one in the above order, with due consideration of the followings. The students had learned the simple fraction equivalence and order such as 1/2=2/4, 1/5<3/5, and 5/7>2/7 at the fourth grade. We expected that students could easily compare two fractions with a common denominator or numerator, 4/5 vs. 3/5 and 3/5 vs. 3/4, and that they might be challenged to compare two fractions 4/5 vs. 3/4. In fact, according to the scheme of difficulty levels (Lesh, Behr, and Post, 1987, pp. 50-51), the comparison of fractions 4/5 vs. 3/4 belongs to the most difficult level 3B, while

the both comparisons of fractions such as 4/5 vs. 3/5 and 3/5 vs. 3/4 belong to the easiest level 1. Moreover, we chose the fraction pair as 4/5 and 3/4, because in the pair numerator and denominator are both one unit away from one, and because these fractions are easily transformed to decimals. This choice of fractions, we expected, might allow students to compare and represent fractions in various and different ways.

The process of teaching and learning in the classroom actually developed as follows. In the following protocol of the lesson, sign *Tn* and sign *Sn* mean a *n*th teacher's utterance and a *n*th student's utterance respectively.

At first, the teacher wrote the symbol 4/5 down on a blackboard and asked "What studies can you do?". A student answered "It (4/5) means four out of five candies". Then, the teacher wrote another symbol 3/5 next to 4/5 and asked the same question again. At that time, many students wanted to do computation with these fractions. When the teacher wrote the third fraction symbol 3/4 next to 3/5 on a blackboard, some students shifted their attention to comparing those fractions as follows.

T3: Now, we have three fractions. What studies can you do?

S6: Order those fractions according to size!

S7: We want to compare fractions by changing denominator and/or numerator, for example, of four-fifths.

As expected, students answered such fraction comparison questions as 4/5 vs. 3/5 and 3/5 vs. 3/4 with, for example, the following relevant justifications.

S8: Four-fifths is larger than three-fifths. Because three-fifths means three pieces if you divide one into five pieces, while four-fifths means four pieces if you divide one into five pieces.

S9: Three-fourths is larger than three-fifths. Because three-fourths means three pieces if you divide one into four pieces, three-fifths means three pieces if you divide one into five pieces, and the size of one piece if you divide one into four pieces is larger.

T10: Now, please anyone say the learning task for this lesson.

S11: Let's investigate which is larger, four-fifths (4/5) or three-fourths (3/4)!

S12: Let's investigate which is larger when the difference between denominator and numerator is one!

S13: I want to make a supplement to S11. Let's compare fractions with different denominators!

S14: We need to investigate how much larger as well as which is larger.

T11: I want to ask you justify your solutions of this learning task in more than three different ways.

Through the above extracted discussion, students and the teacher in this classroom posed the main learning task: investigating which fraction 4/5 or 3/4 is larger and how much larger, and justifying own solutions in more than three different ways. This mutually agreed task shows us that our choice of three different fractions 4/5, 3/5, and

3/4 and presentation of these fractions to students not at the same time but one by one effectively functioned for setting the problematic situation where students could be conscious of the learning task to be challenged.

Students' Representations of Fractions

The students individually had worked on the task for about 15 minutes. During students' individual work, the teacher had circulated among students, helping some students with their works and noting down some students' typical and different ways of justification. Finally he asked each of five students present one of their ways of justification on a large white paper and put it on a blackboard. In this section, we will focus on students' representations of fractions that students made, used, and wrote or drew on their work-sheets as they worked on the fraction comparison task and justified their solutions. The term "representations" here is interpreted as the external and belongs to the five distinct types of representation systems (Lesh, Post, and Behr, 1987, p. 34). We will also focus on students' explanations and discussions among students.

Decimal Type: This representation type is characterized as transforming fractions to decimals (Figure 1). 12 out of 37 students made this type of representation.

Transform 4/5 to a decimal, $4 \div 5 = 0.8$
Transform 3/4 to a decimal, $3 \div 4 = 0.75$
Because 0.8 is larger than 0.75, 4/5 is larger.

Figure 1. Decimal Type

S15: I transform these two fractions to decimals. Four-fifths is 0.8 and three-fourths is 0.75. When I compare these two decimals, 0.8 is larger than 0.75 by 0.05. So, four-fifths is larger than three-fourths by one-twentieth.
T13: Do you agree with S15?
S16: I do not understand why 4/5 is transformed to $4 \div 5$.
S17: Because four-fifths means four pieces if you divide one into five pieces and $5 \div 4$ was larger than one, I think, $4 \div 5$ is right.
T14: S17 made an additional explanation to S15.
S18: If we take 1/2 as an example, we can transform it to a decimal by $1 \div 2 \times 1$, that is 1 ÷denominator ×numerator.
T15: Do you agree with S18? Any comment?
S19: Because 4/5 means four pieces if you divide one into five pieces, $1 \div 5$ is 0.2, and four pieces of 0.2, 0.2×4, is 0.8. So, I think 0.8 is right.

This type is possible because that the students in this classroom had already learned decimals and that both fractions 4/5 and 3/4 are relatively easy for students to transform to decimals. But, as S16 posed a question, the reason of why 4/5 can be transformed to $4 \div 5$ had not yet learned formally in this classroom. Nevertheless, S17, S18, and S19 tried eagerly to explain in their own ways by using the constructed knowledge.
Remainder Type: This representation type is characterized as noticing that smaller remainder means the subtracted is larger (Figure 2). 3 out of 37 students made this type.

$1-4/5=1/5$ $1-3/4=1/4$

The smaller remainder means the fact that the subtracted is closer to 1. When I compare 1/5 with 1/4, 1/5 is smaller. So, 4/5 is closer to 1.

Figure 2. Remainder Type

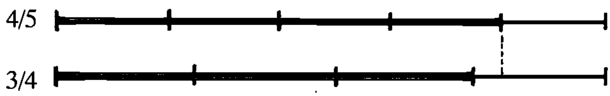
S20: I find out remainders, 1 −4/5 is 1/5 and 1 −3/4 is 1/4. Because the smaller remainder means the fact that the subtracted is closer to one, and 1/5 is smaller than 1/4, so, 4/5 is larger than 3/4.

S21: I want to make an additional explanation. Because in case of four-fifths we divide one as a whole into five pieces, here I think, one as a whole means five-fifths.

S23: I have a comment on the explanation of S20 about remainder. I consider it in the case of "which is closer to ten, eight or seven?". Because 10 −8=2, 10 −7=3, and that eight is closer to ten in which the remainder is smaller, the idea of S20 is right.

When this type of representation was explained, many students admired it as a fine one. Although the possibility of using this idea depends on fractions to be compared and this type of representation is not enough to know how much larger, we might say that three students who made this type have a good number sense and relevant meanings of fractions as a result of their learning experiences.

Line-Segment Picture Type: This representation type is characterized as drawing a line-segment picture (Figure 3). 19 out of 37 students made this type of representation.



The least common multiple of 5 and 4 is 20.

$20\div5=4, 4\times4=16.$ $20\div4=5, 3\times5=15.$ $16/20-15/20=1/20.$

Figure 3. Line-Segment Picture Type

S24: I draw this picture. The dotted line in the picture shows that four-fifths is larger than three-fourths and the difference between them. I use the least common multiple of five and four, that is twenty, to change denominators to the common. Because four-fifths is equal to sixteen-twentieths and three-fourths is equal to fifteen-twentieths, the difference is one-twentieth.

In this type, there was a variety of students' representation. For example, some students represented these two fractions by the line-segment picture of 10 units or 20 units length with or without written language explanations. It, however, should be noted that all students who made a line-segment picture drew two equal length line-segments to represent the whole. Moreover, some students began noticing the similarity and difference between the decimal type and the line-segment type as follows.

S27: I think that the ideas of S15 (Figure 1) and S24 (Figure 3) are similar except for the difference between decimal and fraction.

S28: The unit in case of S15 is 0.1, and the unit in case of S24 is 1/20.

Thin-Rectangle Picture Type: This representation type is characterized as drawing a thin-rectangle picture (Figure 4). 28 out of 37 students made this type of representation.

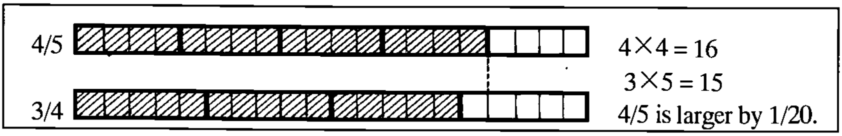


Figure 4. Thin-Rectangle Picture Type

S29: I draw this picture. Because the least common multiple of two denominators five and four is twenty, I divide sections into more small sections. In case of four-fifths, I divide each section into four small sections, and I get 4 ×4, that is 16, small sections. In case of three-fourths, I divide each section into five small sections, and I get 3 ×5, that is 15, small sections. Because 16 −15=1, it means that four-fifths is larger than three-fourths by one-twentieth.

There was also a variety of students' representation in this type and we can see same examples and point out same things as the mentioned above in case of the line-segment picture type except for difference between thin-rectangles and line-segments. When S29 was explained, S30 pointed out similarity and difference between the line-segment picture type and the thin-rectangle picture type as follows.

S30: This idea of S29 (Figure 4) and that of S24 (Figure 3) are the same. We note the difference between them only in their pictures. One is line segments and another is thin rectangles.

Equivalent Fraction Type: This representation type is characterized as making equivalent fractions (Figure 5). 31 out of 37 students made this type of representation.

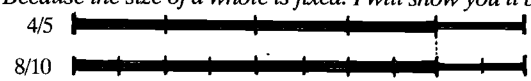
4/5 = 8/10 = 12/15 = 16/20.   3/4 = 6/8 = 9/12 = 12/16 = 15/20.
16/20 − 15/20 = 1/20

Figure 5. Equivalent Fraction Type

S31: I find out fractions that are equal to each of four-fifths and three-fourths like this (Figure 5). The difference is one-twentieth because that four-fifths is equal to sixteen-twentieths and three-fourths is equal to fifteen-twentieths.

S32: I do not understand. Why do you multiply same number, for example two, to both denominator and numerator?

S33: Because the size of a whole is fixed. I will show you it by this picture.



This type was most popular in this classroom and often used with the line-segment picture or thin-rectangle picture type. The reason of the fact is that the students had learned simple fraction equivalence at the fourth grade and the least common multiple of two natural numbers before this lesson at the fifth grade, and that it is included in the learning task in this lesson to know how much larger. Therefore, if the teacher had not asked students justify their solutions in more than three different ways, students' representations might have converged at this type and their active discussion nor meaningful learning might have not occurred.

Conclusion

The study reported in this paper exemplifies the importance of setting a problematic situation in which students are able to be conscious of their own tasks and encouraging students to make various representations for their meaningful learning mathematics. Especially in case of learning fractions at the fifth grade, the choice of different fractions (4/5, 3/5, 3/4) and the presentation of these fractions one by one effectively functioned for setting such a situation. The teacher's activity of encouraging and allowing his students to make, explain, discuss their various representations (Decimal Type, Line-Segment Picture Type, Thin-Rectangle Type, Equivalent Fraction Type, etc.) played an important role for their meaningful learning of fractions. This study also suggests, at least for school mathematics curriculum in Japan, the possibility of changing the sequence of topics related to fraction that is identified in the curriculum by carefully setting a problematic situation in which students might be conscious of and actively work on the fraction comparison tasks, before introducing formal procedures of reduction of fraction(s).

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Number Instantations as Mediators in Solving Word Problems

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Abstract

In this paper, we present a model for solving word problems procedurally. The approach is based on reification theory and is an extension of a model researched in the past for constructing algebraic expressions from verbal expressions. This approach was case studied among average eighth graders at the beginning of the school year. Cognitive processes underlying these translations were explored. It was observed that multiplicative algebraic expressions appear to undergo reification, while the reification of additive algebraic expressions is delayed. When using this model, some common translation errors are avoided.

Many researchers (Chaiklin 1989; Clement, Lochhead & Monk, 1981; Lochhead & Mestre, 1988; Reed et al, 1994; Sutherland & Rojano, 1993) have studied the difficulties encountered when translating from the written language to the language of mathematics. Lochhead and Mestre (1988) and MacGregor and Stacey (1993), among others, studied the reversal error which occurs when two magnitudes are compared. Cortes (1995), documented errors of identifying and writing the relationships between the magnitudes in the word problem. MacGregor and Stacey (1996) found that, in simple word problems, students experienced little difficulty in the actual understanding of the relationships involved; their difficulty was in knowing how to use algebraic notation to express these relationships and integrate them into an equation. Some researchers (e.g. Rojano & Sutherland, 1993; Kutscher, 1996) have used intermediate numerical expressions to generate algebraic ones. These translation methods were used for creating algebraic expressions* (e.g. $10+4\cdot x$) and functional relationships (e.g. $x+5=y$). However, these methods were not applied in the context of translating word problems into equations. The purpose of this study was to apply these ideas in the context of translating word problems into equations. Cognitive processes underlying these translations were investigated within the framework of reification theory.

The theoretical framework

Our method is based on theories which suggest that, initially, most mathematical concepts are grasped as computational processes (Gray&Tall, 1994; Sfard, 1991; Sfard & Linchevski, 1994). These operational perspectives gradually develop into structural conceptions. For example, 5:6 is first grasped as a 'doing process', even when it is written as $\frac{5}{6}$, and then 'reified', or perceived structurally, also as a number. Similarly, expressions using variables, undergo this process: '$x-2$' may be first perceived operationally as the subtraction of 2 from x, while later on a structural perspective emerges. By then, this expression might be also seen as a mathematical object which may serve as a factor in a product (e.g. $5\cdot(x-2)$), as a function, unknown number and the like (Sfard & Linchevski, 1994). The students should eventually acquire a sense of

*When we write 'algebraic expression' in this paper, we refer to expressions like $2x+4$ as opposed to $2x+4=3$, which we refer to as an algebraic equation

duality of these expressions, namely, that these symbols should sometimes be understood operationally, other times structurally, depending on the context.

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Backgronnd: A table-filling method based on reification theory proved to be helpful in translating verbal expressions into algebraic expressions, especially for the average students (Kutscher, 1996). In the present study the above mentioned method for generating algebraic expressions is extended and developed into a process of translating word problems into algebraic equations, through a mediation process of number instantations.

Preliminary considerations: Table I is an example of how the solution of a word problem could be approached in our operational modeling method.

Table I

Danny bought 3 basketballs and 12 T-shirts for his basketball team and paid a total of $243. A ball costs $6 more than a T-shirt. What is the price of a ball and of a T-shirt?

A: Price of a ball	B: Price of a shirt	C: Cost of 3 balls	D: Cost of 12 T-shirts	E: Total paid	F:Is it $243?
10	10-6 ^{4}	3•10 ^{30}	12•(10-6) ^{48}	3•10+12•(10-6) ^{78}	no
14	14-6 ^{8}	3•14 ^{42}	12•(14-6) ^{96}	3•14 +12•(14-6) ^{138}	no
.	.	.	.	.	.
.	.	.	.	.	.
x	x-6	3•x	12•(x-6)	3•x+12•(x-6)	

The equation: 3•x+12•(x-6) =243

Table II

A: Price of a ball	B: Price of a shirt	C: Cost of 3 balls	D: Cost of 12 T-shirts	E: Total paid	F: Is it $243?
10	10-6=4	3•10=30	12•4=48	30+48	no
14.	14-6=8	3•14=42	12•8=96	42+96	no
.	.	.	.	.	.
.	.	.	.	.	.
x	x-6	3•x	12•(x-6)	3•x+12•(x-6)	

When we initially wrote the guidelines for this operational model, we had expected the students to generalize vertically (from top to bottom) in each column of the table. We did not want to write the intermediate results (e.g. 10-6=4) but only the numerical strings (10-6), in order to expose the algebraic structure of the expressions which would lead to generalization (x-6). On the other hand the intermediate calculations were needed in order to check whether the specific number instantation led to the correct solution of the problem. It was decided that these intermediate results would be written above the numerical string, ready for use in calculations in the appropriate stages (Table I, columns D& E). The idea was that the students would pose a number for one of the magnitudes, thereafter filling in the rest of the row. They would get a feel for the relationships between the magnitudes of the problem, would understand the procedure for getting from one magnitude to another, while grounded in numbers. They would learn to check their proposed number instantations against the constraints of the problem (Table I, columns E&F). This was expected to make the problem more meaningful for them. After a few trials of number instantations, the student would then generalize the arithmetical expressions to algebraic expressions. The equation which evolved would be anticipatory, where the student would anticipate that the total paid, represented by 3•x+12•(x-6), would be equal to $243. Thus, initially the format for solving a word problem looked like that displayed in Table I. After piloting this model,

we found that writing complex "procedural" numerical strings such as 3·10+12•(10-6) did not always promote generalization; on the contrary, it often distracted the students' attention from the generalization. Apparently there were just too many details. On the other hand the students, after evaluating the numerical strings, wrote them down spontaneously as numerical "equations" (10-6=4, Table II, column B). They much preferred using the number equivalent, rather than the numerical string, in later stages of constructing the expressions towards generalization (columns D&E). Most importantly, the use of "results" (4) rather than the expression (10-6), did not hinder the students from generalizing correctly to the desired algebraic expressions which would culminate in the required equation. When the students generalized these more complex expressions, they were not solely generalizing vertically, but using a spontaneous combination of both vertical and "horizontal" generalization. For example, in Table II column E, they generalized the additive structure vertically, but the actual terms seem to have been generalized from the earlier columns (columns C&D). Consequently, we decided that the "simpler" type of table (as in Table II) would be presented for the students' problem solving.

Design of the study

Five students, assessed as average mathematics students by their mathematics teacher, were chosen to participate in this study. At the time, the students were at the beginning of the eighth grade. These students all learned in a public school, whose student population reflected the composition of the middle-class district wherein the school was located. These students had completed one semester of basic algebra and were thus acquainted with solution of first-degree linear equations, collecting like terms and the like. They had had no experience in translating any verbal expressions into any algebraic expression. Each student was tutored individually. The first session was devoted to a brief review of basic algebraic terms. The rest of the sessions dealt with problem solving. The learning sessions were audio-taped and transcribed, to allow a closer observation and analysis of the student's reasoning.

Results

Results - Review session:

During the brief review of basic algebraic terms, and the meanings of concepts such as sum, difference, variables and the laws governing the operations involved (commutative, associative etc.) there were indications that the students' thinking processes were still oscillating between the operational and the structural. For example, Inbal was presented with the following table:

		Sum	Difference	Product	Quotient
worked example	3 _ 2	3+2	3-2	3•2	3:2
Inbal's solution	8 _ 4	8+4	8-4	8•4	8:4
worked example	b _ 3	b+3	b-3	b•3	b:3

(R represents the researcher)

R: What is b?

I: Any number?

R: Is also b+3 any number?

I: Yes.

R: Is b+3 the sum of b and 3?

I: No.

R: Why?

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I: Because you don't know what it is.

The teacher now points to the numerical expressions which had been written, correctly, by Inbal herself. The researcher continued:

R: Is 8+4 the sum of 8 and 4?

I: No.

R: So what is the sum?

I: 12

R: And what is 8+4?

I: This is the exercise to get to the sum.

Thus on the one hand she sees b+3 as a number in itself, a structural perception; on the other hand b+3 cannot be the sum - a sum (product, quotient, etc.) has to be the "answer" of an exercise. Similarly, Inbal did not agree that 8:4 was the quotient of 8 and 4. The researcher continued:

R: Are you familiar with the fraction line as an alternative way of writing a quotient?

I: Yes.

R: And if we were to write 5:6 with a fraction line?

I: (She writes) $\frac{5}{6}$

R: Is $\frac{5}{6}$ *the quotient of 5 and 6?*

I: (Hesitantly) *Depends how you look at it, as an exercise or a result.*

According to above mentioned theories (Gray & Tall, 1994; Sfard, 1991; Sfard & Linchevski, 1994), numbers are first seen as processes and then reified to objects. Thus $\frac{5}{6}$ is first the process of division and then the quotient itself. Seemingly the physical, external appearance, is also a factor that assists in the reification of the number. As mentioned above, in similar contexts, "8+4" may not be seen as a number, whereas $\frac{5}{6}$ would be. Our observation is that most students are able to perceive numerical multiplicative expressions structurally, without losing the operational aspect of this expression. In the case of an additive numerical expression, there seems to be a major cognitive obstacle for the student in perceiving it structurally. This phenomenon was initially observed in the review session and appeared again very distinctly during the problem solving sessions. An expression of the type "6x" seemed to be more easily conceptualized as a number than an expression such as "x+7". It seems that multiplicative expressions undergo reification, whereas the reification of additive expressions is delayed. A possible explanation in the case of multiplicative expressions is that 'manipulation' of x·6 or 6·x to 6x assists in the evolving of the structural aspect of the expression. More evidence of this phenomenon will be brought later on.

Results - Problem solving sessions:

In the problem solving sessions the students learned through the guidance of the teacher to create the table themselves. They very quickly gave up looking for the solution of the word problem through improving their choice of number instantations. Their choice of number instantations became almost arbitrary, since they spontaneously understood that the function of the numbers was to expose the algebraic structure of the expressions to be generalized. At the same time the students felt the

need to "execute" all number strings (Table III a&b, columns B&C), even though they knew that the aim was the procedure, not the result. The evaluation of the numerical strings did have merit, however, since the students got accustomed to checking their results against the constraints of the problem.

Very soon after being introduced to problem solving via table-filling, the students felt the cumbersomeness of the repetetiveness of calculations, and wanted to relinquish number instantations. They tried to skip number instantations entirely, to directly express the magnitudes algebraically and to integrate these expressions into an equation. This algebraic modeling of the equation was many times at the expense of writing a correct equation. They were then guided back to the procedural table-filling model which usually led to the correct equation and solution.

The students tried to cut corners in other ways which often resulted in erroneous equation formulation. Table III illustrates Shabi's solution to the following word problem:

Table III

A student bought arithmetic and Hebrew notebooks. The number of arithmetic notebooks he bought was 8 less than the Hebrew notebooks. How many notebooks of each type did he buy if we know that the total number he bought was 42?

A: Number of arithmetic notebooks	B: Number of Hebrew notebooks	C: Total number of notebooks
2	10	12
x-8	x	x-8=42

Shabi first substituted 10 in column B, he mentally subtracted 8 and wrote 2 in column A, and then wrote 12, the total number of books, in column C. He correctly generalized the first two algebraic expressions, writing x, x-8 (Table III, columns A&B). He initially thought that writing the numerical expressions for arriving at each number was redundant. Shabi did not write the procedure, only the "result", for the total number of notebooks (Table III, column C). There was no obvious pattern for him to base the vertical and horizontal generalization and thus he did not generalize correctly. As a result he wrote the incorrect equation, x-8=42 in column C. The latter error, writing an additive algebraic expression as the left-hand-side of an equation when the sum of more terms is called for, is suggestive of known errors which occur in similar two-stage additive arithmetic word problems (Abedel Haleq, 1986). Although this error did not occur at the stage of number instantations, there appeared to be an initial regression at the stage of formulation of algebraic equations. This error occurred at least once for every student when they did not apply the table-filling method fully. Another place for cutting corners was the number of instantations necessary before representing the unknown numbers with variables. Through trial-and-error they found that at least two number instantations were necessary for them to be aware of the patterns in the numerical strings. They learned that the procedures for arriving at the numbers, rather than the numbers themselves, were the key to successful generalization. An error which was avoided throughout the operational modeling problem solving sessions, was the reversal error, reconfirming results found in Kutscher (1996).

In sum, through the procedural table-filling model, the students learned that number instantations indeed led to successful equation formulation. Expressing the relationships between the numerical magnitudes enabled them to generalize, vertically, to the algebraic expressions. These expressions then served as intermediate results to

be integrated, via vertical and horizontal generalization, into the required equation. Ingrained in our problem solving method was constant checking against the constraints of the problem involved. This assisted the students in detecting errors, and consequently enabled them to arrive at a correct and full solution of the word problem at hand. The students learned to enjoy problem solving since they were not frustrated by continual failures, but rather were generally adept and successful in their problem solving endeavors.

Reification of multiplicative versus additive expressions - some examples:

The main purpose of this study was to investigate our procedural model in the context of word problems. However, already during the review session, the phenomenon of multiplicative versus additive expressions was observed, both in the context of numbers and in the context of literal symbols. This phenomenon continued to be observed during problem solving sessions. Our procedural table-filling method seemed to solve most of the difficulties encountered in the formulation of equations from word problems. However, the cognitive obstacle posed by the additive expression (as opposed to the multiplicative expression) emerged frequently, and appeared to demand considerable cognitive effort to overcome it. The following examples are illustrations of some of these problems. Table IV shows two examples worked by the students in the first stages of learning to solve word problems.

Table IVa

Example of reification of multiplicative term:
The sum of two numbers is 80, the second number is 5 times as large as the first number. Find the numbers.

A: Number I	**B**: Number II	**C**: Sum of two numbers
8	8•4=32	32+8=40 no
60	60•4=240	240+60=300 no
x	x•4=4x	4x+x=80

Table IVb

Example of non-reification of additive term
The sum of two numbers is 97. One number is 31 less than the other. Find the numbers.

A: Number I	**B**: Number II	**C**: Sum of numbers
6	6-31= -25	6+(-25)=(-19)
3	3-31= -28	3+(-28)=(-25)
x	x-31	

The first example (Table IVa) illustrates how Yaron extended the reification of multiplicative numerical strings (column B) to an algebraic expression. He expressed x•4 as 4x even though he had learned that 4x was no different from x•4; the change was but cosmetic. Perhaps this cosmetic difference enabled him to relate to the duality of the expression: 4x would represent the structural aspect, x•4 the procedural. The second example (Table IVb), illustrates the cognitive obstacle when an additive expression could not be so easily reified. Maytal, had no trouble filling the table. When she had written "x-31" (column B), she stopped in her tracks. Her difficulty is exposed in the following exchange:

R: Why are you hesitating?

M: A result is needed here (in the cell where "x-31" is written) *and then to add it.*

R: (Teacher points to Table IVb, column C) *Could you have, instead of the sum of 3 and -28* (column C), *written 3 plus 3 minus 31?*

M: Yes

R: Then do you think that it is possible, similarly, to write number II? Can you look at "x-31" as a number in itself, as an algebraic expression which is actually the second number?

M: No

R: If the first number were 61,....., could you write (in column C) *instead of "61+30", "61+(61-31)"?*

M: Yes

R: ....If x is a number, then can you look at x-3 as number? Difficult? Then it is impossible? And if we would put x-31 in brackets ?...(the teacher writes x+(x-31)) *What do you think?*
M: That it is impossible to solve it like this.
R: .... Now I have written an equation (the teacher writes x+(x-31)=97). *What does this tell you?*
M: That the first number and the second number come out to 97.
R: This, we can write?
M: Yes

At this stage, the sum "x+(x-31)" had no meaning for Maytal since it could not be "worked out". However, when this sum was anticipated to "come out to 97", the procedural aspect of the expression was exposed resolving Maytal's cognitive conflict. This resolution is clearly seen in the next, similarly structured, word problem which Maytal solved.

Table V

The sum of two numbers is 53. One number is greater by 7 than the other. Find the numbers.

A: Number I	B: Number II	C: Sum of two numbers
8	8+7=15	8+15=23
5	5+7=12	5+12=17
x	x+7	x+(x+7)

R: (Referring to the above table) *...And what does this actually say to you? What is this "x+(x+7)"?*
M: The sum of two numbers, x is the first number and x+7 is the second.
R: And what is the sum of these two numbers?
M: We shall soon see.

She proceeded to write down x+(x+7)=53 and solve the equation. This suggests that Maytal perceives each, separate, number structurally but the sum of the numbers she does not - in her eyes it is yet to be found out.

Initially, all the students faced the same dilemma - how to reify an additive expression to a number. One student resolved her problem by inserting brackets: she named "x+7" as (x+7). Apparently the brackets helped her see x+7 as a unit, a single number. Another student wrote in the appropriate cell (last row of column B) x+7=x+7, apparently helping himself understand the duality of the situation, that the outcome is the same as the procedure. Eventually all the students were able to use the additive expressions structurally as terms in other additive, or multiplicative algebraic expressions. Inbal had supposedly already understood, from previous examples, that an additive expression may be perceived as both the procedure and the result. Nevertheless, she appeared to regress (Table VI, column B), though she rallied faster than Maytal:

Table VI

A student bought arithmetic and Hebrew notebooks. The number of arithmetic notebooks he bought was 8 less than the Hebrew notebooks. How many notebooks of each type did he buy if we know that the total number he bought was 42?

A: Number of Hebrew notebooks	B: Number of arithmetic notebooks	C: Sum of two types
20	20-8=12	20+12=32
30	30-8=22	22+30=52
x	x-8=	

Inbal, working the solution out loud, says *"x-8, I cannot know"* (what it is equal to - Table VI, column B). As a result, she could not find the sum of the two unknown numbers (x+x-8). She was then shown that 20+12 (Table VI, column C) could be replaced with 20+20-8. She immediately generalized this expression and arrived at the

correct equation. Apparently, Inbal could see x-8 as a number, but the fact that it did not have a "result" initially stumped her.

Discussion and conclusion

This study presented a model for solving problems mediated by number instantations. The operational approach allowed the students to correctly identify the relationships between the magnitudes while grounded in numbers and to generalize, thereafter, to the appropriate algebraic expressions. Correct integration of the obtained algebraic expressions into equations proved to be generally successful. Thus, it seems that this method has solved those difficulties noted by MacGregor and Stacey (1996) and Cortes (1995); the students were both able to recognize and write correctly the relationships between the magnitudes, and to formulate the relevant equation. Our model builds on the students' ability of understanding and expressing of the relationships between magnitudes while grounded in numbers. The numerical pattern which emerges leads the students to the stage of generalization, resulting in a meaningful algebraic equation.

An interesting phenomenon was observed in translating additive, as opposed to multiplicative, expressions. This was found in all contexts of the tutoring sessions: in translating to numerical strings, to algebraic expressions and to algebraic equations. While the multiplicative expressions was perceived both operationally and structurally by the students, the acquisition of the structural perspective of the additive expressions was much more difficult.

Further study is necessary to examine whether the students will eventually be able to translate correctly without the mediation of this method.

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THE MATHEMATICAL KNOWLEDGE AND SKILLS OF CYPRIOT PUPILS ENTERING PRIMARY SCHOOL:
Implications for the development of policy on baseline assessment

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ABSTRACT

This paper presents findings of research investigating the skills and knowledge in Mathematics of Cypriot pupils entering primary school. A performance test, which was designed in order to assess skills and knowledge in Mathematics identified in the Curriculum of Pre-Primary Education of Cyprus, was administered to a representative sample of pupils of Year 1 (n=835). Teachers were asked to complete a report for each pupil. Moreover, questionnaires were administered to teachers to identify their perceptions about baseline assessment. The most important findings were the following. First, the skills included in the curriculum were differentiated into those, which more than 75% of pupils entering primary schools had achieved and those, which more than 30% of pupils entering primary school had not achieved. Second, a correlation was identified between findings gathered from the performance test and from teachers' assessment of pupils' skills in Mathematics. Third, teachers considered baseline assessment as an essential part of teaching but they had not attempted to assess their pupils when they entered the primary school. Fifth, they also considered the performance test as a useful tool for baseline assessment. Implications for the development of a national policy on baseline assessment are drawn.

I) Introduction

The last decade has witnessed a growing recognition of the need for significant changes in educational assessment practices. An important factor contributing to the need for assessment reform involves the relationship between teaching and assessment (Shepard 1989). The assessment process is nowadays seen as an integral part of the educational process. Broadfoot (1986) argues that the curriculum policies of most European countries have promoted a move from summative to formative assessment. Desforges (1989) suggests that formative assessment produces information about what children know and what they do not know in order to help teachers decide how to identify and meet children's learning needs and how to use their teaching time and their resources. An important implication of the identification of learning needs is that decisions about the next learning steps follow from it. A teaching plan, which is organised in such a way, might help teachers to plan class and individual programmes of work according to the different performance level of the pupils.

In Cyprus in 1994, a reform programme was introduced in primary schools which was concerned with content, pedagogy and assessment. Until recently assessment was a neglected issue. This reform can be seen as the first systematic attempt of the Ministry

of Education to establish the base upon which assessment policy in Cyprus could be developed. However, the Ministry of Education has not provided guidance for teachers on how to assess pupils entering the primary school, and there were no instruments which could be used to assess pupils entering the primary school. According to Blatchford and Cline (1992) there are four reasons for which all school systems must have a strategy for finding out about pupils on entry: establishing a basis for measuring future progress, getting a picture of the new intake, getting a profile of the new entrant, and identifying children who may have difficulties in school. This paper is an attempt to present the findings of research investigating the skills and knowledge in Mathematics of Cypriot pupils entering the primary school. A strong priority is given to making use of the initial school-entry record for formative purposes. Thus, the model for baseline assessment which is suggested should not be seen as an attempt to evaluate schools by adopting the business technique of value-added assessment. Since children of similar age are not at the same level and do not progress at the same rate, the main purpose of this model is to help teachers to use the results from baseline assessment in order to organise their teaching programme.The information gathered from each child is expected to be used to match the skills or content of a task to the level of the child.

II) Methodology

Research data were collected by using two different ways of assessment (external assessment and teachers' assessment). A performance test was designed in order to assess knowledge and skills in Mathematics identified in the Cyprus' Pre-Primary Curriculum. Pupils were asked to complete at least two different tasks related to the purposes of teaching Mathematics at first year pupils. Moreover, teachers were asked to complete a report for each pupil indicating whether the child had acquired these skills. Teachers could also respond by indicating that they were not sure as to whether their pupils had acquired a skill. A pilot study was conducted in October 1995. Minor amendments were made in the performance test and in the content of teacher's report in the light of the findings derived from the pilot study. The final versions of the performance test and teacher's report were administered to a representative sample of pupils of Year 1 in October 1996 (n=835). The stratified technique was used for the selection of the sample of pupils. Information about the performance of each pupil was given to his/her teacher in order to use it for formative purposes. Questionnaires were then administered to teachers to identify teachers' perceptions about baseline assessment as well as to find out how they had used the information gathered from the performance test and what their opinions were about using this performance test to assess first year pupils. Semi-structured interviews were, also, conducted with eight teachers in order to test the validity of the findings gathered from the questionnaire.

III) Findings

This section is divided into two parts. The first part deals with the knowledge and skills of pupils in Mathematics as they have been measured by the performance test and

reported by teachers. The second part deals with teachers' responses to the questionnaire and is an attempt to identify teachers' perceptions of baseline assessment.

A) Pupils knowledge and skills in Mathematics

It is, first of all, important to indicate that pupils who took place in the research had similar characteristics with all the Cypriot pupils who enter primary school in 1996. Comparisons of figures for all Cypriot first year pupils during the school year 1996-97 (Ministry of Education 1996) of their sex, their age and size of their classes, with the characteristics of the sample were made. No statistically significant difference was identified between each of the above characteristics of the research sample and the population. It is also important to indicate that almost all of them (91%) live with their parents and had the opportunity to attend nursery school during the previous school year.

A.1) <u>Findings from the performance test</u>

The figures in Table 1 are based on the information derived from pupils' response to the performance test. Percentages of pupils who successfully completed each task of the test and those who did not complete it are shown in Table 1.

Table 1: Percentages of Cypriot pupils who successfully completed the tasks of the performance test related to the following aims of teaching Mathematics and those who did not complete the tasks.

No	Aims of Mathematics (Pupils are able to)	% of pupils who succeeded	% of pupils who not succeeded
1	Compare two objects and find		
	a) the tallest	86.5	13.5
	b) the heaviest	77.5	22.5
	c) the widest	78.8	21.2
2	Compare three sets of objects and find the set which has more objects than the other two	83.5	16.5
3	Understand concepts describing the place of an object (e.g. in, out, under)	79.5	20.5
4	Count up to 5	79.5	20.5
5	Count up to 10	72.5	27.5
6	Read numbers up to 5	61.5	33.5
7	Read numbers up to 10	58.5	41.5
8	Write numbers up to 5	60.0	40.0
9	Write numbers up to 10	56.5	43.5
10	Recognise shapes:		
	a) Circles	92.5	07.5
	b) Squares	88.5	11.5
	c) Triangles	71.5	28.5
	d) Straight line	96.5	03.5
11	Draw shapes:		
	a) Circles	92.5	07.5
	b) Squares	88.5	11.5
	c) Triangles	71.5	28.5

	d) Straight line	92.5	07.5
12	Name shapes:		
	a) Circles	89.5	10.5
	b) Squares	84.5	15.5
	c) Triangles	68.5	21.5
	d) Straight line	92.5	07.5
13	Recognise colours:		
	a) Red	91.5	08.5
	b) Blue	90.5	09.5
	c) Yellow	79.5	20.5
	d) Green	77.5	21.5
14	Name colours:		
	a) Red	91.5	08.5
	b) Blue	89.5	10.5
	c) Yellow	79.5	20.5
	d) Green	76.8	23.2
15	Measure the length of a line by using non-standard metric units	79.5	20.5
16	Measure the length of a line by using metric units (e.g. ruler)	50.5	49.5
17	Match objects and pictures	86.5	13.5
18	Classify objects by using one criterion	73.5	26.5
19	Recognise patterns	48.5	51.5
20	Complete patterns	46.5	53.5

The following observations arise from Table 1. First, more than 75% of the pupils entering primary school were familiar with the meaning of the first mathematical concepts (items 1 - 3) and more than 85% recognised the colours. Second, the pupils had a relatively good background in Geometry since almost all of them recognised and drew shapes. However, 5% of the pupils who recognised and drew shapes did not know their names. Moreover, the percentage of pupils who recognised, drew and named the triangle, is smaller than the percentages of pupils who recognised, drew and named the circle or the straight line. It is, also, important to indicate that less than half of them (48%) could distinguish a rectangle from a square. Third, most of the pupils had some experiences with numbers. Counting is a skill which most of the pupils (75%) seem to have. They had, however, some difficulties in writing and reading numbers. Fourth, almost 80% of them were able to measure the length of a line by using non standard metric units but much fewer were able to measure the length of a line by using metric units. This difficulty seems to arise from the fact that they did not know how to use the ruler. Fifth, although the majority of pupils entering the primary school seems to have some experience with classifying objects, more than 30% of them were not able to identify similar objects or to classify a set of objects into two categories. Moreover, pupils seem to face difficulties in recognising and completing patterns and this may be due to the fact that they had not been involved in such activities either at home or at pre-primary school.

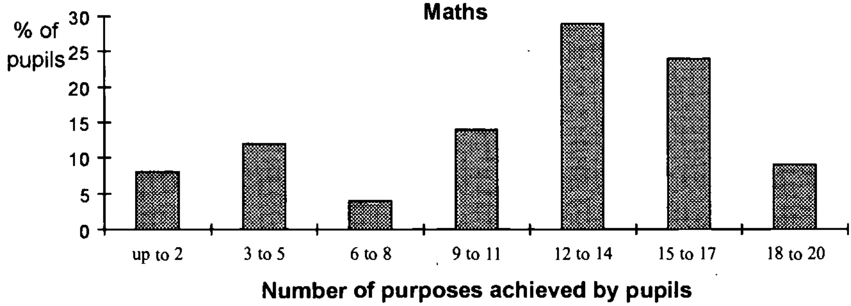
Finally, it was possible to differentiate the skills included in the Mathematics curriculum into two categories. The first category includes the skills which more than 75% of the pupils entering primary schools in Cyprus had achieved. Some of the skills of this category are presented below. Pupils are able to:

a) compare two objects and find the tallest, or the heaviest, or the widest
b) count, read and write numbers up to, at least, 5
c) recognise, draw and name circles, squares and straight lines
d) discriminate between colours and name the red and blue colour
e) match similar objects and pictures

The second category includes all those skills which more than 30% of the pupils had not achieved. Some of them are listed below. Pupils are not able to:

a) recognise, draw and name triangles and distinguish a square from a rectangle
b) measure the length of a line by using metric units
c) recognise and complete patterns

Graph 1: Percentage of pupils who achieved purposes of Maths



Graph 1 shows the number of the above 20 aims of Mathematics (Table 1) which had been achieved by pupils entering primary school. The following observations arise from graph 1. First, 20% of the pupils had not achieved more than 5 purposes. On the other hand, 33% of the pupils had achieved more than 15 purposes. Thus, the distribution, given above, is not a normal one since there are two relatively big groups of pupils at the two extremes of the distribution. The one group consisted of those pupils who achieved most of the purposes whereas the other consisted of pupils who had almost achieved none. Thus, there were significant differences between the skills in Mathematics of these two groups of pupils and these should be taken into account by teachers in order to organise their teaching according to their pupils' needs. Second, the great majority of pupils (60%) had achieved more than half of the above aims of teaching Mathematics. It can, therefore, be claimed that most of the school entrants in Cyprus had a relatively good mathematical background. This argument is also supported by the fact that the mode of the above distribution is represented by the group of pupils who achieved more than half of the purpose (i.e. 12 to 14 purposes).

A.2) Findings from teachers' report

Correlations were identified between findings gathered from the performance test and findings from teachers' assessment of pupils skills in Mathematics ($p<.001$). However, more than 25% of teachers were not able to say whether their pupils had acquired the following skills in Mathematics:

a) measure a straight line by using metric units,
b) recognise and complete a pattern
c) discriminate rectangles from squares

As a consequence, no correlation was identified between teachers' assessment and findings of performance test about the above skills. It is also important to indicate that teachers' report included some aspects of pupils' life which could not be assessed by the performance test. Thus, the findings gathered from teachers' report revealed that most of their pupils had positive attitudes not only towards the school (78%) but also towards Mathematics (69%). It was also found that more than 60% of the pupils could use scissors and other materials in order to do a practical activity but they could not work on an activity in co-operation with another pupil. However, more than 25% of teachers were not able to say whether their pupils are able to work in an activity without asking them. Finally, almost all the teachers (92%) revealed that they had not attempted to systematically assess their pupils when they entered the primary school. Thus, more than half of them mentioned that when they had to complete the report for their pupils' abilities in mathematics, they found out that they did not know what their pupils knew and hence tried to identify not only their pupils' skills in mathematics but also important aspects of their life (e.g. family and health situation, whether the child has school friends, how she/he feels when she/he comes to school).

B) Teachers' perceptions of baseline assessment

One item of the questionnaire administered to teachers asked them to rank the four purposes of baseline assessment, mentioned above, according to their importance. Kendall coefficient of concordance ($W_1=.76$, $Z=6.2$, $V_1=4.0$, $V_2=714$, $p<.005$) shows that Cypriot teachers agreed among themselves in their ranking of the relative importance of the purposes of assessment. Moreover, formative assessment was considered as the most important by almost all the teachers. It is also of interest to emphasise the low rating given to summative purposes of assessment and to the purpose related to the value-added assessment. As far as the purpose related to the summative assessment is concerned, almost all the teachers (95%) saw it as either the least or the second least important purpose. Similarly, 87% saw the purpose concerned with the value-added assessment as the least or the second least important purpose.

The following findings arise from teachers' response to items of the questionnaire concerned with the implementation of policy on baseline assessment in Mathematics. First, the great majority of teachers (85%) considered baseline assessment as an essential part of teaching. They argued that baseline assessment could help teachers to

prepare their programmes more effectively. Second, the great majority of teachers (72%) thought that information gathered from baseline assessment should not be used for labelling children or for early identification of pupils with learning needs since most pupils develop their skills rapidly at this age. Third, more than half of them (64%) considered the performance test as a useful tool for identifying pupils' skills and knowledge when they enter the primary school. However, almost all of them (82%) revealed that it would not be easy for them to use this test since they did not have enough teaching time.

IV) Discussion: Implications of findings for the development of policy on baseline assessment in Mathematics

The evidence presented above can be discussed in terms of its implications for the development of assessment policy in Cyprus. First, it is important to examine policy on baseline assessment in terms of policy on classroom organisation. The fact that significant differences among the skills and knowledge of school entrants have been identified supports both the importance of baseline assessment for formative purposes and that spending most of teaching time working as a whole class, as is the case in Cyprus (Kyriakides 1996), is not an appropriate way of teaching Mathematics to first year pupils. The fact that some school entrants had either achieved most of the aims of teaching Mathematics or had not achieved any one of them, implies that it is not possible to organise teaching Mathematics without taking into account the different Mathematical background of school entrants. Baseline assessment provides teachers with information which help them to respond to the learning needs of each pupil. Thus, the development of policy on baseline assessment may also encourage Cypriot teachers to give more thought to the best way to respond to individual learning needs. Second, almost all the Cypriot teachers revealed that they did not systematically assess their pupils when they entered the school. Thus, their teaching plans are not based on what their pupils know. Moreover, almost all the teachers who were interviewed revealed that they covered topics in Mathematics which, as they found later, most of their pupils had acquired in Pre-Primary school. Thus, developing a policy on baseline assessment in Mathematics may help teachers to cover the first year curriculum in Mathematics which previous research has shown they considered as overloaded (Kyriakides 1994).

Third, Cypriot teachers perceived formative purposes of baseline assessment as more important than the purpose related to the "value-added" assessment or the summative purpose of assessment. This is in line with the argument of Torrance (1986) that teachers consider formative assessment as the most important purpose of assessment. Cypriot teachers would welcome the development of an assessment policy which promoted the formative purposes of baseline assessment, but would be less inclined to support one emphasising summative purposes. Thus, the debate on developing a policy on baseline assessment may not be restricted to workload but raise fundamental issues of educational ideology. Most systems of baseline assessment have strengths and weaknesses, and few meet all possible requirements without being excessively

unwieldly (Tyler 1984). Policy makers in Cyprus must be clear about the objectives for policy on baseline assessment. Teachers suggest that the answer to this question lies on the fact that information provided should be of genuine assistance in determining the appropriate action to be taken in assisting each pupil's development. They do not believe that policy makers should see policy on baseline assessment as an attempt to produce a fairer method of evaluating the work of a school than using outcome data alone. It can, therefore, be claimed that this study does not only reveal the need for developing a policy on baseline assessment but also that this policy should be focused on teachers' perceptions of purposes of baseline assessment since the transformation of curriculum reform into practice depends partly on their perceptions. What is needed is to identify and build upon teachers' perceptions and encourage them to promote curriculum policy at the school level in order to assess their pupils and organise their teaching according to the needs of their school entrants.

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SOME ISSUES IN USING MAYBERRY'S TEST TO IDENTIFY VAN HIELE LEVELS

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In the early 80s Mayberry (1981) developed a diagnostic instrument to be used to assess the van Hiele levels of pre-service teachers. The test which was carried out in an interview situation, was designed to examine seven geometric concepts. There has been no reported attempt to (i) replicate this work in some alternative format, or (ii) analyse the validity of the test questions. To address these issues, a detailed testing and interview program of 60 first year primary-teacher trainees was undertaken at the University of New England. This paper presents a summary of results of the test, relating the levels to the students' geometric background, and considers one aspect of the findings of this study, the potential for certain aspects of Mayberry's work to lead to an incorrect assessment of a student's level of understanding in geometry.

The ability to instruct students at their level of understanding is dependent, in part, on the teacher being able to assess students' levels of understanding. In order to make this assessment, there needs to be available a reliable diagnostic instrument. In the early 80s Mayberry (1981) in her work with pre-service primary teachers, developed such a diagnostic instrument that could be used in an interview situation. While her work has been used as a basis for other research projects (e.g., Denis, 1987), there appears to have been no critical evaluation of the questions used. Before presenting the results of the study and addressing the issue of test validity, a brief background to the important ideas underpinning her work is presented.

Background

The van Hiele Theory

In the 1950s, Pierre van Hiele and Dina van Hiele-Geldof completed companion PhDs which had evolved from the difficulties they had experienced as teachers of Geometry in secondary schools. Whereas Dina van Hiele-Geldof explored the teaching phases necessary in order to assist students to move from one level of understanding to the next, Pierre van Hiele's work developed the theory involving five levels of insight. A brief description of the first four van Hiele levels, ones commonly displayed by secondary students and most relevant to this study, is given:

Level 1 Perception is visual only. A figure is seen as a total entity and as a specific shape. Properties play no explicit part in the recognition of the shape.

Level 2 The figure is now identified by its geometric properties rather than by its overall shape. However, the properties are seen in isolation.

Level 3 The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised.

Level 4 Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions are used with understanding.

The van Hieles saw their levels as forming a hierarchy of growth. A student can only achieve understanding at a level if he/she has mastered the previous level(s) They also saw (i) the levels as discontinuous, i.e., students do not move through the levels smoothly, (ii) the need for a student to reach a 'crisis of thinking' before proceeding to a new level (iii) students at different levels speaking a 'different language' and having a different mental organisation.

Mayberry's Research

JoAnne Mayberry's study (1981) investigated, in part, whether the van Hiele level, at which a student is functioning in geometry, can be discerned. To carry out this investigation, Mayberry created a diagnostic instrument consisting of 62 items (many of them containing separate question parts) designed to the operational definition of each of the levels. The items covered seven geometric concepts, namely, square, right triangle, isosceles triangle, circle, parallel line, congruency, and similarity. These concepts all occur in the elementary curriculum in the USA. A matrix/grid was used to develop questions by level and concept so that the questions would have parallel forms. One or more questions were developed for each cell in the grid. Experts in the fields of mathematics and mathematics education, among them Pierre van Hiele, were asked to validate the items by judging whether the items satisfied certain criteria (Mayberry, 1981, p. 52). The final form of the diagnostic instrument was then used in an interview situation to investigate the understandings of 19 pre-service elementary education students at Georgia College, Milledgeville, Georgia.

Design

In order to consider Mayberry's work in an Australian context, a detailed study of the geometric understanding of 60 first-year primary-teacher trainees was carried out at the University of New England. The study aimed, in part, to provide a written test based on the Mayberry interview schedule. Follow-up interviews were conducted with students to validate the levels of thinking as determined in the written test. Conversion of the Mayberry items to a written test involved some modification of the wording to ensure that the intention of each question was clear. A preliminary study validated the reliability of the written questions. Level 5 items were omitted, hence the written test assessed van Hiele Levels 1 to 4 (Mayberry items 1 to 57).

Results

Every endeavour was taken to replicate Mayberry's evaluation of responses. Her thesis was examined in depth to ascertain her expectations in the responses to the items. However, this was not possible for every item, there being occasions, particularly for the Level 3 and Level 4 items, when insufficient information was to be found in the Mayberry writings. The results of the assessment of the students' levels of understanding are summarised below in Table 1, whilst Table 2 shows the comparable results for the Mayberry subjects. When students failed to identify concepts their result was recorded as No Level. To facilitate comparisons, all results are given as percentages, with the horizontal sums in both tables being 100%.

Table 1
Highest level reached by the Australian students for each concept (% of sample)

Concept	No Level	Level 1	Level 2	Level 3	Level 4
Square	0	3	84	7	7
Right Triangle	3	19	55	19	3
Isosceles Triangle	7	27	43	20	3
Circle	0	13	19	52	16
Parallel Lines	0	17	80	0	3
Congruency	0	32	35	3	29
Similarity	0	43	40	10	7

Table 2
Highest level reached by the Mayberry students for each concept (% of sample)

Concept	No Level	Level 1	Level 2	Level 3	Level 4
Square	0	11	32	26	32
Right triangle	26	21	21	16	16
Isosceles triangle	26	16	11	26	21
Circle	5	11	16	21	47
Parallel lines	26	16	16	37	5
Congruency	0	21	32	21	26
Similarity	5	42	5	21	26

The results show that, for both studies, the majority of students were assessed as having no greater than Level 2 understanding, i.e., they were comfortable recognising concepts, and listing the associated properties, but did not understand the relationships between the properties.

In Australia, most of the mathematics courses offered in senior secondary schools have an integrated syllabus, the geometry segment of which appears generally to be designed for Level 3 and Level 4 instruction. For example, notes on the content of the plane geometry segment of appropriate NSW state mathematics syllabuses includes the development of the understanding of notions of proof, and of the ability

to provide solutions to deductive exercises which rely, for example, on the application of congruency relationships in non-prompted situations and necessary and sufficient conditions, i.e., typical Level 4 competency. In the USA, mathematics is commonly studied in High Schools (Years 9/10 to Year 12), as separate optional courses, e.g., algebra, calculus, geometry. Mayberry's examination of high school geometry textbooks (1983, p.68) showed that "Level 3 thought appears to be needed to begin the course and that Level 4 thought should be developed during the course." In Mayberry's study, 68% of the subjects had taken geometry as a course in High School, and 32% had not. This is similar to the composition of the Australian sample in which 64% of students had completed a senior secondary mathematics course, which included a formal or recognised geometry segment, 23.5% of students had completed a senior secondary mathematics course, which did not contain a formal geometry segment, and 12.5% of students had not completed any senior secondary mathematics. Table 3 compares the van Hiele level achieved with the type of geometric background of the students in the Australian sample.

Table 3
Relationship between the Australian students' geometric backgrounds and their most common van Hiele working level (% of sample)

Geometric Background	van Hiele Level 1	van Hiele Level 2	van Hiele Level 3	van Hiele Level 4
Senior geometry	0	63	14	23
Senior maths but no geometry	8	92	0	0
No senior maths	43	43	14	0

It is significant that 63% of the students who had completed a course in which the instruction is assumed to be at van Hiele Level 3 and 4, could not display overall understanding of Level 3 knowledge in their responses. Further, the students who had completed a senior mathematics course but without a geometry strand had all been exposed to an extensive geometry course in the junior secondary school in which the instruction was addressed at least to Level 3.

Identified Problems

When collating results in the Australian study, inconsistencies in the assignment of van Hiele levels for some students emerged. Overall there were 19 (7.8%) response pattern errors. This meant that a number of students showed up as not validating the level hierarchy. Interviews did not appear to clarify these inconsistencies. On analysis of the results by concept and by level, it was considered that certain aspects of the Mayberry items had the potential to lead to incorrect assessment of a student's level of understanding. In particular, four main features were found to account for major problems to the test validity. They were:

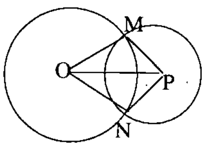
1. incorrect assignation of a level to certain items;
2. unequal treatment of concepts across levels;
3. uneven distribution of questions across levels; and,
4. unbalanced distribution of question focus within levels.

Discussion

Feature 1 (Incorrect assignation of a level to certain items)

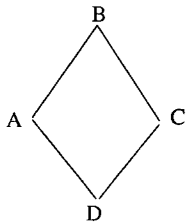
Some items did not appear to be consistent with the level for which they had been designed. This was identified when large differences were exhibited by students on questions supposedly at the same level. It is possible that some teaching effect or rote learning may have influenced these results but this was not confirmed by interview. An example of this phenomenon can be demonstrated by examining and comparing Item 56 and Item 55, two of the twelve items which Mayberry designed to test Level 4.

Item 56



These circles with centres O and P intersect at M and N.
Prove:
Δ OMP $\equiv$ Δ ONP.

Item 55



In this figure AB and CB are the same length.
AD and CD are the same length.
Will $\angle$A and $\angle$C be the same size?
Why or why not?

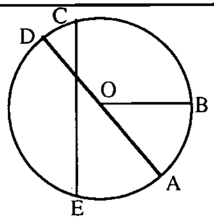
In Item 56, triangles OMP and ONP are clearly delineated. The solution requires identification of three equal pairs of corresponding sides to prove congruency of the triangles. By contrast, Item 55 can be solved by a number of different techniques. One solution to the problem involves the use of congruent triangles. To do this, a decision is needed concerning a suitable construction, i.e., join BD, which will produce the required pair of triangles, (triangles ABD and CBD). The proof of congruency of these triangles then becomes an instrument used within the solution of the problem. In the Australian study, of the nine students who answered Item 56 correctly, only four were also correct for Item 55. No student was incorrect for Item 56 yet correct for Item 55.

The spontaneous recognition of the need to construct triangles before undertaking congruency requires a deeper overview of the power of congruency. This problem begs the question: Is the ability to give a proof of congruency working at Level 4, or only at Level 3? Van Hiele summarises from his dissertation that a student has reached Level 3 thinking “if, on the strength of general congruence theorems, he (she) is able to deduce the equality of angles or linear segments of specific figures”

(1957, p. 239). The very real difference between using the idea when it is apparent and recognising the need to use the idea in a visually unprompted situation is highlighted by the comparison of performances for these two Mayberry items.

Feature 2 (Unequal treatment of concepts across levels)
The seven concepts used in Mayberry's work do not appear to be treated in an equal manner. Investigation of the results across all the concepts reveals that either the students in both USA and Australia had achieved a much greater understanding of the concept circles, or else the items designed for that concept were not true to level descriptions. To explore this issue, two items, Item 35 and Item 52, are examined.

Item 35



This figure is a circle with centre O.
Would the following be:
a) certain b) possible c) impossible
Give reasons for your answer.

1) OB = OA 2) OD = OA 3) 2OB = AD 4) AD = EC

Item 52

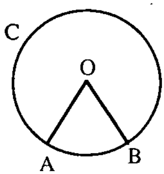


Figure C is a circle.
O is the centre.

Prove that ΔAOB is isosceles.

According to Mayberry, a student answering Item 35 needs to be working at Level 3 in order to answer each of the four parts of the question correctly. It could be argued that the correct answering of the first three parts of the question requires Level 2 knowledge of the properties of a radius, namely, that all radii are of equal length, and that the diameter is equal in length to two radii. Further, there are strong visual clues to support the correct answers. In comparison, the fourth part of this item requires the understanding that a chord, passing through the centre of a circle, is the longest possible chord of a circle, i.e., Level 3 understanding of the relational properties of the diameter.

Mayberry lists Item 52 as requiring Level 4 reasoning for a student to provide the correct solution. A solution of this item requires the identification of equal radii, OA and OB, as equal sides of triangle AOB. It is considered this solution incorporates

the use of the relating of properties in a simple one-step deduction process, i.e., Level 3 thinking.

This focus on one concept at the expense of others raises clear questions about the allocation of levels within this concept. Compounding this problem is the difficulty of considering the growth of student understanding about aspects of a circle. Properties and the relationships of properties of circles at Levels 2 and 3 are not as clear as in the case of quadrilaterals. Further, there is no evidence in van Hiele's writings to provide guidance for the development of circle concepts. The difficulties became obvious when students in the Australian sample were able to score much higher on circle questions than on other concepts. This could not be rationalised in terms of greater experience or familiarity with circles.

Feature 3 (Uneven distribution of questions across levels)
The test items are not evenly distributed throughout the cells of the matrix/grid. This results in an imbalance between levels within a concept, and has the potential to lead to response-pattern errors. This can best be illustrated through the comparison of criteria requirements for Levels 2 and 3. In her design, Mayberry has allocated between three and seven items per concept to test for Level 3, however, she has allocated only one or two items to test for Level 2. Five concepts, right triangle, isosceles triangle, parallel line, similarity and congruency, are tested by a single item at Level 2. For example, the most obvious case concerns the concept isosceles triangle. Whereas seven separate items (Items 28 to 32, 42 and 49) test at Level 3, only a single item (Item 18) determines whether or not a student displays mastery at Level 2. Thus the criteria for attaining Level 2 in isosceles triangle is a perfect score.

Item 18.
What can you tell me about the sides of an isosceles triangle?
What can you tell me about the angles of an isosceles triangle?

Should a student have misunderstood the thrust of this single item, answering, for example, "there are three", or "the angles sum to 180 degrees", or have incorrectly answered "they are all less than 90 degrees" (an answer commonly resulting from frequent exposure to acute-angled triangles), he/she is deemed not to have shown mastery at that level. Often such students can still display mastery of Level 3 items.

Feature 4 (Unbalanced distribution of question focus within levels)
In the Mayberry scoring, it would appear that a subject can be adversely affected through the lack of exposure to a particular aspect of a form of reasoning. In the testing of the square at Level 3, the notion that a square is also a rectangle accounts for three of the nine possible scores, (Items 9a, 25b and 42d). Criteria for this level is a score of six out of nine, hence, a lack of exposure to the above notion means that a

student must score correctly for all other questions in order to register success at Level 3. Should a student not have been exposed to, for example, class inclusion, a Level 3 concept, the Mayberry scoring could assess that student as having mastery only of Level 2. Pegg (1992, p.24) in his investigation of recent research into properties of levels, summarises:

> It is not sufficient to say that a student is not at Level 3 if he/she does not believe a square is a rectangle. Class inclusion is not simply a part of a natural mathematical development. It is linked very closely to a teaching/learning process. It depends upon what has been established as properties. ...The main feature of Level 3 should not, in my view, be the acceptance of class inclusion but the willingness, ability and the perceived need to discuss the issue.

Overall these four features proved to be very important. When the analysis was repeated, taking each feature into account, there was an 68% reduction in the number of error patterns. This meant all but four students' understanding was able to be reconciled. Interestingly, these four students (six error patterns) all exhibited a partial Level 3 understanding and were able to present one acceptable Level 4 response, which satisfied Mayberry's criterion. It is most likely that if *Feature 3* had been addressed, i.e., more Level 4 items, these error patterns would not have occurred.

Conclusion

This analysis not only gives us a clearer perspective about the Mayberry test and the results it generates, but it also allows further insight into the van Hiele Theory. In particular, it provides further empirical evidence about the robust nature of the levels and about what it means to understand at a certain level.

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Defining and Understanding Symmetry

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We propose a definition of Symmetry, which captures the many different aspects of this important concept related to the K-12 mathematics curriculum. Two main types of symmetry are discussed: Geometric Symmetry and Role Symmetry. An investigation of mathematics in-service teachers' understanding of symmetry was conducted in light of this definition, focusing on these two types of symmetry. The findings point to a close connection between the proposed definition and the ways teachers justify whether a given object is symmetrical.

The importance of symmetry in mathematics is well recognized (Alperin, 1978; Browder & MacLane, 1978; Sonin, 1987; Weyl, 1952). Symmetry has an aesthetic value as well as an interdisciplinary nature (Weyl, 1952; Darvas et. al., 1995). It is a powerful tool in solving mathematical problems (Dreyfus & Eisenberg, 1990; Polya, 1973, 1981; Schoenfeld, 1985). Although symmetry is a broad mathematical concept, it is treated in secondary school as a collection of disconnected cases. Teachers do not "think symmetry" and do not use symmetrical considerations in problem solving (Dreyfus & Eisenberg, 1990). Moreover, they often present different definitions for special cases of symmetry, each capturing only some of the various aspects of the concept.

The study reported in this paper is part of a larger study, the purpose of which is to enhance the understanding, the appreciation, and the use of symmetry by mathematics teachers (Leikin, 1997).

What is Symmetry?

We first turn to a comprehensive look at symmetry underlying our study. There are several different approaches to the definition of Symmetry, depending on the perspective taken. According to Lowrey (1989), " ... *the symmetry concept is a basic* ***principle*** *that is useful to explain relationships between aspects of mathematics and physical, biological, and other natural phenomena.*" (p. 485, ibid.). Lowrey claims that the meaning of symmetry is not precisely defined, not even within the mathematics discipline.

Mathematicians treat symmetry in various ways: As a property of an object, as a correspondence between objects, or as a special kind of transformation. In teaching mathematics, teachers and textbooks usually distinguish between symmetry in geometry and symmetry in other branches of mathematics, and even in geometry they deal separately with

different transformations (reflection, rotation, and translation) neglecting to point out the underlying common feature of all these transformations (Eccles, 1972; Fehr, Fey & Hill, 1973; Skopets, 1990; Yaglom, 1962). In algebra and calculus symmetry is defined differently for different kinds of objects (e.g., functions, systems of equations, matrices, groups) (Polya, 1981, 1973; Dreyfus & Eisenberg, 1990). Thus, no wonder symmetry is often viewed as a collection of disconnected concepts.

According to Rosen (1995): *Symmetry is immunity to a possible change* (p.2, ibid.). This is a broad definition that captures the essence of symmetry and applies for each and every instance of symmetry. However, it is much too broad and general for school mathematics. The unifying approach to the definition of symmetry in mathematics that we suggest, is similar to what Rosen (1995) does for symmetry in science. We look at the *immunity* of a *property* of a mathematical *object* with respect to *a possible change*. This *possible change* corresponds to a transformation that can be applied to the *object*. Thus, symmetry has to do with three elements: an object, its property and a transformation, as proposed in the following **DEFINITION**:

> ***SYMMETRY*** *is a TRIPLE consisting of an* ***OBJECT,*** *a specific* ***PROPERTY*** *of the object, and a* ***TRANSFORMATION*** *satisfying the following conditions:*
> *i) The transformation is not the identity;*
> *ii) The object belongs to the domain of the transformation;*
> *iii) Application of the transformation to the object* ***does not change*** *the object's property.*

As examples of the proposed definition, we discuss two main types of symmetry: **Geometric Symmetry** - if the object in the triple is a geometric figure, and **Role Symmetry** - if the transformation is a permutation.

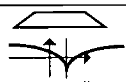
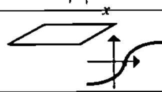
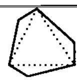
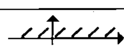
Geometric Symmetry		
Symmetry Transformations	Examples of Symmetrical Objects	
A Reflection (with respect to a line)		An isosceles trapezoid A graph of a function $y=\sqrt{\lvert x-a\rvert}+b$
Central Symmetry (= A Reflection with respect to a point = Rotation by 180°)		A parallelogram A graph of a function $y=(x-a)^3+b$
A Rotation		A polygon built on an equilateral triangle, adding three congruent triangles
Translation		A graph of a periodic function

Figure 1: Examples of different types of Geometric Symmetry

Figure 1 presents examples of different kinds of Geometric Symmetry. In all of these cases **the property** that does not change is the **location** of the geometric figure.

Figure 2 presents a number of examples of the different types of Role Symmetry.

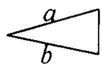
Role Symmetry		
Type of Role Symmetry	Symmetry Transformations	Examples of Symmetrical Objects
Algebraic Role Symmetry	Permutation of Variables	• Certain algebraic expressions, e.g.: $a+b$; • Certain functions, e.g.: $y=8/x$; • Certain systems of equations, e.g.: $\begin{cases} 3x+2y+z=30 \\ x+3y+2z=30 \\ 2x+y+3z=30 \end{cases}$ (Polya, 1981).
Logical Role Symmetry	Permutation of Variables	Symmetrical relations, e.g.: $A \Leftrightarrow B$, $F \cong G$, $a \parallel b$
Geometric Role Symmetry	Permutation of a Triangle's Sides	Isosceles triangle: 

Figure 2: Examples of the two different types of Symmetry of Roles.

What teachers consider symmetrical

The study was designed in order to answer the following **research questions**:

1. What is the relationship between the type of symmetry of a given object, and the ability of teachers to identify it as a symmetrical object?
2. What is the relationship between the representation of a given object, and the ability of the teachers to identify it as a symmetrical object?
3. How do teachers explain that a mathematical object is symmetrical?
4. What typical mistakes do teachers tend to make when determining whether an object is symmetrical?

In order to answer the above questions, a questionnaire was constructed and administered to 36 secondary mathematics teachers. The questionnaire consisted of 34 mathematical objects. For each object the teachers were asked to determine whether it is symmetrical and to justify their answer. The objects included in the questionnaire varied according to their representation and their type of symmetry. All the objects in the questionnaire were divided into four **categories** according to their type of symmetry:

- An asymmetrical object;
- An algebraically symmetrical object;
- A geometrically symmetrical object;
- An object which is both algebraically and geometrically symmetrical.

The set of objects in the questionnaire included: geometric figures, functions, equations, systems of equations and algebraic expressions.

Findings

As mentioned above, each teacher was asked to respond to 34 items. Thus, there were 1224 expected responses. In fact, 1174 responses were actually received. Each response was analyzed with respect to several criteria and coded accordingly.

Correctness of responses

The first level of analysis was done according to the correctness of the statement (regarding whether the object was symmetrical), and according to the correctness of the justification that was provided. Thus, the answers were first classified into three categories with respect to the statement: Correct, partly correct, and incorrect. Then, they were classified with respect to the justification into four categories: Correct, partly correct, incorrect, and unclear. In order to be able to analyze connections between items of the questionnaire, each response was scored.

Figure 3 depicts to distribution of responses according to correctness of the statement and according to the correctness of the justification.

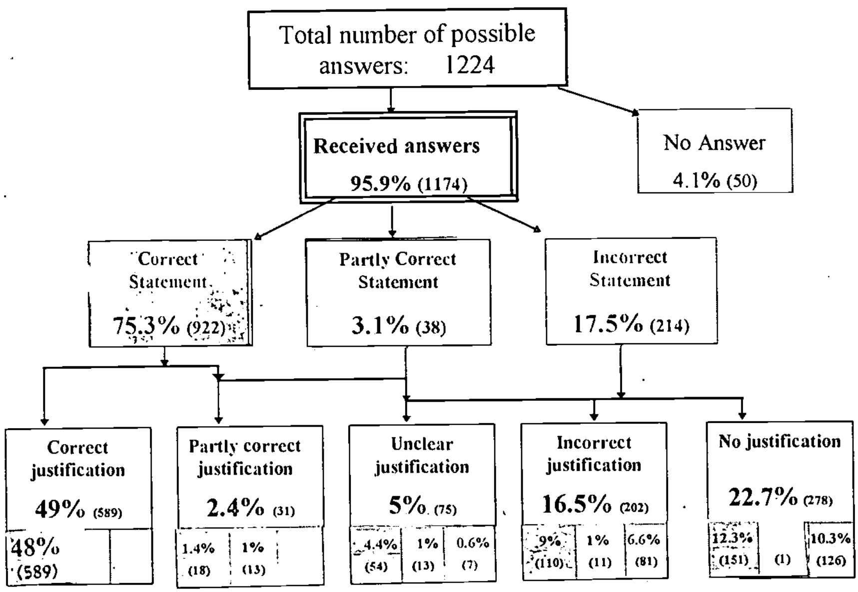


Figure 3. Distribution of the answers according to the degree of correctness.

Explanations justifying why an object is symmetrical

The second level of analysis was done for the 589 fully correct answers. The purpose of this analysis was to characterize the nature of correct justifications (see Research Question 3)

Three types of explanations were identified:

1. *Explanations referring exclusively to the symmetry of the* ***object*** - these explanations explicitly state the fact that the object in question is known as symmetrical in at least one of its representations.

   Example:

$(x-5)^2+(y+3)^2=9$	*Yes, it is symmetrical: This is an equation of a circle and any circle is a symmetrical figure*

2. *Explanations referring to the symmetry* ***transformation*** *of the object* - these explanations either explicitly refer to at least one symmetry transformation that can be applied to the object, or implicitly refer to the transformation by indicating the type of symmetry of the object.

   Examples:

$(x-5)^2+(y+3)^2=9$	*Yes, it is symmetrical: A reflection can be applied.*
$\begin{cases}2x+y=8\\x+2y=8\end{cases}$	*Yes, it is symmetrical: The variables can be switched.*

3. *Explanations referring to the* ***property*** *of the object that does not change under a certain transformation* - these explanations specify certain features of the object which do not change when applying a transformation.

   Example:

$\begin{cases}2x+y=8\\x+2y=8\end{cases}$	*Yes, it is symmetrical: The permutation of variables does not change the solution.*

   This example can be seen as an extension of the previous one. Here, in addition to the transformation (i.e., the permutation) that is referred to, the invariant property (i.e., the solution) is also specified.

The findings related to the type of correct explanations differ according to the type of symmetry of the objects. Thus, in order to prove that an object is *geometrically symmetrical*, in most of the cases (84%) the teachers referred only to *a symmetry transformation* without noting the invariant property of the object. However, when justifying that an object is *algebraically symmetrical* in most of the cases (71%) an explicit connection was made to the relevant *property* of the object.

In many cases, when teachers justified why an object is algebraically symmetrical they did it ***intuitively***. This tendency was identified from video-taped discussions which

were conducted with the teachers after the questionnaires were collected. Some teachers used the term "Role Symmetry", arguing that "...it *seems natural, because what makes the symmetry of the object is the use of variables with identical roles"*.

Analysis of teachers' mistakes

The third level of analysis focused on the identification and classification of mistakes made by the teachers. Most of the mistakes were connected to the definition of symmetry. According to Smith, diSessa & Roschelle (1993), mistakes connected to the definition of a given concept can be divided into two types: mistakes caused by basic misunderstandings of the notion of a definition, or mistakes caused by a misunderstanding of a specific definition of a concept. Some of the findings fall into two similar categories respectively: Mistakes which have to do with the general notion of a definition, and mistakes, which have to do with the specific definition of symmetry.

Misunderstanding of the symmetry concept

In order to analyze teachers' mistakes in identifying symmetrical objects, their incorrect explanations were carefully analyzed. All incorrect and partly correct explanations were divided into the following four main categories:

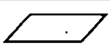
1. Mistakes resulting from the way teachers relate to the ***object***. There are certain sets of objects for which teachers (wrongly) either consider any of their elements to be not-symmetrical or consider symmetry not applicable to any of their elements.

   Example:

$a^2 + b^2 + ab$	*No, it is not symmetrical: This is an algebraic expression.*

2. Mistakes resulting from the way teachers relate to a ***transformation***. A transformation is considered to be a symmetry transformation for certain object without checking whether there is an invariant property of the object with respect to this transformation.

   Example:

	*Yes, it is symmetrical: According to the reflection with respect to the diagonal.*

3. Mistakes resulting from the way teachers relate to a certain ***property*** of the object. Some properties are (wrongly) considered necessary for symmetry to exist, thus, if they do not exist in an object the object is considered not symmetrical.

   Example:

$y = x^3 - x^2$	*No, it is not symmetrical: The function is not odd and not even.*

4. Incomplete interpretation of data. There were cases in which teachers did not make use of implicit information which could be derived from the given object.

Logical mistakes

Logical mistakes are often related to misunderstandings of the notion of a definition. The first type of logical mistakes that teachers made can be attributed to the fact that in order to prove that a given object is symmetrical, the teachers used a necessary, but insufficient condition. In other words, the teachers did not take into account that the condition used to define a concept should be both necessary and sufficient for all objects exemplifying the concept. Responses in which teachers used a necessary but insufficient condition in order to prove that an object is symmetrical, were classified as logical mistakes, which point to basic misunderstanding of the notion of a definition.

A second type of logical mistakes has to do with the fact that the definition of a symmetrical object is an existential definition. An object is symmetrical if there exists a symmetry transformation of the object. Therefore, in order to prove that an object is not symmetrical, it is necessary to prove that any transformation is not a symmetry transformation of the object. Instead, teachers seemed to think that a number of examples of transformations, which are not symmetry transformations of the given object, constitutes a valid justification that the object is not symmetrical.

Conclusion

The type of symmetry of a given object and its representation seem to effect the success in determining whether an object is symmetrical. In general, teachers did not tend to explicitly refer to the invariant property of symmetry. All the cases in which they referred to the invariant property were algebraically symmetrical objects represented symbolically. In addition, geometric figures were easier to identify as symmetrical.

Summary and Discussion

It is interesting to point out that none of the responses, both correct and incorrect, explicitly referred to all three components of the proposed definition of symmetry. Most of the responses referred to only one component. Thus, each of the three categories of correct explanations, as well as those of the incorrect explanations, which were derived from the written responses, relates to one of the three components of our definition of symmetry: to the object (cat. 1), to the transformation (cat. 2) or to the invariant property (cat. 3). In some cases it was possible to respond correctly without referring to all three components. However, there were cases in which it was necessary to consider more than one component of the definition. For example, when justifying that a circle is a symmetrical object, some teachers correctly referred to the reflection as its symmetry transformation, but neglected to consider the invariant property (i.e., the location of the object). A similar response was incorrect when applied to a parallelogram. Although a parallelogram is a symmetrical object, its symmetry transformation is rotation and not reflection. Those who argued that a reflection with respect to the diagonal is its symmetry transformation did not make sure that there exists an invariant property under the proposed transformation.

It is suggested that the proposed definition, which includes all three components, could serve as a guideline for thinking about symmetry, and consequently, for correctly determining whether an object is symmetrical.

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THE PSYCHOLOGY OF MATHEMATICS TEACHERS' LEARNING: IN SEARCH OF THEORY

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Abstract *The research literature on teachers' learning of mathematics and mathematics education often employs terms such as 'teacher change' or 'teacher development' rather than learning. I argue that this is due in large part to rather vague theories of adult learning, dominated as we are by psychological theories of children's learning. Two small-scale research studies are presented in which pre-service and in-service teachers were set a learning activity which brought their naive theories into contact with 'scientific' theories in areas of their own interest. These studies are used to highlight the main issue raised in this paper, that we must pay more attention to theory.*

Introduction

In studying children's learning of mathematics the dominant academic framework upon which researchers draw (Apple, 1995), and the particular perspective of PME of course, is a psychological one. Traditionally, Piagetian theories of cognitive development have been adopted and adapted; more recently Vygotskian theories have been co-opted. Other frameworks are available (Lerman, 1996), including sociological, anthropological, psycho-analytic and post-structuralist and these frameworks are represented increasingly in proceedings of PME (Evans & Tsatsaroni, 1993; Pimm, 1994; Brown, 1994).

In studying pre-service and in-service teachers much of the research emphasis has been on how to describe and analyse teachers' beliefs about mathematics and mathematics teaching, how to analyse and describe teachers' actions in the classroom, and the possible connections between the two. A recurring problem for such research is the apparent mismatch between teachers' stated beliefs and their actions as observed by another. Hoyles (1992) explains this in terms of situated beliefs, that they are "dialectical constructions, products of activity, context and culture" (p. 280). Lerman (1994a) argues similarly, suggesting that there is a confusion between different practices in much descriptive research on teachers' beliefs and actions in which the research tool, be it questionnaire, interview, observation schedule or others, frames the discourse and hence the outcome.

Study has also been made of the connections between teachers' knowledge of mathematics and their knowledge of pedagogy (Shulman, 1986; Even, Tirosh & Markovits, 1996). Ponte (1994) provides one orientation of research on teachers, that of increasingly more elaborate descriptive frameworks. He suggests that a concern with teachers' beliefs and conceptions has been predominant and he offers a

more interactive perspective incorporating teachers' images and their rules of practice.

In studying teachers' learning of mathematics and of mathematics education researchers in PME have often been more vague or perhaps more eclectic in their choice of academic discourse. This may be largely because constructivist theories of cognitive development are concerned with children and children's learning and there is no language for life-long learning available. The empirically identifiable euphemism 'teacher change' is often used in place of learning, which does not engage with the need to elaborate a theoretical perspective on learning. Becker & Pence (1996) found that changes in teachers' beliefs accompany change in classroom practice (p. 115). Of course change is easier to identify when specific pedagogic goals such as those of the US reform of mathematics teaching are set out by an establishment, more difficult when the goals arise from the interests and concerns of individual teachers (Lerman and Scott-Hodgetts, 1991). The problems of associating beliefs and practices as pointed out by Hoyles (1992) and Lerman (1994) are of relevance here too.

Even et al (1996) measure change by teachers' self-reports, and those of supervisors, principals and fellow teachers. They are more specific than most concerning teachers' learning. "Our findings indicate that when asked to respond to specific suggestions made by students, teachers are pushed to articulate their own understanding. Thus, in turn, they provide teacher educators with an opportunity to study adult learners' cognitive processes and conceptions." (p. 128)

In this paper I will discuss briefly the theories that are available to describe mathematics teachers' learning. I have suggested that theories are often lacking but are needed if the research that is presented is to be good research and of use in mathematics teacher education programmes. Research on teachers is used to inform teacher education programmes, although the step from description to prescription requires justification. Without an explicit theory of learning this step can be confused and even incoherent.

I will then present some data from two studies, one of mathematics teachers' learning of issues in mathematics education and the other of pre-service teachers' learning of mathematics, in order to highlight the theme of this paper.

Theories of teachers' learning

Where explicit attention is paid to teachers' learning reflective practice is often indicated as the stimulus which can lead to learning (e.g. Lerman & Scott-Hodgetts, 1991; Mousley 1992). What is not evident in these accounts is the process of cognitive, cultural or social development which is understood to characterise teachers' learning. Reflection on one's own actions presumes a dialogical interaction in which a second voice observes and criticises. In order to lead to learning it would seem that this must be more than the ongoing observation of one's own actions through which one may recognise a satisfactory or unsatisfactory

outcome of an action. An unexpected event, a novel response or question from a student for example, may lead to a question but the learning that is hinted at in the literature on reflective practice assumes a second discursive position within the individual which answers the question. Lerman (1994b) suggests that another critical voice must come from another source, such as peers or literature.

For the most part constructivism is seen as a theory of learning and knowing and appropriate teaching is understood as that which encourages rich constructions by students. Simon (1995), however, sets out a description of constructivist teaching in terms of teachers' hypotheses about children conceptual development which are then tested by further interactions. Simon's work suggests that student teachers and practising teachers learn about teaching through the same process of equilibration as children's learning.

Bruner (1986) has contrasted a propositional mode and a narrative mode of knowing. Researchers draw on this approach in examining mathematics teachers' theories and practices (Burton, 1996) although the move to a theory of learning in a narrative frame is not immediately clear.

Learning about teaching mathematics can also be seen as apprenticeship into a community of practice (Lave & Wenger, 1991). In many ways it may be more appropriate to conceive of learning about teaching as better described in terms of legitimate peripheral participation than learning mathematics in the school classroom. After all children do not choose to go to school, as people do to a large extent when participating in employment practices such as teaching; thus goals and needs are quite different. It is also perhaps inappropriate to describe the practice of school mathematics as leading to school children moving from the periphery to the centre of participation, and becoming the 'masters' whereas it can be applied to learning about teaching.

Activity theory offers a framework that has been used extensively to study teachers' learning. In an attempt to engage student teachers, in the final year of their course, with their still unchallenged assumptions about the role of the teacher, Crawford & Deer (1993) devised an activity in which the students had to work in groups to develop a programme of mathematics which was centred on the children's environment, rather than a prescribed syllabus. The students found this very hard and experienced: "initial ecstasy, shock of recognition, crisis, realism and commitment" (p. 116). The outcome was at least a recognition by the students of having a wider range of skills upon which to draw and in many cases new-found confidence in their ability to create "a very different learning environment ... from the one that they had experienced themselves" (p. 118). Elsewhere Crawford writes:

> The course was designed to create a "zone of proximal development" for student teachers as a way of expanding their knowledge of the dialectic process of

teaching and learning through conscious experience of the process. They were engaged in a learning activity. (Crawford, 1994 p. 6)

Each of these perspectives potentially offers a suitable theoretical framework for learning within which to structure research on mathematics teacher education. **The main theoretical argument of this paper is that researchers' choices of theoretical frameworks for teachers' learning are affected by all sorts of factors, including personal commitments to particular theories, what has served the researcher well, etc. Those choices should be made explicit in research and the research methodologies used and results claimed justified within those frameworks.** The research described here draws on activity theory. I choose the work of Vygotsky and followers because, in my view, it comprises at least three important factors: first it offers a coherent single framework for learning throughout life that applies to young children and equally to mature adults; second it attempts to integrate affect and cognition in focusing on meaning as its unit of analysis; and thirdly it offers a method for rooting knowledge and action in socio-historical-cultural settings. The classroom is a complex site of political and social influences, socio-cultural interactions and multiple positionings involving class, gender, ethnicity, teacher-student relations etc. in which power and knowledge are situated. Vygotsky's psychological theories enable the researcher to accommodate these elements into the analysis.

The study

Two small-scale research studies were carried out in October/November 1996. The first study was with a group of pre-service primary teachers in their first year of the course, during a mathematics class. The aim of the unit is for students to study mathematics at their own level. They are required by UK law to have a minimum qualification in mathematics of a grade C pass at the national examinations at age 16 (or the equivalent). These students have a slightly higher level of certification and have chosen to make a special study of mathematics, but in general their mathematical knowledge and confidence are still not very high. They are all mature students. The second study was with a group of experienced teachers beginning a taught master's degree in mathematics education. The aim of their first unit is to examine psychological and sociological theories of teaching and learning mathematics. Some are secondary mathematics teachers, others are primary teachers with a particular interest in mathematics, and all have taken some course in educational studies, although possibly many years in the past.

The aim of the research was to engage students in their zone of proximal development (zpd) and to draw on their personal goals and needs. In order to focus on the latter the research was designed to confront individual interests in the students' learning. Following the approach of Crawford & Deer (1993) both groups of students were asked to identify areas of study which they wished to learn about but felt that they knew very little. In the first group this was to be a topic in

mathematics and in the second an issue of teaching and learning. They were then given a period of two weeks in which to prepare a short presentation for their colleagues on that topic. The presentation was not part of the formal assessment for either group. The intention of the presentation task was to provide a learning activity in which the students would be 'forced' to contrast their naive notions of their area of study with the literature they had to read to prepare the presentation, which offered 'scientific' notions. The research was concerned with the process the students were going through in their learning and its effects on them. Thus they were asked to keep a diary of their reactions to being given the task, their feelings during their reading and preparing the presentation, their feelings during the presentation and in particular their reflections after the presentation. The students' writings offer authentic and coherent accounts of their experience and feelings. In my view this was the most, perhaps the only, appropriate research method. Learning in the zpd is usually examined in the context of peers working together and slightly less frequently in the context of the teacher and students working together. This research was designed to examine learning in the zpd for each individual student in interaction with texts of their own choosing.

If the students had accepted the task but ultimately come to feel that they had been unable to learn enough to make a presentation this would have made the research hypotheses invalid. It would have suggested that the activity was not a successful learning activity perhaps because the forced choice did not put students into their zpd or perhaps because texts may not function in the same way as peers or a teacher for learning in the zpd. It would not, however, have made the Vygotskian perspective and activity theory invalid for this researcher.

I will present here some extracts from the students' writings and this will be followed by an analysis of their learning and of the research.

<u>Pre-service primary teachers - mathematics</u>

A. (matrices) *I have done them during my school times but I never understood anything... Had I known that I have ... to come back to lecture it to others I wouldn't have mentioned that I have difficulties... I still don't understand. <u>What do I do now?</u> I'm beginning to understand... I have to ... do a few examples and see if I can understand more... After presenting the seminar I thought it wasn't as bad as I thought it was going to be.* (emphasis in original)

P. (calculus) *When I was asked to carry out this mission I was not happy to say the least... Both books required a good knowledge of Algebra, which I am a bit rusty on at the moment... At this stage I began to feel frustrated and pressured... The basis of the calculus began to become apparent to me so I commenced writing some notes on Functions... I believe that if I had teamed up with another member of the group I would have got further.*

S. (bearings) *I can work through problems but I do not understand why. I feel really angry about this and the way I have been taught. How much more of*

mathematics do I not understand?... I felt happy that it was over but disappointed I could not explain it all. I will be looking into history for my own benefit. I really learnt a lot from this about how you should fully understand something before teaching it.

In-service master's - issues in mathematics education

J. (family influences) *It is always interesting to be faced with things you do not know... After having prepared that subject and presented it to the rest of the group I felt that some things were missing. Things I had not thought about before.*

E. (language and mathematics) *At first I felt quite helpless and inadequate. How on earth was I supposed to find answers to a question which I was asking precisely because I found it difficult to answer?... Some background reading helped give me confidence - I felt like I had some 'official' back-up... Once I had rephrased my question and got thinking, reading and writing, I actually enjoyed the task... The enormous complexity of the psychology behind my question struck me, as well as the impossibility of ever knowing that the question has been answered.*

Analysis

The sessions during which the learning activity was set began with a review, in the first case of topics in mathematics that the students wanted to learn and in the second of issues in teaching and learning. They were then set the task. The students in both contexts demonstrated a certain initial shock and reluctance to engage with the task. As is evident from their comments they had no warning that I would come back to them with the topics they had proposed. Whilst this was rather hard on them, I explained that this was by way of an experiment and we would discuss the outcome of the experiment together afterwards. In all cases the students appear to have found a way to overcome the obstacle that their chosen topic had presented and to have been able to offer an appropriate presentation. In the first case the students were revisiting mathematical topics that they had encountered before, at school, and had failed to reach a position in which they felt comfortable with their knowledge and understanding of the topic. The task was perhaps particularly successful for them in that they all felt that they had learnt from their preparations and now felt that they understood their topics much better than before. I believe this to be a result of the learning activity which included as an essential element the requirement to make a public presentation to their colleagues, a requirement which they had accepted although it was not an assessed part of the course. In both cases the choice of topic had been made by the students from their own interests and goals and this too was an important feature of their subsequent engagement and success. Student comments suggest that it may be interesting to give them the opportunity on a future occasion to make paired presentations if they wish, although the topics chosen would have to be of common interest. The whole activity put them into their zpd, leading to each of them confronting their naive notions and partial understandings with written knowledge from texts. The problem that Crawford &

Deer (1993) were facing was that students in initial teacher education programmes students often fail to confront their naive notions of teaching mathematics. They suggest that despite writing good theoretical essays on teaching and learning during the course the students often begin teaching in the same way that they would have without attending the course. I am not claiming here that the students learnt things that they might not learn in other ways; I want instead to illustrate the fruitful outcome of this learning activity.

The research method assumes that students choosing topics which they had stated that they wanted to learn about would engage them in their zpd. Whilst this requires further justification the outcome suggests this assumption is correct, given the theoretical framework. There seems no doubt that learning had taken place for each person, according to their written accounts. Informal subsequent discussion revealed that in many cases they felt pleased that they had been able to learn something difficult, alone, from a textbook.

Ending

I have attempted to make explicit the theory of learning which I have chosen and to set the research method and analysis within it. The more general concern of this paper is that mathematics teacher education research needs to make explicit its theoretical framework, whatever that is.

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On the Difficulties met by Pupils in Learning Direct Plane Isometries (*)

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We report some results of a research of didactical innovation on direct plane isometries, realized using the computer, focusing on some difficulties met by pupils. The hypothesis of the research was that the dynamic visualization of the action of a geometrical transformation on various figures, not necessarily convex or limited, and on sets of loose points, can lead the pupils to: a) construct the appropriate mental images for overcoming well known difficulties met by them in realizing the correspondent of figures according to a certain isometry; b) achieve the meaning of invariant and unite element in a transformation and arrive at the concept of this as a correspondence between points of the plane. The research has evidenciated that, even if resorting to the visualizations on the computer has allowed the pupils to achieve a good interiorisation of the vision of classes of figures united by translation or rotation, several of them met conflicts in representing the correspondent of a translation of a right line according to a vector parallel to it or in realizing the correspondence of a certain couple of figures, such as a circle and an a right line tangent to it, according to particular translations or rotations. Moreover, as to the extension of the transformation to the whole plain, several pupils showed the persitence of a local vision .

Introduction

Geometrical transformations have been inserted in the syllabuses of many countries in the sixties, with a view which reflects the structuralist ideas of that time. But the historical and cultural reasons of this choice are very little known among the teachers and, consequently, this topic is considered by the most part of them as foreign to geometry and often its teaching is reduced to a flat and shorter transmission of the proposals of the textbook (Malara, 1991). For overcoming this situation we have faced with and for the teachers of our research group the problem of the teaching of geometrical transformations and realized various experimental researches framed in organic way in a project for the teaching of geometry for pupils aged 11-14 (Malara 1994, Pincella and Malara 1995, Iaderosa and Malara 1994, 1995).

The researches on the side of the learning of plane isometries are few (Hart 1981, Nasser *et. al.* 1995, Gallou-Domiel 1987, Jaime and Gutierrez 1989, Bartolini Bussi and Mariotti 1996), many of these regard only the axial symmetry and are not centered on the difficulties met by the pupils. As to this last point the study of reference remains the classical one led by Hart (1981), but in it only the difficulties met by the pupils in the construction of the correspondent of a little flag according to particular axial symmetries or rotations are considered. Few or nothing is known as to the ability of the pupils in coordinating the construction of the correspondents of couple of figures, with (or not) some elements in common.

In some more recent researches the positive influence of the computer for the learning of geometrical transformations is stressed (see for instance Clement and Batista, 1992) and our research put itself in this stream. It concerns some results, from the point of view of the difficulties met by the pupils, of a wide and in progress research of didactical innovation on the plane isometries, centered on the visualization through the computer (reserch sketched in Malara 1995b). The results were obtained from a sample of two classes (teacher Rosa Iaderosa), involving 45 pupils aged 12-13, for a period of three mounths.

In the research, analysing behaviour, answers and productions of the pupils, we investigate on the following hypotheses:

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- whether visualising through the computer the effects on various figures of the different isometric transformations promotes the formation of appropriate mental images and can help the pupils overcome well-known difficulties, so that a more correct conceptualization can be produced;
- whether leading the pupils to observe the things that change or keep on many classes of figures, not necessarily limited (polygons, circles, poligonals lines and straight lines) in different positions, lead the pupils to understand the concept of invariant and to see a figure transformed in itself by a certain isometry as a privileged figure in comparison with the others as to this transformation;
- whether and to what extent it is possible to overcome the concept of transformation as action on a figure passing to the concept of transformation as correspondence among all the points of the plane.

These hypotheses were induced from the consideration that a didactical itinerary of static type, that is based only on the images reproduced on the textbook, strengthens the pupils' tendency to conceiving a transformation only as action on close figures or, still more, on the outline of such figures with which the pupils identify the same figure. This way the idea of the extension of a transformation to not limited figures (straight lines, strips or other analogous parts of the plane) and of the contemporary action on all the points of the plane escapes to the pupils.

On the other side, using the computer for giving a dynamic vision of the transformation, even if initially the aspect of "phisical movement" is privileged, enables the pupils to grasp more easily and to elaborate in the most autonomous way possible the aspects which characterize each type of isometry and the concept of invariant. In particular, it is possible to visualize and favour the conceptualization of unite point and unite figure.

In the following paragraphs we trace the main steps of the didactical intervention in classroom for direct plane isometries and the role of the computer, then we describe some of the worksheets on which the pupils have worked and we analyse in details the difficulties met by the pupils for each of them, finally we briefly sketch the general results of our research evidenciating some of its problematic aspects.

The classroom didactical itinerary for each of the isometries can be synthetized as follows:
- moments of visualization through the computer;
- collection of observations and previsions made by the pupils;
- use of worksheets aiming at the construction of concepts and at the evidenciation of possible conflicts between mental images and concepts involved (worksheets elaborated considering the didactical knots and learning difficulties foreseen in an a-priori analysis);
- collective discussion for analyzing with the pupils the results of their work and recovering mistakes and difficulties through the socialization of the cognitions;
- use of worksheets for verifying the interiorization of the activities of visualization, discussion and reflection made (worksheets created on the basis of the emerged difficulties).

An important cultural and didactical choice in this itinerary has been that of the releasing, in the proposed activities, from the metric aspect and from the cartesian reference. We have privileged the use of white sheets and the contruction by straightedge and compasses, also for reflecting on the construction of figures with such tools. or with the computer. We have presented, in the following order, translations, rotations and axial symmetries.

The leitmotiv, in the planning and in the developement of the didactical itinerary on the isometries, has been the association between translation and rotation for analogy between the moving along a straight line and the moving along a circumference. Such analogy has been reinforced also by highlighting, thanks to the visualization, the association between elements united by translation, such as the straigth lines parallel to the vector of translation, and elements united by rotation, such as the circumferences concentrical to the centre of rotation.

3. The worksheets

Here we limit ourselves to describing six worksheets, two specifically devoted to translation, two to rotation and two to both of them.

The two worksheets concerning translation place themselves in a central phase of the didactical itinerary relative to this transformation. In the first one, a couple of little flags corresponding for translation is shown in four different positions. In each of the situations proposed, the little flags are partially superimposed along a side. The pupils are asked to individuate the vector of translation for which the second flag results translated as to the first and then, after they lengthen the couples of correspondent sides through colour pencils (using different colours for each side), they have to express what they think is going to happen for the correspondent couples of straight lines. The objectives of this worksheet regard the control of the conceptualization of the free vector of translation, on which they have already worked, the passage of the observation from segments (limited) to the straight lines of belonging (unlimited) and the intuition of the existence, in a translation, of united straight lines with the same direction of the vector. The foreseen difficulties regard the representation of the vector of translation (applied or free), the inability to distinguish between segments and straight lines of belonging and to recognize the parallelism in the case of superimposed straight lines.

In the second worksheet it is asked, in four different cases, to translate according to a certain vector a given straight line of which two points are evidenciated. Two of the considered cases concern the position of the straight line, respectively horizontal and oblique, the other two concern the direction of the vector as to that of the straight line, respectively parallel and non-parallel. Objective of the worksheet is to verify whether the activities of visualization through the computer, aimed at the pupils' grasping the fact that the straight lines having the direction of the vector of translation are transformed into themselselves by the translation, have brought to such a conceptualization in the pupils. The foreseen difficulties concern the possible conflict between the direction of the vector of translation and the direction of the straight line during the realization of the translation of the straight line and the conceptualization of the fact that a point of the straight line is carried to a point of the same line in the case of the vector parallel to it and also of the sliding of the straight line on itself.

The two worksheets devoted to rotation are posed in the initial phase of the itinerary on this transformation, after a first visualization through the computer of the effects of various rotations on little flags and other limited figures. In both worksheets the visualization through the computer of a little triangular flag is represented, and of the result of its rotation about a point outside it. In the representation are evidenciated four privileged points of the two little flags, constituting the foot and the vertices of the triangle, moreover both feet of the little flags appear connected to the centre of rotation.

The third worksheet shows the outline.of an arc of circumference joining a vertex of the triangle with its correspondent, whereas the fourth -more complex to be read- shows all the arcs of circumference joining respectively the four privileged points with their correspondent and the pairs of radiums joining the corresponding points with the centre of rotation. Aims of the third worksheet are the guided recognition of the characteristic elements of a rotation in a plane and a preliminary inquiry on the pupils'ability of spotting out by themselves some invariant elements. The fourth worksheet is more specifically aimed at the explicitation of the procedure followed in order to rotate a figure on the plane, at the observation of the invariance of the angle individuated by the radiums linking pairs of corresponding points with the centre of rotation. The main difficulty quite consists in recognising the invariance of the angle of rotation as opposed to the variance of the subtendent arc (typical mistake is to consider the width of the angle as dependent on the length of the segments representing the halflines which delimitate it, see for instance Krainer 1991). There is the further difficulty of coordination between the global vision of the figure and the various parts of it, because of the highlighted points.

The remaining worksheets belong to a final test on the two isometries.

The fifth and the sixth are very complex (from a conceptual and a representative point of view). In the fifth we present six situations: four dedicated to the straight line and two to the circle. Precisely, given a straight line of which no point is indicated, it is asked to make: a) its translation as to a vector eiter non-parallel or parallel to it; b) its rotation of 90° clockwise either about a point of its, or about a point outside it; given a circle it is asked to carry out its rotation of 90° clockwise either about its centre, or about a point outside it. The aim is to verify whether the difficulties previously highlighted have been overcome, such as to imagine the translated of a straight line in the case of a vector parallel to it, or the construction of the rotated of a line or of a circle. Such worksheet presents various difficulties connected to the absence of privileged points on the line and on the circle, to the construction of the correspondent of a point as to a 90° clockwise rotation and to imagining the effects on the figures of a rotation about a centre outside them. These worksheets have been also conceived for facilitating the pupils to face the sixth worksheet, separating the difficulties that it presents.

In fact the sixth worksheet is absolutely the most delicate. It shows a figure built from a circle and a straight line tangential to it (the point of tangence is evidenciated). Two cases are presented: in the first it is asked to realize the translation of the figure according to a vector parallel to the tangential line, in the second the request was to realize the 90° clockwise rotation of the same figure about the centre of the circle. In both cases it was also asked to find out the possible united points, straight lines or circumferences and eventually the comparison between the two situations. The aim of the worksheet is to verify the ability of seeing the transformation of compound, not-limited figures, of recognizing in the various cases united elements, analogies and differences, to inquire into the conceptualizations promoted by computer visualization (such as the invariance of sheaves of straight lines individuated by a given straight line and of stripes of plane or families of circumferences concentric to a given one, and of circles). The difficulties presented by this worksheet are manifold and at various levels; there is the difficulty of: a) imagining and building the result of the transformation of a single element of the figure (point, straight line, circumference) in the two transformations; b) coordinating the various elements transformed either in the case of translation or in the case of rotation, for example realize that in order to individuate the transformed figure in the assigned rotation it is enough to find out the result of the transformation of the point of tangence, whereas in the case of the given translation it is enough to individuate the result of the translation of the centre of the circle; c) seeing the tangent line as a united figure in the case of the translation and, which is more difficult, also all the straight lines parallel to it; d) conceive the circumference and all those concentric to it as united figures in the rotation about its centre; e) recognize the centre of rotation as the unique united point in the rotation. The request of comparison between the two situations compelled then the metacognitive control on what was learnt.

Beyond the specific difficulties there are also difficulties of general nature such as those originating from the use of instruments (straightedge and compasses) and those linked to the necessity of expressing observations and considerations.

Difficulties detected in the pupils

The worksheets we have described, allowed us to focus on the learning difficulties, some of which had not been foreseen in an a-priori analysis. For reasons of room here we limit ourselves to describing the main difficulties detected.

As regards specifically translation, we report some productions of the pupils testifying:

- <u>Table 1</u> the difficulties to visualize the vector which generates the translation acting on the flag as indicated in the worksheet and its action on other elements which are not represented on the figure. In particular we can observe some conflicts between the direction of the translation and the one of the flagpole (see fig.1a), the inability to represent the right lenght of the vector of translation (see fig.1b) and the inability to extend the result of the translation to the whole line to which a segment belongs (see also fig.1c);

Table 1

Examples of difficulties in recognizing vectors of translations and correspondent elements in a translation

fig.1a

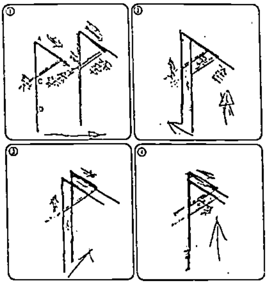


fig.1b

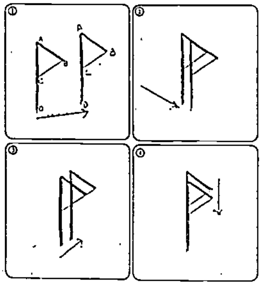


fig.1c

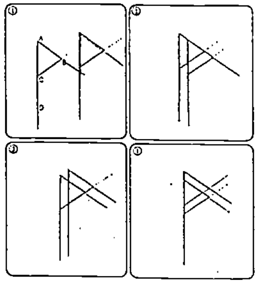


Table 2

Examples of difficulties in translating straight lines

fig.1a

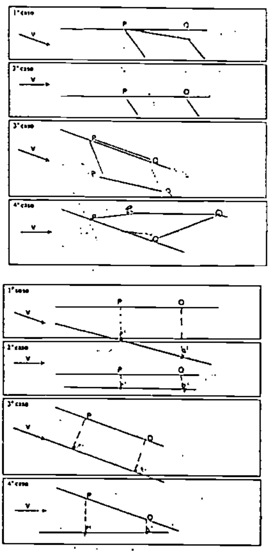


fig.1b

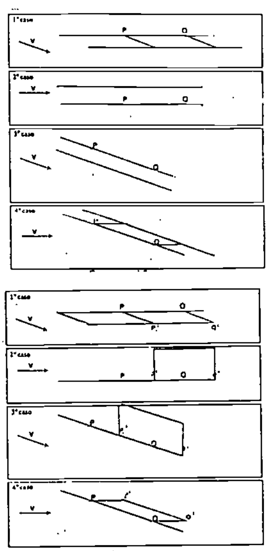
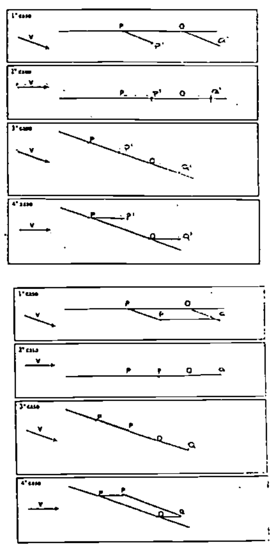


fig.1c



- Table 2 the difficulties to realize the correspondent to a given straight line according to an assigned vector (see fig.2a), specially in the case of parallelism between the vector of translation and to the direction.of the line (see fig.2b) and the inability to represent the correspondent of the straight line after the construction of the correspondents of two its points (see fig.3b).

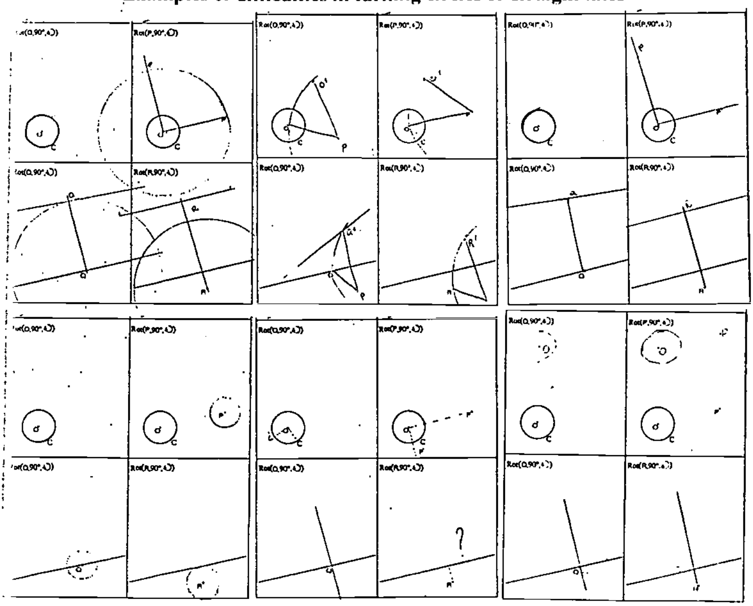
In general we can say that in the various stages of the study, the following aspects have emerged in the pupils:

- the inability of seeing the action of a translation from a segment to the straight line to which the segment belongs;
- the difficulty of spotting out and representing the free vector of a translation from the vector applied and of conceiving in abstracion, given a vector, the translation associated to it;
- the inability of *traslating a straight line* (as we have seen several productions testify the evident and lasting conflict between the direction of the vector of the translation and that of the straight lines on which it acts);
- the difficulty of conceiving the *simultaneous shift* of position of the different points of the plane (it has emerged in particular through another worksheet, conveived for testing the ability to visualize correctly the action of a translation in a situation of distraction, where the pupils had to realize the translation of a square, not in a privileged configuration, according to a vector equipollent to an its side: some pupils have evidenciated a conflict between "point and position of a point", saying united a point "because it overlaps to the correspondent").

Concerning rotation, at the beginning of the study we faced in many pupils the typical wrong association of the width of the angle of rotation to the length of the arcs connecting couples of correspondent points or better to the length of the radiums of such arcs, which did not allow them to seize the invariance of the angle itself.

Table 3

Examples of difficulties in turning circles or straight lines



Another typical difficulty linked to rotation, which makes the pupils interpret the effect on a figure of a rotation according to a point outside it as the result of the composition of a rotation about a privileged point of it and of an opportune translation, was overcome by many pupils: thanks to the visualization on computer it was possible to induce in the pupils the mental image of a figure rotated about a centre external to it.

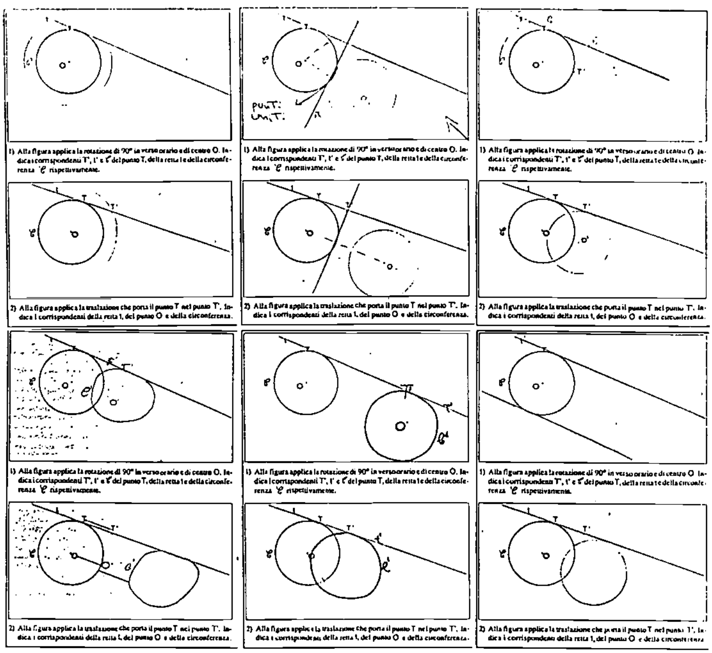
There is however the persistence of the difficulty, assigned a couple of figures corresponding for rotation, of individuating the centre of rotation when it is not in privileged position to them. In some productions we found the prevalence, in the pupils, of visualizations rather than the constructions with ruler and compasses they had learned: the pupils traced all the arcs connecting certain points of the figure with their correspondents and then proceeded by intuition to the individuation of the centre of rotation as the point to which the different radiums of the arcs must converge.

The productions of the pupils on the last two worksheets, concerning either translation or rotation, are extremely interesting. Some productions regarding the fifth worksheet (see table 3) show evident difficulties, even after repeated experiences of visualization: we observed a decrease in the quality of the pupils' performance when they were asked, given a vector, to traslate a given straight line on which no point was evidenciated, or to rotate a circle through 90° about its centre or about a point outside.

However the most difficulties appear in reference to the sixth worksheet where the pupils had to control the simultaneous action of a traslation or rotation on a circle-straight line couple: we have observed that only the thirty percent of the pupils have answered correctly.

Table 4

Examples of difficulties in turning and translating a cirle and a straight line tangent to it



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In table 4 are reported some protocols, through which it is possible to observe the conflict generated in the pupils in facing either translation or rotation, even if most mistakes happen with reference to the rotation. We have to underline that, on applying the rotation, none of the pupils recognises that the centre of rotation is an united point in the transformation.

Brief conclusive remarks

Before finishing we wish to underline that, in spite of the difficulties detected, in general the resort to the visualizations at the computer has allowed the pupils to achieve a good interiorisation of the vision of classes of figures united by translation or rotation and to gain the concept of united figure as to a given isometry, which is essential for giving sense to the study of the problem of characterizing – of a given figure – the isometries as to which it is united. Moreover, in the majority of the pupils a good conceptualisation of trasformations as correspondences has appeared, despite the sequenciality of constructing figures in the visualisation suggested, which we thought might hinder the conception of the simultaneity of the act of transformation on the figures themselves. Problematic has instead been the extention of the transformation to the whole plain, due in our opinion to – beyond the limits of the tool of representation – the persistence in several pupils of a local vision of the facts observed, which would extend according to the cases considered but always far from being global, difficult to achieve owing possibly to the unripeness, at this level of schooling, even of the concepts of straight line and plane.

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THE DIALECTIC RELATIONSHIPS BETWEEN JUDGMENTAL SITUATIONS OF VISUAL ESTIMATION AND PROPORTIONAL REASONING.

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In the following paper we describe a sequence of episodes in which subjects in different ages were involved in judgment of visual estimation of discrete quantities. The analysis of these episodes seems to demonstrate two opposing trends: In the first, situations of visual estimation judgment seem to push young children, (8 to 10 years old) towards proportional reasoning. In the second, the same situations may push subjects (10-11 years old children, student-teachers and teachers) away from a mathematical proportional reasoning towards what might be called "situational reasoning".

Introduction

Estimation is a compound subject involving conceptual components: (i) the recognition that approximate numbers, quantities or measurements are used and that an estimate in itself is an approximation, (ii) the recognition that the appropriateness of an estimate depends on the context and on the desired accuracy, and (iii) the acceptance of multiple values as estimates and multiple processes for obtaining estimates as legitimate (Sowder, 1988). These different processes are due to the type of the estimation involved as well as to the estimator him/herself. Another compound issue that is an integral part of estimation takes place when two or more estimates are given and one has to judge whether one estimate is better than the other. For example, the following is one such task, involving computational estimation (Sowder and Markovits, 1990):

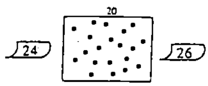
> *"If 34 x 86 is estimated as 30 x 86, then the exact answer is 2924 and the estimate is 2580. The difference between these two numbers is 344. If 496 x 86 is estimated as 500 x 86, the exact answer is 42,565, the estimate is 43,000 and the difference is again 344. Which of these would you choose: (a) the first estimate is better; (b) they are the same; (c) the second estimate is better."* (p. 326).

Mathematical judgment in such situations is based on the ability to see the above differences as relative errors, meaning that the second estimate is better, since 344 out off 42,565 is a smaller error than 344 out off 2580. The understanding and the ability to use relative errors depend of course on the ability to incorporate proportional reasoning.

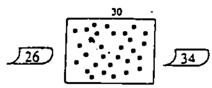
Here one may ask two questions: (i) What is the influence that the context of an estimation judgment task might have on understanding and use of proportional reasoning? (ii) In what ways subjects make use of proportional reasoning in estimation judgment tasks?

We use visual estimation of discrete quantities as a context for investigating the above questions. In a previous study (Hershkowitz and Markovits, 1994; Markovits and Hershkowitz in press), we started our investigations with 3rd graders (stage 1), here we describe the second and third stages of our research on situations of visual estimation judgment. In all three stages we presented subjects, via interviews, with a series of visual estimation judgment tasks. Some of the judgment tasks are shown below:

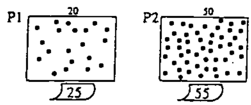
1) Noa and Gal were shown this dot picture for a short period of time and asked how many dots they saw. We know that there are 20 dots in the picture but the children, of course, didn't know it. Noa said that there are 24 dots, Gal said 26 dots. Did one of them give a better answer than the other or were both answers equally good?



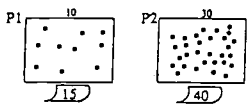
2) A picture, with 30 dots, was shown to Noa and Gal. Gal said that there are 34 dots, Noa said 26. Same question as 1.



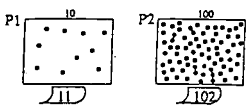
3) Two dot pictures were shown to Noa; 20 in picture 1 (P1) and 50 in picture 2 (P2). Noa said 25 dots in P1 and 55 in P2. Was one of the answers better than the other, or were both answers equally good?



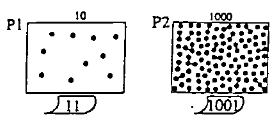
4) Two dot pictures were shown to Gal; 10 dots in P1 and 30 in P2. Gal said 15 in P1 and 40 in P2. Same question as in 3.



5) Two dot pictures were shown to Noa; 10 in P1 and 100 in P2. Noa said that there are 11 in P1 and 102 in P2. Same question as in 3.*



*7) Two dot pictures were shown to Gal; 10 in P1 and 1000** in P2. Gal said 11 for P1 and 1001 for P2. Same question as in 3.*



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Most of the above judgment situations, are **proportional reasoning** situations of comparison type. The situational elements of our proportional reasoning situations, have well-distinguished characteristics:

- Each ratio in the proportion expresses two different variables (the estimate or the error and the quantity to be estimated) which have the same measure (number of dots) and unit (dot);
- The absolute error calculation is an **additive** one. This may cause a delay in the development of the child's multiplicative reasoning;
- The size of the quantities involved (We use "A" and "C" to denote the estimates for P1 and P2 , "B" and "D" for the quantity to be estimated in P1 and P2 respectively.) are quite big, while in many classical situations the numbers are small;
- The two quantities to be estimated are of different order of magnitude (ranging from 5 to 1000 dots), while in the popular research situations, such as the example of Mr. Short and Mr. Tall (Karplus et al., 1974), the numerical values of the variable are of the same order;
- The quantities are discrete, which means that there is no absolute error less than one. etc;
- In addition, in our case the two ratios to be compared differ dramatically one from the other, in contrast to ratios involved in classical proportional tasks. This characteristic may emphasize the need to make the comparison between the ratios, rather than the comparison between one variable only, because the subjects may **see** and **feel** that 1 out of 10 is really different from 1 out of 1000.

In the following we briefly describe results and conclusions from stage 1 and move to describe the results of stage 2 and 3. At the end we discuss the main trends govern the three stages.

Stage I: Towards proportional thinking via judgment of visual estimation

In this stage we first presented twelve third grade students, via interviews, with several visual estimation tasks, and then with a series of the above visual estimation judgment tasks and a few more. We found that all children in this group, except one, started from a pure additive judgment. But, half of them were pushed by the visual component which is integrated with all the above characteristics of the judgment situations, towards qualitative proportional reasoning. These children apply considerations of easy/difficult in their judgment situations, which are quite natural when the quantities became quite big and on different scales. The easy/difficult

considerations led them to see the error in relation to the quantity to be estimate, rather then as absolute difference.

We therefore decided to trace the development of one child (Amir) over a period of three years to see if and how his thinking processes progress in proportional situations of judgment in visual estimation.

Stage II: A longitudinal case study of the development of judgment reasoning

Amir was interviewed five times from the age of eight to the age of eleven. In each interview Amir was first presented with several visual estimation tasks, and then with a series of visual estimation judgment tasks. In the following we present in brief the development of Amir's reasoning over three years.

First interview - age of eight, second grade.

During the entire interview Amir showed an "additive" behavior. For example, in where A=11, B=10, C=101 and D=100, he said that both answers are the same. The interviewer raised the issue of easy/difficult, but Amir was captured by the mathematical fact that 1=1. He answered that *"she missed by the **difference** of 1 in both cases"*, to the interviewer's question *"**from** how many dots did she miss by 1 in each case?"*

Second interview - age of eight and a half, third grade.

Amir started to use arguments of easy/difficult. He did not use the phrase "**difference**" anymore, but rather the phrase "**out of**". He argued that more dots in the quantity to be estimated means "more difficult" and hence a better answer. He systematically based his responses on the above idea and said, for example (when presented with the same task as in the first interview), that P2 is better even if the estimate (C) would be 105, 120 and even 150. Only when we continued to what he considered as very extreme cases he started to consider also the relationships between the error and the quantity to be estimated, and said: *"In the case of 200, P1 is better, because 200 is already 100 too much."*

Third interview - age of nine, third grade.

Amir continued to use easy/difficult arguments. On the whole, he continued to think that more dots means more difficult, thus P2 is better. But he used also mathematical proportion for the first time. For example, in the task where A=11, B=10, C=120 and D=100, Amir said that:

> *"P1 is better. She missed by 1. In P2 she missed by 20, and it is too much. It is 1/5 of the number of dots, of a hundred, although in P2 there are more dots."*

We continued to change quantities, and Amir gave his judgment for each new situation. Sometimes he moved to easy/difficult considerations. He also added that:

"there is a certain point for which one answer is better than the other, a certain point that makes sense."

Fourth interview - age of ten, fourth grade.

Amir exhibited proportional reasoning. To overcome the difficulty in comparing two fractions (ratios), Amir calculated the value of what he called the *"certain point"* which is actually finding a value for C (the forth value in a proportion) that makes the two ratios equal. Then he compared this value to the number given as C in the situation. He was able to use this type of calculation in all the judgment tasks we presented to him. For example in the task where A= 6, B=5, C=110 and D=100, Amir calculated the *"certain point"* - 120, saying that if C is less than 120, P2 is better, but if C is more than 120, then P1 is better. But then he added:

"On a second thought, P2 is better, because P1 is out of 5. You can have one thought, for example, you can multiply and divide, but you can have a second thought ..... that P2 is better, because it is out of 100."

From this interview it is clear that Amir is able to judge the situation using proportion, but it seems that he is not sure whether it is appropriate to use proportional calculations in the situation presented to him, or easy/difficult considerations.

Fifth interview - age of eleven, fifth grade.

Amir demonstrated quantitative proportional reasoning in each of the situations presented to him. He immediately calculated the "certain point" without even being asked. But, again he expressed the dualism of the two parallel lines of reasoning; - The mathematical line, in which one is supposed to calculate, and the situational line. Amir used the same way of calculation as in the fourth interview. He performed the calculations quickly and was very confident. When asked what about 10 dots in P1 where the child said 11, and 100 dots in P2 where the child said 105, the following conversation took place:

Amir: *If he would have said 110, both answers would be the same. But still, P2 is much more difficult, there are more dots, the quantity is much larger. Actually, there is a theory based on calculations, that says that if there are 110 and 11, it is the same. But there are exceptions; even if he would say 150, it would be great. From the mathematical point of view, it is equal. But the eye cannot get the 100 dots, so it is not exactly the same.*

Interviewer: *What is the mistake he could do in P2?*

Amir: *A very large one, a few tens. Even more than 50.*

Interviewer: *And what about the mathematical calculation?*

Amir: *The mathematical calculation does not always apply because the eye is not mathematics. If he was allowed to look at P2 ten times longer than he was*

allowed for P1, than the mathematics would work. There is mathematics and there is reality, unless you allow a different amount of time.

Interviewer: *Isn't it that the mathematics has to be taken into account?*

Amir: *I am not going to explain to you in detail how the eye processes information. But when the eye gets the dots it is difficult to calculate what the number is at the same time. The larger the number, the more difficult it is to understand this number. But this does not work according to mathematics. Our body is not managed by mathematics.*

Over the three year course of the study it was possible to observe two changes in Amir. First of all, it is clear that his thinking has changed with time with respect to the judgment situations. His original thinking classified as additive reasoning, progressed to complete proportional reasoning with time. This was a gradual process whose stages could be traced by the interviews. In addition, Amir's attitude to mathematics and numbers has also changed. In the first interview, the given numbers overcame reality. Amir agreed that P2 was more difficult, but the mathematical fact that 1 = 1 , was much stronger. In the last interview, Amir claimed that mathematics does not always work, and his situational or "real world" considerations overcame mathematics considerations.

Stage III: Proportional reasoning considerations and/or situational considerations in the adults' judgment of visual estimation

In order to deepen our study of judgment situations in the context of visual estimation, we gave a series of such tasks to adults. Ten groups went through this activity. The participants in each group were in service and pre service teachers. The tasks were presented one at a time to the above groups and the participants were asked to write down their answers.

From observing the teachers' work and the analysis of their responses, it became pretty clear that in each group there were two different approaches. The first approach consisted of using proportional reasoning and hence proportional calculations only. The second approach consisted of using proportional calculations up to a certain point, and then abandoning them in favor of situational considerations. These teachers used proportional reasoning when the four different quantities were small and with the same order of magnitude. When the two ratios were of different order of magnitude, they moved towards situational reasoning. This change of strategy was followed by many debates concerning the way they should answer. They asked for more time and it was clear that they were bothered by these tasks. The questionnaire sessions were <u>always</u> followed by a very heated debate among the teachers having different approaches.

As an example we bring here a group of ten pre service and in service teachers. Six teachers calculated the relative errors in all tasks, and relied on these

proportional calculations to decide which answer is better. The other four, used relative error calculations when the numbers of dots in both pictures were close, but used considerations of easy/difficult when the numbers were far apart. For example, one of the teachers, when presented with 5 dots in the first picture where the child said 6 and 100 dots in the second picture where the child said 130, said:

> *"Now the child has an error of 30% in picture 2, but I still think that although the error is big, since there are so many dots it is much more difficult than to say 6 for 5 dots, which is an error of 20%. That's why the answer for P2 is better."*

This teacher continued to use this argument in the task with 10 dots in the first picture where the child said 11, and 1000 dots in the second picture where the child said 1500 but it seemed that she debated the issue with herself:

> *"Even though the relative error in P2 is much bigger than the error in P1, one should not consider the relative errors only. In my opinion, this is true especially when one has to deal with such small numbers as 10, and such big numbers as 1000. That's why it is difficult for me to decide which answer is better. But I still think that the student who was able to estimate the number of dots in the second picture to be around 1000, gave a better answer."*

Immediately after they turned in their questionnaires, the teachers started to argue whether one should use mathematical calculations, or other considerations should be taken into account. The ones that were pro calculations said that since in P2 there are more dots, one is allowed to be off by more dots, while in P1 only by one dot. The others said that one needs others considerations here, not the mathematical ones, since the eye is involved, and you can easily see 5 dots, but the eye cannot capture 100 dots. During this discussion one of the teachers who used calculations was convinced to move to the "opponents" group. But the argument did not end, when the class ended. The teachers continued to argue, trying to convince each other.

Concluding remark

According to Piagetian theory (Inhelder & Piaget, 1958), Amir, in his last interview (the second stage of the study), and teachers (the third stage of the study) were in the last stage of proportional reasoning, the formal operational stage. We saw that at the same time that they could act mathematically, they were bothered by situational considerations, especially when the quantities were far apart. These were considerations of visual nature which in Amir's words belong to the ways in which *"the eye processes information"*, since it takes longer to evaluate pictures with many dots. Amir is intuitively expressing well established research on human perception (Folk et al, 1988). Thus, Amir as well as the teachers have reached the point, in which they had developed a *"criteria by which to judge which of the perspectives is appropriate in a given situation"* (Lamon, 1993). Although this study is not a

longitudinal one (except the three years of Amir), it gives some indications of a developmental process.

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AN ANALYSIS OF THE TEACHER'S ROLE IN GUIDING THE EVOLUTION OF SOCIOMATHEMATICAL NORMS

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The analysis reported in this paper documents the teacher's proactive role in guiding the development of sociomathematical norms. In particular, we will first document how the emergence of the sociomathematical norm of what counts as a different mathematical solution made possible the negotiation of what counts as a sophisticated mathematical solution. We will then document the evolution of what counts as an easy, simple, or clear mathematical contribution from the norms of different and sophisticated mathematical solutions. In the analysis we will also take account of the teacher's learning that occurred as she participated in the negotiations of the sociomathematical norms. The analysis was informed by and builds on Yackel and Cobb's (1996) discussions of sociomathematical norms.

Current American reform efforts aimed at improving students' mathematics education typically characterize the teacher's role as that of a facilitator of learning (National Council of Teachers of Mathematics, 1989, 1991). In such accounts, the teacher is seen to actively guide the development of classroom mathematical practices and individual students' mathematical activity (Cobb, Boufi, McClain, & Whitenack, in press). However, the facilitator metaphor can be interpreted as characterizing the teacher's role in passive terms. In such interpretations, teaching is portrayed in terms of what the teacher does not do when compared with traditional instructional practices. Smith (1996) points out that in the current era of reform, delineating inappropriate teaching practices leads to a sense of loss of efficacy for teachers. Instead, he argues that research needs to offer proactive alternatives that frame a positive vision of reform classrooms. The proactive actions of teachers who view teaching as a problem-solving activity during which they modify their knowledge, instructional practices, and beliefs to resolve situations that they find problematic or surprising in the course of their practice contribute to efforts aimed at improving classroom practice. It therefore seems essential to understand how such teachers proactively support their students' mathematical development.

The purpose of this paper is to document one teacher's role in guiding the development of sociomathematical norms which engendered the mathematical beliefs and values advocated in American reform documents. As a result, this paper should contribute to our understanding of what reform teachers actually do to support their students' mathematical learning. In the course of the analysis, we will document the process by which the sociomathematical norms evolved in one

classroom. In particular, we will first describe how the emergence of the sociomathematical norm of what counts as a *different* mathematical solution made possible the negotiation of what counts as a *sophisticated* mathematical solution. We will then document the evolution of what counts as an *easy, simple*, or *clear* mathematical contribution from the norms of *different* and *sophisticated* mathematical solutions.

The analysis in this paper will build on Yackel and Cobb's (1996) discussion of sociomathematical norms and extends it in two ways that make this paper significant. First, whereas Yackel and Cobb documented the development of sociomathematical norms in a post hoc analysis, this paper focuses on a classroom in which the teacher consciously attempted to guide the development of sociomathematical norms and thus influence her students' beliefs and values about what it means to know and do mathematics. Second, whereas Yackel and Cobb (1996) *identified* particular sociomathematical norms, we document the *process* by which one sociomathematical norm emerged from another in the course of a classroom teaching experiment. Thus, whereas Yackel and Cobb highlighted a phenomenon they considered significant, we analyze the process by which a teacher proactively supported the development of sociomathematical norms.

The sample episodes discussed in this paper are taken from a first-grade classroom with six and seven year old students in which the teacher participated as a collaborating member of a research and development team during a four-month teaching experiment. In the following sections of this paper, we first provide background information about the teacher and her classroom and describe the data corpus. We then document the teacher's proactive role in initiating and guiding the evolution of sociomathematical norms.

Ms. Smith's Classroom

The majority of the eleven girls and seven boys in Ms. Smith's first-grade classroom were from middle or upper middle class American backgrounds. There were no minority students in the classroom, although a small percentage attended the school. Ms. Smith's classroom is of particular interest because an analysis of videorecorded interviews conducted with all students at the beginning and end of the teaching experiment indicated that their mathematical development was substantial. Students who, at the beginning of the year, did not have a way to begin to solve the most elementary kinds of story problems posed with numbers of five or less had, by mid-year, developed relatively sophisticated mental computation strategies for solving a wide range of problems posed with one- and two-digit numbers.

The teacher, Ms. Smith, was a highly motivated and very dedicated teacher in her fourth year in the classroom. She had attempted to reform her practice prior to

our collaboration and voiced frustration with traditional American mathematics textbooks. Although she valued students' ability to communicate, explain, and justify, she indicated that she had previously found it difficult to enact an instructional approach that both met her students' needs and enabled her to achieve her own pedagogical agenda. When we began working with Ms. Smith, it soon became apparent that she continually reflected on and assessed both the instructional activities she used and her own practice. Ms. Smith was seeking guidance with her reform efforts; we were seeking a teacher with whom to collaborate in a developmental or transformational research project.

Data Corpus

Data collected during the four-month teaching experiment consist of daily videotape recordings of 53 mathematics lessons from two cameras. Additional documentation consists of copies of all the students' written work, daily field notes that summarize classroom events, notes from daily debriefing sessions held with Ms. Smith, and videotaped clinical interviews conducted with each student in September, December and January. A method described by Cobb and Whitenack (1996) for conducting longitudinal analyses of videotape sessions guided the analysis. This method is consistent with Glaser and Strauss' (1967) constant comparative methods for conducting ethnographic studies. It involves constantly comparing data as they are analyzed with conjectures and speculations generated thus far in the data analysis. As issues arise while viewing classroom videorecordings, they are documented and clarified through a process of conjecture and refutation.

Sociomathematical Norms

Mathematical Difference

From the beginning of the school year, Ms. Smith encouraged students to offer *different* mathematical solutions during whole class discussions. However, she and the students did not initially appear to have an agreed-upon understanding of what was a difference that made a *mathematical difference*. Further, as Ms. Smith accepted all the students' contributions, the classroom discussions consisted of a sequence of disjoint and sometimes repetitive explanations. From our perspective as observers, there was little reason for students to listen to each others' explanations, and many seemed to be inattentive. After focusing on both the students' activity and the nature of the discussions while viewing classroom videorecordings, Ms. Smith developed a reason and motivation to proactively guide the negotiation of the norm of mathematical difference.

The first occasion when Ms. Smith intervened occurred on September 27. The task involved Ms. Smith showing an arrangement of chips on the overhead projector for two or three seconds and asking the students to tell how many they

saw and to explain how they saw them. In one instance, she showed a row of three chips and a row of two chips beneath. Two students explained that they had seen five as three and two. A third explained he had seen five as four and one. Ms. Smith then asked for a different solution.

Jane: I saw three plus two 'cause. . .

T: (interrupts) Okay, that's the same. . . we've had three plus two. Thanks a lot. We're getting some of the same ways. We're getting some. . . you're telling me some of the ways we've already seen. If you are sure you have another way now I don't mean another way to go 1, 2, 3, 4, (points to the chips). *I don't mean just another way to count but if you grouped them in another way or you saw them in another way that's what will help us* (emphasis added).

In this exchange, Ms. Smith attempted to justify why Jane's solution did not count as different. In doing so, she distinguished between *counting* and *grouping* solutions and tried to clarify that different for her meant grouping the chips in a different way but not counting them in a different order. We should stress that although Ms. Smith was extremely directive in this initial exchange, criteria for what counted as different soon became a topic of genuine negotiation. In addition, we note that Ms. Smith appeared to articulate for herself as well as for students what counted as different as she participated in this negotiation process.

As the semester progressed and the norm of mathematical difference became established, students began to actively think about ways of generating solutions that counted as different. This is illustrated by an episode that occurred on October 20 where a single ten-frame was used to pose problems on the overhead projector. The ten-frame was described to the students as a pumpkin crate and counters were placed in some of the squares to represent designated pumpkins packed in the crate. Ms. Smith showed a single ten-frame on the overhead for two or three seconds with five chips arranged in rows of three and two (see Figure 1).

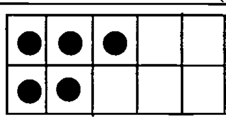

Figure 1. Ten-frame with five chips.

She then asked students to explain how many pumpkins it would take to fill the crate. After Kitty had explained that she saw five as groups of three and two, Dan made the following contribution:

Dan: Um, the way I saw was, I saw four things and another one and I know, okay, five plus five makes ten. . .

T: Okay.

Dan: I had the same theory as Kitty. . .

T: Okay.

Dan: . . . but I did it a different way.

Here Dan both indicated the similarities between his and Kitty's solutions, and justified why his solution was different. In doing so, he was able to judge for himself what was a difference that counted as a mathematical difference in this classroom. In exchanges such as this, the students both acknowledged their obligation to share only different solutions and contributed to the negotiation of the meaning of different in a range of task settings.

Ms. Smith and the students established a basis for communication as they developed a taken-as-shared understanding of mathematical difference. Classroom discussions no longer consisted of a sequence of independent contributions, but instead explanations tended to build on or make reference to others. As a consequence, the students had a reason to actively participate by attempting to understand others' explanations. Further, they became able to judge whether a contribution counted as different as they participated in the negotiation process. This made it possible for them to act as increasingly autonomous members of the classroom community. This devolution of responsibility was pedagogically significant as it constituted a change in the way Ms. Smith perceived her role and that of her students.

<u>Sophisticated Solutions</u>

We have seen that the distinction between counting and grouping solutions emerged relatively early in the school year and became a part of the vocabulary of the classroom. The various counting methods identified by researchers did not count as different in this classroom. Instead, counting was viewed as one way to solve a task that was distinguished from a range of different grouping methods. In the course of classroom discussions, Ms. Smith began to indicate that she particularly valued grouping solutions. Eventually, grouping solutions came to be viewed not only as different but also as more *sophisticated* than counting solutions. Thus, a distinction initially made while negotiating what counted as different subsequently served as a basis for the negotiation of what counted as a sophisticated solution.

As an illustration, consider an episode that occurred on January 10 in which Ms. Smith posed the following task: *Eight cookies are in the cookie jar and I add nine more. How many cookies are there now?*

Jon: See I started with nine and then added the eight.

T: You started with the nine and then you added eight? Did you count up to eight, is that what you did?

Jon: By fours.

T: Okay. Thank you. Jon said he did it by counting. Did someone else figure it out a different way? Bob?

Ms. Smith indicated that counting was a legitimate way to solve the task. Crucially, however, she did not redescribe or notate Jon's solution whereas she did so when each of the other seven students who made contributions in this episode explained thinking strategy or grouping solutions. As an aside, we should stress that Ms. Smith ensured that all students continued to participate by actively soliciting counting solutions from those who she judged were not yet capable of generating thinking strategy solutions. Nonetheless, in treating the two types of solutions differentially, her acts of redescription indicated that grouping solutions were particularly valued.

The emergence of the norm for what counted as a sophisticated solution demonstrates the evolving nature of classroom discourse and offers a counter argument to a belief in a pre-determined set of acceptable responses. This norm had not been discussed by the research team nor had the teacher planned to explicitly encourage grouping solutions in this way. It instead reflected the teacher's understandings of the students' activity in relation to the tasks posed. The establishment of this norm was pedagogically significant in that it enabled students' problem solving efforts to have a sense of directionality. The manner in which Ms. Smith proactively supported its emergence appeared to contribute to her effectiveness in supporting her students' mathematical development. By January, most students used thinking strategies flexibly to solve a range of tasks.

Easy, Simple, or Clear Solutions

As was the case with the negotiation of the norm of sophisticated mathematical explanations and solutions, the norm of an *easy* way to solve tasks also evolved from the prior negotiation of different solutions. Easy and sophisticated designated characteristics of the various solutions that were judged as different. In addition, the notion of a sophisticated or an easy solution also made possible the further elaboration of the norm of mathematical difference. Easy or simple was initially constituted as a characteristic of an arrangement or pattern of items and indicated that it was possible to see how many there were almost immediately without counting. Later, it evolved into a means of discriminating between different types of grouping solutions. Thus, as was the case with sophisticated solutions, the norm of what counted as an easy solution built on the distinction between counting and grouping solutions.

We have already noted that one initial type of instructional activity involved Ms. Smith using an overhead projector to briefly show students arrangements of chips. Later, the students were first shown an organized arrangement (e.g., a domino five pattern) and then a random arrangement containing the same number of chips.

Although the students offered different ways of determining how many they saw, the focus of the discussion soon shifted toward deciding which of the two arrangements was easier to see and why. Most agreed that the organized arrangement was easier because they could readily determine how many chips there were by grouping (e.g., five seen immediately as four and one). It therefore appeared to be taken-as-shared that an organized arrangement was much easier than one that was just "scattered around." Ms. Smith summarized the discussion as follows:

T: An easy way means a way where you don't have to count. Where you don't have to count by ones to find how many. It means you could see a group and you would know how many without counting.

It is important to note that this distinction between easy and hard arose as the students attempted to solve the tasks by grouping. This reflection on their prior activity provided students with the opportunity to not only clarify their understanding of easy in the particular task situation, but to also further elaborate their understanding of different. In doing so, it made it possible for them to not only discriminate between easy and hard patterns, but later to create their own patterns and have the language in which to adequately justify their judgments.

As the semester progressed, easy evolved from a characteristic of tasks into a characteristic of solutions. Ms. Smith could then explicitly evoke it as a criterion of what was valued. Students for their part began to reflect on the explanations of others and judge for themselves whether or not they qualified as easy. This can be seen in an incident that occurred on December 3. Ms. Smith posed the task: *There is fourteen cents in the purse. You spend seven cents. How much is left?* Kitty had solved the task using the arithmetic rack[1] starting with two rows of seven (see Figure 2).

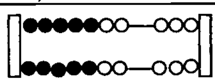

Figure 2. Arithmetic rack showing fourteen cents.

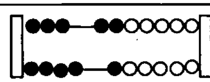

Figure 3. Arithmetic rack after Kitty has taken away seven cents.

She then removed four from one collection of seven and three from the other, leaving three and four respectively (see Figure 3). After she finished, Teri suggested:

Teri: I think I know a way that might be a little easier for Kitty.

T: You think so?

Teri: We know that seven plus seven equals fourteen because we have seven on the top and seven on bottom and (points to rack configured as in Figure 2).

[1] The arithmetic rack is a device composed of two parallel rods. Each rod holds ten beads, five red and five white. Students used the arithmetic rack extensively during the teaching experiment.

. . It might be easier if we just moved one of the sevens on the top or the bottom (points to each group separately).

T: You mean move a whole group of seven altogether?

Teri: (Nods in agreement looking at Kitty.)

At this point, easier had come to mean easy to *comprehend* or *understand* given that there are different ways of grouping. For Teri and most of the students, moving one group of seven was easier than partitioning each of the two collections of seven. We would argue that her ability to make judgments of this type developed as she participated in the interactive constitution of the sociomathematical norms of different, sophisticated, and easy. As a consequence, she could act as an autonomous member of the classroom community.

Conclusion

Throughout this paper, we have attempted to document the evolution of sociomathematical norms in Ms. Smith's classroom. These sociomathematical norms emerged as Ms. Smith reflected on and refined her practice in collaboration with the research team. They were not predetermined criteria introduced into the classroom from outside but were continually re-negotiated in the course of classroom interactions. As students participated in this process, they learned to make judgments about their own and others' solutions. The negotiation of the sociomathematical norms of what counts as different, sophisticated and easy solutions constituted the social situation in which the students developed the beliefs and values that constituted their mathematical dispositions.

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NEGOTIATION OF MEANINGS IN THE MATHEMATICS CLASSROOM

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ABSTRACT

This paper illustrates the practice of algebra teaching and learning as observed in a middle-school classroom. Using a framework of analysis that takes activity as the basic unit, I present a fine-grained study of one videotape segment in order to characterize: (1) how traditionally rooted classroom processes are involved in the transmission of mathematical knowledge; and (2) how teachers and students negotiate the meaning of the mathematical objects (concepts and representations) made available in the classroom.

INTRODUCTION

Issues of learning and teaching mathematical concepts and representations have been traditionally studied within a cognitivist and formalist framework (Putnam, Lampert, and Peterson, 1990). From this perspective, cognitive processes are viewed as pure forms while the environment is factored out as variables only tangentially related to cognitive events. Recent advances in theories of situated knowledge, however, have suggested that the study of cognition ought to reconsider the role of sociocultural contexts and social life in the emergence and evolution of cognitive processes (e.g., Lave, 1988; Lave and Wenger, 1991; Saxe, 1991; Brown, Collins, and Duguid, 1989). According to this perspective, mathematical concepts and representations are constructed and communicated within specific sociocultural practices, while evolve in settings structured by social interactions and material resources.

In mathematics education, this situated approach has raised important questions concerning psychological and didactic principles (Schoenfeld, 1989a; Lampert, 1990; Voigt, 1992, 1993; Lave, Smith and Butler, 1988; Resnick, 1989; NCTM, 1989; Greeno, 1989). Much current research on classroom processes have considered these advances, and a renewed body of challenging findings has been put forth. Cobb, Gravemeijer, Yackel, McClain and Whitenack (in press), Voigt (1993), and Arcavi, Meira, Smith and Cassel (in press) have done extensive analyses of the mathematics classroom culture. These authors emphasize the study of negotiation of meanings in the classroom. Their works bring forth a tension between mathematical conventions and formalisms, and the lively deconstruction and reconstruction of mathematical meanings during classroom activities.

In order to characterize the processes involved with the negotiation of meanings in the mathematics classroom, I discuss below three interrelated themes: (1) individuals' participation in multiple practices; (2) the premises of communication constructed through the cultural history of specific social groups; and (3) the conditions of negotiation and the existing routines of action within specific activities. These analytical themes will then guide the empirical study presented later in this paper.

Individuals' participation in multiple practices. In an investigation of children's discursive practices in several contexts, Walkerdine (1988) observed a nursery teacher as she used the story of Golden Locks and the Three Bears as a way to contextualize and facilitate the study of relational terms such as big, small, bigger, smaller etc. During the lesson, however, the children tended to react negatively to questions such as "Is daddy bear bigger than mummy bear?," even though they could make correct size comparisons in many other tasks in the same lesson, during clinical interviews, and at home. Walkerdine argued that although in the school task the bear-family from the story was intended to instantiate size differences only, the occurrence of relational terms at home was strongly associated with mothers' control of their children's behavior (e.g., in regulating food consumption). The author concluded that those terms embodied for the children unequivalent relations of power within their own families, and which were brought to bear during the classroom activity. By transforming a story about size relations (from an instructional perspective) into one about family relationships, the children developed a cultural interpretation of the bear-narrative. This cultural reading resulted from the children's participation in multiple and shifting practices, and their performance in the classroom could not be accounted for solely in terms of knowledge about size relations. In this sense, analyzing the readings produced by students in the classroom should also involve perceiving them as members of interrelated practices, including but not limited to the mathematics classroom.

Premises of communication. Voigt (1993) argued that the classroom activity gives rise to mathematical themes, or "networks of meanings taken-to-be-shared." (p. 12) Accordingly, Säljö and Wyndhamn (1990) suggested that what becomes a theme in the mathematics classroom is related to historically situated ways of doing, behaving, and communicating in the school setting. Their research shows how expectations developed in the classroom form premises of communication that will constrain and support teachers' and students' activity. In order to evaluate their claims, Säljö and Wyndhamn asked 8th and 9th graders to use a table of postal prices (reprinted in this page) to find the cost of mailing a letter weighing 120g. The frequency of strategy choice among the students (whether they just read off the table or calculated an answer) in two situations (the question was given during a math lesson or a lesson on social studies) revealed that most students in the mathematics class (74.5%) attempted to calculate an answer (e.g., using proportion rules or adding prices), whereas most students in the social studies class (65.9%) simply read the correct answer ($7.50) from the table (a more efficient and adequate approach to this problem). Säljö and Wyndhamn discussed these results in terms of premises for communication, or expectations constructed on the basis of the everyday channels of discourse open for teachers and students in the institutional space of schooling. They argued that "the actions of individuals become subordinated to the 'premises for communication' that people assume to be relevant for [any] particular context." (p. 3) This adds to Voigt's (1993) concept of mathematical themes as niches of action that belong to a specific activity structure and share the same general motive, such as becoming "familiar with the mathematical rationality in the long run." (p. 9)

Maximum weight (*g*)	Price ($)
20	2.10
100	4.00
250	7.50
500	11.50
1,000	15.50

Conditions of negotiation and existing routines of action. Accounting for the production of meanings in school mathematical activity also involves looking at the microculture of specific classrooms, its existing routines and the conditions for negotiating meanings. According to Voigt (1993), "the microculture makes the meanings in the particular interactions understandable, while at the same time, the microculture exists in and through these very interactions." (p. 17) In Arcavi, Meira, Smith and Kessel (in press), we analyzed the teaching activity of Alan Schoenfeld in one version of his well known problem solving course at the University of California at Berkeley. That investigation revealed that the gradual emergence of the classroom as "a microcosm of the mathematical culture" (Schoenfeld, 1989b) depends on a well negotiated system of meanings where the teacher may have a predominant role in establishing "who talks, when, and how" through actions such as discarding or postponing the students' contributions to classroom discourse.

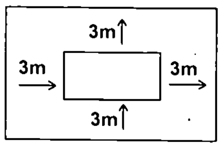
Questions about participation in practices, communication, routines of action, and more generally about the negotiation of meanings in particular settings, have been explored in my research through detailed analyses of episodes from one eighth grade classroom, observed and videotaped during one semester. The videotaped data consist of classroom activities, including the teacher's presentation of content, whole group discussions and small group interactions. In this paper, I will present one illustrative example of the analysis carried out, focusing on how teacher and students handle algebraic representations and procedures in a "traditional" eight-grade classroom. The term traditional is used here to indicate didactic practices based on route learning and rhetorical presentation of content. Of course, this does not mean the absence of negotiation when the production of meaning is at stake; as the study itself attempts to illustrate.

THE ANALYSIS

The sample study presented below discusses the emergence of mathematical representations in classroom activity and illustrates how teachers and students in a traditional classroom attempt to negotiate the meaning of algebraic models and procedures for verbal problems. Additionally, the episode shows the influential nature of representational activity in mathematics, and the problems that arise when the meaning of specific representations are assumed as shared at times when they are in fact idiosyncratic creations of individual problem solvers.

The episode involves the activity of a teacher and his students while they correct a take-home set of problems on algebraic systems. The problem chosen to be discussed in a whole group activity is stated in the worksheet as follows [Notice that the distance between the corresponding sides of the rectangles –3m– is represented in the diagram, but absent in the text of the problem]:

A public park has a rectangular shape, as in the figure. If the gardened area is $2640m^2$, and the park's total area is $3300m^2$, what are the park's dimensions?



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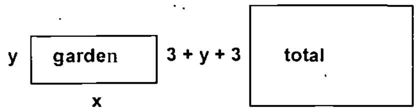
Called to the board to display her solution, a student (Dan) drew a diagram as the one below. Notice that the student's first representation at the board transformed the worksheet diagram in two ways: it produced a figural split of the givens (the rectangles for the garden and total areas of the park), and included the beginnings of a process of mathematization that uses literals for the dimension of various sides (e.g., the height of the larger rectangle is now represented as "3+y+3"). Because the student was not representing the problem but parts of its solution, and the geometrical referent of "3+y+3" is unclear (the distances between the sides of the garden and the park –3m– were omitted), her diagram was not readily accepted by the teacher. As she had just drawn the diagram on the board, the teacher approached her and asked about the meaning of the expression "3+y+3": "This, what is it?". The student gestured on the diagram at the board, indicating the sides of the bigger rectangle and said: "The size of the sides". The teacher replied with agreement ("Okay, nice!"), but did not pass this information on to the class. The student proceeded writing an algebraic system as below:



$$\begin{cases} x.y = 2640 \\ (3+y+3)(3+x+3)=3300 \end{cases}$$

Soon after this algebraic formulation has been displayed, the students begun to raise questions about the meaning of the expression "3+y+3":

S1- Why 3 plus... 3 plus y plus 3? (Pointing to the diagram on the board.)*
S2- [Why is it 3 plus/
T- Can't you see the diagram there (in the worksheet)?
S3- Can you make it two x plus y? (This could mean, for example, 2(x + y), which may be related to the garden's perimeter.)
T- Can you check it out with the diagram there (in the worksheet)?

The teacher assumed Dan's representation of the problem as obvious and, in replying to the students' plea for explanations, simply directed them to the original diagram in the worksheet (recall that the teacher himself did not promptly understand the drawing on the board). After some confusion and inaudible overtalk, Dan resumed her work and developed the procedure to solve the system of equations. A few minutes later, the teacher interrupted and called the students' attention to follow Dan's activity: "Is everything okay?" Several students replied negatively to the teacher's question, saying in chorus that they were "understanding nothing". The teacher's remedial explanations at this point focused on the algebraic procedure of substitution used to solve the system of equations. The student at the board helped out, adding surprised that "this is the kind of system we have been solving for a month!" While some students expressed understanding of the procedure, others returned to a version of the question that caused the original confusion.

* With the exception of the student at the board (Dan), all other students will be referred to as S (plus a number to indicate different individuals); T is the teacher.

S4- What is 3 plus x plus 3? What is that?!
T- Didn't you understand what the x means?
S4- What is 3 plus x plus 3?
S5- No, 3 plus y plus 3!
S4- 3 plus y plus 3.
T- Give me the diagram there (in the worksheet).

S4 gave a copy of the worksheet to the teacher, as he gathered a few students around a desk to answer the question. His explanations were not captured by the camera, and they were presented only to a group of four children at the back of the class. In all, three episodes such as the ones above emerged during the activity involving this problem, none of which seemed to be resolved in any explicit way, or made available to the class as a whole. Although the teacher himself was not clear at the very beginning about the representation proposed by Dan, he made no explicit attempt to discuss its meaning with the students, restraining his action to lecture about the algebraic rules that allowed Dan to manipulate the system of equations.

At this point, we can identify certain mismatches of objectives and communicative resources that emerged in the interaction between the students and the teacher. These mismatches produce regularities in the negotiation process (such as local support to specific students), which might not depend overtly on rational argumentation. In the cases reported above, such mismatches did not favor enhanced understanding of the mathematics on the part of the students. Nevertheless, it served to establish the interaction routines that guide mathematical activity and discourse in this classroom.

It is important to notice that, as mathematical objects, the meanings of "x" and "y" were never explicitly discussed with the students until nearly the end of the episode when values for these unknowns were calculated through the system of equations (x=44 and y=60). Even then, the focus was not on the expression that generated the students' questions ("what's 3+y+3?"), but on the meaning of "x.y" (that appeared in the system of equations but not in the drawings at the board):

T- What is she finding? What is she determining when she solves this (pointing the system of equations)?... What's x and y? ... (No reply from students)... (Notice that, in the system of equations, "x" and "y" do not appear isolated but as, for instance, factors in a multiplication --"x.y") What are they (x and y) representing? The rectangle, let's go back to the geometry (pointing to drawings on the board).
S6- The side.
Dan- Oh god, not the side. X times... The height times the side/
T- The base.
S6- X times y.
T- What is x times y representing?
SSS- The area.

From the perspective developed in this paper, we can identify communicative enclaves that guide the selection of what mathematical objects are elected for discussion and the production of arguments. In this regard, the teacher had a prominent role in electing certain objects and not others (e.g., the meaning of "x", or "x.y", but not of "3+y+3"). At the same

time, the process of negotiating the meanings of these objects involved the students' individual attempts to voice their concerns, even when they elected objects that were underrepresented in the teacher's discourse. Of course, this created a tension that seems unresolved in traditional classrooms where students' voices are neglected (Confrey, 1995).

At the end of this episode, with Dan having arrived at values for x and y (which do not answer the original question about the park's dimensions because these unknowns refers to sides of the garden!), one would think the task was over and the representation displayed on the board was no longer an issue. However, the problem remained until the very end of this session, as illustrated by the following dialogue:

T- Is there any question? (T begins to move on to the following problem in the worksheet.)
S8- Wait, wait, I have... What is that, x+6, y+6? (Expressions that resulted from the simplification of the original expressions "3+x+3" and "3+y+3", written in the system of equations.)
T- (To S8) Here (pointing to the rectangle marked "total" on the board), it's x (the base) plus 6 (gesturing on the rectangle, indicating equivalent dimensions on both sides; notice that the teacher does not use the correct figure --the garden-- to indicate the sum, possibly causing confusion on the part of the student).
S8- Plus 6?
T- Yes (goes to the student's desk, and gestures over his worksheet).
S8- Ah! Now I understand.

There are many instances in the videos analyzed where the uncertainty of mathematical objects is not resolved, causing teacher and students to speak of the meaning of sometimes completely different referents. As Voigt (1993), I suggest that what is seen by teachers and students in the "same" situation is ultimately ambiguous, not readily transparent, and only partially accountable within the classroom. In the episodes above, we can identify a clash of goals emerging from different participants, but also the involvement of teacher and students in an activity where ambiguity is gradually resolved (mainly discarded) though a process of negotiation (mostly implicit) inherent to mathematical teaching and learning.

DISCUSSION AND FINAL REMARKS

Achieving transparency and managing ambiguities in the classroom are complex processes which requires the consideration of multiple viewpoints. As Voigt (1993) put it, "the process of mathematization taken for granted by the experts become problematic when the empirical phenomena are interpreted by subjects whose thinking is not so disciplined by the regulations of a specific classroom culture." (p. 6) Ascribing meaning to mathematical objects involves (beyond pure rational inferences) co-constructing the transparency and collectively dealing with the ambiguities of those objects. For instance, teacher and students in this specific classroom were negotiating over how representations might relate to problem solving, what objects were to be elected in this process, and more generally what types of discursive contributions may be valued. I have suggested elsewhere (Arcavi, Meira, Smith and Kessel, in press) a model of mathematics teaching that takes into account the multiple practices involved in the production of mathematical knowledge, the types of communication

inherent in each of the practices, and the structure of actions that will allow sustained negotiation of meanings in the classroom. (See diagram below.)

Professional Community | **Classroom Community**

Doing Mathematics

is modeled in

Reflective presenting

Teaching mathematics

consists of

Reflective presenting | Rhetorical presenting

Reflective presenting

consists of

Modeling of problem solving | Students' contributions

builds toward

Classroom mathematics doing

The model shows at the top level two *loci* of activity representing a contrast between school and professional mathematics. While the later is characterized by *doing mathematics*, we think of the former as a complex structure where *presenting mathematics* is only a generalized description of what actually happens. In classroom practice, two styles may emerge: reflection and transmission. The reflective mode is constituted by (1) teachers' modeling of discuss patterns and actions selected as inherent features of activities in the mathematical community, and (2) students' contributions to classroom activities. The modeling activities brings to the classroom an idealization of what it means to do mathematics as a participant in the professional community. Modeling activities can emerge in the classroom through (1) *patterns of discourse* (e.g., metacognitive questions through which the students learn to decide what is mathematically acceptable), and (2) *performances*, through which the teacher acts out particular forms of behaving (e.g., as a knowledgeable member of the mathematical community, as a traditional teacher etc.) The transmission mode appears to be an inherent and pervasive feature of classroom life, even in non-traditional approaches where the reflective mode is sometimes laboriously planned and presented to allow only certain kinds of developments to emerge at appropriate moments. Together, reflection and transmission as constructed by the teacher are the basis for creating in the classroom a microcosm of mathematical culture (Schoenfeld, 1989b), represented in the figure above as "school math doing."

Different mathematics classrooms emphasize distinct aspects referred to in the model. In the classroom analyzed in this paper, for example, rhetorical presenting seems to be the main aspect of teaching. However, the teacher is also presenting a mode of problem solving that relates to his own previous experiences as a member of a specific mathematical practice (of professional math educators), in addition to his conceptions about what learning is about. From this perspective, doing mathematics in the classroom (even traditional ones) always bring together the teacher's and students' attempts to coordinate their multiple viewpoints, "even if the participants do not explicitly argue from different points of view" (Voigt, 1993,

p. 10). This process always involves building and negotiating specific forms of activity and communication, and thus the negotiation of the meaning of specific mathematical objects.

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The Use of the Graphing Calculator in Solving Problems on Functions

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Abstract

This paper reports some findings of a study that involved eight students majoring in mathematics education at the University of Georgia. The students worked in pairs and solved two problems that asked for functions that matched some criteria. The students were allowed to use the graphing calculator in one problem but not in the other. The students required more time when the graphing calculator was used. The protocol analysis showed that there were differences in uses of the graphing calculator when it was available, depending on the type of problem. If the problem related to previous knowledge, the students assigned the graphing calculator a verification role; if the problem did not relate to previous knowledge, the students assigned the graphing calculator an exploration role.

Introduction

Graphing calculators appeared in mathematics classrooms rather recently. Detractors of the tool claim that the use of graphing calculators can deprive students of learning basic computational skills that are important for their understanding of mathematical concepts. Those who favor graphing-calculator use see it as a tool that frees students from tedious calculation and lets them focus on more interesting activities, such as exploring, conjecturing, searching, and concluding—activities that have been recognized as critical for gaining a deeper understanding of mathematical concepts (NCTM, 1989). In relation to problem solving and the role of technology, several researchers agree that more important than developing proficiency in solving specific types of problems is encouraging mathematical reasoning and investigation and establishing appropriate ways to think mathematically. The technology needs to be seen as "a tool for problem posing and problem solving [and not as] a tool created to 'teach' links between symbol systems [because] such use can inhibit other kind of understanding" (Williams, 1993, p. 321).

As a cognitive tool, the calculator helps not only to reinforce established modes of thinking, but also to support cognitive growth and change on the part of the user (Ruthven, 1992, pp. 94-95). In a recent study, Dick (1996) has pointed out the capabilities of the zooming option of the graphing calculator for understanding the "holes" in graphs of functions ($y = (x^2 - 1)/(x - 1)$), the local linearity of functions ($\sin x$, very near the origin), and the behavior of slope fields. Other studies that have analyzed the impact of the graphing calculator in the classroom highlight the importance of using the tool not as an add-on element but inside a redefined curriculum. The research

program developed by the "una empresa docente" group in Colombia as part of the PLACEM[1] project studied the effects of the introduction of the graphing calculator in the classroom on different aspects of instruction.[2] The group, for which the more valuable result was the dynamic process between being teachers, curriculum developers and researchers, produced more than 100 problems in different formats–tables, construction of objects, analysis of families of functions, word problems, and investigations–and centered a pre-calculus course on the solving of one to three of these problems in each class period.[3] One of the main results of the study in Colombia was to expose the mathematical, pedagogical, and cognitive complexity that these problems have, and to show that the *micro* level–the level at which the teacher and the students interact in the construction of the mathematical knowledge through the implementation of a curriculum design (Carulla & Gómez, 1996, p. 161)–still needs to be 'split' to understand what happens at the student's level in relation to the graphing calculator and to the mathematical knowledge. The question remains open as to the student's interaction with the graphing calculator and how it relates to the process of solving a problem. In this paper I report some results from a study (Mesa, 1996) addressing that question.

Loci for Research

In any situation involving optional calculator use, there is, on the one hand, a teacher who wants to work with some mathematical content for which he or she has to choose a task and a format. On the other hand, there is a group of students who are given the task and who may use the graphing calculator. The mathematical content and the type of task determine the type of questions and conjectures a student may formulate during the problem-solving session. They also define the teacher's predictions of what the students' performance, possible conjectures, and use of the graphing calculator would be. The questions the students ask during the problem-solving session may require the teacher's intervention. The predictions that he or she made (or did not make) will determine the type of help that the teacher will give the students. In this process there are some key points that can be examined (see numbered boxes in Figure 1):

1. Once the mathematical content and the task have been selected, one locus for research is the predictions that the teacher can make about the performance of a student, the conjectures he or she is going to pose, and the expected uses of the graphing calculator. According to the content and the type of the tasks proposed,

[1] PLACEM Project, *Proyecto Latinoamericano de Calculadoras en Educación Matemática* [Latin American Project of Calculators in Mathematics Education] has been coordinated by Patrick Scott, University of New Mexico, and supported by Texas Instruments. The participant countries are Argentina, Brazil, Chile, Colombia, Mexico, Costa Rica, USA, and Venezuela.

[2] See Carulla & Gómez, 1996; Mesa & Gómez, 1996; and Valero & Gómez, 1996.

[3] See Gómez, P., Mesa, V., Carulla, C., Gómez, C., & Valero, P., 1996.

some differences in these three aspects can be established. An analysis of the teacher's reasons for using specific content is also pertinent.

2. During the process of solving the task, another locus is the type of help the students require from the teacher or from the graphing calculator, the type of questions they ask related to the problem, and the conjectures they pose during the solution process.
3. Once the task is finished, the students' and teacher's performance serve as material for contrasting the teacher's predictions and the students' performance.

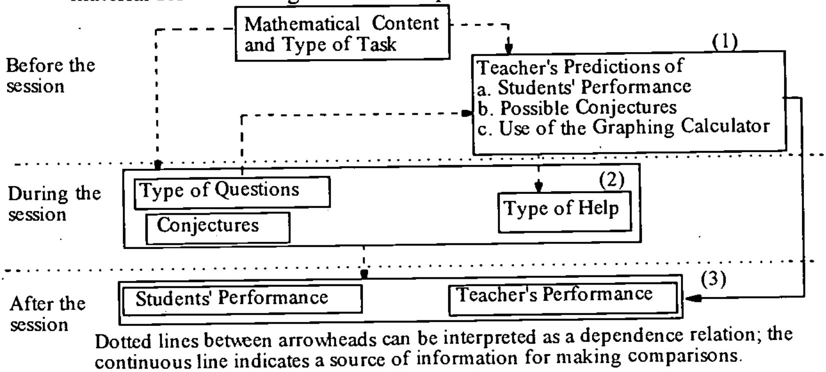


Figure 1. Diagram of problem-solving session with graphing calculator.

In the study reported here I took the role of the teacher for the processes of analyzing the task and of making predictions related to the situation. I assumed the role of researcher during and after the problem-solving session. Acting as a researcher, I took the teacher's predictions as a script for guiding the students in their process. My interest was in what the students did. I was not interested in setting or analyzing learning objectives. The study involved college students majoring in mathematics education working in pairs solving two problems in which they had to find functions that matched some criteria. This paper reports the findings in relation to the following question:

What roles does the graphing calculator play in the problem-solving activities?

a. Do students spend more time when allowed to use the graphing calculator than otherwise?
b. Is the graphing calculator used more in episodes of exploration than in episodes of implementation or verification?

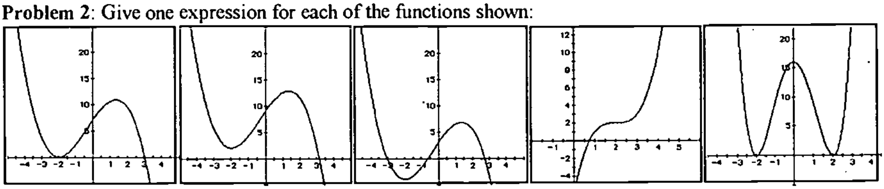
Answering the question demanded the use of the time in minutes that the students spent on the problem and on each episode in problem solving as a necessary variable for analysis. The type of problem and availability of the graphing calculator were the other variables relevant for making a comparison.

METHOD

Four groups each of two undergraduate students majoring in mathematics education at the University of Georgia worked the two problems shown in Figure 2..[4] Two sets of tasks were prepared. In the first set the graphing calculator was allowed for solving the second problem but not for the first. In the second the graphing calculator was allowed for the first but not for the second. Two different groups solved each set. Before the session, the teacher made predictions of students' performance in relation to possible strategies for solving the problems and possible difficulties. Predictions about possible conjectures and uses of the graphing calculator were also made. A set of hints was produced for overcoming the difficulties. The students were asked to work in pairs to make prominent their decision-making processes. Each group worked alone:[5] the researcher monitored the activity and was ready to give the prepared hints when the students were stuck or asked for help. Once the students finished, a short interview was conducted asking the students about the problems and the use of the graphing calculator. The sessions were recorded and videotaped. Each session was transcribed. The product and process variables– achievement of the solution to each problem, the heuristic processes and algorithms used, and the difficulties the students encountered in the process (Kulm, 1984, p. 2)–were made explicit. A framework for analyzing the transcriptions of the students' problem-solving session was adapted from the work of Schoenfeld (1985) and Artzt and Armour-Thomas (1990) and used to parse the protocols into episodes of different types: Read, Analyze, Explore, Plan, Implement, Plan and Implement, Verify, and New Information and Local Assessment. In this paper I report only the results obtained from the protocol analysis.

Problem 1: There are two functions, $f(x) = (x - h)^2 + k$ and $g(x) = a|x - b| + c$ such that the solution to the inequality $f(x)$ ² $g(x)$ is the interval [2, 5]. What are the functions?

Problem 2: Give one expression for each of the functions shown:



[4] Chosen from a larger set of nine problems that were tried out in three different classes.

[5] The students in each group had been working together at least two months.

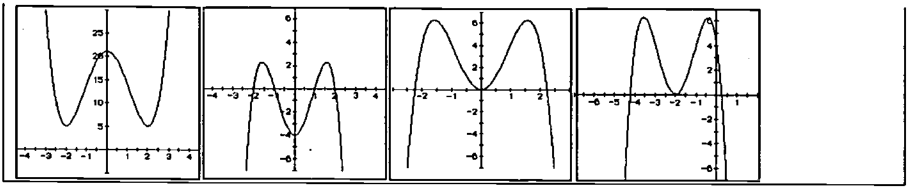


Figure 2: Problems solved by the students.

RESULTS

Table 1 shows the time in minutes each group spent working on each problem. The underlined number corresponds to the session when the graphing calculator was available. Observe that when the students had the graphing calculator, they required more time–in one case twice as much–than the time needed when the graphing calculator was not available.

Table 1: Time in Minutes Used to Solve Each Problem by Each Group.

Group	Problem 1	Problem 2	Total Time
1	14.1	39.3	53.4
2	34.0	21.1	55.1
3	16.3	25.6	41.9
4	26.9	22.3	49.2

Figure 3 shows, by episode type, the total time in minutes in which the graphing calculator was used in Problem 1, and immediately below, the same information for Problem 2. Each bar represents the sum of the time allocated to each type of episode by each pair of groups. Note that the graphing calculator was not used much to solve Problem 1 (21% of the time). In this problem the students did not use the graphing calculator in Analyze or Plan episodes, but used it a little during in the Plan and Implement episode (11%). Observe that the graphing calculator was also not used extensively in the Explore episodes (0.8%). In this problem, a large amount of time was devoted to episodes on Plan and Implement (65%) in comparison to Analyze or Explore episodes. Local Assessments and New Information were done without graphing calculator. In Problem 2 the graphing calculator was used longer (73%) than in Problem 1. Also there were no Plan or Implement episodes that occurred separately. The Local Assessment and New Information episodes did not involve the graphing calculator, whereas Explore episodes were done more than the half of the total time with it (54%). Note that when the Analyze and the Plan and Implement episodes took more time (60% more, and more than 300%, respectively) in Problem 2.

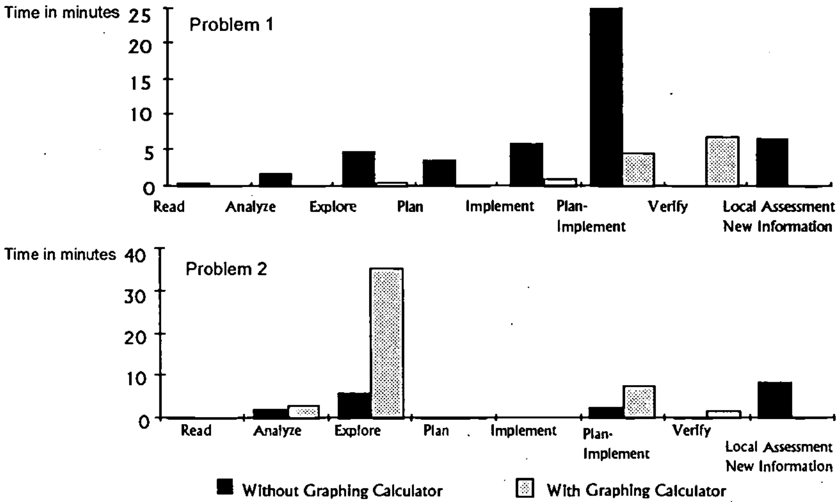


Figure 3: Time spent on each type of episode in Problems 1 and 2, with and without the graphing calculator.

CONCLUSIONS

What probably influenced the time allocation the most was the students' familiarity with the problem, its content, and the procedures involved. The problems were not recognized by the students as 'standard' problems of high school mathematics. The students were more easily engaged in solutions for Problem 1, where the general expressions of the functions were known and the conditions were more familiar. Problem 2 was more difficult for them. The skills needed to solve this problem according to my predictions were in every case beyond their previous knowledge. The hints were totally associated with one specific plan of solution (to find the roots for writing a factored expression using those roots). When given these hints, the students invariably incorporated them into their own solution (translation and dilation) before attempting a radical change in their plan. This phenomenon might be interpreted as a concrete example of what Suydam said in 1982 referring to four-function calculators:

> Calculators are an effective aid in problem solving ... when the problems to be solved are within the scope of the child's ability to solve them using paper and pencil. (cited by Hedrén, 1985, p. 163)

The students were unable to assess the relevance of the given hints because the content of the hints was unfamiliar to them. As a consequence, the need to incorporate

the hints into their solutions added time to their exploratory processes. It is important, nevertheless, to remark that the hints helped them solve Problem 2. The students' greater or lesser familiarity with the content and procedures related to the problems may also explain why in Problem 1 there were almost no exploratory processes. Instead, the students were engaged in producing the solution following their planned steps, and so they used the graphing calculator mainly for verification, as a back-up for their work.

Another interpretation has to do with subjects' experience with the graphing calculator. The problems they solved belonged to a curriculum that incorporated the graphing calculator as an everyday tool. To give the students these problems to solve is, in some sense, like "dropping" a graphing calculator into the classroom, thinking that that action will be enough to foster students' problem-solving skills. The claim is then that a similar experience in which the participants had been exposed to these types of problems would certainly yield different results.

This study provided some evidence related to the way in which problems are posed. It seems to support the claim that if the problem relates to something the students have seen before, they will be willing to ascribe a verification role to the graphing calculator. In this study the graphing calculator proved to be crucial for letting students detect a mismatch. When the students engaged in finding the mistake, the graphing calculator offered them a handy tool for checking an alternative to fix it. If the graphing calculator had not been present, the students might have missed the opportunity to appreciate the problem from another perspective.

We need to let the students play more with the graphing calculator, to learn its potential and its shortcomings so they can gain confidence with it. We need to provide problems that can be solved either with or without a graphing calculator but such that if the graphing calculator is used, the students can pursue different approaches, do more exploration, and make more generalizations. The problems have to provide an environment where exploration is important, but then we, as teachers, need to provide limits to that exploration. The use of the graphing calculator needs to go hand in hand with the teaching of skills in making consistent and systematic explorations.

In the cases in which the students in this study knew how to solve the problem by paper and pencil, they assigned the graphing calculator a verification role. When they did not, they assigned it an exploration role. Whether this phenomenon implies that the graphing calculator is a more effective aid for problem solving in the first case and a less effective tool for problem solving in the second case, I cannot say, for the answer to this question again relies–until research helps us find an answer–on what we consider important in teaching problem solving. As of now, the resolution of the issue has to do with our own view of what is valuable in learning and in mathematics.

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A Hierarchy of Students' Formulation of an Explanation

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Abstract: *The research was a cross-sectional study in Hong Kong. 33 students (aged 12-18) were interviewed. A hierarchy of students' formulation of an explanation was developed by contrasting their responses to an interview task with reference to the SOLO taxonomy. While describing the characteristics in different levels, results also suggested that the levels needed not represent developmental levels. Instead, analysis indicated the key factors towards a higher level was the ability to formulate explaining/proving and versatile and proceptual thinking.*

Introduction

The quest for an explanation is an attempt to find a rationale which may or may not be reduced to a deductive proof (Sierpinska, 1994). The degree of elaboration of an explanation depends on individuals. Reid (1995) distinguishes between formulated and unformulated proving to explain. Formulation describes "the degree to which the proof is thought-of and thought-out" and is related to the articulation and hidden assumptions while proving (p.137). Developing from these ideas, this paper will elaborate the nature of students' formulation by constrasting their responses in a context probing their understanding of the distributive law. In addition, the SOLO taxonomy (Biggs and Collis, 1982, 1991), which has been useful in categorizing observed learning outcomes (e.g., Pegg and Coady, 1993), was used in classifying the students' explanations.

In brevity, the concrete-symbolic mode in the model proposed by Biggs and Collis concerns the use of second order symbol system and responses are linked closely with the students' experiences. In particular, the algebra tasks in the current research probed students working in this mode. The formal mode concerns the use of abstract constructs and works in this mode go beyond empirical observations to various possibilities and alternatives. Moreover, students responses can further be classified as the prestructural, unistructural, multistructural, relational or extended abstract levels. For details of the taxonomy, readers may refer to the works of Biggs and Collis (1982, 1991) and that of Pegg and Coady (1993).

The Research

The distributive law was chosen to be the focus in the present research because that the rule has a pattern commonly found in students' errors, and that it is an algebraic rule used very frequently, thus play a significant role in the empowerment of meaningful algebraic manipulations throughout secondary school mathematics. Consequently, it provides opportunities for designing instruments to differentiate students' thinking.

The results reported here was taken from a cross-sectional study investigating the students' algebraic thinking throughout the secondary schooling in Hong Kong (Mok, 1996). 33 students (aged 12-18) were interviewed. The interview tasks were deliberately open to allow students to give a range of responses. Each interview last for about 30 to 45 minutes and audio-recorded. Analysis of the verbatim transcripts[1] began by a stage of open coding, in which codes, such as "removing brackets", were freely used to indicate students' strategies and characteristics of the responses. At the same time, a SOLO classification was given. In addition, a summary account for each interview was written in order to capture a holistic picture of the student's performance. Afterwards, the codes were further analyzed by looking for similarity/difference and hierarchical relationships, and this created a stage of conceptualization and an evolution of progressive focusing frameworks.

As here is no space for the full report of the research, discussion will focus on a particular interview task (see figure 1). The "a□(b■c)=a□b■a□c" expression consists of two variations: the operations represented in the two squares and the numerical values represented by the letters. The student's answer thus depended on how they took the two relevant clues into consideration. By comparing how the students formulated their explanations and matching their responses with the SOLO taxonomy, a hierarchy was developed. The levels are illustrated with examples in what follows.

Figure 1. The interview task

If a, b, c stand for any numbers, □ and ■ stand for any of the operations +, -, × and ÷, when will a□(b■c)=a□b■a□c be true? always ___ never ___ sometimes when ________________ Reasons:

[1] The interviews were in Cantonese, i.e., the students' mother tongue, and all the excerpts in this paper are translations.

The hierarchy of students' formulation of an explanation

The first level (prestructural) refers to cases in which students do not really engage in the task.

HoHang's answer was "sometimes". On probing, he said that he had not thought of the correct cases. His reason for the "sometimes"-answer was that the operations could be different. Therefore, he did not really engage himself in the task.

HoHang [secondary-2, 13 years old]

HoHang: Yeah. Like this. Here, the squares represent different [operations].

HoHang: [I] don't know the symbol inside the bracket, so [it] may be sometimes correct and sometimes wrong.

The second level (unistructural) refers to responses which uses only one relevant aspect. The explanations are usually in the form of recalling familiar procedures or rules. They are brief, suggesting quick closure and may be inconsistent.

MeiKuen's answer was again "sometimes". She first said that when the white square was multiplication and the black square was addition, the statement was correct for most numbers, then changed her mind and said that the statement was always correct. And when the white square was division, it was sometimes correct and depended on the value of letters. She only considered the cases when the white square was multiplication or division and the black square was addition or subtraction, that is, she gave four cases all together. When asked why she did not consider the cases in which the white square was addition or subtraction, she stated some rules about operations and brackets which was not relevant.

MeiKuen [secondary-4, 15 years old]

MeiKuen: That is, multiplication and addition, or multiplication and subtraction, er, are correct. If division and subtraction, or division, addition, then sometimes allowed, sometimes not allowed. You put the numbers, er, whether the number can divide or not [may mean divisible]. Or that is, minus, afterwards divide and the number is not the same.

MeiKuen: Because, that is, for mathematics, addition, subtraction, multiplication, division. Unless you see a statement. That is, without bracket. Then you should multiply and divide first. If you, the black square, black inside the bracket. Then you, if the statement, how to say? That is, if the statement, multiplication and division inside, even without the bracket, also calculate the multiplication and division first.

The third level (unistructural /multistructural) refers to cases in which the student attempts to elaborate (e.g., in terms of alternative representations) but explanations tend to short and straight-forward.

WingKit's answer was "sometimes". He attempted to justify his answer by giving both correct and sometimes cases. He tried three cases before he said "sometimes". He emphasized that he "calculated" as he rewrote equations in their equivalent forms to check whether they were correct (see figure 2).

WingKit [secondary-2, 13 years old]

WingKit: [explaining his written work.] Because, er, I first put it. Er, the white square represented multiplication, the black square represented addition. Then [I] calculated and obtained that the final answers [which] were the same. The two. Er. Calculated. [I] changed the white square to division. The black square changed to minus. Then the two were calculated and were not the same. Lastly, I calculated. The white square was addition and the black square was multiplication, the answers were also not the same. Therefore, I ticked sometimes.

Figure 2 WingKit's written work (secondary-2)

$$a \times (b+c) = a \times b + a \times c$$
$$ab + ac = ab + ac$$
$$a \div (b-c) = a \div b - a \div c$$
$$\frac{a}{b-c} = \frac{a}{b} - \frac{a}{c}$$
$$a + (b \times c) = a + b \times a + c$$
$$a + bc = (a+b) \times (a+c)$$
$$a + bc = a^2 + ac + ab + bc$$

The same strategy was applied by ManYee who gave a specific format of proving an identity (see figure 3).

[Insert figure 3]

The fourth level (relational) refers to explaining coherently in terms of relevant clues, i.e., the operations and variables in this case. Besides identifying the distributive law in appropriate situations, students can also treat rejected cases as open sentences and identify the valid domains.

For example, from ChungHang's written work (see figure 4), it appeared that he had anticipated that the wrong cases might be correct due to the variation of variables. In particular, he paid special attention to cases when a=0. In probing, believed that he could not exhaust all variations. However, he had given sufficient evidence to support his "sometimes" answer.

[Insert figure 4]

Another student, PuiPui wrote "0+(0+0)=0+0+0+0" and "2+(2+2)=2+2+2+2", and mentioned "a+(b+c)=a+b+a+c" would not be correct for numbers like "3, 4, 5, 6, 7."

PuiPui [secondary-1, 12 years old]

PuiPui: [referring to "a+(b+c)=a+b+a+c".] If zero, then correct. If other numbers, then not correct.

PuiPui: Because, em,.. Think of 2. Then OK. If 2 is not OK, usually other numbers are not OK.

PuiPui: That is, the first one. The addition. If a, b, c become 2, [it is] not correct. Usually the numbers also not correct. Those like 3, 4, 5, 6, 7. [i.e., it is also not correct for the other numbers, such as 3, 4, 5, 6, 7.]

Figure 3 ManYee's written work (secondary-4, 15 years old)

□ = +, ■ = −

$$\begin{aligned} L.H.S. &= a + (b - c) \\ &= a + b - c \\ R.H.S. &= a + b - a + c \\ &= b + c \\ L.H.S. &\neq R.H.S \end{aligned}$$

□ = ×, ■ = ÷

$$\begin{aligned} L.H.S. &= a(b \div c) \\ &= \frac{ab}{c} \\ R.H.S. &= a \times b \div a \times c \\ &= \frac{ab}{ac} \\ &= \frac{b}{c} \\ L.H.S. &\neq R.H.S \end{aligned}$$

□ = ×, ■ = ×

$$\begin{aligned} L.H.S. &= a \times (b \times c) \\ &= abc \\ R.H.S. &= a \times b \times a \times c \\ &= a^2 \times b \times c \end{aligned}$$

□ = ×, ■ = −

$$\begin{aligned} L.H.S. &= a(b - c) \\ &= ab - ac \\ R.H.S. &= ab - a \times c \\ L.H.S. &= R.H.S \end{aligned}$$

□ = ×/÷, ■ = +/−

$$\begin{aligned} L.H.S. &= a \div (b - c) \\ &= \frac{a}{b - c} \\ R.H.S. &= a \div b - a \div c \\ &= \frac{a}{b} - \frac{a}{c} \\ &= a(\frac{1}{b - c}) \\ &= \frac{a}{b - c} \\ L.H.S. &= R.H.S \end{aligned}$$

The fifth level (extended abstract) involves an attempt to prove. At this level, students formulate hypothetical situations, then proceed to justify through a chain of coherent arguments. Via this step of hypothesizing, students no longer rely solely on observed cases. Therefore, their responses extended to the formal mode and are classified as extended abstract.

An example set off from the assumptions about the variables and proceeded by considering the variation of operations.

HiuFung [secondary-6, 18 years old]

HiuFung: I try to see, if b and c are equal to zero, will there be another case? If b equals zero.... If b equals zero, If b equals zero. a is times, I first assume the white square is times or divide, then ..work.. if b is zero , that means a times c or a divided by c. If b equal zero, the left-hand-side ... a times c, the right-hand-side... the white square cannot be divide. Because a is divided by b. If b is zero, then a divided by zero, does not exist, is infinity. So the white square cannot be division. That is proved. [i.e., If b is zero, then the white square cannot be division. See figure 5]

Figure 4 ChungHang's written work (secondary-4, 15 years old)

a÷(b+c)=a÷b+a÷c	√ a=0
	× a≠0
a×(b+c)=a×b+a×c	√
a+(b÷c)=a+b÷a+c	×
a−(b÷c)=a−b÷a−c	×
a+(b÷c)=a+b÷a+c	×
a−(b÷c)=a−b÷a−c	×
a+(b+c)=a+b+a+c	×
	√ a=0
a−(b−c)=a−b−a−c	×
	√ a=0

Figure 5 HiuFung's written work (secondary-6)

If b=0, □=× / ÷ [crossed out ÷ later],	
■= + / − [crossed out − later]	
LHS a•c	RHS a•b + a•c
	0 + ac

Discussion

The nature of the responses depended on how the students thought an explanation should constitute. As described by Mason et al. (1982) in the book, "Thinking Mathematically", there are three levels for the process of verification, namely, convince oneself, convince a friend and convince an enemy. Tall (1991) further argues that the third level of convincing an enemy gets closest to the notion of proof because the argument is intended to be scrutinized and tested, and also the move from elementary to advanced mathematical thinking involves a significant transition from convincing to proving. The more experienced the individual in logical thinking, the more likely they will be to see the need for a more refined argument and attempt to

give one. Therefore, progression is not only exemplified by the production of a refined argument but also by the awareness of the need for a refined argument. If students thought that an explanation was simply recalling facts/rules (e.g., MeiKuen) or carrying out routine manipulations to check whether identical expressions could be obtained (e.g., WingKit), then they would not be likely to give high level explanations. And also they might leave out some relevant clues due to their hidden assumptions. However, some explanations will be well articulated when students gain fluency in handling symbolic works. For example, ManYee gave a better articulated answer as she had a standard format in verifying identity.

Besides articulation via fluent symbolic work, the examples of higher level responses demonstrated a global/holistic grasp of the context of the question or "versatile thinking" (Tall and Thomas, 1991). While seeing that the distributive pattern should not be applied to cases like "a+(b+c)=a+b+a+c", students could expand their explanations by treating the statement as an open sentence, the truth of which depended on the values of a, b and c. In this case, they connected the varying factors in "a□(b■c)=a□b■a□c" (i.e., the operations and variables) coherently and gave a relational responses (e.g., PuiPui and ChungHang). Furthermore, there is a conflict between the idea of an open sentence and a deeply rooted conviction that "a+(b+c)≠a+b+a+c". The conflict exists due to an ambiguity of mathematical symbols which happens throughout mathematics. To handle such ambiguity or accommodate the conflicting ideas, one needs "proceptual thinking" (Gray and Tall, 1994).

When students are fully aware of the impact of the variables and operations on th validity of "a□(b■c)=a□b■a□c", they may pose a range of different cases which need further verifying or proving. In this case, students' responses may then be classified as extended abstract.

To conclude, the aforementioned showed that the SOLO taxonomy has been useful to line up the structure of students' explanations. However, it is important to note that the unistructural-multistructural-relational hierarchy is not necessary an equivalent of the developmental cycle. There is an obvious mismatch between the school years and the level of responses. Although students' response may vary a lot within the same school year and the examples may not be representative of the entire population, the mismatch suggests that the lower level examples are not necessary precedence of the higher level ones. On the other hand, by contrasting the different levels of formulation, students' versatile and proceptual thinking are deemed to be important features in progression. The discrimination illuminates how students may formulate an explanation. Their formulation may be a result of their learning experience under the teacher's expectation in the course of instruction. This may be particularly true in Hong Kong where reception learning is the typically

preferred model and students do not expect the opportunity to articulate their mathematical thoughts (Wong, 1993). Such setting is at the other polar extreme of the collaborative model which has been receiving increasing attention. The latter puts emphasis on students' collaboration and discussion, which naturally provides more opportunities for students to formulate their explanations in different contexts. How the different learning settings will elevate students' formulation of explanations will be a very worthy research question for future work.

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The role of writing to foster pupils' learning about area

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Introduction
In this paper, some findings from a study of seventh grade pupils' writing in solving problems about area are discussed. The purpose of the paper is to illustrate how journal writing can be used in mathematical classrooms to monitor pupils' thinking and learning about mathematical concepts, and, if needed, to orient them towards re-conceptualisation.

The fundamental principle from which our work starts is that "concepts can be abstracted and shaped only by the acting subject's reflection upon an experimental situation and the mental operations it provokes" (Glasersfeld, 1995, p. 188). From this radical constructivist perspective, teaching cannot be seen as 'the traffic of knowledge', but rather a way 'to foster the art of learning'.

We consider such a perspective as being compatible with a phenomenolgical standpoint, similar to the one that appears in Marton and Neuman (1990), according to which both "knowledge is constituted through the internal relation between the knower (object) and the known (object)" and "all mental acts are directed towards something, something beyond themselves" (p. 63). In other words, we see learning as a constructive as well as an interactive activity.

Thus, two ingredients were seen as essential in attempting to promote learning. The first ingredient was pupils' 'journal writing'. The second feature concerned the development and implementation of 12 significant learning activities related to the concept of area. The tasks intended to be within pupils' "zone of proximal development" (Vygotsky, 1978) provided and used various modes of representation about area that were likely to help pupils grasp such a concept . Here, given to space limitations, the focus is only on the first aspect, on locating particulars of the experience and on attempting to explain the relationship between pupils' reactions to writing and events.

Why journal writing?
Following the work of Vygotsky (1976), mathematics education researchers have become increasingly aware, in the last decade or so, of the importance of language in the process of mathematics teaching and learning. Thus, in 1988, at the ICME, in Budapest, a discussion group was launched to study issues and problems concerning the reconciliation of communication in mathematics within a radical constructivist paradigm (Steffe and Wood, 1990). And, in 1996, the central topic of the NCTM Yearbook (Elliot and Kenney, 1996) was, exactly, *Communication in Mathematics*.

One of the main focus of research in this field has been on the role of dialogue and discussion within small group interactions (e.g. Wood and Yackel, 1990; Pirie, 1991; Hoyles, Sutherland and Healy, 1991). Other researchers have highlighted some of the complexities and ambiguities that pupils find in the language used in mathematics classrooms, both in the language of teachers and in textbooks (for some ambiguous English words used commonly in school mathematics, see, for example, Durkin and Shire, 1991). This is a specially important issue, one that, we speculate, many teachers

are not aware of. As Glasersfeld suggests, "because language, by and large, works well in everyday situations, there is the tacit assumption that it must also work in the classroom" (p. 182), but this is not necessarily the case. How can teachers, then, have access to the pupils' interpretations of the classroom discourse? And how can they keep track of each individual's interpretations?

A further issue that has deserved some attention relates to Pimm's (1991) suggestion that "externalising thought through spoken or written language can provide greater access to one's own (as well for others) thoughts, thus aiding the crucial process of reflection" (p. 23). It is relevant that Pimm (1991) speaks explicitly of written language. Indeed, writing is, perhaps, a more challenging journey to foster communication than other forms of building discourses in mathematics classrooms.

Meanwhile, there has also been a growing interest in investigating the use of writing to learn mathematics (and science). Some accounts of how writing has been used and of benefits it brings to learners is to be found, for example, in Connolly and Vilardi (1989).

A brief look at research findings indicates that expressive writing activities (as opposed to transactional ones) and, in particular, journal writing can have a significant impact on learners' cognition and meta-cognition. Specifically, as Powell and López's (1989) case study suggests, pupils (and students) with poor mathematical ability and who are anxious about mathematics are likely to feel more confident in their problem-solving abilities and "understand the material better" as a result of journal writing. This is, of course, a very important aspect, and points to an area of research that deserves to be continued and further developed.

Method

The experience we describe in this paper aimed at examining the following questions

- What are pupils' attitudes and reactions to journal writing in mathematics classrooms?
- What is the nature of pupils' conceptual knowledge about area, as inferred from their writing?
- Does journal writing contributes to improve pupils' mathematical communication and knowledge about area?

The experience took place in a seventh-grade class (13 to 14-year-olds) in a public secondary school of Oporto area, in Portugal, throughout the 1995/96 academic year. The class consisted of 21 pupils, 16 girls and eight boys, all of them of very poor academic ability in the various subjects (and, in particular, in mathematics). Indeed, in the previous academic year, all of them had failed to attain the minimum standards necessary to get a pass to the eighth grade. In turn, the teacher was a woman in her early thirties with ten years of teaching experience, who volunteered to take part in the study.

The experience main phase, during which we observed more closely three pupils while attempting to solve the 12 mathematical activities about area, occurred in April/May 1996. The reasons for selecting these three pupils are in accordance with the basic tenets of the *phenomenographic* (Marton and Neuman, 1990) approach to research which is more easily reconcilable with a phenomenological standpoint. Underlying such an approach is the idea that "whatever phenomena people encounter, there seems to be a limited number of qualitatively different ways in which those phenomena are seen, experienced, or conceptualised" (Marton and Neuman, 1990, p. 64).

In addition to our observations, data included pupils' journal entries, as well as their answers to a written attitude questionnaire (adapted from a previous study)

administered at the end of the experience. In order to evaluate pupils' writing, we categorised the statements they made into four main categories: mathematical communication, mathematical representations, errors and misconceptions.

Furthermore, in order to assess the potential role of writing in learning, we distinguished the four interrelated metacognitive aspects of distancing, *conflict, scaffolding* and *monitoring*, which were used by Hoyles, Sutherland and Healy (1991) as a working framework to analyse pairs of children's discussion while doing mathematics in a computer environment.

Getting started

The problem with journal writing within the context of mathematics classrooms is, perhaps, that it is an alien activity to both mathematics teachers and pupils. One should, therefore, to devise some strategies to assure that this becomes a normal practice in such a class.

Thus, earlier in the academic year, we explained to the pupils that we were conducting a research study, for which they were required to write in their mathematics lessons, and we distributed to each of them a notebook with sheets of loose-leaf paper which were to be used for that purpose. The pupils were made aware that these notebooks (which were called journals) were to be collected by the teacher, but that they were not to be evaluated. The teacher, indeed, would make appropriate comments on their answers, but they were just meant to assist and extend a pupil's response rather than being judgmental

According to our plan, during the first phase of the study, the pupils were to write only once a week, with the first ten minutes of the class being devoted to this task. To start pupils writing, early in the academic year, we asked them to write a word or a statement that they would associate with mathematics. Our second writing assignment consisted of asking pupils to write a paragraph or two that explained the reason why a solution given to a simple mathematical problem was wrong.

The pupils' initial reactions were almost discouraging. They experienced considerable difficulties with these assignments, falling short in giving a proper answer to the questions posed and using imprecise vocabulary.

The situation became even more critical when they were asked to solve a problem and explain their solution. The pupils did not understand what constitutes an explanation or explained things very confusingly. Even with the help of the teacher's typical comments "It seems that you thought well, but we cannot figure out what you want to say with..." or "Would you mind to explain better your idea" things did not get any better.

We realised that pupils, probably, would need a set of step-by-step procedures to be able to write about their thinking while attempting to solve the problems. Kenyon's (1989) *Writing is Problem Solving* gave us a clue. Indeed, it seemed reasonable to speculate that if writing *is* problem solving, then techniques that have been compiled for solving mathematical problems might be used with success throughout the writing process.

We thus adapted Mason, Burton, and Stacey's (1982) model of mathematical problem solving into a four-stage heuristic process (*stuck*, *attack*,*check*, and *reflect*) to solve a problem From the sixth writing session onwards, the sheet in which pupils were to solve a given problem was divided into four parts, each of them corresponding to one of the four stages of that model. The idea, as it was explained to the pupils the first time this method was used, was that they would write as if they were thinking out aloud as they worked on solving the problems.

Problems with the pupils' writing, naturally, were not overcome overnight. For example, in most cases, pupils were not able to write anything under the *'check'* and *'reflect'* headings. By the end of the first academic term, only two girls were coming close to what we called expressive writing.

But as the second term progressed, we could see improvements in the great majority of pupils' writing. Our strategy had made an impact on them. Their writing had improved considerably, as well as their ability to share their thoughts with us. We were then ready to start our main study.

Some research findings

In this section, we focus on three examples of pupils' (two girls and one boy) reactions to journal writing that illuminate different ways in which the different learners interacted with the same activities.

The first example: Catherine

Like most of the pupils in the class, Catherine's previous school record in mathematics looks bleak. Equally bleak is her record in language arts. At the beginning of the study, Catherine was not at ease with putting her thoughts on paper. As she got acquainted with the method proposed, however, Catherine learnt to use it to improve her ability to communicate mathematically, and, most importantly, to detect and correct her own errors and misconceptions.

For Catherine, like for many children, mathematics constituted an alien world designed by and for some people with a special kind of mind. Her experience with writing a journal served to build a bridge between her own world and the mathematics world. Writing allowed Catherine to dissociate mathematics from painful self-exposure.

Here are her comments, at the end of the study:

> To be alone and to be able to reason on my own made me feel secure, to think by myself. It helped not only with my maths, but also in my personal life and in the other subjects. I feel more secure, more confident, I feel much better. It was good to have gone through this experience. I learned that you may be creative in maths.

Perhaps, for the first time, she could feel herself a participant in the maths lessons activities.

Catherine did have some ideas for solving the mathematical problems, but in order to embark on them she needed the reassurance of the teacher. For this reason, at the beginning of the study, she spent much of her time trying to get as much of the teacher's presence and attention as possible.

What is most interesting about Catherine is that in the course of writing about mathematical problems, she was grappling with issues that have nothing to do with mathematics. She used the opportunity as a canvas for personal expression, as the following excerpt illustrates:

> If the boy builds a big fence, the dog will like it. If the boy builds a small fence, then the dog is going to feel bored because he does not have space to run around. But what is important is that the dog may feel happy, because he feels that there is somebody who cares for him.

These comments were related to a problem, involving the concepts of area and perimeter, suggested in the NCTM (1991), which was proposed just before the main phase of the study.

Catherine's reply astounds us. She thought of the boy and his dog as real identities. Her answer is reminiscent of the kind of the difficulties pupils have often with word problems. They cannot draw the line between understanding a 'real life' problem in a way that is appropriate for a maths classroom.

Throughout the main phase of the study, there is something else notable about the way Catherine used her journal. What was different was not just the product, but the way in which she used it. Here, she was using it as a helper, something that she resorts on when she faces any difficulty with a problem proposed. She checks out what she had written before, as well as the remarks made by the teacher. In so doing, she collects evidence for past mistakes and uses them to avoid to making them again. The journal turns into a *'scaffold'* , replacing more and more the role of the teacher, and, indeed, becoming almost a part of herself.

The following excerpts are taken from her reflections with regard to the Activity 8 and the Activity 10, respectively: *"I learned from previous problems that the area of two figures may be the same, but that their perimeter may be different"*, and *"In the latter problems, we saw that two figures can have different perimeters and the same area, and now we see that the perimeter can be the same and the areas can be different"*.

Here, we see how Catherine uses writing to distance herself from her actions in a way that indicates not only that she understood the given problems, but also that she able to make generalisations.

The second example: Mary
Mary is artistic and introspective. She illustrates the case of a pupil who soon adapted herself to write in the context of mathematics lessons, and enjoyed writing her journal. Language arts and writing were probably something that she was fond of, and so, from the very first day, she was meticulous in writing her journal.

Her manner of writing the journal was disciplined and methodical. Her journal is marked by her interest in language. Like in the case of Catherine, the journal represented for Mary a long-waited chance to test her mathematical ability, and, most importantly, seemed to have helped her to do better in the subject.

At the end of the study, she writes:

> *I enjoyed writing the journal very much, I think it was a way of learning mathematics, it was fun. But the most important was to feel more confident in my ability. To have self-confidence is very good.*

Five months later, at the beginning of the current academic year, in expressing (once more) a word or statement that she would associate to mathematics, she reiterates this kind of feeling:

> *Journal of mathematics. I chose these words because last year I loved that kind of work. With it, I developed my mathematical language, both oral and written language. I began to enjoy more the maths lessons and to feel more confident in my ability to do maths. That is why I chose those words.*

For Mary, journal writing mediated a transformation of her relationship with mathematics. Her involvement with writing gave her a sense of accomplishment at being able to find her way around. In contrast to Catherine, in order to solve the mathematical problems, she would not ask for the teacher's attention. She might have to spend considerably more time than her classmates, but she would prefer to do things on her own. For example, in one of the sessions, she spent almost the whole time in

trying to solve the first part of an activity with a tangram. Having succeeded, made her to feeling in control. Mary was delighted, and, it was only then that she called the teacher to show her work. Reflecting on this experience, she writes: *"I was thinking that I would not be able to make it right, it took a lot of time, but I got it. I enjoyed it"*.

Writing about her mathematical activities appeared to have been liberating for Mary. It allowed her to stop thinking of having it right or wrong, and to start thinking of fixing it. Her fixing, however, worked only in a limited way and, sometimes, did not work at all. For example, with regard to one of earlier activities about area, Mary wrote:

> I think that the two figures have the same area. Fig 1 has a C shape, and so has a cut inside and another outside, and Fig 2 has only a cut outside. I think they have the same area. But I am going to see if this is right. I am going to use a ruler to measure the perimeters of the two figures.

In this episode, one can see that Mary's mathematical language is not precise. Moreover, it is noteworthy that Mary writes about what she is going to do to check her intuition, in a way which suggests that she is monitoring her own actions.

After measuring with the ruler, she distances herself from the action and rejects her conjecture: *"Now that I have measured I have the proof that the areas are very different"*. Mary is impressionistic, but she does not trust her impressions. She lets her impressions change as new ideas turn up, but, in so doing, it seems there is very little cognitive conflict.

In the following session, on solving the following activity, she realises that she had made a mistake. She writes: *"Last session, I compared the perimeters of the figures rather than their areas. At the beginning, I was right, the figures seemed to have the same area, but afterwards I got it wrong"*.

However, Mary's realisation of her mistake was not fundamental. At a later stage, an analogous situation emerged, and Mary got it wrong again: *"The two figures have very different shapes, then they must have different areas"*. Afterwards, she realises her mistake again. This time, she worked out an abstracted formulated rule: *"two figures can have very different shapes and have the same area"*.

In the following activities, she constantly uses this belief: *"I know now that two figures can have different shapes and have the same area"*. Her mathematical learning had taken a leap forward.

The third example: Paul

From the beginning, Paul did not care too much about the task of writing. Journal writing was something that he had to do, but that he obviously did not enjoy. Throughout the study initial stage, Paul was not able to uncover his thinking, nor develop any further mathematical communicative skills. Moreover, he had trouble with spelling. This was, perhaps, an important reason to avoid writing.

Unlike Catherine and Mary, Paul's reaction and attitudes to writing the journal, even during the initial phase of the main study, were far from favourable. For example, in one of the earlier activities, the teacher asked him to state the procedures that helped him to solve the problem. Paul limited to give very vague explanations. As the teacher compelled him to describe his thoughts, Paul ended up by writing something of the type "because I was wrong and I did not know what was to be done". Moreover, almost without exception, Paul was not interested in reading the teacher's comments to his previous writings either.

Like both Catherine and Mary, Paul's mathematical ability was far from satisfactory,

but he appeared to be more confident than them. When given a problem, he immediately attempted to solve it, being impatient in getting the work done as soon as possible.

It was a little surprising for us to read, at the end of the study, his comments:

> *I liked writing the journal very much, because it had got nice problems to solve and because these journal also teach us how to learn and reason. I also liked because in spite of the fact that the problems were a little bit difficult, we had to think and ended up by solving them. I learned that I was able to reason. I think that the journal helped us to learn, if we want we are able to think for ourselves.*

These words deserve two commentaries. The first comment relates to the fact that Paul, after using *I*, slips and introduces *we*. This reinforces the idea that he could not identify himself with the thinking agent, the person who wrote the journal.

Second, there seems to be an apparent inconsistency between our previous reference to Paul's reactions to journal writing and his final comment about the task. At first, we interpreted his words as expressing that he enjoyed the mathematical activities rather than the task of journal writing. In retrospect, our interpretation of the situation has evolved. It is possible that a number of small internal changes have taken place within Paul which were not observable at all. Then, at a certain moment, very late in the study, these "small steps of internal reorganisation" (Glasersfeld, 1995) became apparent to Paul himself,.though they might have remained hidden to us.

It is likely, then, that, by the end of the study, Paul's attitude to writing had undergone a modest change. It also seems that journal writing (and the mathematical activities designed) did bring about qualitative advancements in his knowledge about area.

In the past, in his previous experiences with mathematics, Paul had learnt it as a ready made subject and, to a certain extent, to apply, without understanding, ready made algorithms. In regard to area, for example, he had a learnt the formula *length times width,* and this had become his constant way of calculating area.

When Paul is asked to compare the area of two figures, he started by saying that "the two figures had the same area because if I multiply the length by the width, I will find the same area". The teacher writes as a comment: "Do you think that the two figures are rectangles?", to which Paul answers: *"One of the figures resembles a C, whereas the other is like an ice-cream cup, but both of them have the same length and the same width"*. Paul fails to understand the whole idea of the teacher's remark, and reiterates his strategy to calculate area.

In the following activity, however, he sees the same figures tiled with 'small' triangles. The most efficient way of comparing the areas of the two figures becomes that of counting the number of 'small' triangles included in each figure, and this is what Paul does. In his analysis, he could detect immediately a new way of finding the area, discovering, at the same time, his previous misconception: *"As a matter of fact, I was wrong because the reason why [the figures] have the same area is that if I count the triangles that are represented inside the figures then he number is 14 for each figure"*.

Here, we see how the activity provoked Paul to make the transition to a different conceptualisation of area, and how writing highlights this reconceptualisation. Writing served a monitoring function in helping Paul to reflect upon his misunderstanding of the situation.

Another misconception of Paul (and of most pupils in the study) was that figures that have the same are have the same length. In the second part of the Activity 7, in which

pupils had to represent two different figures with the seven pieces of a tangram, Paul wrote *"if we could build the two figures with the 7 pieces, it is because they have the same area, the 7 pieces tiled each of the two figures. They are going to have the same perimeter"*.

On attempting to confirm his conjecture with a ruler, however, he discovered that the two figures had a different perimeter. He writes then: *"I was wrong, I thought that if the two figures had the same area then they would have the same length. It is weird, but it is like that"*. This episode highlights Paul's conflict between two different perceptions. At this stage, however, Paul does not appeared to have undergone further conceptual development. In the following activity, in which pupils were confronted with two different figures with the same area, Paul finds himself confused: *"Things are not as easy as I thought"*.

At a later stage, he seems to have finally come to understand the concepts of area and length, and to distinguish between them. Not only is he no more perceiving area and length uniquely in terms of formulas, but also, he had learned that area and shape are two unrelated concepts. For example, on being asked to compare the area of two different figures, he writes: *"the two figures are very different, and so they should not have the same area. I am going to see which is the area of each one. I will have to count the 'small' squares that tile each figure"*. This was the first occasion in the teaching experiment that Paul monitored his acts. Spontaneously so doing was an accommodation of that latter fact. This accommodation was a permanent change as was confirmed in the next activities.

Concluding remarks

The descriptions given above provide insight into parts of what we think was a successful experience to get pupils writing in a mathematics classroom. They present some evidence that supports the idea that writing can be a window into pupils' thinking and that it can contribute to foster their learning of mathematics. Furthermore, these descriptions also provide examples of the most common errors pupils make about area and misconceptions they have about the concept.

One does not know how pupils abstract such misconceptions from their mathematical experiences, but, surely, it is not something that they learn explicitly. Some authors have already suggested that mixing up area and perimeter of plane figures is formed from the fact that pupils' learning of these concepts takes place almost simultaneously. Given that both are determined in terms of linear measurements (and of the same linear measurements in the case of rectangles), they might to come to see them as indistinguishable.

The second most common misconception concerns the lack of a relationship between area and perimeter. For most pupils in the study, figures with the same perimeter have the same area and reciprocally. It seems that this kind of misconception takes longer to disappear than the previous one. This is probably so because it involves a relationship (or rather, the lack of it) between two different concepts.

Of course, we are not claiming that writing is a panacea for all the problems pupils have with mathematics. First of all, as we suggested at the beginning, we see writing and instructional tasks inextricably intertwined. Our intuitive feeling is that writing without appropriate mathematical tasks would be certainly a less effective and valuable activity than it proved to be here.

Secondly, we cannot forget that, in spite of our efforts, two of the boys in the class did not feel motivated to write their journals at all and felt even reluctant to do so. We can argue that old habits die slowly and that writing in mathematics has naturally the same limits and drawbacks of any innovative programme. Had the initial period of getting

LEARNING PROCESS FOR THE CONCEPT OF AREA OF PLANAR REGIONS IN 12-13 YEAR-OLDS

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ABSTRACT

Our study deals with the learning process of the concept of area in 12-13 year-olds (pupils in "cinquième", i.e. 2nd year in secondary school in France) ; we particularly focused our attention on the way pupils learn how to understand the relationships between lengths and areas, how to coordinate these relationships when learning area formulas and how to distinguish area from perimeter when solving problems. In order to analyze this process, we studied a set of situations based on the concept of area of two-dimensional surfaces, in accordance with the conceptual field theory, and we simultaneously developed a classroom didactic engineering work.

1. INTRODUCTION

The object of this paper is the conception of the notion of area in two-dimensional surfaces and the development of a learning process in 12 to 13 year-olds (pupils in "cinquième", i.e. 2nd year in secondary school in France). Pupils of that age have already gained some knowledge of the concept of area in school but they seldom understand the relationship between geometrical and numerical frameworks.

When analyzing previous studies - (Balacheff, 1988), (Douady and Perrin-Glorian, 1989), (Hart,1981), (Heraud 1989), (Hirstein and al., 1978), (Moreira Baltar and Comiti, 1993), (Nunes and al. 1993), (Tierney and al 1990), (Vinh Bang and Lunzer, 1965) we noted that learning difficulties linked with the ability to distinguish area from perimeter are numerous and persistent.

Douady and Perrin-Glorian (1989) showed that : "*presenting the concept of area as a magnitude can help pupils establish the necessary relationship between both frameworks (geometrical and numerical)*" ; they also conjectured about the necessity of taking the dynamical aspect into account when teaching the concept of area, particularly as far as the distinction between area and perimeter is concerned.

However very few studies have dealt with the subject of formulas for common surfaces. In elementary school, pupils see and use area formulas but the focus is on the change of units and the use of formulas as a means of calculation rather than on the geometrical aspect or magnitude aspect. Moreover, the French national assessments show significant differences in success rates depending on whether a simple calculation task or the geometrical framework is involved. The fact that some pupils lack geometrical knowledge or are unable to articulate the frameworks

is not surprising since areas are dealt with either from a numerical or geometrical point of view without any relationship being established between the two.

This is why we decided to approach the concept of area as a magnitude and to focus our engineering work on the distinction between area and perimeter of planar regions, learning and using area and perimeter formulas for common surfaces, and the consequences of taking into account the dynamical aspect.

As regards distinction between area and perimeter, our engineering work has been organized around three objectives : make pupils able to

- have the area correspond to the inside of a figure (two-dimensional magnitude) and the perimeter correspond to its boundary (one-dimensional magnitude) ;
- get to know the formulas for finding areas and perimeters in association with work in the geometrical field to avoid any confusion when calculating ;
- have the area vary while keeping the same perimeter, modify the perimeter while keeping the same area ; have the area and perimeter vary in opposite direction.

Learning how to distinguish between area and perimeter from these three points of view both depends on and participates in the building of each of the concepts involved. However some distinctions are to be made to better understand the learning process. Our previous studies (Moreira Baltar, 1996) have shown that the knowledge required to distinguish between area and perimeter will vary according to the kind of figures involved. We therefore decided to work on three separate cases : irregular figures, rectangles and parallelograms.

Teaching sequences (concerning the acquisition and use of formulas and the introduction of the dynamical aspect) were conceived and carried out to put the following hypotheses to the test :

H1 : *Studying area and perimeter formulas for common figures together with their geometrical invariants will help build the notion of area as a two-dimensional magnitude.*

H2 : *Taking into account, in teaching, situations involving a dynamical element will help study the geometrical invariants which keep an area constant and will therefore help acquire some knowledge in relation to lengths and areas.*

2. THEORETICAL SCOPE AND METHODOLOGY

In order to test our hypotheses, we studied a set of situations involving the concept of area of planar regions and simultaneously developed a classroom didactic engineering work.

Thanks to the analysis of situations carried out in the light of the theory of the conceptual fields (Vergnaud, 1990), we were able to :

- establish a relationship between the basic aspects of the learning process of the concept of area and the building of the concept of area in general ;

- take into account the role of the various types of situation that can be encountered during the learning process of the concept of area at school ;

- make pertinent decisions aimed at encouraging or stopping some procedures in the course of the learning process.

Didactic engineering as defined by Artigue (1988 : 282) was developed in order to :

- check whether taking into account the various knowledge at stake and separately considering the three kinds of figures (irregular figures, rectangles and parallelograms) make it possible to overcome some of the difficulties in learning how to distinguish area from perimeter ;

- study the building conditions of the knowledge required to link area formulas with geometrical invariants ;

- spot recurrent mistakes.

The preliminary analysis of the set of situations helped to determine the knowledge required for the devolution of the successive situations, which justified the linking of the situations suggested in the scope of our didactic engineering work. The organization of teaching sequences as presented in our study also brings additional material to help analyzing questions related to the construction of the concept of area in children at school.

3. DIDACTIC ENGINEERING

Teaching sequences were carried out in a class ("cinquième") of a school in the Grenoble region. The whole experiment, carried out between March 13th and June 1st 1995 represented some 30 hours of work with the class.

We will explain in our oral presentation how each didactic engineering step is justified in relation to our hypotheses and in relation to the didactic "milieu" (Brousseau, 1986) we wanted to create to favor the devolution (Brousseau, 1986) of each stage to the pupils.

The situations related to distinguishing area from perimeter are transverse to our organization in stages.

Stages 1 to 3, involving work with pen and paper, favored the static point of view whereas stage 4, involving work with the Cabri-géomètre software favored the dynamic point of view.

The main objective of the work with pen and paper was to produce the elements necessary to the understanding of the formulas for finding the areas of rectangles, parallelograms and triangles (geometrical and numerical frameworks in interaction) and to make these formulas available in measurement situations. The aim of the dynamic work with the software was to study the area and perimeter formulas in relation to the geometrical invariants.

To be able to appraise the progress of the pupils' knowledge, we introduced two types of assessment tests. We first organized a "pre-test" prior to the teaching sequences in order to assess the pupils' initial knowledge about the notion of area of

plane surfaces. The pupils' worksheets were gathered up and studied at the end of each sequence. A test was then administered between the paper-pen and the "Cabri-géomètre" stages. The aim of this test was to check that pupils had acquired the knowledge and procedures taught in preliminary activities ; that they had acquired the formulas for finding the area and perimeter of common figures and knew how to use them, as a means of calculation, when dealing with problems of measurement of area and perimeter.

Evolution of the pupils' knowledge was assessed in the course of the dynamic work with the software, from the data collected during this work : recordings, journals, remarks by observers...

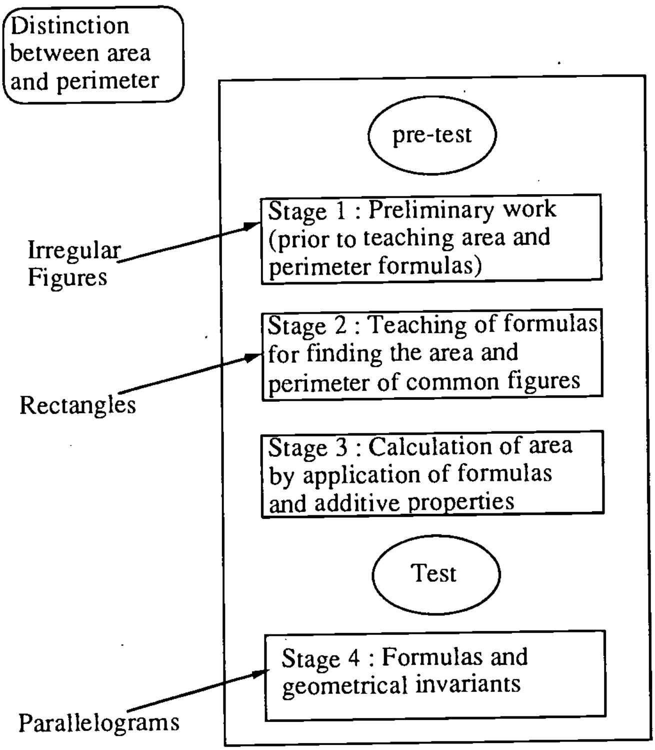

Table recapitulating didactic engineering work

4. CASE STUDY : MEHDI AND KADDA

4.1 Pre-test

During the pre-test, Mehdi's approach was characterized by his resorting to a numerical conception of the area. For him, the area is a number which can be obtained either by calculation (theorem-in-action TC3) or by counting squares (theorem-in-action TC2). Geometrical aspects are ignored in his answers.

Kadda stated that the area is "*what is inside the figure*", which we will relate to theorem-in-action TC1 - according to which the area is the space occupied by the figure - in association with geometrical conceptions. At the same time, when solving problems, he considered as the area either the boundary (problems of comparison for instance) or the number of squares required to tile the area (comparison of tilable areas on squared paper) or the number obtained by calculating a product (comparison of rectangles with parallelograms, measurement of areas, production of a rectangle with the same area as a parallelogram).

Mehdi and Kadda either proceeded from a numerical or geometrical point of view but did not establish any pertinent relations between both frameworks.

During the pre-test, Kadda was able to adequately use the concept of perimeter as the measure of the boundary of the figure and as a length (he always expressed units in centimeters). However he was not able to topologically distinguish the area from the perimeter : he compared boundaries to classify areas ; he did not give any area units when answering measurement questions ; while measuring the area of a complex figure, he changed units to move from a result in centimeters to square centimeters as if centimeters and square centimeters belonged to the same type of units.

Moreover, as regards the area of common figures :

- comparison of areas and perimeters of rectangles and parallelograms : for Mehdi, a rectangle and a parallelogram with equal sides have equal areas ; Kadda calculated and compared the numbers he had obtained.
- calculation of the area of a parallelogram : Mehdi used the cut-and-paste method and calculated the area of the rectangle he had obtained ; Kadda multiplied all the measures indicated on the figure (lengths of the sides and of a height).

Another important aspect in Mehdi's and Kedda's pre-tests was the influence of implicit rules being part of the didactic contract (Brousseau, 1986). For Mehdi, the area is always a number obtained by tiling or calculating : he used the formulas he knew or invented formulas in order to get back to situations he was familiar with (e.g. when comparing the areas of irregular figures, he dealt with the problem as if all areas could be calculated as those of rectangles). For Kadda, areas are calculated by multiplying and all measures indicated on the figures are to be used.

4.2 Test

All along the preparation stage between pre-test and test paper, Mehdi and Kadda worked in pair. Their answers in the test paper showed that, contrary to what they did in the pre-test :

- they were now able to work both in the numerical and geometrical frameworks and could pass from one to the other ;
- they now knew the formulas for finding the areas of a rectangle and a parallelogram and knew how to use them.

However, we noted that Mehdi still had some difficulties with the invariance of the area of a parallelogram whatever base was chosen and also that his knowledge of the formula for finding the area of a triangle was shaky : he properly used the

formula with the help of a drawing but without drawing he multiplied the base by the height (and did not divide by two). For both Mehdi and Kadda we noticed the apparition of a new theorem-in-action, which was not favored by the questions in the pre-test : "the perimeter of the union of two separate areas is the sum of the perimeters of each area". It seems that the property of additivity in areas has been transposed to perimeters.

When analyzing the pre-tests, we had noticed a great influence of the didactic contract on Mehdi's behavior. This hypothesis was comforted by his answers in the test. In the question about the measurement of areas on squared paper for instance, he kept using subtraction procedures even if tiling was possible. Therefore, by analyzing Mehdi's procedures we can get elements to study the evolution of the didactic contract concerning the notion of area between the moment when the pre-test was taken and that when the test was taken.

4.3. Cabri-géomètre environment

Kadda and Mehdi worked together in the Cabri-geomètre environment. Their procedures comforted the persistence of two false theorems-in-action : 1) area and perimeter vary accordingly ; 2) a parallelogram and a rectangle with equal sides have equal areas.

It was still possible to trace these theorems in these pupils' work although they knew the formula for finding the area of a parallelogram and knew how to use it in computation situations. Moreover they did not spontaneously resort to this formula. This comforts our hypothesis that the move from using the formula as a means of calculation (measurement) to using it in other situations (comparison and production) is not automatic.

At the same time, we were also able to observe first signs of the use of formulas as functions of two variables in their justification of variation situations. We could trace the beginning of the construction of a dependence relationship between the invariance of the area and the invariance of the base and height (while sliding one side on its line segment) and the variation of the area according to that of the height (fixed base - rotation about a vertex).

The analysis of their answers confirmed that the questioning of false theorems-in-action and the first step towards a functional use of the formula had been favored by the dynamical point of view and the use of the Cabri-geometre software.

5. GLOBAL RESULTS

The analysis of the global results shows an improvement in the knowledge of the pupils as evidenced in the tables below. The first table concerns the availability of basic knowledge (tiling, cutting and pasting, addition and subtraction of areas) ; the second one concerns the acquisition of formulas for finding the area of parallelograms and triangles and their use as a calculation means.

Basic knowledge concerning areas	pre-test	test
Solid	8	14
Being developed	7	8
None	7	0
Total	22	22

Formulas (calculation means)	pre-test (22)	test (22)	
	right	right	mistake in reading the figure
triangle	0	12	
parallelogram	5	12	3

Generally speaking we can relate this improvement to the teaching organization.

Each stage was conceived so as to involve the building of the knowledge required for the devolution of the following step. The pupils' results show that :

- preliminary activities made it possible for them to acquire the procedures required for the understanding of area formulas for common figures.
- the paper-pen stage made it possible for them to acquire the knowledge required for the devolution of dynamical situations.

Resorting to situations where the dynamical aspect was involved favored the questioning of false theorems-in-action concerning relationships between area and perimeter of parallelograms. It also favored a widened use of formulas not only in calculation but also in measurement situations.

The analysis of the global results highlighted some sources of difficulties such as the notion of base and height in parallelograms and triangles.

The ranking of the various uses of formulas was not as obvious as we had expected. Although the knowledge of the formula and its availability in calculation situations is necessarily prior to its use in comparison situations it appeared that a pupil can be able to use formulas in the latter and fail in the former if his/her knowledge of base and height is shaky enough to prevent him/her from identifying the measures necessary for calculation with a complex figure. Further research on these notions will therefore be necessary.

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STUDY OF THE CONSTRUCTIVE APPROACH IN MATHEMATICS EDUCATION: TYPES OF CONSTRUCTIVE INTERACTIONS AND REQUIREMENTS FOR THE REALIZATION OF EFFECTIVE INTERACTIONS

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⟨ ABSTRACT ⟩

This paper focuses on the interactions in the process of "developing from opposition into agreement" with the aim of elucidating their aspects based on the actual mathematics classes. The first section presents four types of constructive interactions on the basis of the teaching practice. The second section symbolizes major processes of the constructive interactions, and discusses requirements for the realization of effective interactions.

0. INTRODUCTION

The aim of this study is to establish a theory for planning and practicing mathematics class that enables children to actively construct mathematical knowledge. For this aim, the author (Nakahara, 1992) has proposed an approach that is entitled the "Constructive Approach" based on the following five principles.

CA1. Children acquire mathematical knowledge by constructions of their own.

CA2. Basically, children construct and acquire mathematical knowledge in the process of being conscious, operational, mediative, reflective and making agreement.

CA3. In the process in which children are constructing mathematical knowledge, operational activity and reflective thinking play major roles.

CA4. Children construct, criticize and refine mathematical knowledge through constructive interaction with other children or with their teacher and then agree that it is viable knowledge.

CA5. While children are constructing mathematical knowledge, five modes of representation, i.e. realistic representation, manipulative representation, illustrative representation, linguistic representation and symbolic representation play important roles.

In the constructivism-based learning and teaching, social interactions in a class play an extremely important role. So, many researchers have studied them (Cobb, Yackel & Wood, 1992, Cobb & Bauersfeld, 1995 etc.). The author calls them "constructive interactions" based on his understanding that they are interactions for the children to construct their knowledge in a class, and has been studying their aspects, fundamental functions and requirements for their realization (Nakahara, 1992).

This paper should be regarded as a sequel, and focuses on the interactions in the process of "developing from opposition into agreement" with the aim of elucidating their mechanism

based on the actual teaching practice. It further aims to derive factors for generating effective constructive interactions in the teaching.

The first section presents four types of constructive interactions in the process of transition from opposition toward agreement, on the basis of the teaching practice grounded on constructive approach that has been performed by the teachers of author's study group. The second section symbolizes major processes of the constructive interactions, and discusses requirements and factors for the realization of effective interactions.

1. TYPES OF CONSTRUCTIVE INTERACTIONS

As a result of study of the cases of constructive interactions in the process of developing from opposition into agreement so far experienced in the practical study, the interactions are classified into four types:

(1) "From opposition toward decreased opposition type"

< Case 1 > Calculation style for addition by writing

1) Problem: Devise a method of calculating the following (Fig. 1):

$$\boxed{\begin{array}{r} 18 \\ \underline{+13} \end{array}}$$

Fig. 1

2) Initial response:

A: Start calculating from the digit of units. (27 pupils)

B: Start calculating from the digit of tens. (11 pupils)

3) Constructive interaction

C1: If you add the digit of tens at first, you get a result as shown on the right. The answer obtained is 211. This tells that the method is not right (Fig. 2).

$$\begin{array}{r} 18 \\ \underline{+13} \\ 2 \end{array} \Rightarrow \begin{array}{r} 18 \\ \underline{+13} \\ 211 \end{array}$$

Fig. 2

C2: If you do it like this, you get 31.

C3: If you start calculating from the number of tens, you have to erase and rewrite a number. That will take time especially in examination. You'd better make it faster (Fig. 3).

$$\begin{array}{r} 18 \\ \underline{+13} \\ 2 \end{array} \Rightarrow \begin{array}{r} 18 \\ \underline{+13} \\ \cancel{2}1 \end{array} \Rightarrow \begin{array}{r} 18 \\ \underline{+13} \\ 31 \end{array}$$

Fig. 3

T1: Then, I'll show you the method in which you do not have to erase any number (Fig. 4).

$$\begin{array}{r} 18 \\ \underline{+_{2}\,13} \end{array} \Rightarrow \begin{array}{r} 18 \\ \underline{+_{21}13} \\ 1 \end{array} \Rightarrow \begin{array}{r} 18 \\ \underline{+_{21}13} \\ 31 \end{array}$$

C4: No, that isn't good enough.

Fig. 4

4) Concluding response:

A: Start calculating from the digit of units. (33 pupils)

B; Start calculating from the digit of tens. (5 pupils)

In this case, opinions are initially separated into A and B and mutually opposed among the children. After constructive interactions, six pupils changed their method from B to A, but there are five pupils still supporting B. Therefore, this type is characterized as the interaction of "From opposition toward decreased opposition type" (or simply "Decreased Opposition Type"). This type, hereinafter, is represented as "A-B toward A-b". This type tends to appear in cases where various methods or ideas exist leading to the

solution.

(2) "From opposition toward agreement" type

< Case 2 > Linear expression

1) Problem:

Understand the linear expression, which is defined as an expression represented by the sum of character x multiplied by a certain number and a number.
Is the expression "30 - 4x" a linear expression ?

3) Constructive interaction

C1: (B) "30 - 4x" is not a linear expression.

C2: (A) "30 - 4x" is a linear expression, because it may also be represented by "30 + (-4x)".

C3: "30 - 4x" is not a linear expression, while "30 + (-4x)" is a linear expression.

Subsequently, discussions were made about the statement that "30 + (-4x)" is a linear expression and the interpretation of the initially presented definition of the linear expression, and then finally it was agreed that the "30 - 4x" is a linear expression.
In this case 2, the children's views had been split and mutually opposed, and through constructive interactions, agreement was made for A. Therefore, this case may be characterized by the interaction of "From opposition toward agreement type". This type is represented as "A-B toward A". This type tends to be observed when erroneous solutions or ideas are included.

(3) "From opposition toward integral development type"

< Case 3 > Mean velocity

1) Problem: (See Fig.5)

John walks every day from his home in town K to the school in town H. Yesterday, he walked at the speed of 5 kilometers an hour to the point J which situated midway between his home and the school, and at 10 kilometers for the remaining half of the distance to the school. And today, as usual, he hopes to depart his home at the same time as he did yesterday, and to arrive at the school also at the same time. He, however, plans to walk from his home to the school at a constant speed instead of increasing speed midway. At what speed should he walk to school today ?

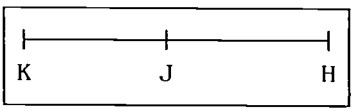

Fig.5

2) Initial response:

A: (solution assuming the distance as 40 kilometers)

$20 \div 5 = 4 \quad 20 \div 10 = 2 \quad 40 / 6 = 6\frac{2}{3}$

B: $(5 + 10) / 2 = 7.5$

3) Constructive interaction

C1: Assuming the distance as 40 kilometers, the required time with B is represented as (40 / 7.5) which does not agree with 6 hours which is obtained by the method A. That indicates B is not right.

C2: You cannot deny it. In some cases, 7.5 may be right...

C3: The ratio of speeds of 5 to 10 translates into 1 to 2. When the distance is the same, ratio of elapsed times is reversed to 2 to 1.

C4: Then, the time required for covering this part is 2 hours and 1 hour for that part, which gives an equation of a total distance $5\times2 + 10\times1 = X\times 3$. Dividing both side by 3 gives the speed value X of $6\frac{2}{3}$.

C5: A peculiar view !

C6: A is assuming a constant distance, while B a constant time. If the distance is constant, time will vary because speed is different. This indicates B is not right.

C7: This may be corrected.

C8: I assumed this distance as 20 kilometers and this as 10 kilometers. Since the speed of 5 kilometers an hour was made for two hours, this may be put as 5 + 5. And since this part is covered at the speed of 10 kilometers an hour, the distance is covered by a single hour. Then, the solution may be obtained by (5 + 5 + 10) / 3 (C).

While, in the constructive interactions in case 3, the idea B was rejected and criticized, the idea was modified by developing the idea A. The finally devised idea C may be considered the integration and development of both ideas created by developing the idea A and by modifying B. Therefore, this type of constructive interaction may be characterized as "From opposition toward integral development type" (or simply "Integral Development Type"), which is represented by "A-B toward B_A° ". B_A° indicates modification of idea B through acceptance of idea A.

This type tends to appear in cases where erroneous methods and ideas are modified and developed by integrating them with other thoughts.

(4) "From opposition toward extensive development type"

< Case 4 > Making quadrilaterals using right-angle manipulation sheets

"The right-angle manipulation sheets" are learning tools which are made of overhead projector transparencies with a right-angle pattern drawn with marker as shown in Fig.6.

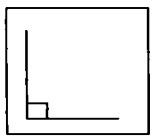

Fig.6

By using two of these sheets in mutually opposed direction, the following quadrilaterals are produced. Translation and rotation of these sheets will highlight the interrelations among these four quadrilaterals

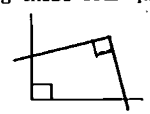

Fig.7 Quadrilateral

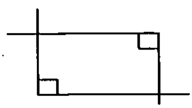

Fig.8 Rectangle

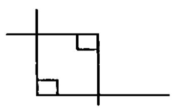

Fig.9 Square

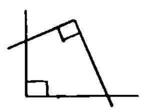

Fig.10 Kite

In this class, the idea of translation that generates a group of similar rectangles was presented by the children. Through the idea, the idea of translation that generates a group of similar squares was derived, and subsequently derived was the incorrect idea of translation that intends to generate a group of similar kite shapes, and finaly its idea has been modified into the correct idea through discussions.

The children gradually expanded and modified the original idea to develop it. For that reason, this constructive interaction is characterized as the "From opposition toward extensive development type" (or simply "Extensive Development Type"), which is denoted as " A-B toward A ° ".

This type appears in cases where an idea is being expanded while correcting and developing erroneous methods and ideas.

2. MECHANISM OF CONSTRUCTIVE INTERACTION

The following discussions try to elucidate the mechanism of above-mentioned constructive interactions, and to extract factors for effective interactions.

(1) Symbolization of constructive interactions

At first, in order to study the mechanism of the constructive interactions, major processes of those interactions are to be symbolized. For the purpose of this section, each of different ideas are given its own character such as A, B etc., similar ideas are identified by A1, A2 etc. Further, the following symbols are used:

- ∼X: Negation of X
- −X: Criticism of X
- +X: Support of X
- X↔Y: Opposition between ideas X and Y or Interaction between ideas X and Y
- X' : Modification of Idea X
- X° : Idea X or X' corrected and agreed upon

Case 1 is to be discussed at first. In this case, the two ideas are mutually opposed:

A : Start calculating from the digit of units.

B : Start calculating from the digit of tens.

C1, C3, and C4 in case 1 were criticism of B. Refutations to them were made to support B. Through all those processes, supporters of A increased while those of B decreased. Hence, the major process is represented as shown in Fig.11.

$$\begin{array}{c} A \longleftrightarrow B \\ \downarrow \\ \boxed{-B \longleftrightarrow +B} \\ \downarrow \\ A \longleftrightarrow b \end{array}$$

Fig.11 Symbolization of Interactions in Case 1

In case 2, the following two ideas were confronted:

−A : "30-4x" is not a linear expression.

A : "30-4x" is a linear expression, because it may be rewritten as "30 + (-40x)".

Afterwards, a compromised idea C3 (half-supporting and half-criticizing A, which is represented by "+A · -A") was presented, which caused the development of discussions leading to the agreement with A. Hence, the process may be represented as shown in Fig.12.

$$\begin{array}{c} A \longleftrightarrow \sim A \\ \downarrow \\ \boxed{+A \cdot -A} \\ \downarrow \\ A \end{array}$$

Fig.12 Symbolization of Interactions in Case 2

In the constructive interactions of case 3, following two ideas

are opposed:

A : $40 / 6 = 6\frac{2}{3}$

B : $(5+10)/2=7.5$

Among various opinions and views including criticisms of B, refutations thereto and defending views, opinions correcting B were presented finally leading to the following solution that developed the idea A.

$B_A°$: $(5+5+10)/3$

Major part of the process may be represented as shown in Fig. 13. The idea A is of course accepted.

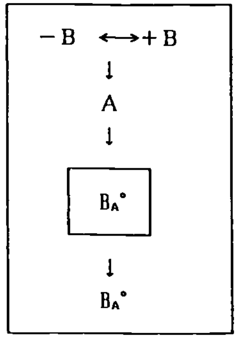

Fig. 13 Symbolization of Interactions in Case 3

In case 4, as mentioned earlier, induced by the translation that generates a group of similar rectangles and squares (represented by a symbol A1), the similar translation was devised for kite shapes. However, the initial translation was erroneous (this is represented as A2). The idea was criticized (-A2), and modified to the idea of "moving while maintaining the right and left equal" (A2'), and finally the right method of moving (A2°) was agreed upon. Therefore, major part of the process is represented by Fig. 14.

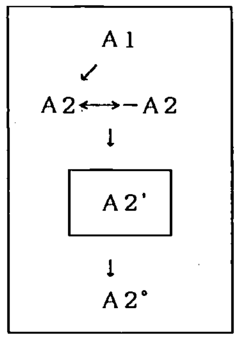

Fig. 14 Symbolization of Interactions in Case 4

(2) Factors for realizing constructive interactions

Subsequently, let us extract the factors for the success of constructive interactions in the transition from opposition toward agreement observed in cases 1-4 based on the above-mentioned symbolization. The author has studied general requirements for the realization of constructive interactions, and identified the following requirements in two categories (Nakahara, 1992) : those concerned with children, and those with the teacher.

< Requirements concerned with children >

C-1. Each individual child has his or her own knowledge and ideas.

C-2. Each individual child is capable of presenting his or her ideas, and of hearing presentations by others.

C-3. The learning group receives presentations of any kind.

C-4. Each individual child is open-minded and has intellectual honesty.

< Requirements concerned with teacher >

T-1. The teacher has been providing the teaching meeting requirements C-1 through C-4.

T-2. The teacher is capable of planning and implementing the classes in which children can construct knowledge and ideas on their own.

T-3. The teacher is capable of organizing children's knowledge and ideas.

T-4. The teacher is capable of generating separation and opposition of views among

children and of leading them to discussions.

In classes as shown in cases 1 through 4, all requirements shown above were basically met, which may be considered a common factor that enabled the successful constructive interactions as shown before. Furthermore, from the phases of symbolized interactions, two common factors may be pointed out in cases 1 through 4.

One is:

① Mutually opposing ideas were presented.

Above-mentioned cases may be different from each other, such as opposing ideas were A and B in a case, and A and ~A in another. However, as understood from Figures 11 through 14, clearly opposing views were presented in any of the cases, and they were suggested by children, which is considered the factor that generated vibrant constructive interactions. In didactics, a teacher's teaching activity called "negational questioning" has been studied in Japan (Yoshimoto, 1981 etc.). It is a teacher's action that negatively jostles children's superficial and flat interpretation of teaching materials, to cause contradictions in children's minds, thus leading them to recognition of higher quality. The above-mentioned opposition functioned as such "negational questioning", and they were presented by the children, which played a significant role beyond the negational questioning.

The other common factor is:

② Before reaching the agreement, mediating ideas were generated.

The routes from opposition leading to agreement are versatile, and the mechanism complicated. None of them goes directly toward the agreement, but includes turns and twists with detours and stops on the way. It should be noted, however, that the idea to be agreed upon never appeared suddenly nor directly in any of the cases. Instead, mediating ideas toward it continued to arise one after another. Typical or decisive mediating idea for each of the cases is as shown below. The parts shown in ▭ in above-mentioned symbolized figures indicate that those ideas took place in the process.

Case 1: That will take time. You'd better make it faster.
Case 2: "30 + (-4x)" is a linear expression.
Case 3: Ratio of elapsed times is reversed to 2 to 1.
Case 4: Moving while maintaining the right and left equal

Then, what are the factors for ① and ② having taken place? In addition to the above-stated general requirements, the following points may be pointed out.

<Factors for mutually opposing ideas having taken place >

①-1 Contents of the problem was suited to the children's level of thinking.
①-2 Contents of the problem inherently included opposing factors.
①-3 Free ideas and thoughts of the children were affirmatively received.
①-4 Agreement or non-agreement was required with respect to others' ideas.

<Factors for mediating ideas having arisen>

②-1 Reasons for agreement or non-agreement were stated.
②-2 Surviving capability was discussed in terms of consistency, rationality, and

REAL WORLD KNOWLEDGE AND MATHEMATICAL KNOWLEDGE

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480 children of primary schools solved 6 non-standard problems with different number of constrain. Their solutions and drawings demonstrated real world considerations.

Introduction

Recent research reports emphasize the importance of exposing children to non-standard problems. The problems are non-standard in the sense that the solver has to take into consideration real world constraints, such as: balloons cannot be cut into pieces and still remain balloons, or one cannot order half a bus, etc. (Reusser, 1996, Wyndham & Saljo, 1996, Verschaffel et al., 1994)). These reports are very important in stressing that we all expect the knowledge of mathematics to be applied properly, and that the way we now teach mathematics at schools sometimes violates it (Greer, 1996, Nesher, 1980).

Greer (1996) suggests that when children ignore their real world knowledge, this is not because of a cognitive deficit, but rather because of the "didactical contract" of schools, or because children understand well the special "language game" of math word problems in school (Greer, 1996, Nehser, 1980).

There is , however, another aspect that should concern math educators. When students are confronted with non-standard problems, and do notice the realistic constraints, how do they cope with them mathematically? What are the mathematical tools that they bring with them for modeling under such conditions? How do they cope with mathematization of non-standard problems? Unfortunately, we lack this part of the story. We have plenty of evidence of students' unreasonable replies in non-stereotyped situations, but we have much less documentation about their actual performance in modeling such situations.

We report here a study of children attempting to solve non-standard problems while we are controlling the degree of constraints appearing in the problems. The attempt was to document how children, who take into account the real world constraints, behave from the mathematization

functionality.

②-3 Representational modes were utilized. (Cases 1 and 2 involved symbolic representation, case 3 illustrative representation, and case 4 manipulative representation.)

②-4 Efforts were made to utilize ideas that contained problems and errors.

②-5 Discussions were made patiently.

The points shown above suggest means for the teacher to devise in order to realize the constructive interactions leading from opposition toward agreement.

3. CONCLUSION

In this paper, four types of constructive interactions for the transition from the opposition toward agreement have been presented based on the analyses through practices of teaching. Subsequently discussed have been two characteristics that are common to constructive interactions. They are critical factors for the constructive interactions having worked effectively. For that reason, requirements and factors for realizing the constructive interactions have been summarized including the above-shown factors.

In General, realizing the constructive interactions leading from opposition toward agreement enables the following points:

(a) Construction of knowledge by individual child on his or her own

(b) Creation of classes that help understand mutually

(c) Construction of concepts, and enhanced/evolved understanding of meanings

(d) Nurturing mathematical constructive capability

To realize the above significance, it should be hoped that requirements and factors for realizing the constructive interactions, that have been summarized in this paper, are effectively utilized.

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point of view. How do they work systematically, in situations that are becoming less and less constrained, which is in many cases the real life situation.

Non-standard problems are multifaced.Usually when people speak about non-standard word problems, they might be thinking of:
a) Problems to which the child was not exposed to at school.
b) Problems with many solutions.
c) Problems that require attention to real world knowledge and constraints.

For example, the following well studied problem:
Carl has five friends and George has six friends. Carl and George decide to give a party together. They invite all their friends. All friends are present. How many friends are there at the party?
(Nelissen, 1987, taken from Reusser, 1996)

The party problem has each of the above three characteristics. Not only it is not given usually at school, we also want the child to notice that there are several possible replies, and also, real life knowledge should inform the child that Carl and George might have some common friends. This particular problem is phrased in an ambiguous manner in regard to whether Carl and George themselves are to be counted, but this can be clarified.
The main issue from the mathematical point of view is that the solver should move from the domain of adding two disjoint sets (where the + sign is applicable) to the domain of uniting two sets, not necessarily disjoint. Thus, the solver has to hesitate about the correct mathematical model. Should we tell him in math lessons the difference between the above two models, or do we leave it to him to invent it?

Do we also want him to approach all the possibilities in the above situation in a systematic manner, through a methodical inquiry? Is this also part of our mathematical goals? These were the questions we bore in mind in planning our study.

The experiment

In order to understand children's ways of modeling non-standard problems, we constructed a set of problems all derived from the same context, but differing in their level of openness (or constraints). Though

the first problem (P1) could be regarded as a standard algebra problem, it was given to elementary school children, grades 4 to 6, who are unfamiliar with the language of algebra, and for whom it was a non-standard arithmetic problem. There were six problems graded from P1 which has just one solution to P6 the most open with an infinite number of solutions.

The general context for all six problems was: ordering pizzas for children in a summer camp.

Problem P1:

For a dinner in a summer camp some large pizzas and some small pizzas were ordered. Altogether 17 pizzas. Each large pizza was divided among four children, and each small pizza was divided between two children. There were 40 children in the camp. How many large and how many small pizzas were ordered?

Problem P2:

For a dinner in a summer camp 17 pizzas were ordered. Some were large, and others were small. Each large pizza was divided among four children, and each small pizza was divided between two children. How many children were in the camp, and how many pizzas of each kind were ordered?

Take note that problem P2 does not mention how many children are in the camp and leaves it open for many solutions, as long as the child observes the constraint of 17 pizzas altogether and how theyare divided. The six experimental problems differ in the number of constraints given the children and as a result in, the amount of possible correct solutions. Table 1 details the characteristics of the six problems in the experiment:

Table 1

Description of the Six Experimental Problems

	P1	P2	P3	P4	P5	P6
Number of pizzas	17	17		17		
Large pizzas						
Small pizzas						
Division of large pizza	4	4	4			
Division of small pizza	2	2	2			
Number of children	40		40		40	

The above problem set was given to 480 children of grades 4, 5 and 6, from 15 schools of heterogeneous population. The problems were distributed randomly among the children, each child solving just one problem.

The problems were solved by each child individually. In each problem, when appropriate, the children were instructed to find different possibilities for solution. The full instructions were:
a) To draw the story described in the text .
b) To explain in detail the line of thinking, either verbally, by mathematical sentences, or by drawing.
c) To write how would they explain their solution to another friend.

Findings:

We report here two aspects of their performance: their actual solution and their visual presentation.

The Solutions:

Table 2 presents the distribution of solvers among the 6 problems and their ability to cope with this tipe of problem. By 'coping' we mean, children who understood the task and offered all kinds of solutions, though not necessarily the complete solutions.

Table 2
Number of Solvers for Each Problem

	P1	P2	P3	P4	P5	P6
N	114	84	91	75	74	42
coped with the problem	66 58%	66 79%	76 84%	47 63%	57 77%	36 86%
Could not cope with the problem	48 42%	18 21%	15 16%	28 37%	17 23%	6 14%

As can be seen from Table 2, most children could cope with the situation and offered some solutions. We were mainly interested **in the way** children cope with non-standard problems that have different degrees of

constraints. In analyzing the solutions we relied on their explicit verbal explanations and used the following four categories:

1. Could not cope with the situation.
2. Gave one correct solution.
3. Gave several correct solutions.
4. Gave a systematic solution, exhausting all possibilities.

Table 3 presents the distribution of the above four categories for each of the problems, P1 to P6.

Table 3

Distribution of the Four Solutions Categories among P1 to P6 (in percentage)

	P1	P2	P3	P4	P5	P6
could not cope	42	21	16	37	23	14
One correct solution	66	47	51	29	39	21
several correct solutions	na*)	17	24	15	17	15
Systematic	na*)	2	1	4	1	-

*) "na" means, not relevant to this problem.

As can be seen from Table 3, most children could offer at least one solution, although they were requested to give several solutions. Some 20% of the children offered several solutions. However, only a few children of this age demonstrated a systematic method of inquiry.

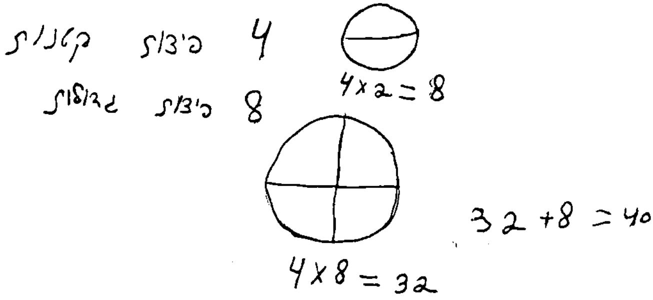
The Drawings

The drawings were very informative regarding the children's modes of thinking. Our analysis of the drawings was for all children whether they could or could not cope with the situation.

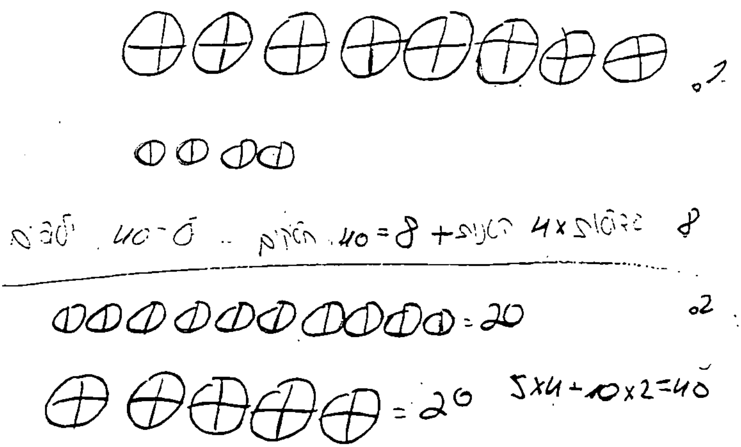
In analyzings the drawing we employed the following categories:

1. No drawing at all.

2. Schematic drawing to support their thinking.



3. A full drawing of all parts to be counted.



4. Drawing of the "real situation"as described.

Table 4 presents the distribution of the children's drawings among the four categories.

Table 4
Distribution of Drawings (in percentage)

	P1	P2	P3	P4	P5	P6
No drawing	15	23	14	23	23	2
Schematic drawings	20	24	34	24	26	40
Drawing for counting	57	45	46	44	38	52
Drawing "real situation"	13	8	5	11	15	21

As can be seen from Table 4 the different degrees of constraints, apparent in problems P1 to P6, did not yield different patterns in the categories of drawings.

We would add an observation which may be important, andwe are still looking into it while analyzing our data:

The most advanced solutions from the mathematical point of view were accompanied by no drawings at all, or by schematic drawings. The children who drew for counting also exhibited less advanced (from the mathematical point of view) solutions. Most children who drew the "real situation" were among the children who could not offer any solution at all. This was one of the most impressive findings and one that calls for further elaboration.

Conclusions and Discussion

From the part of the study reported here, we learned that young children in primary grades are able to cope with non-standard problems with various degrees of constraints.

The children did bring many real life considerations into their solutions. Where information was missing, many of them did not think of all theoretical possibilities, or even about some possibilities, but rather

employed their every day knowledge to supplement the missing information. For example, when the number of pizza portions was unknown, they assumed, without hesitation, the normal dividing of pizza in Israel (they divided large ones into 8 pieces and small ones into 4). They assumed that some children would get more than others (as they probably know from their experience), Sometimes they dealt with different flavors of the pizzas, etc., adding, more pieces to each child. In short, we think that they brought into the solutions so much of their everyday knowledge that we could observe some tension between the abstract thinking about all possibilities and their everyday knowledge. This was especially conspicuous in their drawings. Solving the problem more elegantly from the mathematical modeling standpoint, came together with a more abstract drawing, rather than a more realistic drawing.

We suspect, without probing it as yet, that the reason so many children gave only one solution, although we asked for several solutions, can be attributed to the fact that in real life one orders just one order, and once they gave one possibility, it was unrealistic for them to superficially add other possibilities. Thus, in this context we could not expect to have a real inquiry of all possible theoretical solutions. We plan to replicate the study with different contexts, where probing all possibilities is a realistic demand and to report on it at the conference.

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IMMEDIATE AND SEQUENTIAL EXPERIENCES OF NUMBERS

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The motive of the study presented here was to promote change in the way that primary school children who were unable to learn even the simplest numerical skills within the 1-10 range experienced numbers. It was related to a more large-scale project conducted by researchers representing different research approaches: phenomenography as well as Vygotskyan approaches and based on theories of early number sense indicating that children who experience numbers in an integral and immediate way, as 'structured', develop a sense of numbers, in which 'number facts' become an integrated part, while sequential experiences of 'unstructured' numbers lead children into a 'blind alley'. It utilised computer games which, when related to auxiliary stimuli, gave low achievers experiences of structured numbers and knowledge of number facts.

Introduction

The study presented here was related to a project called IDM (Interactive Didactical Milieus), which was concerned with the development of computer games (Lindström & Ekeblad, 1989, Neuman, 1990, Ekeblad, 1996). One of its motives was to help primary school students who were unable to learn the so called 'number facts', experience numbers in a manner that would result in a better sense of number perception. The project was based on the phenomenographic approach, which considers human consciousness to be a relationship between the individual and the world created through experiences (Marton and Neuman, 1996) and on the theories of Vygotsky (1978), according to which consciousness has a social origin and is mediated through the 'tools' – i.e., the words and signs – used in personal interaction. The word 'interactive' in IDM, signifies a triangular interaction between child, computer and teacher.

The study described here also made use of theories related to a phenomenographic study of how 105 7-year old Swedish school beginners perceived numbers (Neuman, 1987).

The School Beginner Study and a Study of Pupils with Difficulties in Mathematics

In the study concerning primary school students two ways of perceiving numbers were distinguished:

structured ways – where a number presented in a word problem was experienced in an integral and immediate way;

unstructured ways – where such a number was experienced as 'a manifold' of elements and perceived through estimation, or sequentially through counting, often with fingers used for keeping track ('double-counting').

Before the study concerning the school beginners began, a pilot study had been carried out attempting to map differences between pupils who experienced difficulties in mathematics and pupils who did not (Neuman, 1987). In this study 59 pupils aged 8 – 13 were interviewed and given simple addition and subtraction problems (within the numerical range of 1–20) to solve. Thirty one of these pupils were receiving special education in mathematics. These students seemed to percieve numbers as unstructured and could barely solve any of the tasks in the study (not even within the range 1–10) without 'double counting'. Conversely, their 28 class-mates seemed to experience the numbers as

structured. They solved the problems immediately, and chose freely from a wide range of mental calculation strategies.

Gray and Tall (1994) reported similar observations from their research on differences between strategies used by 72 pupils aged 7–12, 1/3 of them low achievers and 1/3 high achievers. They denote the techniques used by the low achievers as 'procedural' and the ones used by the high achievers as 'proceptual'. Pupils displaying procedural behaviour almost always solved problems with the help of counting strategies, in the way I would interpret as related to an experience of unstructured numbers. Pupils with proceptual behaviour, on the other hand, mostly answered with 'known' or 'derived' facts, in the way I would have seen as related to the experience of structured numbers.

At 7 years of age all pupils – those displaying proceptual thinking, as well as those displaying procedural behaviour – knew only a few 'number facts' within the numerical range of 1–10, Grey et al. pointed out. Yet contrary to pupils exhibiting procedural behaviour, those displaying proceptual thinking used the facts they already knew to help them derive new facts. To solve the problem 3 + 4, for example, they could think '7, since 3 + 3 = 6' or '7, since 2 + 5 = 7'. In this way all the facts within the range of 1–10 gradually became known by them, and they could then use these facts to derive facts within higher number ranges. Their procedures seemed to be gradually encapsulated into procepts, i. e. into objects possible to use in new and more complex processes.

As Grey and Tall emphasise, it is not correct to refer to low achievers as 'slow learners'. Their problem is not that they learn more slowly than other pupils, but that they use qualitatively different techniques, which forces them into a 'cul de sac'.

The Design of the Computer Games

For a procedure to be encapsulated into an object – or a number – children must abandon the sequential behaviour related to counting strategies. To do that, we assumed, they must receive concrete experiences of how numbers can be perceived as 'structured' in integral, immediate ways. The 7-year old school beginners in my study illustrated that they were able to experience numbers as structured in three ways, using as aids either:

> *'A multiple of 2, 3 or 4'* (to which one single unit could be added or subtracted) Example: '3 + 4 = 7 since 3 + 3 = 6' or 'since 4 + 4 = 8'.
>
> *'An undivided 5'* (as the first part of the number, structuring the larger part of the number as well as the whole number). Example: 9 – 7 = _ and 2 + _ = 9 were experienced as (5 + 2) + 2 (later as 7 + 2), from which 7 or two could be separated, or as (the hand plus 2 fingers) + 2 fingers (with the two last fingers put aside)
>
> *'A known number combination which could be transformed'*. Example: '4 + 6 = 10, since 5 + 5 = 10', or 'the 5 + 5 fingers on the two hands with one thumb moved from one hand to the other'.

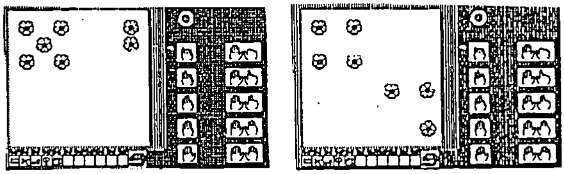
Concrete situations where children perceived 'numbers with an undivided five' and where they 'transformed number combinations' had been identified, when children formed 'finger numbers' to represent a number in a word problem. The notion 'finger number' (subsequently referred to as 'fn') was used by Neuman (1987) to signify fingers put up in

a row beginning with one hand and ending on a finger of the second hand. The 'fn' for 8, thus, began with the one hand and ended on the middle finger of the next hand.

Concrete actions through which children obtain experiences of numbers as 'doubles +/– one' however, were difficult to observe in the interview study. One assumption was that ideas of this kind can be formed when children play dice or domino games. Yet, the structured ways of perceiving numbers most suited to computer games were the ones related to 'doubles'. Experiences related to 'fn:s' were thought to be as much of a tactile-kinesthetic as of a visual nature, and 'fn:s' drawn on the screen were not considered to provide the children with the same 'body-anchored' experience of numbers they got when they used their own fingers. In variations of the games we still decided to draw pictures of 'fn:s', but then in order to use them as a substitute for the digits required for the response. Our intention in using 'fn:s' in this way was to give the children an opportunity to identify isomorphic structures, such as between a pattern of a 5-group plus a 2-group, and a 'fn' constituted by the hand plus two fingers. Freudenthal (1983) regarded children's identification of isomorphic structures to be important in their development of number sense.

The groupings of the flowers, sweets, etc., that appeared on the screen in the games were done in a lot of different ways, two of them shown in figure 1. It was possible to begin with small numbers, and to gradually extend the number range.

In the lower part of the screen rewards of some kind, e. g., sweets or flowers, appeared each time a correct answer had been given



Examples of pattern variations in one of the games. *fig.1*

The patterns were flashed on the screen for only a few seconds. This short time exposure was thought to make the tasks exceed the children's present capabilities. According to Vygotsky (1978) two principles are of importance to researchers wanting to use an 'experimental-developmental method' (p 61).[1] Firstly, children should not be able to solve the problems in the experiment by using existing skills. Secondly, a neutral object should be introduced as an 'auxiliary stimuli' (p 71). We could say, according to Vygotsky, that 'when difficulties arise, neutral stimuli take on the function of a sign and from that point on the operation's structure assumes a totally different character' (p 74). However, in the beginning we did not find any ideas of auxiliary stimuli suitable for the games.

Trying Out the Games

In a first 'trial-run' of the games we also became aware of the fact that the short amount of time that the patterns appeared on the screen rarely made the tasks exceed the pupils' capabilities, at least not initially, when the numbers in question were small. The children saw the patterns as two sets, each of them holding a number of elements, small enough to

[1] Vygotsky (1978) illustrated that this method – first introduced by Werner (1948) – provides an approach to dynamic analyses, with the aim of studying not only a final effect, but processes in their entire structure.

subitize. Their 'natural imagery' (Vygotsky, 1978, p39) thus made the visualised objects as easily countable on the screen as they were when they appeared concretely.

In a study with five pre-schoolers, 5-6 years old, an attempt was made to solve this problem. The study was carried out by the designer of the programs, Eva Ekeblad, and myself. The children played – one at a time – the games during two approximately 30 minute sessions using variations where digits were replaced by 'fn:s'. The sessions were tape-recorded and transcribed and notes were taken of the children's behaviour. As expected, the children counted the visualised patterns pointing on the screen with their fingers. When they had done this for a while however, we told them that the screen would become smeary from all this pointing, and asked them to count in other ways.

Since the person running the experiment was expected to interact and to solve the problems 'in conjunction' with the children – working with them in a way which, as Vygotsky (1978) says, 'is not differentiated with respect to the roles played by the child and his helper' (p 29) – we immediately intervened when the child became confused, posing questions of the kind (for example, if a 3 + 3 pattern had just appeared): 'Did you see how many flowers there were in the top?' When the child answered 'Three', we would continue: 'Then you might perhaps want to put up three fingers to help you remember that.' If the child held up three fingers, but not the 'fn' for three, we said: 'Yes, good, three fingers, but it might be easier for you in this game to put three fingers on the desk in the way the three fingers are held up on the screen.' The child would change the configuration of fingers and we continued questioning: 'Did you also see how many there were in the bottom?' When the child again answered 'three', then, 'why not put up three more fingers!', we would suggest. If the child then held up three fingers on the other hand and began to count, we again pointed out that it might be easier to put down the fingers in the way they appeared on the screen, i. e. 'with no gap' between the 3 + 3 fingers. The children extended three fingers to the three on the table and were happy to see the 3 + 3 pattern transformed into the finger group they used to call 'six': one hand and the thumb of the other hand.

Four of these five children recognised all 'finger numbers' without counting. The fifth child, however, was not aware that her hand had five fingers, when she first put up the 'fn' for five. Yet as the games went on she learned all her 'fn:s'. Table 1 illustrates what the children knew at the beginning and at the end of the two game playing sessions.

Table 1. Number of children, knowing 'fn:s' and patterns on the screen at start and end of the sessions

	All 'fn:s'	The 5-patterns	The 3+3-pattern	The 5+1-	5+2-	5 + 3- and 5 + 4-patterns	
Beginning	4	4	2	1	1	1	1
End	5	5	3	4	3	2	1

Four Boys with Mathematical Difficulties Play the Games

These ideas were then used in a more extensive study of four primary graders, all of them boys, who were taking part in special education courses in mathematics. These boys had great difficulties in learning the 'number facts' that children are expected to learn during the primary grades. A special education teacher Mrs Young (all names are fictious) promised to play the games with the boys for 15–20 minutes sessions at a time, once or twice a week during the seven weeks left before the summer vacation.

In the first session Mrs Young and I met the boys individually and let them try the games, which they liked very much indeed. We had chosen the version where 'fn:s' replaced the digits, so there was nothing reminding them of the 'school mathematics' they hated so. This first session was tape recorded and transcribed, and notes were taken of the boys' behaviour in order to make an account of their knowledge at the beginning of the experiment.

Before the games were started I told the boys that – as in all games – these games had rules. The two rules they had to remember were: 1) pointing to the screen was not allowed, and 2) those who wanted to use their own fingers had to put them up as the 'fn:s' pictured on the screen. The boys were then asked to put up some 'fn:s' I pointed to. We discussed and tested concretely how a number of fingers, six fingers for example, could be put up in several ways – e.g. as three fingers on each hand or as two on one hand and four on the other. 'Yet, the pictures of finger groups in the games' we explained, 'are called 'fn:s', not just fingers, because they are all formed in the same way: beginning with the left little finger and then extended one finger at a time'.

Then we started the games. If the boys needed help I – and later Mrs Young – talked with them in the same way that I had earlier talked with the pre-schoolers. The four boys understood the rules rather quickly and could then play the games unassisted.

At the end of the 7th week, a tape-recorded and transcribed evaluation of the experiment was done. Yet before this final evaluation is described, I will make a brief account of the pre-evaluation carried out during the first session, of how each boy behaved in the beginning of this session, and of how their behaviour changed towards its end.

The First Session

At the beginning of this first session two of the four children, Andy(11) [2] and Brent(12), already knew all their 'fn:s', and one, Sam(8), knew all his 'fn:s' except the one representing 8. The fourth boy, Al(8), however, knew only the 'fn' for five. [3] All the boys could also recognise patterns on the screen that represented numbers less than five. The five pattern formed as a dice pattern, was also known by all the boys, except Al.

Andy, an 11 Year Old Third-grader

Initially, Andy forgot about the rule to not to put up fingers in any other way than as 'fn's. He tried three fingers on one hand and two on the other to represent a 3 + 2 pattern on the screen, and began to count. He was then reminded of the rule, changed the pattern and said, astonished and happy, 'Five!', without any counting, discovering that he got 'the hand'. This immediate and satisfactory recognition of a known 'fn' appeared again, when he later formed larger 'fn:s'.

During this first session, however, Andy never managed to directly recognise '5 + some number' patterns when they flashed on the screen, in spite of the fact that they were isomorphic to the 'fn:s'. He continually needed to represent even these simpler patterns with his own fingers, before he could mark the correct 'fn' on the screen.

[2] The digits within brackets refer to the children's ages.
[3] Children rarely put up 'fn:s' for one, two and three spontaneously. These f:ns had to be shown to all pupils.

Brent, a 12 Year Old Fourth-grader

Brent initially forgot the rule about not pointing at the screen and started to count the 2 + 3 objects he visualised before I reminded him of the rules. He then took unifix cubes, put them up in a 2 + 3 pattern and counted, before he marked the 'fn' for 5 on the screen. The next time it was '2 + 3' that appeared, and when once again he reached for the unifix cubes, I suggested that he replace them with the 'fn' for 2 extended by 3 fingers. He did that, saw the hand and cried out 'Five!', astonished and relieved, and then proceeded to mark the correct symbol.

After that Brent put up his hand directly when he saw patterns for five. Yet, initially he exactly as Andy always put up his own fingers before he could mark any of the'fn:s' on the screen. Later on in the game however, we observed on two occasions that he directly recognised the 5 + 2 and 5 + 3 patterns without first using his own fingers. His behaviour illustrated how he began to feel more and more sure of himself and more contented with this feeling.

Sam, an 8 Year Old First-grader[4]

Sam was the boy who did not know the 'fn' for eight without counting and he was annoyed to have to count the fingers every time the 'fn' he formed for a pattern ended on the middle finger of the second hand. Conversely, when he saw 4 and 3 on the screen, had put up the 'fn' for four, and then had extended it with three fingers, he smiled and said: 'Seven!' After this first occasion he immediately recognised 4 and 3 (and also 5 and 2) as 7 without any concrete use of fingers. The pattern showing 3 threes and one single was difficult for him to construct as a 'fn', but when it was done he immediately recognised the 'fn' as 10. The next time this pattern appeared he immediately said 'ten'. Yet, he explained that he had seen 'six in the left hand top and one in the right hand bottom' of the screen. Thus, he had to put up the 'fn' for ten and analyze the number of threes he could find in it, and then we again showed the10-pattern on the screen and let him see and analyze that.

Al, an 8 Year Old First-grader

Al was the boy who only knew the 'fn' for five. At the start of the experiment he had to count the fingers for all other 'fn:s'. This made the games difficult for him. So, initially, we let him use a variant where there was no limitation on the time that the patterns could be observed and he was allowed to count on the screen. Yet, I tried to draw his attention to the fact that the objects formed groups, and asked him to tell me about the number in each group before he began to count. He immediately said '4 + 1' for the dice pattern and '3 + 2' for the other 5-patterns. The numbers in the subsets were small enough to be subitized. Yet, since he did not know any 'number facts', he still had to count in order to carry out the addition of 4 + 1 or 2 + 3.

After he had played the first part of one game in this way, we introduced the 'flashing' variant, and the dice pattern for five appeared. He said '4 and 1', and – after being informed of the two rules – then put up the 'fn' for four and extended it with one finger. He was very happy to see that this pattern was 'the hand', which he called 'five'. Yet, except for 'the hand', he initially had to count all the 'fn:s' he held up.

[4] In Sweden children begin school in August of the year they turn 7. This was the second term of Sam's school year.

After some time had passed however, we saw that he did not count his 'fn:s' any more. He said 'Six' directly when he had put up 'the hand + one finger' for the patterns 5 + 1 or 3 + 3, and 'Seven', when he had put up 'the hand + two fingers'. Towards the end of the session he did not even need to put up any fingers for the patterns 2 + 3, 5 + 1, and 5 + 2, but immediately recognised them when they flashed on the screen.

The Final Evaluation Seven Weeks Later

In the final evaluation seven weeks later we met again with the four boys individually. At this point all of them knew all of the 'fn:s', but none had to use them any more. They now recognised all the patterns on the screen immediately and could also, with great assurance, describe the composition of the patterns long after they had disappeared. When marking the 'fn' 5 + 2 for 7, for example, they could correctly explain that they had seen 3 + 4 objects, and when they had marked the 'fn' 5 + 3 for 8, that they had seen 4 + 4. When the games were finished, Mrs Young asked the boys to explain different ways in which numbers could appear. All the boys gave several examples of how numbers could be compound,for example, that 3 + 5 makes 8, but that 8 also can be 4 + 4, 2 + 6 or 1 + 7. They could also illustrate with 'fn:s' how transformations between number combinations could be done.

The boys' teachers met us when the evaluation was finished. All of them assured us that something seemed to have happened to these boys during the last weeks, that a change had taken place. Their self esteem was better and they were not nervous in the way they had been earlier.

Conclusion

According to phenomenographic assumptions our consciousness consists of relations created between the individual and the world through experiences. What people experience in a given situation depends on what they have earlier experienced in similar situations.

When the four boys started to play the games, their earlier experiences prompted them to act – i.e., to count. Through the games, however, they garnered experience which obstructed the immediate impulse to count and instead prompted them to first think about how to re-group the patterns into a structure which made the numbers well known – or easily learned – by them. In the final evaluation not even this prompt to think and to 're-structure' patterns seemed to be consciously experienced by the boys. Now the procedures had been encapsulated into objects in the form of known sums.

The 'auxiliary stimuli' – the children's own 'fn:s' – became naturally introduced as a result of our decision to solve the problems in 'conjunction' with the children. They provided new perceptual experience – experience in how to immediately grasp numbers by transforming them into numbers with a '5' structure.

Discussion

The hand seems to represent a natural number-area, Werner (1973) states, adding that his research indicated 'a definite relationship between the ability to articulate the fingers and the early development of number concept' (p 296). He also refers to Werner and Strauss (in Werner, 1973, p 297) who reported on the relationship between difficulties in grasping optical configurations constructed of discrete elements, e. g., dots, and deficiencies in the development of number concepts. Pupils who have not developed more advanced

methods for conceiving and dealing with optical numerical forms are not able to deal with abstract number-concepts either, Werner concludes, referring to Brownell (in Werner, 1973, p 297).

The ability to experience the sum of an addition – the addition of 3 + 4, for example – as a 'number fact' or sum', seems to require the encapsulation of two procedures. First, the numbers within each addend – 3 and 4 in the example – must have been encapsulated into (subitized as) two immediately experienced objects. Secondly, to make an encapsulation possible of the two added numbers (3 + 4) – i. e. to make it possible to experience their sum as related to one single symbol (7) without counting – the number combination must be re-structured, given a structure common to all numbers (in the example [3 + 2] + 2).

In our culture we have chosen to structure numbers with the help of tens and multiples of tens. Yet to begin with, the most important thing is to give the smaller numbers outside the subitizing range, but within the basic number range of 1–10, such a common structure. If sums within this number range are not automatized and experienced as objects, possible to divide up into two parts in all possible ways, they can not be used as thought tools in more complex addition and subtraction, for instance over 10-borders.

Once we began to picture our fingers as 'Roman numerals' these basic numbers gained a common semi decimal structure. In a similar way the children in the study presented here re-grouped all the different patterns appearing on the computer screen and provided them with a semi-decimal structure. The auxiliary stimuli in the form of 'fn:s' made this possible. Auxiliary stimuli can, as Vygotsky says, take on the function of a sign, when difficulties arise. And those signs can make the operation's structure assume a totally different character.

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MICROANALYSIS OF THE WAYS OF USING SIMPLER PROBLEMS IN MATHEMATICAL PROBLEM SOLVING [1]

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The aim of this paper is to examine how solvers use solutions of simpler problems to explore original problems. According to analysis of data by two problem solvers, it will be stated that; (i) solutions of simpler problems can suggest some aspects to which solvers should pay attention in exploring an original problem situation, and can support importance of some elements in the situation; (ii) it is an important factor in using solutions of simpler problems to explore original situations and get information about it ; (iii) although solvers' making-sense of solutions of simpler problems plays a crucial role, inappropriate making-sense does not need to lead to failure and can promote solvers' activities.

1. Introduction

It is widely recognized as a strategy to use similar and simpler problems in solving mathematical problems. Yokoyama (1991) found, however, that teaching of this strategy did not have so much effect on children as teaching of other strategies such as guess-and-check, making-lists, and working-backward. While Schoenfeld (1985) showed that students could use solutions of simpler problems effectively in solving original ones, Tsukahara (1991) reported students' difficulty in using those solutions.

Such research did not focus on processes themselves in which students investigate original problems taking advantage of solutions of their simpler versions. This paper will attempt to examine these processes as such by analyzing the protocols of the actual mathematical problem solving, to understand roles of simpler problems better.

2. Gathering Data and the Outlines of the Solutions

2.1 Gathering Data

The problem solving processes of two solvers (call them the subjects S and T in the rest of this paper) will be treated in this paper. Each of these solvers participated in the problem solving experiments consisting of nine sessions. Both of them were graduate school students studying mathematics education, and S is the same person as the subject in Nunokawa (1994b). What will be analyzed here are the data of the third session of each experiment. In this session, the following problem was tackled;

> Prove that if a, b, and c are positive real numbers, then
> $$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$$
> (Klamkin, 1988, p. 5).

The subjects were asked to solve this in the think-aloud fashion. The whole solving processes and interviews were recorded by ATR and VTR. The transcriptions of

these records, answer sheets written by the solvers, and memos taken by the experimenter are used for analysis (for detail, see Nunokawa (1994b)).

2. 2 The Outline of the S's Solving Process

(i) He transformed the right-side of the inequality to be proved, and got $a^{a/3}a^{b/3}a^{c/3}b^{a/3}b^{b/3}b^{c/3}c^{a/3}c^{b/3}c^{c/3}$. Then he took the logarithms of the both-sides and multiplied them by 3 to get $3a\log a+3b\log b+3c\log c$ and $(a+b+c)\log a+(a+b+c)\log b+(a+b+c)\log c$, respectively. He introduced the condition $a>b>c>0$ by himself, saying "since it does not lose generality." Here, he mentioned $3a-(a+b+c)>0$ and $3c-(a+b+c)<0$, but said that he could not decide whether the rest (i.e.$3b-(a+b+c)$) was positive or negative. Saying "I would try to subtract [$(a+b+c)$ from $3a$, and so on]," he wrote $2a-(b+c)$, $2b-(a+c)$, and $2c-(a+b)$, and added the mark "O "(indicating 'OK') to the first one, "×"('not OK') to the second and third ones.

(ii) He examined $0.5^{0.5}$ saying "Does the greater-less relation reverses at 1?" Then he drew a graph of $y=x^x$. Based on the fact that the minimum of $y=x^x$ was $(e^{-1})^{e^{-1}}$, he estimated the left-side as $a^a b^b c^c \geq (e^{-1}e^{-1}e^{-1})^{e^{-1}}=e^{-3e^{-1}}$, saying "it cannot be less than this." After that, he began to search for an alternative approach.

(iii) Saying "I'd try another idea," he examined the one-letter and two-letter cases of the given inequality. He took the logarithms of the both sides of $a^a b^b \geq (ab)^{(a+b)/2}$, and subtracted its right-side from the left-side. After transforming it into $(a-b)\log a+(b-a)\log b$, and into $(a-b)(\log a-\log b)$, he noted that the two-letter case had been proved because $(a-b)$ and $(\log a-\log b)$ had the same sign.

(iv) Adopting "the same policy," he wrote $3a\log a+3b\log b+3c\log c-(a+b+c)\{\log a+\log b+\log c\}$ and transformed this into $(2a-(b+c))\log a+(2b-(a+c))\log b+(2c-(a+b))\log c$. He transformed $2a-(b+c)$, $2b-(a+c)$, and $2c-(a+b)$ into $a+a-b-c$, $b-a+b-c$, and $c-a-b+c$ respectively, and examined them. During this examination, he marked the sign "+" on the first expression, "?" on the second, and "−" on the third.

(v) He transformed $a+a-b-c$, $b-a+b-c$, and $c-a-b+c$ into $a+(a-b-c)$, $b-(a-b+c)$, and $c-(a+b-c)$ respectively. Then he tried another transformations like $a+a-b-c=(a-c)+(a-b)$, $b-a+b-c=(b-c)-(a-b)$, $c-a-b+c=-(a-c)-(b-c)$, and said "How about this combination?" After that, he transformed the expression obtained at the stage (iv) into $\{(a-c)+(a-b)\}\log a+\{(b-c)-(a-b)\}\log b+\{-(a-c)-(b-c)\}\log c$, then into $(a-c)\{\log a-\log c\}+(b-c)\{\log b-\log c\}+(a-b)\{\log a-\log b\}$. He added "$\geq 0$" to this expression and said "so, this is positive." The solver closed this solving process by himself, which took 43 minutes.

2. 3 The Outline of the T's Solving Process

(i) After trying some numbers for a, b, c and checking whether the given inequality held, he listed up several ideas for a proof, and mentioned the idea to find C which satisfied $A \geq C \geq B$ (here, A and B may refer to the left-side and right-side of the inequality respectively). Then he began to prove the two-letter case of the inequality. After he tried to apply the relation between arithmetic and geometric

means and examined a graph of $y=a^x$, he began to consider the difference between the left-side and right-side in the two-letter case. He consequently proved it by transforming this difference into $a^{(a+b)/2}b^{(a+b)/2}((a/b)^{(a-b)/2}-1)\geq 0$.

(ii) Returning to the three-letter case, he tried to apply the result of the two-letter case to a part of the given inequality, $b^b c^c$. Since a^a can be seen as $a^{(a+b+c)/3}a^{(2a-b-c)/3}$, he attempted to search for α satisfying $a^{(2a-b-c)/3}\geq(bc)^{\alpha}$ and $(bc)^{\alpha+(b+c)/2}\geq(bc)^{(a+b+c)/3}$ (Indeed, if such α were found, he could have showed $a^a b^b c^c\geq a^{(a+b+c)/3}(bc)^{\alpha}(bc)^{(b+c)/2}\geq(abc)^{(a+b+c)/3}$, which proved the given inequality). However, he gave up this search.

(iii) He tried to show $(a^a b^b c^c)/(abc)^{(a+b+c)/3}\geq 1$. This time, he searched for α which satisfied $a^{(2a-b-c)/3}\geq(bc)^{\alpha}$ and $b^{\alpha+b-(a+b+c)/3}c^{\alpha+c-(a+b+c)/3}\geq 1$. He derived the condition $\alpha\geq(b+a-2c)/3$ from the latter requirement. Then he focused on one letter b to reduce the complexity, and searched for α which satisfied $a^{(2a-b-c)/3}c^{(2c-a-b)/3}\geq b^{\alpha}$ and $b^{\alpha+b-(a+b+c)/3}\geq 1$.

(iv) He wrote new expressions saying "That may be what I wanted," and reached the following; $(a/c)^{(a-c)/3}\{a^{(a-b)/3}/c^{(b-c)/3}\}b^b\geq b^{(a+b+c)/3}$. He looked for α satisfying $(a/c)^{(a-c)/3}\{a^{(a-b)/3}/c^{(b-c)/3}\}\geq b^{\alpha}$ and $b^{\alpha}b^b\geq b^{(a+b+c)/3}$ (Finding such α is not necessary for the solution in fact). But he could not find such α, partly because of the mistakes in his calculation. The solving process, which took 101 minutes, was closed by the intervention of the researcher.

3. Impact of the Solutions of the Simpler Problems

3.1 *Impacts Observed in the S's Solution*

Here will be analyzed two stages, (iv) and (v), of the S's solution, which were directly related to his proof of the given inequality. Some activities in these stages are similar to or corresponding to the activities done before tackling the two-letter case. In spite of those similarities, however, there are differences between the activities before and after tackling the two-letter case. I would explore impact of the solution of the two-letter case on the solving process for the original problem, by considering such differences.

(1) Taking the logarithms of the both sides of the inequality had been done even before he tackled the two-letter case. But, in the stage (ii), he rejected the ideas of considering the difference between the cubes of the both sides and of considering $a^a b^b c^c/(abc)^{(a+b+c)/3}$, because these ideas were essentially the same as taking the logarithms. That is, he was not confident of the effectiveness of taking the logarithms. After he had proved the two letter-case, he immediately began to take the logarithms of the both sides of the three-letter case again and did not change this direction. This implies that the solution of the simpler problem had shown the validity of the idea of taking the logarithms of the both sides.

(2) The way of investigating the expressions such as $3a-(a+b+c)$ and $2a-(b+c)$ had changed after tackling the two-letter case. At the stage (i), his attention was paid

only to whether each expression was positive or negative, and was not paid to relations among those expressions. He added the sign "o" to positive expressions and "×" to negative ones and said "This can't make it go well." He seemed to assume that log$a$ was positive, and try to show that the difference between the left-side and right-side of the given inequality was positive based on the following facts; (a) both of log$a$ and $2a-(b+c)$ were positive and their product was also positive; (b) similar facts worked for log$b(2b-(a+c))$ and log$c(2c-(a+b))$; (c) the difference between the left-side and right-side was expressed as the sum of these three terms.

At the stages (iv) and (v), he treated such expressions in the context of "exchange [the letters] and factorize it well, " and so investigated them relating them to each other. This attempt was clearly observed in his behavior that, after transforming some expressions into $a+(a-b-c)$, $b-(a-b+c)$, $c-(a+b-c)$ at (v), he pointed to three a's of $(a-b-c)$, $(a-b+c)$, and $(a+b-c)$ with his finger saying "a's are arranged well, but others are not." This idea occurred naturally, in the two-letter case, during transforming the expressions, because, in the two-letter case, $(a-b)$ and $(b-a)$ had occurred and it was easier to see their relation. So his attention to the relation among the expressions can be considered an impact of the solution of the simpler problem.

(3) The idea of gathering common factors, in relating to (2), was observed only after his solution of two-letter case. Before that, he tried to transform the difference between the left-side and right-side of the given inequality (after taking their logarithms) into a certain sum of positive terms. Emphasis was put on determining whether each appearing term was positive or negative. After the two-letter case, by contrary, he intended to factorize that difference, and emphasis was put on finding common factors in different terms. Although he investigated the same difference before and after the two-letter case, what he tried to find or construct in it had changed. His new intention can be seen an impact of the solution of the two-letter case.

(4) His attention to certain forms of expressions, e.g. $(a-b)$ and $(a-c)$, can be also considered an impact of the solution of the simpler problem. When he made a transformation like $a+a-b-c= (a-c)+(a-b)$, he said "So, I can use a very analogy with this." This transformation was done, however, without a sufficient prospect of a final solution, since he said "What can I get by approaching in such a way?" during this transformation. Only after he wrote $(a-c)\{\log a-\log c\}+ (b-c)\{\log b-\log c\}+$ as the transformation proceeded, he said "I've got it." This suggests that his previous utterance about utility of an analogy meant that he could then begin a transformation similar to the two-letter case. In other words, the transformation like $a+a-b-c =(a-c)+(a-b)$ was justified not because it could produce a proof of the given inequality, but because it could make it easier to relate the original and simpler problems and make it possible to proceed a transformation similar to the two-letter case. The solution of the simpler problem had presented a context where the factors that would play an essential role in the later activities could be supported when they occurred.

3.2 *Impacts Observed in the T's Solution*
Although the subject T did not reach a complete proof, he obtained the following expression during his solving process;

$$\left(\frac{a}{c}\right)^{\frac{a-c}{3}} \bullet \frac{a^{\frac{a-b}{3}}}{c^{\frac{b-c}{3}}} \bullet b^{b} \geq b^{\frac{a+b+c}{3}} \quad \cdots (*).$$

Dividing the both sides of this by $b^{(a+b+c)/3}$ and transforming the left-side can lead to a proof of the given inequality. Thus, this (*) means the considerable progress of T's solving process, and so the part of obtaining this would be analyzed here.

At the stage (iii), when T tried to find α such that $a^{(2a-b-c)/3}c^{(2c-a-b)/3} \geq b^{\alpha}$ and $b^{\alpha+b-(a+b+c)/3} \geq 1$, he said "Doing this breaks the attempt." But when he transformed the left-side of the former condition to $a^{\{(a-b)+(a-c)\}/3}c^{\{(c-a)+(c-b)\}/3}$, he said "No, it doesn't break." He wrote newly $\{a^{(a-b+a-c)/3}b^{b}\}/c^{\{(a-c)+(a-c)\}/3} \geq b^{(a+b+c)/3}$ saying "That may be what I wanted" and transformed the left-side of this expression into $(a/c)^{\{(a-c)+(a-c)\}/3}b^{b}$, which includes his mistakes. Correcting the mistakes in the exponents, he reached the expression (*). He said "Doing this breaks the attempt" in writing $a^{(2a-b-c)/3}c^{(2c-a-b)/3}$, but he said "It doesn't break" when he modified it into $a^{\{(a-b)+(a-c)\}/3}c^{\{(c-a)+(c-b)\}/3}$. This suggests that transforming the exponent $(2a-b-c)/3$ into $\{(a-b)+(a-c)\}/3$ was a clue to the expression (*).

Since he immediately proceeded to $(a/c)^{\{(a-c)+(a-c)\}/3}$ saying "That may be what I wanted," he might say "It doesn't break" with such a transformation in his mind. Taking account of the fact that he had proved the two-letter case by making the form like $(a/b)^{(a-b)/2}$ and that that form of expressions had never appeared elsewhere, it can be said that the solution of the two-letter case might show the possibility and validity of the transformation into that form. Just before the end of the process, he pointed to $a^{(a-b)/3}$, $c^{(b-c)/3}$, $b^{(b-c)/3}$, $b^{(a-b)/3}$, which appeared as a result of a certain transformation, and said "They seem similar to..." This can also be considered to show his orientation to a similar transformation.

4. Importance of Exploration of the Original Problem Situation

As shown in the previous section, in the cases of the both subjects, the transformation of $2a-b-c$ into $(a-b)+(a-c)$ and other similar ones were the important clues to the progresses of their solving processes. This appearing form $(a-b)$, a difference of two letters, is certainly easy to be found in solving simpler problem. In fact, in the S's solving activities, this element of $(a-b)$ was naturally generated by taking logarithms of the both sides of the two-letter case and ordering them with respect to $\log a$ and $\log b$. In the T's solution, the terms $a^{(a-b)/2}$ and $b^{(b-a)/2}$ appeared through factorization of $a^{a}b^{b} - (ab)^{(a+b)/2}$ by the common factors $a^{(a+b)/2}$ and $b^{(a+b)/2}$, and they had the forms $(a-b)$ and $(b-a)$ in their exponents. The transformations could be furthered, in the processes of the both subjects, by interpreting this $(b-a)$ as $-(a-b)$. In this sense, the solution of the simper problem can be considered to have shown the validity of such forms of expressions.

Such forms as $(a-b)$ can be appear in the three-letter case only when some expressions like $2a-b-c$ and $a+a-b-c$ are transformed appropriately. But the solution of the simpler case cannot give information about those appropriate transformations. Analyzing S's and T's processes with respect to this point, it can be noted that activities with such transformations had been done in other contexts.

The subject T subtracted $(a+b+c)/3$ from $(b+c)/2$ in order to check which was bigger, in the context of finding an appropriate α at the stage (ii), and tried to determine whether the numerator $b+c-2a$ of their difference was positive or negative. In doing that, he transformed it into $(b-a)+(c-a)$ and said that it was absolutely negative. The transformation necessary for the activities at the later stages did appear here. His utterance that $(b-a)+(c-a)$ was absolutely negative might be supported by the fact that $(b-a)<0$ and $(c-a)<0$, which implies his attention to these differences of the pairs of two letters. In the earlier part of (iii), he calculated the difference of the exponents $\{(2a-b-c)/3\}-\{(a+b-2c)/3\}$ to get $(a-b)+(c-b)$, and checked which of $(a-b)$ and $(b-c)$ was bigger. He had invented the transformation which would become necessary later, in his attempt to determine which exponent was bigger.

The subject S checked whether $3a-(a+b+c)$ or $2a-b-c$ was positive or negative at the stage (i), before tackling the simpler problem. At the stage (iv) (after solving the simpler problem), he searched for the common factors to factorize in the three-letter case and checked whether some expressions were positive or negative because, in the two-letter case, interpreting the negative term $(b-a)$ as $-(a-b)$ made a factorization possible;

> S (37:33): This is...at least...this [$a+a-b-c$] is positive, this [$b-a+b-c$] is undecided, this [$c-a-b+c$] is also undecided, ah, this [$c-a-b+c$] is negative...

It can be said that, in doing this, he made the differences of two letters and determined whether each expression was positive or negative based on positiveness or negativeness of those differences, just as T did. That is, S had paid attention to the form of differences of two letters in the context of checking whether some expressions were positive or negative.

Here, the expressions $2a-b-c$ and $a+a-b-c$ were generated through operations on the problem situation, i.e. the given inequality, and can be regarded as new elements of this situation. So, the fact that they could be transformed into $(a-b)+(a-c)$ etc. is new information about the problem situation. The above discussion in this section can be restated as follows; the information about the problem situation obtained by the activities which were not directly related to the final solution, played a critical role in applying the solution of the simpler problem to search for a solution of the original problem. This coincides with the discussions of some researchers (Terada, 1991; Tsukahara, 1991) that applying solutions of simpler problems requires understanding of original problems.

Indeed, in the S's solving process, the solution of the two-letter case provided him with the idea of factorization essential to the final solution. But the final solution of the original problem was not constructed by translating the solution of the two-letter case into the three-letter case. What he aimed at first according to the

two-letter-case solution was the organization of the situation in the form of (expressions of used letters without log)× (expressions of used letters with log). This is reflected in that he made at stage (v) $a+(a-b-c)$, $b-(a-b+c)$, and $c-(a+b-c)$, all of which included similar forms like (a*b*c) (* is + or −). On the other hand, the organization in the final solution was the sum of the terms in the form of (difference between two letters)× (expressions of two letters with log). The latter form of organization was not found by examining various ways of organization referring to the two-letter-case solution (i.e. giving new senses to the two-letter-case solution), but by aiming at the former organization, investigating the relations among expressions without log, paying attention to differences between two letters like $(a-b)$, and transforming expressions based on those differences. In other words, it was found by his exploration of the problem situation aiming at the former organization. His report in the interview supports this;

> But during separating the letters, I noted there were two *a*'s, like *a* minus *a* minus *c*, so combine *a* and this, another one...Since there are two, so try to separate them, separate them further. I must have another $a-c$ elsewhere, so I've done that, then it worked well.

This utterance implies that the transformation into $(a-c)$ was continued based on a characteristic of the problem situation that two *a*'s existed in one term and on an attempt to treat them separately and combine them to other letters, rather than on an effort to make the form of $(a-c)$ or $(b-c)$ because the term $(a-b)$ became the common factor in the two-letter case. The final organization of the problem situation seems a result of such transformation. That is, the solution of the original problem was not attained by, in the original problem situation, searching elements which were needed to solve the simpler problem (see Polya, 1973, p. 111). During his activities with an attempt to make correspondence of the original problem with the simpler, he found new unexpected elements in the original situation, and importance of these elements was supported by the solution of the simpler problem. Organizing the original situation based on those elements, as a result, a structure of the situation different from the expected one occurred and it led to the solution of the original problem.

5. Utility of Simpler Problems and Giving Senses

To sum up the above discussion, according to the examples analyzed here, contributions of the solution of the simpler problem are suggesting some aspects to which solvers should pay attention in exploring the original problem situation and supporting importance of some elements in the situation (which may be obtained through activities not directly related to the final solution), rather than presenting the very procedure for the solution or the results available for it.

While the importance of selecting appropriate simpler problems has been emphasized in the previous research (e.g. Polya, 1973, pp. 52-53; Schoenfeld, 1985, pp. 84-96), little attention has been paid to how solvers use solutions of simpler problems to tackle the original problems. The above analysis shows, however, that it is not always simple to use those solutions to tackle the original problems. One of its reasons may be that it is a solvers' role to make sense of the solution of the simpler problem and decide how to apply that sense-making to the original problem

situation. It is difficult to decide which may be better to make sense of the S's two-letter-case solution as (expression of the used letters without log) × (expression of the used letters with log) or as (the difference of the two letters) × (the difference of logs of the letters), referring only to the solution of the two-letter case. Like as utility of diagrams (Nunokawa, 1994a), senses given by the solvers are important factors in utility of simpler problems.

The above analysis also shows that failure of making-sense dose not necessarily mean the failure of using simpler problems. Even making-sense which was inappropriate to the final solution promoted exploring the problem situation and made it possible for the solvers to generate new information. If an appropriate making-sense cannot be determined uniquely, in using solutions of simpler problems, it seems important not only to translate procedures or results of simpler problems to original problems, but also to continue to explore the problem situation following information obtained by tentative senses of the simpler-case solutions.

6. Concluding Remarks

In this paper, the actual problem solving processes were analyzed and one aspect of the utility of simpler problems, that is, how solutions of the simpler problems can *indirectly* promote the solvers' exploration of the problem situation, was found. Taking account of this aspect, we can introduce "Using Simpler Problems" strategy, in the problem solving strategy instruction, in a little different way. The point emphasized in the introduction may be what kind of exploration can be continued according to solutions of simpler problems.

While using simpler problems can change solver's structures of a problem situation (Nunokawa, 1994b) in above-mentioned ways, making these simpler problems may be influenced by the solver's structure at that time. Interactions between used simpler problems and solver's structures are to be investigated in future research.

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PUPILS' PERCEPTION OF PATTERN IN RELATION TO SHAPE

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This paper reports on some aspects of a study of pattern perception conducted with pupils aged 9 to 16. Tests were administered to explore the possible relationship between pattern recognition and general ability. Issues related to the mental manipulation of 2-D shapes were also considered. The relative difficulty of transformations was investigated together with the influence of frames of reference. Developmental stages have been suggested in terms of the test items.

Introduction

Any study of pattern in relation to shape ought to start from a clear idea of what is meant by pattern. Providing a definition is not easy. Grünbaum and Shephard (1986), claim that they have been unable to find one that is satisfactory. The patterns of fabrics or wallpapers would suggest that the idea of repetition is important but to Sawyer (1963) pattern "is any kind of regularity that can be recognised by the mind" and the word 'pattern' is also used to refer to "a configuration consisting of several elements that somehow belong together" (Zusne, 1970, in Reed, 1973). This variety of ideas about pattern led to a broad view being adopted in this study. Pattern was not seen as confined to repeating patterns but included ideas about shape recognition, congruence and symmetry. The aim was to investigate how easily pupils 'see' patterns and the main instrument was a pattern recognition test.

The recognition of congruent shapes in different orientations involves some form of mental transformation but is the image that is transformed propositional or pictorial? For example, in recognising two different rotations of a triangle as congruent does a pupil's mental activity use a set of propositions (expressing the properties of each triangle) or has one triangle been mentally rotated to match the other? Those who accept the idea of a mental (pictorial) image refer to a close relationship between imagery and perception. Clements (1982) gives a good summary of theories supporting and opposing visual (pictorial) images and Cooper (1990) provides more recent evidence in support of mental representations of 3-D objects. Solano and Presmeg (1995) focus on the relationship among images rather than mental manipulation in their work on visualisation. Krutetskii's distinction between different types of mathematical mind (Krutetskii, 1979), however, suggests that the thinking of some pupils ('analytic' types) may be without pictures or entirely non-visual. The present study sought enlightenment on whether (and, if so, how) pupils use pictorial images in the context of pattern recognition.

Work by the APU (1980) and Küchemann (1980) indicate children's difficulties with reflection in an oblique mirror line and the CSMS results (Hart, 1981) point to further difficulty in rotating a shape. Chipman and Mendelson (1979) tested children's sensitivity to different types of visual structure and suggested the chronological order

double symmetry, vertical symmetry, horizontal symmetry, diagonal symmetry and rotational symmetry and this largely agrees with cross-cultural studies in pattern perception (Bentley, 1977 in Deregowski, 1980). Bryant (1974) suggests that a perceptual framework like the side of a page can help children distinguish between a vertical and an oblique line. The influence of a frame of reference featured in some of the pattern questions.

The Research Study

The pattern recognition test was conducted with nearly 300 pupils aged 9 to 16, taken from Years 5, 7, 9 and 11 (i.e.ages 9/10, 11/12, 13/14, and 15/16). Additional information was obtained from individual interviews conducted with 12 of the pupils. Each pupil was also given the AH4 Test of General Ability.

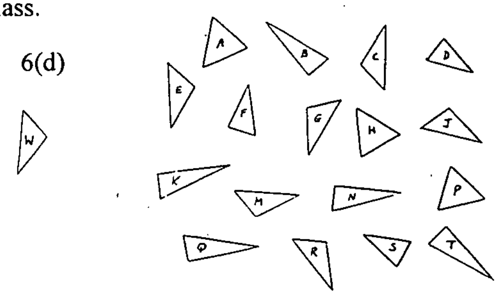
Values of Pearson's product moment correlation coefficient, r, were calculated for the AH4 test totals and the pattern recognition totals and are shown in Table 1. The

	Boys			Girls		
	n	r	p	n	r	p
Y5	44	0.60	<0.001	36	0.53	<0.01
Y7	47	0.70	<0.001	39	0.87	<0.001
Y9	51	0.37	<0.01	37	0.26	
Y11	27	0.44	<0.05	23	0.20	

Table 1 - Correlation between Pattern Recognition and General Ability

correlation coefficients for Y7 were very highly significant and in general the results mostly show correlation. The relationship between general ability and recognition of pattern, however, did not seem to be a simple one. Scatter diagrams showed that there was more correlation in the lower half of the ability range than the upper and, although correlation was generally evident over the whole ability range, the results suggest that it might not be detected in a setted class.

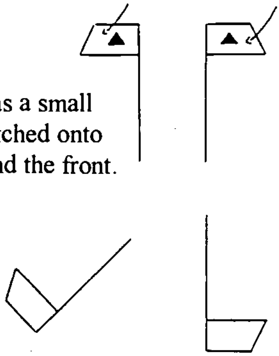
In Question 6(d) (Figure 1) interviews revealed that some pupils assumed that only one shape was required so the scores were perhaps not as meaningful as had been hoped. Another problem for the pupils was deciding whether reflection was allowed. Finally during individual interviews an attempt was made to time the pupils as they wrote down their answers and there was some evidence that the length of time decreased with age.



Which triangles could be cut out and placed on top of the triangle W?

Figure 1

4(a)

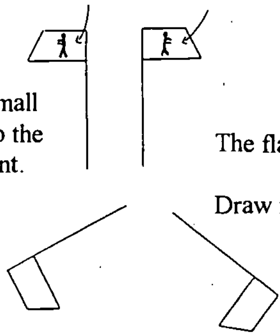


The flag has a small triangle stitched onto the back and the front.

The flag is moved.

Draw in the triangle.

4(b)



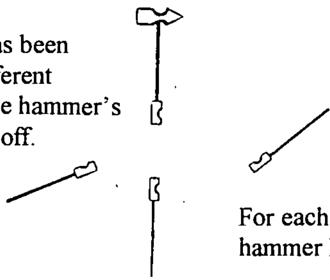
The flag has a small figure stitched to the back and the front.

The flag is moved.

Draw in the figure.

4(c)

The hammer has been moved to 3 different positions but the hammer's head has come off.



For each position draw in the hammer head.

Figure 2

Question 4 (Figure 2) was designed to explore the possible influence of a frame of reference when transforming a shape. Analysis of the results had to take account of the fact that the drawings of the backs of the flags allowed ambiguity of interpretation with the transformed flags. For example, in (a) one transformed flag is either a reflection of the front of the flag in a horizontal axis or a rotation of the back of the flag, and the other could be either a reflection of the back of the flag in a diagonal axis or a rotation of the front of the flag. Individual interviews revealed that some pupils completed 4(a) and 4(b) using the outline of the flags as a frame of reference and filled in the triangles and figures in relation to the top and bottom of the flag perhaps without any mental transformation being involved. Indeed many of the pupils physically moved the answer paper round to align the flagpoles with the 'vertical' of the desktop. Table 2 shows the percentage of the total marks available scored by pupils in each part of the question at the different age levels.

Age	Yr.	4(a)	4(b)	4(c)
9/10	Y5	71.3	45.6	46.3
11/12	Y7	87.8	73.3	63.6
13/14	Y9	92.6	86.4	74.6
15/16	Y11	89.0	90.0	76.0

Table 2 - Results for Question 4

Question 4(a) was generally found to be easier than 4(b). The asymmetrical figure of 4(b) required more consideration of direction, and caused difficulty especially with younger pupils. Older pupils seemed to make more errors with 4(c) than with the other parts. No frame of reference was available in this part and the orientation of the handle was often ignored and all the transformations taken as rotations. It had been anticipated that the horizontal reflection would be found easier than rotation but this was not found to be so in either 4(c) or in 4(a). The presence of a frame of reference seems to remove the demand for visualisation and enables pupils to transform a shape by applying certain rules.

Question 7 (see Figure 3) was amongst the hardest questions on the pattern recognition test and involved the recognition of rotation. D was the most common response. Perhaps some pupils thought the question involved shape matching and matched the first shape with D. One Y5 pupil, when interviewed, explained it differently:
"because that's where it starts again. The three shapes form the pattern and then it starts again, so D."

7.

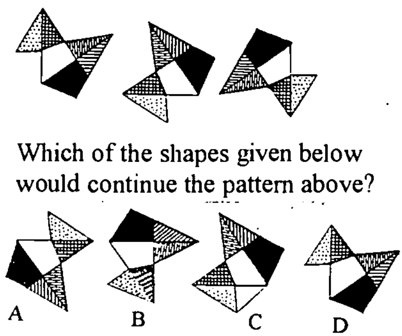


Figure 3

Individual interviews revealed that children were very good at seeing a different pattern from the one intended, as this explanation for the choice of C in question 7 reveals:

> *"The three shapes are like ducks. It must be C because it's not like a duck. It has an extra line."*

It was also clear from the interviews that some children recognised the rotation but still gave a 'wrong' answer.

It had been hoped that Question 10 (Figure 4) would be a fruitful question for comparing the difficulty of mental transformations. Each part of the question starts with a model shape which is transformed.

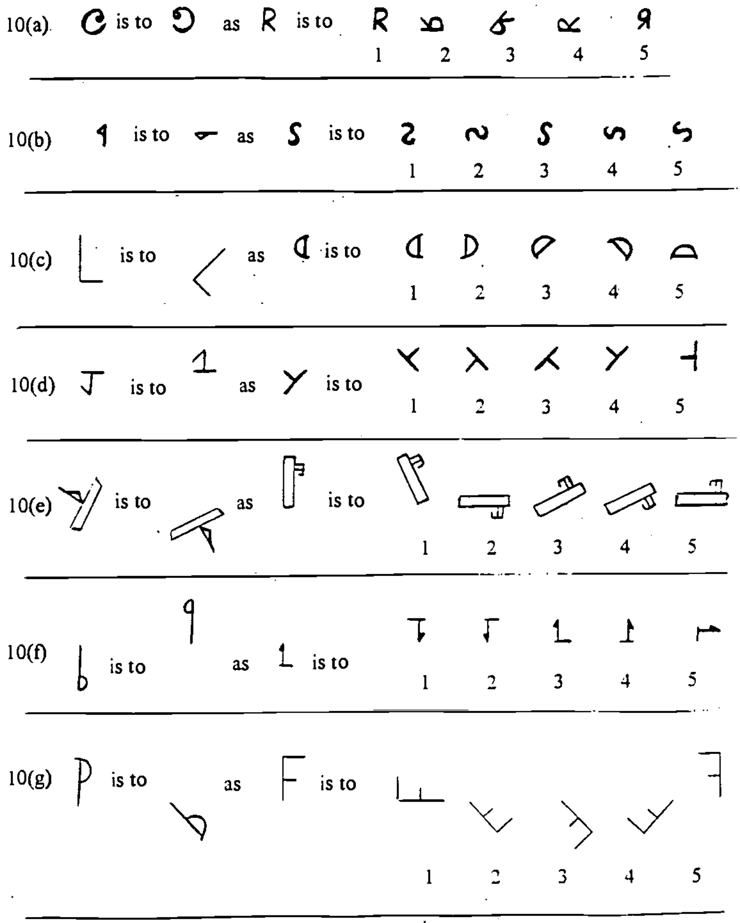


Figure 4.

Table 3 shows the mean scores for each part of the question at the different age levels.

	a	b	c	d	e	f	g
	Ref. |	Rot. ↖ 90°	Rot. ↗ 45°	Ref. —	Ref. /	Rot. 180°	Ref. /
Y5	0.70	0.31	0.43	0.40	0.04	0.33	0.48
Y7	0.76	0.52	0.65	0.53	0.07	0.47	0.70
Y9	0.97	0.63	0.86	0.69	0.08	0.69	0.75
Y11	0.96	0.64	0.90	0.58	0.00	0.62	0.72

Table 3 - Results for Question 10

Clearly (a) which involves reflection in a vertical axis was the easiest and (e), a reflection in a diagonal was the most difficult. Part (g) was also a reflection in a diagonal but this was generally found no more difficult than other parts [with the exception of (a)]. The difference between (e) and (g) was that in (e) the orientation of the shape to be transformed was different from the orientation of the model shape. It was no longer possible to use the orientation of a significant line of the transformed model shape as a guide (or frame of reference) in choosing the answer. An analysis of error responses revealed that 3 and 4 were the most popular choices for (e), supporting this hypothesis.

Parts (c), (f) and (g) all have a vertical line of the shape to be transformed matching the orientation of a line in the model shape, enabling the answer to be selected partly using this line as a frame of reference. It is possible that some pupils used this method without any mental transformation. The method would give 3 as a clear answer for (c), would give 2 or 3 for (g) but would only eliminate 5 for (f). Certainly the popular choice of the wrong answer 3 for (g) would support this and it could help to explain why (c), a rotation question, was not found as difficult as expected and why (f) was found more difficult. Pupils' explanations during individual interviews give added support; for example:

"That one's swapped around to that position so F will swap around too."

It had been expected that (d), a reflection in the horizontal might be an easy question (see Chipman and Mendelson, 1979) but this was not the case. The most popular wrong answer was 3 suggesting recognition that reflection in the horizontal was required but inaccuracy in the mental transformation. The relative complexity of the shape may well have had some effect too. Part (b), a rotation, was not expected to be easy and this was true particularly for younger pupils. It is possible that pupils imagined a vertical line through the S to compare with the example ("standing upright, moving onto its side" as one pupil explained it) or that rotation was recognised but inaccurately performed. The most common error 2 would be expected in both cases. Pupil explanations revealed that reflection in a diagonal line, parts (e) and (g), was not generally recognised. The transformation was seen as a combination of turning over and rotating:

"Gone round and then over"; "Move round and then flip over"; "Turn and then tilt"; "Reflection in what would be the x-axis and then gone anti-clockwise a bit".

From that point of view (e) becomes an unreasonable question. The size of rotation is considered important and none of the options shows the correct angle of rotation. Whether the pupils' explanations are a result of a transformation involved in their thinking is not clear.

Individual interviews revealed variation not only in understanding but also in mastery of mathematical language (see also Orton, 1993).

Mean scores were calculated, for the ten questions in the pattern recognition test, for each age group. Questions 5 and 6, were similar questions and taken together. Mean totals were also computed (see Table 4).

Q.	1	2	3	4	5+6	7	8	9	10	Total
Y5	4.53	1.56	1.56	3.73	14.04	0.3	0.59	0.3	2.68	29.25
Y7	4.66	1.71	1.70	5.13	14.68	0.56	0.60	0.44	3.70	33.12
Y9	4.83	1.91	1.92	5.82	16.56	0.68	0.70	0.55	4.63	37.58
Y11	4.84	1.82	1.94	5.86	13.46	0.64	0.58	0.56	4.42	34.12

Table 4 - Mean scores for the pattern recognition test

The mean totals show some clearly increasing values with age and these increases are significant except between Y9 and Y11 (where a large number of less able girls in the Y11 sample appear to have affected the results).

Three developmental stages are suggested in terms of the items of the pattern recognition test. Their content includes:

Stage 1: Copying a shape; detection of embedded pictures; simple completion of pattern; matching picture shapes; recognition of reflection in a 'vertical' axis; simple rotation and reflection completion tasks with a frame of reference.

Stage 2: Matching of embedded shapes; matching of simple geometric shapes in different orientations; more complex rotation and reflection tasks with a frame of reference.

Stage 3: Matching of more complex shapes in different orientations; more complex completion of pattern tasks including rotation; recognition of most reflection and rotation.

Conclusions

- There seems to be a clear body of pattern recognition 'knowledge' established by age 9/10 (Stage 1, above) of which teachers should take advantage.
- Individual interviews showed that pupils often recognise pattern but lack the vocabulary to fully explain what they perceive.
- Rotational symmetry may not always be more difficult than reflective symmetry (compare Chipman and Mendelson, 1975). The angle of rotation and complexity of the shape are important additional factors.

- It was not always clear from pupils' responses what mental transformations had taken place.
- There was considerable evidence of a frame of reference being used to simplify a transformation.

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EARLY REPRESENTATIONS OF TILING AREAS

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Students' early area concepts were investigated by an analysis of responses to a worksheet of items that involved visualising the tiling of given figures with different-shaped tiles. Students aged 7 to 10 attempted the items on three occasions. About half the students had difficulty; some who participated in spatial activities after the first occasion seemed to be more successful in determining the number of tiles. Students who drew the tilings were more successful on the trapezia items but drew too many tiles for larger shapes; some took account of limitations of their own drawings. An analysis of students' drawings suggested that there was development from beginning tiling from the sides and corners to an awareness of having no gaps, regular patterns, alignment of tiles, and consistency of tile size.

Introduction

Learning to count involves more than just reciting number words in the correct order. Similarly, an adequate understanding of *area* is expected to involve several ideas which students may gain from early experiences. If teachers are to overcome the commonly reported problem of students calculating areas without really knowing what they are calculating then we need a better appreciation of young students' understanding of *area*; some aspects are discussed in this research report.

Owens (1993b) found that students in Years 2 and 4 at school did not spontaneously refer to area when asked what was the same about all the different pentomino shapes and they also had difficulty in estimating the number of small triangles needed to cover the larger ones (Figure 1).

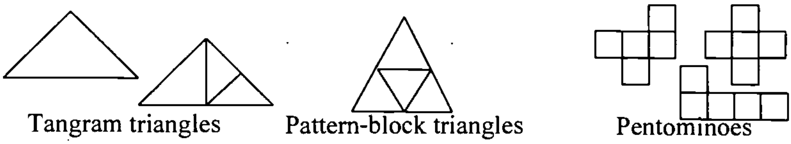


Figure 1. Shapes made during spatial activities.

Observations of pre-school children covering squares, rectangles, and triangles with smaller cut-out shapes have shown that students vary in their ability to choose shapes, in their persistence, and in their turning and flipping tactics (Mansfield & Scott, 1990). The most difficult shape to cover was an equilateral triangle with a point facing down. Familiarity with the shape to be covered seemed to be important. In a study by Wheatley and Cobb (1990), students were asked to cover a square selecting from a square, several triangles, and a parallelogram. Some students chose just the parallelogram. Wheatley and Cobb considered that this approach suggested students were matching lengths but students may have chosen the shape that

appeared to be largest. Other responses involved leaving gaps, especially on the sides, and overlapping pieces or the sides of the square.

Practice in tiling with blocks may not help an understanding of area as the materials may structure the tessellation (Doig, Cheeseman, & Lindsey, 1995; Outhred, 1994). The use of paper squares gave information about children's inadequate understandings of area as they were likely to leave gaps or overlap the paper tiles (Doig, Cheeseman, & Lindsey, 1995) It seems to be important that concrete experiences of covering areas also engage students' visual imagery and analysis and involve student-student and student-teacher interaction about the ideas needing development if mathematical concepts are to emerge (Hart & Sinkinson, 1988; Owens, 1993a).

Drawing may be one way of linking experiences with concrete materials to students' mental models of tessellations. Outhred (1994) found that many students had difficulties drawing tilings of squares, particularly for rectangles with large dimensions. Some students' drawings suggested that they did not understand what features of arrays were important to construct tessellations of squares. Owens (1992a) also found students had difficulties imagining tilings of squares, rectangles, and triangles. There were similarities between drawings produced by students in the studies undertaken by Owens and Outhred that warranted a further investigation of the data collected by Owens. These data comprised responses to items in which students were required to visualise tessellations of units for different figures.

Research Questions

The investigation into students' early development of the concept of area proceeded with the following questions:

1. How difficult do students find different tiling items?
2. What are the effects of (a) prior attempts at the items and (b) prior attempts plus a series of concrete spatial problem-solving activities on students' responses?
3. What are the characteristics of students' spontaneous drawings and what are the changes to these drawings over three attempts?
4. What are the effects of spontaneously drawing on students' responses?

Method

Tiling Items

The tiling items were developed as part of the test *Thinking about 2D Shapes* (Owens, 1992a). Students were introduced to the tiling items by the teacher discussing the idea of covering shapes with tiles without cutting or overlapping, illustrating with large cardboard cut-outs. Students then tried a practice example. Two different forms of the worksheet were used. Form S is shown in Figure 2; the same items were presented in a different order on the other form. A score of one was given for answering correctly whether the shape could be made with the given tile. A score of two was given if the student also gave the correct number of tiles.

The instructions for the worksheet items were as follows:

Suppose you had some **tiles** like the shape that is under the face. **Without** cutting or overlapping, could you fit them together to make the shape.

Circle Yes or No.

If Yes write the number of tiles you need.

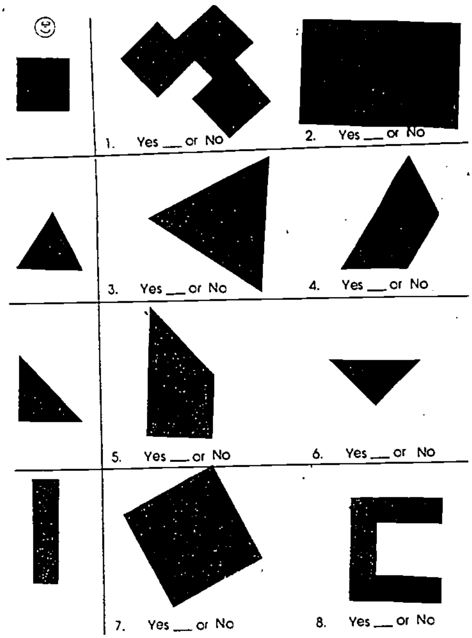


Figure 2. Reduced copy of A4 coloured worksheet on tiling (Form S).

Sample and Procedure

Each form of the test *Thinking about 2D Shapes* was attempted by two hundred students in Years 2 and 4 (aged 7 to 10) in five multicultural schools in Sydney (Owens, 1992a) in order to assess item difficulty and fit on an underlying trait.

Over 170 of these students in four of the schools participated in a further study of the effect of spatial activities on students' spatial thinking (Owens, 1992b). As part of this study, students attempted the worksheet of tiling items as part of the test on three occasions over a three-month period. The students were matched on school, year, class, and initial test scores and randomly allocated to either a group of non-participants or participants (either working individually or in small groups) in a series of spatial problem-solving activities based on tangrams, pattern blocks, pentominoes, and matchstick designs (Owens, 1995). These activities were not

specifically designed to train students to answer the tiling items, nor were students given feedback on the correct answers for the items.

Method of Analysis

The research questions were answered in turn by the following analyses of data:

1. A Rasch analysis (Andrich, 1988) of all items of the test was used to analyse the level of difficulty of the items. The percentages of participants and non-participants giving different responses to the items were examined. Eighteen students (nine in each Year across the ability range) were interviewed immediately after they completed the worksheet and the information from this immediate recall of how they were thinking illuminated the data from students' responses to the worksheet items.

2. The differences in percentages of participants and non-participants giving different responses to the items on the first and last attempts were used to show the effects of the worksheet plus spatial activities.

3. The drawings of all 62 students who drew (21 on more than one attempt) were considered for similarities and differences between students and over time.

4. The effect of drawing was investigated by comparing the results on each item at the last attempt for those who spontaneously drew and those who did not.

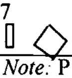
Results

1. The Rasch analysis of all test items indicated that all the items were testing the underlying trait called *2D Spatial Thinking* except Item 8, the C shaped item. This item was technically the hardest but some students said in interview or showed in drawings that the figure could be made with the tiles even though they were aware that the tiles would overlap. The order of increasing difficulty was: 1 & 6, 2, 7, 3 & 4, 5, 8 (Owens, 1992a). Differences between the two forms may be due to proximity of tile to figure and previous items attempted. On the last attempt, students found three staggered squares (Item 1) and a right-angled triangle in a turned position (6) to be easy; 83% and 80% of students respectively gave the correct number of tiles.

The items of particular interest are the other items requiring tessellations. The students' results for initial and final testing on these items are shown in Table 1. The percentages of incorrect responses indicate that the use of triangular tiles are found to be more difficult than the use of rectangular (including square) tiles, particularly when the shape to be covered was not triangular. On the first attempt, more than half the students thought the uncommon shapes, the trapezia, could not be made by tessellating the tiling unit and less than a third could give the correct number of tiles.

Although many students seemed to realise that the square, the non-square rectangle, and the equilateral triangle could be made by tessellating the unit, many students were unable to visualise and work out how many units would fit. For both the equilateral triangle and the square, many students wrote 3 or 5 tiles; for the rectangle, common answers were 8 and 9 but larger answers were also given suggesting some students disregarded size, especially if they tried to draw the tiling.

Table 1
Percentages of Students giving Different Responses on First and last Attempts

Item	Group	Incorrect Response 'No'			Incorrect Number of Tiles			Correct Number of Tiles		
Tile & Figure		First	**Last**	*Difference*	First	**Last**	*Difference*	First	**Last**	*Difference*
2	P	30	**21**	*-9*	42	**39**	*-3*	29	**40**	*11*
	NP	20	**17**	*-3*	44	**38**	*-6*	37	**45**	*8*
3	P	45	**35**	*-10*	27	**32**	*5*	28	**32**	*4*
	NP	42	**36**	*-6*	35	**36**	*1*	23	**29**	*6*
4	P	52	**36**	*-16*	19	**16**	*-3*	28	**48**	*20*
	NP	48	**49**	*1*	17	**17**	*0*	36	**34**	*-2*
5	P	57	**53**	*-4*	15	**17**	*2*	27	**30**	*3*
	NP	56	**50**	*-6*	17	**19**	*2*	27	**31**	*4*
7	P	33	**26**	*-7*	28	**25**	*-3*	39	**48**	*9*
	NP	35	**21**	*-14*	21	**37**	*16*	44	**42**	*-2*

Note: P ~ 130 students who participated in a series of spatial activities.
NP~ 42 students who did not participate in the series of spatial activities.

2. Overall, students' responses to the items seemed to improve with experience on the test (Table 1). Students who participated in the spatial activities improved slightly more than the non-participants, especially for Items 4 and 7, as indicated by the percentage differences in students giving the correct number of tiles. One reason that participants improved in giving the number of equilateral triangles for the trapezium might be their making of pattern-block shapes and enlargements (including an equilateral triangle and isosceles trapezium). In the interviews, several students who had been involved in the activities spontaneously noted that they had made the isosceles trapezium from equilateral triangles in class. The improvement in the use of square units for a rectangle might be associated with the test itself which included items involving designs made from squares. The spatial activities encouraged students to visualise grids of squares during the pentomino and matchstick-design activities and to build rectangles with squares from the pattern-block sets. Students had right-angled triangles in the tangram sets and most made a square, but they did not have enough triangles to make a trapezium; this may be a reason for the small improvement in Item 5. The activities used concrete materials and only some students drew copies of their designs whereas the worksheet required visualisation, some students using drawing.

The improvement in responses by participants is supported by the larger study (Owens, 1992b, 1993b) which showed that participants did improve significantly more on the delayed posttest (*Thinking about 2D Shapes)* than non-participants when pretest scores were taken into account as a covariate.

3. Students' drawings showed the following approaches: tiling around sides, filling from a corner, drawing individual tiles in rows (often sloping or getting smaller), representing rows by lines but marking off individual tiles (like a grid), and maintaining good size (see Figure 3 and Owens & Outhred, 1996). Outhred's (1994) study identified similar response categories for rectangular items. There was reasonable consistency in the way students drew on each occasion. Three case studies (Figure 3) are representative and show the development from individual tiles to rows and patterns The size of tiles improved in some instances and the use of mental imagery increased with incomplete drawing or pointing. Drawing difficulties led to uncertainties (e.g., Student 3).

Comments

Student 1 (Year 2)

First Attempt
Does not attempt rectangle
Individual tiles

Second Attempt
Attempts rectangle
Joins triangles
Notes triangle at top
Counts incorrectly for square

Third Attempt
Slopes squares from bottom
Rows of triangles
Relies on diagram

Student 2 (Year 2)

First Attempt
Points out squares
Rows of triangles
Trouble with size of equilateral triangle

Second Attempt
Draws grid for rectangle
Good size for equilateral triangle

Third Attempt
Needs only draw part of grid
Good equilateral triangles
No need to draw right-angled triangles

Student 3 (Year 4)

First Attempt
Reasonable size, individual but good line
Uses point for triangle
Halves square

Second Attempt
Concerned about gap
Imagines but uneasy about drawing triangles

Third Attempt
Concerned about gap
Hesitates with triangles

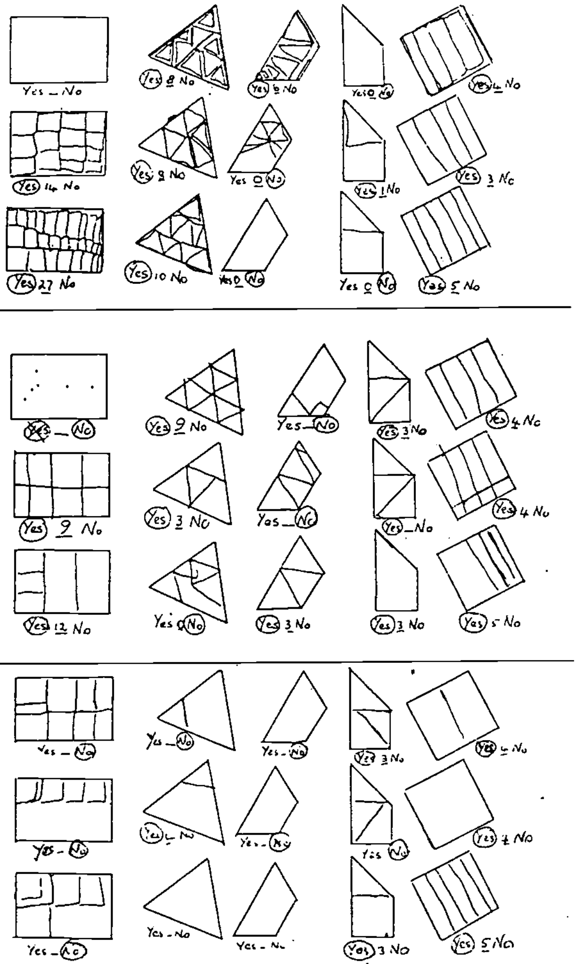
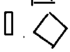


Figure 3. Examples of drawing over the three occasions.

4. The results presented in Table 2 indicate that drawing the tessellation seemed to make little difference to correct responses, except for Items 4 and 5 (the trapezia) for which students who drew seemed to be more successful. For larger items, students drew too many tiles. More students solved the problem mentally than drew solutions.

Table 2
Percentage of Responses (with Drawing or with No Drawing) on Last Attempt

Item		Number of Responses		Incorrect Response "No" (%)		Incorrect Number of Tiles (%)		Correct Number of Tiles (%)	
		ND - No Drawing	**D - Drawing**	ND	**D**	ND	**D**	ND	**D**
2		*148*	**32**	20	**19**	38	**41**	42	**41**
3		*154*	**27**	38	**22**	31	**48**	31	**30**
4		*154*	**18**	43	**11**	14	**15**	44	**67**
5		159	**20**	56	**25**	18	**15**	26	**60**
7		157	**23**	27	**4**	27	**61**	46	**35**

Conclusion

The development of the concept of area is complex and this study has provided insights into components of the area concept that make it so difficult for young students. This study suggests that students' responses were influenced by their cognisance of the following: (a) tile size, gaps, and overlaps; (b) features of tiles such as type of angle or part that matches the figure; (c) the relevant pattern for tessellating, and alignment of tiles, (d) the row and column structure of rectangles, and (e) the limitations of their own drawings. It seems, from this study, that students first consider covering an area with tiles by filling in from the sides and corners. Gradually they become more systematic by drawing in rows and more aware of features such as size and alignment of tiles.

Activities have the potential to improve area concepts. The use of non-square units in activities would highlight the importance of covering without gaps or overlap. Making tessellations with tiles of various shapes and drawing tessellations should assist students to recognise composite units and the patterns of lines and grids formed by tiling. Similar results have been found by Wheatley and Reynolds (1996) in considering students abstraction of units. The use of drawing to develop area concepts would seem worthwhile because drawings can be used to develop abstractions. However, drawing difficulties need to be discussed in order to prevent students thinking that shapes cannot be tessellated because of poor drawing skills. Students need investigative learning experiences that will engage them in noting features of shapes, in analysing tiling patterns, and in assessing their drawings

adequately. Such experiences will promote understanding of key attributes of tessellations, that is the units are all the same size and are aligned in a regular pattern. For area concepts, students have to consider units, composite units such as rows, and fractional units. Structured materials (tiles) might reduce non-investigative area tasks to counting tasks. There seems to be a need for greater emphasis on students who are learning about area to be able to transform shapes to other orientations, recognise and partition shapes, and identify key features of shapes, for example, matching parts such as right angles or lengths.

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WHAT CAN BE DONE TO OVERCOME THE MULTIPLICATIVE REVERSAL ERROR?

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Two studies are reported in this paper. The first investigates the rate of incidence of the multiplicative-reversal error among Australian high-school teachers and finds patterns similar to those found in the USA for teachers with "math" and "non-math" backgrounds. The second is an experimental study in which three different "worked example" methods were used in an attempt to reduce the incidence of the error among grade 8 and grade 9 Australian students. While all three methods, taken together, proved superior to "conventional" (the control), the "no-checking" method was the only method that was alone superior to the control condition.

BACKGROUND

The "multiplicative-reversal phenomenon" is an error commonly made in the process of translation of sentences into equations, or equations into sentences. The error is made in cases where two variables are related to each other such that one is a multiple of the other (*eg* $x = 2y$).

The classical "students and professors" problem (Kaput & Clement, 1979; Clement, Lochhead & Monk, 1981) has the following form: "Write an equation for the following statement: 'There are six times as many students as professors at this university.' Use S for the number of students and P for the number of professors." With this problem, 37% of 1st year engineering students gave incorrect answers, $^2/_3$ of the errors being reversals; for students doing a college algebra course but studying non-science majors, the error rate was 57% (Clement, Lochhead & Monk, 1981).

Whereas this problem tests equation formation, the "assemblers and solderers" problem [see Study 1, Item 2] tests equation interpretation. Using this problem, Lochhead (1980) found that 12% of physical-sciences unversity staff and 53% of "other" staff were incorrect, with 28% and 60% of high-school teachers incorrect. Using the same problem with MBA students, Cooper (1984) found that 31% of the "science/technology" graduates and 63% of the graduates in "other" areas exhibited reversal. In a study of final-year secondary-teacher trainees, (Cooper, 1986b) found that 27% of math/science teacher trainees and 48% of trainees for other subjects reversed in equation interpretation. For high-school students, the proportion of reversals for grade 9 was 80% on equation interpretation, falling with each extra year to 59% in grade 12, while for equation formation it was 92% for grade 9 falling to 73% for grade 12 (Cooper, 1986b).

The incidence of the multiplicative-reversal error is thus very high, even among graduates in the physical sciences. This points to the difficulty of translating between words and equations, and the fact that many people do not learn the skills

needed for this process of translation. Many writers (Lochhead, 1980; Rosnick & Clement, 1980; Clement, Lochhead & Monk, 1981) suggest that greater emphasis should be placed on developing translation skills.

Early studies suggested that the error is due to one of two factors: "word-order matching" or "syntactic translation" consisting of a left-to-right process of replacing the words with mathematical symbols (Rosnick & Clement, 1980; Clement, Lochhead & Monk; 1981; Wollman 1983), and "set matching" or "static comparison", consisting of matching a "set" of six students with a "set" of one professor, saying "six students for every professor" and then representing the set of six students by the expression 6S, using the equals sign as meaning "for every" or "is associated with", and using P to represent the "set" of one professor — so arriving at the reversed equation 6S=P. This would involve misconceptions about the use of letters in equations, and the use and meaning of the equals sign (Rosnick & Clement, 1980; Rosnick, 1981; Clement, Lochhead & Monk, 1981; Wollman,1983). The letters S and P are used as labels, suggesting that there is a "units or labels frame" and a "numerical variables equation frame" that may be retrieved from memory, each appropriate for different uses, the problem of reversal arising when the labels frame is inappropriately applied when the variables frame should be used (Davis, 1980). Cooper (1986a), however, found that replacing the letters S and P with x and y did not help, but that the inclusion of a multiplication sign between the numeral and the letter standing for the number of objects (eg, 6 and P) did reduce the proportion of reversals (Cooper, 1986a).

MacGregor and Stacey (1993) set out to "test the sufficiency of the published explanations" by designing test items to "eliminate the possibility of translation errors from all known causes", and found that, even with all these causes eliminated, there was still a high percentage of reversal errors. The suggestion that syntactic trans-lation is a common procedure was therefore not supported by the results. The theory put forward by MacGregor and Stacey is that, in the process of understanding the text, students construct a mental model in which both the quantities are viewed simultaneously. In the translation process, this information is accessed in a random order, not necessarily the order in which it occurred in the original sentence. They suggest that the reversal error occurs in the attempt to represent on paper these cogni-tive models of compared unequal quantities, and suggest that these models do not conform to algebraic notation because they do not centre around the concept of equal-ity. They point out that even though this model is not correctly translated into alge-braic notation, it may be adequate as a basis for reasoning and making inferences. Seeger (1990) found that students had actually comprehended the meaning and were able to solve problems, even though they were unable to use algebraic forms.

Crowley, Thomas and Tall (1994) explored the differences between proceptual thinkers (having a flexible use of symbolism in algebra) and procedural thinkers (who try to give the expressions a process meaning); these two approaches result in a

different order for the symbols in an equation. They found that the proportion of errors in translation from words to equation was greater for procedural thinkers.

Bloedy-Vinner (1995) found evidence for the previously suggested explanations of the reversal error, but also for a new explanation. She introduced the concept of the "analgebraic mode of thinking" (Bloedy-Vinner, 1994) and argued that translation errors occur when students attempt to translate natural-language predicates or relations which do not exist in algebraic language by erroneously enriching "their" algebraic language. Thus, by what they write they attempt to convey a "meaning" that makes sense to them, but does not conform to the normal mathematical meaning of the symbols. She suggests that "in 6S the origin S and the image 6S are conceived as one entity, the number of students, which is changing and becoming six times larger. This leads to the interpretation of 6S as the predicate 'S is six times larger' ". More evidence for this mode of thinking is provided in Bloedy-Vinner (1996). She concluded that errors were due to failure in three skill components: analysis of the problem and domain related knowledge, knowledge of algebraic language, and management of the solution.

What can be done to counter the error?

Rosnick and Clement (1980) tried different teaching strategies in taped interviews with nine students who had initially reversed in the students and professors problem. They followed this with a written teaching unit for six other students enrolled in a calculus course for engineers, scientists and mathematics majors. Their conclusion was that "though students' behavior for the most part was changed,..... their conceptual understanding of equation and variable remained....unchanged". Cooper (1984a), however, found with MBA students, of whom 49% initially made the reversal error, that by teaching about proportion and a constant of proportionality, and reference to problems, only about 4% reversed afterwards.

Davis (1980, p 192) suggested that an instructional program should make sure that the students "are aware of the likelihood of an incorrect choice, and form the habit of checking to see if they have in fact chosen correctly". Wollman (1983) notes that "experienced individuals consciously check their results" (p 170), and concludes that the inclusion of a check that the equation produced is correct is the "crucial step from a pedagogical point of view". In a further study, Wollman (1983) demonstrated the beneficial effect of including an explicit checking question in a set of items.

In summary, it appears that two approaches could possibly reduce the tendency to make the error. One is to pay far more attention to actually teaching students how to translate from sentences to equations and *vice versa*, ensuring a better understanding of algebraic language than appears to be common, as shown by studies to date. The other is explicitly to teach methods of checking until these become as automatic as the other aspects of problem solution, since conscious checking seems to be necessary even for experienced mathematicians.

STUDY 1 — Teachers

Responses of high-school teachers to one equation-formation item and one equation-interpretation item were examined, with the expectation that the results would follow patterns similar to those obtained by Lochhead (1980) for high-school teachers and university faculty, Cooper (1984a) for MBA students and Cooper, (1986b) for teacher trainees in which a smaller proportion of reversal errors was made by those trained in "scientific" disciplines than those trained in other disciplines.

The items were as follows:

1. For every Packard machine in a particular office, there are four Canon machines. Using the letter P to represent the number of Packard machines in the office, and the letter C to represent the number of Canon machines, write a simple equation corresponding to the above statement (Cooper, 1984).
2. Write *one* sentence in English that gives the same information as the following equation: A=7S. The letter A represents the number of assemblers in a factory; S is the number of solderers in the factory (Lochhead, 1980).

For consistency with the classification adopted by Lochhead (1980), biology and geology were counted as "non-mathematical" rather than as "math/science" subjects (mathematics, science and computing studies), since Lochhead's classification was labelled "physical sciences". In this analysis, non-reversal errors were disregarded, and an *a priori* comparison was made between the number correct and the number reversing for "non-mathematics" and "math/science" in both the equation-formation and equation-interpretation tasks (see summary in Table 1).

	equation formation		**equation interpretation**	
	non-math	math/science	non-math	math/science
sample size	37	16	44	16
proportion reversing	0.676	0.250	0.591	0.188
difference of proportions	**0.426**		**0.403**	

Table 1. Summary of proportions reversing

In each case, a large-sample normal approximation of the Fisher exact test was used. The values of the test statistic were z=2.860 (equation formation) and z=2.762 (equation interpretation). Both are significant at the 0.05 level on a one-tailed test, supporting the expectation that "non-math" teachers would tend to reverse more than "math/science" teachers. For the equation-interpretation task, the results were similar to those obtained by Lochhead (1980) using the same task with US teachers:

Australia:	*math/science/computing*	23.5% incorrect	*others*	59.1% incorrect
USA:	*physical sciences*	27.8% incorrect	*others*	59.8% incorrect

Wollman (1983) found that a proportion of those with the correct answer had initially reversed and then self-corrected. For the equation-formation task in the present study, three of the 12 correct in the math/science group and two of the 12 correct in the non-science group had initially reversed and then self-corrected. For the equation-interpretation task, one of the 13 correct in the math/science group and five of the 18 correct in the non-science group exhibited spontaneous self-correction. This indicates that teaching a checking method would probably have positive results.

STUDY 2 — High School Students

In the second study, multiplicative-reversal differences were examined among four ability-matched groups of 293 grade 8 and grade 9 students: a control group and three experimental groups. Each of the experimental groups was given worked examples of translation tasks but differed in that the **"no-checking" group** was taught no checking method, the **"comparison" group** was taught to check by asking which quantity is larger in the sentence and in the equation and to make sure they are the same, and the **"substitution" group** was taught to check by substituting numbers for the variables in the sentence and then to use these numbers in the equation, and make sure the equation then "works". It was expected that the proportion reversing in the experimental groups, would be smaller than in the **control** group, for which introductory material identical to that for the experimental groups was followed by word problems to solve which avoided comparison of quantities, so they were not practising problems that were likely to produce reversal. These problems were taken directly from the textbook used by grade 8 students.

A post-treatment test, identical to that administered to the teachers in Study 1, was administered at the end of the treatment. The proportion reversing in each group are shown for each ability level and for each type of task in Table 2.

	control	experimental			
		equ-check	compare	substitute	average
equation formation					
grade 8 lower	1.00	0.68	0.91	0.65	0.75
grade 8 higher	0.90	0.45	0.65	0.55	0.55
grade 9 lower	0.93	0.60	0.81	0.81	0.74
grade 9 higher	0.94	0.53	0.29	0.65	0.49
equation interpretation					
grade 8 lower	0.83	0.63	0.90	0.65	0.73
grade 8 higher	0.90	0.59	0.48	0.65	0.57
grade 9 lower	0.87	0.67	0.75	0.67	0.70
grade 9 higher	0.88	0.29	0.47	0.41	0.39

Table 2. Proportions of students reversing

These results were first analyzed to test the expectation that, at each level, the control group would make a greater proportion of reversal errors than the experimental groups combined. In this context, the proportions reversing were with reference to those who presented reversed or correct answers, "other" errors being disregarded. For each ability level, *a priori* tests of comparison were carried out between the control-group proportion and the average of the proportions for the three experimental groups in both the equation-formation and equation-interpretation tasks, using the test of homogeneity of binomial proportions (Marascuilo & McSweeney, 1967). Table 3 shows the value of X^2 for each task-level combination.

	equation formation X^2	equation interpretation X^2
grade 8 lower	10.64*	0.63
grade 8 higher	14.71*	13.05*
grade 9 lower	3.82	1.32
grade 9 higher	26.29*	11.54*

*$p<0.05$

Table 3. ***A priori*** **comparisons between control-group mean and average of experimental-group means**

For equation formation, the grade 8 higher, grade 8 lower and grade 9 higher ability levels showed significant differences between the control-group proportion and the mean of the three experimental group proportions, but for the grade 9 lower ability level the difference just failed to reach significance [X^2 =3.82, critical value =3.84].

For equation interpretation, both higher-ability levels showed significant differences between the control group and the mean of the three experimental groups, but the differences were not significant for either lower-ability level, although there was a trend in the expected direction.

A priori tests were carried out for each ability level, comparing control-group and "no-checking" group proportions for both the equation-formation and equation-interpretation tasks, using in each case a large-sample normal approximation to the Fisher exact test. For the equation-formation task, the proportion reversing in the "no-checking" group proportions is significantly smaller than that in the control group for each ability level [see Table 4], as expected. For the equation-interpretation task, the expectation was supported only at the higher-ability levels, although the results for both lower-ability levels indicate a trend in the expected direction. These results have the same pattern as those given in Tables 2 and 3.

	equation formation		equation interpretation	
	difference in proportions	z	difference in proportions	z
grade 8 lower	0.33	2.47*	0.20	1.29
grade 8 higher	0.45	3.06*	0.31	2.28*
grade 9 lower	0.36	2.18*	0.19	1.09
grade 9 higher	0.41	2.72*	0.57	3.26*

*p<0.05

Table 4. Results of *a priori* comparisons of control-group and "no-checking"-group proportions, using large-sample approximations of Fisher's exact test

Exploratory, *post hoc* tests were carried out for each ability level, comparing the control-group proportion and each of the "comparison" and "substitution" group proportions on both equation formation and equation interpretation, again using tests of homogeneity of binomial proportions (Marascuilo & McSweeney, 1967). There were significant differences from the control-group only for grade 8 lower "substitution"-group and grade 9 higher "comparison"-group for equation formation, and grade 8 higher "comparison"- and grade 9 higher "substitution"-group for equation interpretation. There were no significant differences between different experimental-group proportions at any level .

Conclusions

In the initial study, which examined the responses of high-school teachers to equation-formation and equation-interpretation item, the results were similar to those obtained by Lochhead (1980) for high-school teachers and university faculty, Cooper (1984a) for MBA students and Cooper, (1986b) for teacher trainees, in that persons with a math-science background tended to reverse less (but still substantially) than those with a "non-math" background. In common with Wollman (1983), it was found that a proportion of those with the correct answer had exhibited spontaneous self-correction.

In the experimental study with grade 8 and grade 9 students, the three experimental "worked example" methods ("no-checking", "comparison" and "substitution") were generally found to result in a significantly smaller proportion of reversals than the "conventional" method, although this result did not extend to lower-ability students in the case of equation interpretation, for whom there was a trend in the expected direction. Taken singly, only the "no-checking" method produced a significantly better result than the "conventional" method, while no consistent difference among the methods was apparent. This result demonstrates the effectiveness of worked examples in leading to an understanding of the translation process.

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